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COMPARISON PRINCIPLE AND CONSTRAINED RADIAL SYMMETRY FOR THE SUBDIFFUSIVE p-LAPLACIAN

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Abstract: A comparison principle for the subdiffusive p-Laplacian in a possibly nonsmooth and unbounded open set is proved. The result requires that the involved sub and supersolution are positive, and the ratio of the former to the latter is bounded. As an application, constrained radial symmetry for overdetermined problems is obtained. More precisely, both Dirichlet and Neumann conditions are prescribed on the boundary of a bounded open set, and the Neumann condition depends on the distance from the origin. The domain of the problem, unknown at the beginning, turns out to be a ball centered at the origin if a positive solution exists. Counterexamples are also discussed.

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1. Introduction

This paper deals with the *subdiffusive* p-Laplacian, i.e., with the equation

$$-\Delta_p u = f(x,u), \tag{1}$$

where the function f is dominated by the power u^{p-1} as $u \to +\infty$. As usual, Δ_p denotes the p-Laplace operator $\Delta_p u = \operatorname{div}(|Du|^{p-2}Du)$, with $p \in (1,+\infty)$. The first result is the following weak comparison principle between a subsolution u and a supersolution v. The composite functions $f(x,u(x))$ and $f(x,v(x))$ are assumed to be measurable, and the products $f(x,u(x))\varphi(x)$ and $f(x,v(x))\varphi(x)$ summable for all nonnegative $\varphi \in W_0^{1,p}(\Omega)$, so that the notions of weak sub and supersolution make sense.

Theorem 1.1 (Weak comparison principle)**.** *Let Ω be an open set in $\mathbb{R}^N$, $N \geq 2$, possibly non-smooth and unbounded. Let $u, v \in W^{1,p}(\Omega)$ be a weak subsolution and a weak supersolution, respectively, to equation* (1) *in Ω, where f is a real-valued function of the variables $x \in \Omega$ and $y \in (0, +\infty)$ having the properties indicated above.*

(i) In the case when

$$\text{(2)} \qquad \textit{the ratio } \frac{f(x,y)}{y^{p-1}} \textit{ is strictly decreasing in } y \in (0,+\infty) \quad \textit{for a.e. } x \in \Omega,$$

assume that all of the following conditions are satisfied: $u, v > 0$ a.e. in Ω, the ratio u/v belongs to $L^\infty(\Omega)$, and $(u-v)^+ \in W_0^{1,p}(\Omega)$. Then $u \leq v$ a.e. in Ω.

(ii) In the case when

$$\text{(3)} \qquad \textit{the ratio } \frac{f(x,y)}{y^{p-1}} \textit{ is non-increasing in } y \in (0,+\infty) \quad \textit{for a.e. } x \in \Omega,$$

assume that all of the following conditions are satisfied: $u > 0$ a.e. in Ω, $\operatorname*{ess\,inf}_{x\in\Omega} v(x) > 0$, the ratio u/v belongs to $L^\infty(\Omega)$, and $(u-v)^+ \in W_0^{1,p}(\Omega)$. Then $u \leq v$ a.e. in Ω.

Of course, there is no hope to obtain a comparison principle under assumption (3) alone: a counterexample is readily obtained with the equation $-\Delta u = \lambda_1 u$, where λ_1 is the first eigenvalue of the Dirichlet–Laplacian in Ω, assuming that Ω is a bounded, smooth domain. A similar situation also occurs with the first eigenfunctions of the p-Laplace operator: see, for instance, [**4, 22**] and the references therein.

It is well known, by contrast, that for linear equations the existence of a supersolution lying far above zero yields a maximum principle (the *generalized* maximum principle in [**26**, Section 5]).

Theorem 1.1 is proved in Section 2. The proof is based on the somehow *weird* inequality (8), which is in turn a consequence of the convexity of the function $|\xi|^p$, $p \in (1, +\infty)$, with respect to the variable $\xi \in \mathbb{R}^N$. Several related results are found in the literature: in particular, Díaz–Saá's inequality [**5, 10**] and Picone's identity [**2**]. An elegant uniqueness result exploiting the strict convexity of the associated functional $H_p(u)$ with respect to the function u^p (*hidden* convexity) is found in [**4**]. The fundamental comparison principle between a p-subharmonic and a p-superharmonic function is proved in [**19, 24**].

A weak comparison principle for the subdiffusive p-Laplacian involving a subsolution $u > 0$ and a supersolution $v > 0$ both belonging to $W_0^{1,p}(\Omega)$ is found in [**1**, Section 4.1] together with some extensions and applications. In particular, Theorem 1.1 in the present paper extends [**1**, Lemma 4.1] to the case of non-vanishing boundary values, at the cost of assuming the ratio u/v bounded.

In the case when $f(x, u)$ if monotone in u, the strong comparison principle for positive $u, v \in W_0^{1,p}(\Omega)$ is proved in [**7**]. If, instead, $f = f(u)$ and $p > (2N + 2)/(N + 2)$, a strong comparison principle is found in [**9**] (see also [**27**]). More precisely, assuming $u \leq v$ in a bounded smooth domain Ω, Theorem 1.4 of [**9**] states that either $u < v$ in Ω or $u \equiv v$. For the strong minimum principle, which is a strong comparison principle against the null function, see [**25, 30, 31**]. Comparison principles for more general equations under different assumptions are found, for instance, in [**6, 8, 29**].

Concerning the interior (respectively, global) $C^{1,\alpha}$-regularity of bounded weak solutions to (1), the reader may consult the classical paper [**29**] (respectively, [**23**]). However, regularity is not involved in the proof of Theorem 1.1.

As an application, the symmetry result stated in Theorem 1.2 is established. As before, the open set Ω is not required to be *a priori* smooth, but now it must be bounded and containing the origin. Moreover, the function f depends on x through $r = |x|$, and is denoted by $f(r, y)$. It is defined for all $r \in (0, +\infty)$ and $y \in [0, +\infty)$; takes on positive values whenever $y > 0$, and it is subject to conditions (H_1)–(H_3) and (4) listed below. Such conditions are inspired by [**10**], with some modifications. In particular, the set Ω will be treated as an unknown of the problem, and therefore we need the assumptions to hold independently of its size. We also require $f(r, y)$ to be monotone in r:

(H_1) For almost every $r \in (0, +\infty)$, the function $y \mapsto f(r, y)$ is a continuous function of $y \in [0, +\infty)$. Furthermore, for every fixed $y \in [0, +\infty)$, the function $r \mapsto f(r, y)$ is bounded and non-increasing.

(H_2) There exists a constant $\varepsilon_0 \in (0, p-1]$ such that for a.e. $r \in (0, +\infty)$ the function $y \mapsto f(r, y)/y^{p-1-\varepsilon_0}$ is monotone non-increasing with respect to $y \in (0, +\infty)$. The constant ε_0 is independent of r. Since

$$\frac{f(r, y)}{y^{p-1}} = \frac{f(r, y)}{y^{p-1-\varepsilon_0}} \frac{1}{y^{\varepsilon_0}},$$

it follows that the ratio in the left-hand side is strictly decreasing: this is assumption (H_2) in [**10**].

(H_3) For every bounded interval $(0, r_0)$ there exists a constant $C(r_0)$ such that $f(r, y) \le C(r_0)(y^{p-1} + 1)$ for a.e. $r \in (0, r_0)$ and for all $y \in [0, +\infty)$. We also assume that

$$\lim_{y \to 0^+} f(r, y)/y^{p-1} = +\infty \quad \text{for almost every } r \in (0, +\infty). \tag{4}$$

Provided that such assumptions are satisfied, we have:

Theorem 1.2 (Constrained radial symmetry)**.** *Let Ω be a bounded open set in $\mathbb{R}^N$, $N \ge 2$, containing the origin, and let $q(r)$ be a real-valued function of $r \in (0, +\infty)$ such that*

$$\text{the ratio } \frac{q(r)}{r^{\frac{p}{\varepsilon_0} - 1}} \text{ is strictly increasing in } r, \tag{5}$$

where ε_0 is the constant in (H_2). If there exists a weak solution $u \in C^1(\overline{\Omega})$, positive in Ω, to the overdetermined problem

$$\begin{cases} -\Delta_p u = f(|x|, u) & \text{in } \Omega, \\ u(x) = 0,\ |Du(x)| = q(|x|) & \text{for } x \in \partial\Omega, \end{cases}$$

then Ω is a ball centered at the origin.

Note that condition (H_2) is in competition with (5): we are led to search for a small ε_0 in order to facilitate (H_2), but not so small to prevent (5) from holding.

Let us compare the result to the celebrated achievements of Serrin [**28**] and Weinberger [**32**]. Remarkably, they showed that if there exists a solution u to $-\Delta u = 1$ in a sufficiently smooth domain Ω satisfying both $u = 0$ on $\partial\Omega$ and $|Du| = $ constant on $\partial\Omega$, then Ω is a ball.

Contrary to what one may expect, if we replace the boundary condition $|Du| = $ constant with $|Du(x)| = q(|x|)$ for $x \in \partial\Omega$, then the corresponding overdetermined problem may well be solvable even though the domain Ω is not a ball. A counterexample is readily obtained by letting $\Omega \subset \mathbb{R}^2$ be an ellipse in canonical position. Indeed, the solution $u_0 \in H_0^1(\Omega)$ to $-\Delta u = 1$ is symmetric with respect to both axes. Denoting by $a > b$ the semi-axes, for every $r \in [b, a]$ the intersection $S_r = \partial\Omega \cap \{|x| = r\}$ has the same kind of symmetry as u, and therefore we are legitimated to define $q(r) = |Du_0(x)|$ where x is any point in S_r. Thus, we end up with an overdetermined problem

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ u(x) = 0 \text{ and } |Du(x)| = q(|x|) & \text{for } x \in \partial\Omega, \end{cases} \tag{6}$$

which is solvable although Ω is not a disc. Another counterexample is constructed with an ellipse having one focus at the origin: this example, worked out in [**18**, Section 5], relies on the fact that the boundary of such a domain intersects any circle centered at O in at most two points, the other two being imaginary.

What are convenient properties of the function q, capable of implying that problem (6) is solvable only if Ω is a ball? A sufficient condition is indicated in [**18**, Theorem 3.1]: more precisely, if the ratio $q(r)/r$ is monotone non-decreasing, then Ω must be a ball *centered at the origin* in order that problem (6) is solvable. Such a condition has an optimality character: indeed, if we allow

$$\frac{d}{dr}\frac{q(r)}{r} \geq -\varepsilon_0$$

for some constant $\varepsilon_0 > 0$, then we may always construct a counterexample (the one just mentioned) no matter how small ε_0 is.

Theorem 1.2 shows that condition (5) suffices to obtain radial symmetry for problems governed by the subdiffusive p-Laplacian. Such kind of symmetry is told *constrained* because it occurs with respect to a prescribed point (the origin) due to the structure of the Neumann data. For *free* radial symmetry, i.e., symmetry for translation-invariant problems involving the p-Laplacian, see [**12**] and the references therein.

The novelty of Theorem 1.2 lies in the fact that here $f(r, y)$ is not assumed to have the special form $f(r, y) = y^m$ as in [**18**, Theorem 3.1], nor to be non-increasing in y as in [**16**, Theorem 1.1]. Instead, the weaker condition (H_2) is required. The proof is given in Section 4, after a preliminary investigation of the positive, radial solution $u_R \in W_0^{1,p}(B_R)$ to

$$-\Delta_p u = f(|x|, u). \tag{7}$$

More precisely, by means of a scaling technique, the rate of increase of the boundary gradient $|Du_R(x)|_{|x|=R}$ with respect to the radius R of the ball B_R is estimated (see Section 3). Then, using a comparison argument, we show that assumption (5) on the rate of increase of $q(r)$ prevents an either non-spherical, or non-centered-at-O open set Ω from having a solution to the given problem.

The method described above was outlined in [**18**]. It is a refinement of the one used in [**21**], which required more restrictive assumptions. Further applications are found in [**3, 14, 15, 16, 17, 18, 20**]. In particular, the paper [**21**] that started the series deals with domains with cavities: these are also considered in [**15, 16, 17, 20**]. Boundary conditions leading Ω to be an ellipsoid, instead of a ball, are considered in [**14, 17, 18, 21**]. More general equations have been taken into account: for instance, degenerate operators including the p-Laplacian are considered in [**16, 18, 20**]; the equation of constant mean (respectively, Gaussian) curvature has been investigated in [**18**]; the case when the right-hand side is Dirac's delta function is considered in [**3**].

2. Weak comparison principle

This section is devoted to the proof of Theorem 1.1. The result extends [**18**, Theorem 7.1], which applies to the special case $f(x,y) = a(x)y^{p-1}$, $a \in L^\infty(\Omega)$. As there, the argument is based on inequality (8) below. Other strategies may be taken in consideration: for instance, in the case (2) of strict monotonicity, the result would follow by extending Díaz–Saá's inequality [**10**] to the possibly unbounded, quasi-open set $\Omega^+ = \{x \in \Omega \mid u(x) > v(x)\}$. In fact, an extension of Díaz–Saá's inequality to the whole $\mathbb{R}^N$ is given in [**5**].

Lemma 2.1 (Weird inequality)**.** *Extend the function $|\xi|^{p-2}\xi$ at $\xi = 0$ by continuity in case $p \in (1,2]$. Then, for every $\xi, \eta \in \mathbb{R}^N$, $\lambda \in [0,1]$ and $p \in (1,+\infty)$ we have:*

$$\begin{aligned} |\xi|^p - p|\eta|^{p-2}\eta\cdot\xi + (p-1)|\eta|^p \\ \geq \lambda\Big(|\xi|^{p-2}\xi\cdot\eta + (p-2)|\eta|^p - (p-1)|\eta|^{p-2}\eta\cdot\xi\Big). \end{aligned} \tag{8}$$

Equality holds if and only if $\xi = \eta$.

Proof: The inequality appears in the proof of [**18**, Theorem 7.1]. By exploiting the convexity of $|\xi|^p$, a different proof is given here (and a typo in the last term is corrected). The tangent hyperplane to the graph of $|\xi|^p$ at some point $\xi = \eta$ is the graph of the function $\xi \mapsto p|\eta|^{p-2}\eta \cdot (\xi - \eta) + |\eta|^p$. Hence we have

$$|\xi|^p \geq p|\eta|^{p-2}\eta\cdot(\xi-\eta) + |\eta|^p \tag{9}$$

and (by strict convexity) equality holds if and only if $\xi = \eta$. Inequality (9) may be rewritten as $A(\xi,\eta) \geq 0$, where $A(\xi,\eta) = |\xi|^p - p|\eta|^{p-2}\eta \cdot \xi + (p-1)|\eta|^p$ denotes the left-hand side of (8). Consequently, it is immediate to recognize that

$$(1-\lambda)p\,A(\xi,\eta) + \lambda\Big(A(\xi,\eta) + A(\eta,\xi)\Big) \geq 0,$$

with equality if and only if $\xi = \eta$. By inserting the definition of $A(\xi,\eta)$ into the inequality above we obtain (8). □

We can now prove the weak comparison principle.

Proof of Theorem 1.1: Most of the argument is common to both claims. Differences are pointed out at the end of the proof. Since u is a subsolution to (1), we have

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, dx \leq \int_\Omega f(x, u(x))\varphi(x) \, dx$$

for all non-negative $\varphi \in W_0^{1,p}(\Omega)$. The function $\varphi = (u - v)^+$ is an admissible test function by assumption, and vanishes outside the set $\Omega^+ = \{x \in \Omega \mid u(x) > v(x)\}$. Hence we have

$$\int_{\Omega^+} |Du|^{p-2} Du \cdot D(u - v)^+ \, dx \leq \int_{\Omega^+} f(x, u)(u - v)^+ \, dx.$$

In the case (3) of mild monotonicity, and *a fortiori* if (2) holds, we may write

$$\int_{\Omega^+} |Du|^{p-2} Du \cdot D(u-v)^+ \, dx \leq \int_{\Omega^+} f(x, v) \left(\frac{u}{v}\right)^{p-1} (u-v)^+ \, dx. \tag{10}$$

More precisely, if strict monotonicity (2) holds, and if the set Ω^+ has a positive measure, then the preceding inequality is also strict: this fact will be used in the conclusion. To proceed in the estimate, we are led to use

$$\psi = \left(\frac{u}{v}\right)^{p-1} (u - v)^+ \tag{11}$$

as a test function in

$$\int_\Omega f(x, v(x))\psi(x) \, dx \leq \int_\Omega |Dv|^{p-2} Dv \cdot D\psi \, dx. \tag{12}$$

Indeed, with such a ψ the left-hand side of (12) coincides with the right-hand side of (10). The definition (11) is well posed because $v > 0$ a.e. in Ω by assumption. Let us check that ψ is an admissible test function. Clearly, $\psi \in L^p(\Omega)$ because the ratio $\frac{u}{v}$ is bounded by assumption. To see that $\psi \in W_0^{1,p}(\Omega)$, and its gradient is given by

$$D\psi = (p-1)\left(\tfrac{u}{v}\right)^{p-2} \left(\tfrac{u}{v} - 1\right)^+ \left(Du - \tfrac{u}{v} Dv\right) + \left(\tfrac{u}{v}\right)^{p-1} D(u - v)^+, \tag{13}$$

recall that weak differentiability is (up to the choice of a suitable representative) equivalent to absolute continuity along almost all lines parallel to the coordinate axes, together with local summability of the partial derivatives with respect to the N-dimensional Lebesgue measure: see [**13**, Section 7.3 and Problem 7.8] for details. Concerning the first term in the right-hand side of (13), note that the singularity of t^{p-2} at $t = 0$ when $p \in (1, 2)$ plays no role because when $t = \frac{u}{v} < 1$ the factor $(\frac{u}{v} - 1)^+$ vanishes. Using the function ψ defined in (11) as a test function in the inequality (12), we may estimate the right-hand side of (10) and find

$$\int_{\Omega^+} \Big(|Du|^{p-2} Du \cdot D(u-v)^+ - |Dv|^{p-2} Dv \cdot D\psi \Big)\, dx \le 0.$$

The remainder of the proof consists in proving that the function under the sign of integral is non-negative, hence it must vanish almost everywhere. Now the weird inequality (8) comes into play. Define ξ, η by letting $\xi = \frac{Du}{u}$ and $\eta = \frac{Dv}{v}$, and observe that the ratio $\lambda(x) = v(x)/u(x)$ belongs to the interval $(0,1)$ for every $x \in \Omega^+$ by definition. With this notation, and taking (13) into account, the preceding inequality is shortened as follows:

$$\int_{\Omega^+} u^p B(\xi, \eta, \lambda)\, dx \le 0,$$

where $B(\xi, \eta, \lambda)$ is given by

$$\begin{aligned} B(\xi,\eta,\lambda) = {} & |\xi|^p - p|\eta|^{p-2}\eta \cdot \xi + (p-1)|\eta|^p \\ & - \lambda\Big(|\xi|^{p-2}\xi \cdot \eta + (p-2)|\eta|^p - (p-1)|\eta|^{p-2}\eta \cdot \xi \Big). \end{aligned}$$

By (8) it follows that $\xi = \eta$ a.e. in Ω^+, and an equality sign holds in (10). Thus, if strict monotonicity (2) holds, we deduce that $|\Omega^+| = 0$ and Claim (i) follows. If, instead, v keeps far from zero, then the ratio $\rho = (u-v)^+/v$ belongs to $W_0^{1,p}(\Omega)$. The equality $\xi = \eta$ a.e. in Ω^+ implies $D\rho = 0$ a.e. in the whole set Ω, hence $\rho = 0$ a.e. in Ω. The last equality proves Claim (ii). $\square$

Uniqueness results may be derived from Theorem 1.1. For instance, in the case (3) of mild monotonicity, we have:

Corollary 2.2 (Uniqueness). *Let Ω be an open set in $\mathbb{R}^N$, $N \ge 2$, possibly non-smooth and unbounded, and let $u_1, u_2 \in W^{1,p}(\Omega)$ be two solutions to*

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega, \\ u - g \in W_0^{1,p}(\Omega), \end{cases}$$

where the function f satisfies (3), and g is a prescribed function in $W^{1,p}(\Omega)$. If there exist two positive constants $c_1, c_2 > 0$ such that $c_1 < u_i(x) < c_1$ for $i = 1,2$ and for a.e. $x \in \Omega$, then $u_1 = u_2$ a.e. in Ω.

3. Radial solutions

As mentioned in the introduction, the proof of Theorem 1.2 relies on a preliminary investigation of the radial case. From now on, assumptions (H_1)–(H_3) and (4) are in effect. We make use of the known existence and uniqueness results [**4, 10**] for bounded, smooth domains under homogeneous boundary conditions in order to ensure that equation (7) has a unique, radial, positive solution $u_R \in W_0^{1,p}(B_R)$ in the ball B_R centered at the origin. Such a solution is also bounded, and belongs to the class $C^{1,\alpha}(\overline{B}_R)$ [**10, 11, 23, 29**].

The main purpose of the following lemma is to estimate the rate of increase of the boundary gradient $|Du_R(x)|_{|x|=R}$ when the radius R is let vary.

Lemma 3.1 (Rate of increase of the boundary gradient)**.** *For every given $R \in (0,+\infty)$, the gradient $Du_R(x)$ vanishes if and only if $x = 0$; when R is let vary, the product $R^{1-(p/\varepsilon_0)}|Du_R(x)|_{|x|=R}$ is monotone nonincreasing in R.*

Proof: Writing $u_R(x) = v(|x|)$ for a convenient function $v(r)$ of the real variable r, and integrating equation (7), we arrive at

$$|v'(r)|^{p-2}v'(r) = \frac{-1}{r^{N-1}} \int_0^r t^{N-1} f(t, v(t))\, dt \quad \text{for } r \in (0, R].$$

From the representation above we see that the gradient $Du_R(x)$ is continuous in x and does not vanish outside the origin, which proves the first claim.

In order to investigate the product $R^{1-(p/\varepsilon_0)}|Du_R(x)|_{|x|=R}$, we follow the scaling procedure used in [**16**, Lemma 4.2]. Consider $0 < R_1 < R_2$ and denote by u_{R_i} the positive solution to (7) in $W_0^{1,p}(B_{R_i})$, $i = 1, 2$. Define $t = R_2/R_1 > 1$. Let us check that the rescaled function $w(x) = t^{-p/\varepsilon_0}u_{R_2}(tx)$, which is defined in the ball B_{R_1} and vanishes on ∂B_{R_1}, is a subsolution to (7). By computation we have $\Delta_p w(x) = t^{1+(p-1)(1-(p/\varepsilon_0))}(\Delta_p u_{R_2})(tx)$ for $x \in B_{R_1}$. Using the equation for u_{R_2}, and the definition of $w(x)$, we may write

$$-\Delta_p w(x) = t^{1+(p-1)(1-(p/\varepsilon_0))} f(|tx|, t^{p/\varepsilon_0}w(x)). \tag{14}$$

Since assumption (H_2) holds with a positive constant ε_0, we have $t^{p/\varepsilon_0}w > w$ and we may write

$$\frac{f(|tx|, t^{p/\varepsilon_0}w(x))}{(t^{p/\varepsilon_0}w(x))^{p-1-\varepsilon_0}} \le \frac{f(|tx|, w(x))}{(w(x))^{p-1-\varepsilon_0}}.$$

By ruling out the term $w(x)$ in the denominators, we arrive at

$$f(|tx|, t^{p/\varepsilon_0} w(x)) \le t^{(p-1-\varepsilon_0)p/\varepsilon_0} f(|tx|, w(x)).$$

The power of t multiplying f in the last term cancels with the one in (14), and therefore we obtain $-\Delta_p w(x) \le f(|tx|, w(x))$. Finally, since $f(r, u)$ is non-increasing in r by assumption, we conclude that $-\Delta_p w(x) \le f(|x|, w(x))$, hence w is a subsolution to (7) in $W_0^{1,p}(B_{R_1})$, as claimed.

Let us compare $u_{R_1}(x)$ and $w(x)$. Since the gradients $Du_{R_1}(x)$ and $Dw(x) = t^{1-(p/\varepsilon_0)}(Du_{R_2})(tx)$ do not vanish along ∂B_{R_1}, the ratio w/u_{R_1} is bounded in B_{R_1} and Claim (i) of Theorem 1.1 applies. It follows that $w \le u_{R_1}$ in B_{R_1}, and since both functions vanish on ∂B_{R_1} we also have $|Dw(x)|_{|x|=R_1} \le |Du_{R_1}(x)|_{|x|=R_1}$. Now, recalling the expression of Dw given before, and since $t = R_2/R_1$, the last inequality becomes $R_2^{1-(p/\varepsilon_0)}|Du_{R_2}(x)|_{|x|=R_2} \le R_1^{1-(p/\varepsilon_0)}|Du_{R_1}(x)|_{|x|=R_1}$, which is what we intended to prove. □

Remark. If $f(r, y)$ is non-increasing in y then we may relax the monotonicity assumption of f in r as in [**16**, Lemma 4.2]. There, it is required that the product $r^{1+(p-1)\sigma} f(r, y)$ is non-increasing in r for some $\sigma \in (-\infty, 1]$, and it turns out that

$$R^{\sigma}|Du_R(x)|_{|x|=R} \text{ is non-increasing in } R. \tag{15}$$

The two results agree in the special case when $f(x, y)$ is non-increasing both in r and in y. Indeed, in such a case we may take $\varepsilon_0 = p - 1$ in the preceding lemma and get that $R^{-1/(p-1)}|Du_R(x)|_{|x|=R}$ is non-increasing in R, which corresponds to (15) with $\sigma = -1/(p-1)$.

4. Constrained radial symmetry

Combining the weak comparison principle and the radial estimate established in the preceding sections, we may finally prove Theorem 1.2.

Proof of Theorem 1.2: Let x_1 (respectively, x_2) be a point on the boundary $\partial\Omega$ that minimizes (resp. maximizes) the distance to the origin, and define $R_1 = |x_1| \le |x_2| = R_2$. The aim of the proof is to show that $R_1 = R_2$. Let us compare the solution u with the positive radial solutions $u_{R_i} \in W_0^{1,p}(B_{R_i})$, $i = 1, 2$, of equation (7). More precisely, let us check that

$$u_{R_1} \le u \text{ in } B_{R_1} \quad \text{and} \quad u \le u_{R_2} \text{ in } \Omega. \tag{16}$$

It suffices to verify that the assumptions of Theorem 1.1, Claim (i), are satisfied. Condition (2) holds true as a consequence of (H_2). Furthermore, $u > 0$ in Ω by assumption. It remains to check that the ratios u_{R_1}/u and u/u_{R_2} are bounded. To this end, recall that $u \in C^1(\overline{\Omega})$ by assumption, and $u_{R_i} \in C^{1,\alpha}(\overline{B}_{R_i})$, $i = 1, 2$ as mentioned at the beginning of Section 3. Furthermore, by Lemma 3.1 the gradient Du_{R_i} does not vanish along ∂B_{R_i}. Let us prove that Du does not vanish along $\partial\Omega$. Observe that the set Ω obviously satisfies the interior sphere condition at $x_1 \in \partial\Omega \cap \partial B_{R_1}$. Since $-\Delta_p u > 0$ in Ω, and since u is positive in Ω and vanishes at x_1, by Hopf's lemma we have $|Du(x_1)| > 0$, hence

$$q(R_1) > 0. \tag{17}$$

This and (5) imply $q(r) > 0$ for all $r \geq R_1$. Thus, Du does not vanish along $\partial\Omega$, as claimed, and therefore the ratio u_{R_1}/u is bounded in B_1, and u/u_{R_2} is bounded in Ω. Consequently, Claim (i) of Theorem 1.1 applies, and we arrive at (16). Taking into account that $u_{R_1}(x_1) = u(x_1)$ and $u(x_2) = u_{R_2}(x_2)$, inequalities (16) imply

$$|Du_{R_1}(x_1)| \leq |Du(x_1)| \quad \text{and} \quad |Du(x_2)| \leq |Du_{R_2}(x_2)|. \tag{18}$$

According to Lemma 3.1, the product $R^{1-(p/\varepsilon_0)}|Du_R(x)|_{|x|=R}$ is non-increasing with respect to the radius R, hence we may write $R_2^{1-(p/\varepsilon_0)}|Du_{R_2}(x_2)| \leq R_1^{1-(p/\varepsilon_0)}|Du_{R_1}(x_1)|$. By combining the last inequality with (18), we obtain

$$R_2^{1-(p/\varepsilon_0)}|Du(x_2)| \leq R_1^{1-(p/\varepsilon_0)}|Du(x_1)|,$$

from which we deduce $R_1^{1-(p/\varepsilon_0)}q(R_1) \leq R_2^{1-(p/\varepsilon_0)}q(R_2)$. By (5), we then have $R_1 = R_2$, and the conclusion follows. $\square$

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