

Publ. Mat. **60** (2016), 265–272

DOI: 10.5565/PUBLMAT_60116_10

SOME LOCAL PROPERTIES DEFINING \mathcal{T}_0 -GROUPS AND RELATED CLASSES OF GROUPS

A. BALLESTER-BOLINCHES, J. C. BEIDLEMAN,
R. ESTEBAN-ROMERO, AND M. F. RAGLAND

Abstract: We call G a $\text{Hall}_{\mathcal{X}}$ -group if there exists a normal nilpotent subgroup N of G for which G/N' is an \mathcal{X} -group. We call G a \mathcal{T}_0 -group provided $G/\Phi(G)$ is a \mathcal{T} -group, that is, one in which normality is a transitive relation. We present several new local classes of groups which locally define $\text{Hall}_{\mathcal{X}}$ -groups and \mathcal{T}_0 -groups where $\mathcal{X} \in \{\mathcal{T}, \mathcal{PT}, \mathcal{PST}\}$; the classes \mathcal{PT} and \mathcal{PST} denote, respectively, the classes of groups in which permutability and S-permutability are transitive relations.

2010 Mathematics Subject Classification: Primary: 20D10, 20D20, 20D35.

Key words: Subnormal subgroup, \mathcal{T} -group, \mathcal{PST} -group, finite solvable group.

1. Introduction

All groups considered will be finite. All unexplained notation or terminology can be found in [3] or [8].

There is a well-known theorem of Philip Hall which states that if a group G possesses a normal nilpotent subgroup N such that G/N' is nilpotent, then G is nilpotent. It has been the subject of several papers to consider what can be said about the class of groups G possessing a normal nilpotent subgroup N such that G/N' is an \mathcal{X} -group for various classes \mathcal{X} . In particular, the fourth author in [6], for the solvable case, answered this question when \mathcal{X} is any one of the classes \mathcal{T} , \mathcal{PT} , or \mathcal{PST} , that is, respectively, the classes where normality, permutability, and S-permutability are transitive relations. Let us call G a $\text{Hall}_{\mathcal{X}}$ -group if there exists a normal nilpotent subgroup N of G for which G/N' is an \mathcal{X} -group. Since the Fitting subgroup of a group G contains all normal nilpotent subgroups one can simply replace N in the definition just given with F , the Fitting subgroup of G .

It is a classical problem to determine the structure of a group G such that $G/\Phi(G)$ is an \mathcal{X} -group for various classes \mathcal{X} . For a class \mathcal{X} , let us call G an \mathcal{X}_0 -group if $G/\Phi(G)$ is an \mathcal{X} -group. The fourth author in [6] characterized the solvable \mathcal{X}_0 -groups when \mathcal{X} is one of the classes \mathcal{T} , \mathcal{PT} ,

or \mathcal{PST} . In particular, it was shown that within the class of solvable groups, the classes \mathcal{T}_0 , \mathcal{PT}_0 , and \mathcal{PST}_0 coincide.

In this paper, we turn to the concept of a “local” characterization theorem for solvable \mathcal{T}_0 -groups and Hall $_{\mathcal{X}}$ -groups for $\mathcal{X} \in \{\mathcal{T}, \mathcal{PT}, \mathcal{PST}\}$. That is, we are interested in characterizations in terms of the different prime divisors of the orders of such groups.

2. Preliminaries

The classes \mathcal{C}_p , \mathcal{X}_p , and \mathcal{Y}_p , and the theorems that follow, will be critical in giving the desired local characterization theorems.

Definition.

- (1) \mathcal{C}_p is the class of groups G for which each subgroup of a Sylow p -subgroup P of G is normal in the normalizer, $N_G(P)$, of P .
- (2) \mathcal{X}_p is the class of groups G for which each subgroup of a Sylow p -subgroup P of G is permutable in the normalizer, $N_G(P)$, of P .
- (3) \mathcal{Y}_p is the class of groups G where $H \leq S \leq P$ with P a Sylow p -subgroup of G , always implies H is S -permutable in the normalizer, $N_G(S)$, of S .

Theorem 1. *Let G be a group.*

- (1) (Robinson [7], see also [3, 2.2.2]) *G is a solvable \mathcal{T} -group if and only if G is a \mathcal{C}_p -group for all primes p .*
- (2) (Beidleman, Brewster, and Robinson [5], see also [3, 2.2.3]) *G is a solvable \mathcal{PT} -group if and only if G is an \mathcal{X}_p -group for all primes p .*
- (3) (Ballester-Bolinches and Esteban-Romero [2], see also [3, 2.2.9]) *G is a solvable \mathcal{PST} -group if and only if G is a \mathcal{Y}_p -group for all primes p .*

We will need some results on the classes \mathcal{C}_p , \mathcal{X}_p , and \mathcal{Y}_p to aid in our characterization theorems.

Lemma 1 (Ballester-Bolinches and Esteban-Romero [2], see also [3, 2.2.13]). *A group G is a \mathcal{Y}_p -group if and only if G is p -nilpotent or G is a \mathcal{C}_p -group with abelian Sylow p -subgroups.*

Lemma 2 (Ballester-Bolinches and Esteban-Romero [2], see also [3, 2.2.4]). *A group G is an \mathcal{X}_p -group (respectively, \mathcal{C}_p -group) if and only if G is a \mathcal{Y}_p -group and the Sylow p -subgroups of G are Iwasawa (respectively, Dedekind) groups.*

In the previous lemma, Iwasawa groups (respectively, Dedekind groups) are the groups for which every subgroup is permutable (respectively, normal).

Lemma 3. *Let G be a group with N a normal p' -subgroup of G and M a normal p -subgroup of G .*

- (1) *If G is a \mathcal{Y}_p -group, then G/M is a \mathcal{Y}_p -group.*
- (2) *If G is an \mathcal{X}_p -group, then G/M is an \mathcal{X}_p -group.*
- (3) *If G is a \mathcal{C}_p -group, then G/M is a \mathcal{C}_p -group.*
- (4) *G is a \mathcal{Y}_p -group if and only if G/N is a \mathcal{Y}_p -group.*
- (5) *G is an \mathcal{X}_p -group if and only if G/N is an \mathcal{X}_p -group.*
- (6) *G is a \mathcal{C}_p -group if and only if G/N is a \mathcal{C}_p -group.*

Proof: (1), (2), and (3) follow directly from the definitions of \mathcal{Y}_p , \mathcal{X}_p , and \mathcal{C}_p .

(4) This can be found in [4] as Lemma 6.

(5) This can be found in [1] as Lemma 4.1.

(6) Let G be a \mathcal{C}_p -group. By Lemma 2, this is equivalent to affirming that G is a \mathcal{Y}_p -group with Dedekind Sylow p -subgroups. Since the Sylow p -subgroups of G and G/N are isomorphic, by (4) we have that this is equivalent to stating that G/N is a \mathcal{Y}_p -group with Dedekind Sylow p -subgroups. By Lemma 2, this is to say that G/N is a \mathcal{C}_p -group. This completes the proof. \square

Remark. In fact, the classes \mathcal{C}_p , \mathcal{X}_p , and \mathcal{Y}_p are quotient closed in general, however, this is a much more difficult fact to deduce. For our purposes, quotient closure for normal p' -subgroups and p -subgroups is sufficient.

Lastly, we record some results on \mathcal{T}_0 -groups which will be needed.

Theorem 2.

- (1) (van der Waall and Fransman [9]) *The class of solvable \mathcal{T}_0 -groups is a quotient closed class of groups.*
- (2) (van der Waall and Fransman [9]) *A solvable \mathcal{T}_0 -group is supersolvable.*
- (3) (Ragland [6]) *A group G is a solvable Hall $_{\mathcal{PST}}$ -group if and only if G is a solvable \mathcal{T}_0 -group.*
- (4) (Ragland [6]) *A group G is a solvable \mathcal{PST}_0 -group if and only if G is a solvable \mathcal{T}_0 -group.*
- (5) (Ragland [6]) *A group G is a solvable \mathcal{T}_0 -group with abelian nilpotent residual if and only if G is a solvable \mathcal{PST} -group.*

3. Main results

In this section we provide local characterization theorems for the solvable \mathcal{T}_0 -groups, Hall \mathcal{T} -groups, Hall \mathcal{PT} -groups, and Hall \mathcal{PST} -groups. We will need the following definitions:

Definition. Let p be a prime. For a group G , let $\Phi(G)_p$ denote the Sylow p -subgroup of the Frattini subgroup of G . Let F'_p denote the derived subgroup of the Sylow p -subgroup of the Fitting subgroup, F , of G . Let L'_p denote the derived subgroup of a Sylow p -subgroup of the nilpotent residual, L , of G . If p does not divide the order of $\Phi(G)$ then let us denote by $\Phi(G)_p$ the identity subgroup. Likewise, $F'_p = 1$ if p does not divide the order of F' and $L'_p = 1$ if p does not divide the order of L' . Let \mathcal{Z} be one of the classes \mathcal{C}_p , \mathcal{X}_p , or \mathcal{Y}_p .

- (1) Define $\Phi_{\mathcal{Z}}$ to be the class of groups G for which $G/\Phi(G)_p$ is a \mathcal{Z} -group.
- (2) Define $\mathcal{F}_{\mathcal{Z}}$ to be the class of groups G for which G/F'_p is a \mathcal{Z} -group.
- (3) Define $\mathcal{L}_{\mathcal{Z}}$ to be the class of groups G for which L is p' -nilpotent and G/L'_p is a \mathcal{Z} -group.

Since the Frattini factor group of a solvable \mathcal{T}_0 -group is a \mathcal{T} -group, and \mathcal{T} -groups can be locally characterized using the class \mathcal{C}_p , it stands to reason that \mathcal{T}_0 -groups can be locally characterized using the class \mathcal{C}_p with certain conditions involving the Frattini subgroup. This is indeed the case, however, recall that $\mathcal{T}_0 = \mathcal{PT}_0 = \mathcal{PST}_0$ in the class of solvable groups. So it also stands to reason that one could replace \mathcal{C}_p with \mathcal{X}_p or \mathcal{Y}_p . In view of Theorem 2, part (3), one should be able to characterize the solvable \mathcal{T}_0 -groups using \mathcal{Y}_p with certain conditions placed upon the Fitting subgroup. The details can be found in the following theorem:

Theorem 3. *The following are equivalent for a group G .*

- (1) G is an $\mathcal{L}_{\mathcal{Y}_p}$ -group for all primes p .
- (2) G is an $\mathcal{F}_{\mathcal{Y}_p}$ -group for all primes p .
- (3) G is a solvable Hall \mathcal{PST} -group.
- (4) G is a $\Phi_{\mathcal{C}_p}$ -group for all primes p .
- (5) G is a $\Phi_{\mathcal{X}_p}$ -group for all primes p .
- (6) G is a $\Phi_{\mathcal{Y}_p}$ -group for all primes p .
- (7) G is a solvable \mathcal{T}_0 -group.

Proof: Throughout the proof let us denote the nilpotent residual of G by L and the Fitting subgroup of G by F . Also, for any group H , H_p will denote a Sylow p -subgroup of H .

(1) \Rightarrow (2) If G is an $\mathcal{L}_{\mathcal{Y}_p}$ -group for all primes p then L is p' -nilpotent for all primes p and hence nilpotent. Thus $L \leq F$ so that for any prime p we have $L'_p \leq F'_p$. Now since G/L'_p is a \mathcal{Y}_p -group for all primes p , we have, by Lemma 3, part (1), that $(G/L'_p)/(F'_p/L'_p) \simeq G/F'_p$ is a \mathcal{Y}_p -group for all primes p .

(2) \Rightarrow (3) If G is an $\mathcal{F}_{\mathcal{Y}_p}$ -group for all primes p , then G/F'_p is a \mathcal{Y}_p -group for all primes p . Using Lemma 3, part (4), we have $(G/F'_p)/(F'/F'_p) \simeq G/F'$ is a \mathcal{Y}_p -group for all primes p . Hence by Theorem 1, part (3), we have that G/F' is a solvable \mathcal{PST} -group so that G is a solvable Hall $_{\mathcal{PST}}$ -group.

(3) \Rightarrow (4) If G is a solvable Hall $_{\mathcal{PST}}$ -group, then, by Theorem 2, part (3), we have that G is a solvable \mathcal{T}_0 -group. Let $\bar{G} = G/\Phi(G)_p$. Then \bar{G} is a solvable \mathcal{T}_0 -group by Theorem 2, part (1). By induction, if $\Phi(G)_p \neq 1$, we have $\bar{G}/\Phi(\bar{G})_p$ is a \mathcal{C}_p -group. But $\Phi(\bar{G})_p = 1$ and so \bar{G} is a \mathcal{C}_p -group. If $\Phi(G)_p = 1$ then one only needs G itself to be a \mathcal{C}_p group. Now $G/\Phi(G)$ is a \mathcal{T} -group and hence a \mathcal{C}_p -group by Theorem 1, part (1). Lemma 3, part (6), gives us that G is a \mathcal{C}_p -group since $\Phi(G)$ is a p' -group.

Both (4) \Rightarrow (5) and (5) \Rightarrow (6) follow directly from the definitions of \mathcal{C}_p , \mathcal{X}_p , and \mathcal{Y}_p .

(6) \Rightarrow (7) If G is a $\Phi_{\mathcal{Y}_p}$ -group for all primes p , then $G/\Phi(G)_p$ is a \mathcal{Y}_p -group for all primes p . Since $\Phi(G)/\Phi(G)_p$ is a p' -group for all primes p , we have, by Lemma 3, part (4), that $(G/\Phi(G)_p)/(\Phi(G)/\Phi(G)_p) \simeq G/\Phi(G)$ is a \mathcal{Y}_p -group for all primes p . Thus $G/\Phi(G)$ is a \mathcal{PST} -group by Theorem 1, part (3). Hence G is a solvable \mathcal{PST}_0 -group and thus a solvable \mathcal{T}_0 -group by Theorem 2, part (4).

(7) \Rightarrow (1) If G is a solvable \mathcal{T}_0 -group, then G is supersolvable by Theorem 2, part (2). Thus G' is nilpotent in which case so is L . Hence L is p' -nilpotent for all primes p . Note that if L is abelian then, by Theorem 2, part (5), G is a \mathcal{PST} -group and hence a \mathcal{Y}_p -group for all primes p by Theorem 1, part (3). From Lemma 3, part (1), it would follow that G is a $\mathcal{L}_{\mathcal{Y}_p}$ -group for all primes p if L is abelian. So let us assume L is not abelian.

Let $\bar{G} = G/L'_p$ and let \bar{L} be the nilpotent residual of \bar{G} . Now \bar{G} is a solvable \mathcal{T}_0 -group by Theorem 2, part (1). By induction, if $L'_p \neq 1$, we have \bar{G}/\bar{L}'_p is a \mathcal{Y}_p -group. But $\bar{L}'_p = 1$ and so \bar{G} is a \mathcal{Y}_p -group.

Assume that $L'_p = 1$. Since L is not abelian there exists a prime $q \neq p$ for which $L'_q \neq 1$. Now let \bar{G} denote G/L'_q with \bar{L} the nilpotent residual of \bar{G} and note \bar{G} is a \mathcal{T}_0 -group by Theorem 2, part (1). So, by induction, \bar{G}/\bar{L}'_p is a \mathcal{Y}_p -group. Thus \bar{G} is a \mathcal{Y}_p -group and hence G is a \mathcal{Y}_p -group by Lemma 3, part (4). It now follows that G is a $\mathcal{L}_{\mathcal{Y}_p}$ -group for all primes p completing the proof. \square

The classes of solvable Hall \mathcal{T} -groups and Hall \mathcal{PT} -groups admit similar local characterizations which we provide now.

Theorem 4. *The following are equivalent for a group G .*

- (1) G is a $\mathcal{F}_{\mathcal{X}_p}$ -group for all primes p .
- (2) G is a solvable Hall \mathcal{PT} -group.

Proof: (1) \Rightarrow (2) The proof is similar to the proof of (2) \Rightarrow (3) from Theorem 3. One only needs to use invoke the appropriate results corresponding to \mathcal{X}_p from Lemma 3 and Theorem 1.

(2) \Rightarrow (1) If G is a solvable Hall \mathcal{PT} -group, then it is a solvable Hall \mathcal{PST} -group and hence by Theorem 3 we have G/F'_p is a \mathcal{Y}_p -group for all primes p .

Now G/F' a solvable \mathcal{PT} -group and thus an \mathcal{X}_p -group for all primes p by Theorem 1, part (2). Hence the Sylow subgroups of G/F' are Iwasawa. Let P denote a Sylow p -subgroup of G . Then $PF'/F' \simeq P/(P \cap F') = P/F'_p$ and thus P/F'_p is an Iwasawa group.

Now one can conclude from Theorem 2 that G/F'_p is an \mathcal{X}_p -group for all primes p so that G is a $\mathcal{F}_{\mathcal{X}_p}$ -group for all primes p . \square

Theorem 5. *The following are equivalent for a group G .*

- (1) G is a $\mathcal{F}_{\mathcal{E}_p}$ -group for all primes p .
- (2) G is a solvable Hall \mathcal{T} -group.

Proof: Using the appropriate theorems and lemmas, the proof is quite similar to that of Theorem 4 and so we omit the proof. \square

The next theorem shows that the difference between solvable \mathcal{T}_0 -groups and Hall \mathcal{PT} -groups (Hall \mathcal{T} -groups) amounts to the Sylow structure of G/F'_p for all primes p .

Theorem 6. *Let G be a solvable group with $F = \text{Fit}(G)$. Then G is a \mathcal{T}_0 -group where the Sylow p -subgroups of G/F'_p are Iwasawa (Dedekind) groups for all primes p if and only if G is a Hall \mathcal{PT} -group (Hall \mathcal{T} -group).*

Proof: Suppose G is a \mathcal{T}_0 -group where the Sylow p -subgroups of G/F'_p are Iwasawa (Dedekind) groups. By Theorem 3, we have G/F'_p is a

\mathcal{Y}_p -group for all primes p . By Lemma 2, we must have G/F'_p is an \mathcal{X}_p -group (\mathcal{C}_p -group) for all primes p . We can now deduce from Theorem 4, part (2), that G is a Hall \mathcal{PT} -group (Hall \mathcal{T} -group).

Suppose G is a Hall \mathcal{PT} -group (Hall \mathcal{T} -group). Then Theorem 4, part (5), says that G/F'_p is an \mathcal{X}_p -group (\mathcal{C}_p -group) for all primes p . Using Lemma 2, we can deduce that G/F'_p is a \mathcal{Y}_p -group for all primes p and that the Sylow p -subgroups of G/F'_p are Iwasawa (Dedekind) groups. That G is a \mathcal{T}_0 -group follows from Theorem 3. \square

Acknowledgments

The first and the third authors have been supported by the research project MTM2010-19938-C03-01 from the *Ministerio de Ciencia e Innovación*, Spain. The first author has also been supported by the National Natural Science Foundation of China (Grant no. 11271085).

References

- [1] M. ASAAD, Finite groups in which normality or quasinormality is transitive, *Arch. Math. (Basel)* **83(4)** (2004), 289–296. DOI: 10.1007/s00013-004-1065-4.
- [2] A. BALLESTER-BOLINCHES AND R. ESTEBAN-ROMERO, Sylow permutable subnormal subgroups of finite groups, *J. Algebra* **251(2)** (2002), 727–738. DOI: 10.1006/jabr.2001.9138.
- [3] A. BALLESTER-BOLINCHES, R. ESTEBAN-ROMERO, AND M. ASAAD, “*Products of finite groups*”, de Gruyter Expositions in Mathematics **53**, Walter de Gruyter GmbH & Co. KG, Berlin, 2010. DOI: 10.1515/9783110220612.
- [4] A. BALLESTER-BOLINCHES, R. ESTEBAN-ROMERO, AND M. RAGLAND, A note on finite \mathcal{PST} -groups, *J. Group Theory* **10(2)** (2007), 205–210. DOI: 10.1515/JGT.2007.016.
- [5] J. C. BEIDLEMAN, B. BREWSTER, AND D. J. S. ROBINSON, Criteria for permutability to be transitive in finite groups, *J. Algebra* **222(2)** (1999), 400–412. DOI: 10.1006/jabr.1998.7964.
- [6] M. F. RAGLAND, Generalizations of groups in which normality is transitive, *Comm. Algebra* **35(10)** (2007), 3242–3252. DOI: 10.1080/00914030701410302.
- [7] D. J. S. ROBINSON, A note on finite groups in which normality is transitive, *Proc. Amer. Math. Soc.* **19** (1968), 933–937. DOI: 10.1090/S0002-9939-1968-0230808-9.
- [8] D. J. S. ROBINSON, “*A course in the theory of groups*”, Second edition, Graduate Texts in Mathematics **80**, Springer-Verlag, New York, 1996. DOI: 10.1007/978-1-4419-8594-1.

- [9] R. W. VAN DER WAALL AND A. FRANSMAN, On products of groups for which normality is a transitive relation on their Frattini factor groups, *Quaestiones Math.* **19(1–2)** (1996), 59–82. DOI: 10.1080/16073606.1996.9631826.

A. Ballester-Bolinches:
Department d'Àlgebra
Universitat de València
Dr. Moliner, 50
E-46100 Burjassot, València
Spain
E-mail address: Adolfo.Ballester@uv.es

J. C. Beidleman:
Department of Mathematics
University of Kentucky
Lexington, KY 40605-0027
USA
E-mail address: james.beidleman@uky.edu

R. Esteban-Romero:
Institut Universitari de Matemàtica Pura i Aplicada
Universitat Politècnica de València
Camí de Vera, s/n
E-46022 València
Spain
E-mail address: resteban@mat.upv.es

Current address:
Departament d'Àlgebra
Universitat de València
Dr. Moliner, 50
E-46100 Burjassot, València
Spain
E-mail address: Ramon.Esteban@uv.es

M. F. Ragland:
Department of Mathematics
Auburn University at Montgomery
P.O. Box 244023
Montgomery, AL 36124-4023
USA
E-mail address: mragland@aum.edu

Primera versió rebuda el 4 de juny de 2014,
darrera versió rebuda el 12 de gener de 2015.