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SMOOTH POTENTIALS WITH PRESCRIBED BOUNDARY BEHAVIOUR

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Abstract

This paper examines when it is possible to find a smooth potential on a C^1 domain D with prescribed normal derivatives at the boundary. It is shown that this is always possible when D is a Liapunov-Dini domain, and this restriction on D is essential. An application concerning C^1 superharmonic extension is given.

1. Results

Let D be a C^1 domain in Euclidean space $\mathbb{R}^n$, where $n \geq 2$. Thus D is bounded, ∂D can be represented locally as the graph of a C^1 function of $n-1$ variables, and there is a uniquely defined inward normal n_z at each point z of ∂D. We denote by $C^1(\overline{D})$ the collection of continuous functions on $\overline{D}$ which possess a continuous gradient on D that extends continuously to $\overline{D}$.

This paper is concerned with whether it is possible to find a smooth potential on D with prescribed normal derivatives on the boundary. More precisely, given a continuous function $g\colon \partial D \to (0,+\infty)$, we ask if there is a function $v \in C^1(\overline{D})$ which is superharmonic on D and satisfies the boundary conditions

$$v(z) = 0 \quad \text{and} \quad \frac{\partial v}{\partial n_z} = g(z) \qquad (z \in \partial D), \tag{1}$$

where $\partial/\partial n_z$ denotes differentiation in the direction of the inward normal at z. The answer will be given in Theorem 1 below.

By a *Dini function* we mean an increasing continuous function $\varepsilon\colon (0,+\infty) \to (0,+\infty)$ such that $\varepsilon(t)/t^\gamma$ is decreasing on $(0,1)$ for some $\gamma \in (0,1)$ and

$$\int_0^1 \frac{\varepsilon(t)}{t}\, dt < +\infty. \tag{2}$$

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A C^1 domain D is called a *Liapunov-Dini domain* (cf. [**11**]) if there is a Dini function ε such that the angle between the normals n_y and n_z at any two points $y, z \in \partial D$ does not exceed $\varepsilon(\|y-z\|)$. Examples include the $C^{1,\alpha}$-domains ($0 < \alpha < 1$), which correspond to the case where $\varepsilon(t) = t^\alpha$.

Theorem 1. *Let D be a Liapunov-Dini domain. Then, for each continuous function $g\colon \partial D \to (0, +\infty)$, there is a function $v \in C^1(\overline{D})$ which is superharmonic on D and satisfies* (1)*.*

The function v of Theorem 1 is certainly not unique: as will be clear from the proof it can be chosen to be harmonic on any predetermined open subset U of D which satisfies $\overline{U} \subset D$. We remark that Theorem 1 is related to work of Wallin [**10**] on the extension, in the form of potentials, of continuous functions from compact polar sets.

The example below shows the relevance of condition (2) to Theorem 1.

Example 1. Let $\varepsilon\colon [0, +\infty) \to [0, +\infty)$ be an increasing continuous function such that $\varepsilon(0) = 0$ and (2) fails to hold. (For example, we could choose $\varepsilon(t) = \{1 + \log^+(e/t)\}^{-1}$.) Further, let D be a C^1 domain such that

$$D \cap \{\|x\| < 1\} = \bigl\{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > -\psi(\|x'\|)\bigr\} \cap \{\|x\| < 1\}$$

where $\psi(t) = \int_0^t \varepsilon(s)\, ds$. Then the only function v in $C(\overline{D})$ which is superharmonic on D, valued 0 on ∂D and has a finite normal derivative at 0, is the zero function.

We give below an application of Theorem 1 to superharmonic extension.

Corollary 1. *Let D be a Liapunov-Dini domain (with Dini function ε) such that $\mathbb{R}^n \backslash \overline{D}$ is connected. Suppose that $u \in C^1(\overline{D})$, where $u\big|_D$ is superharmonic and, for each $z \in \partial D$, there is a linear polynomial L_z such that*

$$|u(x) - L_z(x)| \le \varepsilon(\|x-z\|)\, \|x-z\| \qquad (x \in \partial D). \tag{3}$$

Then there is a superharmonic function $\overline{u} \in C^1(\mathbb{R}^n)$ such that $\overline{u} = u$ on $\overline{D}$.

Corollary 1 is related to a question raised by Verdera, Mel'nikov and Paramonov [**9**] concerning C^1 extension of superharmonic functions. We do not know if condition (3) can be omitted.

We will establish Theorem 1, Example 1 and Corollary 1 in Sections 3–5 respectively, following some preliminary material in Section 2.

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2. Preliminaries

We write C_a for a positive constant, depending at most on a, not necessarily the same on any two occurrences, and assume, without loss of generality, that $0 \in D$. We write $\delta(x)$ for the distance of a point x from ∂D, denote the Green function for D by $G_D(\cdot,\cdot)$, and define

$$M(z,y) = \lim_{x\to z,\, x\in D} \frac{G_D(x,y)}{G_D(x,0)} \qquad (z \in \partial D;\ y \in D).$$

(This is the "Martin kernel" for D; see [**2**, Chapter 8].)

Lemma A. *Let D be a Liapunov-Dini domain. Then:*

(i) $G_D(x,0) \le C_D \delta(x) \|x\|^{1-n} \quad (x \in D)$;

(ii) $G_D(\cdot,y) \in C^1(\overline{D}\backslash\{y\}) \quad (y \in D)$;

(iii) *for each $y \in D$ the function $z \mapsto \dfrac{\partial}{\partial n_z} G_D(\cdot,y)$ is positive and continuous on ∂D;*

(iv) $M(x^*,x) \ge C_D\{\delta(x)\}^{1-n} \quad (x \in D)$, *where x^* is any point of ∂D satisfying $\|x-x^*\| = \delta(x)$.*

When $n \ge 3$ assertions (i)–(iii) above may be found in Theorems 2.3–2.5 of [**11**], and (iv) follows from an estimate on p. 28 of that paper. In two dimensions the lemma can be verified using a conformal mapping argument, even under somewhat weaker hypotheses on D (cf. [**7**, Theorem 3.5] for the case where D is simply connected).

The next result is a special case of Theorem 1 of [**1**]. As usual, $B(x,r)$ denotes the open ball of centre x and radius r in $\mathbb{R}^n$.

Lemma B (Boundary Harnack Principle)**.** *There are constants $R > 0$, $a_0 > 1$ and $c_0 > 1$, depending only on D, with the following property: if $z \in \partial D$ and $0 < r < R$, and h_1, h_2 are positive harmonic functions on $D \cap B(z,a_0 r)$ that vanish continuously on $\partial D \cap B(z,a_0 r)$, then*

$$\frac{h_1(x)}{h_2(x)} \le c_0 \frac{h_1(y)}{h_2(y)} \qquad (x,y \in D \cap \overline{B(z,r)}).$$

3. Proof of Theorem 1

For each $y \in D$ let $B_y = B(y,\delta(y)/2)$, and let

$$D(r) = \{x \in D : \delta(x) < r\} \qquad (r > 0).$$

We note that

$$\|x-y\| \geq \frac{\delta(x)}{3} \qquad (x \in D\backslash B_y), \tag{4}$$

for otherwise there exists $x \in D\backslash B_y$ such that

$$\|z-x\| - \|z-y\| < \frac{\delta(x)}{3} \qquad (z \in \mathbb{R}^n),$$

whence

$$\|z-y\| > \frac{2\delta(x)}{3} \qquad (z \in \partial D)$$

and we obtain the contradictory conclusion that

$$\delta(y) \geq \frac{2\delta(x)}{3} > 2\,\|x-y\|.$$

Now let R, a_0 and c_0 be as in Lemma B (we choose R small enough so that $2a_0R < \delta(0)$), and let $y \in D(R/2)$. We claim that

$$\frac{G_D(x,y)}{G_D(x,0)} \leq C_D M(x^*,y) \qquad (x \in D(R)\backslash B_y), \tag{5}$$

where x^* denotes any point of ∂D satisfying $\|x-x^*\| = \delta(x)$. To see this we define

$$\rho = \min\left\{\delta(x), \frac{\|x^*-y\|}{a_0}\right\},$$

whence $\rho < R$. The choice of ρ and R ensure that the functions $G_D(\cdot,y)$ and $G_D(\cdot,0)$ are harmonic on $D \cap B(x^*, a_0\rho)$, so we can apply the boundary Harnack principle to see that

$$\frac{G_D(z,y)}{G_D(z,0)} \leq c_0 M(x^*,y) \qquad (z \in D \cap \overline{B(x^*,\rho)}). \tag{6}$$

If $\delta(x) \leq \|x^*-y\|/a_0$, then $\rho = \delta(x)$ and the inequality in (5) clearly holds. It remains to consider the case of (5) where $\delta(x) > \|x^*-y\|/a_0$, and so

$$\|x-y\| > \frac{\|x^*-y\|}{3a_0} = \frac{\rho}{3}, \tag{7}$$

by (4). Let

$$z_1 \in D \cap B(x^*,\rho) \quad \text{and} \quad z_2 \in D \cap \partial B(y,\rho/3).$$

Then

$$\|z_1-z_2\| \leq \|z_1-x^*\| + \|x^*-y\| + \|y-z_2\| \leq (a_0+4/3)\rho$$

and

$$\|z_1-y\| \geq \|y-x^*\| - \|z_1-x^*\| > (a_0-1)\rho,$$

whence $z_1 \notin B(y, \rho/3)$ provided we arrange that $a_0 > 4/3$. We note that $\delta(z_1), \delta(z_2) \in (0, \rho(a_0 + 1/3)]$, so $z_1, z_2 \notin B(0, R)$ in view of our choice of R. Since D is C^1, we can join z_1 to z_2 by a curve γ in $D\setminus[B(y,\rho/3)\cup B(0,R)]$, of length at most $C_D\rho$. Further, we can choose $c_1 > 0$, depending only on D, such that, for each $z \in \gamma$, either

$$B(z, 2c_1\rho) \subset D\backslash\{0, y\}$$

or

$$B(z, c_1\rho) \subset B(z^*, 3c_1\rho) \quad \text{and} \quad 0, y \notin B(z^*, 3a_0c_1\rho).$$

Thus (6), together with repeated use of Harnack's inequalities and the boundary Harnack principle as appropriate, yields

$$\frac{G_D(z_2, y)}{G_D(z_2, 0)} \le C_D M(x^*, y) \qquad (z_2 \in D \cap \partial B(y, \rho/3)),$$

and it follows from the minimum principle that

$$C_D M(x^*, y) G_D(\cdot, 0) - G_D(\cdot, y) > 0 \quad \text{on} \quad D\backslash B(y, \rho/3).$$

The claim (5) now holds in view of (7).

Using (4) and a well known consequence of Harnack's inequalities (see [**2**, Corollary 1.4.2]), we observe that

$$\|\nabla_x G_D(x, y)\| \le \frac{3n}{\delta(x)} G_D(x, y) \qquad (x \in D\backslash B_y), \tag{8}$$

and hence

$$\|\nabla_x G_D(x, y)\| \le C_D \frac{G_D(x, y)}{G_D(x, 0)} \qquad (x \in D(R)\backslash B_y), \tag{9}$$

by Lemma A(i). Now let v_y denote the (Green) potential on D of normalized Lebesgue measure on B_y. By the mean value property of harmonic functions, (9) and then (5) we see that

$$\|\nabla_x v_y(x)\| = \|\nabla_x G_D(x, y)\| \le C_D M(x^*, y) \qquad (x \in D(R)\backslash B_y). \tag{10}$$

Further, if we define $v_y = 0$ on ∂D, then it follows from Lemma A(ii) (and [**2**, Theorem 4.5.3]) that $v_y \in C^1(\overline{D})$, and

$$\|\nabla_x v_y(x)\| \le C_n \{\delta(y)\}^{1-n} \qquad (x \in \partial B_y)$$

in view of (8). Since $\Delta v_y = -C_n\{\delta(y)\}^{-n}$ on B_y, the components of $\nabla_x v_y$ are harmonic there, and so

$$\|\nabla_x v_y(x)\| \le C_n \{\delta(y)\}^{1-n} \qquad (x \in B_y).$$

From Lemma A(iv) and Harnack's inequalities we see that

$$\|\nabla_x v_y(x)\| \le C_D M(x^*, x) \le C_D M(x^*, y) \qquad (x \in B_y).$$

Combining this with (10) we obtain

$$\|\nabla_x v_y(x)\| \le C_D M(x^*, y) \qquad (x \in D(R)), \tag{11}$$

whence

$$v_y(x) \le C_D \delta(x) M(x^*, y) \qquad (x \in D(R)). \tag{12}$$

Now let $g\colon \partial D \to (0, +\infty)$ be continuous. By Lemma A(iii) and a special case of Theorem 3 in [**6**] (cf. [**3**, Theorem 10]), there are sequences (y_k) in $D(R/2)$ and (a_k) in $[0, +\infty)$ such that

$$\frac{g(z)}{\frac{\partial}{\partial n_z} G_D(\cdot, 0)} = \sum_{k=1}^{\infty} a_k M(z, y_k) \qquad (z \in \partial D), \tag{13}$$

and the convergence is uniform on ∂D in view of Dini's theorem. It follows from (12) that the series $\sum a_k v_{y_k}$ converges (uniformly) on $\overline{D(R)}$, and hence on $\overline{D}$ by the maximum principle (each v_{y_k} is harmonic on $D \backslash \overline{D(R)}$). We denote the sum of this series by v. By (11) the series $\sum a_k \|\nabla v_{y_k}\|$ also converges uniformly on $\overline{D}$. It follows that $v \in C^1(\overline{D})$ and that

$$\begin{aligned} \frac{\partial v}{\partial n_z} &= \sum a_k \frac{\partial}{\partial n_z} v_{y_k} \\ &= \sum a_k \frac{\partial}{\partial n_z} G_D(\cdot, y_k) \\ &= \left\{ \sum a_k M(z, y_k) \right\} \frac{\partial}{\partial n_z} G_D(\cdot, 0) \\ &= g(z) \qquad\qquad \text{when } z \in \partial D, \end{aligned}$$

by (13). Thus (1) holds, since clearly $v = 0$ on ∂D in view of (12). Finally, each v_{y_k} is superharmonic on D, so the same is true of v. Theorem 1 is now proved. □

4. Details of Example 1

Let D be as stated in Example 1 and let $z_t = (0, \ldots, 0, t)$. The failure of (2) to hold implies that

$$\frac{G_D(z_t, z_{1/2})}{t} \to +\infty \qquad (t \to 0+) \tag{14}$$

(see [**4**, Corollary 4.3]; cf. [**8**, p. 377] when $n = 2$). Now suppose that $v \in C(\overline{D})$ and that v is superharmonic on D and valued 0 on ∂D. By the Riesz decomposition theorem v is of the form $v(x) = \int G_D(x, \cdot)\, d\mu$

on D for some measure μ. If $\mu \neq 0$, then Harnack's inequalities, applied to $G_D(x, \cdot)$, show that there are positive constants a, c such that

$$v(z_t) \geq cG_D(z_t, z_{1/2}) \qquad (0 < t < a),$$

and it follows from (14) that v does not have a finite normal derivative at 0.

5. Proof of Corollary 1

Let D and u be as in the statement of Corollary 1, and let $\Omega = \mathbb{R}^n \backslash \overline{D}$. By hypothesis Ω is connected. In view of condition (3) and [**11**, Theorem 2.4], the solution w to the Dirichlet problem in Ω, with boundary data u on ∂D and 0 at ∞, satisfies $w \in C^1(\overline{\Omega})$, where $w = u$ on ∂D.

If $n \geq 3$, then we define h_0 to be the harmonic measure of $\{\infty\}$ in Ω; if $n = 2$, then we define h_0 to be the Green function for $\Omega \cup \{\infty\}$ with pole at ∞. In either case we define $h_0 = 0$ on ∂D and note from Lemma A and the Kelvin transform that $h_0 \in C^1(\overline{\Omega})$ and $-\partial h_0/\partial n_z$ is a positive continuous function of z in ∂D. (We always use n_z to denote the inward normal at z relative to D.)

We now choose $a > 0$ large enough so that the continuous function

$$g(z) = \frac{\partial w}{\partial n_z} - \frac{\partial u}{\partial n_z} - a\frac{\partial h_0}{\partial n_z} \qquad (z \in \partial D) \tag{15}$$

is positive on ∂D. By Theorem 1 and inversion there is a function $v \in C^1(\overline{\Omega})$ such that $v\big|_\Omega$ is superharmonic on Ω and

$$v(z) = 0 \quad \text{and} \quad -\frac{\partial v}{\partial n_z} = g(z) \qquad (z \in \partial D). \tag{16}$$

For each $b \geq a$, let

$$u_b(x) = \begin{cases} u(x) & (x \in \overline{D}) \\ w(x) + v(x) - bh_0(x) & (x \in \Omega) \end{cases}.$$

Since $v - bh_0 = 0$ on ∂D, the functions u_b are continuous on $\mathbb{R}^n$. Further, by (16) and then (15),

$$\frac{\partial}{\partial n_z}(w + v - ah_0) = \frac{\partial w}{\partial n_z} - g(z) - a\frac{\partial h_0}{\partial n_z} = \frac{\partial u}{\partial n_z} \qquad (z \in \partial D),$$

so $u_a \in C^1(\mathbb{R}^n)$. It remains to establish the superharmonicity of u_a. Clearly, it will be enough to check the superharmonicity of u_b when $b > a$, and then let $b \to a+$. Further, since we know that u_b is superharmonic both on D and on Ω, we need only verify the superharmonic mean value inequality at points of ∂D.

We will do this using an argument of Carroll [**5**], which we include here for the sake of completeness. Let $z \in \partial D$ and $r > 0$, and let h be the harmonic extension of u_b from $\partial B(z,r)$ to $\overline{B(z,r)}$. Further, let c denote the minimum value of $u_b - h$ on $\overline{B(z,r)}$, and suppose, for the sake of contradiction, that $c < 0$. Then the value c is attained by $u_b - h$ at some point $y \in B(z,r)$. The minimum principle, applied on $B(z,r)\backslash\partial D$, shows that $y \in \partial D \cap B(z,r)$. By considering $u_b - h$ separately on $\overline{D}$ and on $\overline{\Omega}$, we obtain

$$\frac{\partial}{\partial n_y}(u_a - h) \geq 0 \geq \frac{\partial}{\partial n_y}(u_a - h) - (b-a)\frac{\partial h_0}{\partial n_y},$$

which contradicts the fact that $\partial h_0/\partial n_y < 0$. Thus $c = 0$, and $u_b \geq h$ on $B(z,r)$, whence $u_b(z) \geq h(z)$, as required. Corollary 1 is now established. □

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