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REPRESENTATION OF ALGEBRAIC DISTRIBUTIVE LATTICES WITH $\aleph_1$ COMPACT ELEMENTS AS IDEAL LATTICES OF REGULAR RINGS

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Abstract

We prove the following result:

Theorem. *Every algebraic distributive lattice D with at most $\aleph_1$ compact elements is isomorphic to the ideal lattice of a von Neumann regular ring R.*

(By earlier results of the author, the $\aleph_1$ bound is *optimal*.) Therefore, D is also isomorphic to the congruence lattice of a sectionally complemented modular lattice L, namely, the principal right ideal lattice of R. Furthermore, if the largest element of D is compact, then one can assume that R is unital, respectively, that L has a largest element. This extends several known results of G. M. Bergman, A. P. Huhn, J. Tůma, and of a joint work of G. Grätzer, H. Lakser, and the author, and it solves Problem 2 of the survey paper [**10**].

The main tool used in the proof of our result is an amalgamation theorem for semilattices and *algebras* (over a given division ring), a variant of previously known amalgamation theorems for semilattices and *lattices*, due to J. Tůma, and G. Grätzer, H. Lakser, and the author.

Introduction

It is a well-known and easy fact that the lattice of ideals of any (von Neumann) regular ring is algebraic and distributive. In unpublished notes from 1986, G. M. Bergman [**2**] proves the following converse of this result:

Bergman's Theorem. *Every algebraic distributive lattice D with countably many compact elements is isomorphic to the ideal lattice of a regular ring R, such that if the largest element of D is compact, then R is unital.*

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On the negative side, the author of the present paper, in [**20**], using his construction in [**19**], proved that Bergman's Theorem cannot be extended to algebraic distributive lattices with $\aleph_2$ many compact elements (or more). This left a gap at the size $\aleph_1$, expressed by the statement of the following problem:

Problem 2 of [**10**]**.** *Let D be an algebraic distributive lattice with at most $\aleph_1$ compact elements. Does there exist a regular ring R such that the ideal lattice of R is isomorphic to D?*

In this paper, we provide a positive solution to this problem, see Theorem 5.2. Of independent interest is an amalgamation result of ring-theoretical nature, mostly inspired by the lattice-theoretical constructions in [**17**] and [**13**], see Theorem 4.2. This result is the main tool used in the proof of Theorem 5.2.

Once the Amalgamation Theorem (Theorem 4.2) is proved, the representation result follows from standard techniques, based on the existence of lattices called 2-*frames* in [**6**], or *lower finite* 2-*lattices* in [**5**]. Such a technique has, for example, been used successfully in [**14**], [**15**], where A. P. Huhn proves that every distributive algebraic lattice D with at most $\aleph_1$ compact elements is isomorphic to the congruence lattice of a lattice L. Theorem 5.2 provides a strengthening of Huhn's result, namely, it makes it possible to have L *sectionally complemented* and *modular*, see Corollary 5.3. Note that we already obtained L relatively complemented with zero (thus sectionally complemented), though not modular, in [**13**].

We do not claim any originality about the proof methods used in this paper. Most of what we do amounts to translations between known concepts and proofs in universal algebra, lattice theory, and ring theory. However, the interconnections between these domains, as they are, for example, presented in [**10**], are probably not well-established enough to trivialize the results of this paper.

1. Basic concepts

Lattices, semilattices. References for this section are [**3**], [**11**], [**12**].

Let L be a lattice. We say that L is *complete*, if every subset of L has a supremum. An element a of L is *compact*, if for every subset X of L such that the supremum of X, $\bigvee X$, exists, $a \leq \bigvee X$ implies that there exists a finite subset Y of X such that $a \leq \bigvee Y$. The *unit* of a lattice is its largest element, if it exists.

We say that L is *algebraic*, if it is complete and every element is the supremum of compact elements. If L is an algebraic lattice, then the set S of all compact elements of L is closed under the join operation, and it contains 0 (the least element of L) as an element. We say that S is a $\{\vee, 0\}$-*semilattice*, that is, a commutative, idempotent monoid (the monoid operation is the join).

If S is a $\{\vee, 0\}$-semilattice, an *ideal* of S is a nonempty, hereditary subset of S closed under the join operation. The set $\operatorname{Id} S$ of all ideals of S, partially ordered by containment, is an algebraic lattice, and the compact elements of $\operatorname{Id} S$ are exactly the principal ideals $(s] = \{t \in S \mid t \leq s\}$, for $s \in S$. In particular, the semilattice of all compact elements of $\operatorname{Id} S$ is isomorphic to S. Conversely, if L is an algebraic lattice and if S is the semilattice of all compact elements of L, then the map from L to $\operatorname{Id} S$ that with every element x of L associates $\{s \in S \mid s \leq x\}$ is an isomorphism. It follows that algebraic lattices and $\{\vee, 0\}$-semilattices are categorically equivalent. The class of homomorphisms of algebraic lattices that correspond, through this equivalence, to $\{\vee, 0\}$-homomorphisms of semilattices, are the *compactness preserving, $\bigvee$-complete homomorphisms* of algebraic lattices. (We say that a homomorphism $f\colon A \to B$ of algebraic lattices is $\bigvee$-*complete*, if $\bigvee f[X] = f(\bigvee X)$, for every (possibly empty) subset X of A.) We observe that if B is finite, then any homomorphism from A to B is compactness-preserving.

A $\{\vee, 0\}$-semilattice S is *distributive*, if its ideal lattice $\operatorname{Id} S$ is a distributive lattice. Equivalently, S satisfies the following statement:

$$(\forall a, b, c)\big(c \leq a \vee b \Rightarrow (\exists x, y)(x \leq a \text{ and } y \leq b \text{ and } c = x \vee y)\big).$$

For a lattice L, the set $\operatorname{Con} L$ of all congruences of L, endowed with containment, is an algebraic lattice. Its semilattice of compact elements is traditionally denoted by $\operatorname{Con_c} L$, the semilattice of *finitely generated congruences* of L.

We say that L is

— *modular*, if $x \wedge (y \vee z) = (x \wedge y) \vee z$ for all x, y, $z \in L$ such that $x \geq z$;

— *complemented*, if it has a least element, denoted by 0, a largest element, denoted by 1, and for all $a \in L$, there exists $x \in L$ such that $a \wedge x = 0$ and $a \vee x = 1$;

— *sectionally complemented*, if it has a least element, denoted by 0, and for all a, $b \in L$ such that $a \leq b$, there exists $x \in L$ such that $a \wedge x = 0$ and $a \vee x = b$.

We denote by **2** the two-element lattice.

Rings, algebras. All the rings encountered in this work are associative, but not necessarily unital. A ring R is *regular* (in von Neumann's sense), if it satisfies the statement $(\forall x)(\exists y)(xyx = x)$. If R is a regular ring, then the set of all principal right ideals of R, partially ordered by containment, is a *sectionally complemented modular lattice*, see, for example, [**7**, page 209].

If R is a ring, then we denote by $\operatorname{Id} R$ the set of all two-sided ideals of R, partially ordered by containment. Then $\operatorname{Id} R$ is an algebraic modular lattice, which turns out to be distributive if R is regular. We denote by $\operatorname{Id}_{\mathrm{c}} R$ the semilattice of all compact elements of $\operatorname{Id} R$, that is, the finitely generated two-sided ideals of R. It is to be noted that $\operatorname{Id}_{\mathrm{c}}$ can be extended to a *functor* from rings and ring homomorphisms to $\{\vee, 0\}$-semilattices and $\{\vee, 0\}$-homomorphisms, and that this functor preserves direct limits.

If K is a division ring, a *K-algebra* is a ring R endowed with a structure of two-sided vector space over K such that the equalities

$$\lambda(xy) = (\lambda x)y, \quad (x\lambda)y = x(\lambda y), \quad (xy)\lambda = x(y\lambda)$$

hold for all x, $y \in R$, and $\lambda \in K$. Such a structure is called a *K-ring* in [**4**, Section 1]. Most of the rings that we shall encounter in this work are, in fact, algebras.

2. Embedding into V-simple algebras

Definition 2.1. A unital, regular ring R is *V-simple*, if R_R is isomorphic to all its nonzero principal right ideals, and there are nonzero principal right ideals I and J of R such that $I \oplus J = R_R$.

It is obvious that if R is V-simple, then it is simple. The converse is obviously false, for example, if R is a field.

Notation. Let κ be an infinite cardinal number, let U be a two-sided vector space over a division ring K. For example, if I is any set, then the set $K^{(I)}$ of all I-families with finite support of elements of K is endowed with a natural structure of two-sided vector space over K.

Let $\mathrm{N}_\kappa(U)$ be the subset of the algebra $\operatorname{End}_K(U)$ of *right* K-vector space endomorphisms of U defined by

$$\mathrm{N}_\kappa(U) = \{f \in \operatorname{End}_K(U) \mid \dim_K \operatorname{im} f < \kappa\}.$$

It is obvious that $\mathrm{N}_\kappa(U)$ is a two-sided ideal of $\operatorname{End}_K(U)$. We define a K-algebra, $\mathrm{E}_\kappa(U)$, by

$$\mathrm{E}_\kappa(U) = \operatorname{End}_K(U)/\mathrm{N}_\kappa(U).$$

The idea behind the proof of Lemma 2.2 and Proposition 2.3 is old, see, for example, Theorem 3.4 in [**1**]. For convenience, we recall the proofs here.

Lemma 2.2. *Let κ be an infinite cardinal number, let U be a two-sided vector space of dimension κ over a division ring K. Then $\mathrm{E}_\kappa(U)$ is a unital, regular, V-simple K-algebra.*

Proof: Note that $\mathrm{E}_\kappa(U)$ is nontrivial, because $\kappa \leq \dim_K U$. Since the endomorphism ring $\mathrm{End}_K(U)$ is regular, so is also the quotient ring $\mathrm{E}_\kappa(U) = \mathrm{End}_K(U)/\mathrm{N}_\kappa(U)$, see Lemma 1.3 in [**8**].

Furthermore, the principal right ideals of $R = \mathrm{End}_K(U)$ are exactly the ideals of the form

$$I_X = \{f \in R \mid \operatorname{im} f \subseteq X\},$$

for a subspace X of U. If X and Y are subspaces of U, then $X \cong Y$ implies that $I_X \cong I_Y$. If X is a subspace of dimension κ of U, then X can be decomposed as $X = X_0 \oplus X_1$, where $\dim_K X_0 = \dim_K X_1 = \kappa$, hence

$$[I_X] = [I_{X_0}] + [I_{X_1}] = 2[I_X],$$

where $[I]$ denotes the isomorphism class of a right ideal I. Since $X \cong U$, $[I_X] = [I_U] = [R]$. However, if X is a subspace of U of dimension $< \kappa$, then the image of I_X in $\mathrm{E}_\kappa(U)$ is the zero ideal. The conclusion follows. $\square$

Proposition 2.3. *Let K be a division ring. Every unital K-algebra has a unital embedding into a unital, regular, V-simple K-algebra.*

Proof: Let R be a unital K-algebra. Put $\kappa = \aleph_0 + \dim_K R$, where $\dim_K R$ denotes the *right dimension* of R over K, and $U = R^{(\kappa)}$, the R-algebra of all κ-sequences with finite support of elements of R. We put

$$S = \mathrm{E}_\kappa(U).$$

Since the dimension of U over K equals κ, it follows from Lemma 2.2 that S is a unital, regular, V-simple K-algebra.

Define a map $\varphi\colon R \to \mathrm{End}_K(U)$, by the rule

$$\varphi(a)\colon U \to U, \quad x \mapsto ax,$$

for all $a \in R$. It is easy to see that φ is a unital ring homomorphism from R to $\mathrm{End}_K(U)$.

Let $a \in R \setminus \{0\}$. If $\langle e_\xi \mid \xi < \kappa \rangle$ denotes the canonical basis of U over R, then the range of $\varphi(a)$ contains all the elements ae_ξ, for $\xi < \kappa$, thus its dimension over K is greater than or equal to κ. Hence, $\varphi(a)$ does not belong to $\mathrm{N}_\kappa(U)$. Therefore, the map ψ from R to S defined by the rule

$$\psi(a) = \varphi(a) + \mathrm{N}_\kappa(U),$$

for all $a \in R$, is a unital K-algebra embedding from R into S. □

3. Amalgamation of algebras over a division ring

The following fundamental result has been proved by P. M. Cohn, see Theorem 4.7 in [**4**]. We also refer the reader to the outline presented in [**16**, page 110], in the section "Regular rings: AP".

Theorem 3.1. *Let K be a division ring, let R, A, and B be unital K-algebras, with R regular. Let $\alpha\colon R \hookrightarrow A$ and $\beta\colon R \hookrightarrow B$ be unital embeddings. Then there exist a unital, regular K-algebra C, unital embeddings $\alpha'\colon A \hookrightarrow C$, and $\beta'\colon B \hookrightarrow C$, such that $\alpha' \circ \alpha = \beta' \circ \beta$.*

By combining Theorem 3.1 with Proposition 2.3, we obtain immediately the following slight strengthening of Theorem 3.1:

Lemma 3.2. *Let K be a division ring, let R, A, and B be unital K-algebras, with R regular. Let $\alpha\colon R \hookrightarrow A$ and $\beta\colon R \hookrightarrow B$ be unital embeddings. Then there exist a unital, regular, V-simple K-algebra C, unital embeddings $\alpha'\colon A \hookrightarrow C$, and $\beta'\colon B \hookrightarrow C$, such that $\alpha' \circ \alpha = \beta' \circ \beta$.*

The following example partly illustrates the underlying complexity of Theorem 3.1, by showing that even in the case where A and B are finite-dimensional over R, one may *not* be able to find a finite-dimensional solution C to the amalgamation problem:

Example 3.3. Let K be any division ring. We construct unital, matricial extensions A and B of the regular K-algebra $R = K^2$ such that the amalgamation problem of A and B over R has no finite-dimensional solution.

Proof: We put $A = \mathrm{M}_2(K)$ (resp., $B = \mathrm{M}_3(K)$), the ring of all square matrices of order 2 (resp., 3) over K, endowed with their canonical K-algebra structures. We define unital embeddings of K-algebras $f\colon R \hookrightarrow A$ and $g\colon R \hookrightarrow B$ as follows:

$$f(\langle x, y\rangle) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \quad \text{and} \quad g(\langle x, y\rangle) = \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix}, \tag{1}$$

for all x, $y \in K$.

Suppose that the amalgamation problem of A and B over R (with respect to f and g) has a solution, say, $f'\colon A \hookrightarrow C$ and $g'\colon B \hookrightarrow C$, where C is a *finite-dimensional*, not necessarily unital K-algebra and f', g' are embeddings of unital K-algebras. For a positive integer d, we denote by $e^d_{i,j}$, for $1 \leq i, j \leq d$, the canonical matrix units of the matrix ring $\mathrm{M}_d(K)$. So, (1) can be rewritten as

$$(2)\quad f(\langle x, y\rangle) = e^2_{1,1}x + e^2_{2,2}y, \quad \text{and} \quad g(\langle x, y\rangle) = (e^3_{1,1} + e^3_{2,2})x + e^3_{3,3}y,$$

for all x, $y \in K$. Put $u_{i,j} = f'(e^2_{i,j})$, for all i, $j \in \{1, 2\}$, and $v_{i,j} = g'(e^3_{i,j})$, for all i, $j \in \{1, 2, 3\}$. Then apply f' (resp. g') to the first (resp., second) equality of (2). Since $f' \circ f = g' \circ g$, we obtain that the equality

$$u_{1,1}x + u_{2,2}y = (v_{1,1} + v_{2,2})x + v_{3,3}y,$$

holds, for all x, $y \in K$. Specializing to x, $y \in \{0, 1\}$ yields the equalities

$$(3)\qquad u_{1,1} = v_{1,1} + v_{2,2},$$

$$(4)\qquad u_{2,2} = v_{3,3}.$$

However, the elements $u_{i,j}$ of C, for i, $j \in \{1, 2\}$, satisfy part of the equalities defining matrix units in C, namely, $u_{i,j}u_{k,l} = \delta_{j,k}u_{i,l}$, for all i, j, k, $l \in \{1, 2\}$ (δ_{-} denotes here the Kronecker symbol). Similarly, the elements $v_{i,j}$ of C, for i, $j \in \{1, 2, 3\}$, satisfy the equalities $v_{i,j}v_{k,l} = \delta_{j,k}v_{i,l}$, for all i, j, k, $l \in \{1, 2, 3\}$. Therefore, by (3) and (4), $\dim_K(u_{1,1}C) = \dim_K(v_{1,1}C) + \dim_K(v_{2,2}C) = 2\dim_K(v_{3,3}C) = 2\dim_K(u_{2,2}C) = 2\dim_K(u_{1,1}C)$. But then, since C is finite-dimensional, $\dim_K(u_{1,1}C) = 0$, so $\dim_K(f'A) = \dim_K(g'B) = 0$, a contradiction. $\square$

For a further discussion of Example 3.3, see the comments following the statement of Problem 1 in Section 6.

4. The amalgamation theorem

Definition 4.1. Let K be a division ring. A K-algebra R is *V-Boolean*, if it is isomorphic to a finite direct product of V-simple K-algebras.

Theorem 4.2. *Let K be a division ring, let R_0, R_1, and R_2 be unital K-algebras, with R_0 regular, let S be a finite Boolean lattice. For $k \in \{1, 2\}$, let $f_k\colon R_0 \to R_k$ be a homomorphism of unital K-algebras and let $\psi_k\colon \operatorname{Id} R_k \to S$ be a unit-preserving $\bigvee$-complete homomorphism, such that $\psi_1 \circ \operatorname{Id} f_1 = \psi_2 \circ \operatorname{Id} f_2$. Then there exist a unital, regular, V-Boolean K-algebra R, homomorphisms of unital K-algebras $g_k\colon R_k \to R$, for*

$k \in \{1,2\}$, and an isomorphism $\alpha\colon \operatorname{Id} R \to S$ such that $g_1 \circ f_1 = g_2 \circ f_2$ and $\alpha \circ \operatorname{Id} g_k = \psi_k$ for $k \in \{1,2\}$ (see Figure 1).

$$\begin{tikzcd}
 & R & \\
R_1 \arrow[ur, "g_1"] & & R_2 \arrow[ul, "g_2"'] \\
 & R_0 \arrow[ul, "f_1"] \arrow[ur, "f_2"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \operatorname{Id} R \arrow[d, "\alpha"] & \\
\operatorname{Id} R_1 \arrow[ur, "\operatorname{Id} g_1"] \arrow[r, "\psi_1"] & S & \operatorname{Id} R_2 \arrow[ul, "\operatorname{Id} g_2"'] \arrow[l, "\psi_2"'] \\
 & \operatorname{Id} R_0 \arrow[ul, "\operatorname{Id} f_1"] \arrow[ur, "\operatorname{Id} f_2"'] \arrow[uu, "\psi_0"] &
\end{tikzcd}$$

Figure 1

Proof: We adapt to the context of regular rings the lattice-theoretical proof of Theorem 1 of [**13**].

We put $\psi_0 = \psi_1 \circ \operatorname{Id} f_1 = \psi_2 \circ \operatorname{Id} f_2$. We start with the following case:

Case 1: $S \cong \mathbf{2}$.

We put $I_k = \{x \in R_k \mid \psi_k(R_k x R_k) = 0\}$, for $k \in \{0,1,2\}$. Since ψ_k is a $\bigvee$-complete homomorphism, I_k is the largest ideal of R_k whose image under ψ_k is zero. Furthermore, since ψ_k is unit-preserving, I_k is a *proper* ideal of R_k.

Next, we put $\overline{R}_k = R_k/I_k$, and we denote by $p_k\colon R_k \twoheadrightarrow \overline{R}_k$ the canonical projection.

For $k \in \{1,2\}$, the equivalence $x \in I_0 \Leftrightarrow f_k(x) \in I_k$ holds for all $x \in R_0$, thus there exists a unique unital *embedding* $\overline{f}_k\colon \overline{R}_0 \hookrightarrow \overline{R}_k$ such that $p_k \circ f_k = \overline{f}_k \circ p_0$, see Figure 2.

$$\begin{tikzcd}
R_0 \arrow[r, "f_k"] \arrow[d, two heads, "p_0"'] & R_k \arrow[d, two heads, "p_k"] \\
\overline{R}_0 \arrow[r, hook, "\overline{f}_k"'] & \overline{R}_k
\end{tikzcd}
\qquad
\begin{tikzcd}
 & R & \\
\overline{R}_1 \arrow[ur, hook, "\overline{g}_1"] & & \overline{R}_2 \arrow[ul, hook, "\overline{g}_2"'] \\
 & \overline{R}_0 \arrow[ul, hook, "\overline{f}_1"] \arrow[ur, hook, "\overline{f}_2"'] &
\end{tikzcd}$$

Figure 2

Since $\overline{R}_0$ is regular and $\overline{f}_0$, $\overline{f}_1$ are unital embeddings, there exist, by Lemma 3.2, a unital, regular, V-simple K-algebra R and embeddings $\overline{g}_k\colon \overline{R}_k \hookrightarrow R$, for $k \in \{1,2\}$, such that $\overline{g}_1 \circ \overline{f}_1 = \overline{g}_2 \circ \overline{f}_2$, see Figure 2.

We put $g_k = \overline{g}_k \circ p_k$, for $k \in \{1,2\}$, see Figure 3.

$$\begin{tikzcd}
 & & R & & \\
 & \overline{R}_1 \arrow[ur, hook, "\overline{g}_1"] & & \overline{R}_2 \arrow[ul, hook', "\overline{g}_2"'] & \\
 & R_1 \arrow[u, two heads, "p_1"] \arrow[uur, bend left, "g_1"] & \overline{R}_0 \arrow[ul, hook', "\overline{f}_1"'] \arrow[ur, hook, "\overline{f}_2"] & R_2 \arrow[u, two heads, "p_2"'] \arrow[uul, bend right, "g_2"'] & \\
 & & R_0 \arrow[ul, "f_1"] \arrow[ur, "f_2"'] \arrow[u, two heads, "p_0"] & &
\end{tikzcd}$$

Figure 3

Therefore,

$$g_1 \circ f_1 = \overline{g}_1 \circ p_1 \circ f_1 = \overline{g}_1 \circ \overline{f}_1 \circ p_0 = \overline{g}_2 \circ \overline{f}_2 \circ p_0 = g_2 \circ f_2.$$

Since R is V-simple, it is simple, thus, since $S \cong \mathbf{2}$, there exists a unique isomorphism $\alpha\colon \operatorname{Id} R \to S$. To verify that $\alpha \circ \operatorname{Id} g_k = \psi_k$, for $k \in \{1,2\}$, it suffices to verify that $(\operatorname{Id} g_k)(I) = 0$ iff $\psi_k(I) = 0$, for every $I \in \operatorname{Id} R_k$. We proceed:

$$\begin{aligned}
(\operatorname{Id} g_k)(I) = 0 \quad &\text{iff} \quad g_k[I] = 0 \\
&\text{iff} \quad \overline{g}_k \circ p_k[I] = 0 \\
&\text{iff} \quad p_k[I] = 0
\end{aligned}$$

(because $\overline{g}_k$ is an embedding)

$$\begin{aligned}
&\text{iff} \quad I \subseteq I_k \\
&\text{iff} \quad \psi_k(I) = 0,
\end{aligned}$$

which concludes Case 1.

Case 2: General case, S finite Boolean.

Without loss of generality, $S = \mathbf{2}^n$, with $n < \omega$. For $i < n$, let $\pi_i\colon S \twoheadrightarrow \mathbf{2}$ be the projection on the i-th coordinate. We apply Case 1 to

the maps $\pi_i\psi_k$, for $k \in \{1,2\}$, see Figure 4.

$$\begin{tikzcd}
 & R^{(i)} & \\
R_1 \arrow[ur, "g_{1,i}"] & & R_2 \arrow[ul, "g_{2,i}"'] \\
 & R_0 \arrow[ul, "f_1"] \arrow[ur, "f_2"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \operatorname{Id} R^{(i)} \arrow[d, "\alpha_i"] & \\
\operatorname{Id} R_1 \arrow[ur, "\operatorname{Id} g_{1,i}"] \arrow[r, "\pi_i\psi_1"] & \mathbf{2} & \operatorname{Id} R_2 \arrow[ul, "\operatorname{Id} g_{2,i}"'] \arrow[l, "\pi_i\psi_2"'] \\
 & \operatorname{Id} R_0 \arrow[ul, "\operatorname{Id} f_1"] \arrow[ur, "\operatorname{Id} f_2"'] \arrow[u, "\psi_0"] &
\end{tikzcd}$$

Figure 4

We obtain a unital, regular, V-simple K-algebra $R^{(i)}$, K-algebra homomorphisms $g_{k,i}$, for $k \in \{1,2\}$, an isomorphism $\alpha_i\colon \operatorname{Id} R^{(i)} \to \mathbf{2}$, such that $g_{1,i} \circ f_1 = g_{2,i} \circ f_2$ and $\alpha_i \circ \operatorname{Id} g_{k,i} = \pi_i \circ \psi_k$, for $k \in \{1,2\}$, see Figure 4.

Now we put $R = \bigoplus_{i<n} R^{(i)}$, with the componentwise ring structure. So R is a unital, regular, V-Boolean K-algebra. For $k \in \{1,2\}$, we define a unital homomorphism $g_k\colon R_k \to R$ by the rule

$$g_k(x) = \langle g_{k,i}(x) \mid i < n\rangle, \quad \text{for all } x \in R_k.$$

It is immediate that $g_1 \circ f_1 = g_2 \circ f_2$.

Furthermore, observe that $\prod_{i<n} \operatorname{Id} R^{(i)} \cong \operatorname{Id} R$, *via* the isomorphism that sends a finite sequence $\langle I_i \mid i < n\rangle$ to $\bigoplus_{i<n} I_i$. Define an isomorphism $\alpha\colon \operatorname{Id} R \to S$ by the rule

$$\alpha\left(\bigoplus_{i<n} I_i\right) = \langle \alpha_i(I_i) \mid i < n\rangle, \quad \text{for all } I \in \operatorname{Id} R.$$

For $k \in \{1,2\}$ and $I \in \operatorname{Id} R_k$, we compute:

$$\begin{aligned}
\alpha \circ (\operatorname{Id} g_k)(I) &= \alpha\left(\bigoplus_{i<n} (\operatorname{Id} g_{k,i})(I)\right) \\
&= \langle \alpha_i \circ (\operatorname{Id} g_{k,i})(I) \mid i < n\rangle \\
&= \langle \pi_i \circ \psi_k(I) \mid i < n\rangle \\
&= \psi_k(I),
\end{aligned}$$

so $\alpha \circ \operatorname{Id} g_k = \psi_k$. $\square$

5. The representation theorem

We first state a useful lemma, see [**5**] and [**6**]:

Lemma 5.1. *There exists a lattice F of cardinality $\aleph_1$ satisfying the following properties:*

(i) *F is* lower finite*, that is, for all $a \in F$, the principal ideal*

$$F[a] = \{x \in F \mid x \leq a\}$$

is finite.

(ii) *Every element of F has at most two immediate predecessors.*

Theorem 5.2. *Let D be an algebraic distributive lattice with at most $\aleph_1$ compact elements, and let K be a division ring. Then there exists a regular K-algebra R satisfying the following properties:*

(i) $\operatorname{Id} R \cong D$.

(ii) *If the largest element of D is compact, then R is a direct limit of unital, regular, V-Boolean K-algebras and unital embeddings of K-algebras. In particular, R is unital.*

Proof: A similar argument has already been used, in different contexts, in such various references as [**6**], [**14**], [**15**], [**13**].

We first translate the problem into the language of semilattices. This amounts, by defining S as the $\{\vee, 0\}$-semilattice of compact elements of D, to verifying the existence of a regular K-algebra R satisfying the following condition

(i') $\operatorname{Id}_{\mathrm{c}} R \cong S$,

along with (ii). We do this first in the case where the largest element of D is compact, that is, where S has a largest element. It is proved in [**10**] that S is a direct limit of finite Boolean $\{\vee, 0\}$-semilattices and $\{\vee, 0, 1\}$-homomorphisms, say,

$$\langle S, \varphi_i \rangle_{i \in I} = \varinjlim \langle S_i, \varphi^i_j \rangle_{i \leq j \text{ in } I},$$

where I is a directed partially ordered set, and $\langle S_i, \varphi^i_j \rangle_{i \leq j \text{ in } I}$ is a direct system of finite Boolean $\{\vee, 0\}$-semilattices and $\{\vee, 0, 1\}$-homomorphisms (in particular, $\varphi^i_j \colon S_i \to S_j$ and $\varphi_i \colon S_i \to S$, for $i \leq j$ in I). Furthermore, one can take I countably infinite if S is finite, and $|I| = |S|$ if S is infinite. In particular, $|I| \leq \aleph_1$.

Let F be a lattice satisfying the conditions of Lemma 5.1. Since $|F| \geq |I| > 0$, there exists a surjective map $\nu_0 \colon F \twoheadrightarrow I$. Since F is lower finite, it is well-founded, so we can define inductively an order-preserving,

cofinal map $\nu\colon F \to I$, by putting

$$\nu(x) = \text{any element } i \text{ of } I \text{ such that } \nu_0(x) \leq i \text{ and } \nu(y) \leq i, \quad \text{for all } y < x,$$

for all $x \in F$. This is justified because F is lower finite, and this does not use part (ii) of Lemma 5.1. As a conclusion, we see that we may index our direct system by F itself, that is, *we may assume that I satisfies the conditions (i), (ii) of Lemma 5.1.*

We shall now define inductively unital, regular, V-Boolean K-algebras R_i, unital homomorphisms of K-algebras $f^i_j\colon R_i \to R_j$, and isomorphisms $\varepsilon_i\colon \operatorname{Id_c} R_i \to S_i$, for $i \leq j$ in I.

Let $\varrho\colon I \to \omega$ be the natural rank function, that is,

$$\varrho(i) = \sup\{\varrho(j) \mid j < i\} + 1,$$

for all $i \in I$. For all $n < \omega$, we put

$$I_n = \{i \in I \mid \varrho(i) \leq n\}.$$

By induction on $n < \omega$, we construct V-Boolean K-algebras R_i (note then that $\operatorname{Id} R_i = \operatorname{Id_c} R_i$), maps $\varepsilon_i\colon \operatorname{Id} R_i \to S_i$, and unital homomorphisms of K-algebras $f^i_j\colon R_i \to R_j$, for all $i \leq j$ in I_n, satisfying the following properties:

(a) $f^i_i = \mathrm{id}_{R_i}$, for all $i \in I_n$.
(b) $f^i_k = f^j_k \circ f^i_j$, for all $i \leq j \leq k$ in I_n.
(c) ε_i is a lattice isomorphism from $\operatorname{Id} R_i$ onto S_i, for all $i \in I_n$.
(d) The following diagram is commutative, for all $i \leq j$ in I_n:

$$\begin{tikzcd} \operatorname{Id} R_i \arrow[r, "\operatorname{Id} f^i_j"] \arrow[d, "\varepsilon_i"'] & \operatorname{Id} R_j \arrow[d, "\varepsilon_j"] \\ S_i \arrow[r, "\varphi^i_j"'] & S_j \end{tikzcd}$$

For $n = 0$, it suffices to construct a V-Boolean K-algebra R_0 such that $\operatorname{Id} R_0 \cong S_0$. This is easy: if p is the number of atoms of S_0, take any V-simple K-algebra R, and put $R_0 = R^p$.

Suppose having done the construction on I_n, we show how to extend it to I_{n+1}. Let $i \in I_{n+1}$ such that $\varrho(i) = n + 1$. Denote by i_0 and i_1 the two immediate predecessors of i in I. Note that i_0 and i_1 do not need

to be distinct.

$$\begin{tikzcd}
 & R_i & \\
R_{i_0} \arrow[ur, "g_0"] & & R_{i_1} \arrow[ul, "g_1"'] \\
 & R_{i_0\wedge i_1} \arrow[ul, "f_{i_0}^{i_0\wedge i_1}"] \arrow[ur, "f_{i_1}^{i_0\wedge i_1}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \operatorname{Id} R_i \arrow[d, "\varepsilon_i"] & \\
\operatorname{Id} R_{i_0} \arrow[ur, "\operatorname{Id} g_0"] \arrow[r, "\varphi_i^{i_0}\varepsilon_{i_0}"] & S_i & \operatorname{Id} R_{i_1} \arrow[ul, "\operatorname{Id} g_1"'] \arrow[l, "\varphi_i^{i_1}\varepsilon_{i_1}"'] \\
 & \operatorname{Id} R_{i_0\wedge i_1} \arrow[ul, "\operatorname{Id} f_{i_0}^{i_0\wedge i_1}"] \arrow[ur, "\operatorname{Id} f_{i_1}^{i_0\wedge i_1}"'] &
\end{tikzcd}$$

Figure 5

By Theorem 4.2, there exist a unital, regular, V-Boolean K-algebra R_i, unital homomorphisms of K-algebras $g_k\colon R_{i_k} \to R_i$, for $k < 2$, and an isomorphism $\varepsilon_i\colon \operatorname{Id} R_i \to S_i$ such that the following equalities hold:

$$g_0 \circ f_{i_0}^{i_0\wedge i_1} = g_1 \circ f_{i_1}^{i_0\wedge i_1}; \tag{5}$$

$$\varepsilon_i \circ \operatorname{Id} g_k = \varphi_i^{i_k}\varepsilon_{i_k}, \quad \text{for } k \in \{1,2\}, \tag{6}$$

see Figure 5. If $i_0 = i_1$, we may replace g_1 by g_0: the diagrams of Figure 5 remain commutative and (5), (6) remain valid. Thus we may define $f_i^{i_0} = g_0$ and $f_i^{i_1} = g_1$, and (5), (6) are restated as

$$f_i^{i_0} \circ f_{i_0}^{i_0\wedge i_1} = f_i^{i_1} \circ f_{i_1}^{i_0\wedge i_1}; \tag{7}$$

$$\varepsilon_i \circ \operatorname{Id} f_i^{i_k} = \varphi_i^{i_k}\varepsilon_{i_k}, \quad \text{for } k \in \{1,2\}. \tag{8}$$

At this point, we have defined f_i^j, if $i \in I_{n+1} \setminus I_n$ and j is an immediate predecessor of i. If $i \in I_{n+1} \setminus I_n$ and $j < i$, then the only possibility is to put $f_i^j = f_i^{i_\nu} \circ f_{i_\nu}^j$, where $\nu < 2$ is such that $j \le i_\nu$. For this to be possible, we need to verify that if $j \le i_0 \wedge i_1$, then $f_i^{i_0} \circ f_{i_0}^j = f_i^{i_1} \circ f_{i_1}^j$. This follows from (7), along with the following sequence of equalities:

$$\begin{aligned}
f_i^{i_0} \circ f_{i_0}^j &= f_i^{i_0} \circ f_{i_0}^{i_0\wedge i_1} \circ f_{i_0\wedge i_1}^j \\
&= f_i^{i_1} \circ f_{i_1}^{i_0\wedge i_1} \circ f_{i_0\wedge i_1}^j \\
&= f_i^{i_1} \circ f_{i_1}^j.
\end{aligned}$$

At this point, we have defined f_i^j, if $i \in I_{n+1} \setminus I_n$ and $j < i$. We extend this definition by putting $f_i^i = \mathrm{id}_{R_i}$. The verification of conditions (a)–(d) above is then straightforward.

Let R be the direct limit of the R_i, with the transition maps f^i_j for $i \leq j$ in I. Since the $\mathrm{Id_c}$ functor preserves direct limits, $\mathrm{Id_c}\, R$ is isomorphic to the direct limit of the $\mathrm{Id_c}\, R_i$, with the transition maps $\mathrm{Id_c}\, f^i_j$, for $i \leq j$ in I. By (c) and (d), it follows that $\mathrm{Id_c}\, R$ is isomorphic to the direct limit of all the S_i, with the transition maps φ^i_j, for $i \leq j$ in I; whence $\mathrm{Id_c}\, R \cong S$. This settles part (ii) of the statement of Theorem 5.2.

Let now S be a distributive $\{\vee, 0\}$-semilattice of cardinality at most $\aleph_1$. Adjoin a new largest element to S, forming $T = S \cup \{1\}$. Then T is a distributive $\{\vee, 0\}$-semilattice with a largest element and of cardinality at most $\aleph_1$, thus, by what we just proved, it is isomorphic to $\mathrm{Id_c}\, R$, for some regular unital K-algebra R. Let $\varepsilon\colon \mathrm{Id_c}\, R \to T$ be an isomorphism. We define an ideal I of R, by

$$I = \{x \in R \mid \varepsilon(RxR) \in S\}.$$

Then S is the image of $\mathrm{Id_c}\, I$ under ε. In particular, $\mathrm{Id_c}\, I \cong S$. Since R is regular and I is an ideal of R, I is regular, see [**8**, Lemma 1.3]. □

Corollary 5.3. *Let D be an algebraic distributive lattice with at most $\aleph_1$ compact elements. Then there exists a sectionally complemented modular lattice L such that $\mathrm{Con}\, L \cong D$. Furthermore, if the largest element of D is compact, then one can take L with a largest element.*

Proof: By Theorem 5.2, there exists a regular ring R such that $\mathrm{Id}\, R \cong D$, and R is unital if the largest element of D is compact. Let L be the lattice of all principal right ideals of R. Then L is a sectionally complemented modular lattice, see Section 1. Note that if D has a largest element, then R is unital, thus L has a largest element. By [**20**, Theorem 4.3], $\mathrm{Con}\, L$ is isomorphic to $\mathrm{Id}\, R$. Hence $\mathrm{Con}\, L \cong D$. □

6. Problems and comments

The main amalgamation result of this paper, Theorem 4.2, is easily seen to imply that for every finite diagram $\mathcal{D}$ of finite Boolean $\{\vee, 0\}$-semilattices, if $\mathcal{D}$ is indexed by the *square* $\mathbf{2}^2$, then $\mathcal{D}$ has a lifting, with respect to the $\mathrm{Id_c}$ functor, by regular rings. The analogue of this result in case $\mathcal{D}$ is indexed by the *cube*, $\mathbf{2}^3$, does *not* hold, by the results of [**18**]. A $\mathbf{2}^3$-indexed diagram $\mathcal{D}$ is produced there, that cannot be lifted, with respect to the $\mathrm{Con_c}$ functor, by lattices with permutable congruences. Since for a regular ring R, the ideal lattice of R is isomorphic to the congruence lattice of the principal right ideal lattice L of R, and since L has permutable congruences (because it is sectionally complemented), $\mathcal{D}$ cannot be lifted, with respect to the $\mathrm{Id_c}$ functor, by regular rings as well.

The regular rings obtained in the underlying construction of Bergman's Theorem are *locally matricial*, that is, direct limits of matricial rings (a ring is *matricial* over a division ring K, if it is isomorphic to a direct product of finitely many matrix rings over K). Quite to the contrary, the rings that we obtain in the proof of Theorem 5.2 are not locally matricial. In fact, if R is one of those rings, then the monoid $V(R)$ of isomorphism classes of finitely generated projective right R-modules is a *semilattice*, as opposed to the case of a locally matricial ring R, for which $V(R)$ is *cancellative*. Our proof cannot be extended to provide R locally matricial, because the finite-dimensional analogue of Theorem 4.2 fails. This suggests the following problem:

Problem 1. Let D be a distributive algebraic lattice with at most $\aleph_1$ compact elements. Does there exists a locally matricial ring R such that $\operatorname{Id} R \cong D$?

Problem 1 is equivalent to the particular instance of [**10**, Problem 1] obtained by taking $|S| \leq \aleph_1$.

Because of Example 3.3, Theorem 4.2 does not extend to matricial algebras over a division ring. However, this does not rule out *a priori* the possibility of an extension of Theorem 4.2 to a sufficiently large subcategory of the category of matricial K-algebras and unital homomorphisms of algebras.

Problem 2. Let D be an algebraic distributive lattice with at most $\aleph_1$ compact elements. Does there exist a *locally finite*, complemented, modular lattice L such that $\operatorname{Con} L \cong D$?

(A lattice L is *locally finite*, if every finitely generated sublattice of L is finite.) It can be easily proved, by using some results of [**9**] and [**21**], that a positive answer to Problem 2 would imply a positive answer to Problem 1. Conversely, if one obtains a positive answer to Problem 1 with R locally matricial over a *finite* field K, then the lattice of principal right ideals of R is locally finite, thus providing a positive answer to Problem 2.

References

[1] P. Ara, On $\aleph_0$-injective regular rings, *J. Pure Appl. Algebra* **42(2)** (1986), 109–115.

[2] G. M. Bergman, Von Neumann regular rings with tailor-made ideal lattices, unpublished notes, 1986.

[3] G. Birkhoff, *"Lattice theory"*, Third edition, American Mathematical Society Colloquium Publications, Vol. XXV, American Mathematical Society, Providence, R.I., 1967.

[4] P. M. Cohn, On the free product of associative rings, *Math. Z.* **71** (1959), 380–398.

[5] S. Z. Ditor, Cardinality questions concerning semilattices of finite breadth, *Discrete Math.* **48(1)** (1984), 47–59.

[6] H. Dobbertin, Vaught measures and their applications in lattice theory, *J. Pure Appl. Algebra* **43(1)** (1986), 27–51.

[7] K. D. Fryer and I. Halperin, The von Neumann coordinatization theorem for complemented modular lattices, *Acta Sci. Math. (Szeged)* **17** (1956), 203–249.

[8] K. R. Goodearl, *"Von Neumann regular rings"*, second ed., Robert E. Krieger Publishing Co. Inc., Malabar, FL, 1991.

[9] K. R. Goodearl and D. E. Handelman, Tensor products of dimension groups and K_0 of unit-regular rings, *Canad. J. Math.* **38(3)** (1986), 633–658.

[10] K. R. Goodearl and F. Wehrung, Representations of distributive semilattices by dimension groups, regular rings, C^*-algebras, and complemented modular lattices, Preprint, 1997.

[11] G. Grätzer, *"Lattice theory. First concepts and distributive lattices"*, W. H. Freeman and Co., San Francisco, Calif., 1971.

[12] G. Grätzer, *"General lattice theory"*, second ed., Birkhäuser Verlag, Basel, 1998.

[13] G. Grätzer, H. Lakser and F. Wehrung, Congruence amalgamation of lattices, *Acta Sci. Math. (Szeged)* **66** (2000), 3–22.

[14] A. P. Huhn, On the representation of distributive algebraic lattices. II, *Acta Sci. Math. (Szeged)* **53(1-2)** (1989), 3–10.

[15] A. P. Huhn, On the representation of distributive algebraic lattices. III, *Acta Sci. Math. (Szeged)* **53(1-2)** (1989), 11–18.

[16] E. W. Kiss, L. Márki, P. Pröhle and W. Tholen, Categorical algebraic properties. A compendium on amalgamation, congruence extension, epimorphisms, residual smallness, and injectivity, *Studia Sci. Math. Hungar.* **18(1)** (1982), 79–140.

[17] J. Tůma, On the existence of simultaneous representations, *Acta Sci. Math. (Szeged)* **64(3-4)** (1998), 357–371.

[18] J. Tůma and F. Wehrung, Simultaneous representations of semilattices by lattices with permutable congruences, *Internat. J. Algebra Comput.*, (to appear).

[19] F. Wehrung, Non-measurability properties of interpolation vector spaces, *Israel J. Math.* **103** (1998), 177–206.

[20] F. Wehrung, A uniform refinement property for congruence lattices, *Proc. Amer. Math. Soc.* **127(2)** (1999), 363–370.

[21] F. Wehrung, The dimension monoid of a lattice, *Algebra Universalis* **40(3)** (1998), 247–411.

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