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THE MAXIMAL QUOTIENT RINGS OF REGULAR GROUP RINGS III

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To the memory of Andreu Pitarch

Abstract

We give a new proof of the main result of [**1**] which does not use the classification of the finite simple groups.

Introduction. In [**1**] the proof of the main result, [**1**, Theorem 2.3], is based implicitly on the classification of the finite simple groups through the use of [**6**, Theorem]. We give here a proof of [**1**, Theorem 2.3] which does not use [**6**, Theorem].

The proof. Let G be a group. Recall that $\Delta(G) = \{g \in G \mid [G : C_G(g)] < \infty\}$ is a characteristic subgroup of G. A group G is an FC-group (finite conjugate group) if $G = \Delta(G)$. A group G satisfies Min (minimal condition on subgroups) if every non-empty set of subgroups of G, partially ordered by inclusion, has a minimal element. A group satisfies Min-p for the prime p if each of its p-subgroups satisfies Min.

Throughout, K denotes a field and G a locally finite group with no elements of order char(K). Thus $K[G]$ is a regular group ring (cf. [**9**, Theorem 3.1.5]).

Lemma 1. *Suppose that G is an FC-group satisfying Min-p for all primes p and that the type I_f part of $Q^r(K[G])$ is non-zero. Then G is abelian-by-finite.*

Proof: Suppose that G has no abelian subgroup of finite index. By [**9**, Lemma 6.3.3] and [**1**, Lemma 1.3(ii)], we may assume that G is countable. By [**10**, Lemma 6] and [**1**, Lemma 1.3(ii)], we may assume

$$G = \left(\prod_{i=1}^{\infty} H_i\right) / H,$$

where $\prod_{i=1}^{\infty} H_i$ is a (weak) direct product of finite groups H_i. We may also assume that $H \cap H_i = \langle 1 \rangle$ for all i. Since H and H_i are disjoint normal subgroups of $\prod_{i=1}^{\infty} H_i$, they commute and we have

$$H \leq Z\left(\prod_{i=1}^{\infty} H_i\right) = \prod_{i=1}^{\infty} Z(H_i).$$

By [**1**, Lemmas 1.3(ii), 1.4 and Proposition 1.1], we may assume that the H_i are not nilpotent and every proper subgroup of H_i is abelian.

By [**8**], $H_i = P_i \langle t_i \rangle$, where P_i is a normal Sylow p_i-subgroup of H_i and $t_i \in H_i$ is an element of order $q_i^{n_i}$ ($n_i \geq 1$), where q_i is a prime. Furthermore, $\langle t_i \rangle$ is not normal in H_i and $\langle t_i^{q_i} \rangle = Z(H_i)$.

Since the type I_f part of $Q^r(K[G])$ is non-zero, by [**3**, Lemma 4.2], there exists a non-zero abelian idempotent $e \in K[G]$. Thus there exists an integer n, such that

$$\langle \operatorname{Supp} e \rangle \leq \left(\prod_{i=1}^{n} H_i\right) H/H \leq G.$$

Since G satisfies Min-p for all primes p, it is easy to see that there are infinitely many distinct q_i. Let q_j be such that $j > n$ and $\langle \operatorname{Supp} e \rangle$ has no element of order q_j. Let $\tilde{t}_j = 1 + t_j + \cdots + t_j^{q_j^{n_j}-1}$. Note that $\tilde{t}_j / q_j^{n_j}$ is idempotent. Since $(x, t_j) = 1$ for all $x \in \operatorname{Supp} e$, $e\tilde{t}_j / q_j^{n_j} \in eK[G]e$ is idempotent. Since e is abelian and $(x, h) = 1$ for all $x \in \operatorname{Supp} e$ and $h \in H_j$, we have

$$e\tilde{t}_j h = eh\tilde{t}_j$$

for all $h \in H_j$. Using the fact that $\sum_{i=0}^{q_j^{n_j}-1} K[\langle \operatorname{Supp} e \rangle] t_j^i$ is a direct sum, we deduce that $xt_j \in \operatorname{Supp} e\tilde{t}_j$ for all $x \in \operatorname{Supp} e$. Thus, given $x \in \operatorname{Supp} e$ and $h \in H_j$, there exist $y \in \operatorname{Supp} e$ and m such that $xt_j h = yht_j^m$. Hence $t_j h t_j^{-m} h^{-1} = x^{-1} y \in \langle \operatorname{Supp} e \rangle \cap H_j = \langle \operatorname{Supp} e \rangle \cap Z(H_j)$. Since $Z(H_j)$ is a q_j-group and $\langle \operatorname{Supp} e \rangle$ has no element of order q_j, we see $h^{-1} t_j h = t_j^m$. But $\langle t_j \rangle$ is not normal in H_j, which is a contradiction, so the lemma is proved. ■

Let J be a right ideal of $K[G]$. As in [**3**], we define $\alpha(J, F) = \dim(J \cap K[F])/|F|$ for each finite subgroup F of G, and $\alpha(J) = \sup \alpha(J, F)$, where F ranges over all finite subgroups of G.

We denote by $\pi(G)$ the set of all primes p such that G has an element of order p. If π is a set of primes we say that G is a π-group if $\pi(G) \subseteq \pi$.

Lemma 2. *Suppose that G satisfies Min-p for all primes p and that the type I_f part of $Q^r(K[G])$ is non-zero. Then G is abelian-by-finite.*

Proof: By [**9**, Lemma 6.3.3] and [**1**, Lemma 1.3(ii)], we may assume that G is countable. By [**1**, Lemma 1.3(ii)] and Lemma 1, $\Delta(G)$ is abelian-by-finite. By [**9**, Lemma 12.1.2], $\Delta(G)$ has a characteristic abelian subgroup A of finite index. Let $\sigma = \pi(\Delta(G)/A)$.

Since the type I_f part of $Q^r(K[G])$ is non-zero, by [**3**, Lemma 4.2], there exists a non-zero abelian idempotent $e \in K[G]$. Let $H = \langle \operatorname{Supp} e \rangle$ and $\tau = \pi(H)$. Hence $\pi = \sigma \cup \tau$ is a finite set of primes.

Now A is the direct product $A = \prod A_p$ of its p-primary parts. Let $A_{\pi'} = \prod_{p \notin \pi} A_p$. Then $A_{\pi'}$ is characteristic in G. Consider $\bar{G} = G/A_{\pi'}$. Let $\delta \in K[G]$. We denote by $\bar{\delta}$ the image of δ in $K[\bar{G}]$. By [**4**, Lemma 7.6], $\bar{e}$ is an abelian idempotent, and clearly it is non-zero. Let p be a prime such that $\alpha(\bar{e}K[\bar{G}]) > p^{-1}$ and $p \notin \pi$. We shall see that $p \notin \pi(\bar{G})$.

Suppose that $p \in \pi(\bar{G})$. Then there exists $g \in G \backslash \Delta(G)$ with $o(g) = p^n$ and $o(\bar{g}) = p$. Let $\tilde{g} = 1 + g + \cdots + g^{p^n-1}$. Since $K[G]$ is regular, there exists $\beta \in K[G]$ such that

$$\tilde{g}e = \tilde{g}e\beta\tilde{g}e.$$

By squaring it, we see that $e\beta\tilde{g}e$ is an idempotent in $eK[G]e$. By [**2**, Lemma 2.1], $\operatorname{Supp} e\beta\tilde{g}e \subseteq \Delta(G)H$. Let $H_1 = \langle \operatorname{Supp} e\beta\tilde{g}e \cup \operatorname{Supp} e \rangle$. Thus $p \notin \pi(\bar{H}_1)$. Using the fact that we have a direct sum $\sum_{i=0}^{p-1} \bar{g}^i K[\bar{H}_1]$, we deduce that

$$p^{n-1}\bar{e} = p^{n-1}\overline{e\beta\tilde{g}e}.$$

Since $p \neq 0$ in K, $\bar{e} = \overline{e\beta\tilde{g}e}$. Thus $\bar{e}K[\bar{G}] \cong \overline{\tilde{g}e\beta}K[\bar{G}]$. By [**3**, Lemma 1.2(iv)], $\alpha(\bar{e}K[\bar{G}]) \leq \alpha(\bar{\tilde{g}}K[\bar{G}])$. But an easy calculation shows that $\alpha(\bar{\tilde{g}}K[\bar{G}]) = p^{-1}$. This contradicts the choice of p. Thus $p \notin \pi(\bar{G})$. Hence $\pi(\bar{G})$ is finite.

Since $\bar{e}$ is a non-zero abelian idempotent, by [**3**, Lemma 4.2], the type I part of $Q^r(K[\bar{G}])$ is non-zero. By [**1**, Proposition 1.2], $[\bar{G} : \Delta(\bar{G})] < \infty$ and $\Delta(\bar{G})'$ is finite. By [**7**, Theorem 3.13], $\Delta(\bar{G})$ satisfies Min-p for all primes p. By [**1**, Lemma 1.4], $[\Delta(\bar{G}) : Z(\Delta(\bar{G}))] < \infty$. Thus $\bar{G}$ is abelian-by-finite. By [**1**, Lemma 1.3(ii)], we may assume that $\bar{G}$ is abelian.

Let $\pi_1 = \pi \cup \pi(\bar{G})$ and let $A_{\pi_1'} = \prod_{p \notin \pi_1} A_p$. Then $\pi(G/A_{\pi_1'}) \subseteq \pi_1$. By [**9**, Lemma 12.4.12], there exists a π_1-subgroup Q of G with $G = A_{\pi_1'}Q$. By [**1**, Lemma 1.3(ii), Proposition 1.2 and Lemma 1.4], Q is abelian-by-finite. By [**1**, Lemma 1.3(ii)], we may assume that Q is abelian. Since $\pi(Q)$ is finite and satisfies Min-p for all primes p, Q has a minimal subgroup of finite index. Thus by [**1**, Lemma 1.3(ii)], we may assume that Q contains no proper subgroup of finite index.

We shall see that G is abelian. Let q be a prime such that $q \notin \pi_1$. By [**1**, Lemma 1.3(ii), Proposition 1.2 and Lemma 1.4], $A_q Q$ has an abelian normal subgroup B of finite index. Now $[Q : Q \cap B] < \infty$, thus $Q \leq B$. Now B is the direct product $B = \prod B_p$ of its p-primary parts. Let $B_{\pi_1} = \prod_{p \in \pi_1} B_p$. Since Q is a π_1-group, $Q \leq B_{\pi_1}$. Thus $A_q Q = A_q B_{\pi_1}$. Since B_{π_1} is a normal subgroup of $A_q B_{\pi_1}$, $A_q Q$ is abelian. Hence $G = A_{\pi_1'} Q$ is abelian. ■

Theorem 3 ([**1**, Theorem 2.3])**.** *The type I_f part of $Q^r(K[G])$ is non-zero iff $[G : \Delta(G)] < \infty$ and $\Delta(G)'$ is finite. Furthermore, in this case the type I_f part of $Q^r(K[G])$ is isomorphic to $Q^r(K[G/M])$, where $M = \cap L'$ and the intersection is over all subgroups L of G of finite index.*

Proof: The proof of the "if" part and the second part is as in [**1**].

Suppose that the type I_f part of $Q^r(K[G])$ is non-zero. Suppose that $[G : \Delta(G)] = \infty$ or $|\Delta(G)'| = \infty$. By [**1**, Lemma 1.3(i)], there exists a non-zero central idempotent $u \in K[G]$ such that $uK[G]$ has bounded index of nilpotence. Thus, by [**9**, Theorem 5.3.15], $uK[G]$ does not satisfy any polynomial identity. By [**1**, Lemma 2.2], there exists an irreducible $uK[G]$-module V with representation $\rho\colon K[G] \to \operatorname{End} V$ such that $\rho(G)$ has no abelian subgroup of finite index.

Since $uK[G]$ has bounded index of nilpotence, by [**4**, Corollary 7.10], V is finite dimensional over its commuting ring. By [**5**, Lemma 2.4], $\rho(G)$ satisfies Min-p for all primes p. Let $\varphi\colon K[G] \to K[\rho(G)]$ the natural projection. It is easy to see that $\varphi(u)$ is a non-zero central idempotent of $K[\rho(G)]$. By [**4**, Proposition 7.7], $\varphi(u)K[\rho(G)]$ has bounded index of nilpotence. By [**4**, Corollary 7.4 and Theorems 7.20 and 10.24], the type I_f part of $Q^r(K[\rho(G)])$ is non-zero. By Lemma 2, $\rho(G)$ is abelian-by-finite, a contradiction, thus $[G : \Delta(G)] < \infty$ and $|\Delta(G)'| < \infty$. ■

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References

1. F. Cedó, On the maximal quotient ring of regular group rings, *J. Algebra* **115** (1988), 164–174.
2. F. Cedó, The maximal quotient ring of regular group rings. II, *Proc. Amer. Math. Soc.* **104** (1988), 357–362.

3. F. CEDÓ AND B. HARTLEY, Maximal quotient rings and essential right ideals in group rings of locally finite groups, *Publ. Mat.* **36** (1992), 473–480.

4. K. R. GOODEARL, "*Von Neumann Regular Rings*," Pitman, London, 1979.

5. B. HARTLEY, Injective modules over group rings, *Quart. J. Math. Oxford Ser. (2)* **28** (1977), 1–29.

6. B. HARTLEY AND M. A. SHAHABI SHOJAEI, Finite groups of matrices over division rings, *Math. Proc. Cambridge Philos. Soc.* **92** (1982), 55–64.

7. O. H. KEGEL AND B. A. F. WEHRFRITZ, "*Locally Finite Groups*," North-Holland, Amsterdam, London, 1973.

8. G. A. MILLER AND H. MORENO, Nonabelian groups in which every subgroup is abelian, *Trans. Amer. Math. Soc.* **4** (1903), 398–404.

9. D. S. PASSMAN, "*The Algebraic Structure of Group Rings*," Wiley-Interscience, New York, 1977.

10. R. L. SNIDER, Group algebras whose simple modules are finite dimensional over their commuting rings, *Comm. Algebra* **2** (1974), 15–25.

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