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ALGEBRAIC CHARACTERISTIC CLASSES FOR IDEMPOTENT MATRICES

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To the memory of Pere Menal

Abstract

This paper contains the algebraic analog for idempotent matrices of the Chern-Weil theory of characteristic classes. This is used to show, algebraically, that the canonical line bundle on the complex projective space is not stably trivial. Also a theorem is proved saying that for any smooth manifold there is a canonical epimorphism from the even dimensional algebraic de Rham cohomology of its algebra of smooth functions onto the standard even dimensional de Rham cohomology of the manifold.

1. Introduction

In this paper I comment about algebraic characteristic classes and some related problems.

The idea for defining characteristic classes for finitely generated projective modules comes from the equivalence between vector bundles and finitely generated projective modules given by the cross section functor, see Swan [**12**]. These classes were defined by Ozeki, [**9**], by imitating the *Chern Weil* construction via principal connections. The classes introduced by Ozeki belonged to a cohomology of the ring based on the derivations of such a ring. Later Kong, [**7**], defined the *Euler* class for inner product projective modules by using linear connections on the module and the algebraic *de Rham* cohomology of the ring, i.e. the cohomology of the exterior algebra of the *Kähler* differentials with the canonical extension of the universal derivative. See also Karoubi, [**6**], for a generalization to the noncommutative case and its relation with the corresponding theory using cyclic homology, see Connes, [**2**].

The functors $A \to P_A$, $A \to I_A$ are naturally equivalent, where $A \to P_A$ associates to each commutative ring with unit, A, the isomorphism class of finitely generated projective modules over A, and $A \to I_A$

associates to such a ring A the equivalence class of idempotent matrices with entries in A, where an idempotent matrix φ is equivalent to an idempotent matrix φ' if and only if there exist matrices M and N such that $MN = \varphi$, $NM = \varphi'$. This equivalence is obtained by sending the class of φ in I_A to the class of the image of φ in P_A. The above equivalence induces one between the abelian group $\tilde{K}_0(A)$ of stably isomorphism classes of finitely generated projective modules and the corresponding abelian group obtained from the idempotent matrices. The opposite of the class represented by an idempotent matrix φ is clearly the class represented by $I_n - \varphi$, where I_n is the $n \times n$ identity matrix.

2. Examples.

a) Consider the idempotent matrix

$$\varphi = \begin{pmatrix} 1-X^2 & -XY & -XZ \\ -XY & 1-Y^2 & -YZ \\ -XZ & -YZ & 1-Z^2 \end{pmatrix}$$

with entries in $A = R[X,Y,Z]/(X^2+Y^2+Z^2-1)$, where R is any commutative ring with unit.

The class φ in $\tilde{K}_0(A)$ is zero. In fact the opposite class is represented by the idempotent matrix

$$\begin{pmatrix} X^2 & XY & XZ \\ XY & Y^2 & YZ \\ XZ & YZ & Z^2 \end{pmatrix} = MN$$

with $M = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$, $N = (X\ Y\ Z)$ and we have $NM = (1)$.

If A is the ring of C^∞ functions on the 2-sphere S^2, the above matrix φ represents the tangent bundle of S^2 and this example is the algebraic version of the well kown fact that the tangent bundle of the 2-sphere is stably trivial.

b) If $R = \mathbf{R}$(real numbers) or $R = \mathbf{C}$ (complex numbers), the *Grassmannian* of r-planes in R^n, $G_r(R^n)$, is diffeomorphic to the manifold whose points are the $n \times n$ idempotent matrices $\varphi = (\varphi_i^j)$ such that $\bar{\varphi} = \varphi^t$ and having rank r ($\bar{\varphi}$ denotes conjugate and φ^t transpose). The canonical r-plane bundle γ is then represented by the $n \times n$ idempotent matrix $\gamma = (\gamma_i^j)$ where $\gamma_i^j : G_r(R^n) \to R$ is given by $\gamma_i^j(\varphi) = \varphi_i^j$. Therefore if $R = \mathbf{R}$(resp.$R = \mathbf{C}$) the idempotent matrix γ makes sense in the quotient of the polynomial ring $R[X_i^j]_{i,j=1,\dots,n}$(resp. $R[i][X_i^j, Y_i^j]_{i,j=1,\dots,n}$) by the ideal generated by the polinomial identities for the conditions

$\gamma^2 = \gamma$, $\bar{\gamma} = \gamma^t$ and $rank\gamma = r$. This allows us to define what a *canonical r-plane bundle* should be in general for any commutative ring with unit R (resp. $R[i]$ with $i^2 = -1$).

For instance if $r = 1, n = 2$ the ring is the quotient of $R[i][X_i^j, Y_i^j]_{i,j=1,2}$ by the ideal generated by Y_1^1, Y_2^2, $X_1^2 - X_2^1$, $Y_1^2 + Y_2^1$, $X_1^1 + X_2^2 - 1$, $X_1^1(1 - X_1^1) - ((X_1^2)^2 + (Y_1^2)^2)$; which is isomorphic to

$$R[i][X_1^1, X_1^2, Y_1^2]/(X_1^1(1 - X_1^1) - ((X_1^2)^2 + (Y_1^2)^2)$$

and the algebraic version of the canonical complex line bundle should be the idempotent matrix

$$\gamma = \begin{pmatrix} X_1^1 & X_1^2 + iY_1^2 \\ X_1^2 - iY_1^2 & 1 - X_1^1 \end{pmatrix}$$

If $\frac{1}{2} \in R$ we can write $X = 2X_1^2$, $Y = 2Y_1^2$, $Z = 2X_1^1 - 1$ and the ring above is isomorphic to $A = R[i][X, Y, Z]/(X^2 + Y^2 + Z^2 - 1)$, where R is any integral domain containing $\frac{1}{2}$.

Choose now any commutative differential graded algebra (Ω, d) with $\Omega^0 = A$. For instance the algebraic *de Rham* complex, i.e. $\Omega^0 = A$, Ω^1 are the *Kähler* differentials with the universal derivative $d : \Omega^0 \to \Omega^1$, Ω^p is the p-th exterior power of Ω^1 and $d : \Omega^p \to \Omega^{p+1}$ is the canonical extension of $d : A \to \Omega^1$. In our case is clear that $\Omega^p = 0$ for $p \geq 3$ and $\Omega^2 = A.\omega$ is a free module of rank one, where $\omega = XdYdZ + YdZdX + ZdXdY$.

It makes sense now to consider the matrix $\gamma(d\gamma)^2$ with entries in Ω^2 and a straightforward computation yields $trace(\gamma(d\gamma)^2) = -\frac{i}{2}\omega$.

Observe that for a matrix α with entries in $R[i]$ one has $d\alpha = 0$ and for a matrix $\psi = \begin{pmatrix} \gamma & 0 \\ 0 & I_m \end{pmatrix}$ where I_m is the $m \times m$ identity matrix, one has $trace\psi(d\psi)^2 = trace(\gamma(d\gamma)^2)$. These two observations and the following lemma tell us that to prove that γ does not represent the zero class in $\tilde{K}_0(A)$ we must show that $trace(\gamma(d\gamma)^2)$ is not of the form $d\alpha$ for some $\alpha \in \Omega^1$. In our case we must show that $\omega \notin Im(d)$.

Lemma. *If φ and φ' are idempotent matrices with $MN = \varphi$, $NM = \varphi'$ for some matrices M, N; then $trace(\varphi'(d\varphi')^2) - trace(\varphi(d\varphi)^2) = d(trace(\varphi M dN))$.*

Proof:

$$trace(\varphi'(d\varphi')^2) - trace(\varphi(d\varphi)^2) =$$
$$= trace(NM(dN.M + N.dM)^2 - MN(dM.N + M.dN)^2) =$$
$$= trace(\varphi dMdN - \varphi' dNdM - 2\varphi dM\varphi' dN).$$

The last equality comes from the observation that for α, β matrices with entries in Ω^p, Ω^q respectively one has $trace(\alpha\beta) = (-1)^{pq} trace(\beta\alpha)$. In particular $trace(\alpha^2) = 0$ for p odd.

On the other hand

$$d(trace(\varphi M dN)) = trace(d(MNMdN)) =$$
$$= trace(dM.NMdN + MdN.MdN + MNdM.dN) =$$
$$= trace\varphi dMdN - trace\varphi' dNdM.$$

Therefore to conclude the proof of the lemma one must show that $trace(\varphi dM\varphi' dN = 0$.

But we have

$$trace(\varphi dM\varphi' dN) = trace(\varphi^2 dM\varphi' dN) =$$
$$= trace(M\varphi' NdM\varphi' dN) = trace(\varphi' NdM\varphi' dN.M) =$$
$$= trace(\varphi' NdM\varphi' d\varphi') - trace(\varphi' NdM\varphi' NdM) =$$
$$= trace(\varphi' NdM\varphi' d\varphi') = trace(NdM\varphi' d\varphi'.\varphi') = 0,$$

because $\varphi' d\varphi'.\varphi' = 0$.

In fact, $d\varphi' = d(\varphi')^2 = d\varphi'.\varphi' + \varphi' d\varphi'$ implies $d\varphi'.\varphi' = d\varphi'.\varphi' + \varphi' d\varphi'.\varphi'$

Hence $\varphi' d\varphi'.\varphi' = 0$. ■

If *we assume* that R contains the rational numbers, then we may show that $\omega \notin Im(d)$ by considering the $R[i]-$linear map given by Kong, cf. page 297 of [7], $\rho : \Omega^2 \to R[i]$ given by $\rho(\omega) = 1, \rho(X^\alpha Y^\beta.\omega) = 0$ if α or β are odd, $\rho(X^\alpha Y^\beta Z.\omega) = 0, \rho(X^{2n} Y^{2m}.\omega) = \frac{1}{2(n+m)+1} \cdot \frac{(2m)!(2n)!}{2(m+n)!} \cdot \frac{(m+n)!}{m!n!}$ and check that ρ vanishes on the image of d.

Actually it is not difficult to give a direct proof showing that ω is not in the image of d for any ring $R[i]$ as we have considered, i.e. an integral domain containing $\frac{1}{2}$.

3. Characteristic classes for idempotent matrices

The examples above suggest how to define characteristic classes for idempotent matrices φ having entries in an R-algebra A where both R and A are commutative rings with the same unit element. One simply chooses a commutative graded differential algebra (Ω, d) with $\Omega^0 = A$. For instance take (Ω, d) as the algebraic *de Rham* complex of A. Then $det(\varphi(d\varphi)^2 + I_n) = 1 + c_1(\varphi) + \cdots + c_n(\varphi)$, where I_n is the identity $n \times n$ matrix and $c_p(\varphi) \in \Omega^{2p}$ is the p-th characteristic coefficient of φ.

Define also $Tr_p(\varphi) = trace(\varphi(d\varphi)^2)^p \in \Omega^{2p}$ for $p \geq 1$ (the trace coefficient of φ). They are related by $Tr_p = Q_p(C_1, \ldots, C_p)$, $p!C_p = P_p(Tr_1, \ldots, Tr_p)$ for polynomials P_i, Q_i in $\mathbf{Z}[X_1, \ldots, X_i]$.

Lemma. *$dTr_p(\varphi) = 0$. Thus if R contains the rationals one has $dc_p(\varphi) = 0$.*

Proof: We have already seen that $\varphi d\varphi.\varphi = 0$ and $\varphi(d\varphi)^2 = (d\varphi)^2\varphi$. Observe that we have $trace(\varphi(d\varphi)^{2p+1}) = 0$.
In fact,

$$\varphi(d\varphi)^{2p+1} = \varphi^2(d\varphi)^{2p+1}$$

Thus

$$trace(\varphi(d\varphi)^{2p+1}) = trace(\varphi(d\varphi)^{2p+1}\varphi) = trace(\varphi d\varphi.\varphi.(d\varphi)^{2p}) = 0.$$

But

$$(d\varphi)^{2p+1} = d\varphi.\varphi.(d\varphi)^{2p} + \varphi(d\varphi)^{2p+1}.$$

Therefore

$$trace(d\varphi)^{2p+1} = trace(d\varphi.\varphi.(d\varphi)^{2p}) = trace(\varphi(d\varphi)^{2p+1}) = 0. \quad \blacksquare$$

As a consequence of this lemma $c_p(\varphi)$ represents a cohomology class in $H^{2p}(\Omega, d)$ called the *pth-Chern class* of φ. In the example above we have computed the first *Chern class* of γ.

If $R = \mathbf{R}$, $A = C^\infty(X)$ and φ is the idempotent matrix corresponding to a vector bundle ξ over the C^∞ manifold X, one has that the cohomology class represented by $c_p(\varphi)$ is zero for p odd and the cohomology class represented by $c_{2q}(\varphi)$ equals $(2\pi)^{2q}.p_q(\xi)$, where $p_q(\xi)$ is the q-th *Pontrjagin* class of ξ. If $R = \mathbf{C}$, $A = C^\infty(X; \mathbf{C})$ and φ corresponds to a complex vector bundle ξ, the cohomology class represented by $c_p(\varphi)$ equals $(-2\pi i)^p c_p(\xi)$, where $c_p(\xi)$ is the p-th *Chern* class of ξ.

The next proposition shows that actually the p-th *Chern* class is defined for elements in $\tilde{K}_0(A)$.

Proposition. *If φ and φ' are equivalent idempotent matrices, then $c_p(\varphi') - c_p(\varphi)$ belongs to the image of d.*

Proof: Suppose $MN = \varphi, NM = \varphi'$. We proceed as follows. Observe first that the images of φ and φ' are finitely generated projective modules and that $N : Im(\varphi) \to Im(\varphi')$ is an isomorphism with inverse M. Define then linear connections $\nabla : Im(\varphi) \to Im(\varphi) \otimes \Omega^1$, $\nabla' : Im(\varphi') \to Im(\varphi') \otimes \Omega^1$ and $\tilde{\nabla} : Im(\varphi') \to Im(\varphi') \otimes \Omega^1$ as follows:

$$\nabla x = \sum_{i=1}^{n} dx_i \otimes \varphi(e_i)$$

for $x = \sum_{i=1}^{n} x_i e_i \in Im(\varphi) \subset A^n$, where $e_1, \dots, e_n$ is the canonical basis of A^n.

$$\nabla' x' = \sum_{i=1}^{m} dx'_i \otimes \varphi'(e'_i)$$

for $x' = \sum_{i=1}^{m} x'_i . e'_i \in Im(\varphi) \subset A^m$, where $e'_1, \dots, e'_n$ is the canonical basis of A^m.

$\tilde{\nabla}$ such that the following diagram commutes

$$\begin{array}{ccc} Im(\varphi) & \xrightarrow{\nabla} & Im(\varphi) \otimes \Omega^1 \\ \downarrow N & & \downarrow N \otimes id \\ Im(\varphi') & \xrightarrow{\tilde{\nabla}} & Im(\varphi') \otimes \Omega^1 \end{array}$$

Denote by R_∇, $R_{\nabla'}$ and $R_{\tilde{\nabla}}$ the corresponding curvatures for ∇, ∇', $\tilde{\nabla}$. ∎

Remark. Recall that if $\nabla : M \to \Omega^1 \otimes M$ is a linear connection for a finitely generated projective A-module M, then $\nabla^2 : M \to \Omega^2 \otimes M$ is A-linear and so it can be regarded as an element R_∇ of $\Omega^2 \otimes L_M$ which is called the curvature of ∇. If ∇ denotes also the induced linear connection on L_M and the corresponding covariant exterior derivative (just imitating the usual definitions in Differential Geometry) one has the *Bianchi identity* $\nabla R_\nabla = 0$.

An easy computation shows that $trace(R_\nabla \circ \cdots \circ R_\nabla) = trace(\varphi(d\varphi)^{2p})$, $trace(R_{\nabla'} \circ \cdots \circ R_{\nabla'}) = trace(\varphi'(d\varphi')^{2p})$ and $trace(R_\nabla \circ \cdots \circ R_\nabla) = trace(R_{\tilde{\nabla}} \circ \cdots \circ R_{\tilde{\nabla}})$.

Therefore to finish the proof we must check that for any two linear connections ∇_1, ∇_2 on a finitely generated projective module, we have that $trace(R_{\nabla_1} \circ \cdots \circ R_{\nabla_1}) - trace((R_{\nabla_2} \circ \cdots \circ R_{\nabla_2})$ belongs to the image of d. But since $\nabla_2 - \nabla_1 \in \Omega^1 \otimes L_M$, it is enough to do it for $\nabla_2 = \nabla_1 + \alpha \otimes \psi$ with $\psi \in L_M, \alpha \in \Omega^1$. In this case $R_{\nabla_2} = R_{\nabla_1} + \nabla(\alpha \otimes \psi)$ and the proof is an easy consequence of *Bianchi identity* and the relation $\nabla \circ trace = trace \circ d$.

4. Some questions

a) Suppose $A = C^\infty(X)$, where X is a C^∞−manifold, we have then a canonical homomorphism $H^*_{dR}(C^\infty(X)) \to H^*_{dR}(X)$ where $H^*_{dR}(C^\infty(X))$ denotes the algebraic *de Rham* cohomology of $C^\infty(X)$ and $H^*_{dR}(X)$ denotes the usual *de Rham* cohomology of X.

It is well known that the above homomorphism is not, in general, an isomorphism, see proposition 6, page 143 of [**10**], but we have the following theorem

Theorem. *The homomorphism $H^*_{dR}(C^\infty(X)) \to H^*_{dR}(X)$ is an epimorphism in even dimensions.*

Proof: Observe that if ξ is a vector bundle over X the algebraic *Chern* classes of the corresponding idempotent matrix are mapped canonically to the corresponding characteristic classes of ξ. Then we use the fact that any even dimensional de Rham class, with coefficients in $\mathbf{Q}$,$\mathbf{R}$ or $\mathbf{C}$ is the characteristic class of some complex vector bundle over X. This is true because of the isomorphism given by the *Chern character* $ch : K(X) \otimes_{\mathbf{Z}} \mathbf{Q} \to \prod_{p \geq 0} H^{2p}(X; \mathbf{Q})$, see page 119 of [**3**].

We can pose then the question:*what happens in odd dimensions?* ∎

b) It seems natural to consider for a topological space X the algebraic *de Rham* cohomology of its algebra of real or continuous functions. *What can be said of such a cohomology?.* Except for H^0 or trivial cases, see [**4**], I do not know anything else.

c) It should be interesting to find, for a given space, commutative differential graded algebra (Ω, d) such that $\Omega^0 = A$ is some subalgebra of functions of X such that we have isomorphisms $H_{dR}(\Omega, d) \cong H^*(X)$ and $\tilde{K}_0(A) \cong \tilde{K}_0(X)$. See also Carral [**1**],Lønsted [**8**] and Swan [**11**] for some other related problems.

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