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THE p-PERIOD OF AN INFINITE GROUP

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Abstract

For Γ a group of finite virtual cohomological dimension and a prime p, the p-period of Γ is defined to be the least positive integer d such that Farrell cohomology groups $\hat{H}^i(\Gamma; M)$ and $\hat{H}^{i+d}(\Gamma; M)$ have naturally isomorphic p-primary components for all integers i and $Z\Gamma$-modules M.

We generalize a result of Swan on the p-period of a finite p-periodic group to a p-periodic infinite group, i.e., we prove that the p-period of a p-periodic group Γ of finite *vcd* is $2LCM(|N(\langle x\rangle)/C(\langle x\rangle)|)$ if the Γ has a finite quotient whose a p-Sylow subgroup is elementary abelian or cyclic, and the kernel is torsion free, where $N(-)$ and $C(-)$ denote normalizer and centralizer, $\langle x\rangle$ ranges over all conjugacy classes of Z/p subgroups. We apply this result to the computation of the p-period of a p-periodic mapping class group. Also, we give an example to illustrate this formula is false without our assumption.

For Γ a group of virtual finite cohomological dimension (*vcd*) and a prime p, the p-period of Γ is defined to be the least positive integer d such that the Farrell cohomology groups $\hat{H}^i(\Gamma; M)$ and $\hat{H}^{i+d}(\Gamma; M)$ have natually isomorphic p-primary components for all $i \in Z$ and $Z\Gamma$-modules M [**3**].

The following classical result for a finite group G was showed by Swan in 1960 [9].

Theorem (Swan).

a) *If a 2-Sylow subgroup of G is cyclic ($\neq \{1\}$), the 2-period of G is 2. If a 2-Sylow subgroup of G is a (generalized) quaternion group, the 2-period of G is 4.*

b) *Suppose p an odd prime and a p-Sylow subgroup of the finite group G is cyclic ($\neq \{1\}$). Let S_p denote the p-Sylow subgroup and A_p the group of automorphisms of S_p induced by inner automorphism of G. Then the p-period of G is twice the order of A_p.*

Remark.

The group A_p above is isomorphic to $N(S_p)/C(S_p)$, where $N(-)$ and $C(-)$ denote the normalizer and centralizer of S_p in G.

It is very natural to ask a question: If Γ is a p-periodic group of finite *vcd*, is a similar result still true? In other words, is it possible to describe the p-period of a p-periodic group Γ of finite *vcd* by an algebraic *non-homological* invariant of the group Γ itself?

In this paper, we generalize the result of Swan for a finite group to a p-periodic group Γ of finite *vcd* which has a finite quotient whose a p-Sylow subgroup is elementary abelian or cyclic, and the kernel is torsion-free, i.e., we prove that the p-period of a p-periodic group Γ of finite *vcd* is twice the least common multiple of $\{|N(\langle x\rangle)/C(\langle x\rangle)|\}$ in these two cases, where $\langle x\rangle$ ranges over all conjugacy classes of Z/p subgroups of Γ. On the other hand, we give a group Γ_0 of finite *vcd* whose only finite subgroup is a $Z/2$, but the 2-period of Γ_0 is greater than $2|N(Z/2)/C(Z/2)|$. Finally, an application will be made for calculating the p-period of a mapping class group.

The following four theorems are our main results of this paper.

Theorem 1. *Assume that Γ is p-periodic. If Γ has a normal subgroup of finite cohomological dimension so that the associated quotient is a finite group whose a p-Sylow subgroup is elementary abelian, then the p-period of Γ is twice the least common multiple of $\{|N(\langle x\rangle)/C(\langle x\rangle)|\}$, where $\langle x\rangle$ ranges over all conjugacy classes of Z/p subgroups of Γ.*

Theorem 2. *Let Γ be a group which has a normal subgroup of finite cohomological dimension so that the associated quotient is a finite group whose a p-Sylow subgroup is cyclic, then the p-period of Γ is twice the least common multiple of $\{|N(\langle x\rangle)/C(\langle x\rangle)|\}$, where $\langle x\rangle$ ranges over all conjugacy classes of Z/p subgroups of Γ.*

Theorem 3. *There is a group Γ_0 of finite vcd whose only finite subgroup is a $Z/2$, but the 2-period is greater than $2|N(Z/2)/C(Z/2)|$.*

Theorem 4. *If the mapping class group Γ_g is a p-periodic group and $g < p(p-1)/2$, then the p-period of Γ_g is $2LCM\{\gcd(p-1, b_i)\}$, where $b_i \in B_{g,p}$ (cf. section 3).*

The rest of this paper is organized as follows. In section 1, we prove Theorems 1 and 2. In section 2, we provide an example illustrating Theorem 3. Finally in section 3, we give a formula for the calculation of the p-period of a p-periodic mapping class group Γ_g.

1. Proof of Theorems 1 and 2

Lemma 1.1. *Let $H = \langle x, y / x^p = 1, yxy^{-1} = x^r \rangle$, where $q = 0$ or $q \neq 0$ mod(p). If d is the minimal positive integer such that $r^d = 1$ mod(p), then the p-period of H equals $2d$.*

Proof: If $q \neq 0$, H is a finite group, the proof is immediate by Swan Theorem. Otherwise, if $q = 0$, H is infinite and we look at the short exact sequence $1 \to Z/p \to H \to Z \to 1$. The spectral sequence of Farrell cohomology associated to the exact sequence converges in the following way: $E_2^{i,j} = H^i(Z; \hat{H}^j(Z/p; Z)) \to \hat{H}^{i+j}(H; Z)$ [2]. This spectral sequence collapses since $H^i(Z; \hat{H}^j(Z/p; Z)) = 0$ when $i < 0$ or $i > 1$. Therefore, $1 \to \hat{H}^{n-1}(Z/p; Z)_Z \to \hat{H}^n(H; Z) \to \hat{H}^n(Z/p; Z)^Z \to 1$ is an exact sequence. By looking at the Z action on the subgroup $Z/p, u^d \in \hat{H}^{2d}(Z/p; Z)$ is an invariant element of the Z action on $\hat{H}^{2d}(Z/p; Z)$. Here u is a generator of $\hat{H}^2(Z/p, Z)$. Therefore, there exists an element $h \in \hat{H}^{2d}(H; Z)$ such that $\mathrm{Res}(h) = u^d \neq 0$ on $\hat{H}^{2d}(Z/p; Z)$. By Brown-Venkov theorem [2] and $\hat{H}^{2kd}(H; Z) = Z/p$, $\hat{H}^{2kd+1}(H; Z) = Z/p$, $\hat{H}^i(H; Z) = 0$ for other i's, the p-period of H is $2d$. ∎

Lemma 1.2. *Let Z/p be a normal subgroup of a group Γ of finite vcd, and let M be a finite quotient of Γ with torsion free kernel. Then $\Gamma / C_\Gamma(Z/p) = N_\Gamma(Z/p)/C_\Gamma(Z/p) = N_M(Z/p)/C_M(Z/p) = M/C_M(Z/p)$. Here we still use Z/p to stand for the image of Z/p in M.*

Proof: Let $pr : \Gamma \to M$ be the natural projection map. The map pr maps $N_\Gamma(Z/p)$ onto $N_M(Z/p)$ and $C_\Gamma(Z/p)$ to $C_M(Z/p)$, so induced map $pr_* : N_\Gamma(Z/p)/C_\Gamma(Z/p) \to N_M(Z/p)/C_M(Z/p)$ is a well-defined surjective homomorphism. Let $\langle x \rangle = Z/p$, if $yxy^{-1} = x^r$, then $pr(y)xpr(y)^{-1} = x^r$, i.e., pr_* is an injective. ∎

Lemma 1.3. *Suppose a group M contains a cyclic subgroup $Z/p^n \supset Z/p$ and $|N(Z/p^n)/C(Z/p^n)|$ is prime to p, then the homomorphism induced by inclusion $i_* : N(Z/p^n)/C(Z/p^n) \to N(Z/p)/C(Z/p)$ is injective.*

Proof: Notice $N(Z/p) \supset N(Z/p^n)$ and the inclusion i maps $C(Z/p^n)$ to $C(Z/p)$, i.e., the induced map by inclusion $i_* : N(Z/p^n)/C(Z/p^n) \to N(Z/p)/C(Z/p)$ is a well-defined homomorphism. Now let $\langle x \rangle = Z/p^n$, then $\langle x^{p^{n-1}} \rangle = Z/p$, if $y \in C(Z/p)$, $yxy^{-1} = x^k$, then $yx^{p^{n-1}}y^{-1} = x^{kp^{n-1}} = x^{p^{n-1}}$, so $(k-1)p^{n-1} = 0 \bmod(p^n)$, i.e., $k = 1 \bmod(p)$. Let $k = Ap^m + 1$, A is prime to p and $1 \leq m < n$, $k^d = 1 \bmod(p^n)$, d divides $p-1$

by assumption. Hence $k^d = (Ap^m + 1)^d = B + Adp^m + 1 = 1 \bmod(p^n)$, where p^{2m} divides B. This implies $Ad = 0 \bmod(p)$, a contradiction unless $A = 0$. ■

Lemma 1.4 (Swan) [9]. *Suppose the p-Sylow subgroup S_p of a finite group M is abelian. Let A_p be the group of automorphisms of S_p induced by inner automorphisms of M. Then an element $a \in H^i(S_p; Z)$ is stable if and only if it is fixed under the action of A_p on $H^i(S_p; Z)$.*

Proof: See [9]. ■

Proof of Theorem 1: A theorem of Brown [**3**, p. 293] states that if Γ is p-periodic, then $\hat{H}^*(\Gamma; Z)_{(p)} = \Pi_{P_j \in S} \hat{H}^*(N(P_i); Z)_{(p)}$, where S is the set of all conjugacy classes of Z/p of Γ. Therefore, the p-period of Γ is the least common multiple of the p-periods of $N_\Gamma(P_i)$.

1) Lower bound. Let $|N_\Gamma(P_i)/C_\Gamma(P_i)| = d_i$, $\langle x \rangle = P_i$. There exists $y \in \Gamma$, such that $yxy^{-1} = x^r$, $r^{d_i} = 1 \bmod(p)$. Let $H = \langle x, y \rangle$ be a subgroup of Γ generated by elements x and y. Then the p-period of H is $2d_i$ by Lemma 1.1, i.e., the p-period of $N_\Gamma(P_i)$ is a multiple of $2d_i$.
2) Upper bound. Let $pr : \Gamma \to M$ be a projection onto the finite quotient M whose a p-Sylow subgroup is elementary abelian, and $pr_i : N_\Gamma(P_i) \to M_i$ be the restriction map of pr, where M_i is the image of pr_i. Then $M_i = Im N_\Gamma(P_i) = N_{M_i}(P_i)$ normalizes P_i (P_i also denotes the image of P_i), the group A_p of automorphisms of S_p induced by inner automorphisms of M_i maps P_i to itself.

Let $u \in H^2(S_p; Z) = \text{Hom}(P_i \times Z/p \times \ldots\ldots Z/p, C^*)$ be a cohomology element such that $u(x) \neq 1$ and $u(y) = 1$ if $\langle x \rangle = P_i$, $\langle y \rangle = Z/p$, where C^* is the multiple group of nonzero complex numbers. Then $\text{Res}(u) \neq 0$ in $H^2(P_i; Z)$. Now we claim that $u^{d_i} \in H^{2d_i}(S_p; Z)$ is a stable element for S_p in M_i. In fact, $d_i = |N_{M_i}(P_i)/C_{M_i}(P_i)|$ by Lemma 1.2, and A_p fixes the element $u^{d_i} \in H^{2d_i}(S_p; Z)$ since $N_{M_i}(P_i)/C_{M_i}(P_i)$ fixes the element u^{d_i}. By Lemma 1.4 [9], u^{d_i} is a stable element for S_p in M_i, i.e., there exists an element $v \in H^{2d_i}(M_i; Z)$ such that $Res\, s_{P_i}^{Mi}(v) = Res\, s_{P_i}^{Sp}(u^{d_i}) = [Res\, s_{P_i}^{Sp}(u)]^{d_i} \neq 0$. If we apply the canonical homomorphism g^* from ordinary cohomology to Farrell cohomology [**3**, p. 278] we have $Res\, s_{P_i}^{Mi}(g^*(v)) = Res\, s_{P_i}^{Sp}(g^*(u^{d_i})) = Res\, s_{P_i}^{Sp}(g^*(u))^{d_i} \neq 0$, i.e., there exists an element $pr_i^* g^*(v) \in \hat{H}^{2d_i}(N_\Gamma(P_i); Z)$ such that $Res\, s_{P_i}^{N(Pi)}(pr_i^* g^*(v)) \neq 0$ in $\hat{H}^{2d_i}(P_i; Z)$, by Brown-Venkov theorem [**2**] and the fact that $N_\Gamma(P_i)$ has only one order p subgroup, the p-period of

$N_\Gamma(P_i)$ divides $2d_i$. See following diagram.

$$
\begin{array}{ccccc}
\hat{H}^{2d_i}(N_\Gamma(P_i);Z) & \xrightarrow{\text{Res}} & & & \hat{H}^{2d_i}(P_i;Z) \\
{\scriptstyle pri^*}\uparrow & & & & \uparrow\| \\
\hat{H}^{2d_i}(M_i;Z) & \xrightarrow{\text{Res}} & \hat{H}^{2d_i}(S_p;Z) & \xrightarrow{\text{Res}} & \hat{H}^{2d_i}(P_i;Z) \\
\|{\scriptstyle g^*}\uparrow & & \|{\scriptstyle g^*}\uparrow & & \|{\scriptstyle g^*}\uparrow \\
H^{2d_i}(M_i;Z) & \xrightarrow[\text{Res}]{} & H^{2d_i}(S_p;Z) & \xrightarrow[\text{Res}]{} & H^{2d_i}(P_i;Z)
\end{array} \blacksquare
$$

Proof of Theorem 2: is basically a similar argument except for the upper bound part. In fact, if Γ has a finite p-periodic quotient M with torsion free kernel, then Γ is p-periodic and the p-period of Γ divides the p-period of M. This is because the inflation map $\hat{H}^*(M) \to \hat{H}^*(\Gamma)$ maps an invertible element of $\hat{H}^*(M)$ to an invertible element of $\hat{H}^*(\Gamma)$. Using Swan Theorem, we obtain that the p-period of $N_\Gamma(P_i)$ divides the p-period of M_i, which is $2|N_{M_i}(Z/p^n)/C_{M_i}(Z/p^n)|$. Also, by Lemma 1.3, the number $2|N_{M_i}(Z/p^n)/C_{M_i}(Z/p^n)|$ divides $2|N_{M_i}(P_i)/C_{M_i}(P_i)| = 2|N_\Gamma(P_i)/C_\Gamma(P_i)|$. $\blacksquare$

2. An example

Lemma 1.3, Lemma 1.1 and Swan Theorem imply that the equality $|N(S_p)/C(S_p)| = |N(Z/p)/C(Z/p)|$ holds in the case of a finite group G whose a p-Sylow subgroup is cyclic, here Z/p is the order p subgroup of S_p. Therefore, Theorems 1 and 2 are generalizations of Swan Theorem.

In the case of a group Γ of finite *vcd*, in general, $|N(S_p)/C(S_p)| \neq |N(Z/p)/C(Z/p)|$ even if all maximal p-subgroups S_p of Γ are cyclic. For example, let $\Gamma^* = \langle x, y | x^{p^2} = 1,\ yxy^{-1} = x^{p+1}\rangle$, and d is the minimal positive integer such that $(p+1)^d = 1 \bmod (p^2)$. Then $|N(\langle x\rangle)/C(\langle x\rangle)| = d = p$, but $|N(\langle x^p\rangle)/C(\langle x^p\rangle)| = 1$. A similar argument to Lemma 1.1 shows the p-period of Γ^* above equals $2p$. This trivial example shows that the p-period of an infinite group Γ can not be only described in the form $2LCM\{|N(Z/p)/C(Z/p)|\}$ in general.

The example Γ^* above could lead us to think that the p-period of a p-periodic group Γ equals $2LCM\{|N(C(p))/C(C(p))|\}$, where $C(p)$ ranges over all conjugacy classes of maximal p-cyclic subgroups of Γ. Recall in the case of a finite group G, Swan Theorem can be also stated in the different form: the p-period of G equals $2|N(C(p))/C(C(p))|$ (including the case $p = 2$), where $C(p)$ is a maximal p-cyclic subgroup of G.

Unfortunately, the next example shows that this is not true.

Example. Let $\Gamma_{n,m}$ denote the congruence subgroup of $SL(n,Z)$ of level m, i.e., the kernel of the surjective homomorphism $r_m : SL(n,Z) \to SL(n,Z/m)$ induced by the reduction mod(m) (m may not be prime). It is well-known that the group $\Gamma_{n,m}$ is always torsion free when $n \geq 1$ and $m \geq 3$. A result of Charney [4] states that the group $\Gamma_{n,p}$ is cohomology stable with $Z/2$ coefficient for any odd prime p. Define $\Gamma_p = \lim_n \Gamma_{n,p}$, then $H^i(\Gamma_{n,p}; Z/2) = H^i(\Gamma_p; Z/2)$ for $n \geq 2i+5$.

Let $GL(Z)$ be the infinite general linear group of Z and $w_i \in H^i(GL(Z); Z/2)$ the i-th Stiefel-Whitney class of the inclusion $GL(Z) \to GL(R)$ for $i \geq 1$. We still denote by w_i the image of w_i under the restriction $H^i(GL(Z); Z/2) \to H^i(SL(Z); Z/2) \to H^i(\Gamma_m; Z/2)$.

The calculation in [1] by Arlettaz gives following results: for any odd prime p

a) $w_1(\Gamma_p) = 0$
b) $w_2(\Gamma_p) \neq 0$
c) $w_3(\Gamma_p) = 0$ if and only if $p = 7$ mod(8).

Also, we know from Wu formula for the Steenrod square $Sq^1(w_2) = w_1 w_2 + w_0 w_3 = w_3$ in $H^3(\Gamma_p; Z/2)$. Again, denote by w_i the image of w_i under the restriction $H^i(\Gamma_5; Z/2) \to H^i(\Gamma_{11,5})$. Combining both results of Charney and Arlettaz above, we have $w_1 = 0$, $w_2 \neq 0$ and $Sq^1(w_2) = w_3 \neq 0$ in $H^*(\Gamma_{11,5}; Z/2)$ (in fact, these are all true for $H^*(\Gamma_{n,5}; Z/2)$ as long as $n \geq 11$.)

Let Γ_0 denote the group of the extension $1 \to Z/2 \to \Gamma_0 \to \Gamma_{11,5} \to 1$ which corresponds to the non-trivial cohomology element $w_2 \in H^2(\Gamma_{11,5}; Z/2)$. Obviously, the group Γ_0 contains only one 2-subgroup $Z/2$, and the extension is central. Next, we check that the group Γ_0 is of finite *vcd*, then show that the 2-period of Γ_0 is greater than 2.

Consider the following commutative diagram, where all maps R_1, R_2, R_3 and R_4 are restriction maps.

$$\begin{array}{ccc} H^2(\Gamma_{11,4}; Z/2) & \xrightarrow{R_3} & H^2(\Gamma_{11,20}; Z/2) \\ \uparrow{\scriptstyle R_1} & & \uparrow{\scriptstyle R_2} \\ H^2(SL(11,Z); Z/2) & \xrightarrow[R_4]{} & H^2(\Gamma_{11,5}; Z/2) \end{array}$$

In fact, the map $R_1 = 0$ is a special case of the result by Millson [7, p. 85] which states that for any $n \geq 3$ the map $r^* : H^2(SL(n, Z/4); Z/2) \to H^2(SL(n,Z), Z/2)$ induced by the reduction mod(4) is an isomorphism.

Thus, we obtain the nontrivial second Stiefel-Whitney class w_2 in $H^2(\Gamma_{11,5}; Z/2)$, but the restriction of w_2 into the cohomology of the finite index subgroup $H^2(\Gamma_{11,20}; Z/2)$ is 0. This actually proves that the group Γ_0 is finite vcd and the $vcd(\Gamma_0) = cd(\Gamma_{11,20}) = vcd(SL(11, Z)) = 55$ [**3**, p. 229].

In order to find a lower bound on the 2-period of Γ_0, consider two spectral sequences as follows:

1. The Lyndon-Hochschild-Serre spectral sequence of the group extension $1 \to Z/2 \to \Gamma_0 \to \Gamma_{11,5} \to 1$ with $Z/2$ coefficient. This takes the form $E_2^{i,j} = H^i(\Gamma_{11,5}; H^j(Z/2; Z/2)) \Rightarrow H^{i+j}(\Gamma_0; Z/2)$.
2. The Farrell cohomology spectral sequence [**2**] of the group extension $1 \to Z/2 \to \Gamma_0 \to \Gamma_{11,5} \to 1$ with $Z/2$ coefficient. This takes the form $E_2^{i,j} = H^i(\Gamma_{11,5}; \hat{H}^j(Z/2; Z/2)) \Rightarrow \hat{H}^{i+j}(\Gamma_0; Z/2)$.

Let $u \in H^1(Z/2; Z/2)$ be the generator of the cohomology ring $H^*(Z/2; Z/2) = F_2[u]$, and $d_2(u) = w_2 \in H^2(\Gamma'; Z/2)$ be the second Stiefel-Whitney class corresponding to the extension $1 \to Z/2 \to \Gamma_0 \to \Gamma_{11,5} \to 1$. Then u is transgressive, $d_2(u) = \tau(u) = w_2$, where τ is the transgression. The element $u^2 = Sq^1(u)$ is also transgressive [**8**, p. 81], and $d_3(u^2) = \tau(u^2) = \tau(Sq^1(u)) = Sq^1(\tau(u)) = Sq^1(w_2) = w_3 \neq 0$ in E_3 because $H^1(\Gamma_{11,5}; Z/2)$ is trivial.

Consider a commutative diagram involving in both spectral sequences as follows:

$$\begin{array}{ccc} H^0(\Gamma_{11,5}; \hat{H}^2(Z/2; Z/2)) & \xrightarrow{d_3} & H^3(\Gamma_{11,5}; \hat{H}^0(Z/2; Z/2)) \\ \uparrow \| g^* & & \uparrow \| g^* \\ H^0(\Gamma_{11,5}; H^2(Z/2; Z/2)) & \xrightarrow{d_3} & H^3(\Gamma_{11,5}; H^0(Z/2; Z/2)) \end{array}$$

The nontriviality of d_3 in the second row implies the nontriviality of d_3 in the first row. This shows Res:$\hat{H}^2(\Gamma_0; Z/2) \to \hat{H}^2(Z/2; Z/2)$ is trivial since the map Res factors through $E_\infty^{0,2} = 0$. Therefore, there is no invertible element in $\hat{H}^2(\Gamma_0; Z/2)$. By the fact that the reduced map $\hat{H}^2(\Gamma_0; Z)_{(2)} \to \hat{H}^2(\Gamma_0; Z/2)$ is ring homomorphism, there is no invertible element in $\hat{H}^2(\Gamma_0; Z)_{(2)}$, i.e., the 2-period of Γ_0 is greater than 2. We have proved our Theorem 3.

3. The p-period of the mapping class group Γ_g

The p-periodicity of the mapping class group is studied in a different paper of the author [**11**]. As an application of the theorem 1, we obtain the p-period of a p-periodic mapping class group Γ_g when $g < p(p-1)/2$.

Recall that the mapping class group Γ_g is defined to be the group of path components of orientation preserving diffeomorphisms of the closed orientable surface S_g of genus $g > 1$. Next, we define a set $B_{g,p}$ for surface S_g and a prime p.

Definition. For p odd, let $2g-2 = mp - i, 0 \leq i \leq p-1$.

$$B_{g,p} = \{i, i+p, i+2p, \ldots\ldots i + ([2g/(p-1)] - m)p\} \text{ if } i \neq 1.$$
$$B_{g,p} = \{1+p, 1+2p, \ldots\ldots 1 + ([2g/(p-1)] - m)p\} \text{ if } i = 1.$$

And for $p = 2$,

$$B_{g,2} = \{0, 4, 8, \ldots\ldots 2g+2\} \text{ if } g \text{ is odd.}$$
$$B_{g,2} = \{2, 6, 10, \ldots\ldots 2g+2\} \text{ if } g \text{ is even.}$$

Remarks.

1. The notation $[-]$ here means the integer part.
   In case $i \neq 1$, $2g/(p-1) < m$, define $B_{g,p} = \emptyset$.
   In case $i = 1$, $2g/(p-1) < m+1$, define $B_{g,p} = \emptyset$.
2. It is proved in [11] that the set $B_{g,p}$ is exactly the set of all possible number of fixed points when an order p diffeomorphism acts on the surface S_g.

Lemma 3.1. *For the mapping class group Γ_g, there is a formula $LCM\{|(N(\langle x\rangle)/C(\langle x\rangle)|\} = LCM\{\gcd(p-1, b_i)\}$, where $\langle x\rangle$ ranges over all conjugacy classes of Z/p in Γ_g, b_i ranges over all $b_i \in B_{g,p}$.*

Proof: 1) Assume $|N(\langle x\rangle)/C(\langle x\rangle)| = d$. Then there exists an integer r such that $x \approx x^r \approx \cdots\cdots \approx x^{r^{d-1}}$ ($\approx$ means "is conjugate to" in Γ_g) so that d is the minimal positive integer satisfying $r^d = 1 \bmod(p)$. The d divides $p-1$ obviously. Let b be the number of fixed points of the x action on S_g, $\sigma(x) = (\beta_1, \beta_2, \ldots\ldots \beta_b)$ the fixed point datum, where $\beta_i \in Z/p - \{0\}$ (cf. **[10]**).

Let us define a permutation r^* on the ordered b_i-tuple $(\beta_1, \beta_2, \ldots, \beta_{b_i})$. Set $r^*(\beta_1, \beta_2, \ldots\ldots \beta_{b_i}) = (r\beta_1, r\beta_2, \ldots\ldots r\beta_{b_i})$, $(r^*)^2 = (r^2)^* \ldots \ldots (r^*)^{d-1} = (r^{d-1})^*$. It is well-defined since $\sigma(x) = \sigma(x^{r^2}) = \cdots\cdots = \sigma(x^{r^{d-1}})$ as an unordered b-tuples [**12**]. We can decompose $r^* = (\beta_{i_1}, \beta_{i_2}, \ldots\ldots \beta_{i_s})(\beta_{j_1}, \beta_{j_2}, \ldots\ldots \beta_{j_t})\ldots\ldots(\beta_{k_1}, \beta_{k_2}, \ldots\ldots \beta_{k_u})$, a product of cyclic permutations. Notice that permutations r^*, $(r^*)^2, \ldots \ldots (r^*)^{d-1}$ do not have fixed points. Otherwise, there exists β_i such that $rj\beta_i = \beta_i \bmod(p)$, $1 \leq j \leq d-1$. This forces $rj = 1 \bmod(p)$, a contradiction. But, of course, $(r^*)^d = (r^d)^* = \mathrm{Id}$. These imply

$s = t = \cdots\cdots = u = d$, i.e., the number $|N(\langle x\rangle)/C(\langle x\rangle)| = d$ divides the number b_i of fixed points of the x action on the surface S_g. We have showed that $LCM\{|N(\langle x\rangle)/C(\langle x\rangle)|\}$ divides $LCM\{\gcd(p-1, b_i)\}$, where $\langle x\rangle$ ranges over all conjugacy classes of Z/p in Γ_g, b_i ranges over all $b_i \in B_{g,p}$.

2) Conversely, assume $\gcd(p-1, b_i) = d$. Then there is a mod(p) integer r so that d is a minimal positive integer satisfying $r^d = 1 \bmod(p)$.

Case 1. $b_i \neq 0$. If $d \neq 1$, then $r \neq 1$. Consider the unordered b_i-tuples $\sigma = (1, r, r^2, \ldots\ldots r^{d-1}, 1, r, r^2, \ldots\ \ldots r^{d-1}, \ldots\ldots 1, r, r^2, \ldots\ldots, r^{d-1})$. Since $(b_i/d)(1+r+r^2+\ldots\ldots r^{d-1}) = 0 \bmod(p)$. There exists an element $x \in \Gamma_g$, $x^p = 1$, and the it's representive fixed point datum $\sigma(x)$ is σ, i.e., the unordered b_i-tuples σ can be realized as a fixed point datum of an order p element in Γ_g [6]. Obviously, $\sigma(x) = \sigma(x^r) = \sigma(x^{r^2}) = \cdots\cdots = \sigma(x^{r^{d-1}})$ or $x \approx x^r \approx x^{r^2} \approx \cdots\cdots \approx x^{r^{d-1}}$ in Γ_g. This implies that the number d divides the order $|N(\langle x\rangle)/C(\langle x\rangle)|$. If $\gcd(p-1, b_i) = d = 1$, for any order p element x in Γ_g with the number of fixed points b_i, obviously 1 divides $|N(\langle x\rangle)/C(\langle x\rangle)|$.

Case 2. $b_i = 0$. On the one hand, we have $\gcd(p-1, b_i) = p-1$. On the other hand, the x acts on S_g freely. All order p free actions are conjugate by [5], this implies $|N(\langle x\rangle)/C(\langle x\rangle)| = p-1$.

So, $LCM\{\gcd(p-1, b_i)\}$ divides $LCM\{|N(\langle x\rangle)/C(\langle x\rangle)|\}$. ■

Proof of Theorem 4: Let $\mu : \Gamma_g \to Sp(2g, Z)$ be the canonical homology representation and $p : Sp(2g, Z) \to Sp(2g, F_q)$ be the reduction map. Here q can be chosen a primitive root of mod(p) such that $q \geq 3$, and q^{p-1} is not congruent to 1 mod(p^2) (by the Dirichlet theorem).

Now $\mathrm{Ker}(p\mu) = N$ is a torsion free, normal, finite index subgroup of Γ_g and a p-Sylow subgroup of the finite quotient $\Gamma_g/N = Sp(2g, F_q)$ is elementary abelian if $2g < p(p-1)$. Then we can use Theorem 1 and Lemma 3.1 to finish the proof. ■

A list of the p-period of a p-periodic mapping class group Γ_g can be also found in the Appendix C of the author's thesis [12].

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