Publicacions Matemàtiques, Vol 35 (1991), 517–525.

ON THE COMPLETE DIGRAPHS WHICH ARE SIMPLY DISCONNECTED

Davide C. Demaria and J. Carlos S. Kiihl

Abstract

Homotopic methods are employed for the characterization of the complete digraphs which are the composition of non-trivial highly regular tournaments.

1. Introduction

It is known that one can also construct a homotopy theory for categories of spaces having a structure weaker than a topology. For example, one can take the category of *prespaces* or *Čech closure spaces*.

To every digraph D one can associate, in a natural way, two finite prespaces $P(D)$ and $P^*(D)$; and vice versa, to every finite prespace one can associate two digraphs G and G^*, dually oriented. Hence one can relate the category of the digraphs with that of the finite prespaces. Therefore a homotopy theory can be defined for digraphs, by setting the regular homotopy group $Q_n(D)$ of D to be the homotopy group $\pi_n(P(D))$ of the associated prespace (see [**6**]).

In [**3**] Burzio and Demaria proved that the groups $Q_n(D)$ are isomorphic to the classical homotopy groups $\pi_n(|K_D|)$, where $|K_D|$ is the polyhedron of a suitable simplicial complex K_D associated with the digraph D. Then in [**5**] they obtained, as an application of the regular homotopy of digraphs, a structural characterization of tournaments T, called *simply disconnected tournaments*, whose fundamental group $Q_1(T)$ is non trivial. In [**4**], they have obtained another characterization for the simply disconnected tournaments by using coned 3–cycles.

In this paper we extend those results to the case of digraphs D which are complete, and we get analogous results if $Q_1(D) \neq 0$. First of all, we must generalize the concept of simple quotient for every type of digraph and we prove the following theorem:

Theorem 4.4. *Every (non-trivial) digraph has a unique simple quotient.*

In this way we obtain the following theorems:

Theorem 5.3. *A complete digraph D is simply disconnected if and only if its simple quotient is a highly regular tournament.*

Theorem 5.10. *A complete digraph D_n is simply disconnected if and only if:*

(a) there exists in D_n a non coned 3–cycle;

(b) every symmetric pair and every 3–cycle in D_n are shrinkable in D_n.

2. Some Definitions and Notations

Definition 2.1. Let V be a finite non–empty set and E a set of ordered pairs $(u, v) \in V \times V$, such that $u \neq v$. We call the pair $D = (V, E)$ a *directed graph* or *digraph*. The elements of V are the *vertices* of D, the cardinality of V the *order* of D, and the elements of E the *arcs* of D. Moreover, we write $u \to v$ instead of (u, v), and we call u a *predecessor* of v and v a *successor* of u.

Remark 1. Given two distinct vertices u and v, we have *a priori* four possibilities, and four types of arc:

(1) there is no oriented arc between u and v – we denote by $u|v$ the *null arc*;

(2) there is the oriented arc (u, v), but not the arc (v, u) – we denote the *simple arc* by $u \to v$;

(3) there is the oriented arc (v, u), but not the arc (u, v) – we denote the *simple arc* by $u \leftarrow v$;

(4) there are both oriented arcs (u, v) and (v, u) – we denote the *double arc* by $u \leftrightarrow v$. (A double arc is also called a *symmetric pair*.)

Definition 2.2. A digraph is called *oriented* if, between two distinct vertices, there is at most one ordered arc – that is, the possible arcs are either simple arcs or null arcs. A digraph is called a *non-oriented graph* if, between two distinct vertices, there is either a double arc or a null arc. A digraph is called *complete* if, between two distinct vertices, there is at least one ordered arc; the possible arcs in this case are either simple or double arcs.

Definition 2.3. A digraph T is a *tournament* if, between every pair of distinct vertices, there is one and only one arc. A tournament T is called *hamiltonian* if it contains a spanning cycle – that is, a cycle passing through all the vertices of T.

Definition 2.4. Let $D = (V, E)$ and $D' = (V', E')$ be digraphs. A function $f : V \to V'$ is a *homomorphism* between D and D' if, for every $u, v \in V$, $u \to v$ implies either $f(u) \to f(v)$ or $f(u) = f(v)$.

Remark 2. We can consider two kinds of dualities for a given digraph D:

(a) the first one is the dually oriented digraph $\overline{D}$, which is obtained by changing the orientation of the arcs; in this case, both digraphs are of the same type.

(b) the second kind is the digraph $\overline{\overline{D}}$, which is obtained by maintaining the simple arcs and by changing the null arcs into double arcs, and vice–versa. In this case, oriented digraphs become complete digraphs, and vice–versa. On the other hand, tournaments and non-oriented graphs do not change.

It is known that, given a tournament T, we can associate an algebraic structure to T in a natural way. In fact we have (see [7]):

Proposition 2.5. *A tournament T becomes a commutative groupoid $A(T)$ if we define the following binary operation $*$:*

$$\textit{for all } u, v \in T,\ u * v = v * u = \begin{cases} u, & \textit{if } u \to v \textit{ or } u = v; \\ v, & \textit{if } v \to u. \end{cases}$$

Remark 3. Similarly we can associate with T the dual commutative groupoid $A'(T)$, by defining:

$$\text{for all } u, v \in T\ u * v = v * u = \begin{cases} u, & \text{if } v \to u \text{ or } u = v; \\ v, & \text{if } u \to v. \end{cases}$$

Remark 4. Every homomorphism between two tournaments T and T' is also an algebraic homomorphism between the commutative groupoids $A(T)$ and $A(T')$ (*or* $A'(T)$ and $A'(T')$), and vice–versa.

The same definitions can be applied to the case of a digraph D of any type, and then we have the associated groupoids $A(D)$ and $A'(D)$, which are dual. In this case, we set for $A(D)$:

$$\text{for all } u, v \in D,\ u * v = \begin{cases} u, & \text{if } u \to v \text{ or } u = v; \\ v, & \text{if } v \not\to u; \end{cases}$$

and for $A'(D)$:

$$\text{for all } u, v \in D,\ u * v = \begin{cases} u, & \text{if } u \not\to v \text{ or } u = v; \\ v, & \text{if } u \to v. \end{cases}$$

Remark 5. In general the two groupoids are not commutative. In the first case, from $u \leftrightarrow v$ we get

$$u * v = u \text{ and } v * u = v;$$

on the other hand from $u|v$ we get:

$$u * v = v \text{ and } v * u = u.$$

Remark 6. Whereas the homomorphisms between two tournaments coincide with the algebraic homomorphisms between the associated groupoids, this is not true in the general case, for there are homomorphisms between digraphs which are not algebraic homomorphisms between the associated groupoids.

For example, given the 3–cycle $C : u \to v \to w \to u$ and the symmetric pair $D : x \leftrightarrow y$, if we define $f : C \to D$ by $f(u) = x$, $f(v) = f(w) = y$, then in $A(C)$ and $A(D)$ we have

$$u * w = w, \quad f(u) * f(w) = x * y = x \text{ and } f(w) = y.$$

Remark 7. We can still associate other two groupoids $\overline{A}(D)$ and $\overline{A}'(D)$ to the digraph D, in the following way:

$$\text{for all } u,\, v \in D,\ u * v = \begin{cases} u, & \text{if } u = v \text{ or } u \leftarrow v; \\ v, & \text{if } u \not\leftarrow v. \end{cases}$$

and

$$\text{for all } u,\, v \in D,\ u * v = \begin{cases} u, & \text{if } u = v \text{ or } u \not\leftarrow v; \\ v, & \text{if } u \leftarrow v. \end{cases}$$

We observe that, if we change the orientation of the arcs, then $A(D)$ becomes $\overline{A}(D)$ and $A'(D)$ becomes $\overline{A}'(D)$; in changing the double arcs into null arcs and the null arcs into double arcs, $A(D)$ becomes $\overline{A}'(D)$, and $A'(D)$ becomes $\overline{A}(D)$.

3. Quotient Digraphs

We say that a subset X of vertices of a digraph D is a *set of equivalent vertices* if for any vertex u in $D - X$ the oriented arcs from u to any vertex v in X are all of the same type. Of course the type of oriented arc can change if we vary the vertex u in $D - X$.

If $p : A(D) \to A(Q)$ is a surjective algebraic homomorphism between the groupoids which are associated to the two digraphs D_n and Q_m, of order n and m, respectively, then we see that the groupoid $A(Q)$ is isomorphic to the quotient groupoid $A(D)/p$. For, if we consider the m pre–images of the vertices $v_1, \ldots, v_m$ in Q_m, and we set $S^{(i)} = p^{-1}(v_i)$, $i = 1, \ldots, m$, then we can subdivide the n vertices of D in m disjoint subdigraphs $S^{(1)}, \ldots, S^{(m)}$ of equivalent vertices, because the type of arc which joins the vertex v_i to v_j is of the same type as the arcs which join every vertex in $S^{(i)}$ to every vertex in $S^{(j)}$.

If the above conditions hold, then we write

$$D_n = Q_m(S^{(1)}, \ldots, S^{(m)}),$$

and we say that the digraph D_n is the *composition* of the m digraphs $S^{(1)}, \ldots, S^{(m)}$. The subdigraphs $S^{(1)}, \ldots, S^{(n)}$ are called the *components* of the digraph D_n, and Q_m is the *quotient* of the digraph D_n.

We say that a digraph is *simple* if the composition

$$D_n = Q_m(S^{(1)}, \ldots, S^{(m)})$$

implies that $m = 1$ or $m = n$ – that is, if the quotient Q_m or the components $S^{(i)}$ coincide with the trivial digraph of order 1.

4. Properties of the Quotient Digraphs

Proposition 4.1. *Let D be a digraph, Q be one of the quotient digraphs of D, and S be one of the components of D with respect to Q; then S is a set of equivalent vertices. Conversely, if X is a set of equivalent vertices, then X is a component of D.*

Proof: The first statement is obvious. For the second, we consider the partition of D in the subset X and the singular subsets of $D - X$ ■

Proposition 4.2. *Let X and Y be two sets of equivalent vertices of a digraph D. If $X \cap Y \neq \emptyset$ and $X \cup Y \neq D$, then $X \cup Y$ is a set of equivalent vertices.*

Proof: Let u be a vertex in $X \cap Y$ and let v be a vertex in $D - (X \cup Y)$; for each vertex w in $X \cup Y$, the oriented arc between v and w is of the same type as the oriented arc between v and u ■

Proposition 4.3. *Let D be a digraph and Q one of its quotient digraphs; then Q is isomorphic to a subdigraph E of D.*

Proof: In fact we can construct E by choosing one vertex in each components of D ■

Theorem 4.4. *Every non-trivial digraph has a unique simple quotient.*

Proof: We argue by contradiction. Let $P = [S^{(1)}, \dots, S^{(h)}]$ and $Q = [T^{(1)}, \dots, T^{(k)}]$ be two different partitions of the digraph D into components of equivalent vertices, such that the quotient digraphs P_h and Q_k are non-trivial and simple. Then we have:

$$D = P_h(S^{(1)}, \dots, S^{(h)}) = Q_k(T^{(1)}, \dots, T^{(k)}).$$

Suppose that $h > 2$. Since the two partitions are distinct, there must exist two distinct components S and T with non empty intersection. If S is not contained in T and the other components T have non-empty intersection with S, then at least one of them cannot be contained in S, for otherwise we can replace these particular components T in the partition Q by the unique component S; this contradicts the simplicity of Q_k.

Therefore there exist a component S and a component T such that $S \cap T \neq \emptyset$, $S \not\subset T$ and $T \not\subset S$, and we choose them to be $S^{(1)}$ and $T^{(1)}$. We now number the components S in such a way that $S^{(1)}, S^{(2)}, \dots, S^{(r)}$ intersect $T^{(1)}$, while the rest $S^{(r+1)}, S^{(r+2)}, \dots, S^{(h)}$ do not intersect $T^{(1)}$.

We distinguish two cases:

(a) $1 < r < h$.

Since the components $S^{(1)}, S^{(2)}, \ldots, S^{(r)}$ intersect $T^{(1)}$, their union $U = S^{(1)} \cup \cdots \cup S^{(r)}$ is a subset of equivalent vertices. Hence, to the partition $P' = [U, S^{(r+1)}, S^{(r+2)}, \ldots, S^{(h)}]$, we can associate a new composition of D, which will induce a composition of P_h. But this contradicts the simplicity of P_h.

(b) $r = h$.

The oriented arcs between a vertex u in $S^{(1)} - T^{(1)}$ and any vertex in the union $U = S^{(2)} \cup S^{(3)} \cup \cdots \cup S^{(h)}$ are all of the same type, because every component $S^{(i)}$, $i = 2, 3, \ldots, h$ intersects the component $T^{(1)}$. On the other hand, since u is a vertex in $S^{(1)}$, the oriented arcs between any vertex in $S^{(1)}$ and any vertex in U are of the same type. This means that the partition $[S^{(1)}, U]$ is a composition of D, which is a contradiction, for we have supposed that the quotient P_h is simple of order $h > 2$.

Finally, if $h = 2$, then k must be also equal to 2, for otherwise it would be sufficient to interchange the two compositions and repeat the previous argument. On the other hand, the digraphs P_2 and Q_2 must be isomorphic, for if P_2 is the null arc, then the digraph D is disconnected, if P_2 is the simple arc, then the digraph D is weakly (but not strongly) connected; and if P_2 is the double arc, then the digraph D is strongly connected ■

Remark 1. It follows from the proof that, for $h > 2$, we have not only a unique simple quotient, but also a unique partition into components. On the other hand, for $h = 2$, then we may have more partitions. For example, if $D = [u, v, w;\ u \leftrightarrow v,\ u \leftrightarrow w,\ v \leftrightarrow w]$ we have the following partitions

$$P = [[u, v], w], \quad Q = [u, w], v] \text{ and } R = [[v, w], u].$$

Remark 2. We recall that a digraph is *hamiltonian* if there is a cycle passing through all the vertices. We shall consider a *symmetric pair* as a hamiltonian cycle.

Proposition 4.5. *A complete digraph is hamiltonian if and only if each of its (non-trivial) quotients is hamiltonian.*

Proof: From results due to Rado (1943), Roy (1958) and Camion (1959), we know that *a complete digraph D is hamiltonian if and only if the tournament T_2 (the simple oriented arc) is not the simple tournament related to D.* We now observe that either the initial digraph or any of its quotient digraphs have the same simple quotient. The assertion follows ■

5. Complete Digraphs which are Simply Disconnected

Definition 5.1. A tournament T is *regular* if, for each vertex $v \in T$, the numbers of predecessors and successors of v are the same (and hence the order of T is odd). A tournament T_{2m+1} is *highly regular* if there exists a cyclical ordering $v_1, \ldots, v_{2m+1}, v_1$ on the vertices of T_{2m+1} such that $v_i \to v_j$ if and only if v_j is one of the first m successors of v_i in the cyclical ordering of T_{2m+1}.

Definition 5.2. A digraph D is *simply connected* if its first homotopy group $Q_1(D)$ is trivial. A digraph D is *simply disconnected* if $Q_1(D)$ is non-trivial.

We have the following theorem:

Theorem 5.3. *A complete digraph D is simply disconnected if and only if its simple quotient is a highly regular tournament.*

This theorem is a generalization of the analogous theorem for tournaments (see Theorem 3.9, of [**5**]). For the proof, we need the following lemmas.

Lemma 5.4. *A complete digraph D is simply connected if and only if each of its non-trivial quotient digraphs D^* is simply connected.*

Proof: This is analogous to the proof of Proposition 2.1 in [**5**] for tournaments ■

Lemma 5.5. *For every complete digraph D_n of order n, there exists at least one tournament of order n which is a subdigraph of D_n.*

Proof: It is sufficient to eliminate one oriented arc from each symmetric pair ■

Lemma 5.6. *Let D_n and F_n be two given complete digraphs, each of order n, such that F_n is a subdigraph of D_n. Then if F_n is simply connected, so is D_n.*

Proof: Each edge-loop in the polyhedron associated to the digraph D_n is null–homotopic, since it is null-homotopic in the sub–polyhedron associated to the digraph F_n ■

We now prove Theorem 5.3:

(a) If the simple quotient of a complete digraph D is a highly regular tournament, then D is simply disconnected.

This follows directly from Lemma 5.4 and the fact that a highly regular tournament is simply disconnected.

(b) Let D_n be a complete digraph which is simply disconnected; then D_n has a highly regular tournament as a simple quotient.

A tournament T_n which is a subdigraph of D_n is also simply disconnected, and therefore is the composition of a highly regular tournament, by the analogous theorem for tournaments.

It is now sufficient to prove that the simple arcs which ought to be replaced by double arcs in order to pass from the tournament T_n to the initial complete digraph D_n, all belong to subtournaments which are components of T_n.

From Lemma 5.4, we see that the replacement of simple arcs by double arcs in the components of T_n, do not change the first homotopy group of the digraphs which are obtained one by one.

On the other hand, if we assume there exists a double arc with vertices u and v, which belongs to two different components of T_n, then:

1) if we construct a 3–cycle C in T_n using the vertices u and v, then the edge–loop determined by C is not null-homotopic in the polyhedron associated to T_n, as it is a generator of $Q_1(T_n)$ (see [**5**, Proposition 3.6]);

2) on the other hand, if we replace the simple arc between u and v by a double arc, then the same loop becomes null–homotopic in the polyhedron associated with the complete digraph obtained in such a manner from T_n.

Therefore the first homotopy group of such a digraph is trivial, and hence by Lemma 5.6 the group $Q_1(D_n)$ is also trivial. Hence the result follows ■

Corollary 5.7. *A simply disconnected complete digraph is hamiltonian.*

Proof: This follows easily from Proposition 4.5 ■

Before we obtain a second characterization for the simply disconnected complete digraphs, we need to introduce some further definitions.

Definition 5.8. A subdigraph F of a digraph D is *coned* if there is at least one vertex u in $D - F$, such that u is either a predecessor or a successor of the vertices in F; otherwise, the subdigraph E is *non–coned.*

Definition 5.9. A subdigraph F of a digraph D is *shrinkable* if there exists a proper subset of D consisting of equivalent vertices, and containing F.

Theorem 5.10. *A complete digraph D_n is simply disconnected if and only if:*

(a) there exists in D_n a non–coned 3–cycle;

(b) every symmetric pair and every coned 3–cycle in D_n are shrinkable in D_n.

Proof: By Theorem 5.3, it is sufficient to show that a complete digraph whose simply quotient is a highly regular tournament, is characterized by the conditions (a) and (b).

1) The proof that conditions (a) and (b) are necessary is analogous to that given for tournaments in [**4**, Th. 7].

2) If we suppose that conditions (a) and (b) hold for the complete digraph D_n, then by (b) the simple quotient digraph Q of D_n is a tournament, such that each of its 3–cycle is non–coned. Hence by the analogous theorem for tournaments which was mentioned above and by the simplicity of Q, we see that the tournament Q is highly regular. The result is proved ■

Aknowledgements. This work was performed under the auspices of the Consiglio Nazionale delle Ricerche (CNR, GNSAGA) and of FAPESP (Proc. Nº 89/2042-1).

References

1. BEINEKE, L. W. AND REID, K. B., Tournaments, in *"Selected Topics in Graph Theory,"* Edited by L. W. Beineke and R. J. Wilson, Academic Press, New York, 1978.
2. BURZIO, M. AND DEMARIA, D. C., Duality theorem for regular homotopy of finite directed graphs, *Rend. Circ. Mat. Palermo* **(2)**, **31** (1982), 371–400.
3. BURZIO, M. AND DEMARIA, D. C., Homotopy of polyhedra and regular homotopy of finite directed graphs, Atti II° Conv. Topologia, *Suppl. Rend. Circ. Mat. Palermo* **(2)**, **12** (1986), 189–204.
4. BURZIO, M. AND DEMARIA, D. C., Characterization of tournaments by coned 3–cycles, *Acta Univ. Carol. Math. Phys.* **28** (1987), 25–30.
5. BURZIO, M. AND DEMARIA, D. C., On simply disconnected tournaments, Proc. Catania Conf., *Ars Combinatoria* **24 A** (1988), 149–161.
6. DEMARIA D. C. AND GARBACCIO BOGIN R., Homotopy and homology in pretopological spaces, Proc. 11^{th} Winter School, *Suppl. Rend. Circ. Mat. Palermo* **(2)**, **3** (1984), 119–126.
7. MÜLLER, V., NEŠETŘIL J. AND PELANT J., Either tournaments or algebras?, *Discrete Math.* **11** (1975), 37–66.

Davide C. Demaria: Dipartimento di Matematica
Universitá di Torino
via Principe Amedeo 8
10123 Torino
ITALIA

J. Carlos S. Kiihl: Departamento de Matemática
IMECC-UNICAMP
Caixa Postal 6065
13081 - Campinas,
SP BRASIL

Rebut el 28 de Gener de 1991