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BOUNDARY VALUE PROBLEMS FOR ELLIPTIC EQUATIONS

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In this note I will describe some recent results, obtained jointly with R. Fefferman and J. Pipher [**RF-K-P**], on the Dirichlet problem for second-order, divergence form elliptic equations, and some work in progress with J. Pipher [**K-P**] on the corresponding results for the Neumann and regularity problems.

Let us start by recalling some classical results for the Laplacian $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ in the unit ball $B = \{X = (x_1, \dots, x_n) \,|\, |X| < 1\}$. The Dirichlet problem is to solve

$$\text{(D)} \qquad \begin{cases} \Delta u = 0 \text{ in } B \\ u|_{\partial B} = f \end{cases}$$

while the Neumann problem is to solve

$$\text{(N)} \qquad \begin{cases} \Delta u = 0 \text{ in } B \\ \frac{\partial u}{\partial N}\big|_{\partial B} = f, \quad \int_{\partial B} f \, d\sigma = 0, \end{cases}$$

where $\frac{\partial}{\partial N}$ denotes differentiation in the direction of the normal to ∂B. As is well known, there are explicit formulas for the solutions of the above problems, and one can then give a very careful analysis of the solutions when, say $f \in L^p(\partial B, d\sigma)$, $1 < p < \infty$. In both cases, the boundary values are taken in the sense of non-tangential convergence, i.e., if $Q \in \partial B$, and $\Gamma(Q) = \Gamma_\alpha(Q) = \{X \in B \,|\, |X - Q| < (1+\alpha)\text{dist}(X, \partial B)\}$, $\alpha > 0$, in the case of (D) we have $\lim_{\substack{X \to Q \\ X \in \Gamma(Q)}} u(X) = f(Q)$ for a.e. $Q(d\sigma)$, and in the case of (N) we have $\lim_{\substack{X \to Q \\ X \in \Gamma(Q)}} \nabla u(X) N_Q = f(Q)$ for a.e. $Q(d\sigma)$. Moreover, the Hardy-Littlewood maximal theorem, and in the case of (N), this theorem and the Calderón-Zygmund theory of singular integrals give, denoting by $u^*(Q) = \sup_{X \in \Gamma(Q)} |u(X)|$, the estimates

$$\|u^*\|_{L^p(\partial B, d\sigma)} \leq C_p \|f\|_{L^p(\partial B, d\sigma)} \quad 1 < p \leq \infty \text{ for (D) and}$$
$$\|(\nabla u)^*\|_{L^p(\partial B, d\sigma)} \leq C_p \|f\|_{L^p(\partial B, d\sigma)} \quad 1 < p < \infty \text{ for (N)},$$

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so that the non-tangential convergence is in fact "dominated" in the sense of Lebesgue's dominated convergence theorem.

Finally, the regularity problem (R) is the Dirichlet problem (D), but where the boundary data $f \in L_1^p(\partial B, d\sigma)$, i.e., it has tangential derivatives, which belong to $L^p(\partial B, d\sigma)$. In this case, the Hardy-Littlewood maximal theorem, and the Calderón-Zygmund theory give the estimate

$$\|(\nabla u)^*\|_{L^p(\partial B, d\sigma)} \leq C_p \|f\|_{L_1^p(\partial B, d\sigma)} \quad 1 < p < \infty$$

In **[RF-K-P]** and **[K-P]** we considered elliptic operators $L = \frac{\partial}{\partial x_i} a_{ij}(X) \frac{\partial}{\partial x_j}$, where $A(X) = (a_{ij}(x))$ is a real, symmetric matrix which is bounded, i.e., $\|A\|_{L^\infty(\mathbb{R}^n)} \leq \lambda^{-1}$ and elliptic, i.e.,

$$\langle A(X)\xi, \xi\rangle \geq \lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^n,$$

where $\lambda > 0$. The purpose of these investigations was to understand to what extent the above results extended to

$$(\mathrm{D}_L)_{p_0} \qquad \begin{cases} Lu = 0 \text{ in } B \\ u|_{\partial B} = f \in L^{p_0}(\partial B, d\sigma) \end{cases}$$

$$(\mathrm{N}_L)_{p_0} \qquad \begin{cases} Lu = 0 \text{ in } B \\ \frac{\partial u}{\partial v}\big|_{\partial B} = \langle A\nabla u, N\rangle|_{\partial B} = f \in L^{p_0}(\partial B, d\sigma), \quad \int_{\partial B} f = 0 \end{cases}$$

and $(R_L)_{p_0}$, which is the Dirichlet problem for L with data in $L_1^{p_0}(\partial B, d\sigma)$.

The operators L is described above have been extensively studied, an important results were obtained in the pioneering work of DeGiorgi **[DeG]**, Nash **[N]**, Moser **[Mo]**, Stampacchia **[S]**, and Littman-Stampacchia and Weinberger **[L-S-W]**.

In order to give some perspective into the understanding of the problems mentioned before, it is worthwhile to consider $(\mathrm{D})_{p_0}$, $(\mathrm{N})_{p_0}$ and $(\mathrm{R})_{p_0}$, for the Laplacian Δ, on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, i.e., on domains given locally as $\{X = (x, y) : y > \varphi(x)\}$, $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$, $|\varphi(x) - \varphi(x')| \leq M|x - x'|$. Such domains verify (and in fact are characterized by) uniform interior and exterior cone conditions, and hence the study of non-tangential convergence is meaningful. Moreover, they have a tangent plane at almost every boundary point, and hence normal and tangential differentiation make sense. The above problems are very well understood in this context, and we have the following result.

Theorem. *There exists $\varepsilon = \varepsilon(\Omega) > 0$ such that*

(a) *$(D)_{p_0}$ holds for $2 - \varepsilon < p_0 \leq \infty$, i.e., if u solves (D), then $\|u^*\|_{L^{p_0}(\partial\Omega, d\sigma)} \leq C\|f\|_{L^{p_0}(\partial\Omega, d\sigma)}$ (Dahlberg* **[D1]**).

(b) *$(N)_{p_0}$ holds for $1 < p_0 < 2+\varepsilon$, i.e.,*

$$\|(\nabla u)^*\|_{L^{p_0}(\partial\Omega, d\sigma)} \le C\|f\|_{L^{p_0}(\partial\Omega, d\sigma)}, \quad 1 < p_0 < 2+\varepsilon$$

(Jerison-Kenig **[J-K1]***) for $p_0 = 2$, Dahlberg-Kenig* **[D-K]** *for the general case).*

(c) *$(R)_{p_0}$ holds for $1 < p_0 < 2+\varepsilon$, i.e., if u solves the Dirichlet problem with data in $L_1^{p_0}(\partial\Omega)$, then*

$$\|(\nabla u)^*\|_{L^{p_0}(\partial\Omega, d\sigma)} \le C\|f\|_{L^{p_0}(\partial\Omega)}, \quad 1 < p_0 < 2+\varepsilon$$

(Jerison-Kenig **[J-K1]** *for $p_0 = 2$, Verchota* **[V]** *for the general case, also, Dahlberg-Kenig* **[D-K]** *gave a new proof of the general case).*

Moreover, in (a), (b), and (c) the range of p_0's is sharp, as simple examples show.

The connection of the above results with our problems comes from the fact that if we make the change of variables $(x, y) \mapsto (x, y - \varphi(x))$, the domain Ω is mapped to the upper half-plane, but the Laplacian is mapped to an operator L whose coefficients depend on the Jacobian of the change of variables, and hence are bounded measurable, but not any more regular. It is easy to see that L is nevertheless, elliptic.

Let me begin by describing the results for the Dirichlet problem, which are the best understood ones. First, recall that Littman, Stampacchia and Weiberger **[L-S-W]** showed that the classical Dirichlet problem for L (i.e., with data $f \in C(\partial\Omega)$ and solution u in $C(\overline{\Omega})$) has a solutions in Ω if and only if it has a solution for Δ in Ω. In particular, this can be done in B. Thus, the mapping $f \mapsto u(X)$, where u is the solution of the classical Dirichlet problem with data f, for L in B defines a continuous, positive linear functional on $C(\partial B)$, and hence, there exists a probability measure $d\omega_L^X$ such that $u(X) = \int_{\partial B} f\, d\omega_L^X$. $\{d\omega_L^X\}$ is called L-harmonic measure. The members of this family of measures are mutually absolutely continuous by Moser's **[Mo]** Harnack principle. We sometimes call $\omega_L = \omega_L^0$ the L-harmonic measure. In **[C-F-M-S]**, Caffarelli, Fabes, Mortola and Salsa established an analogue of the classical theorem of Fatou, for L. They showed that if $Lu = 0$ in B, $u \ge 0$, then u has finite non-tangential limits at a.e. $Q \in \partial B(d\omega_L)$. This generalized the corresponding result for harmonic functions on Lipschitz domains, obtained by Hunt and Wheeden **[H-W]**. The basic estimates obtained in **[C-F-M-S]** to prove this Fatou theorem were:

(1) $\omega_L(\Delta_{2r}) \le C\omega_L(\Delta_r)$ (doubling), where $\Delta_r = B(Q, r) \cap \partial B$, $Q \in \partial B$

(2) If $f \ge 0$, and u solves the Dirichlet problem for L with data f, then

$$u^*(Q) \simeq M_{\omega_L}(f)(Q),$$

where

$$M_{\omega_L}(f)(Q) = \sup_{\Delta_r \ni Q} \frac{1}{\omega_L(\Delta_r)} \int_{\Delta_r} f\, d\omega_L.$$

Thus, a natural question is whether ω_L and σ are mutually absolutely continuous, and also whether $\omega_L \in A_\infty(d\sigma)$, where A_∞ is the classical class of weights introduced by Muckenhoupt [**Mu**], and Coifman and Fefferman [**Co-F**]. From (1) and (2) it is easy to see that

Theorem 1. *The following are equivalent*

(i) $\omega_L \in A_\infty(d\sigma)$

(ii) $(D_L)_{p_0}$ *holds for some* $p_0 > 1$, *i.e.*,

$$\|u^*\|_{L^{p_0}(\partial B, d\sigma)} \leq C\|f\|_{L^{p_0}(\partial B, d\sigma)}$$

(iii) $k_L = \frac{d\omega_L}{d\sigma}$ *exists and belongs to* $B_{q_0}(d\sigma)$, *where* $\frac{1}{p_0} + \frac{1}{q_0} = 1$, *i.e.*,

$$\left(\frac{1}{\sigma(\Delta_r)} \int_{\Delta_r} k_L^{q_0}\, d\sigma\right)^{1/q_0} \leq C\left(\frac{1}{\sigma(\Delta_r)} \int_{\Delta_r} k_L\, d\sigma\right).$$

Thus we would like to know when any of the equivalent conditions in Theorem 1 are verified.

In 1981, Caffarelli, Fabes and Kenig [**C-F-K**], and independently, Modica and Mortola [**M-M**], found examples of operators L, with continuous coefficients, such that ω_L and σ are mutually singular, and hence for which none of the conditions in Theorem 1 hold. The examples in [**C-F-K**] were based on the work of Beurling-Ahlfors [**B-A**] on quasi-conformal mappings. The natural question is what is better about the L's that arise from Lipschitz domains. The point is that the Jacobian of the change of variables $(x, y) \mapsto (x, y - \varphi(x))$ is constant in y, and y is the normal direction. In fact, in [**J-K1**], Jerison and Kenig show that if $A(X)$ is C^1 in the radial direction near ∂B, then $(D_L)_2$ holds. In light of this, the search was for the "sharp" smoothness condition in the radial variable, at the boundary, which guarantees $(D_L)_{p_0}$ for some $p_0 > 1$. All the subsequent work was inspired by classical results in the theory of differentiation of functions (see [**RF-K-P**] to understand that the analogy is in fact precise).

Let $g : \mathbb{R} \to \mathbb{R}$ be a measurable function. When is g differentiable a.e.? The necessary condition

$$|g(x+t) + g(x-t) - 2g(x)| = O(|t|) \tag{3}$$

is not sufficient, as the Weierstrass nowhere differentiable function shows. On the other hand, if g is differentiable at x, the expression on the left in (3) is in fact $o(|t|)$. Thus, we may ask, when is g differentiable a.e., if it satisfies

$$|g(x+t) + g(x-t) - 2g(x)| \leq |t|\eta(t) \tag{4}$$

where $\eta(t)$ verifies $\eta(0) = 0$, η continuous, $\eta \downarrow$. The answer is provided in the work of Calderón-Zygmund **[C-Z]**, John-Nirenberg **[Jo-N]**, and Stein-Zygmund **[St-Z]**: if $\int_0 \eta^2(t)\frac{dt}{t} < \infty$, g' exists a.e. and is in L^p_{loc} for all $p < \infty$. Moreover, given such an η, with $\int_0 \eta^2(t)\frac{dt}{t} = +\infty$, there exists a g verifying (4) which is nowhere differentiable.

A sharper result, which is necessary and sufficient for differentiability for a given g, an which involves integral conditions instead of the pointwise condition (4) is due to Marcinkiewicz **[M]**, and Stein-Zygmund **[St-Z]**. They showed that, under the necessary condition (3), a necessary and sufficient condition for the a.e. differentiability of g is

$$(5) \qquad \int_{|t|<1} \frac{|g(x+t)+g(x-t)-2g(x)|^2}{|t|^2}\frac{dt}{|t|} < \infty \text{ for a.e. } x.$$

In our analogy, we think of (3) as an "additive" analogue of the "multiplicative" condition (1), and seek the corresponding analogues of (4) and (5). See **[RF-K-P]** for the details of this argument.

Continuing with our description of further developments, in **[F-J-K]**, Fabes, Jerison and Kenig showed that if $A \in C(\overline{B})$ and

$$\eta(t) = \sup_{\substack{0<s<t \\ |Q|=1}} |A((1-s)Q) - A(Q)|$$

verifies $\int_0 \eta^2(t)\,dt/t < \infty$, then $(\mathrm{D}_L)_{p_0}$ was valid for every $1 < p_0 < \infty$. Moreover, the counterexamples in **[C-F-K]** showed that given any η with $\eta(0) = 0$, $\eta \downarrow$, and $\int_0 \eta^2(t)\,dt/t = +\infty$, we could find $A \in C(\overline{B})$, as above, with ω_L and σ mutually singular.

In **[D3]**, B.E.J. Dahlberg extended this result, with a condition in the spirit of (5), and which extended the Lipschitz domain result. He introduced the quantities

$$(6) \qquad \varepsilon(X) = \sup_{Y \in B(X,\delta(X)/2)} |A(Y) - A(Y/|Y|)|$$

and

$$(7) \quad h(r,Q) = \left(\frac{1}{r^{n-1}}\iint_{B(Q,r)\cap\partial B} \frac{\varepsilon^2(X)}{|1-|X||}\,dX\right)^{1/2} \quad 0<r<1,\ Q\in\partial B$$

The quantity $h(r,Q)$ expresses an integral way of measuring smoothness in the radial direction, at ∂B, in a "dilation invariant" manner. Dahlberg's result was that, if

$$(8) \qquad \overline{\lim_{r\to 0}} \sup_{|Q|=1} h(r,Q) = 0,$$

then there exists $\varepsilon > 0$ such that $(\mathrm{D}_L)_{p_0}$ holds for $2-\varepsilon < p_0 \le \infty$.

If then becomes of interest to investigate to what extent the $\overline{\lim}$ in (8) being 0 was necessary. The first progress in this direction was made by R. Fefferman, **[RF]**, who introduced the quantity

$$A(\varepsilon)(Q) = \left(\iint_{\Gamma(Q)} \frac{\varepsilon^2(X)}{|1-|X||^n}\, dx \right)^{1/2} \tag{9}$$

Note that $h(r,Q) \le \frac{C}{r^{n-1}} \int_{B(Q,r)\cap\partial B} A(\varepsilon)^2\, d\sigma$. R. Fefferman **[RF]** showed that, if $\|A(\varepsilon)\|_{L^\infty(\partial B)} \le C$, then $\omega_L \in A_\infty(d\sigma)$ and hence $(\mathrm{D}_L)_{p_0}$ holds for some $p_0 > 1$. In **[RF-K-P]**, R. Fefferman, Kenig and Pipher prove the following

Theorem 2.

(i) *If* $\sup_{0<r<1} \sup_{|Q|=1} h(r,Q) < +\infty$, *then* $\omega_L \in A_\infty(d\sigma)$, *and hence* $(D_L)_{p_0}$ *holds for some* $p_0 > 1$.

(ii) *If* $E \subset \partial B$ *is closed, and* $A(\varepsilon)(Q) < +\infty$ *for each* $Q \in E$, *then* ω_L *and* σ *are mutually absolutely continuous on* E.

This theorem is sharp in a number of ways.

Theorem 3. *If* $n=2$, *and* $\omega \in A_\infty(d\sigma)$, *there exists* $A(X)$ *such that* $A(Q) = I$ *for* $Q \in \partial B$, *and such that* $\omega_L \simeq \omega$ *and* $\sup_{0<r<1}\sup_{|Q|=1} h(r,Q) < \infty$, *i.e., every* A_∞ *measure arises in the manner described by Theorem 2.*

Theorem 4. *If* $n=2$, *and* $a(X)$ *is any bounded function on* B, *which is "slowly varying" (see* **[RF-K-P]** *for the precise definition) such that*

$$\left(\sup_{0<r<1} \sup_{|Q|=1} \frac{1}{r^{n-1}} \iint_{B(Q,r)\cap B} \frac{a^2(X)}{|1-|X||}\, dX \right)^{1/r} = +\infty$$

we can find $A(X)$ *with* $A(Q) = I$ *for all* $Q \in \partial B$, *with*

$$\frac{1}{r^{n-1}} \iint_{B(Q,r)\cap B} \frac{\varepsilon^2(X)}{|1-|X||}\, dX \le C \left\{ \frac{1}{r^{n-1}} \iint_{B(Q,2r)\cap B} \frac{a^2(X)}{|1-|X||}\, dX + 1 \right\}$$

for all $0 < r < 1/4$, $Q \in \partial B$, *and such that* $\omega_L \notin A_\infty(d\sigma)$, *i.e., the Carleson measure condition* $\sup_{0<r<1}\sup_{|Q|=1} h(r,Q) < +\infty$ *is the weakest condition of its kind that guarantees that* $\omega_L \in A_\infty(d\sigma)$.

To establish Theorems 3 and 4 we use the Beurling-Ahlfors results, as in **[C-F-K]**, together with some new characterizations of A_∞ weights. (See **[RF-K-P]** for the details.)

I will now turn my attention to work in progress [**K-P**], with J. Pipher on the regularity and Neumann problems.

First, notice that in the context that we are working in, $(\nabla u)^*$ may be identically infinite, even if $A(X) = A(X/|X|)$. This is because ∇u need not have pointwise values in the interior, when there is no further regularity on the coefficients. The most that can be said in general is that it is in $L^2_{\mathrm{loc}}(B)$. Thus, we need a new definition

$$(10) \quad (\nabla u)^{**}(Q) = \sup_{X\in\Gamma(Q)} \left(\frac{1}{|1-|X||^n} \iint_{B(X,(1-|X|)\alpha/4)} |\nabla u(Y)|^2 \, dY \right)^{1/2}$$

Note that when we are working with the Laplacian on a Lipschitz domain, $(\nabla u)^{**}$ and $(\nabla u)^*$ are comparable, if we use wider cones.

Moreover, the examples in [**C-F-K**] can be used, in conjunction with well known properties on the boundedness of the Hilbert transform with weights ([**H-M-W**]) to show that there exists L with continuous coefficients such that the estimates $(\mathrm{N}_L)_{p_0}$ and $(\mathrm{R}_L)_{p_0}$:

$$\|(\nabla u)^{**}\|_{L^{p_0}(\partial B, d\sigma)} \le C\|f\|_{L^{p_0}(\partial B, d\sigma)} \qquad f = \left.\frac{\partial u}{\partial v}\right|_{\partial B}$$

and

$$\|(\nabla u)^{**}\|_{L^{p_0}(\partial B, d\sigma)} \le C\|f\|_{L^{p_0}(\partial B)}, \quad u|_{\partial B} = f,$$

hold for no p, $1 \le p_0 \le \infty$.

We have been able to establish the following positive result:

Theorem 5. *Assume that (8) holds. Then, there exists $\varepsilon > 0$ such that, for $1 < p_0 < 2+\varepsilon$, $(N_L)_{p_0}$ and $(R_L)_{p_0}$ hold.*

What happens when merely $\sup_{0<r<1} \sup_{|Q|=1} h(r,Q) < \infty$, in this situation remains an open problem.

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