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# REMOVABLE SETS FOR HOLOMORPHIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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*Abstract*

We show that every closed subset of $C^N$ that has finite $(2N-2)$-dimensional measure is a removable set for holomorphic functions, and we obtain a related result on the ball.

## 1. Introduction

A colleague has remarked that *Everybody knows that a set too small to be a variety is removable.* The present paper is devoted to an explication of certain cases of this general philosophy, which are motivated by a result of Shiffman [**11**], [**12**], to the effect that a closed subset $E$ of a domain $\Omega$ in $\mathbb{C}^N$ is removable for holomorphic functions in the sense that if $f \in \mathcal{O}(\Omega \backslash E)$, then $f$ extends holomorphically to an $\tilde{f} \in \mathcal{O}(\Omega)$ provided $\Lambda^{2N-2}(E) = 0$, $\Lambda^{2N-2}$ denoting $(2N-2)$-dimensional Hausdorff measure.† Because of the Hartogs phenomenon, this result is of interest only in the case that the set $E$ is not compact. Our principal result is an extension of this theorem, in the case that $\Omega$ is $\mathbb{C}^N$ itself, that replaces the hypothesis that $\Lambda^{2N-2}(E) = 0$ by the hypothesis that $\Lambda^{2N-2}(E)$ be finite.

## 2. The main result

We shall prove the following result.

**1. Theorem.** *If $E \subset \mathbb{C}^N$, $N \geq 2$, is a closed set with $\Lambda^{2N-2}(E) < \infty$, then $E$ is removable.*

This is a global theorem in that the conclusion fails for closed sets in bounded domains. For example, if $\Omega$ is a bounded domain that contains the origin, and if $E = \Omega \cap \{z_N = 0\}$, then $\Lambda^{2N-2}(E) < \infty$, but $E$ is not removable, as the function $f(z) = z_n^{-1}$ shows.

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†A version of the result of Shiffman had been found earlier by Caccioppoli [**3**].

*Proof of the Theorem:* We give a direct proof in the case of $\mathbb{C}^2$ and then argue by induction.

The proof in $\mathbb{C}^2$ depends on a lemma, which is based on work of Alexander [1].

Denote by $\mathbb{B}_N$ the unit ball in $\mathbb{C}^N$ and by $r\mathbb{B}_N$ the set $\{rz : z \in \mathbb{B}_N\}$ when $r \in (0, \infty)$. The boundary $br\mathbb{B}_N$ is the sphere in $\mathbb{C}^N$ of radius $r$ centered at the origin.

The referee has kindly drawn the author's attention to Théorème 4, p. 309, of Sibony's paper [**15**], which contains this lemma, with the constant 2 rather than the constant $\sqrt{2}\pi$, as a special case. It would be of interest to what the best value of the constant is.

**2. Lemma.** *If $Y$ is a closed subset of $br\mathbb{B}_2$ and if the polynomially convex hull of $Y$ contains the origin, then $\Lambda^1(Y) \geq \sqrt{2}\pi r$.*

*Proof:* First, let $X \subset b\mathbb{B}_2$ be a compact set with $0 \in \hat{X}$, $\hat{X}$ the polynomially convex hull of $X$. According to Theorem 1 of [1], if $\pi_j : \mathbb{C}^2 \to \mathbb{C}$ is the projection given by $\pi_i(z_1, z_2) = z_i, i = 1, 2$ then

$$\Lambda^2(\pi_1(\hat{X})) + \Lambda^2(\pi_2(\hat{X})) \geq \pi, \tag{1}$$

whence one of the summands, say the first, in (1) is at least $\pi/2$.

Let $Z$ denote the polynomially convex hull of the set $\pi_1(\hat{X})$, i.e., the union of $\pi_1(\hat{X})$ and the bounded components of $\mathbb{C}\backslash\pi_1(\hat{X})$. The boundary of $Z$ is the boundary of the unbounded component of the set $\mathbb{C}\backslash\pi_1(\hat{X})$, and the set $Z$ does not disconnect the plane. According to the isoperimetric inequality [**2**, §§14.3, 14.6]

$$\Lambda^1(bZ) \geq 2\sqrt{\pi}\,[\Lambda^2(Z)]^{\frac{1}{2}}.$$

Every point of $bZ$ is a peak point for the algebra $\mathcal{P}(Z)$,* and so for every point $p \in bZ$, the set $\pi_1^{-1}(p) \cap \hat{X}$ is a peak set for the algebra $\mathcal{P}(\hat{X})$, which can be identified with $\mathcal{P}(X)$. Consequently, the set $\pi_1^{-1}(p)$ meets the Šilov boundary for $\mathcal{P}(X)$, i.e., the set $X$ : We have that $\pi_1(X) \supset bZ$. As $\pi_1$ is a Lipschitz map with Lipschitz constant one, we must have $\Lambda^1(X) \geq \Lambda^1(bZ)$. As $\Lambda^1(bZ) \geq \sqrt{2}\pi$, we have $\Lambda^1(X) \geq \sqrt{2}\pi$.

If now $Y \subset br\mathbb{B}_2$, define $T : \mathbb{C}^2 \to \mathbb{C}^2$ by $Tz = r^{-1}z$, and set $X = TY$. If $0 \in \hat{Y}$, then $0 \in \hat{X}$, so $\Lambda^1(X) \geq \sqrt{2}\pi$ whence $\Lambda^1(Y) \geq \sqrt{2}\pi r$, and the lemma is proved.

The theorem, in case $N = 2$, is proved as follows. Fix a point $z_0 \in E$; we prove that if $f \in \mathcal{O}(\mathbb{C}^2\backslash E)$, then $f$ extends holomorphically into a neighborhood

*We use the customary notation that $\mathcal{P}(S)$ denotes the algebra of continuous functions on the compact set $S$ that can be approximated uniformly by holomorphic polynomials.

of $z_0$. Without loss of generality, we can take $z_0$ to be the origin. Let $\rho : \mathbb{C}^2 \to [0,\infty)$ be the map $\rho(z) = \|z\| = \sqrt{|z_1|^2 + |z_2|^2}$. According to [2, §13.3; 4], we have

$$\int^*_{[0,\infty)} \Lambda^1 (E \cap \rho^{-1}(z))\, dt \leq \text{const.}\Lambda^2 (E) < \infty.$$

This implies the existence of $t_j \in (0,\infty)$ with $t_1 < t_2 < \ldots, t_j \to \infty$, such that

$$\lim_{j\to\infty} \Lambda^1 (E \cap bt_j\mathbb{B}_2) = 0.$$

Fix a value of $j$ large enough that $t_j > 1$ and $\Lambda^1 (E \cap bt_j\mathbb{B}_2) < 1$.

The lemma implies that the origin does not lie in the polynomially convex hull of the set $E \cap bt_j\mathbb{B}_2$. If $\Phi_j$ denotes the restriction to $bt_j\mathbb{B}_2 \backslash E$ of the function $f$, then $\Phi_j$ satisfies the tangential Cauchy-Riemann equations and so ([6],[7],[8, Appendix]) continues holomorphically into $t_j\mathbb{B}_2\backslash(E \cap bt_j\mathbb{B}_2)^\wedge$, which is a neighborhood of the origin. Denote this extension by $\widetilde{\Phi}_j$. That $\widetilde{\Phi}_j$ is an extension of $f$ follows from the fact that $f$ and $\widetilde{\Phi}_j$ agree on an open subset of $bt_j\mathbb{B}_2$.

The theorem is proved now in the two–dimensional case. We next assume it proved in the N–dimensional case and derive the $(N+1)$–dimensional case. To this end, it is of some importance to notice that the argument just given works equally well granted only that $\Lambda^2 (E \cap \{z : |z| > 1\})$ is finite.

We consider in $\mathbb{C}^{N+1}$ a closed subset $E$ with $\Lambda^{2N}(E) < \infty$. Let $f \in \mathcal{O}(\mathbb{C}^N\backslash E)$. Fix a point $z \in \mathbb{C}^{N+1}$ and denote by $\mathcal{G}_{N+1,N}(z)$ the Grassmannian of all complex affine N–planes in $\mathbb{C}^{N+1}$ that pass through the point $z$. There is a natural invariant measure on $\mathcal{G}_{N+1,N}(z)$, which we shall denote by $d\mu(\Pi)$. We assume this measure to be normalized so that it has total mass one. We have by [12] that if $\widetilde{E} = E \cap \{|z| \geq 1\}$, then

$$\int^*_{\mathcal{G}_{N+1,N}} \Lambda^{2N-2} (\widetilde{E} \cap \Pi) d\mu(\Pi) < c_N \Lambda^{2N} (\widetilde{E}) < \infty$$

for a fixed constant $c_N$. In particular, for almost every $\Pi \in \mathcal{G}_{N+1,N}(z), \Lambda^{2N-2}$ $(\widetilde{E} \cap \Pi) < \infty$. Thus, for almost every $\Pi$, $f|(\Pi\backslash E)$ extends holomorphically through all of $\Pi$. Denote this extension by $f_{\Pi,z}$. We define

$$F(z) = f_{\Pi,z}(z).$$

This gives a well–defined value for $F(z)$, because $f_{\Pi,z}(z)$ is independent of the choice of $\Pi$: Two $\Pi$'s, say $\Pi_1$ and $\Pi_2$, in $\mathcal{G}_{N+1,N}(z)$ intersect in an affine subspace of $\mathbb{C}$ of positive dimension on which $f_{\Pi_1,z}$ and $f_{\Pi_2,z}$ agree. Thus they agree at $z$. The function $F$ defined in this way is defined on all of $\mathbb{C}^{N+1}$, and it agrees with $f$ on $\mathbb{C}^{N+1}\backslash E$.

We have to see that $F$ is holomorphic, and for this, it suffices to show that it is continuous. To do this, let $\{z_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{C}^N$ that converges to $z_0$; we shall show that $F(z_n) \to F(z_0)$. Fix a $\Pi_0 \in \mathcal{G}_{N+1,N}(z_0)$ with $\Lambda^{2N-2}(\Pi_0 \cap E) < \infty$ and such that $F$ is holomorphic on $\Pi_0$. For each $n$, choose $\Pi_n \in \mathcal{G}_{N+1,N}(Z_n)$ such that $\Lambda^{2N-2}(\Pi_n \cap E) < \infty$, such that $F$ is holomorphic on $\Pi_n$, and such that $\Pi_n \to \Pi_0$.

If $z \in \mathbb{C}^N$ and $\Pi \in \mathcal{G}_{N+1,N}(z)$, denote by $\mathbb{P}_z(\Pi)$ the projective space of all complex lines in $\Pi$ through the point $z$. We have $\dim_{\mathbb{R}} \mathbb{P}_z(\Pi) = 2N-2$. There are large values of $R$ such that $\Lambda^{2N-3}(b\mathbb{B}_{N+1}(z,R) \cap E \cap \Pi_0) < \infty$, so if $\pi : \Pi_0 \setminus \{z_0\} \to \mathbb{P}_{z_0}(\Pi_0)$ is the standard projection, then $\pi(b\mathbb{B}_{N+1}(z_0,R) \cap E \cap \Pi_0)$ is a set of measure zero in $P_{z_0}$. Thus, there is a complex line $\lambda_0$ with $z_0 \in \lambda_0 \subset \Pi_0$ and with $\lambda_0 \cap E \cap b\mathbb{B}_{N+1}(z_0,R) = \emptyset$. We may choose $\lambda_n \in \mathbb{P}_{z_n}(\Pi_n)$ so that $\lambda_n \to \lambda_0$. For large values of $n$, $\lambda_n \cap E \cap b\mathbb{B}_{N+1}(z_0,R) = \emptyset$. If we apply the Cauchy integral formula in $\lambda_n$ and $\lambda_0$ to represent $F(z_n)$ and $F(z_0)$ as the Cauchy integral of $f$ over the circle $\lambda_n \cap b\mathbb{B}_{n+1}(z_0,R)$ and of $\lambda_0 \cap b\mathbb{B}_{N+1}(z_0,R)$, respectively, we find that as $n \to \infty$, $F(z_n) \to F(z_0)$ as desired.

Thus, $F$ is continuous and so necessarily holomorphic.

This completes the proof of the theorem.

## 3. Variations on the theme

The first variation is to the effect that there is an analogue of the result for submanifolds of $\mathbb{C}^N$: *Let $\mathcal{M}$ be a $k$-dimensional complex submanifold of $\mathbb{C}^N$, and let $E \subset \mathcal{M}$ be a closed subset with $\Lambda^{2k-2}(E) < \infty$.* If $f \in \mathcal{O}(\mathcal{M} \setminus E)$, then $f$ continues holomorphically into all of $M$.*

In the case that $\mathcal{M}$ is an algebraic manifold, we can invoke [**10**, Th. 10, p. 52] to find a projection $\pi : \mathbb{C}^N \to \mathbb{C}^k$ that exhibits $\mathcal{M}$ as an analytic cover over $\mathbb{C}^k$. Using symmetric functions and applying the result already established in $\mathbb{C}^k$, we can derive the result on $\mathcal{M}$.

In the case of a general $\mathcal{M}$, there will be no such projection, and, in essence, it is necessary simply to rewrite the proof given above. The case $n = 2$ proceeds as before: Fix $z_0 \in E$. For certain large values of $t$, $\Lambda^1(E \cap b\mathbb{B}_N(z_0,t))$ will be small and $b\mathbb{B}_N(z_0,t) \cap \mathcal{M}$ will be a smooth $(2k-1)$-dimensional real hypersurface that bounds the domain $\Delta(t,z_0) = \mathbb{B}_N(z_0,t) \cap \mathcal{M}$. By Lemma 2, the polynomially convex hull of $E \cap b\mathbb{B}_N(z_0 t)$ does not contain $z_0$, and by the extension theorem given by Laurent–Thiebeaut [**6**], $f|b\Delta(t,z_0) \setminus E$ extends holomorphically into a neighborhood of $z_0$. The rest of the argument in the two–dimensional case is as before.

For the induction step we replace the affine hyperplanes used in the proof of the theorem by intersections $\mathcal{M} \cap \Pi$, $\Pi$ a codimension one affine hyperplane

*Here, as above, we are computing Hausdorff measures with respect to the Euclidean metric on $\mathbb{C}^N = \mathbb{R}^{2N}$. Below we shall consider the Hausdorff measures associated to certain other metrics, but there we shall be quite explicit about the metrics involved.

in $\mathbb{C}^N$ that is transverse to $\mathcal{M}$. The generic $\Pi$ is transverse to $\mathcal{M}$ and so, generically, $\mathcal{M} \cap \Pi$ is a codimension one submanifold of $\mathcal{M}$. In a bit more detail, if $z \in \mathcal{M}$, then almost every $\Pi \in \mathcal{G}_{N,N-1}(z)$ is transverse to $\mathcal{M}$ and, by [**12**, Lemma 5] almost every $\Pi$ also satisfies $\Lambda^{2k-4}(\Pi \cap E) < \infty$. Thus, the induction hypothesis applies to extend $f|(\mathcal{M} \cap \Pi \backslash E)$ to an $f_\Pi \in \mathcal{O}(\mathcal{M} \cap \Pi)$. We define $F(z) = f_\Pi(z)$; this is well-defined and gives the desired extension of $f$ throughout $\mathcal{M}$.

A second variation of the theme is that the hypothesis that $\Lambda^{2N-2}(E)$ be finite can be replaced by the condition that $\Lambda^{2N-2}(E \cap r\mathbb{B}_N)$ not grow too rapidly as a function of $r$, $r \to \infty$. In fact *if $E$ is a closed subset of $\mathbb{C}^2$ that satisfies $\Lambda^2(E \cap r\mathbb{B}_N) < \alpha r^2$ for all large $r$, then $E$ is removable provided* $\alpha < \frac{\pi^2}{4\sqrt{2}}$.

That the desired conclusion can be drawn may be seen as follows. Notice first that $\Lambda^{2N-2}(E \cap r\mathbb{B}_N) < \alpha r^2$ for large $r$ implies that $\Lambda^{2N-2}(E \cap \mathbb{B}_N(p,r)) < \alpha r^2$ for large $r$, no matter what center $p$ is chosen. Next, we have by [**2**, 4] that

$$\alpha r^2 > \Lambda^2(E \cap r\mathbb{B}_N) \geq \frac{\pi}{4} \int^*_{[o,r]} \Lambda^1(E \cap \{|z| = t\})\, dt. \tag{2}$$

Consequently,

$$\Lambda^1(E \cap \{|z| = t\}) < \sqrt{2}\pi t$$

for infinitely many arbitrarily large values of $t$, and this implies that the origin is not in the polynomially convex hull of $E \cap \{|z| = t\}$ for such values of $t$. Thus, by arguments we have used already, $f$ continues holomorphically into a neighborhood of the origin. Similarly, it continues holomorphically into a neighborhood of every point of $\mathbb{C}^2$, and the result is established.

The example $E = \{(z_1, 0) : z_1 \in \mathbb{C}\}$ shows that the result just derived cannot be obtained under the hypothesis that $\Lambda^2(E \cap r\mathbb{B}_2) \leq \pi r^2$. It seemes probable that if $\Lambda^2(E \cap r\mathbb{B}_2) < \pi r^2$ for all large values of $r$ then $E$ is removable, but no proof has presented itself. The discrepancy between $\frac{\pi^2}{4\sqrt{2}}$ here arises in part from the integral geometric inequality (2) and in part from Lemma 2.

## 4. A result on the ball

We now turn to a result on the ball that is an analogue in the Bergman geometry of the result we obtained above for $\mathbb{C}^N$.

The Bergman kernel on the ball in $\mathbb{C}^N$ is given by

$$K(z,\zeta) = \frac{N!}{\pi^N} \frac{1}{(1 - \langle z, \zeta \rangle)^{N+1}} \tag{3}$$

if $\langle\ ,\ \rangle$ denotes the Hermitian inner product on $\mathbb{C}^N$, and the Bergman metric is given by

$$ds^2 = \sum_{j,k=1}^{N} T_{jk} dz_j \otimes d\bar{z}_k$$

with coefficients $T_{jk}$ given by

$$\begin{aligned} T_{jk} &= \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \ln K(z,z) \\ &= N(1-\langle z,z\rangle)^{-2}\{(1-\langle z,z\rangle)\delta_{jk} + z_k \bar{z}_j\} \end{aligned}$$

We shall denote by $\Lambda_B^\alpha$ the $\alpha$-dimensional Hausdorff measure computed with respect to the distance function on $\mathbb{B}_N$ derived from the Bergman metric. We shall prove the following analogue of Theorem 1.

**3. Theorem.** *If $E \subset \mathbb{B}_N$ is closed set with $\Lambda_B^{2N-2}(E) < \infty$, then $E$ is removable.*

**4. Corollary.** *If $E \subset \mathbb{B}_N$ is a subvariety of codimension one, then $E$ has infinite area, area computed with respect to the Bergman metric.*

The corollary follows from the theorem, for codimnension-one subvarieties of the ball are not removable: If $V$ is such a variety, then as we can solve the second Cousin problem on $\mathbb{B}_N$, there is $f \in \mathcal{O}(\mathbb{B}_N)$ with $V$ as its zero set. The reciprocal of $f$ shows $V$ not to be removable.

As we shall see below, there is a straightforward calculation that shows that if $V \subset \mathbb{B}_N$ is a k-dimensional variety, then $\Lambda_B^{2k}(V)$ is infinite.

*Proof of the Theorem:* The proof follows the general lines of the proof in the case of $\mathbb{C}^N$, but certain integral-geometric details require attention. We start with the case that $N = 2$.

Let $\mathrm{dist}_B(z,w)$ denote the Bergman distance between the points $z, w \in \mathbb{B}_2$.

Fix a point $z_0 \in \mathbb{B}_2$, and define $\rho : \mathbb{B}_2 \to [0,\infty)$ by $\rho(z) = \mathrm{dist}_B(z, z_0)$. The triangle inequality in the Bergman distance yields that $\rho$ is a Lipschitz function:

$$|\rho(z) - \rho(z')| \leq \mathrm{dist}_B(z,z').$$

As $\rho$ satisfies a Lipschitz condition and $\Lambda_B^2(E) < \infty$, we have that

$$\infty > \Lambda_B^2(E) > \mathrm{const.} \int_{[0,\infty)}^{*} \Lambda_B^1\left(E \cap \{z \in \mathbb{B}_2 : \rho(z) = t\}\right) dt.$$

This yields a sequence $\{t_1\}_{i=1}^{\infty}$ with $t_j \to \infty$ and with

$$\Lambda_B^1\left(E \cap \{z \in \mathbb{B}_2 : \rho(z) = t_j\}\right) \to 0,$$

and this implies that

$$\Lambda^1\left(E \cap \{z \in \mathbb{B}_2 : \rho(z) = t_j\}\right) \to 0.$$

(NB. As before, $\Lambda^1$ denotes the 1-dimensional Hausdorff measure computed with respect to the Euclidean metric.)

Let $D(t, z_0) = \{z \in \mathbb{B}_2 : \rho(z, z_0) = t\}$. This is a ball in the Bergman metric, and its boundary is smooth.

Granted that $f \in \mathcal{O}(\mathbb{B}_2 \backslash E)$, we know that $f|bD(t_j, z_0)\backslash E$ continues holomorphically into a neighborhood of $z_0$, at least when $j$ is large, so the result in the two–dimensional case is obtained as before.

To make the induction step work as before, we need two facts. First, we need to know that if $E \subset \mathbb{B}_{N+1}$ satisfies $\Lambda_B^{2N}(E) < \infty$, then for almost every $\Pi \in \mathcal{G}_{N+1,N}$, $\Lambda_B^{2N-2}(\Pi \cap \widetilde{E}) < \infty$ where we denote by $\widetilde{E}$ the set $E \cap \{z : \operatorname{dist}_B(z, 0) > 1\}$. ($\Lambda_B^{2N-2}(\Pi \cap E)$ denotes the Hausdorff measure computed with respect to the Bergman metric on $\mathbb{B}_{N+1}$.) The second point we need is that the finiteness of the quantity $\Lambda_B^{2N-2}(\Pi \cap E)$ implies the finiteness of the $(2N-2)$–dimensional Hausdorff measure of the set $\Pi \cap E$ *computed with respect to the Bergman metric on the $N$-dimensional ball* $\Pi \cap \mathbb{B}_{N+1}$.

The latter point is straightforward though, for the metric induced on $\Pi \cap \mathbb{B}_{N+1}$ from the Bergman metric on $\mathbb{B}_{N+1}$ differs only by a constant factor from the Bergman metric on the $N$–ball $\Pi \cap \mathbb{B}_{N+1}$.

That $\Lambda_B^{2N}(E) < \infty$ implies $\Lambda_B^{2N-2}(\Pi \cap \widetilde{E}) < \infty$ for almost all $\Pi$'s is an analogue in the Bergman metric of the result of Shiffman used above. We prove the following integral–geometric fact.

**5. Lemma.** *There is a constant $c_N$ such that if $S \subset \mathbb{B}_N \backslash \{z : \operatorname{dist}_B(z, 0) < 1\}$, then*

$$c_N \Lambda_B^{2N-2}(S) \geq \int_{\mathcal{G}_{N,N-1}} \Lambda_B^{2N-4}(S \cap \Pi) d\mu(\Pi).$$

The proof of this lemma follows precisely the lines of the proof of Shiffman's Lemma 5 in [**12**] once we have the following estimate.

**6. Lemma.** *There is a constant $k_N$ such that for small $\delta > 0$ if $T \subset \mathbb{B}_N \backslash \{z : \operatorname{dist}_B(0, z) < 1\}$ and $T$ has diameter less than $\delta$ in the Bergman distance, then $\mu(\{\Pi \in \mathcal{G}_{N,N-1} : \Pi \cap T \neq \emptyset\}) < k_N \delta^2$.*

For the convenience of the reader, we recall the argument in [**12**] that proves Lemma 5. Denote by $\delta(E)$ the diameter of the subset $E$ of $\mathbb{B}_N$ computed with respect to the Bergman distance. If $E \subset \mathbb{B}_N$, then

$$\int^*_{\mathcal{G}_{N,N-1}} \delta^{2N-4}(E \cap \Pi) d\mu(\Pi) \leq \delta^{2N-4}(E) \mu\{\Pi : \Pi \cap E \neq \emptyset\}.$$

If $E \subset \mathbb{B}_N \backslash \{z : \operatorname{dist}_B(z, 0) < 1\}$, and if $\delta(E)$ is small, then Lemma 6 implies the estimate

$$\mu(\{\Pi : \Pi \cap E \neq \emptyset\}) \leq \text{const.} \delta^2(E),$$

so, for such an $E$, we have

$$\int^{*} \delta^{2N-4}(E \cap \Pi) d\mu(\Pi) < \text{const.} \delta^{2N-2}(E).$$

Now given $S$ as in Lemma 6, assume $\Lambda_B^{2N-2}(S) < \infty$. Fix a small $\varepsilon > 0$, and choose a covering of $S$ by a sequence $\{S_n\}_{n=1}^{\infty}$ of sets with $\delta(S_n) < \varepsilon$ and

$$\sum_n \delta^{2N-2}(S_n) < \Lambda_{B,\varepsilon}^{2N-2}(S) + \varepsilon.$$

Here

$$\Lambda_{B,\varepsilon}^{2N-2}(S) = \inf \{ \sum_n \delta^{2N-2}(Q_n) : S \subset \cup Q_n \text{ and } \delta(Q_n) < \varepsilon \}.$$

We have then that

$$\begin{aligned}
\int^{*}_{\mathcal{G}_{N,N-1}} \Lambda_B^{2N-4}(S \cap \Pi) d\mu(\Pi) &\le \int^{*}_{\mathcal{G}_{N,N-1}} \sum_n \Lambda_B^{2N-4}(S_n \cap \Pi) d\mu(\Pi) \\
&\le \sum_n \int^{*}_{\mathcal{G}_{N,N-1}} \Lambda_B^{2N-4}(S_n \cap \Pi) d\mu(\Pi) \\
&\le \text{const.} \sum_n \int^{*}_{\mathcal{G}_{N,N-1}} \delta^{2N-4}(S_n \cap \Pi) d\mu(\Pi) \\
&\le \text{const.} \sum_n \delta^{2N-2}(S_n) \\
&\le \text{const.} (\Lambda_{B,\varepsilon}^{2N-2}(S) + \varepsilon).
\end{aligned}$$

As this is true for all $\varepsilon$ and as $\Lambda_B^{2N-2}(S) = \lim_{\varepsilon} \Lambda_{B,\varepsilon}^{2N-2}(S)$, we have the desired inequality.

Lemma 6 is a consequence of the corresponding Hermitian result. The Bergman diameter of a set is not smaller than the Euclidean diameter. Thus, if $T$ has small Bergman diameter $d$ and is included in $\mathbb{B}_N \backslash \{z : \operatorname{dist}_B(0,z) < 1\}$, then $T$ is contained in a Euclidean ball $B$ of Euclidean diameter $2d$. As $d$ is small, $B$ can be choose to lie in $\{z : |z| > 1 - d\}$. Everything follows from the estimate:

$$\mu(\{\Pi : \Pi \cap \mathbb{B}(p_0, R) \neq \emptyset\}) \le \text{const.} \left( \frac{R}{|p_0|} \right)^2, \tag{4}$$

which is established in the next section.

It is worth noting that our Theorem 3 implies Shiffman's result that for domains in $\mathbb{C}^N$, closed sets of vanishing $(2N-2)$-dimensional measure are removable. Shiffman's result is local, and if $\Lambda^{2N-2}(E) = 0$, then for every $p \in E$ and every ball $\mathbb{B}_N(p,r)$ centered at $p$, the set $E \cap \mathbb{B}_N(p,r)$ has zero $(2N-2)$-dimensional measure with respect to the Bergman metric on $\mathbb{B}_N(p,r)$. Thus, $E$ is locally removable and so removable.

## 5. An integral–geometric computation

In the analysis above, we need to know the measure of the set of $(N-1)$-dimensional subspaces of $\mathbb{C}^N$ that meet a ball. In (4) we stated an estimate that suffices; in this section, we shall evaluate this volume precisely. We shall, in fact, work in a slightly more general context. (It seems probable that the result obtained here exists somewhere in the published literature, but we know no reference.)

We are denoting by $\mathcal{G}_{N,k}$ the Grassmannian of all $k$-dimensional complex subspaces of $\mathbb{C}^N$. (Thus, the elements of $\mathcal{G}_{N,k}$ pass through the origin). The manifold $\mathcal{G}_{N,k}$ is a homogeneous space of the unitary group $\mathcal{U}(N)$ : If $g \in \mathcal{U}(N)$ and $\Pi \in \mathcal{G}_{N,k}$, then $g \cdot \Pi = g(\Pi) \in \mathcal{G}_{N,k}$. There is a unique measure $\mu_k$ on $\mathcal{G}_{N,k}$ with $\mu_k(\mathcal{G}_{N,k}) = 1$ that is invariant under the action of $\mathcal{U}(N)$. If we denote by $\Pi_0$ the element

$$\{z \in \mathbb{C}^N : z_{k+1} = \cdots = z_N = 0\}$$

of $\mathcal{G}_{N,k}$ and if $\pi : \mathcal{U}(N) \to \mathcal{G}_{N,k}$ is the map given by $\pi g = g \cdot \Pi_0$, then $\mu_k$ can be calculated by

$$\mu_k(E) = v(\pi^{-1}(E))$$

if $v$ denotes the normalized Haar measure on $\mathcal{U}(N)$.

Our problem, precisely formulated, is the following: To determine

$$\mu_k\left(\{\Pi \in \mathcal{G}_{N,k} : \Pi \cap \mathbb{B}_N(z_0, R) \neq \emptyset\}\right),$$

or, equivalently, to determine

$$v\left(\{g \in \mathcal{U}(N) : g(\Pi_0) \cap \mathbb{B}_N(z_0, R) \neq \emptyset\}\right).$$

Here, $z_0 \in \mathbb{C}^N$ and $R > 0$. If $|R| > z_0$, then $0 \in \mathbb{B}_N(z_0, R)$, so the measure in question is one. In general, the answer will be a function of $z_0$ and $R$. The problem is plainly invariant under the action of $\mathcal{U}(N)$, so without loss of generality, we may suppose that $z_0 = \boldsymbol{\rho} = (\rho, 0, \ldots, 0)$ with $\rho = |z_0|$.

We have that $g(\Pi_0) \cap \mathbb{B}_N(\boldsymbol{\rho}, R) \neq \emptyset$ if and only if the distance $d(\boldsymbol{\rho}, g(\Pi_0))$ is less than $R$.

We denote by $\{e_1, \ldots, e_N\}$ the standard orthonormal basis for $\mathbb{C}^N$. Then $\{e_1, \ldots, e_k\}$ is an orthonormal basis for $\Pi_0$ and $\{e_{k+1}, \ldots, e_N\}$ is an orthonormal basis for the orthogonal complement, $\Pi_0^\perp$, of $\Pi_0$. Consequently, if $(,)$ denotes the standard Hermitian inner product on $\mathbb{C}^n$ then

$$\begin{aligned} d(\boldsymbol{\rho}, g(\Pi_0)) &= \Big(\sum_{j=k+1}^{N} |(\boldsymbol{\rho}, g(e_j))|^2\Big)^{\frac{1}{2}} \\ &= \Big(\sum_{j=k+1}^{N} |(g^{-1}\boldsymbol{\rho}, e_j)|^2\Big)^{\frac{1}{2}}. \end{aligned}$$

We identify $\mathcal{U}(N)$ with the group of unitary $N \times N$ matrices $U = (a_{r,s})_{r,s=1,\dots,N}$. If under this identification $g$ corresponds to $U$, then

$$\langle g^{-1}\rho, e_j\rangle = \rho\bar{a}_{1j},$$

so

$$d(\boldsymbol{\rho}, g(\Pi_0)) = (\sum_{j=k+1}^{N} \rho^2|a_{1j}|^2)^{\frac{1}{2}}.$$

If we set

$$\mathcal{E}(k;c) = \{g \in \mathcal{U}(N) : \sum_{j=1}^{N} |a_{1j}|^2 < c\},$$

then we have to determine $v(\mathcal{E}(k;c))$. For $c > 1, \mathcal{E}(k;c) = \mathcal{U}(N)$; in general it is an open set.

For the computation of $v(\mathcal{E}(k;c))$ we need to recall the explicit form of the measure $v$. An invariant volume form on $\mathcal{U}(N)$ is the form $\Omega$ given by

$$\Omega = (\bigwedge_{1\le i<j\le N} \omega_{ij} \wedge \bar{\omega}_{ij}) \wedge \bigwedge_{1\le k\le N} \omega_{kk}$$

where

$$\omega_{ij} = \sum_{k=0}^{N} \bar{a}_{ki} da_{kj}.$$

The forms $\omega_{ij}$ are left–invariant on $\mathcal{U}(N)$. For the construction of $\Omega$, see [**9**]. In particular, one finds there the evaluation

$$\int_{\mathcal{U}(N)} \Omega = \prod_{j=0}^{N-1} \frac{(2\pi i)^{j+1}}{j!}.$$

We shall denote this value by $v(N)$. It follows then that the normalized Haar measure $v$ on $\mathcal{U}(N)$ is the measure derived from the form $v(N)^{-1}\Omega$.

Introduce the forms $\omega'(z)$ and $\omega(\bar{z})$ on $\mathbb{C}^N$ by

$$\omega'(z) = \sum_{j=1}^{N} (-1)^{j-1} z_i dz_1 \wedge \cdots \wedge [j] \wedge \cdots \wedge dz_N$$

and

$$\omega(\bar{z}) = d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_N.$$

If $T : \mathbb{C}^N \to \mathbb{C}^N$ is a linear transformation, then $T^*\omega(\bar{z}) = (\overline{\det T})\omega(\bar{z})$ and $T^*\omega'(z) = (\det T)\omega$. Consequently, the form $\widetilde{\omega}(z) = \omega'(z) \wedge \omega(\bar{z})$ is unitarily invariant.

We define a map $\eta : \mathcal{U}(N) \to S^{2N-1}$ by $\eta g = g\mathbf{1}$ where by $\mathbf{1}$ we mean the north pole $\mathbf{1} = (1, 0, \ldots 0)$. The fiber $\eta^{-1}(\mathbf{1})$ is the subgroup of $\mathcal{U}(N)$ isomorphic to $\mathcal{U}(N-1)$ that consists of the matrices of the form

$$\begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & a_{22} & \ldots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{N2} & \ldots & a_{NN} \end{bmatrix}.$$

The form $\widetilde{\omega}$ is invariant under the action of $\mathcal{U}(N)$ on $S^{2N-1}$, and it follows that $\eta^*\widetilde{\omega}$ is a left-invariant form on $\mathcal{U}(N)$.

At the identify of $\mathcal{U}(N)$, we have

$$\eta^*\widetilde{\omega} = d\alpha_{21} \wedge \cdots \wedge d\alpha_{N1} \wedge d\bar{\alpha}_{11} \wedge d\bar{\alpha}_{21} \wedge \cdots \wedge d\bar{\alpha}_{N1},$$

and

$$\begin{aligned}(\bigwedge_{i<j\le N} (\omega_{1j} \wedge \bar{\omega}_{1j}) \wedge \omega_{11} &= \{ \bigwedge_{1<j\le N} (\sum_{k=1}^{N} \delta_{k1} d\alpha_{kj}) \wedge (\sum_{k=1}^{N} \delta_{k1} d\bar{\alpha}_{kj})\} \wedge d\alpha_{11} \\ &= (\bigwedge_{1<j\le N} (d\alpha_{1j} \wedge d\bar{\alpha}_{1j})) \wedge d\alpha_{11}\end{aligned}$$

For a unitary matrix $A$, we have $\bar{A}^{\tau} A = 1$, i.e.,

$$\sum_{r=1}^{N} \bar{a}_{rk} a_{rj} = \delta_{jk}$$

whence

$$0 = \sum_{r=1}^{N} \bar{a}_{rk} da_{rj} + \sum_{r=1}^{N} a_{rj} d\bar{a}_{rk}.$$

Thus, $\omega_{kj} = -\bar{\omega}_{jk}$. In particular, at the identity, $d\alpha_{1j} = -d\bar{\alpha}_{j1}$. This implies that at the identity

$$(\bigwedge_{1<j\le N} (\omega_{1j} \wedge \bar{\omega}_{1j})) \wedge \omega_{11} = -(\bigwedge_{1<j\le N} d\bar{\alpha}_{j1} \wedge d\alpha_{j1}) \wedge d\bar{\alpha}_{11}$$

We see then that for a suitable choice of constant $\epsilon_N = \pm 1$, at the identity of $\mathcal{U}(N), \eta^*\widetilde{\omega}$ and $\epsilon_N(\bigwedge_{1<j\le N} (\omega_{1j} \wedge \bar{\omega}_{1j})) \wedge \omega_{11}$ coincide. As each is left invariant, they coincide on the whole $\mathcal{U}(N)$: On $\mathcal{U}(N)$,

$$\eta^*\widetilde{\omega} = \epsilon_N(\bigwedge_{1<j\le N} (\omega_{1j} \wedge \bar{\omega}_{1j})) \wedge \omega_{11}.$$

We now proceed to the computation of $v(\mathcal{E}(k;c))$. For this purpose, it is convenient to notice that if $\mathcal{E}'(k;c) = \{q \in \mathcal{U}(N) : \sum_{j=k+1}^{N} |a_{j1}|^2 < c\}$ then $v(\mathcal{E}'(k;c)) = v(\mathcal{E}(k;c))$. Under the projection $\eta$, $\mathcal{E}'(k;c)$ goes onto the subset

$$\sum(k;c) = \{z \in S^{2N-1} : |z_{k+1}|^2 + \cdots + |z_N|^2 < c\}$$

of $S^{2N-1}$. It follows from Fubini's theorem–see **[14]** for a version suitable for our purposes–that

$$\int_{\mathcal{E}'(k;c)} \Omega = \pm \int_{\Sigma(k;c)} \{ \int_{\eta^{-1}(z)} \bigwedge_{2\le i<j\le N} \omega_{ij} \wedge \bar{\omega}_{ij} \wedge \bigwedge_{2\le r\le N} \omega_{rr} \} \widetilde{\omega}.$$

Each of the fibers $\eta^{-1}(z)$ is a coset of the subgroup $\eta^{-1}$ $(1)$ of $\mathcal{U}(N)$, and accordingly, for each $z$,

$$\int_{\eta^{-1}(z)} \cdots = v(N-1).$$

Thus,

$$\int_{\mathcal{E}'(k;c)} \Omega = \pm v(N-1) \int_{\Sigma(k;c)} \widetilde{\omega}.$$

It remains for us to evaluate the integral on the right. Let us call it $I(k;c)$. It will be convenient to introduce the notation that for $z \in \mathbf{C}^N$, $z' = (z_1, \ldots, z_k)$, $z'' = (z_{k+1}, \ldots, z_N)$. By Stokes's theorem we have

$$I(k;c) = \int_{\mathbf{B}_N \cap \{|z''|^2 < c\}} d\widetilde{\omega} \quad - \quad \int_{\mathbf{B}_N \cap \{|z''|^2 = c\}} \widetilde{\omega}$$

Call the first of these integrals $I'$, the second $I''$. If $\gamma_N = \frac{1}{2}(N^2 - N)$, then

$$\begin{aligned}
I' &= N \int_{\mathbf{B}_N \cap \{|z''|^2 < c\}} dz_1 \wedge \cdots \wedge dz_N \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_N \\
&= (-1)^{\gamma_N} N \int_{\mathbf{B}_N \cap \{|z''|^2 < c\}} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_N \wedge d\bar{z}_N \\
&= (-1)^{\gamma_N} N \int_{\{|z''|^2 < c\}} \{ \int_{\{|z'|^2 < 1 - |z''|^2\}} dz_1 \wedge \cdots \wedge d\bar{z}_k \} \, dz_{k+1} \wedge \cdots \wedge d\bar{z}_N \\
&= (-1)^{\gamma_N} N \frac{(2\pi i)^k}{k!} \int_{\{|z''|^2 < c\}} (1 - |z''|^2)^k dz_{k+1} \wedge \cdots \wedge d\bar{z}_N \\
&= (-1)^{\gamma_N} N \frac{(2i)^N \pi^k}{k!} S_{2k-1} \int_0^{\sqrt{c}} (1-\rho^2)^k \rho^{2N-2k-1} d\rho
\end{aligned}$$

where $S_{2k-1}$ denotes the area of the unit sphere in $\mathbb{R}^{2k}$ so that $S_{2k-1} = \frac{2\pi^k}{(k-1)!}$. If we expand $(1-\rho^2)^k$ with the binomial theorem and integrate term–by–term, we reach

$$I' = (-1)^{\gamma_N} N \frac{(2i)^N \pi^k}{k!} \sum_{r=0}^{k} (-1)^r \binom{k}{r} \frac{c^{N-k+r}}{2(N-k+r)}.$$

For $I''$ we compute as follows: On the path of integration, $\Gamma$, for the integral $I''$, we have $|z_{k+1}|^2 + \cdots + |z_N|^2 = c$, so there

$$\sum_{r=k+1}^{N} z_r d\bar{z}_r + \bar{z}_r dz_r = 0.$$

Off the set where $z_N \neq 0$, we can solve this for $dz_N$:

$$dz_N = -\bar{z}_N^{-1}\Big(\sum_{r=k+1}^{N-1} \bar{z}_r dz_r + \sum_{r=k+1}^{N} z_r d\bar{z}_r\Big).$$

This leads to the expression

$$\begin{aligned}
\widetilde{\omega} &= -\sum_{j=1}^{N-1} (-1)^{j-1} z_i dz_1 \wedge \cdots \wedge [j] \wedge \cdots \wedge dz_{N-1} \wedge \Big(\bar{z}_N^{-1} \sum_{r=k+1}^{N-1} \bar{z}_r dz_r\Big) \wedge \omega(\bar{z}) \\
&\quad + (-1)^{N-1} z_N dz_1 \wedge \cdots \wedge dz_{N-1} \wedge \omega(\bar{z}). \\
&= -\sum_{j=r+1}^{N-1} (-1)^{j-1+N-j-1} z_j dz_1 \wedge \cdots \wedge dz_{j-1} \wedge (\bar{z}_N^{-1} \bar{z}_j dz_j) \wedge dz_{j+1} \\
&\quad \wedge \cdots \wedge dz_{N-1} \wedge \omega(\bar{z}) + (-1)^{N-1} z_N dz_1 \wedge \cdots \wedge dz_{N-1} \wedge \omega(\bar{z}) \\
&= (-1)^{N-1} \bar{z}_N^{-1} \Big(\sum_{j=r+1}^{N} z_j \bar{z}_j\Big) dz_1 \wedge \cdots \wedge dz_{N-1} \wedge \omega(\bar{z}) \\
&= c(-1)^{N-1+\gamma_{N-1}} \bar{z}_N^{-1} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{N-1} \wedge d\bar{z}_{N-1} \wedge d\bar{z}_N
\end{aligned}$$

The path of integration in $I''$ is specified by

$$|z'|^2 = |z_1|^2 + \cdots + |z_k|^2 < 1 - c$$

and

$$|z''|^2 = |z_{k+1}|^2 + \cdots + |z_N|^2 = c,$$

so we reach

$$\begin{aligned}
I'' = c(-1)^{N-1+\gamma_{N-1}} &\Big(\int_{\{|z'|^2 < 1-c\}} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_k \wedge d\bar{z}_k\Big) \\
&\Big(\int_{\{|z''|^2 = c\}} \bar{z}_N^{-1} dz_{k+1} \wedge d\bar{z}_{k+1} \wedge \cdots \wedge dz_{N-1} \wedge d\bar{z}_{N-1} \wedge d\bar{z}_N\Big).
\end{aligned}$$

The value of the first integral on the right is $\frac{(2\pi i)^k}{k!}(1-c)^k$. The second integral is evaluated as follows

$$\int_{\{|z''|^2=c\}} \bar{z}_N^{-1} dz_{k+1} \wedge d\bar{z}_{k+1} \wedge \cdots \wedge dz_{N-1} \wedge d\bar{z}_{N-1} \wedge d\bar{z}_N$$

$$= \int_{|z_{k+1}^2|+\cdots+|z_{N-1}|^2<c} \Big\{ \int_{|z_N|^2=c-|z_{k+1}|^2-\cdots-|z_{N-1}|^2} \bar{z}_N^{-1} d\bar{z}_N \Big\} dz_{k+1} \wedge d\bar{z}_{k+1} \wedge \cdots \wedge dz_{N-1} \wedge d\bar{z}_{N-1}$$

For every choice of $c$, the inner integral has the value $-2\pi i$. and

$$\int_{|z_{k+1}|^2+\cdots+|z_{N-1}|^2<c} dz_{k+1} \wedge d\bar{z}_{k+1} \wedge \cdots \wedge dz_{N-1} \wedge d\bar{z}_{N-1}$$

$$= \frac{(2\pi i)^{N-1-k}}{(N-1-k)!} c^{N-1-k}$$

Thus,

$$I'' = (-1)^{N+\gamma_{N-1}} \frac{(2\pi i)^N}{k!(N-1-k)!} (1-c)^k c^{N-k}.$$

The quantity we are interested in is $v(\mathcal{E}(k;c))$, which is given by

$$\begin{aligned} v(\mathcal{E}(k;c)) &= v(N)^{-1} \int_{\mathcal{E}'(k;c)} \Omega \\ &= \pm v(N)^{-1} v(N-1)[I'(k;c) - I''(k;c)]. \end{aligned}$$

Thus,

$$\begin{aligned} v(\mathcal{E}(k;c)) = \\ \pm N! \left(\frac{1}{2\pi i}\right)^N \Bigg[ (-1)^{\gamma_N} N \frac{(2i)^N \pi^k}{k!} \sum_{r=0}^{k} (-1)^r \tbinom{k}{r} \frac{c^{N-k+r}}{2(N-k+r)} \\ (-1)^{N+\gamma_{N-1}} \frac{(2\pi i)^N}{k!(N-1-k)!} (1-c)^k c^{N-k} \Bigg]. \end{aligned}$$

In the preceding section, we needed the special case of this in which $k = N-1$ and $c$ is small. we see that in this case,

$$\begin{aligned} v(\mathcal{E}(N-1;c)) &= \pm \frac{N!}{\pi^N} \left[ (-1)^{\gamma_N} \frac{N\pi^{N-1}}{(N-1)!2} + 0(c^2) - (-1)^{N+\gamma_{N-1}} \frac{\pi^N}{(N-1)!} c + O(c^2) \right] \\ &= 0(c), \end{aligned}$$

and this gives us what we needed.

## 6. Concluding remarks

The results on removable singularities we have obtained above are surely not the end of the story. The two results are of the general form: $\Omega$ is a domain in an $N$-dimensional complex manifold, $ds^2$ is a Hermitian metric on $\Omega$ and $\Lambda_s^{2N-2}$ denotes the $(2N-2)$-dimensional measure derived from $ds^2$. In two special cases, we have that a closed set $E$ in $\Omega$ is removable provided $\Lambda_s^{2N-2}(E)$ is finite. One may pose the question: What conditions on the metric $ds^2$ suffice for us to draw this conclusion? In particular, is it sufficient for $ds^2$ to be a complete Kähler metric? Do the metrics of Carathéodory or Kobayashi play a rôle here?

Another problem that arises is to stablish a projective version of the result valid for meromorphic functions. Consider the Fubini–Study metric on the complex projective space $P^N$. With respect to this metric, the volumes of the subvarieties of $P^N$ form a countable set; the volume of a variety in $\mathbf{P}^N$ is, to within a normalizing constant, its degree. If $E$ is a compact subset of $P^N$ that has $(2N-2)$-dimensional measure (with respect to the Fubini–Study metric) less than the smallest of the volumes of codimension–one hypersurfaces in $P^N$, does it follow that $E$ is removable for meromorphic functions in the evident sense that it $F$ is a function meromorphic on $P^N \backslash E$, then $F$ extends through $E$ to be meromorphic on the whole on $P^N$?

Another question that is suggested by what we have done is the following: If $D$ is a pseudoconvex domain in $C^N$, must $bD$ have dimension at least $2N-2$? The removable singularity theorem of Shiffman implies that the Hausdorff dimension or metric dimension is at least $2N-2$, and our Theorem 1 implies that $bD$ must have infinite $(2N-2)$-dimensional measure. The present question understands dimension in the sense of the topological theory of dimension for which one may consult [5].

## Appendix

**The Bergman area of varieties.**

We now take up a matter to which we adverted above, the fact that subvarieties of the ball have infinite area in the Bergman metric.

Given a domain $D$ in $\mathbf{C}^N$, the Bergman metric on $D$ is given by

$$ds^2 = \sum_{i,j=1}^{N} T_{ij} dz_i \otimes d\bar{z}_j$$

with

$$T_{ij} = \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K(z,z)$$

if $K$ denotes the Bergman kernel function. The associated *fundamental form* is the $(1,1)$–form $\omega$ given by

$$\omega = \sum_{i,j} T_{ij} dz_i \wedge d\bar{z}_j = \partial\bar{\partial} \log K(z,z),$$

and if $V \subset D$ is a $k$–dimensional variety, then the *Bergman volume of* $V$ is given by the integral $\left(\frac{1}{2i}\right)^k \int_V \omega^k$.

We fix a bounded domain $V_0 \subset V$ with $bV_0$ smooth enough that Stokes's theorem holds on it.*Then

$$\begin{aligned}\int_{V_0} \omega^k &= \int_{V_0} d\left\{(\bar{\partial} \log K(z,z)) \wedge (\partial\bar{\partial} \log K(z,z)^{k-1}\right\} \\ &= \int_{bV_0} \bar{\partial} \log K(z,z) \wedge (\partial\bar{\partial} \log K(z,z))^{k-1}.\end{aligned}$$

We have

$$\partial\bar{\partial} \log K = \frac{K\partial\bar{\partial}K - \partial K \wedge \bar{\partial}K}{K^2}.$$

As the exterior product of a 1–form with itself is zero, we find that

$$\int_{bV_0} \bar{\partial} \log K \wedge (\partial\bar{\partial} \log K)^{k-1} = \int_{bV_0} K^{-k}\bar{\partial}K \wedge (\partial\bar{\partial}K)^{k-1}.$$

If $V_r = \{z \in D : K(z,z) < r\}$, then we find

$$\begin{aligned}\int_{bV_R} \omega^k &= r^{-k} \int_{bV_r} \bar{\partial}K \wedge (\partial\bar{\partial}K)^{k-1} \\ &= r^{-k} \int_{V_r} (\partial\bar{\partial}K)^k.\end{aligned}$$

In the case of the ball, where $K$ is given by (3), a computation shows that with $c_N = \frac{N!}{\pi^N}$,

$$\bar{\partial}K \wedge (\partial\bar{\partial}K(z,z))^{k-1} = \frac{c_N^k (N+1)^k}{(1-\langle z,z\rangle)^{k(N+2)}} (\sum_{j=1}^N z_j d\bar{z}_j) \wedge (\sum_{j=1}^N dz_j \wedge d\bar{z}_j)^{k-1}.$$

If $W_r = \{z \in V : |z| < r\}$, then on $bW_r$, $K(z,z) = c_N(1-r^2)^{-(N+1)}$, so

$$\begin{aligned}\int_{\partial W_r} \omega^k &= (N+1)^k (1-r^2)^{-k} \int_{W_r} (\sum dz_i \wedge d\bar{z}_i)^k \\ &= (2i)^k (N+1)^k (1-r^2)^{-k} \Lambda^{2k}(W_r),\end{aligned}$$

where, as before, $\Lambda^{2k}$ denotes the $2k$–dimensional Hausdorff measure computed with respect to the Euclidean metric.

We have reached the result that if $V \subset \mathbb{B}_N$ is a $k$–dimensional variety, then

$$\Lambda_B^{2k}(V \cap r\mathbb{B}_N) = \frac{(N+1)^k}{(1-r^2)^k} \Lambda^{2k}(V \cap r\mathbb{B}_N).$$

In particular, $V$ has infinite volume in the Bergman metric.

*A discussion of a version of Stokes's theorem sufficient for our present needs is given in [13].

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