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# INFINITE LOCALLY FINITE GROUPS OF TYPE $PSL(2,K)$ OR $Sz(K)$ ARE NOT MINIMAL UNDER CERTAIN CONDITIONS

JAVIER OTAL AND JUAN MANUEL PEÑA

*Abstract*

In classifying certain infinite groups under minimal conditions it is needed to find non-simplicity criteria for the groups under consideration. We obtain some of such criteria as a consequence of the main result of the paper and the classification of finite simple groups.

**Introduction.** If $X$ is a class of groups by a *minimal non-$X$- group* we mean a group $G$ which is not a $X$-group but in which every proper subgroup is a $X$-group. In [**1**], [**2**] and [**3**] Bruno and Phillips have studied various minimality conditions, which have been extended in [**7**], [**8**] and [**9**] by the authors of the present paper. Roughly speaking, our extension has consisted in replacing the term "finite group" in Bruno-Phillips cases by "Černikov group" in ours, although the results we have found have a rather different nature. In order to obtain the main results of the mentioned papers it has been needed to give some non-simplicity criteria for the groups under consideration. These criteria are now a consequence of some results obtained with the aid of the classification of finite simple groups (see [**6**] and [**12**]), which reduce the simple cases to projective special linear groups $PSL(2,K)$ and Suzuki groups $Sz(K)$, and of the main result of this paper, which we state as follows:

**Theorem.** *Let $K$ be an infinite locally finite field, and let $C$ be one of the following classes of groups: (i) $CC$-groups; (ii) Černikov-by-hypocentral groups; (iii) hypercentral-by-Černikov groups. Then no group of type $PSL(2,K)$ or $Sz(K)$ is a minimal non-$C$-group.*

If $X$ is one of the classes of groups considered in [**1**], [**2**], [**3**], [**7**], [**8**] or [**9**] and $G$ is a locally graded minimal non-$X$-group, then it is proved that $G$ is locally finite and, in the Černikov-by-nilpotent and nilpotent-by-Černikov cases, the proper subgroups of $G$ are soluble-by-finite, so that [**6**] or [**12**] and the present Theorem ensure that $G$ cannot be simple. Another application of our Theorem also gives information on the structure of a (locally graded) minimal non-$CC$-group (see [**8**, 4.2]).

Throughout the paper we use the standard group-theoretic notation from [**11**]. The proof of our Theorem will be carried out in sections 3 and 4, where we shall construct proper subgroups of such simple group which are not $C$-groups for each $C$ considered. Some of these examples are similar to those given in [**10**] and in Bruno-Phillips papers. However our cases need somewhat more than the infinitude of the field $K$, namely the structure of its multiplicative group $K^*$, which will be given in section 1, and, moreover, we have needed to look for certain Frobenius subgroups in such simple groups in order to solve the hypercentral-by-Černikov case, whose details will be done in section 2.

**1. The multiplicative group of a locally finite field $K$.** We recall that $K$ is exactly an algebraic extension of a finite field. If $p$ is the characteristic of $K$, then the additive group $K^+$ of $K$ is $p$-elementary abelian whereas the multiplicative group $K^*$ is locally cyclic and can be embedded in a direct product of Prüfer groups, one for each prime different of $p$ (see [**4**]). The key point in what follows is the following well-known fact, of which we give a proof for the reader's convenience.

**(1.1).** *$K^*$ is a Černikov group if and only if $K$ is finite.*

Its proof needs an auxiliary result:

**(1.2).** *If $p$ is a prime, $m \geq 1$ is an integer and $q$ is either an odd prime or 4, then the factorization of $p^{mq}-1$ at least contains one more prime than that of $p^m - 1$.*

*Proof:* Suppose that the result is false. If $r$ is a prime divisor of $N :=$ $= p^{m(q-1)} + p^{m(q-2)} + \cdots + p^m + 1$, then $r$ divides $p^m - 1$. Thus $r$ divides each one of $p^{m(q-1)} - 1, \ldots, p^m - 1$ so that $r$ divides $q$. Therefore $r = q$, if $q$ is odd, or $r = 2$, otherwise.

In the first case $N = q^s$, $s \geq 1$, $p^m - 1 = q^t n$, $t \geq 1$ and $(n, q) = 1$. Thus $q^{t+s}n = p^{mq} - 1 = (q^t n+1)^q - 1 = q^{t+1}n(a+1)$ and so $a+1 = q^{s-1}$, where $a = \sum_{i=0}^{q-2} \binom{q}{i} q^{t(q-i-1)} n^{(q-i-1)}$. Since $q \neq 2$, we have $a > 1$ and $s - 1 \geq 1$. Clearly $q$ divides $a$ and so $q$ divides 1, a contradiction.

If $r = 2$, then $(p^{2m} + 1)(p^m + 1) = N = 2^s$ and so $p^m + 1 = 2^t$ and $p \neq 2$. Thus $(p^m + 1)^2 - 2p^m = p^{2m} + 1 = 2^{s-t}$ and it follows that $s - t \geq 2$. Therefore 4 divides $2p^m$, a contradiction. ■

*Proof of (1.1):* Suppose that $K$ is infinite. Let $K_1 < K_2 < \ldots$ be an infinite ascending chain of finite subfields of $K$. If $|K_i| = p^{r_i}$, then $r_i$ divides $r_{i+1}$ for every $i \geq 1$, so that $r_{i+2} = r_i q n_i$ where $n_i \geq 1$ and $q$ is either an odd prime or 4. By applying the Lemma we deduce that the set of all primes occurring as divisors of the $|K_i^*|$, $i \geq 1$, is infinite and therefore $K^*$ cannot be Černikov. ■

**2. Locally finite hypercentral-by-Černikov Frobenius groups.** We refer to the section 1.J of [5], where the abstract definition and the basic properties of a locally finite Frobenius group can be found. In what follows, we shall frequently use the following characterization of Frobenius groups ([5, 1.J.3]), which is an extension of that of the finite case: *If $G$ is a locally finite group, then $G$ is a Frobenius group if and only if $G$ has a proper normal subgroup $N$ such that $C_G(x) \leq N$ for every $1 \neq x \in N$.* In fact, it is not hard to show that such $N$ is exactly the Frobenius kernel of $G$.

The result we shall need in the next sections is the following:

**(2.1).** *Let $G$ be a locally finite Frobenius group with complement $C$. Then the following are equivalent: (1) $C$ is Černikov. (2) $G$ is nilpotent-by-Černikov. (3) $G$ is hypercentral-by-Černikov.*

*Proof:* Let $N$ be the kernel of $G$. Since $N$ is always nilpotent, (1) implies (2). That (2) implies (3) is trivial so that it suffices to show that (3) implies (1).

Let $L$ be a hypercentral normal subgroup of $G$ such that $G/L$ is Černikov. Clearly, we may assume that $L \neq 1$. If $L \cap N = 1$, by [5, 1.J.2] we have that $L$ is contained in some conjugate of $C$ so that $L$ is contained in every conjugate of $C$, which gives $L = 1$. Therefore $L \cap N \neq 1$ and so $\varsigma(L) \cap N \neq 1$ since $L$ is hypercentral. If $1 \neq x \in \varsigma(L) \cap N$, then $L \leq C_G(x)$ and, since $C_G(x) \leq N$, it follows that $L \leq N$. Therefore $G/N \simeq C$ is a Černikov group. ∎

**3. The $PSL(2, K)$ case, $K$ infinite locally finite.** Let $H$ be the image in $PSL(2, K)$ of the group of lower triangular matrices in $SL(2, K)$. $H = UD$ is the semidirect product of the image $U$ of the group of lower unitriangular matrices by the image $D$ of the group of diagonal matrices in $SL(2, K)$. We remark that $U \simeq K^+$ and that $D$ is isomorphic to a quotient of $K^*$ by a subgroup of order at most two so that $D$ is not a Černikov group by (1.1).

If $a \in K$ and $b \in K^*$, then we have

$$(1) \qquad \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b^2 a & 1 \end{pmatrix}.$$

Thus $x^H$ is infinite for every $1 \neq x \in U$ and, since $x^H \leq U$, it follows that $H$ is not a $CC$-group.

Moreover,

$$(2) \qquad \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ b^2 a & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a(b^2 - 1) & 1 \end{pmatrix}.$$

From (1) and (2) we readily obtain that $U = [U, D] = [U, H] = H'$. By induction it is easy to show that $U = \gamma_\alpha(H)$ for every ordinal $\alpha \geq 2$. Therefore $H$ is not Černikov-by-hypocentral.

Finally, from (1) we obtain $C_H(x) \leq U$ for every $1 \neq x \in U$. Therefore $H$ is a Frobenius group with complement $D$ and so $H$ cannot be hypercentral-by-Černikov by (2.1).

**4. The $Sz(K)$ case, $K$ infinite locally finite of characteristic 2.** As in the finite case (see [**13**]) the group $Sz(K)$ is a subgroup of $SL(4,K)$ defined in terms of an automorphism $\theta$ of $K$ satisfying $a^{\theta^2} = a^2, a \in K$, and generated by:

(i) a group $Q$ of matrices having the form

$$(a,b) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^{1+\theta} + b & a^\theta & 1 & 0 \\ a^{2+\theta} + ab + b^\theta & b & a & 1 \end{pmatrix} \qquad a, b \in K.$$

(ii) the group $D$ of diagonal matrices of the form

$$\bar{f} := diag[f^{1+\theta^{-1}}, f^{\theta^{-1}}, f^{-\theta^{-1}}, f^{-1-\theta^{-1}}] \qquad f \in K^*.$$

(iii) the permutation matrix

$$\tau = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Clearly $D \simeq K^*$, so that $D$ is not Černikov by (1.1), and $\tau$ is an involution inverting every element of $D$. Thus $L := < D, \tau >$ is metabelian and $L' \leq D$. If $\bar{f} \in D$, then $[\bar{f}, \tau] = \bar{f}^{-2}$ and since $(|\bar{f}|, 2) = 1$, it follows that $D = [D, \tau] = [L, \tau]$. Hence $L$ is not a $CC$-group. Moreover $D = [D, L] = D'$ and by induction it is easy to show that $D = \gamma_\alpha(L)$ for every ordinal $\alpha \geq 2$. Therefore $L$ is not Černikov-by-hypocentral.

Let $G := QD$ be the semidirect product of $Q$ by $D$. Clearly we have that $(a_1, b_1)(a_2, b_2) = (a_1 + a_2, a_1 a_2^\theta + b_1 + b_2)$ and $(a,b)^{\bar{f}} = (af, bf^{1+\theta})$. Let $1 \neq x = (a,b) \in Q$. If $\bar{f}(c,d) \in C_G(x)$, then $\bar{f}(c + a, ca^\theta + d + b) = \bar{f}(af + c, afc^\theta + bf^{1+\theta} + d)$ so that $af = a$ and $ca^\theta + b = afc^\theta + bf^{1+\theta}$. If $f \neq 1$, then $f^{1+\theta} \neq 1$ and we find $a = 0$ and $b = 0$, a contradiction. Therefore $\bar{f} = 1$ and our argument has just showed that $C_G(x) \leq Q$ for every $1 \neq x \in Q$. Thus $G$ is a Frobenious group with complement $D$ and hence $G$ cannot be hypercentral-by-Černikov by (2.1).

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Departamento de Matemáticas
Facultad de Ciencias
Universidad de Zaragoza
50009–Zaragoza, SPAIN.

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