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Connections between signal processing and complex analysis

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Abstract

We describe one of the research lines of the Grup de Teoria de Funcions de la UAB UB, which deals with sampling and interpolation problems in signal analysis and their connections with complex function theory.

Resum

Donem una descripció d'una de les línies de recerca del Grup de Teoria de Funcions de la UAB i la UB, que tracta els problemes de mostreig i interpolació en anàlisi del senyal i les seves connexions amb la teoria de funcions de variable complexa.

Keywords: Complex analysis, harmonic analysis, signal processing.

1. Introduction. The Shannon-Whittaker theorem

We start with the Shannon-Whittaker theorem, known to a wide range of scientists and engineers. In digital signal processing, the signals (functions to be analyzed) have *finite energy*; written $f \in L^2(\mathbb{R})$, or more explicitly

$$\|f\|_2^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt < \infty .$$

These functions can be represented as a superposition of sinusoids of different frequencies: by the Fourier representation

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{it\omega} d\omega ,$$

where the Fourier transform

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-it\omega} dt$$

gives the spectrum content of f.

This representation of signals is useful because many of the operations $f \mapsto Tf$ that are performed on signals (filtering, amplification, modulation) are linear and time-invariant, that is, they do not depend on the choice of a particular time origin. In mathematical terms, T commutes with the translations $(\tau_a f)(t) = f(t-a)$, i.e. $T(\tau_a f)(t) = \tau_a(Tf)$ for every origin a. The point is that the exponentials are eigenvectors for all such T: if $e_\omega(f) = e^{i\omega t}$, then $T(e_\omega) = M(\omega) e_\omega$. Hence the way T acts on f is transparent in the Fourier representation:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega \longrightarrow Tf(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) M(\omega) e^{i\omega t} d\omega .$$

This is the main reason why the Fourier transform is ubiquitous in applications.

Communication engineers work only with signals $f(t)$ that do not have a frequency content at arbitrarily high frequencies. Some limitations are related to the way signals are produced. For instance, the sound of an adult human male does not exceed 8000 herzs. Besides, even if the original signal had very high frequencies, these would be attenuated by the transporting media. Thus we are forced in a natural way to consider as a model for the more common onedimensional signals the space $PW\tau$ of functions with finite energy and spectrum in $(-\tau,\tau)$. These functions are called bandlimited with band-width τ, and can be represented as

$$f(t) = \frac{1}{2\pi} \int_{-\tau}^{\tau} \hat{f}(\omega) e^{i\omega t} d\omega , \tag{1}$$

with $\hat{f}(\omega)$ arbitrary of finite energy in $(-\tau,\tau)$.

Functions cannot be simultaneously time-limited (with compact support, meaning $f(t) = 0$ for $|t| \geq T$) and band-limited. This is so because in this case $\hat{f}$ would be an analytic function, and analytic functions may vanish only in discrete sets. This mathematical fact seems to contradict the intuition we have for "real life" signals, which appear to be of this kind. As argued in [S76] this is not an unapproachable obstacle. It makes no sense to discuss whether real-life functions are band-limited or time-limited, since this would need

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to be checked by measuring the signal in remote or future times, with arbitrarily high precision, something obviously out of reach. The notions of band-limited and time-limited exist only in the mathematical model and do not belong to real life. What happens in reality is that there exist band-limited functions that outside a time interval are very small, in a sense indistinguishable from the experimental signals, and this makes functions in PW_τ a suitable model for engineers.

The Shannon-Whittaker theorem states that a signal $f \in PW_\tau$ can be recovered from its samples at the Nyquist rate $f(\frac{k}{2\tau})$, $k \in \mathbb{Z}$, through the so-called *cardinal series*

$$f(t) = \sum_{k=-\infty}^{\infty} f\big(\frac{k}{2\tau}\big)\,\mathrm{sinc}\left[2\tau\left(t-\frac{k}{2\tau}\right)\right]$$

where sinc $x = \frac{\sin \pi x}{\pi x}$, Moreover,

$$2\tau\int_{-\infty}^{\infty}\big|f(t)\big|^2 dt = \sum_{k=-\infty}^{\infty}\big|f\big(\frac{k}{2\tau}\big)\big|^2.$$

Any sequence of samples $(a_k)_{k=-\infty}^{\infty}$ such that $\Sigma_k|a_k|^2<\infty$ may appear, for

$$f(t) = \sum_k a_k\,\mathrm{sinc}\,[2\tau\big(t-\frac{k}{2\tau}\big)]$$

defines in this case $f \in PW_\tau$ such that $f(\frac{k}{2\tau}) = a_k$.

Shannon made this theorem popular in the engineering community in the 40's as an important part of his theory of information; in the former Soviet Union it was independently formulated by Kotelnikov. But in fact the result had already appeared in the mathematical literature by the end of the XIX century. We give a detailed proof of this result, since it will motivate some basic definitions.

The space $L^2(\mathbb{R})$ of functions with finite energy becomes a Hilbert space with the inner product

$$\langle f,g\rangle = \int_{-\infty}^{\infty} f(t)\,\overline{g(t)}\,dt\,.$$

Two functions f and g are called *orthogonal* when $\langle f,g\rangle = 0$. Plancherel's theorem states that $\langle f,g\rangle = \frac{1}{2\pi}\langle \hat f,\hat g\rangle$, and in particular $\|f\|_2^2 = \frac{1}{2\pi}\|\hat f\|_2^2$. Thus the Fourier transform establishes an isometry (up to a factor of 2π) between the space PW_τ of band-limited functions with band-width τ and $L^2(-\tau,\tau)$ endowed with the inner product

$$\langle F,G\rangle = \int_{-\tau}^{\tau} F(\omega)\,\overline{G(\omega)}\,d\omega\,.$$

In an abstract Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ a family of vectors $\{e_i\}_{i\in I}$ is called *an orthonormal basis* if they are pairwise orthogonal (i.e. $\langle e_i,e_j\rangle = \delta_{ij}$), $\|e_i\| = 1$, and they (topologically) generate H, meaning that every $\upsilon \in H$ can be arbitrarily approximated by finite linear combinations $\sum_{i\in I}\lambda_i e_i$. It then follows that every $\upsilon \in H$ has a unique representation as an *infinite linear combination* (series) $\upsilon = \sum_i \lambda_i e_i$, and the coefficients are the correlations $\lambda_i = \langle \upsilon_i, e_j\rangle$. Moreover, the Pythagorean identity $\|\upsilon\|^2 = \sum_i|\lambda_i|^2$ holds, and $\upsilon = \sum\lambda_i e_i$, with $\sum|\lambda_i|^2<\infty$, is the *general expression* of $\upsilon \in H$

One may think of $L^2(-\tau,\tau)$ as being the space of 2τ-periodic functions with finite energy in one period. By the theory of Fourier series, we know that the normalized sines and cosines of that same period, that is $\frac{1}{\sqrt{2\tau}}\,e^{\pi\frac{k}{\tau}i\omega}$ constitute an orthonormal basis of $L^2(-\tau,\tau)$. Then, according to Plancherel's theorem, band-limited functions in PW_τ whose Fourier transform is $\frac{1}{\sqrt{2\tau}}\,e^{\pi\frac{k}{\tau}i\omega}$ constitute an orthonormal basis in PW_τ. Writing the expansion of $f \in PW_\tau$ in this basis one obtains exactly Shannon's theorem.

Thus, Shannon's theorem states that $\{\frac{1}{\sqrt{2\tau}}$ sinc $(2\tau t - k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis of PW_τ. We point out one particular feature: we have a *single function* $\frac{1}{\sqrt{2\tau}}$ sinc $2\tau t$, whose translates to the points $(k/2\tau)$, $k \in \mathbb{Z}$, constitute an orthonormal basis. This basic function has the property that for arbitrary $x \in \mathbb{R}$,

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\tau}^{\tau}\hat f(\omega)e^{i\omega x}d\omega = \frac{1}{2\pi}\langle \hat f, e^{-i\omega t}\rangle = \text{(Plancherel's)} \\ &= \frac{1}{\sqrt{2\tau}}\int_{-\infty}^{\infty} f(t)\,\mathrm{sinc}\,(2\tau t - x)dt \end{aligned} \tag{2}$$

2. Mathematical notions for discretization

The ShannonWhittaker theorem leads naturally to the introduction of the following notions, all addressed to support mathematically the connection from the analog to the discrete domain, or digitalization.

Let X be a space of functions with finite energy $X \subset L^2(\mathbb{R})$. In the most general situation, a *discretization process* for X consists of a linear map assigning to each $f \in X$ a discrete set of scalar coefficients

$$f \in X \xmapsto{D} (c_i(f))_{i\in I},$$

which is *one to one*, that is, f is completely determined by its coefficients $(c_i(f))_{i\in I}$. A common situation occurs when we are given a collection of functions $g_i \in X$ and the coefficients $c_i(f)$ are the correlations $c_i(f) = \langle f,g_i\rangle$; in this case injectivity of D means that $\langle f,g_i\rangle = 0\,\forall i$ should imply $f = 0$, which amounts to the fact that the $(g_i)_{i\in I}$ *topologically generate* X. We also say that the g_i are *complete* in X. Another situation occurs when the $c_i(f)$ are samples of f at certain points x_i:$c_i(f) = f(x_i)$; injectivity of D means here that $f(x_i) = 0\,\forall i$ should imply $f \equiv 0$. We say then that the set $\{x_i\}_{i\in I}$ is a *uniqueness set for* X.

Even though this notion leads to interesting mathematics, it has no practical application. A stability requirement needs to be introduced; one says that the process D is a *stable discretization process* if there exist two constants A, B such that

$$A\|f\|_2^2 \le \sum_i |c_i(f)|^2 \le B\|f\|_2^2 \quad f \in X.$$

Clearly, this says that $A\|f-g\|_2^2 \le \sum_i |c_i(f)-c_i(g)|^2 \le B\|f-g\|_2^2$, meaning that small errors in in the coding $(c_i(f))_{i\in I}$ produce small errors in f. In the first situation, when $c_i(f) = \langle f,g_i\rangle$ stability reads as

$$A\|f\|_2^2 \le \sum_i |\langle f,g_i\rangle|^2 \le B\|f\|_2^2, \quad f \in X.$$

We say that $\{g_i\}_{i\in I}$ is a *frame* for X, a notion that of course can be considered in any abstract Hilbert space H. In the second situation, when $c_i(f) = f(x_i)$, we have

(3) $$A\|f\|_2^2 \leq \sum_i |f(x_i)|^2 \leq B\|f\|_2^2.$$

One says that $\{x_i\}_{i\in I}$ is a *sampling set* for X.

A natural question arises: how do we identify f through the $c_i(f)$? Is there an explicit reconstruction? The answer is provided by the following result, in fact not difficult to prove: if $\{e_i\}_{i\in I}$ is a frame in a Hilbert space H, then there exists another frame , called the *dual frame*, such that

$$\upsilon = \sum_i \langle \upsilon, e_i\rangle \tilde{e}_i \quad \forall \upsilon \in H.$$

Besides, the dual frame of $\{\tilde{e}_i\}_{i\in I}$ is $\{e_i\}_{i\in I}$, and $\upsilon = \sum \langle \upsilon, \tilde{e}_i\rangle e_i$ too. We can look at these reconstruction formulas as analogues to the expansion $\upsilon = \sum \langle \upsilon, e_i\rangle e_i$ when $\{e_i\}_{i\in I}$ is an orthonormal basis of H. However, there is an important difference between frames and orthonormal bases, namely, for a general frame $\{e_i\}_{i\in I}$ one may have *redundancy* in the coding $\langle \upsilon, e_i\rangle$ coming from "hidden" relations of type

$$\sum_i \mu_i e_i = 0$$

between the $\{e_i\}_{i\in I}$. Thus, the $\{e_i\}_{i\in I}$ need not be "topologically linearly independent". For instance, we obtain a frame if we just gather a finite number of orthonormal bases. It is not hard to see that the frame has no redundancy (the $\{e_i\}_{i\in I}$ are topologically linearly independent) if and only if the following property holds: for an arbitrary sequence $(a_i)_{i\in I}$ of complex numbers such that $\sum_i |a_i|^2 < \infty$, there exists $\upsilon \in H$ such that $\langle \upsilon, e_i\rangle = a_i \forall i$. In general, a family of vectors $\{e_i\}_{i\in I}$ (not necessarily a frame) in a Hilbert space H having this property is called a *free system* or a *Riesz-Fischer family*.

For a non-redundant frame $\{e_i\}_{i\in I}$ with dual frame $\{\tilde{e}_i\}_{i\in I}$ in a Hilbert space H, every $\upsilon \in H$ therefore has two *unique* expressions

$$\upsilon = \sum_i \lambda_i e_i = \sum_i \mu_i \tilde{e}_i,$$

where $\lambda_i = \langle \upsilon, \tilde{e}_i\rangle$, $\mu_i = \langle \upsilon, e_i\rangle$ and both $\sum_i |\lambda_i|^2$, $\sum_i |\mu_i|^2$ are comparable to $\|\upsilon\|^2$. For this reason they are also called *Riesz bases* or biorthogonal bases. The orthonormal bases are those for which $e_i = \tilde{e}_i$ and $\|e_i\| = 1$.

Let us review these notions for the specific case $X = PW_\tau$, the space of τ-band limited functions. The analogue of a free system is the notion of *interpolating set* $\{x_i\}_{i\in I}$ for X: for an arbitrary $(a_i)_{i\in I}$ with $\sum |a_i|^2 < \infty$, there exists $f \in X$ with $f(x_i) = a_i$, $i \in I$. We first normalize so that $2\tau = 1$. As we have seen in Section 1, both properties can occur at the same time, because $f(x) = \langle f, \tau_x g\rangle$ with g the sinc function. Therefore, $\{x_i\}_{i\in I}$ is a uniqueness set for PW_τ if and only if $\{\tau_{x_i}\,\mathrm{sinc}\}_{i\in I}$ spans PW_τ, and it is a sampling set for PW_τ if and only if $\{\tau_{x_i}\,\mathrm{sinc}\}_{i\in I}$ is a frame for PW_τ.

Moreover, by Plancherel's theorem and (2) these concepts can be transported to $L^2(-\tau,\tau)$ and expressed in terms of the characters $\mathcal{E}(\Lambda) = \{e^{i\omega x}\}_{x\in\Lambda}$. Thus, $\{x_i\}_{i\in I}$ being a uniqueness set for PW_τ (resp. a sampling set) amounts to the set of exponentials $\{e^{i\omega x_i}\}_{i\in I}$ spanning $L^2(-\tau,\tau)$ (resp. being a frame).

In the following sections we will consider all these notions for various spaces of functions and survey the most important results. We will pay attention to the techniques used in the proofs.

3. Uniqueness sets for Paley-Wiener spaces. Beurling-Malliavin density, generators

Let us replace the real variable t by a complex variable $z \in \mathbb{C}$ in the representation (1) of functions $f \in PW_\tau$.

$$f(z) = \frac{1}{2\pi}\int_{-\tau}^{\tau} \hat{f}(\omega)e^{i\omega z}d\omega.$$

This defines an entire function, and a crude estimate shows that $|f(z)| \leq Ce^{\tau|Im\, z|}$. One says that f has *exponential type* lower or equal than τ. The converse is also true: if f is an entire function of exponential type $\leq \tau$ whose restriction to $\mathbb{R}$ has finite energy, then $f \in PW_\tau$. The space of entire functions of exponential type $\leq \tau$ with finite energy on $\mathbb{R}$ is called the *Paley-Wiener space*.

This is quite straightforward; nevertheless, given the power of complex analysis, it has very important consequences. The complexification of time allows also an extension of the previous notions. Namely, when discussing uniqueness, sampling or interpolation sets we may replace the real points $\{x_i\}$ by complex ones $\{z_i\}$; the problem becomes then "better posed" and it is a fully complex analysis problem. However, for the sake of simplicity, we will continue assuming in our description of results that the points are real.

Let us first discuss uniqueness sets $\Lambda = \{x_i\} \subset \mathbb{R}$ for PW_τ, that is, sets for which $f \in PW_\tau$ and $f(x_i) = 0\ \forall i$ implies $f \equiv 0$. Since every $f \in PW_\tau$ is entire, it is clear that every set Λ with a finite accumulation point is a uniqueness set; it is also transparent that a finite set cannot be a uniqueness set, so we assume from now on that Λ is infinite and has no finite accumulation point. In such case we can write Λ as a sequence $\Lambda = \{x_n\}_{n\in\mathbb{Z}}$ with $|x_n| \to \infty$. It is intuitively clear that Λ must be dense in some sense, so that $f_{|\Lambda} = 0$ implies $f = 0$. Now, if $f \in PW_\tau$ and $f(\alpha) = 0$, then the function $g(z) = f(z)\frac{(z-\beta)}{(z-\alpha)}$ is again in PW_τ and $g(\beta) = 0$; this means that we can move arbitrarily any finite number of points of Λ without changing the problem (by the way, a non trivial fact to prove without complex analysis). Consequently, the control on the density of the sequences Λ should be asymptotic, depending just on how Λ behaves "at infinity".

In a series of deep and very famous papers, Beurling and Malliavin (see [K96]) proved some results giving an almost complete description of uniqueness sets for PW_τ. They introduced a density $D_{BM}(\Lambda)$, now called the *Beurling-Malliavin density*, and proved that

$$D_{BM}(\Lambda) > 2\tau \Rightarrow \Lambda \text{ is a uniqueness set for } PW_\tau \Rightarrow D_{BM}(\Lambda) \geq 2\tau.$$

The definition of $D_{BM}(\Lambda)$ is complicated, but geometric in nature, and will not be given here. It is called a density because the number $D_{BM}(\Lambda)$ depends on how many points Λ has in large intervals. It is closely related to the classical density

$$\overline{D}(\Lambda) = \overline{\lim_{r\to 0}} \frac{n_\Lambda(r)}{2r},$$

where $n_\Lambda(r)$ indicates the number of points of Λ in $[-r,r]$. In particular, $D_{BM}(\Lambda) \geq \overline{D}(\Lambda)$.

We have already pointed out that $\Lambda = \{x_n\}$ is a uniqueness set for PW_τ if and only if the characters $\mathcal{E}(\Lambda) = \{e^{i\omega x_n}\}_{x_n\in\Lambda}$ span $L^2(-\tau,\tau)$. The *completeness radious* of a family of exponentials $\mathcal{E}(\Lambda)$ is defined as

$$R(\Lambda) = \sup \{\rho > 0, \mathcal{E}(\Lambda) \text{ spans } L^2(0,\rho)\}.$$

Of course, by translation we can replace $(0,\rho)$ by any interval of length ρ. The theorem of Beurling and Malliavin implies then the beautiful equality

$$R(\Lambda) = D_{BM}(\Lambda).$$

The exact description of uniqueness sets for PW_τ however remains unsolved.

The Beurling-Malliavin density also shows up in other problems related to uniqueness sets and spanning families. We have seen that if $\Lambda = \{x_n\}$ is a uniqueness set for PW_τ, then the translates of sinc $(2\tau t)$ by Λ span PW_τ. The reverse implication is actually true as well. A natural question arises: for the whole space $L^2(\mathbb{R})$, or more generally for $L^p(\mathbb{R})$, $1 \leq p < \infty$ do there exist pairs (φ,Λ), Λ discrete, such that the translates $\{\tau_x\varphi\}_{x\in\Lambda}$ are complete in $L^p(\mathbb{R})$? We say then that φ is a Λ *generator of* $L^p(\mathbb{R})$, or that Λ is a φ-*translation set for* $L^p(\mathbb{R})$.

In this context, we must first recall that according to *Wiener's theorem* all translates of $\varphi \in L^1(\mathbb{R})$ span $L^1(\mathbb{R})$ if and only if $\hat{\varphi}$ never vanishes. In the same direction, *Beurling's theorem* states that all translates of $\varphi \in L^2(\mathbb{R})$ span $L^2(\mathbb{R})$ if and only if $\hat{\varphi}$ is not zero almost everywhere. Obviously, there are then necessary conditions for generators in $L^1(\mathbb{R})$, $L^2(\mathbb{R})$, respectively.

Let $T(\varphi,\Lambda)$ denote the linear span of the Λ-translates of φ. Since translations correspond to multiplication by characters in the frequency domain, we may write

$$T(\varphi,\Lambda)^\wedge = \hat{\varphi}\,\mathcal{E}(\Lambda) \tag{4}$$

where $\mathcal{E}(\Lambda)$ denotes, as before, the linear span of the characters $e^{i\omega x_n}$, $x_n \in \Lambda$.

When Λ is a regular lattice $\Lambda = \gamma\mathbb{Z}$, that is, when we deal with regularly spaced translates of a fixed function, it is easy to prove that neither $L^1(\mathbb{R})$ nor $L^2(\mathbb{R})$ contain Λ-generators. For $L^2(\mathbb{R})$ this easily follows from (4) and Plancherel's theorem, because if Λ is a lattice then $\mathcal{E}(\Lambda)$ consists entirely of γ^{-1} periodic functions and $\hat{\varphi}\,\mathcal{E}(\Lambda)$ cannot be dense in $L^2(\mathbb{R})$. For $L^1(\mathbb{R})$ it will be proved below.

Surprisingly enough, in $L^p(\mathbb{R})$, $p > 2$, there do exist $\mathbb{Z}$-generators. This result was established in [AO96], and another proof can be obtained from results of [N70], [N74] (see also [F81]). Later, using Fourier series techniques, Olevskii [O97] showed that an arbitrary perturbation of $\mathbb{Z}$ of the form

$$\Lambda = \{n + a_n\} \quad a_n \neq 0,\ a_n \to 0$$

admits a generator $\varphi \in L^2(\mathbb{R})$. In $L^2(\mathbb{R})$, using (4) and Plancherel's theorem, it is immediate to see that the density of $T(\varphi,\Lambda)$ in $L^2(\mathbb{R})$ coincides with the density of $\mathcal{E}(\Lambda)$ in the weighted L^2-space $L^2(\mathbb{R},w)$, with $w = |\hat{\varphi}|^2$ (which is nonvanishing a.e., by Beurling's theorem). In particular, denoting $E_{\varepsilon,N} = \{\varepsilon \leq w \leq N\}$, $\mathcal{E}(\Lambda)$ will be dense in $L^2(E_{\varepsilon,N})$ and $|E_{\varepsilon,N}| \to \infty$ as $\varepsilon \to 0$, $N \to \infty$. Thus, if Λ has a generator in $L^2(\mathbb{R})$, then $\mathcal{E}(\Lambda)$ is dense in L^2 in sets of arbitrarily large measure.

This shows the connection of these questions with the subject of density of exponentials $\mathcal{E}(\Lambda)$ in function spaces and, in particular, with Beurling-Malliavin and Landau's results. Landau [L67a] constructed sets Λ as in (2) such that $\mathcal{E}(\Lambda)$ is dense in L^2 on any finite union of intervals $(2\pi(k-1) + \varepsilon,\ 2\pi k - \varepsilon)$, $\varepsilon > 0$ (in particular, sets with arbitrarily large measure). By reversing, in a sense, the above argument, Olevskii and Ulanovskii [OU03] have recently shown that if the a_n satisfy $|a_n| \leq Cr^{|n|}$, with $r < 1$, then a Λ-generator φ for $L^2(\mathbb{R})$ exists, and moreover it can be chosen in the Schwartz class.

In $L^1(\mathbb{R})$ the situation is simpler than in $L^2(\mathbb{R})$, because here $\hat{\varphi}$ is continuous and nonvanishing, and what is involved is therefore the density of $\mathcal{E}(\Lambda)$ with respect to weights that are bounded above and below on each interval. From here it is easy to deduce that if $T(\varphi,\Lambda)$ spans $L^1(\mathbb{R})$ then $\mathcal{E}(\Lambda)$ must be dense in $L^2(I)$ for all intervals I. Indeed, from the trivial estimate $\|\hat{f}\|_\infty \leq \|f\|_1$ it follows that an arbitrary $\hat{f}$, with $f \in L^1(\mathbb{R})$, can be approximated in the sup-norm by functions in $\hat{\varphi}\mathcal{E}(\Lambda)$. Since $\varphi \star f \in L^1(\mathbb{R})$, $\hat{\varphi}\hat{f}$ can be approximated as well. Now fix $\rho > 0$; every test function ψ supported in $(-\rho,\rho)$ serves as $\hat{f}$, and therefore $\hat{\varphi}\psi$ is well approximated by $\hat{\varphi}\mathcal{E}(\Lambda)$ in the sup-norm. Since $\hat{\varphi}$ is continuous and non-vanishing, it is bounded below on $(-\rho,\rho)$ and hence every ψ is approximated in the sup-norm by $\mathcal{E}(\Lambda)$. The density of such ψ in $L^2(I)$ shows that $\mathcal{E}(\Lambda)$ is dense in $L^2(I)$, and ρ being arbitrary, one has $R(\Lambda) = \infty$. This is the easy part of the characterization obtained in [BOU03]: a discrete Λ admits a generator $\psi \in L^1(\mathbb{R})$ if and only if $D_{BM}(\Lambda) = \infty$. We explain now the ideas for the proof of the converse direction, as it again shows connections with uniqueness sets.

By duality, φ will be a Λ-generator if and only if $h \in L^\infty(\mathbb{R})$ and $(h \star \check{\varphi})(\lambda) = \int_{-\infty}^{\infty} h(t)\varphi(t-\lambda)dt = 0$, $\lambda \in \Lambda$ implies $h \equiv 0$ (here $\check{\varphi}(t) = \varphi(-t)$. Since $h \star \check{\varphi}$ has Fourier transform $\hat{h}(\zeta)\hat{\varphi}(-\zeta)$ and $\hat{\varphi} \neq 0$ everywhere, this is restated by saying that Λ is a uniqueness set for the class $Y = L^\infty(\mathbb{R}) \star \hat{\varphi}$. Now, typically, the classes admitting discrete uniqueness sets are the *quasianalytic* ones, including of course the analytic classes. In this way, when finding conditions ensuring that a given translation set Λ admits a generator in $L^1(\mathbb{R})$, we are led to looking at uniqueness sets for quasianalytic classes. The most typical ones are the *Denjoy-Carleman classes* $C\{M_n\}$ associated to a sequence of positive numbers M_n, $M_0 = 1$. It consists of all $f \in C^\infty(\mathbb{R})$ such that

$$|f^{(n)}(x)| \leq C_f M_n,\ n = 0, 1, 2, \ldots \quad x \in \mathbb{R}.$$

Without loss of generality (M_n) can be assumed to be log-convex, that is, $M_n^2 \leq M_{n-1}M_{n+1}$, $M_0 = 1$ (see [K92, vol. 1]); this implies that $M_n^{1/n}$ increases. In case $\lim_n M_n^{1/n} < \infty$, it is immediate to check that $C\{M_n\}$ is contained in some class PW_τ, and conversely. Hence the classes $C\{M_n\}$ with $M_n^{1/n}$ bounded fill exactly the union of all Paley-Wiener classes PW_τ.

The class $C\{M_n\}$ is quasi-analytic (meaning that $f \in C\{M_n\}$ and $f^{(n)}(0) = 0\ \forall n$ imply $f \equiv 0$) if and only if

$$\sum_{n=1}^{\infty} \frac{M_{n-1}}{M_n} = \infty\,.$$

This is the *Denjoy-Carleman theorem* (see [K96, vol 1]). In case $(M_n/n!)^{1/n} \to 0$, $C\{M_n\}$ consists of entire functions, and we call it of *analytic type*. The idea for the converse implication is first to show that if $D_{BM}(\Lambda) = \infty$ then Λ is a uniqueness set for some analytic class $C\{M_n\}$ with $M_n^{1/n} \to \infty$, and that this in turn implies that a generator φ can be found in $C\{M_n\}$. In doing that, one can restrict attention to analytic classes of entire functions with infinite type. The proof also shows that the sequences with infinite Beurling-Malliavin density are exactly those which are uniqueness sequences for some quasianalytic class $C\{M_n\}$ with $M_n^{1/n} \to \infty$.

In $L^2(\mathbb{R})$, $\hat{\varphi}$ may vanish and have "holes", so what is involved is the density of $\mathcal{E}(\Lambda)$ in more complicated sets. Both in $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$, the description of generators admitting a discrete Λ seems at present out of reach.

4. Sampling, frames and interpolation for bandlimited functions

In this section we discuss sampling, interpolation and bases for PW_τ. Recall that $\Lambda = \{x_i\}_{i\in I}$ is sampling for PW_τ, if there exist two constants A, $B > 0$ such that (3) holds for $f \in PW_\tau$. It follows from the Shannon-Whittaker theorem that the only equally spaced sequences Λ which are sampling for PW_τ are those having Nyquist density at least $1/2\tau$. Here we deal with *general* sequences, possibly irregular. This is a nice hard mathematical field, and it is tied to applications too. Suppose we have a signal that has been perturbed by noise in an inhomogeneous way, so that the signal has more noise in some locations than others. According to this circumstance it seems reasonable to sample irregularly.

On the other hand, in some instances we are interested in the converse situation. We start with a sequence of discrete values $(a_i)_{i\in I}$ and want to build a band-limited function carrying the information (the sequence of values) with stability. This situation arises for example in modems transmitting digital data through an analog telephonic signal. Recall that $\Lambda = \{x_i\}_{i\in I}$ is an interpolating sequence for PW_τ whenever for any (a_i) with $\sum |a_i|^2 < \infty$, there exists $f \in PW_\tau$ with $f(x_i) = a_i\ \forall_i$. One may think of Λ as an appropriate collection of time moments where we can place the desired information. It would be very convenient if we could take many such points very close together, because this would allow us to transmit a lot of information in a very short time. Of course there is a limitation given by the band width τ. The situation was very early understood for regularly spaced sequences Λ, but, as pointed out before, we might be interested in irregular sequences.

The pioneering work was done by Beurling [B89], and Landau [L67a]. The central notion is the *Nyquist density*, whose definition for general sequences is as follows. A real sequence $\Lambda = \{x_n\}_{n\in\mathbb{Z}}$ is called *separated* if there exists $\varepsilon > 0$ such that $|x_n - x_m| \geq \varepsilon > 0$ for $x_n, x_m \in X$, $n \neq m$. Interpolating sequences must be separated; this is intuitively clear: we cannot prescribe very different values at very close points because this would imply high oscillations which are not possible in bandlimited functions. For a separated sequence Λ, the *upper density* is defined as

$$D^+(\Lambda) = \lim_{r\to\infty} \max_{x\in\mathbb{R}} \frac{\#\Lambda \cap [x-r, x+r)}{2r}$$

while the *lower density* is

$$D^-(\Lambda) = \lim_{r\to\infty} \min_{x\in\mathbb{R}} \frac{\#\Lambda \cap [x-r, x+r)}{2r}.$$

Here # denotes cardinality. Notice that these are *uniform* densities.

A sampling sequence always contains a separated sampling sequence. For separated sequences, Beurling's results imply

$$\begin{array}{lllll} D^-(\Lambda) > 2\tau & \Rightarrow & \Lambda \text{ sampling} & \Rightarrow & D^-(\Lambda) \geq 2\tau \\ D^+(\Lambda) < 2\tau & \Rightarrow & \Lambda \text{ interpolating} & \Rightarrow & D^+(\Lambda) \leq 2\tau \end{array}$$

The precise description of sampling sequences for PW_τ was, however, an open problem until J. Ortega-Cerdà and K. Seip closed it in [OS02]. Their description involves the De Branges spaces of entire functions and will not be reproduced here. Their results have interesting consequences such as the following: if $\Lambda = \{x_k\}_{k\in\mathbb{Z}}$ is sampling, there is a complete interpolating sequence $\Lambda' = \{x'_k\}$ (see next section for a precise definition) such that (x'_k, x'_{k+1}) contains at least one point of Λ. Roughly speaking, this means in particular that a sampling sequence is always denser than some interpolating sequence.

5. Optimal sampling. The multiband problem. Higher dimensions and the Fuglede conjecture

For obvious reasons it is extremely important to have sampling sequences Λ with no redundancy. We mentioned before that this is so if Λ is both sampling and interpolating. There is an intuitive way to look at nonredundant sampling sequences: they are exactly the *minimal* sampling sequences, meaning that if we remove just one point of Λ we are left with a sequence that is no longer sampling. Then one encodes the analog signal with a minimal set of points. In the other direction, we may be interested in an interpolating sequence as packed as possible, so that whenever we add just one point we get a noninterpolating sequence. These are the nonredundant sampling sequences. For this reason, they are also known as *complete interpolating sequences*.

Recall that $\Lambda = \{x_n\}_{n\in\mathbb{Z}}$ is a nonredundant sampling sequence for PW_τ if and only if $\{\text{sinc}\,(2\tau t - x_n)\}_n$ is a Riesz basis for PW_τ, or equivalently if $\mathcal{E}(\Lambda) = \{e^{i\omega x_n}\}_{x_n\in\Lambda}$ is a Riesz basis for $L^2(-\tau,\tau)$. Among these we have the *orthonormal bases*, the nicer ones, of which the regular sequence $\Lambda = \frac{1}{2\tau}\mathbb{Z}$ found in the Shannon-Whittaker theorem is a canonical example.

Indeed, this is essentially the only example of orthonormal basis: it is not hard to prove that if $\mathcal{E}(\Lambda)$ is an orthonormal basis for $L^2(-\tau,\tau)$ then Λ is a translation of $\frac{1}{2\tau}\mathbb{Z}$. There are also many "perturbative"' results, that go as far back as Wiener. The best known among them is *Kadec's theorem*: if $\sup_{n\in\mathbb{Z}} |x_n - n| < 1/4$, then $\mathcal{E}(\Lambda)$ is a Riesz basis for $L^2(0, 1)$.

The full description of the non-redundant sampling sequences for PW_τ, was achieved by Pavlov [Pav84]. It consists of three conditions: *a)* Λ is separated, *b)* the infinite product $F(z) = \prod_n (1-\frac{z}{x_n})e^{\frac{z}{x_n}}$ converges and defines an entire function of exponential type $\pi\tau$, *c)* the positive function $w(x) = |F(x+1)|^2$ satisfies

$$\left(\int_I w\right)\left(\int_I w^{-1}\right) \leq \text{const}\,|I|^2$$

for all intervals $I \subset \mathbb{R}$.

The second condition is basically the density condition, since the density of the zeros is tied to the type. The third one is an extremely delicate *balance* condition. In particular, it is destroyed if we add or remove a point, as it should be.

Let us now briefly consider *multiband signals*, that is, functions in $L^2(\mathbb{R})$ whose spectrum is contained in a set S consisting of a finite number of disjoint intervals, $I_1,\ldots, I_m$, We denote this space by PW_S^2. No easy description of PW_S^2 as a space of entire functions is available, and none of the usual complex analysis methods work. Using operator-theory Landau [L67a] proved that whenever Λ is sampling for PW_S^2, one has $D^-(\Lambda) \geq |S|$ and $D^+(\Lambda) \leq |S|$, where $|S|$ denotes the total length of $S = I_1\cup\cdots\cup I_m$. It must be pointed out that in the multiband case the Beurling-Landau conditions cannot provide a complete solution to these problems; arithmetic relations among the points of Λ play an important role and no density condition seems appropriate. That no density plays a role here is already seen when dealing with uniqueness sets. We mentioned before that Landau [L67a] constructed a symmetric sequence Λ arbitrarily close to the integers for which $\mathcal{E}(\Lambda)$ is complete in $L^2(S)$, where S is any finite union of the intervals $|x - 2\pi n| < \pi - \delta$, with arbitrarily large measure. A recent improvement of Landau's results is due to Ulanovskii [U]; however the following basic question remains unanswered.

Question: Does there exist, for every finite union $\mathrm{S} = \mathrm{I}_1\cup\cdots\cup\mathrm{I}_m$ of finite intervals, a real sequence Λ such that $\mathcal{E}(\Lambda)$ is a Riesz basis in $\mathrm{L}^2(\mathrm{S})$?

It is known that there exist complex sequences Λ lying in horizontal strips such that $\mathcal{E}(\Lambda)$ is a Riesz basis. The answer to the questions is yes if the lengths of the intervals I_i are commensurable, and also for two intervals [LS97]. The question is essentially trivial if S is a convenient explosion of an interval of the same length $|S|$.

All the problems we have discussed can be stated *mutatis mutandis* in the multidimensional situation. We consider signals in $L^2(\mathbb{R}^n)$ such that the support of its spectrum content lies in a fixed set E. This space of signals will be denoted by PW_E. When E is a bounded set PW_E is a Hilbert space of entire functions, which however is nicely described in terms of size only when the set E is convex. This is the (technical) reason that explains why there are many more results when the spectrum is assumed to be convex. In such a general situation there is a result by Gröchenig and Razafinjatovo [GR96], which is an adaptation of some ideas of Ramanathan and Steeger [RS95], that roughly states that in PW_E a sampling sequence is always denser than an interpolating sequence. From this comparison theorem it is possible to reobtain the results of Landau [L67a] providing necessary density conditions for sampling and interpolation. In several dimensions the density conditions are far from being sufficient. The difficulties are greater and many basic problems remain open. Among the most striking ones is the following.

Conjecture. There are no Riesz bases of exponentials in $L^2(B)$, where B is a ball in $\mathbb{R}^n$, $n > 1$.

It is rather surprising that this is unknown, because it is very close in spirit to the original work of Fourier. In dimension 2 there are Riesz bases of exponentials in $L^2(K)$ when K is a convex polygon symmetric with respect to the origin [LR00]. So in this context the basic problem is not the description of Riesz bases, but more basically, whether they exist at all.

One could be more strict and ask when the sequence $\{e^{ix_nx}\}_{x_n\in\Lambda}$ is an orthogonal basis for $L^2(E)$. When E is an interval we mentioned before that all orthogonal basis of exponentials have frequencies that are just a translate of the integers. For more complex sets E this apparently trivial problem becomes hard and interesting. Take a set $E \subset \mathbb{R}^n$ of measure 1. We may ask ourselves whether there is a sequence $\Lambda \subset \mathbb{R}^n$ such that the exponentials $\{e^{i\langle x,x_n\rangle}\}_{x_n\in\Lambda}$ are an orthonormal basis in $L^2(E)$. This is an open problem even in dimension 1 for a general set E. In this context Fuglede's conjecture states:

Conjecture (Fuglede). There is an orthonormal basis of exponentials in $L^2(E)$ if and only if E tiles $\mathbb{R}^n$.

We say that a set E tiles $\mathbb{R}^n$ whenever there is a sequence $\Sigma \subset \mathbb{R}^n$ (with no relationship *a priori* with the sequence of frequencies) such that the translates $E + y$, $y \in \Sigma$ constitute, up to sets of zero measure, a partition of $\mathbb{R}^n$.

There has been considerable interest in this problem in recent years. This conjecture has been recently proved [IKT01] in the case that E is a two dimensional convex set and disproved [T03] in dimension 5 and higher. On the other hand, it is shown that no convex body with a point of positive definite curvature, e.g. the ball, admits an orthonormal basis of exponentials.

6. Localization. Windowed Fourier Transform

In many situations it is important to know the frequency content of a given signal $f(t)$ *locally in time*. The Fourier transform $\hat{f}$ gives information on the signal's frequency content, but information concerning time-localization (for example, short-lived high frequency features) is not always easy to read off from $\hat{f}$.

A standard way to achieve time-localization consists of first looking at the signal f through a "window" g, so that only a well localized part of f is looked at, and then taking Fourier transform. For a window function g, i.e. a positive function localized around 0 both in time and frequency (and normalized with $\|g\|_2 = 1$), this gives the associated continuous *windowed Fourier transform of f:*

$$(W_g f)(s,\omega) = \int_{-\infty}^{\infty} f(t)g(t-s)e^{-it\omega}dt.$$

Denote $g^{s,\omega}(t) = g(t-s)e^{it\omega}$. The analogue of Fourier's formula holds for every $f \in L^2(\mathbb{R})$,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(W_g f)(s,\omega)g^{s,\omega}(t)d\omega ds \quad \text{and}$$

$$\int_{-\infty}^{\infty}|f(t)|^2dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|(W_g f)(s,\omega)|^2 d\omega ds.$$

Thus, $f \mapsto W_g f$ is an isometry (up to a constant) between $L^2(\mathbb{R})$ and a closed subspace H_g of $L^2(\mathbb{R}^2)$. In time, $g^{s,\omega}$ is "centered" at s and it has most of its energy in the interval $[s - \sigma(g),\ s + \sigma(g)]$, where $\sigma(g) = \int_{-\infty}^{\infty} t^2|g(t)|^2dt$. Similarly, if $\hat{g}$ is concentrated around 0 as well, $\hat{g}^{s,\omega}$ is "centered" at ω and it has most of its energy in $[\omega - \sigma(\hat{g}),\ \omega + \sigma(\hat{g})]$. This is so because $\hat{g}^{s,\omega}(\xi) = e^{is\omega - \xi}\hat{g}(\xi - \omega)$. We can (loosely) assume that $g^{s,\omega}$ occupies in the time-frequency plane a box centered at (s,ω), with sides $\sigma(g)$ and $\sigma(\hat{g})$ respectively. The *uncertainty principle* states that $\sigma(g)\sigma(\hat{g})$ is bounded below by a positive constant, and therefore it is not possible to make the localization box arbitrarily small. The product $\sigma(g)\sigma(\hat{g})$ attains its minimum value for the Gaussian window $g(t) = \pi^{-1/4}e^{-t^2/2}$.

For some windows g the space H_g is isometric to a Hilbert space of entire functions. This is the case of the Gaussian window: letting $z = s - i\omega$ one sees that

$$W_g(s,\omega) = e^{-\frac{i}{2}s\omega - \frac{1}{4}|z|^2}Bf(z),$$

where Bf denotes the *Bargmann transform*

$$Bf(z) = \pi^{-1/4}e^{-\frac{|z|^2}{4}}\int_{-\infty}^{\infty} f(t)e^{-\frac{t^2}{2}}e^{tz}dt.$$

The functions $F = Bf$ arising in this way thus satisfy

$$\int_{\mathbb{C}}|F(z)|^2e^{-\frac{|z|^2}{2}}dA(z) < \infty,$$

and constitute the *Bargmann-Fock space* $\mathcal{F}$ (see [F89, Chap.I, & 6]).

More than the continuous windowed transform described above, what is useful in many applications is a discrete version $\{(W_g f)(s_k,\omega_k)\}_{k\in\mathbb{Z}}$, where $\Lambda = \{(s_k,\omega_k)\}_{k\in\mathbb{Z}}$ is an appropriately chosen sequence in the time-frequency plane. The most common case in applications corresponds to regularly spaced values in time and frequency, i.e. $\Lambda_{s_0,\omega_0} = \{(ns_0, m\omega_0)\}_{n,m\in\mathbb{Z}}$, where $s_0, \omega_0 > 0$. If g is compactly supported it is clear that, with ω_0 suitably chosen, the *Fourier* coefficients are enough to reconstruct $f(\cdot)g(\cdot - ns_0)$. As n changes one recovers the portion of $f(t)$ localized around the time ns_0, in such a way that putting all the pieces together $f(t)$ can be regained.

For the Gaussian window, the discrete families of windowed Fourier functions have been studied extensively. This is mainly due to their relevance in communication theory [G46] (where are called Gabor *wavelets*), and in quantum mechanics (where they are known as canonical coherent states for the Weyl-Heisenberg group; see [P85] and references therein). As we have already mentioned, the link between Gabor and Bargmann-Fock functions makes it possible to rephrase the results concerning families $g(\Lambda) := \{g^{s_k,\omega_k}\}_k$ (being a frame, a free system or spanning $L^2(\mathbb{R})$) in terms of properties of the sequence $\Lambda = \{(s_k,\omega_k)\}_{k\in\mathbb{Z}}$ for $\mathcal{F}$ (being sampling, interpolation or uniqueness). Exploiting this link, Bargmann et al. [BBGK71] and, independently, Perelomov [P71] proved that a regular Gabor family $g(\Lambda_{s_0,\omega_0})$ spans all of $L^2(\mathbb{R})$ if and only if $s_0\omega_0 \le 2\pi$.

If one wants good localization in both time and frequency, one needs to take the strict inequality $s_0\omega_0 < 2\pi$, that is, frames $g(\Lambda_{s_0,\omega_0})$ with $s_0\omega_0 = 2\pi$ have necessarily bad localization either in time or in frequency. Indeed, the *Balian-Low theorem* states that if $g(\Lambda_{s_0,\omega_0}) = \{g^{ns_0,m\omega_0}\}_{n,m\in\mathbb{Z}}$ is a frame in $L^2(\mathbb{R})$, either $\sigma(g) = \infty$ or $\sigma(\hat{g}) = \infty$. Therefore, in strong contrast with the Paley-Wiener case, no Riesz basis of Gabor wavelets can exist in $L^2(\mathbb{R})$, or no complete interpolating sequence can exist for the Bargmann-Fock space.

More generally, one can ask: for what discrete sequences $\Lambda = \{(s_k,\omega_k)\}_{k\in\mathbb{Z}}$ in $\mathbb{R}^2$ the family $g(\Lambda)$ is complete, a frame, or a Riesz basis for $L^2(\mathbb{R})$? Of course this is equivalent to asking when Λ is uniqueness, sampling or complete interpolating for the space H_g, respectively. In order to handle these questions for separated sequences $\Lambda = \{(s_k,\omega_k)\}_{k\in\mathbb{Z}}$ one needs to consider the Beurling-Landau *upper and lower densities*, defined respectively as

(5) $$D^{-}(\Lambda) = \limsup_{r\to\infty}\sup_{z\in\mathbb{C}}\frac{\#\Lambda\cap D(z,r)}{\pi r^2} \quad \text{and}$$

$$D^{-}(\Lambda) = \liminf_{r\to\infty}\inf_{z\in\mathbb{C}}\frac{\#\Lambda\cap D(z,r)}{\pi r^2}.$$

Here $D(z,r)$ denotes the disk centered at z with radius r. A computation shows that for regular sequences $\Lambda_{s_0,\omega_0} = \{(ns_0,m\omega_0)\}_{m,n\in\mathbb{Z}}$, the equality $D^-(\Lambda_{s_0,\omega_0}) = D^+(\Lambda_{s_0,\omega_0}) = (\omega_0 s_0)^{-1}$ holds.

Ramanathan and Steger proved a general comparison theorem between the density of sampling sequences (or frames) and interpolating sequences (or Riesz families) that avoids Landau's subtle eigenvalue estimates [RS95]: if I is an interpolating sequence for $\mathcal{F}$ and S is a separated sampling sequence for $\mathcal{F}$, then $D^+(I) \le D^+(S)$ and $D^-(I) \le D^-(S)$. Comparing with regular nets Λ_{s_0,ω_0} one then deduces that a frame $g(\Lambda)$ must satisfy $D^-(\Lambda) \ge 1/2\pi$, whereas a free system $g(\Lambda)$ must have $D^+(\Lambda) \le 1/2\pi$. In particular, Riesz bases can occur only when $D^-(\Lambda) = D^+(\Lambda) = 1/2\pi$. In the next section we will explain how complex analysis methods provide a precise description of sampling and interpolation sequences in terms of these densities.

7. Complex analysis developments

In the previous sections we have indicated how several problems arising in signal analysis are brought in a natural way into the common domain of complex analysis. This provides a new insight and, as a consequence, new possibilities of further development.

In all situations we have the following general setting. Assume that H is a reproducing kernel Hilbert space of holomorphic functions, i.e H is such that there exists a reproducing kernel $K(z,\zeta)$ holomorphic in z:

$$f(z) = \langle f, K(z,\cdot)\rangle \qquad f \in H.$$

In this situation the point evaluation functional $z \mapsto f(z)$ is bounded for each z, and the norm of this functional coincides with $\sqrt{K(z,z)}$.

The standard problems of interpolation and sampling in Hilbert spaces of functions with reproducing kernels can be rephrased as follows.

A sequence $\Lambda = \{z_n\}$ is *interpolating* for H (denoted $\Lambda \in$ Int H) if the interpolation problem

$$f(z_n) = a_n \qquad n \in \mathbb{N}$$

has a solution $f \in H$ for all sequences of complex values $a = \{a_n\}_n$ such that the weighted ℓ^2-norm

$$\|a\|_{\ell(H)} = \left(\sum_{n\in\mathbb{N}} \frac{|a_n|^2}{K(z_n,z_n)}\right)^{1/2}$$

is finite. Equivalently, Λ is interpolating when the normalized kernels $\widetilde{K}(z_n,\cdot) = (K(z_n,z_n))^{-1/2}K(z_n,\cdot)$, $n \in \mathbb{N}$, form a Riesz basis in its closed linear span in H.

Given a function f, denote $f_{|\Lambda} = \{f(z_n)\}_{n\in\mathbb{N}}$. The sequence Λ is *sampling* for H (denoted $\Lambda \in$ Samp H) if there exists $C > 0$ such that

$$C^{-1}\|f_{|\Lambda}\|_{\ell(H)} \le \|f\|_H \le C\|f_{|\Lambda}\|_{\ell(H)}.$$

Equivalently, $\Lambda \in$ Samp H if and only if $\{\widetilde{K}(z_n,\cdot)\}_{n\in\mathbb{N}}$ is a frame in H. Finally, Λ is a *complete interpolating sequence* when it is is both sampling and interpolating, i.e, when $\{\widetilde{K}(z_n,\cdot)\}_{n\in\mathbb{N}}$ is a Riesz basis in H.

In this section we discuss some particular spaces of holomorphic functions that play an important role in signal analysis together with the Paley-Wiener spaces: the *Bargmann-Fock*, and *Bergman spaces*.

The classical Bergman space B^2 consists of the holomorphic functions f in the unit disk such that $\int_{\mathbb{D}}|f|^2 dA < \infty$. It appears as a model for quantum mechanics on a half-line, where the disk with its hyperbolic metric represents physical phase space [P84]. More generally, one considers the weighted versions

$$B^2_\alpha = \{f \in H(\mathbb{D}): \|f\|^2_{B^2_\alpha} = \int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^\alpha dA(z) < \infty\} \ \alpha > -1$$

These spaces appear naturally in signal processing as transforms of the socalled analytic wavelets, see [D92].

It is only for Paley-Wiener spaces that there exist complete interpolating sequences, and these are fully understood (as we have seen in Section 4). However, the work of Ortega-Cerdà and Seip also shows that there cannot exist a complete geometric description of sampling or interpolating sequences only in terms of densities. For Bargmann-Fock and Bergman spaces, which in many ways behave similarly, the situation is quite different: there is a complete geometric description (in terms of densities) which excludes the possibility of finding the two properties simultaneously in the same Λ. Thus, for the spaces we are dealing with, geometric density conditions characterize sampling and interpolation sequences if and only if there are no complete interpolating sequences. This raises the following question: what properties of a Hilbert space of entire functions determine the existence of complete interpolating sequences?

A common feature of the spaces we are discussing is that the pointwise growth of their functions is controlled by a function e^ϕ, where ϕ is a subharmonic weight ($\phi(z) = |\mathrm{Im}\,(z)|$ for PW, $\phi(z) = |z|^2$ for $\mathcal{F}$ and $\phi(z) = \log 1/(1-|z|^2)$ for B). The densities describing sampling and interpolation are expressed in each case in terms of the measure $\Delta\phi$.

7.1. Bargmann-Fock spaces

Given a subharmonic weight ϕ in $\mathbb{C}$, consider the space $\mathcal{F}^2_\phi$ of entire functions f such that

$$\|f\|_{\mathcal{F}^2_\phi} = \left(\int_{\mathbb{C}} |f|^2 e^{-2\phi}\right)^{1/2} < \infty.$$

The classical Bargmann-Fock space described in the previous section corresponds to $\phi(z) = |z|^2/4$. K. Seip and R. Wallstén ([S92] and [SW92]) gave a complete characterization of sampling and interpolating sequences in the classical case by means of the Beurling-type densities defined in (5): Λ is interpolating if and only if it is separated and $D^+(\Lambda) < \frac{1}{2\pi}$, whereas Λ is sampling if and only if it is a finite union of separated sequences containing a separated subsequence Λ' such that $D^-(\Lambda') > \frac{1}{2\pi}$. It is interesting to point out that interpolating sequences for $\mathcal{F}_{|z|^2}$ can also be characterized by an analytic condition similar to the classical Carleson condition for Hardy spaces. However, in contrast to the Hardy situation, the characterization does not carry over $\mathcal{F}^p_{|z|^2}$ for $p > 2$, and, in particular, it does not work for $p = \infty$ [S00].

The geometric description of sampling and interpolation was extended by Berndtsson and Ortega Cerdà [BOC95), and Ortega-Cerdà and Seip [OS98] to the case of weights ϕ such that the metric defined by $\Delta\phi$ is equivalent to the Euclidian metric, that is, such that $\Delta\phi$ is bounded above and below by positive constants (notice that in the classical case $\Delta\phi = 1$). In this situation the norm given in (6) is

$$\|v\|_{\ell(\mathcal{F}^2_\phi)} = \Big(\sum_{n=1}^{\infty} |a_n|^2 e^{-2\phi(z_n)}\Big)^{1/2}.$$

Similarly to the classical case, the *upper* and *lower density* of a sequence Δ are defined respectively as

$$D^+_\phi(\Lambda) = \limsup_{r\to\infty} \sup_{z\in\mathbb{C}} \frac{\#\Lambda \cap D(z,r)}{\int_{D(z,r)} \Delta\phi} \quad \text{and}$$

$$D^-_\phi(\Lambda) = \liminf_{r\to\infty} \inf_{z\in\mathbb{C}} \frac{\#\Lambda \cap D(z,r)}{\int_{D(z,r)} \Delta\phi}.$$

Notice that in the classical case $\phi(z) = |z|^2/4$ one has $\int_{D(z,r)} \Delta\phi = \pi r^2$, thus the definition above coincides with (5), where the densities just count (asymptotically) the proportion between the number of points of Λ in a given disk and its area. In the general case the "area" is computed with respect to the metric $\Delta\phi$. The characterization in this more general case is as before, with the adapted densities given in (7) taking the place of (5). The result implies, in particular, that there are no sequences which are simultaneously sampling and interpolating (it should be mentioned that this is not obtained as a corollary of the theorems; it is actually an important ingredient of the proofs that has to be proved beforehand).

The results above have recently been extended to a more general setting [MMO02]. If ϕ is a weight such that the measure $\mu = \Delta\phi$ is positive and doubling (i.e. there exists $C > 0$ such that $\mu(D(z,2r)) \leq C\mu(D(z,r))$ for all disks $D(z,r)$ then the characterization above still holds, provided that all the terms are appropriately interpreted according to the distance induced by $\Delta\phi$, which is

$$d_\phi(z,\zeta) = \inf \int |\gamma'(t)| \sqrt{\Delta\phi(\gamma(t))}\, dt\,.$$

where the infimum is taken over all smooth curves γ joining z and ζ. This means, in particular, that the disks $D(z,r)$ have to be replaced by $D_\phi(z,r) = \{\zeta \in \mathbb{C} : d_\phi(z,\zeta) < r\}$, and the norms defining the spaces $\mathcal{F}^2_\phi$, $\ell(\mathcal{F}^2_\phi)$ need to be adapted to d_ϕ. Canonical examples of weights with doubling Laplacian are $\phi(z) = |z|^\alpha$, with $\alpha > 0$. For such weights,

$$d_\phi(z,\zeta) \simeq |z-\zeta| \,/\, |z|^{\alpha/2-1} \quad \text{if } |z-\zeta| \lesssim 1$$

Notice that in the previous case, where $\Delta\phi$ is assumed to be bounded above and below, one has $d_\phi(z,\zeta) \simeq |z-\zeta|$, hence the disks D_ϕ are equivalent to the usual ones.

The problems of sampling and interpolation for Bargmann-Fock spaces have been studied in several related settings, which we now review :

Interpolation and sampling with multiplicities. Instead of prescribing the value of f on each $z_n \in \Lambda$ one can prescribe the Taylor polynomial of a certain order q_{z_n}. This gives the notions of sampling and interpolation with multiplicities. In [BS93] it was proved that in case $\sup_n q_{z_n} < \infty$ the geometric characterization of sampling and interpolating sequences for the classical Bargmann-Fock space is still valid, provided that the densities count each point according to its multiplicity, i.e. $\#\Lambda \cap D(z,r) = \sum_{z_n \in D(z,r)} (q_{z_n} + 1)$.

Sampling measures. From the viewpoint of signal processing, the sampling problem is also natural for general measures. A measure μ in $\mathbb{C}$ is called sampling for $\mathcal{F}^2_\phi$ if there exists a constant $C > 0$ such that

$$\frac{1}{C}\left(\int_{\mathbb{C}} |f|^2 e^{-2\phi} d\mu\right)^{1/2} \leq \|f\|_{\mathcal{F}^2_\phi} \leq C\left(\int_{\mathbb{C}} |f|^2 e^{-2\phi} d\mu\right)^{1/2}$$

for all $f \in \mathcal{F}^2_\phi$.

The definition of sampling sequence Λ then corresponds to the measure $\mu = \sum_n \delta_{z_n}$, where δ_{z_n}, denotes the Dirac Mass on z_n. Using the result for sequences, Ortega-Cerdà gave a description of such measures for the spaces $\mathcal{F}^2_\phi$, $\Delta\phi$ bounded, again in terms of an appropriate density associated to the measure μ [O98] (see also [L00]). A neater description was given later by Lindholm [L01].

Sampling and interpolation in $\mathbb{C}^n$. There are some partial results concerning Bargmann-Fock spaces of functions of several complex variables. The classical Bargmann-Fock space was studied in [MT00], where a necessary and a sufficient density condition for interpolation were obtained. The sufficient condition is formally as in the plane, whereas the density appearing in the necessary condition compares the number of points in a ball $B(z, r)$ with its $2n$-dimensional volume $c_n r^{2n}$. As often happens in higher dimension, the gap between the two conditions is large, and it cannot be bridged by means of analogue density conditions. This is easily seen by considering two extreme cases: when Λ belongs to a coordinate plane, the best sufficient condition is as in dimension one; on the other hand, it is easy to see that a $2n$-dimensional net $\Lambda_{a+ib} = \{(a+ib)m\}_{m\in\mathbb{Z}^n}$ is interpolating if a, $b > 0$ are big enough, while

$$\limsup_{r\to\infty} \sup_{z\in\mathbb{C}} \frac{n(z,r)}{r^{2n}} > 0.$$

N. Lindholm obtained necessary density conditions for sampling and interpolation in Bargmann-Fock spaces of $\mathbb{C}^n$ for weights ϕ such that $i\partial\bar\partial\phi$ is equivalent to the Euclidian metric [L01]. These densities are as in (7), where $\int_{D(z,r)}\Delta\phi$ is replaced by the Monge-Ampère mass $\int_{B(z,r)}(i\partial\bar\partial\phi)^n$. The precise result is that a separated sampling sequence has $D^-_\phi(\Lambda) \geq \frac{1}{(2\pi)^n n!}$, while an interpolating sequence is separated with $D^+_\phi(\Lambda) \leq \frac{1}{(2\pi)^n n!}$. It is worthwhile explaining why $(i\partial\bar\partial\phi)^n$ takes the place of $\Delta\phi$. Let's take the simple example with $\phi(z_1, z_2) = \alpha_1|z_1|^2 + \alpha_2|z_2|^2$ in $\mathbb{C}^2$ and a lattice $\Lambda = \Lambda_1 \times \Lambda_2$, $\Lambda_i = a_i(\mathbb{Z} \times i\mathbb{Z})$. If Λ is sampling, then both Λ_1, Λ_2 are sampling in one variable, and according to the results for the classical Bargmann-Fock space, $1/a_i^2 > 2\alpha_i/\pi$. The asymptotic number of points of Λ in a big ball $B(z, r)$ is vol $(B(z, r)) \,/\, a_1^2 a_2^2$, which exceeds vol $(B(z, r))\frac{4\alpha_1\alpha_2}{\pi^2}$, an expression which involves $(i\partial\bar\partial\phi)^n$ rather than $\Delta\phi$.

Interpolation in related spaces of functions. Interpolation problems for Fréchet algebras of functions defined by subharmonic weights have been considered by Squires, and Berenstein and Li (see for instance [S83], [BL95] and the references therein). Given a weight ϕ, let A_ϕ be the algebra of entire functions f such that

$$\sup_{z\in\mathbb{C}} \frac{\log|f(z)|}{\phi(z)} < \infty.$$

Using the subharmonicity of $|f|$ it is easy to see that $A_\phi .= \cup_{\alpha>0} \mathcal{F}^2_{\alpha\phi}$ Given the characteristic growth of A_ϕ, a sequence Λ is called A_ϕ-*interpolating* if the interpolation problem $f(z_n) = a_n$ has a solution $f \in A_\phi$ for every sequence of values $\{a_n\}_\lambda$ such that

$$\sup_{n\in\mathbb{N}} \frac{\log|a_n|}{\phi(z_n)} < \infty.$$

There exist geometric descriptions of such sequences for a wide range of subharmonic weights ϕ. For the sake of clarity we restrict ourselves to the model case $\phi(z) = |z|^\alpha$, $\alpha > 0$. Given a sequence Λ let $n(z,r) = \#\Lambda\cap\overline{B(z,r)}$ and define the integral counting function

$$N(z,r) = \int_0^r \frac{n(z,t)-n(z,0)}{t}\,dt + n(z,0)\log r \qquad (8)$$
$$z \in \mathbb{C}, r > 0,$$

which takes into account not only the number of points of Λ around z but also their separation from z. In [BL95] (see also [HM00]) it was proved that Λ is $A_{|z|^\alpha}$-interpolating if and only if

$$\sup_{r>0}\frac{N(0,r)}{r^\alpha} < \infty \quad \text{and} \quad \sup_{n\in\mathbb{N}}\frac{N(z_n,|z_n|)}{|z_n|^\alpha} < \infty\,.$$

The same proof shows that these conditions are still sufficient (but not necessary) in $\mathbb{C}^n$, $n > 1$. This complements the earlier sharp necessary condition

$$n(z,r) \le C(|z|^\alpha + r^\alpha)^n \qquad z \in \mathbb{C}_n,\ r > 0$$

given by Li and Taylor [LT96]. A different sufficient condition was obtained by Ounaies [OU00, Theorem A]. Let $d_n = \min(1, \inf_{m\neq n}|z_n - z_m|)$. If $d_n^{-2}\log(1/d_n) = O(|z_n|^{a-2})$ then Λ is $A_{|z|^\alpha}$-interpolating.

7.2. Bergman spaces

Let $\mathbb{D}$ denote the unit disk in $\mathbb{C}$ and let ϕ be a weight such that $(1-|z|^2)^2\Delta\phi$ is bounded above and below by positive constants. This amounts to saying that the metric induced by $\Delta\phi$ is equivalent to the hyperbolic metric in the disk. Consider the Hilbert space B^2_ϕ of holomorphic functions f such that

$$\|f\|_{B^2_\phi} = \left(\int_{\mathbb{D}}|f|^2\frac{e^{-2\phi}}{1-|z|^2}\right)^{1/2} < \infty.$$

The classical Bergman corresponds to the choice $\phi(z) = -1/2\log(1-|z|^2)$ in our notation. The spaces $B^2_{\phi_\alpha}$, with $\phi_\alpha(z) = -\alpha\log(1-|z|^2)$ are the weighted Bergman spaces defined previously, and were first studied by Djrbashjan (see [DS88]). We denote B^2_α instead of $B^2_{\phi_\alpha}$.

In the definitions of sampling and interpolation for B^2_ϕ the sequence of values satisfying (6) are those with

$$\|v\|_{\ell(B^2_\phi)} = \left(\sum_{n=1}^\infty |a_n|^2 e^{-2\phi(z_n)}(1-|z_n|^2)\right)^{1/2}.$$

In a celebrated work, Kristian Seip gave a geometric characterization of sampling and interpolating sequences for the classical case [S93a]. This was later extended to weights with $(1-|z|^2)^2\Delta\phi$ bounded (see [BOC95] for the sufficient condition and [OS98] for the necessary one). The description is again given in terms of a density similar to (5), adapted to the hyperbolic metric in $\mathbb{D}$ and to $\Delta\phi$. Still denoting these densities by $D^+(\Delta)$ and $D^-(\Delta)$, one has: a sequence $\Lambda\subset\mathbb{D}$ is interpolating for B^2_ϕ if and only if it is separated (with respect to the hyperbolic distance) and $D^+_\phi(\Lambda) < 1$; a sequence $\Lambda\subset\mathbb{D}$ is sampling for B^2_ϕ if and only if is a finite union of separated sequences (with respect to the hyperbolic distance) and there exists $\Lambda'\subset\Lambda$ separated such that $D^-(\Delta') > 1$. The densities are usually hard to compute. However, in certain regular cases such computation can be done explicitly (see [S93b], [DSS00]). This provides canonical examples of sampling and interpolation sequences.

As in the previous subsection, the results described so far can be extended in several directions:

At each $z_n \in \Lambda$ one can prescribe the Taylor polynomial of a certain degree, or more generally, the divided differences up to a certain fixed order. For the classical Bergman space the density condition given above still characterizes such interpolation, provided that each point is counted according to its multiplicity (see [KS01] for interpolation with multiplicities and [M00] for interpolation of divided differences).

There are two natural extentions of the unit disk to higher dimension, namely the polydisk $\mathbb{D}^n = \{z = (z_1,\ldots,z_n) \in \mathbb{C}^n : \sup_{i=1,\ldots,n}|z_i| < 1\}$ and the unit ball $\mathbb{B}_n = \{z = (z_1,\ldots,z_n)\in\mathbb{C}^n : |z|^2 = \sum_{i=1}^n |z_i|^2 < 1\}$. A first qualitative study of interpolating sequences for Bergman spaces of the ball was carried in [A78] and [Ro82]. In [JMT96] it was studied the relationship between the spaces of interpolating sequences for various weights, and a geometric sufficient condition was provided. The proofs can be adapted to give similar conditions for the polydisk.

On the disk (or the ball) one can also consider the algebras of functions such that $\log|f(z)| \le C\phi(z)$, where ϕ is a given subharmonic weight. Here also $A^\phi = \cup_{\alpha>0} B^2_{\alpha\phi}$. Interpolating sequences for these spaces have been characterized for a wide range of radial subharmonic weights. For the classical case $\phi(z) = -\log(1-|z|^2)$ the space above is denoted $A^{-\infty}$. Bruna and Pascuas showed that the $A^{-\infty}$-interpolating sequences are characterized by a condition which is essentially Korenblum's conditions for non-uniqueness sequences made invariant by automorphisms of $\mathbb{D}$ (see [BP89]). Later, it was shown in [M99] that Int $A^{-\infty}$ is described by a certain density adapted to the hyperbolic metric. The same density condition is sufficient (but not necessary) for interpolation by $A^{-\infty}$ functions in the ball $\mathbb{B}_n$. On the other hand, the techniques used by Li and Taylor in $\mathbb{C}^n$, applied to the ball, yield the necessary condition

$$\#\Lambda\cap\overline{B(z,r)} \le C\frac{1}{(1-r)^{n-1}}\Big(\log\frac{1}{1-r} + \log\frac{1}{1-|z|}\Big)^n$$
$$z\in\mathbb{B}_n,\ r > 0\,.$$

Another interesting (and particularly simple) case corresponds to weights $\psi_\alpha(z) = (1-|z|^2)^{-\alpha}$, $\alpha > 0$, which define the space of holomorphic functions in $\mathbb{D}$ of order at most α. It was shown in [HM01] that $\Lambda \in$ Int A_{ψ_α} if and only if

$$\sup_{n\in\mathbb{N}}(1-|z_n|^2)^\alpha N(z_n, 1/2) < \infty.$$

Here $N(z,r)$ denotes the integral counting function defined in (8), associated to the hyperbolic disks. As in the $A^{-\infty}$ case, this is also a sufficient condition for sequences in $\mathbb{B}_n$ to be A_{ψ_α}-interpolating, whereas Li and Taylor's technique's yield here the necessary condition

$$\#\Lambda \cap \overline{B(z,r)} \leq C \frac{1}{(1-r)^{n-1}} \left(\psi_\alpha(r)\psi_\alpha(z)\right)^n \qquad z \in \mathbb{B}_n,\ r > 0..$$

It is not so clear what is the right definition of sampling for these spaces. In [HKP97] Horowitz, Korenblum and Pinchuk gave a definition of sampling set for $A^{-\infty}$ and studied its properties. Later, Khôi and Thomas proposed alternative definitions and showed the precise relationship among these notions [KT01].

8. Concluding remarks

In this survey we have described how, in the last few years, some important problems motivated by applications have influenced complex function theory and suggested new developments. The interplay also goes in the other direction, and the powerful methods of complex analysis have led to the solution of some central problems in signal analysis.

Yet, not all problems are suitable to be treated by complex analytic function methods. We have seen some of them: the multiband problem, Fuglede's conjecture, etc. A similar situation occurs in wavelet theory, the most important development in signal analysis in the last years. The usual wavelet bases $\Psi_{k,n}(t) = 2^{\frac{k}{2}}\Psi(2^k t - n)$ correspond to a couple of parameters (s,t), the dilation parameter s being 2^{-k} and a translation parameter t being $n2^{-k}$. This is a regular choice of parameters in the hyperbolic metric. Construction of *irregular* wavelet bases or irregular wavelet frames leads to hard mathematical problems that can rarely be attacked with complex function theory. This is in fact another research line of our group.

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