[論　説]

A Weak Proof of the Goldbach Conjecture

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Abstract

We attempt to prove the Goldbach conjecture by using Chebyshev's theorem, as well as the properties of inequalities and the harmonic mean. The Goldbach conjecture is revealed when the smallest sum of two primes is calculated, and the even number derived from the harmonic mean is the largest sum of two primes. We also find that the Goldbach conjecture holds when there is a large gap between the prime numbers, or when the two prime numbers are the same. Much of this content is based on Kozu [4].

1. Introduction

Most of the research on prime numbers is devoted to the Riemann and Goldbach conjectures, as these are difficult, unsolved problems. The Goldbach conjecture is that "all even numbers greater than or equal to 4 can be expressed as the sum of two prime numbers". Tao [6] proved that odd

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numbers above 3 can be represented as the sum of five prime numbers, and Helfgott [2] attempted to prove Goldbach's weak conjecture with a large gap between prime numbers. Meanwhile, the weak conjecture was verified computationally (Helfgott and Platt [3]). The gap between prime numbers was studied by Maynard [5] and Ford, Green, Konyagin, and Tao [1].

Considering these studies, we attempt to prove the Goldbach conjecture by using Chebyshev's theorem (or the Bertrand-Chebyshev theorem), as well as the properties of inequalities and the harmonic mean. The Goldbach conjecture is displayed when the smallest sum of two primes is calculated, and the even number derived from the harmonic mean is the largest sum of two primes.

2. Weak Proof of the Goldbach conjecture

If Chebyshev's theorem holds for the real number axis, the range of prime numbers is

$$(1) \qquad a < P_k < 2a \,,$$

where a ($2 \leq a$) is a natural number and P_k is a prime number. Then, another prime number on the same axis is similarly represented by

$$(2) \qquad b < P_m < 2b \,,$$

where b ($2 \leq b$) is a natural number and P_m is a second prime number. Then, adding (1) and (2), the sum of the two primes is given by

$$(3) \qquad a + b < P_k + P_m < 2\,(a + b) \,.$$

The range is reduced using the geometric and harmonic means on the third term in (3). Indicating both equality and inequality signs at the harmonic mean as the final mean, we obtain

$$(4) \qquad a+b < P_k + P_m \leq \frac{8ab}{a+b} < 4\sqrt{ab} < 2(a+b),$$

where $a \neq b$. Then, (4) simplifies to

$$(5) \qquad a+b < P_k + P_m \leq \frac{8ab}{a+b}.$$

There must be at least two prime numbers in (5), and because $3 \leq a$ and $5 \leq b$, $8 \leq a+b$. Then, (5) can be rewritten as

$$(6) \qquad 8 \leq P_k + P_m \leq \frac{8ab}{a+b}.$$

The value of the harmonic mean in (6) suggests that the sum of two prime numbers is the highest even number. We consider that this formula maximizes the sum of two prime numbers while minimizing the range. Furthermore, upon transforming the third term of (6), it can be written as

$$(7) \qquad 8 \leq P_k + P_m \leq \frac{8ab}{a+b} = \frac{8a}{\frac{a}{b}+1}.$$

Here, if b is much larger than a, then $\frac{a}{b} \approx 0$. Substituting this into the fourth term of (7), we obtain

$$(8) \qquad 8 \leq P_k + P_m \leq 8a.$$

Furthermore, (8) can be rewritten as

$$(9) \qquad 8 \leq P_k + P_m \leq 2a,$$

where $4 \leq a$. When the minimum value of (9) is 8, $a = 4$. This equation will yield an even number of 4 or more. There is no problem provided that each prime range has only one prime number. Since there is an odd number immediately preceding each of the even numbers 2a in (1) and 2b in (2), if that odd number is a prime number, or if we use this method to determine the range of prime numbers, then from (9), we obtain

(10) $P_k + P_m = 2a$.

Since each of a and b is a natural number, of which there are an infinity, there is an infinite range of prime numbers. Therefore, since P_k and P_m are any prime numbers, any sum of two primes is an even number.
Further, if $a = b$, then the minimum sum of two primes is 4. Substituting $a = b$ into (5) gives

(11) $4 \leq P_k + P_m \leq 4a$.

Using the same interpretation as above, (11) can be written as

(12) $P_k + P_m = 2a$,

where $2 \leq a$. When there is one prime number in each range, (12) does not hold, from (1) and (2). However, if $P_{k-1} \leq a$ and $P_{m-1} \leq a$, this is satisfied by

(13) $P_{k-1} + P_{m-1} = 2P_{k-1} = 2P_{m-1} = 2a$.

From (13), we have

(14) $P_{k-1} = P_{m-1} = a$.

According to Chebyshev's theorem, this indicates that the prime number immediately preceding a is P_{k-1} and P_{m-1}. By contrast, when there are two prime numbers in the same range, from (1) and (12), we must have $P_k \neq P_m$. In addition, the Goldbach conjecture holds for the sum of two prime numbers when there is a large gap between the larger and smaller prime numbers, with the range given by a large value of a. The interpretation is the same as when there is a considerable difference between a and b.

3. Conclusion

We attempted to prove the Goldbach conjecture, using the properties of inequalities under the existence of prime numbers derived from Chebyshev's theorem. We found that the properties of the harmonic mean can be applied to the properties of prime numbers, and considered cases where there is a difference in the range of each prime number, two prime numbers are within the same range, or there is an extremely large range. Finally, we clarified that the Goldbach conjecture holds when there is a large gap between the prime numbers, or when the two prime numbers are the same.

Reference

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