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TAYLOR'S THEOREM FOR FUNCTIONALS ON BMO WITH APPLICATION TO BMO LOCAL MINIMIZERS

By

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Abstract. In this note two results are established for energy functionals that are given by the integral of $W(\mathbf{x}, \nabla\mathbf{u}(\mathbf{x}))$ over $\Omega \subset \mathbb{R}^n$ with $\nabla\mathbf{u} \in \mathrm{BMO}(\Omega; \mathbb{R}^{N\times n})$, the space of functions of Bounded Mean Oscillation of John and Nirenberg. A version of Taylor's theorem is first shown to be valid provided the integrand W has polynomial growth. This result is then used to demonstrate that every Lipschitz-continuous solution of the corresponding Euler-Lagrange equations at which the second variation of the energy is uniformly positive is a strict local minimizer of the energy in $W^{1,\mathrm{BMO}}(\Omega; \mathbb{R}^N)$, the subspace of the Sobolev space $W^{1,1}(\Omega; \mathbb{R}^N)$ for which the weak derivative $\nabla\mathbf{u} \in \mathrm{BMO}(\Omega; \mathbb{R}^{N\times n})$.

1. Introduction.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a Lipschitz domain. Suppose that $\mathbf{d} : \mathcal{D} \to \mathbb{R}^N$, $N \geq 1$, is a given Lipschitz-continuous function, where $\mathcal{D} \subset \partial\Omega$, the boundary of Ω. We herein consider functionals of the form

$$\mathcal{E}(\mathbf{u}) = \int_\Omega W\big(\mathbf{x}, \nabla\mathbf{u}(\mathbf{x})\big)\,\mathrm{d}\mathbf{x} \tag{1.1}$$

for W that satisfy, for some $a > 0$ and $r > 0$,

$$|\mathrm{D}^3 W(\mathbf{x}, \mathbf{K})| \leq a(1 + |\mathbf{K}|^r),$$

for all real N by n matrices $\mathbf{K}$ and almost every $\mathbf{x} \in \Omega$. We take $\mathbf{u} = \mathbf{d}$ on $\mathcal{D}$ and $\mathbf{u} \in W^{1,\mathrm{BMO}}(\Omega; \mathbb{R}^N)$, the subspace of the Sobolev space $W^{1,1}(\Omega; \mathbb{R}^N)$ for which the weak derivative $\nabla\mathbf{u}$ is of Bounded Mean Oscillation (BMO). Our main result shows that any Lipschitz-continuous weak solution $\mathbf{u}_\mathrm{e}$ of the corresponding Euler-Lagrange equations:

$$0 = \delta\mathcal{E}(\mathbf{u}_\mathrm{e})[\mathbf{w}] = \int_\Omega \mathrm{D}W\big(\mathbf{x}, \nabla\mathbf{u}_\mathrm{e}(\mathbf{x})\big)\big[\nabla\mathbf{w}(\mathbf{x})\big]\,\mathrm{d}\mathbf{x} \quad \text{for all } \mathbf{w} \in \mathrm{Var}, \tag{1.2}$$

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at which the second variation of $\mathcal{E}$ is uniformly positive: for some $b > 0$ and all $\mathbf{w} \in \mathrm{Var}$,

$$\delta^2\mathcal{E}(\mathbf{u}_{\mathrm{e}})[\mathbf{w},\mathbf{w}] = \int_\Omega \mathrm{D}^2W\big(\mathbf{x},\nabla\mathbf{u}_{\mathrm{e}}(\mathbf{x})\big)\big[\nabla\mathbf{w}(\mathbf{x}),\nabla\mathbf{w}(\mathbf{x})\big]\,\mathrm{d}\mathbf{x} \geq b\int_\Omega |\nabla\mathbf{w}(\mathbf{x})|^2\,\mathrm{d}\mathbf{x}, \quad (1.3)$$

will satisfy, for some $c > 0$,

$$\mathcal{E}(\mathbf{w}+\mathbf{u}_{\mathrm{e}}) \geq \mathcal{E}(\mathbf{u}_{\mathrm{e}}) + c\int_\Omega |\nabla\mathbf{w}(\mathbf{x})|^2\,\mathrm{d}\mathbf{x}$$

for all $\mathbf{w} \in W^{1,\mathrm{BMO}}(\Omega;\mathbb{R}^N)\cap\mathrm{Var}$ whose gradient has sufficiently small *norm* in $\mathrm{BMO}(\Omega)$. Here

$$\mathrm{D}^jW(\mathbf{x},\mathbf{K}) = \frac{\partial^j}{\partial\mathbf{K}^j}W(\mathbf{x},\mathbf{K}), \qquad \mathrm{Var} := \{\mathbf{w} \in W^{1,2}(\Omega;\mathbb{R}^N) : \mathbf{w} = \mathbf{0} \text{ on } \mathcal{D}\},$$

$$\|\nabla\mathbf{u}\|_{\mathrm{BMO}} := [\![\nabla\mathbf{u}]\!]_{\mathrm{BMO}} + \big|\langle\nabla\mathbf{u}\rangle_\Omega\big|,$$

$[\![\cdot]\!]_{\mathrm{BMO}}$ denotes the standard semi-norm on $\mathrm{BMO}(\Omega)$ (see (2.1)), and $\langle\nabla\mathbf{u}\rangle_\Omega$ denotes the average value of the components of $\nabla\mathbf{u}$ on Ω.

The above result extends prior work[1] of Kristensen and Taheri [19, §6] and Campos Cordero [4, §4] (see, also, Firoozye [8]) who showed that, for the Dirichlet problem, if $\mathbf{u}_{\mathrm{e}}$ is a Lipschitz-continuous weak solution of the Euler-Lagrange equations, (1.2), at which the second variation of $\mathcal{E}$ is uniformly positive, (1.3), then there is a neighborhood of $\nabla\mathbf{u}_{\mathrm{e}}$ in $\mathrm{BMO}(\Omega)$ in which all Lipschitz mappings have energy that is greater than the energy of $\mathbf{u}_{\mathrm{e}}$.

Our proof of the above result makes use of a version of Taylor's theorem on $\mathrm{BMO}(\Omega)$ that is established herein: Let W satisfy, for some $a > 0$, $r > 0$, and integer $k \geq 2$,

$$|\mathrm{D}^kW(\mathbf{x},\mathbf{K})| \leq a(1+|\mathbf{K}|^r),$$

for all real N by n matrices $\mathbf{K}$, and almost every $\mathbf{x} \in \Omega$. Fix $M > 0$ and $\mathbf{F} \in L^\infty(\Omega;\mathbb{R}^{N\times n})$. Then there exists a constant $c = c(M, ||\mathbf{F}||_\infty) > 0$ such that every $\mathbf{G} \in \mathrm{BMO}(\Omega;\mathbb{R}^{N\times n})$ with $||\mathbf{G}-\mathbf{F}||_{\mathrm{BMO}} < M$ satisfies

$$\int_\Omega W(\mathbf{G})\,\mathrm{d}\mathbf{x} \geq \int_\Omega W(\mathbf{F})\,\mathrm{d}\mathbf{x} + \sum_{j=1}^{k-1}\frac{1}{j!}\int_\Omega \mathrm{D}^jW(\mathbf{F})\big[\mathbf{H},\mathbf{H},\ldots,\mathbf{H}\big]\,\mathrm{d}\mathbf{x} - c\int_\Omega |\mathbf{H}|^k\,\mathrm{d}\mathbf{x}, \quad (1.4)$$

where $\mathbf{H} = \mathbf{G}-\mathbf{F}$, $\mathbf{F} = \mathbf{F}(\mathbf{x})$, $\mathbf{G} = \mathbf{G}(\mathbf{x})$, and, e.g., $W(\mathbf{F}) = W(\mathbf{x},\mathbf{F}(\mathbf{x}))$.

The key ingredient in our proof of (1.4) is the *interpolation inequality* [22, Theorem 2.5]: If $1 \leq p < q < \infty$, then there is a constant $C = C(p,q,\Omega)$ such that, for all $\psi \in \mathrm{BMO}(\Omega)$,

$$\int_\Omega |\psi(\mathbf{x})|^q\,\mathrm{d}\mathbf{x} \leq C\Big([\![\psi]\!]_{\mathrm{BMO}} + \big|\langle\psi\rangle_\Omega\big|\Big)^{q-p}\int_\Omega |\psi(\mathbf{x})|^p\,\mathrm{d}\mathbf{x}. \quad (1.5)$$

When $\Omega = \mathbb{R}^n$ and $\langle\psi\rangle_{\mathbb{R}^n} = 0$ this inequality is due to Fefferman and Stein [7, p. 156], although it is clear from [16, pp. 624–625] that Fritz John was aware of (1.5) when $[\![\psi]\!]_{\mathrm{BMO}}$ was sufficiently small and $\langle\psi\rangle_\Omega = 0$ (for domains Ω of bounded eccentricity).

We mention that our main result assumes that the solution $\mathbf{u}_{\mathrm{e}}$ of the Euler-Lagrange equations (1.2) is Lipschitz continuous and has uniformly positive second variation (1.3). It follows that $\mathbf{u}_{\mathrm{e}}$ is a *weak relative minimizer* of the energy (1.1), that is, a minimizer

[1] The result in [19, §6] has been extended to the Neumann and mixed problems in [22, §3].

with respect to perturbations that are small in $W^{1,\infty}$. Grabovsky and Mengesha [11, 12] give further conditions[2] that they prove imply that $\mathbf{u}_e$ is then a *strong relative minimizer* of $\mathcal{E}$, that is, a minimizer with respect to perturbations that are small in L^∞, whereas our result only changes $W^{1,\infty}$ to $W^{1,\mathrm{BMO}} \subset\subset L^\infty$. However, as Grabovsky and Mengesha have noted, their results require that $\mathbf{u}_e$ be C^1. Examples of Müller and Šverák [21] demonstrate that not all Lipschitz solutions of (1.2) need be C^1. Also, the Lipschitz example of Kristensen and Taheri [19, §7] satisfies both (1.2) and (1.3).

Finally, we note that although BMO has become a standard tool in analysis, it appears that only Fritz John (see, e.g., [16]) has made use of this space to investigate applied problems.[3] However, it appears to us that the interpolation inequality (1.5) should allow other researchers in Applied Mathematics to make use BMO in their analysis. In particular, (1.5) has allowed us to extend and strengthen the results presented in [19, §6] and [4, §4].

2. Preliminaries. For any *domain* (nonempty, connected, open set) $U \subset \mathbb{R}^n$, $n \geq 2$, we denote by $L^p(U;\mathbb{R}^N)$, $p \in [1,\infty)$, the space of (Lebesgue) measurable functions $\mathbf{u}$ with values in $\mathbb{R}^N$, $N \geq 1$, whose L^p-norm is finite:

$$||\mathbf{u}||_p^p = ||\mathbf{u}||_{p,U}^p := \int_U |\mathbf{u}(\mathbf{x})|^p \,\mathrm{d}\mathbf{x} < \infty.$$

$L^\infty(U;\mathbb{R}^N)$ shall denote those measurable functions whose essential supremum is finite. We write $L^1_{\mathrm{loc}}(U;\mathbb{R}^N)$ for the set of measurable functions that are integrable on every compact subset of U.

We shall write $\Omega \subset \mathbb{R}^n$, $n \geq 2$, to denote a *Lipschitz domain*, that is a bounded domain whose boundary $\partial\Omega$ is (strongly) Lipschitz. (See, e.g., [6, p. 127], [20, p. 72], or [14, Definition 2.5].) Essentially, a bounded domain is Lipschitz if, in a neighborhood of every $\mathbf{x} \in \partial\Omega$, the boundary is the graph of a Lipschitz-continuous function and the domain is on "one side" of this graph. $W^{1,p}(\Omega;\mathbb{R}^N)$ will denote the usual Sobolev space of functions $\mathbf{u} \in L^p(\Omega;\mathbb{R}^N)$, $1 \leq p \leq \infty$, whose distributional gradient $\nabla\mathbf{u}$ is also contained in L^p. Note that, since Ω is a Lipschitz domain, each $\mathbf{u} \in W^{1,\infty}(\Omega;\mathbb{R}^N)$ has a representative that is Lipschitz continuous. We shall write $\mathbb{R}^{N\times n}$ for the space of real N by n matrices with inner product $\mathbf{A} : \mathbf{B} = \mathrm{trace}(\mathbf{A}\mathbf{B}^{\mathrm{T}})$ and norm $|\mathbf{A}| = \sqrt{\mathbf{A} : \mathbf{A}}$, where $\mathbf{B}^{\mathrm{T}}$ denotes the transpose of $\mathbf{B}$.

2.1. *Bounded Mean Oscillation.* The BMO-seminorm[4] of $\mathbf{F} \in L^1_{\mathrm{loc}}(U;\mathbb{R}^{N\times n})$ is given by

$$[\![\mathbf{F}]\!]_{\mathrm{BMO}(U)} := \sup_{Q\subset\subset U} \fint_Q |\mathbf{F}(\mathbf{x}) - \langle\mathbf{F}\rangle_Q| \,\mathrm{d}\mathbf{x}, \tag{2.1}$$

[2]The most significant are quasiconvexity in both the interior and at the boundary. See Ball and Marsden [1].

[3]John showed that small nonlinear strain, $(\nabla\mathbf{u})^{\mathrm{T}}\nabla\mathbf{u} - \mathbf{I}$, in L^∞ yields a small deformation gradient, $\nabla\mathbf{u}$, in BMO. A result similar to (1.5) then yields uniqueness, for the displacement problem, in Nonlinear Elasticity for deformations with small strain. See [22, §6] for the mixed problem.

[4]See Brezis and Nirenberg [2, 3], John and Nirenberg [17], Jones [18], Stein [23, §4.1], or, e.g., [13, §3.1] for properties of BMO.

where the supremum is to be taken over all nonempty, bounded (open) n-dimensional hypercubes Q *with faces parallel to the coordinate hyperplanes*. Here

$$\langle \mathbf{F} \rangle_U := \fint_U \mathbf{F}(\mathbf{x})\,\mathrm{d}\mathbf{x} := \frac{1}{|U|}\int_U \mathbf{F}(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

denotes the average value of the components of $\mathbf{F}$, $|U|$ denotes the n-dimensional Lebesgue measure of any bounded domain $U \subset \mathbb{R}^n$, and we write $Q \subset\subset U$ provided that $Q \subset K_Q \subset U$ for some compact set K_Q.

The space $\mathrm{BMO}(U;\mathbb{R}^{N\times n})$ (Bounded Mean Oscillation) is defined by

$$\mathrm{BMO}(U;\mathbb{R}^{N\times n}) := \{\mathbf{F} \in L^1_{\mathrm{loc}}(U;\mathbb{R}^{N\times n}) : [\![\mathbf{F}]\!]_{\mathrm{BMO}(U)} < \infty\}. \tag{2.2}$$

One consequence of (2.1)–(2.2) is that $L^\infty(U;\mathbb{R}^{N\times n}) \subset \mathrm{BMO}(U;\mathbb{R}^{N\times n})$ with

$$[\![\mathbf{F}]\!]_{\mathrm{BMO}(U)} \le 2\|\mathbf{F}\|_{\infty,U} \quad \text{for all } \mathbf{F} \in L^\infty(U;\mathbb{R}^{N\times n}). \tag{2.3}$$

We note for future reference that if $U = \Omega$, a Lipschitz domain, then a result of P. W. Jones [18] implies, in particular, that

$$\mathrm{BMO}(\Omega;\mathbb{R}^{N\times n}) \subset L^1(\Omega;\mathbb{R}^{N\times n}).$$

It follows that[5]

$$\|\mathbf{F}\|_{\mathrm{BMO}} := [\![\mathbf{F}]\!]_{\mathrm{BMO}(\Omega)} + |\langle \mathbf{F} \rangle_\Omega| \tag{2.4}$$

is a *norm* on $\mathrm{BMO}(\Omega;\mathbb{R}^{N\times n})$.

Remark 2.1. The standard example of a function $\phi \in \mathrm{BMO}(\mathbb{R}^n)$ that is not bounded is $\phi(\mathbf{x}) = \ln|\mathbf{x}|$.

2.2. *Further Properties of* BMO. The main property of BMO that we shall use is contained in the following result. Although the proof can be found in [22], the significant analysis it is based upon is due to Fefferman and Stein [7], Iwaniec [15], and Diening, Růžička, and Schumacher [5].

Proposition 2.2. Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a Lipschitz[6] domain. Then, for all $q \in [1,\infty)$,

$$\mathrm{BMO}(\Omega;\mathbb{R}^{N\times n}) \subset L^q(\Omega;\mathbb{R}^{N\times n})$$

with continuous injection, i.e., there are constants $J_1 = J_1(q,\Omega) > 0$ such that, for every $\mathbf{F} \in \mathrm{BMO}(\Omega;\mathbb{R}^{N\times n})$,

$$\left(\fint_\Omega |\mathbf{F}|^q\,\mathrm{d}\mathbf{x}\right)^{1/q} \le J_1\|\mathbf{F}\|_{\mathrm{BMO}}. \tag{2.5}$$

Moreover, if $1 \le p < q < \infty$ then there exists constants $J_2 = J_2(p,q,\Omega) > 0$ such that every $\mathbf{F} \in \mathrm{BMO}(\Omega;\mathbb{R}^{N\times n})$ satisfies

$$||\mathbf{F}||_{q,\Omega} \le J_2\Big(||\mathbf{F}||_{\mathrm{BMO}}\Big)^{1-p/q}\Big(||\mathbf{F}||_{p,\Omega}\Big)^{p/q}. \tag{2.6}$$

Here $\|\cdot\|_{\mathrm{BMO}}$ is given by (2.1) and (2.4).

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[5]If $\mathbf{F} = \nabla\mathbf{w}$ with $\mathbf{w} = \mathbf{0}$ on $\partial\Omega$ then $\|\nabla\mathbf{w}\|_{\mathrm{BMO}} = [\![\nabla\mathbf{w}]\!]_{\mathrm{BMO}(\Omega)}$ since the integral of $\nabla\mathbf{w}$ over Ω is then zero.

[6]This result, as stated, is valid for a larger class of domains: Uniform domains. (Since $\mathrm{BMO} \subset L^1$ for such domains. See P. W. Jones [18], Gehring and Osgood [10], and e.g., [9].) A slightly modified version of this result is valid for John domains. See [22] and the references therein.

REMARK 2.3. Proposition 2.2 together with (2.3) shows that, for every $p \in [1, \infty)$,

$$L^\infty(\Omega) \subset \mathrm{BMO}(\Omega) \subset L^p(\Omega).$$

Thus, BMO is a space that is "between" L^∞ and all of the other L^p-spaces. However, researchers in Harmonic Analysis make use of BMO as a replacement for L^∞. See, e.g., [23, §4.5].

3. An Implication of Taylor's Theorem for a Functional on BMO.

HYPOTHESIS 3.1. Fix $k, N \in \mathbb{Z}$ with $k \geq 2$ and $N \geq 1$. We suppose that we are given an *integrand* $W : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ that satisfies:

(H1) $\mathbf{K} \mapsto W(\mathbf{x}, \mathbf{K}) \in C^k(\mathbb{R}^{N\times n})$, for *a.e.* $\mathbf{x} \in \Omega$;

(H2) $(\mathbf{x}, \mathbf{K}) \mapsto \mathrm{D}^j W(\mathbf{x}, \mathbf{K})$, $j = 0, 1, \ldots, k$, are each (Lebesgue) measurable on their common domain $\Omega \times \mathbb{R}^{N\times n}$; and

(H3) There are constants $c_k > 0$ and $r > 0$ such that, for all $\mathbf{K} \in \mathbb{R}^{N\times n}$ and *a.e.* $\mathbf{x} \in \Omega$,

$$|\mathrm{D}^k W(\mathbf{x}, \mathbf{K})| \leq c_k(1 + |\mathbf{K}|^r).$$

Here, and in the sequel,

$$\mathrm{D}^0 W(\mathbf{x}, \mathbf{K}) := W(\mathbf{x}, \mathbf{K}), \qquad \mathrm{D}^j W(\mathbf{x}, \mathbf{K}) := \frac{\partial^j}{\partial \mathbf{K}^j} W(\mathbf{x}, \mathbf{K})$$

denotes j-th derivative of $\mathbf{K} \mapsto W(\cdot, \mathbf{K})$. Note that, for every $\mathbf{K} \in \mathbb{R}^{N\times n}$, *a.e.* $\mathbf{x} \in \Omega$, and $j = 1, 2, \ldots, k$,

$$\mathrm{D}^j W(\mathbf{x}, \mathbf{K}) \in \mathrm{Lin}(\overbrace{\mathbb{R}^{N\times n} \times \mathbb{R}^{N\times n} \times \cdots \times \mathbb{R}^{N\times n}}^{j \text{ copies}}; \mathbb{R}),$$

that is, $\mathrm{D}^j W(\mathbf{x}, \mathbf{K})$ can be viewed as a multilinear map from j copies of $\mathbb{R}^{N\times n}$ to $\mathbb{R}$.

REMARK 3.2. Hypothesis (H3) implies that $\mathrm{D}^j W$, $j = 0, 1, \ldots, k-1$, each satisfy a similar growth condition, i.e., $|\mathrm{D}^j W(\mathbf{x}, \mathbf{K})| \leq c_j(1 + |\mathbf{K}|^{r+k-j})$. It follows that each of the functions $\mathrm{D}^j W$ is (essentially) bounded on $\Omega \times \mathcal{K}$ for any compact $\mathcal{K} \subset \mathbb{R}^{N\times n}$.

LEMMA 3.3. Let W satisfy Hypothesis 3.1. Fix $M > 0$ and $\mathbf{F} \in L^\infty(\Omega; \mathbb{R}^{N\times n})$. Then there exists a constant $c = c(M, ||\mathbf{F}||_\infty) > 0$ such that every $\mathbf{G} \in \mathrm{BMO}(\Omega; \mathbb{R}^{N\times n})$ with $||\mathbf{G} - \mathbf{F}||_{\mathrm{BMO}} < M$ satisfies

$$\int_\Omega W(\mathbf{G})\,\mathrm{d}\mathbf{x} \geq \int_\Omega W(\mathbf{F})\,\mathrm{d}\mathbf{x} + \sum_{j=1}^{k-1} \frac{1}{j!} \int_\Omega \mathrm{D}^j W(\mathbf{F})\big[\mathbf{H}, \mathbf{H}, \ldots, \mathbf{H}\big]\,\mathrm{d}\mathbf{x} - c\int_\Omega |\mathbf{H}|^k\,\mathrm{d}\mathbf{x}, \quad (3.1)$$

where $\mathbf{H} = \mathbf{G} - \mathbf{F}$, $\mathbf{F} = \mathbf{F}(\mathbf{x})$, $\mathbf{G} = \mathbf{G}(\mathbf{x})$, and, e.g., $W(\mathbf{F}) = W(\mathbf{x}, \mathbf{F}(\mathbf{x}))$.

Proof. Fix $M > 0$ and $\mathbf{F} \in L^\infty(\Omega; \mathbb{R}^{N\times n})$. Let $\mathbf{G} \in \mathrm{BMO}(\Omega; \mathbb{R}^{N\times n})$ satisfy $||\mathbf{G} - \mathbf{F}||_{\mathrm{BMO}} < M$. We first note that (2.5) in Proposition 2.2 yields

$$\mathbf{H} := \mathbf{G} - \mathbf{F} \in L^q(\Omega; \mathbb{R}^{N\times n}) \;\text{ for every } q \geq 1, \quad (3.2)$$

while (H3) together with the fact that $\mathbf{F}$ is in L^∞ yields (see Remark 3.2), for some $C > 0$ and *a.e.* $\mathbf{x} \in \Omega$,

$$\big|\mathrm{D}^j W\big(\mathbf{x}, \mathbf{F}(\mathbf{x})\big)\big| \leq C, \quad j = 0, 1, \ldots, k-1. \quad (3.3)$$

Consequently, (3.2) and (3.3) yield, for every $q \geq 1$,

$$\mathbf{x} \mapsto \mathrm{D}^j W\big(\mathbf{x}, \mathbf{F}(\mathbf{x})\big)\big[\mathbf{H}(\mathbf{x}), \mathbf{H}(\mathbf{x}), \dots, \mathbf{H}(\mathbf{x})\big] \in L^q(\Omega; \mathbb{R}^{N\times n}), \tag{3.4}$$

for $j = 0, 1, \dots, k-1$.

Next, by Taylor's theorem for the function $\mathbf{A} \mapsto W(\cdot, \mathbf{A})$, for almost every $\mathbf{x} \in \Omega$,

$$\begin{aligned} W(\mathbf{G}) &= W(\mathbf{F}) + \sum_{j=1}^{k-1} \frac{1}{j!} \mathrm{D}^j W(\mathbf{F})\big[\mathbf{H}, \mathbf{H}, \dots, \mathbf{H}\big] + R(\mathbf{F}; \mathbf{H}), \\ R(\mathbf{F}; \mathbf{H}) &:= \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} \mathrm{D}^k W(\mathbf{F} + t\mathbf{H})\big[\mathbf{H}, \mathbf{H}, \dots, \mathbf{H}\big]\,\mathrm{d}t. \end{aligned} \tag{3.5}$$

We note that hypothesis (H3) together with the inequality $|a+b|^r \leq c_r(|a|^r + |b|^r)$, $c_r = \max\{1, 2^{r-1}\}$, and the fact that $t \in [0,1]$ gives us

$$|\mathrm{D}^k W(\mathbf{F} + t\mathbf{H})| \leq c_k\big(1 + |\mathbf{F} + t\mathbf{H}|^r\big) \leq c_k + c_k c_r\big(|\mathbf{F}|^r + |\mathbf{H}|^r\big) \tag{3.6}$$

and hence the absolute value of the integrand in $(3.5)_2$ is bounded by $c_k/(k-1)!$ times

$$|\mathbf{H}|^k\big(1 + c_r||\mathbf{F}||_\infty^r\big) + c_r|\mathbf{H}|^{k+r}. \tag{3.7}$$

We next integrate $(3.5)_1$ and $(3.5)_2$ over Ω to get, in view of (3.4), (3.6), and (3.7),

$$\int_\Omega W(\mathbf{G})\,\mathrm{d}\mathbf{x} = \int_\Omega W(\mathbf{F})\,\mathrm{d}\mathbf{x} + \sum_{j=1}^{k-1} \frac{1}{j!} \int_\Omega \mathrm{D}^j W(\mathbf{F})[\mathbf{H}, \mathbf{H}, \dots, \mathbf{H}]\,\mathrm{d}\mathbf{x} + \int_\Omega R(\mathbf{F}; \mathbf{H})\,\mathrm{d}\mathbf{x} \tag{3.8}$$

and

$$\begin{aligned} \int_\Omega R(\mathbf{F}; \mathbf{H})\,\mathrm{d}\mathbf{x} &\leq C_1 \int_\Omega |\mathbf{H}|^k\,\mathrm{d}\mathbf{x} + C_2 \int_\Omega |\mathbf{H}|^{k+r}\,\mathrm{d}\mathbf{x} \\ &\leq \big(C_1 + C_2 J_2^{k+r}||\mathbf{H}||_{\mathrm{BMO}}^r\big) \int_\Omega |\mathbf{H}|^k\,\mathrm{d}\mathbf{x}, \end{aligned} \tag{3.9}$$

where we have made use of (2.6) of Proposition 2.2 with $p = k$ and $q = k + r$, $C_2 := c_k c_r/(k-1)!$, and $C_1 := c_k(1 + c_r||\mathbf{F}||_\infty^r)/(k-1)!$. The desired result, (3.1), now follows from (3.8) and (3.9). □

4. The Second Variation and BMO Local Minimizers.

We take

$$\partial\Omega = \overline{\mathcal{D}} \cup \overline{\mathcal{S}} \ \text{ with } \mathcal{D} \text{ and } \mathcal{S} \text{ relatively open and } \mathcal{D} \cap \mathcal{S} = \varnothing.$$

If $\mathcal{D} \neq \varnothing$ we assume that a Lipschitz-continuous function $\mathbf{d} : \mathcal{D} \to \mathbb{R}^N$ is prescribed. We define

$$W^{1,\mathrm{BMO}}(\Omega; \mathbb{R}^N) := \{\mathbf{u} \in W^{1,1}(\Omega; \mathbb{R}^N) : \nabla\mathbf{u} \in \mathrm{BMO}(\Omega; \mathbb{R}^{N\times n})\} \tag{4.1}$$

and denote the set of *Admissible Mappings* by

$$\mathrm{AM} := \{\mathbf{u} \in W^{1,\mathrm{BMO}}(\Omega; \mathbb{R}^N) : \ \mathbf{u} = \mathbf{d} \text{ on } \mathcal{D} \text{ or } \langle\mathbf{u}\rangle_\Omega = \mathbf{0} \ \text{ if } \ \mathcal{D} = \varnothing\}.$$

The *energy* of $\mathbf{u} \in \mathrm{AM}$ is defined by

$$\mathcal{E}(\mathbf{u}) := \int_\Omega W\big(\mathbf{x}, \nabla\mathbf{u}(\mathbf{x})\big)\,\mathrm{d}\mathbf{x}, \tag{4.2}$$

where W is given by Hypothesis 3.1 with $k = 3$. We shall assume that we are given a $\mathbf{u}_{\mathrm{e}} \in \mathrm{AM}$ that is a weak solution of the *Euler-Lagrange equations* corresponding to (4.2), i.e.,

$$0 = \int_\Omega \mathrm{D}W\big(\mathbf{x}, \nabla\mathbf{u}_{\mathrm{e}}(\mathbf{x})\big)[\nabla\mathbf{w}(\mathbf{x})]\,\mathrm{d}\mathbf{x}, \tag{4.3}$$

for all *variations* $\mathbf{w} \in \mathrm{Var}$, where

$$\mathrm{Var} := \{\mathbf{w} \in W^{1,2}(\Omega;\mathbb{R}^N) : \mathbf{w} = \mathbf{0} \text{ on } \mathcal{D} \text{ or } \langle\mathbf{w}\rangle_\Omega = \mathbf{0} \text{ if } \mathcal{D} = \varnothing\}.$$

THEOREM 4.1. Let W satisfy Hypothesis 3.1 with $k = 3$. Suppose that

$$\mathbf{u}_{\mathrm{e}} \in \mathrm{AM} \cap W^{1,\infty}(\Omega;\mathbb{R}^N)$$

is a weak solution of (4.3) that satisfies, for some $a > 0$,

$$\int_\Omega \mathrm{D}^2W(\nabla\mathbf{u}_{\mathrm{e}})\big[\nabla\mathbf{z}, \nabla\mathbf{z}\big]\,\mathrm{d}\mathbf{x} \geq 4a\int_\Omega |\nabla\mathbf{z}|^2\,\mathrm{d}\mathbf{x} \;\text{ for all } \mathbf{z} \in \mathrm{Var}. \tag{4.4}$$

Then there exists a $\delta > 0$ such that any $\mathbf{v} \in \mathrm{AM}$ that satisfies

$$||\nabla\mathbf{v} - \nabla\mathbf{u}_{\mathrm{e}}||_{\mathrm{BMO}} < \delta \tag{4.5}$$

will also satisfy

$$\mathcal{E}(\mathbf{v}) \geq \mathcal{E}(\mathbf{u}_{\mathrm{e}}) + a\int_\Omega |\nabla\mathbf{v} - \nabla\mathbf{u}_{\mathrm{e}}|^2\,\mathrm{d}\mathbf{x}. \tag{4.6}$$

In particular, any $\mathbf{v} \not\equiv \mathbf{u}_{\mathrm{e}}$ that satisfies (4.5) will have strictly greater energy than $\mathbf{u}_{\mathrm{e}}$.

REMARK 4.2. 1. The theorem's conclusions remain valid if one subtracts $\int_\Omega \mathbf{b}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x}$ and $\int_{\mathcal{S}} \mathbf{s}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x})\,\mathrm{d}\mathcal{S}_{\mathbf{x}}$ from $\mathcal{E}$. 2. Fix $q > 2$. Then inequality (2.6) in Proposition 2.2 together with (4.6) yields a constant $\hat{j} = \hat{j}(q)$ such that any $\mathbf{v} \in \mathrm{AM}$ that satisfies (4.5) will also satisfy

$$\mathcal{E}(\mathbf{v}) \geq \mathcal{E}(\mathbf{u}_{\mathrm{e}}) + \hat{a}\hat{j}\delta^{2-q}\int_\Omega |\nabla\mathbf{v} - \nabla\mathbf{u}_{\mathrm{e}}|^q\,\mathrm{d}\mathbf{x}.$$

REMARK 4.3. The conclusions of Theorem 4.1 remain valid if we replace the assumption that $\mathbf{u}_{\mathrm{e}}$ is a weak solution of (4.3) by the assumption that $\mathbf{u}_{\mathrm{e}}$ is a weak relative minimizer of $\mathcal{E}$, i.e., $\mathcal{E}(\mathbf{v}) \geq \mathcal{E}(\mathbf{u}_{\mathrm{e}})$ for all $\mathbf{v} \in \mathrm{AM} \cap W^{1,\infty}(\Omega;\mathbb{R}^N)$ with $\|\nabla\mathbf{v} - \nabla\mathbf{u}_{\mathrm{e}}\|_\infty$ sufficiently small.

Proof of Theorem 4.1. Let $\mathbf{u}_{\mathrm{e}} \in \mathrm{AM}$ be a weak solution of the Euler-Lagrange equations, (4.3), that satisfies (4.4). Suppose that $\mathbf{v} \in \mathrm{AM}$ satisfies (4.5) for some $\delta > 0$ to be determined later and define $\mathbf{w} := \mathbf{v} - \mathbf{u}_{\mathrm{e}} \in \mathrm{Var} \cap W^{1,\mathrm{BMO}}$. Then Lemma 3.3 yields a constant $c > 0$, such that

$$\mathcal{E}(\mathbf{v}) \geq \mathcal{E}(\mathbf{u}_{\mathrm{e}}) + 2\hat{k}\int_\Omega |\nabla\mathbf{w}|^2\,\mathrm{d}\mathbf{x} - c\int_\Omega |\nabla\mathbf{w}|^3\,\mathrm{d}\mathbf{x}, \tag{4.7}$$

where we have made use of (4.2)–(4.4).

We next note that inequality (2.6) in Proposition 2.2 (with $q = 3$ and $p = 2$) gives us

$$J^3||\nabla\mathbf{w}||_{\mathrm{BMO}}\int_\Omega |\nabla\mathbf{w}|^2\,\mathrm{d}\mathbf{x} \geq \int_\Omega |\nabla\mathbf{w}|^3\,\mathrm{d}\mathbf{x}. \tag{4.8}$$

The desired inequality, (4.6), now follows from (4.5), (4.7), and (4.8) when δ is sufficiently small. Finally, $\mathcal{E}(\mathbf{v}) > \mathcal{E}(\mathbf{u}_{\mathrm{e}})$ is clear from (4.6) since Ω is a connected open region and either $\langle \mathbf{w} \rangle_\Omega = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$ on $\mathcal{D} \subset \partial\Omega$. □

5. Comparison with Prior Results. Given a Lipschitz-continuous (equivalently, $W^{1,\infty}$), weak solution of the Euler-Lagrange equations, $\mathbf{u}_{\mathrm{e}}$, Theorem 4.1, as well as the comparable results in [4, 19, 22], yields a neighborhood of $\mathbf{u}_{\mathrm{e}}$ in the space $W^{1,\mathrm{BMO}}(\Omega;\mathbb{R}^n)$ (see (4.1)) in which certain competitors have strictly greater energy than the energy of $\mathbf{u}_{\mathrm{e}}$. In [4, 19, 22] such competitors must be Lipschitz, while Theorem 4.1 allows such mappings to be contained in the larger space $W^{1,\mathrm{BMO}}(\Omega;\mathbb{R}^n)$. However, our results, as well as the result in [4], require the polynomial growth of W (see (H3)), which is incompatible with $W(\mathbf{F}) \to \infty$ as the determinant of $\mathbf{F}$ approaches 0, as is usually assumed in Nonlinear Elasticity.[7] Finally, we note that the results in [4, 19], for the Dirichlet problem, are valid for W that are C^2 rather than C^3 as required here and in [22]. It appears that an extension of our results to C^2 integrands will necessitate a generalization of Proposition 2.2 to certain Orlicz spaces.

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