

Dynamics of bubbles under stochastic pressure forcing

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Several studies have investigated the dynamics of a single spherical bubble at rest under a non-stationary pressure forcing. However, attention has almost always been focused on periodic pressure oscillations, neglecting the case of stochastic forcing. This fact is quite surprising, as random pressure fluctuations are widespread in many applications involving bubbles (e.g., hydrodynamic cavitation in turbulent flows or bubble dynamics in acoustic cavitation) and noise, in general, is known to induce a variety of counter-intuitive phenomena, in non-linear dynamical systems such as bubble oscillators. In order to shed light on this unexplored topic, here we study bubble dynamics as described by the Keller Miksis equation, under a pressure forcing described by a Gaussian colored noise modeled as an Ornstein-Uhlenbeck process. Results indicate that, depending on noise intensity, bubbles display two peculiar behaviors: when intensity is low, the fluctuating pressure forcing mainly excites the free oscillations of the bubble, and the bubble's radius undergoes small amplitude oscillations with a rather regular periodicity. Differently, high noise intensity induces chaotic bubble dynamics, whereby non linear effects are exacerbated and the bubble behaves as an amplifier of the external random forcing.

I. INTRODUCTION

Over the last decades, the dynamics of gas-bubbles (also referred to as cavities) in liquids has attracted a lot of interest in the scientific community [e.g., 1–3]. This paper focuses on the canonical case of a spherical bubble subjected to a prescribed external forcing which drives variations in the bubble's radius. The problem has been extensively addressed [e.g., see 4–7] and can be mathematically described by ordinary differential equations, which, depending upon different simplifying assumptions, can take different forms [8–12]. Despite such differences, all these equations share the common feature of retaining strongly non-linear terms which make gas-bubbles in liquids dynamically-rich systems [13].

One of the attractive features of bubble dynamics involves the possibility of cavities to undergo abrupt variations in size. In particular, due to the high inertia of the liquid hosting the cavities, bubbles, if properly excited, can be subjected to abrupt collapses that generate intense pressure and temperature peaks, which, in turn, are associated with the generation of shock waves and the emission of light and sound [14–16].

The attractiveness of such extreme pressure and temperature events stems from the fact that they can be exploited in several technological applications. For instance, in medicine, bubble collapses are used to break liver and kidney stones and cancer cells [17, 18]. In the water industry, bubbles' collapses physically inactivate bacteria and the free-radicals generated by the temperature peaks reached during the collapsing phase are used

to oxidize pollutants for waste-water treatment purposes [19–22]; in geophysics, bubble implosions are useful for sub-sea geological explorations [23, 24].

Several factors influence bubble dynamics. The most relevant are the properties of the liquid hosting the gas bubble [25], the presence of solid boundaries close to the bubble [26–28], the interaction with other proximal gas cavities [29, 30], and the action of an external forcing that alters the bubble equilibrium conditions. Two classes of forcing are commonly considered. The first one consists in the alteration of the bubble size in a liquid at rest (with time-invariant pressure) using either laser beams or sparks [31, 32]. The second class involves variations of the static pressure of the liquid hosting the bubble [33, 34]. Static pressure variations are usually induced by ultrasound waves traveling within a volume of liquid at rest [35, 36] or by alterations of the liquid velocity (e.g., geometrical constrictions like orifice plates or Venturi tubes) in a pressurized system of conduits [37, 38].

The pressure forcing – especially, the case of pressure fluctuations in a liquid at rest – has been the focus of a great deal of studies and will be considered also in the present paper. The largest part of previous works have generally explored the effects of sinusoidal pressure oscillations on the bubble's radius [e.g., 39]. In spite of the simple and regular temporal structure of the forcing, the response of the bubble turned out to be very rich, exhibiting period-doubling bifurcations and period-doubling cascades that can ultimately lead to a chaotic behavior [40–45].

Other studies have investigated the forced dynamics of bubbles when the pressure of the hosting liquid is perturbed by a bi-harmonic signal obtained as the sum of two sinusoidal signals [46, 47]. It was found that such a combined signal induces significant alterations in the

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66 thresholds of period-doubling bifurcations and period- 99
 67 doubling cascades. It was therefore suggested to adopt 100
 68 bi-harmonic pressure signals to control chaos inception 101
 69 and to give a more controlled and predictable bubble 102
 70 behavior [48]. Finally, some theoretical and experimen- 103
 71 tal studies have focused on the transient phase occurring 104
 72 during the inception of an ultrasound field and in pulsed 105
 73 ultrasound fields [49–51]. Results showed that the col- 106
 74 lapse of bubbles was more intense in the transient phases, 107
 75 rather than during the regular sinusoidal phase of pulsed 108
 76 ultrasound fields. 109

77 To the authors’ opinion, the aforementioned results 110
 78 from the literature suggest that transients and irregu- 111
 79 larities of the external forcing can lead to yet unexplored 112
 80 bubbles’ responses. This should not be entirely surpris- 113
 81 ing because it is well known that many interesting and 114
 82 unexpected phenomena emerge from the stochastic forc- 115
 83 ing (i.e., a form of irregular forcing) of strongly non-linear
 84 systems (i.e., the so called noise-induced phenomena, see
 85 [52–55]). It is within this context that the aim, novelty 116
 86 and relevance of the present paper are cast. The aim is in-
 87 deed to explore the response of a single bubble to random
 88 fluctuations of the external pressure. The study is novel 117
 89 because, to the best of the authors’ opinion, it has never
 90 been addressed before. Its relevance lies in the fact that 118
 91 stochastic pressure-forcing are important for a number 119
 92 of applications and are encountered in a number of envi- 120
 93 ronments. Notable examples include: (i) hydrodynamic 121
 94 cavitation reactors (mainly used for water-treatment pro- 122
 95 cesses) where the pressure fluctuations imposed by tur- 123
 96 bulence and by the geometry of the reactor are known 124
 97 to heavily influence bubbles’ dynamics [30, 43, 56] and, 125
 98 ultimately, bubble’s efficiency in oxidation and disinfec- 126

tion processes; and (ii) acoustic cavitation reactors where
 bubbles’ dynamics is influenced by the interactions be-
 tween the sinusoidal pressure-waves generated by ultra-
 sound transmitters and the random shock pressure waves
 generated by imploding bubbles [38, 57, 58].

In order to fulfill the aim of the paper, we chose to
 adopt a modeling approach whereby the dynamics of
 bubble was investigated through numerical integration
 of the Keller-Miksis equations [40]. The pressure of the
 fluid hosting the bubble (i.e., the pressure forcing) was
 assumed to undergo stochastic fluctuations which were
 simulated using the Ornstein-Uhlenbeck model [59]. This
 model is well-established and represents a wide number
 of random processes in nature [60–63]. More importantly,
 it is characterized by only two free parameters that al-
 low for a systematic exploration of noise-intensity and
 noise-autocorrelation effects on bubble dynamics.

II. METHODS

A. Mathematical modeling of bubble dynamics

We focus on a single bubble located far from solid
 boundaries or liquid surfaces. The bubble is assumed
 to be spherical, positionally stable, and its surface not
 to be affected by instability mechanisms [64]. In order
 to study the dynamics of this type of cavity, it is stan-
 dard to focus on the temporal evolution of the bubble
 radius, $R(t)$, where t is time. A well-known mathemati-
 cal framework for the modeling of $R(t)$ is given by the
 Keller-Miksis equation [40]

$$\left(1 - \frac{\dot{R}}{c}\right) R\ddot{R} + \left(1 - \frac{\dot{R}}{3c}\right) \frac{3}{2}\dot{R}^2 = \left(1 + \frac{\dot{R}}{c}\right) \left(\frac{p_w - p(t)}{\rho}\right) + \frac{R}{c} \frac{d}{dt} \left(\frac{p_w - p(t)}{\rho}\right), \quad (1)$$

127 where dots denote time derivation, c is the speed of 144
 128 sound, ρ is the liquid density, $p(t)$ is the (possibly time 145
 129 dependent) liquid pressure indefinitely far from the bub- 146
 130 ble wall (often indicated in the literature also as p_∞), and 147
 131 p_w is the liquid pressure at the bubble wall. We chose 148
 132 the Keller-Miksis equation in place of more simplified for- 149
 133 mulations (e.g., the Rayleigh-Plesset equation) in order 150
 134 to properly model large and fast temporal variations of 151
 135 the radius $R(t)$ [40]. In the following, we will show that 152
 136 long-lasting and large increments of the bubbles’ radius 153
 137 play a key role in determining chaos in the radius dynam- 154
 138 ics. In this regard, Nazari-Mahroo *et al.* [69] compared 155
 139 the Keller-Miksis, Gilmore, and Lezzi-Prosperetti mod-
 140 els, and showed that – during the radius expansion stage
 141 – they behave very similarly. This means that the results
 142 presented herein are robust and overall insensitive to the
 143 choice of the specific bubbles’ dynamics model. It should 156
 157

also be noted that during radius expansion stage, the
 bubble remains spherical. This is confirmed for instance
 by the experiments reported by Löfstedt *et al.* [70].

The bubble is assumed to contain a mixture of liq-
 uid vapor and non-condensable gas and to be submerged
 within a liquid at constant temperature. If this mix-
 ture behaves as an ideal gas, the total pressure inside the
 bubble can be evaluated as $p_G + p_v$, where p_G and p_v
 are the gas and vapor partial pressure inside the cavity,
 respectively. Under this assumption, the pressure at the
 bubble wall, p_w , can be derived by a force balance at the
 gas-liquid interface, reading

$$p_w = p_G + p_v - \frac{2S}{R} + 4\mu \frac{\dot{R}}{R}, \quad (2)$$

where S is the surface tension, and μ is the liquid dy-
 namic viscosity. Provided that the liquid that hosts the

TABLE I. Physical parameters adopted for the liquid hosting the bubble. Data refer to water at 293 K.

ρ	S	μ	c	p_v
($\text{kg} \cdot \text{m}^{-3}$)	($\text{N} \cdot \text{m}^{-1}$)	($\text{Pa} \cdot \text{s}$)	($\text{m} \cdot \text{s}^{-1}$)	(Pa)
998	$73 \cdot 10^{-3}$	$1.00 \cdot 10^{-3}$	1481	2338

bubble is kept at constant temperature, the vapor pressure inside the cavity, p_v , is also constant. The gas pressure inside the bubble, instead, can be evaluated according to the polytropic relationship

$$p_G = p_{G,eq} \left(\frac{R_{eq}}{R} \right)^{3k}, \quad (3)$$

where $p_{G,eq}$ and R_{eq} are the gas pressure inside a bubble and the bubble radius in equilibrium conditions, respectively, and k is the so-called polytropic exponent. In this study, bubbles are supposed to undergo adiabatic volume changes, i.e., $k = 1.4$. This is consistent with several studies [3, 65, 66] that have shown that bubble dynamics – as predicted by adiabatic mathematical models – matches experimental observations. Finally, the pressure inside a bubble in steady conditions, $p_{G,eq}$, is evaluated from (1) and (2) setting $\dot{R} = \ddot{R} = 0$ as

$$p_{G,eq} = p - p_v + \frac{2S}{R_{eq}}, \quad (4)$$

where p is the pressure of the liquid far from the bubble. A key parameter is the period of bubble free oscillations [3, 25]

$$T_n = 2\pi \left(\frac{3k(p - p_v)}{\rho R_{eq}^2} + \frac{2(3k - 1)S}{\rho R_{eq}^3} \right)^{-\frac{1}{2}}. \quad (5)$$

This parameter will be crucial in the interpretation of the temporal evolution of the cavity radius $R(t)$. In the following, water at 293 K is assumed as hosting liquid, and Table I reports the corresponding physical parameters.

B. The stochastic forcing

The pressure of the liquid hosting the bubble is supposed to evolve over time as

$$p(t) = \bar{p} + p'(t), \quad (6)$$

where \bar{p} is the mean pressure experienced by the cavity, and $p'(t)$ is the time-dependent fluctuation around \bar{p} . The fluctuations $p'(t)$ are modeled as an Ornstein-Uhlenbeck process [59, 63].

The Ornstein-Uhlenbeck process is a stationary correlated Gaussian-Markov process with the following characteristics: (i) the probability density function of the realizations $p'(t)$ is a normal distribution with zero

mean and standard deviation σ_p ; (ii) the stochastic process is exponentially autocorrelated as $\overline{p'(t)p'(t+\tau)} = \sigma_p \exp[-\tau/\tau_p]$, where τ_p is the autocorrelation time-scale; and (iii) the process is stationary, namely σ_p and τ_p do not change over time.

We have chosen the Ornstein-Uhlenbeck process as the random pressure forcing due to its simplicity, mathematical tractability and the possibility of changing its variance and (linear) memory by acting on only two parameters, namely the standard deviation σ_p and the autocorrelation time-scale τ_p .

From a numerical point of view, the realizations of the pressure fluctuations, $p'(t)$, are evaluated by the so-called “exact update formula” provided by Gillespie [67], namely

$$p'(t + \Delta t) = p'(t) \cdot \zeta + \sigma_p \cdot \sqrt{1 - \zeta^2} \cdot n, \quad (7)$$

where n is a unit normal random number, Δt is the time-step of the process and $\zeta = \exp[-\Delta t/\tau_p]$. Since (7) provides an exact update for $p'(t)$, the actual value of the time-step of the process is arbitrary, and $\Delta t = \tau_p/50$ was chosen in this study.

C. Simulation of bubble radius dynamics

In order to investigate the effect of the stochastic pressure forcing on the dynamics of a bubble (i.e., on the time-series of the bubble radius $R(t)$), a number of numerical simulations was performed. Each numerical simulation consisted of two steps. Firstly, a random pressure forcing $p(t)$ was simulated according to (7). Secondly, Equation (1) was forced with $p(t)$ and numerically solved to obtain the response of the bubble, namely the time-series of the radius $R(t)$.

Simulations of $p(t)$ were performed setting $\bar{p} = 100 \cdot 10^3$ Pa. Three correlation times $\tau_p = [0.5, 1, 2]T_n$ were considered, and the standard deviation of the pressure was changed in the range $[0, 120] \cdot 10^3$ Pa. The duration of the simulations was set equal to $4000T_n$. This duration guaranteed a robust estimation of all the statistical properties of $R(t)$, for all the investigated conditions.

In order to obtain $R(t)$ from the numerical integration of (1) with the forcing (7), the initial conditions $R(0) = R_{eq} = 5 \cdot 10^{-6}$ m and $\dot{R}(0) = \ddot{R}(0) = 0$ were imposed and the time step $\Delta t = 10^{-8}$ s was adopted. $R(t)$ was normalized with the equilibrium radius R_{eq} [25, 40, 45] to better quantify the dynamics of the bubble radius.

Fig. 1b, reports the time-series of the normalized radius $R(t)/R_{eq}$ as obtained from integration of Eq. 1 when forced with the pressure reported in Fig. 1a. In Fig. 1c – d the pdfs of the time-series $p(t)$ and $R(t)$ (partially reported in Panels a – b) illustrate the variability of $p(t)$ and $R(t)$. Similarly, Figs. 1e – f report the autocorrelation functions, and illustrate how the correlation time is evaluated.

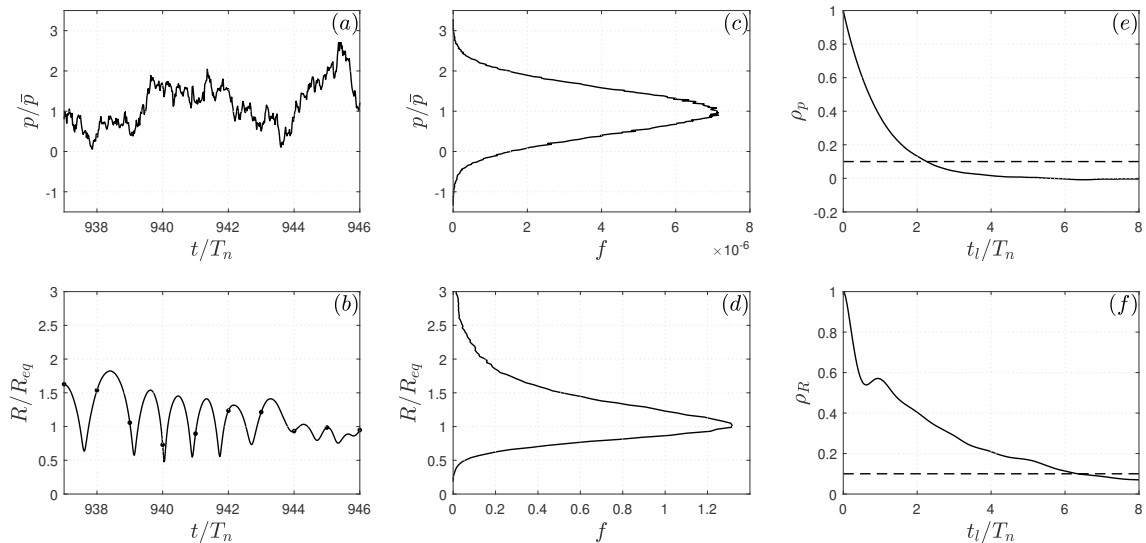


FIG. 1. (a) Example of a time-series of the normalized stochastic pressure forcing $p(t)/\bar{p}$. (b) Time-series of the normalized radius $R(t)/R_{eq}$ of the bubble forced by the pressure reported in (a). The black dots in (b) highlight the bubble radius attained at the instants nT_n , where n is an integer and T_n is the natural oscillation period of the bubble (see Section III A for the explanation). (c, d) Probability density functions and (e, f) autocorrelation functions of the time-series (partially) reported in (a, b). The dashed lines in (e, f) mark the level $\rho_p = \rho_R = 0.1$. It should be noted that the time-lag \hat{t}_l such that $\rho(\hat{t}_l) = 0.1$ is defined as the correlation time of the time-series. The times in (a, b) and the time lags reported in (e, f) are normalized by T_n . The adopted parameters are $\sigma_p = 60 \cdot 10^3$ Pa and $\tau_p = 2.0T_n = 2.8 \cdot 10^{-6}$ s.

241 The interested reader can find in Appendix C further 266
 242 details about the numerical techniques adopted to solve 267
 243 (1) and a sensitivity analysis of the solution with respect
 244 to: the time-step adopted for the numerical solution; the 268
 245 duration of the simulations; and the number of realiza- 269
 246 tions adopted for the statistical analyzes. 270

247 III. RESULTS 277

248 Four complementary perspectives are adopted to study 278
 249 the behavior of $R(t)/R_{eq}$. The first (Sec. A) is based on 279
 250 bifurcation diagrams and presents a way to identify the 280
 251 onset of chaos in the $R(t)/R_{eq}$ time-series. The second 281
 252 (Sec. B) investigates the physical mechanisms underpin- 282
 253 ning the onset of chaotic fluctuations. The third (Sec. 283
 254 C) is a detailed statistical analysis of $R(t)/R_{eq}$, with 284
 255 a particular emphasis on the dependence of $R(t)/R_{eq}$ 285
 256 statistical-moments on various combinations of noise in- 286
 257 tensity and correlation time scales. Finally, Sec. D digs 287
 258 deeper into second order statistics and investigates dom- 288
 259 inant modes and characteristic time scales of $R(t)/R_{eq}$ 289
 260 time-series. This provides hints about the random vs 290
 261 organized temporal structure of $R(t)$. 291

262 All the results are wrapped up in Sec. IV, which pro- 292
 263 vides an overview of bubbles' behavior under stochastic 293
 264 pressure forcing, using and harmonizing all the results 294
 265 obtained from Sec. III A, B, C and D. 295

A. Assessment of the temporal pattern and bifurcation diagram

We begin the results section by discussing the temporal dynamics exhibited by $R(t)/R_{eq}$. To this aim, the values $R(t = nT_n)/R_{eq}$ with $n = 1, 2, \dots$ were extracted from $R(t)/R_{eq}$ (see dot-symbols in Fig. 1b). If the bubble radius oscillation exhibits a period T_n , $R(t)$ takes the same value at instants that are multiples of T_n . Conversely, if $R(t)$ is not periodic (or when the period of oscillations is different from T_n) then $R(nT_n)/R_{eq}$ exhibits a variability. 276

Figs. 2a, b show results associated with the analysis of $R(nT_n)/R_{eq}$ in the form of noise-intensity bifurcation diagrams. These graphs report on the x-axis the noise intensity σ_p/\bar{p} and on the y-axis the values of $R(nT_n)/R_{eq}$ extracted from the corresponding time-series $R(t)$. The gray and red dots in Panels 2a and 2b refer to different correlation times τ_p . The noise-intensity-bifurcation diagrams obtained in Figs. 2a, b align with those obtained from other studies that considered a sinusoidal forcing [25, 40, 45], but key differences can be observed.

In the case of a sinusoidal forcing with amplitude A_p and period T_n , the noise-intensity-bifurcation diagrams exhibit two different zones. When A_p is lower than a threshold $A_{p,c}$, the metric $R(nT_n)/R_{eq}$ is perfectly constant for any n . This can be seen, for example, in Figs. 3a, b that report the radius dynamics forced by the sinusoidal pressure with amplitude $A_p < A_{p,c}$ shown in Figs. 3d, e. Differently, for $A_p > A_{p,c}$ the metric $R(nT_n)/R_{eq}$ exhibits a large variability for a fixed value of A_p and for

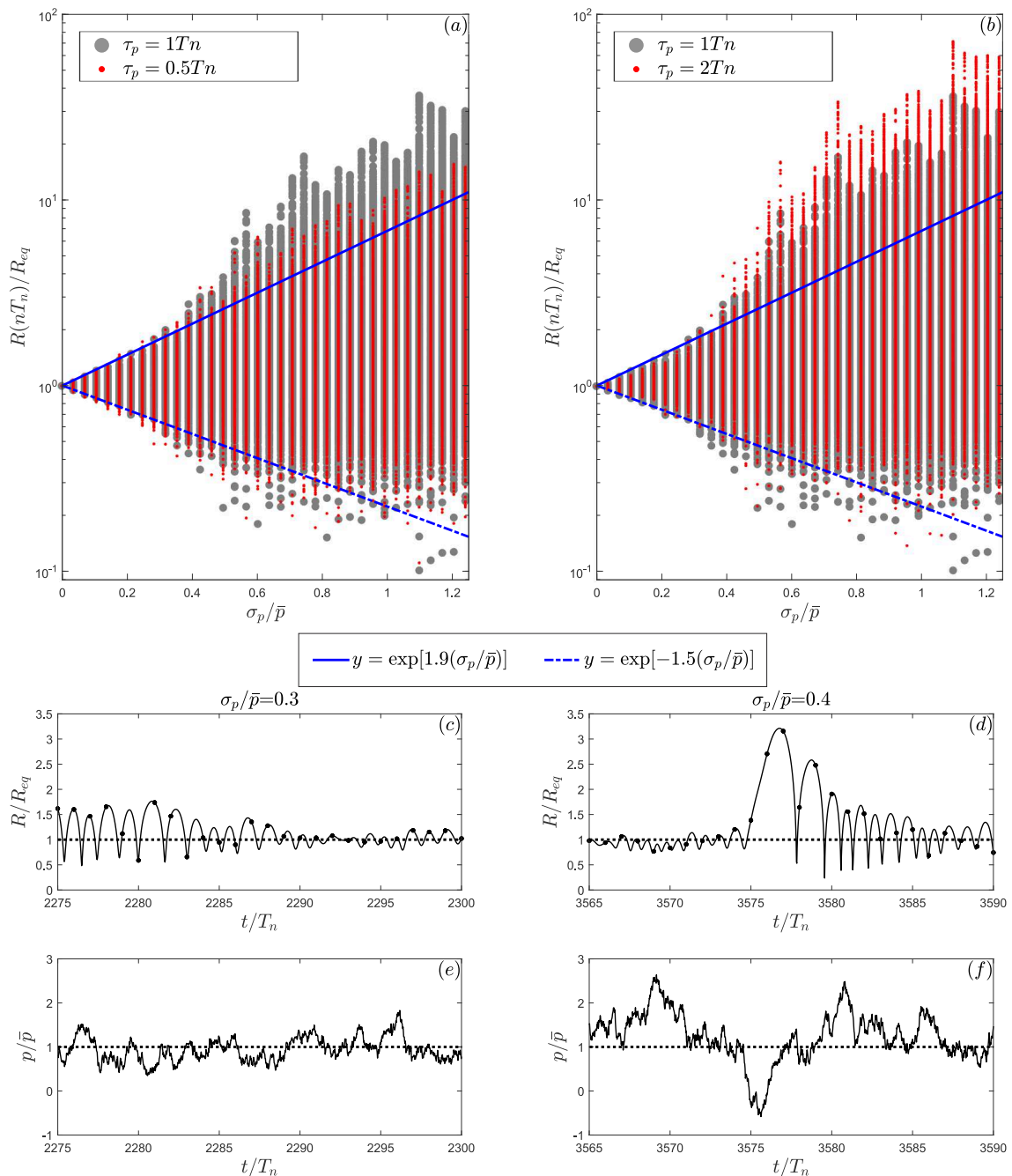


FIG. 2. (a, b) Noise-intensity bifurcation diagrams. For a given value of σ_p/\bar{p} , the dynamics of $R(t)$ is simulated for $4000T_n$. From this simulation, only the values $R(nT_n)/R_{eq}$ are selected, and are reported in the vertical axis for the given σ_p/\bar{p} . In both Panels, the gray circles refer to $\tau_p = T_n$. In Panel (a) and (b) the red dots refer to $\tau_p = T_n/2$ and $\tau_p = 2T_n$, respectively. (c–f) Time segments of the time-series $R(t)/R_{eq}$ and $p(t)/\bar{p}$. The horizontal dotted lines mark the equilibrium radius $R(t)/R_{eq} = 1$ and the mean pressure $p(t)/\bar{p} = 1$. The black dots in (c–d) highlight the bubble radius attained at the instants nT_n . Panels (c, e) refer to $\sigma_p/\bar{p} = 0.3$ and Panels (d, f) refer to $\sigma_p/\bar{p} = 0.4$; in both cases, $\tau_p = T_n$.

296 different values of n . This non-regular behavior is exem-302
 297 plified in Fig. 3c, which shows the radius dynamics under303
 298 the sinusoidal pressure forcing with amplitude $A_p > A_{p,c}$ 304
 299 of Fig. 3f. Therefore, in the case of a sinusoidal forc-305
 300 ing, $A_{p,c}$ represents an amplitude threshold that sharply306
 301 separates the non chaotic and chaotic regimes. 307

When stochastic fluctuations of pressure are consid-
 ered, the variability of $R(nT_n)/R_{eq}$ increases with in-
 creasing σ_p/\bar{p} (Figs. 2a, b). This is consistent with
 the case of a sinusoidal forcing. However, while σ_p/\bar{p}
 increases, a clear threshold that separates regular os-
 cillations from chaotic fluctuations does not emerge.

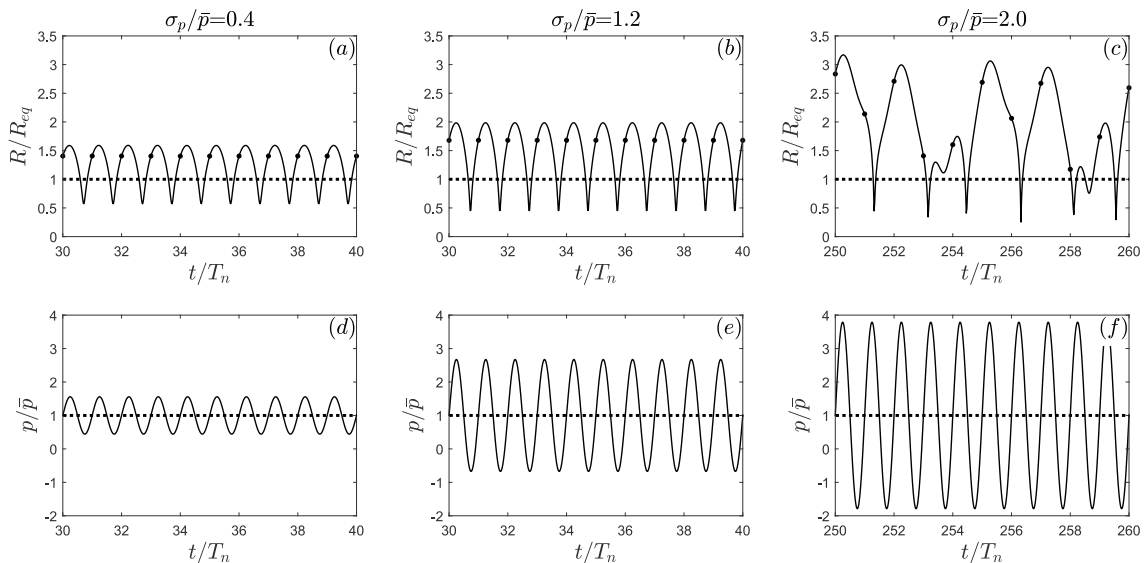


FIG. 3. Time segments of the time-series $R(t)/R_{eq}$ (top row) and $p(t)/\bar{p}$ (bottom row) when a sinusoidal external pressure – with amplitude $A_p = \sqrt{2}\sigma_p$ and period $\tau_p = T_n$ – is applied. The dotted lines mark the equilibrium radius $R(t)/R_{eq} = 1$ and the mean pressure $p(t)/\bar{p} = 1$. The black dots in (a – c) highlight the bubble radius attained at the instants nT_n . Panels (a, d), (b, e) and (c, f) refer to $\sigma_p/\bar{p} = 0.4$, $\sigma_p/\bar{p} = 1.2$ and $\sigma_p/\bar{p} = 2.0$, respectively.

Moreover, even for very low values of σ_p/\bar{p} , the metric $R(nT_n)/R_{eq}$ does show some level of variability and hence it is not constant.

A more careful inspection shows that a change in bubble dynamics occurs at $\sigma_p/\bar{p} \approx 0.30$: for $\sigma_p/\bar{p} \lesssim 0.30$ the normalized radius oscillates around 1 and is confined by the almost symmetrical curves $\exp[1.9(\sigma_p/\bar{p})]$ and $\exp[-1.5(\sigma_p/\bar{p})]$ (these curves were obtained by fitting the maximum and minimum values attained by $R(nT_n)/R_{eq}$ for $\sigma_p/\bar{p} < 0.30$); differently, for $\sigma_p/\bar{p} \gtrsim 0.30$ the variability of the radius suddenly increases and $R(nT_n)/R_{eq} \in [0.01, 50]$.

B. Physics of chaos inception

In order to elucidate the physical behavior behind the inception of chaos in the dynamics of $R(t)$ occurring for $\sigma_p/\bar{p} > 0.3$, Panels 2c – d report two exemplifying portions of time-series $R(t)/R_{eq}$. To relate the bubble radius dynamics to the pressure fluctuations, the corresponding time-series $p(t)/\bar{p}$ are reported in Panels 2e – f. These pressure time-series are obtained setting the same noise time-scale $\tau_p = T_n$ but different noise intensities. The dotted lines mark the threshold $p(t)/\bar{p} = 1$, and help to discern the instants when the instantaneous forcing pressure is below average (i.e., $p(t)/\bar{p} < 1$) or above average (i.e., $p(t)/\bar{p} > 1$). We recall that when the instantaneous pressure is below/above average, the bubble radius tends to increase/decrease.

Panels 2c, e refer to the noise intensity $\sigma_p/\bar{p} = 0.3$ (i.e., just below the threshold that separates the non-chaotic/chaotic behaviors). In this case, the pressure

oscillates slightly around the mean value (Panel 2e) and the bubble radius does not undergo large increments ($R(t)/R_{eq}$ never exceeds the value 2, see Panel 2c). It follows that during the small radius increments little energy is stored in the bubble. As a consequence of this: (i) the subsequent rebound is mild ($R(t)/R_{eq}$ remains close to unity); and (ii) the radius growth that follows the rebound is mild as well. The radius dynamics is therefore characterized by a sequence of modest increments of radius intercut with mild rebounds. At this conditions, the period of the oscillations is very close to the natural oscillation period of the bubble and no chaos is detected.

In contrast, Panels 2d, f focus on the noise intensity $\sigma_p/\bar{p} = 0.4$ (i.e., above the no-chaos/chaos threshold). In this case, the pressure may deviate significantly from the mean value (e.g., see immediately after $t/T_n = 3575$ in Panel 2f). As a result, large increments in the bubble radius occur, that may last a few times the natural period T_n . For instance, this can be seen in Panel 2d, where the radius growth starting at $t/T_n \approx 3575$ lasts about $3T_n$, and $R(t)/R_{eq}$ eventually exceeds the value 3. During these large increments of radius, a significant amount of energy is stored in the bubble. Consequently: (i) the subsequent rebound is violent ($R(t)/R_{eq}$ is much lower than unity); and (ii) the radius growth that follows the rebound may be considerable and long lasting (this is exemplified in in Panel 2d, where the radius growth that begins after the rebound at $t/T_n \approx 3578$ lasts about $2T_n$). The radius dynamics is therefore characterized by a sequence of significant and long lasting increments of radius (the duration of these phases exhibit a wide variability) intercut with violent rebounds. At these conditions, $R(t)/R_{eq}$ deviates significantly from unity, and

the period of the oscillations varies significantly from the natural oscillation period of the bubble. Accordingly, chaotic behavior is detected. It should be noted that the behaviors reported in the time segments of Panels 2c-f are not rare, but are detected in a large number of time segments in the time-series simulated in this work.

The examples previously reported depict a picture where bubble chaotic dynamics is characterized by long lasting and large radius increments, induced by coherent negative pressure fluctuations. It follows that chaos occurs when downcrossing events in the pressure signal exceed suitable thresholds; namely, the duration and the magnitude of the negative pressure fluctuations (with respect to the pressure mean value) become sufficiently high. In the cases investigated in this work, such downcrossing analysis gives that bubble chaotic dynamics occurs when: (i) the duration of pressure reduction events exceeds the threshold $1.5T_n$; and (ii) the corresponding mean value of the pressure reduction during this negative pressure events is greater than $0.6\bar{p}$. However, it should be noted that the bubble response to pressure forcing depends on the physical properties of fluid and the initial size of the bubble. Therefore, the physics of chaos inception previously described (i.e., interplay between long lasting, intense pressure fluctuations and nonlinear bubble dynamics) is of general validity. However, the exact threshold values dictating the transition to chaos detected here are surely dependent on the fluid characteristics (see Table I). The precise determination of this dependence is beyond the scope of the present work, and will be the subject of future work.

We now briefly highlight the key role of pressure stochasticity in the inception of chaos in bubbles' dynamics. To this aim, we evaluated the response of a bubble to three sinusoidal pressure forcing $p(t)/\bar{p} = 1 + (A_p/\bar{p}) \sin(2\pi t/T_n)$, and compared it against the behavior depicted in Panels 2d,f. Three relevant values of the oscillation amplitude, A_p , were tested: (i) $A_p/\bar{p} = \sqrt{2} \cdot 0.4$, such that the standard deviation of the sinusoidal signal is $\sigma_p = 0.4 \cdot \bar{p}$, and the resulting radius dynamics can be compared with Panel 2d (that refers to a stochastic pressure forcing with $\sigma_p/\bar{p} = 0.4$); (ii) $A_p/\bar{p} = \sqrt{2} \cdot 1.2$, (i.e., the sinusoidal forcing is characterized by $\sigma_p/\bar{p} = 1.2$) such that the minimum pressure attained by the sinusoidal forcing is the same typically attained by the stochastic forcing of Panel 2f; and (iii) $A_p/\bar{p} = \sqrt{2} \cdot 2.0$, inducing pressure oscillations with $\sigma_p/\bar{p} = 2.0$, i.e., much higher than 0.4. Results on $R(t)/R_{eq}$ are reported in Fig. 3. The noise intensities $\sigma_p/\bar{p} = 0.4$ and $\sigma_p/\bar{p} = 1.2$ (Panels 3d-e) did not lead to inception of chaos: the radius time-series were very regular and exhibited fluctuations with the constant period T_n (Figs. 3a-b). Differently, for $\sigma_p/\bar{p} = 2.0$ (Panel 3f), a chaotic behavior of the bubble radius occurred (Panel 3c).

The comparison of results shown in Fig. 2 (related to random forcing) and in Fig. 3 (corresponding to sinusoidal forcing) clearly shows that stochasticity promotes

the chaos inception. Although sinusoidal pressure signals have the same standard deviation ($\sigma_p/\bar{p} = 0.4$, Panels 3a,d) or the same typical minimum values (Panel 3b,e) of the stochastic forcing, sinusoidal pressure forcing do not lead to chaotic bubble dynamics, while random forcing does. Only the increment of the oscillation amplitude of the sinusoidal pressure to $A_p/\bar{p} = \sqrt{2} \cdot 2.0$ eventually lead to the inception of chaos. Namely, the noise intensity of the sinusoidal pressure should be five times larger than that of the stochastic case, in order to observe a similar pattern of chaotic radius fluctuations.

The role of the correlation time of the forcing, τ_p , was also explored. Red dots in Figs. 2a,b correspond to $\tau_p = 0.5T_n$ and $\tau_p = 2T_n$, respectively; in each panel data pertaining to $\tau_p = T_n$ (gray circles in both panels) are kept to allow for comparisons. It emerges that variations of τ_p are relevant only for $\sigma_p/\bar{p} \gtrsim 0.30$ (i.e., above the threshold identified before) and positively correlated with the variability of the bubble radius. This behavior is in accordance with the physical explanation of the inception of chaos described so far. Higher values of correlation time of the forcing entail longer periods over which the pressure fluctuation has a constant sign. Hence, longer periods of pressure below average can be observed. These, in turn, promote large radius increments and thus the inception of chaos. This analysis is performed in more details in the Appendix A.

C. Statistical analysis

The analysis of Fig. 2 reveals that $R(t)$ deviates significantly from its equilibrium value and the behavior of $R(t)$ can be very irregular. In order to better quantify the deviations of $R(t)$ from R_{eq} , the probability density functions (pdf) and the cumulative distribution functions (cdf) of the metric R/R_{eq} were evaluated. Details about this statistical analysis are given in the Appendix B where we report that changes in both σ_p/\bar{p} and τ_p induce significant alterations in the pdf of the bubble radius $R(t)$. However, σ_p/\bar{p} -effects seems to be stronger. For this reason, the effect of σ_p/\bar{p} was systematically explored in the relatively large range $[0, 1.20]$ for only three values of the noise correlation time $\tau_p = [0.5, 1, 2]T_n$.

For the sake of clarity, the corresponding effects on the pdfs of $R(t)$ are then expressed in terms of four relevant statistical parameters, reported in Fig. 4: (i) the mean value of the normalized bubble radius, \bar{R}/R_{eq} ; (ii) the coefficient of variation of $R(t)$, i.e., $c_{V,R} = \sigma_R/\bar{R}$; (iii) the skewness s_R of the time-series; and (iv) the kurtosis k_R of $R(t)$.

The noise intensity σ_p/\bar{p} has a strong effect on the mean value of the bubble radius (Fig. 4a). In particular, σ_p/\bar{p} is positively correlated with \bar{R} . This is a key point: the mean value of the bubble radius depends not only on the mean pressure, \bar{p} , but also on the noise intensity, σ_p . Therefore, in the case of a stochastic pressure forcing, it can be misleading to estimate the mean value of the

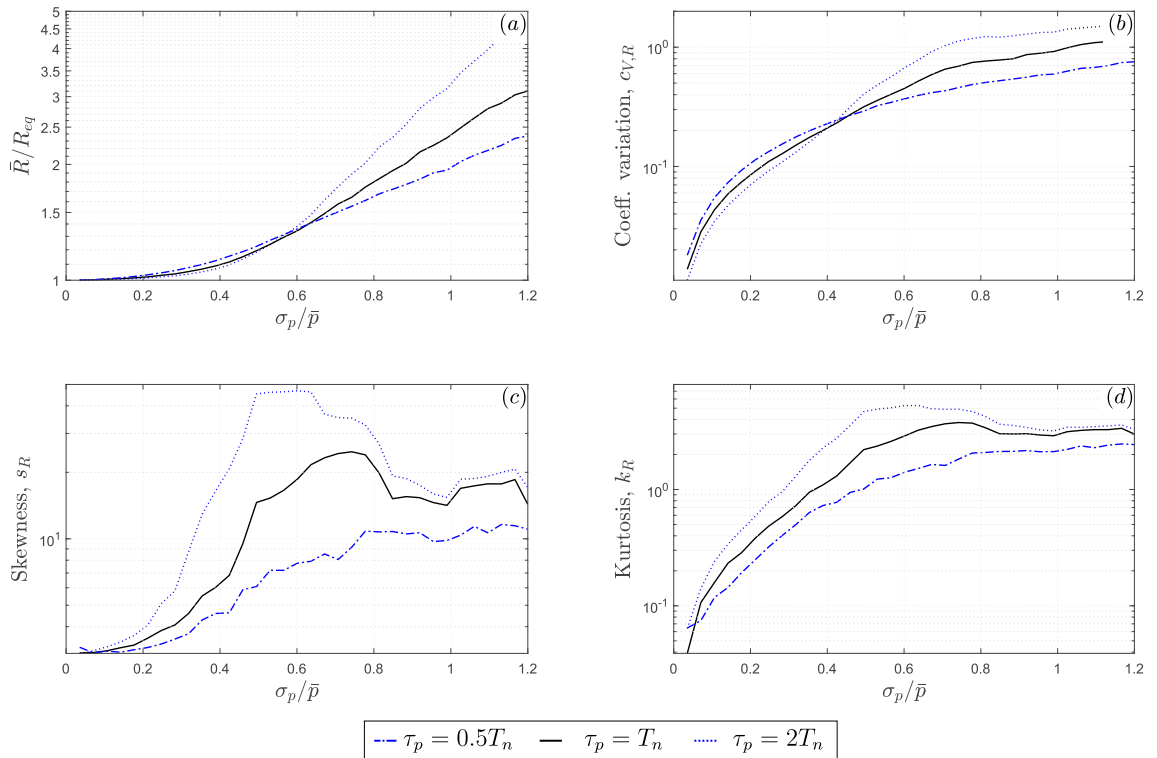


FIG. 4. Effect of σ_p/\bar{p} on some relevant statistical parameters that describe the time-series $R(t)$.

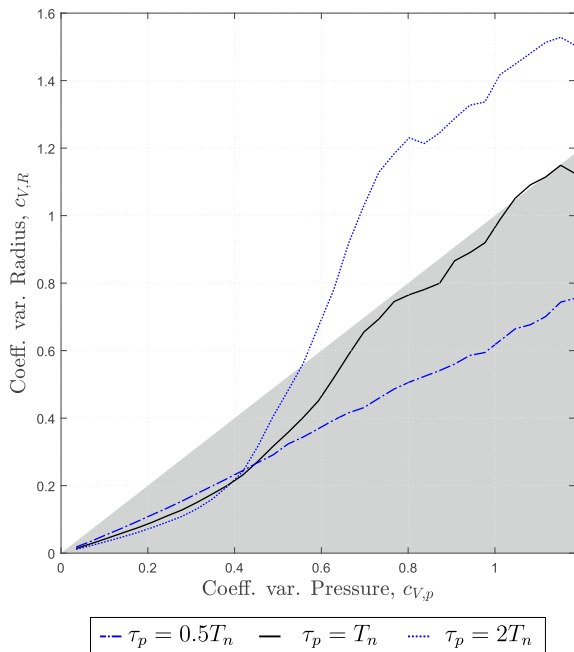


FIG. 5. Effect of the coefficient of variation of the pressure, $c_{V,p}$, on the coefficient of variation of the bubble radius, $c_{V,R}$. The shaded zone highlights the lower half plane bounded by the bisector, where bubble exhibits the “damper” behavior. In the upper half plane bubble behaves as “amplifier”.

484 bubble radius just from the mean (background) pressure.

485 When σ_p/\bar{p} exceeds 0.60, different curves \bar{R}/R_{eq} are
 486 observed for different values of τ_p . This can be explained
 487 as follows. According to the analysis presented in Section
 488 III A, the deviation of \bar{R} from R_{eq} is due to the nonlin-
 489 ear nature of the bubble dynamics and, in particular, it
 490 is ascribable to the effect of time segments during which
 491 the instantaneous pressure is below average (i.e., when
 492 $p(t) < \bar{p}$). When the pressure is below average, the bub-
 493 ble radius undergoes a strong increment and deviates sig-
 494 nificantly from R_{eq} (i.e., the equilibrium radius attained
 495 at $p(t) = \bar{p}$, see Panels 2c, d). This, clearly, contributes to
 496 increase \bar{R} . It was also pointed out that, the higher τ_p ,
 497 the longer the duration of time segments during which
 498 the instantaneous pressure is below average (see the Ap-
 499 pendix A), and thus the stronger the increments of the
 500 bubble radius and, consequently, of \bar{R} from R_{eq} . Besides
 501 \bar{R} , the other statistical parameters are all also strongly
 502 affected by the noise intensity (see Panels 4b – d).

503 The correlation time τ_p does not change the qualitative
 504 behavior of the the curves presented in Fig. 4, however,
 505 some peculiarities do occur: (i) the effect of τ_p on the
 506 mean value and on the coefficient of variation of $R(t)$
 507 is most relevant for high values of σ_p/\bar{p} (Panels 4a, b);
 508 (ii) the skewness and the kurtosis are affected by τ_p the
 509 most when σ_p/\bar{p} is in the range [0.4, 0.8] (see Panels 4c, d),
 510 instead the curves tend to merge for higher values of the
 511 correlation time of the pressure forcing.

512 The behavior of skewness and kurtosis shows other

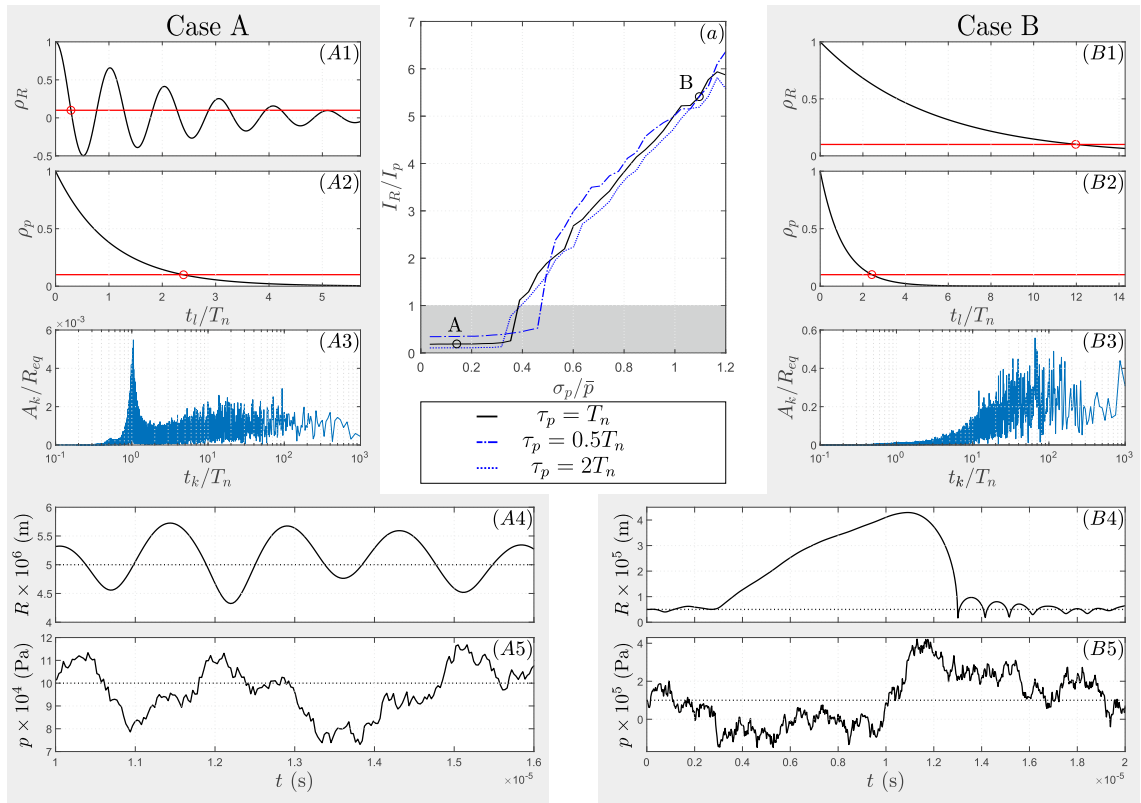


FIG. 6. (a) Effect of the noise intensity σ_p/\bar{p} on the ratio between the integral scale of the radius time-series, I_R , and the integral scale of the pressure forcing, I_p . The gray zone highlights the condition $I_R < I_p$. Autocorrelation diagrams of $R(t)$ (Panels A1, B1) and $p(t)$ (Panels A2, B2). The red lines mark the level where the autocorrelation function is 0.1. (A3, B3) Power (amplitude) spectrum of $R(t)$. It should be noted that the horizontal axis reports the period of the k -th harmonics (rather than its frequency). (A4 – B5) Relevant time segment of the time-series $R(t)$ and $p(t)$. The dotted lines mark the equilibrium radius R_{eq} and the mean pressure \bar{p} .

513 interesting aspects. For all investigated values of τ_p ,⁵³⁶
 514 they increase with increasing σ_p/\bar{p} within the range⁵³⁷
 515 $\sigma_p/\bar{p} = [0, 0.60]$. For $\sigma_p/\bar{p} \gtrsim 0.60$, instead, they seem⁵³⁸
 516 to tend monotonically (kurtosis) or non-monotonically⁵³⁹
 517 (skewness) to an asymptotic value (Panels 4c, d). In-⁵⁴⁰
 518 terestingly, the kurtosis tends to its Gaussian value of⁵⁴¹
 519 three. In summary, the trends observed in Fig. 4 indicate⁵⁴²
 520 that increments in the noise intensity tend to increase⁵⁴³
 521 the mean radius of the bubble as well as the intensity⁵⁴⁴
 522 of its variations (Panels a and b). The positive value of⁵⁴⁵
 523 the skewness indicates that it is more probable to have⁵⁴⁶
 524 $R(t) > R_{eq}$ than $R(t) < R_{eq}$. This asymmetry increases⁵⁴⁷
 525 with increasing σ_p/\bar{p} but saturates for $\sigma_p/\bar{p} \gtrsim 0.60$. The
 526 behavior depicted by kurtosis indicates that the occur-
 527 rence of extreme events (i.e., intermittency) in $R(t)$ in-
 528 creases with increasing noise intensity, but, as per the
 529 skewness, it saturates for $\sigma_p/\bar{p} \gtrsim 0.60$.⁵⁴⁸

530 An important aspect in studies about nonlinear oscil-
 531 lators is to evaluate whether the system behaves as a
 532 “damper” or as an “amplifier” of the external forcing⁵⁵¹
 533 [54]. To this end, the variability of the bubble radius⁵⁵³
 534 was compared to the variability of the forcing pressure⁵⁵⁴
 535 forcing (see Fig. 5). The gas bubble can be classified⁵⁵⁵

as a “damper” when the coefficient of variation of the
 fluctuating pressure forcing is larger than the coefficient
 of variation of the fluctuating bubble radius (i.e., $c_{V,p} >$
 $c_{V,R}$, gray zone in Fig. 5). On the other end, if $c_{V,p} <$
 $c_{V,R}$ (white zone in Fig. 5) the gas bubble behaves as a
 noise “amplifier”. The correlation time of the noise, τ_p
 is a key parameter in determining the amplifier/damper
 behavior of the bubble oscillator. For $\tau_p \leq T_n$ the bubble
 dynamics usually exhibits a “damper” behavior. Differ-
 ently, when $\tau_p = 2T_n$, the bubble behaves as a noise
 “amplifier” for $c_{V,p} \gtrsim 0.5$.

D. Temporal correlation

It is now instructive to analyze the correlation time-
 scale of the radius signal $R(t)$. To this end, we evaluate
 the autocorrelation function $\rho_R(t_l)$ (see the examples re-
 ported in Panels 6A1, B1). Then, we select the turnover
 time-lag $\hat{t}_{l,R}$ so that $\rho_R(\hat{t}_{l,R})=0.1$ (red circles in Figs.
 6A1, B1). Finally, the integral scale of the signal is eval-
 uated as $I_R = \int_0^{\hat{t}_{l,R}} \rho_R(t_l) dt_l$. If the same procedure is
 applied to the time-series $p(t)$ (see Panels 6A2, B2), the

integral scale of the noise $I_p = \tau_p$ is obtained. In order to highlight the non-linear behavior of the bubble oscillator, we focus on the ratio between the integral scale of the bubble radius and the integral scale of the pressure, namely I_R/I_p (Fig. 6a). Note that the definition of the crossover time scale based on the $\rho_R = 0.1$ is arbitrary. Note also, that any other value of ρ_R reasonably close to 0 proved to lead to almost identical results and trends presented in Fig. 6a, meaning that the results discussed in what follows are essentially independent on the exact definition of the crossover time scale.

Fig. 6a shows the effect of the noise intensity σ_p/\bar{p} on I_R/I_p and two contrasting behaviors are observed. When the noise intensity σ_p/\bar{p} is lower or greater than ≈ 0.30 (this value depends slightly on τ_p), then $I_R \ll I_p$ (gray zone in Fig. 6a) and $I_R \gg I_p$, (white zone in Fig. 6a) respectively.

In order to investigate the physical processes underpinning this sharp change in the behavior of I_R/I_p , we select two values of σ_p/\bar{p} for which these contrasting behaviors are observed (see points A and B in Fig. 6a). For both cases, the radius signal $R(t)$ (Figs. 6A4, B4) and the pressure signal $p(t)$ (Figs. 6A5, B5) are also reported over a significant time interval. Moreover, the power spectrum of $R(t)$ is evaluated (Figs. 6A3, B3).

Case A. For low values of the noise intensity, the only effect of pressure fluctuations is to excite the free oscillations of the bubble. For instance, when $\sigma_p/\bar{p} = 0.14$, the bubble radius oscillates with a varying amplitude (see Fig. 6A4), but the oscillation period is almost constant, and close to the natural period of oscillation of the bubble, T_n . This is confirmed by: (i) the peak in the power spectrum of $R(t)$ (Fig. 6A3); and (ii) the shape of the autocorrelation function (Fig. 6A1), which resembles that of a periodic signal with period equal to T_n . Therefore, for low noise intensity levels, pressure variations are not able to significantly alter the free oscillations of the bubbles and induce chaos.

Case B. For high values of the noise intensity, pressure fluctuations drive the bubble dynamics. In the considered case (the noise intensity is $\sigma_p/\bar{p} = 1.10$), the bubble exhibits oscillations that attain large amplitudes (Fig. 6B4). Differently from Case A, the oscillation period undergoes strong variations in the range $[0.5, 10]T_n$. As a result, the power spectrum of $R(t)$ (see Fig. 6B3) does not show any clear peak, and harmonics with periods in the wide range $[10^1, 10^3]T_n$ are characterized by comparable amplitudes. The signal portions reported in Fig. 6B4, B5 show that pressure variations alter to a major extent the dynamics of the bubble – according to the physical mechanisms explained in Sec. III B – and free oscillations with period T_n are rarely observed. For instance, during the very long time segment from $t \approx 2 \cdot 10^{-6}$ s to $t \approx 10 \cdot 10^{-6}$ s, the bubble radius comes very large (≈ 10 times the equilibrium value, see Fig. 6B4). After this long growth phase, oscillations with a period slightly higher than T_n are observed. The high values of I_R observed for high values σ_p are therefore in-

duced by the long periods over which a constant growth of $R(t)$ takes place. Note that, these long lasting growth phases are followed by rebounds exhibiting a period comparable to the bubble natural period. It follows that the increment of I_R due to long lasting radius growth phases cannot be balanced by phases during which the bubble oscillates with a period close to T_n .

The behavior previously described justifies the negligible effect of noise correlation time on bubbles' dynamics observed when the noise intensity is below the no-chaos/chaos threshold. This result was detected in the Fig. 2a, b (see Sec. III B). When the noise intensity is below the no-chaos/chaos threshold, bubbles oscillate at their natural frequency, and the only role of pressure fluctuation is to provide energy to sustain this motion. The characteristics of such pressure fluctuations are irrelevant in determining the frequency of vibration of the bubble. At most, they slightly alter the amplitude of the radius oscillation. Differently, when the noise intensity is above the no-chaos/chaos threshold, the bubble's dynamics are strongly driven by the pressure forcing. Hence, key characteristics of the pressure fluctuation – such as the noise correlation time – become important in determining bubble dynamics. In particular, longer correlation times – according to the mechanisms illustrated in Section III B – are associated with a more chaotic bubble response.

IV. CONCLUSIONS

The response of a single bubble to a stochastic pressure forcing was investigated. The motivation underpinning this study lies: (i) in the occurrence of random pressure fluctuations in many applications exploiting bubble dynamics; and (ii) in the strong nonlinearities affecting the deterministic bubble dynamics, which suggests the possible occurrence of non-trivial noise-induced phenomena.

Two key parameters control stochastic bubble dynamics: the ratio between the standard deviation and the mean value of the forcing pressure (σ_p/\bar{p}), and the ratio between the noise correlation time-scale and the period of bubble free oscillations (τ_p/T_n). Two typical behaviors were detected. The first one occurs when σ_p/\bar{p} is lower than a threshold value around 0.3; namely, when pressure fluctuates with small amplitudes. In this case, the random pressure forcing mainly excites the free oscillations of the bubble whose radius undergoes small amplitude oscillations and exhibits a rather regular periodicity. Moreover, we observed that (i) the effect of τ_p/T_n is small, (ii) the mean value of the background pressure can be adopted to estimate the mean value of the bubble radius, and (iii) bubble always behaves as a damper of external noise.

The second behavior occurs when the fluid hosting the bubble experiences large-amplitude pressure fluctuations (i.e., $\sigma_p/\bar{p} > 0.3$). At these conditions, pressure stochasticity is able to trigger a chaotic bubble dynamics. Time-series of the bubble radius exhibit large amplitude fluctu-

tuations and no evident periodicities occur, not even at the bubble natural frequency. The parameter τ_p/T_n now significantly affects the bubble dynamics. In particular, when τ_p/T_n is high, long time intervals during which the instantaneous pressure is below the mean pressure appear; these intervals entail large increments of $R(t)$ and are usually followed by cavities' collapses and rebounds. A strong variability of the $R(t)$ time-series occurs and the bubble behaves as a nonlinear oscillator that amplifies the external noise. Consequently, the mean value of the background pressure cannot be adopted to estimate the mean value of the bubble radius; in doing so, the mean radius of the bubble can be underestimated of a factor five. It should be finally remarked the key role of stochasticity in triggering chaos in bubble's radius dynamics. Two pressure forcing – one stochastic, one sinusoidal – characterized by the same noise intensity σ_p/\bar{p} behave very differently: the stochastic pressure forcing is more prone to trigger strong chaotic radius fluctuations than its sinusoidal counterpart.

In this work, we have demonstrated that stochastic forcing can induce interesting and unexpected bubble behaviors, presumably induced by the strongly non-linear nature of the bubble oscillator. This paves the way to study other type of noises (e.g., dichotomous or shot noises) and to investigate how random forcing could be conveniently exploited in various applications. For example, noise-induced violent cavities implosions – attained when intensity and correlation of pressure fluctuations are high – can be used to make water disinfection processes based on hydrodynamic cavitation and sonochemical reactions more energy efficient.

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Appendix A: Role of correlation time

In order to elucidate the increment of variability of $R(t)$ with τ_p , Panels 7a,b report some exemplifying time-series $R(t)/R_{eq}$ obtained with the same noise intensity $\sigma_p/\bar{p}=0.70$ (chosen in order to be in the chaos domain) but different noise time-scales, namely $\tau_p = T_n/2$ and $\tau_p = 2T_n$, respectively. To relate the bubble radius dynamics to the pressure fluctuations, red dots plotted in Panels 7a,b (along the line $R(t)/R_{eq} = 1$) mark the instants when the instantaneous forcing pressure is below average (i.e., $p(t) < \bar{p}$). From a physical point of view, when the instantaneous pressure is below average the bubble radius tends to increase; on the contrary, radius contractions are promoted when the instantaneous pressure is above average (i.e., $p(t) > \bar{p}$), identified by red dots at $R(t)/R_{eq} = 1$.

Panel 7a shows that, when the correlation time of the pressure forcing, τ_p , is low, time segments with pressure below average ($p(t) < \bar{p}$) and time segments with pressure above average ($p(t) > \bar{p}$) alternate fairly regularly: the red dots plotted at $R(t)/R_{eq} = 1$ are grouped in short time segments, and are followed by short segments where no dots are reported. A key consequence of short time segments with pressure below average ($p(t) < \bar{p}$) is that the bubble radius cannot attain large increments (see the black time-series in Panel 7a).

In contrast, for high values of the correlation time, time segments with pressure below average ($p(t) < \bar{p}$) persist for long time and are followed by long-lasting time intervals with pressure values above average ($p(t) > \bar{p}$): Fig. 7b shows, indeed, that long sets of red dots alternate with long sets without dots. In this case, time segments in which the pressure is below average ($p(t) < \bar{p}$) last so long that very large radius increments are attained (e.g., see the strong growth of $R(t)$ occurring at $t/T_n \approx 255$ in the second time segments of Panel 7b). *Vice versa*, when the condition $p(t) > \bar{p}$ is restored, the bubble collapses. As explained in Section III B, the occurrence of these phases of remarkable radius expansion contributes to trigger the irregularity of $R(t)$.

Appendix B: Statistical analysis – pdf and cdf

In Fig. 8, we show some exemplifying cases, in order to discuss the effect of σ_p/\bar{p} and τ_p on the the probability density function (pdf) and cumulative density function (cdf) of the bubble radius. To this end, it is useful to define a benchmark case (see the thick black lines). We selected the benchmark correlation time $\tau_p = T_n$. This choice was based on past studies that considered sinusoidal pressure oscillations. These studies found that complex dynamics occurs when the period of the sinusoidal forcing is equal to the natural oscillation period of the bubble [13, 25, 41, 42, 45]. Therefore, we expect bubbles to exhibit interesting dynamics when the correlation time of the noise signal is equal to the natural oscillation period of the bubble. On the other hand, we selected the benchmark noise intensity $\sigma_p/\bar{p} = 0.60$. This choice was based on the results reported in Figs. 2a,b, showing chaotic dynamics of the bubble radius in the σ_p/\bar{p} -range [0.30,1.10]. We wanted to focus on bubble exhibiting a chaotic behavior, so we chose a value of noise intensity in this chaos range.

The noise intensity (in terms of σ_p/\bar{p}) was then altered, keeping $\tau_p = T_n$ (broken lines in Panels 8a,c,d). Finally, τ_p was also changed while σ_p/\bar{p} was kept at its benchmark value (broken lines in Panel 8b). The dotted (dash-dot) lines refer to a parameter higher (lower) than the benchmark value.

Irrespectively of the noise parameters $\{\sigma_p/\bar{p}, \tau_p\}$, the quantity R/R_{eq} exhibits a unimodal pdf (Panels 8a,b), whose shape, though, depends significantly on the noise intensity (Fig. 8a). In particular, increments of σ_p/\bar{p}

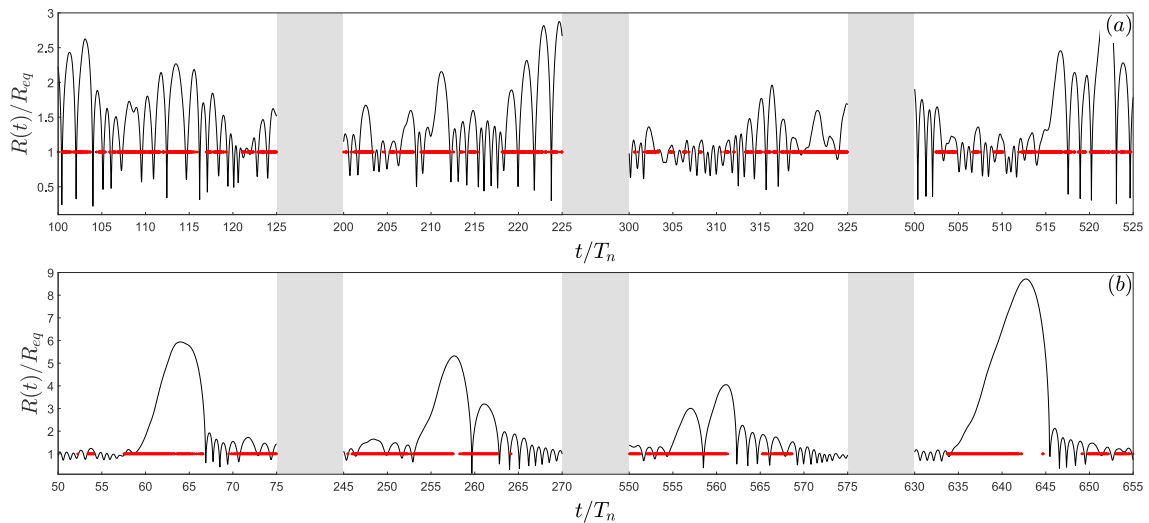


FIG. 7. Time-series of $R(t)/R_{eq}$ in four relevant time segments in the case of $\tau_p = T_n/2$ (a) and $\tau_p = 2T_n$ (b). In both cases $\sigma_p/\bar{p}=0.70$. The red dots plotted at $R(t)/R_{eq} = 1$ mark the instants when $p(t) < \bar{p}$, and should not be confused with the dynamics of $R(t)/R_{eq}$ reported by the black line. Panels (a) and (b) report different ranges in the vertical axis.

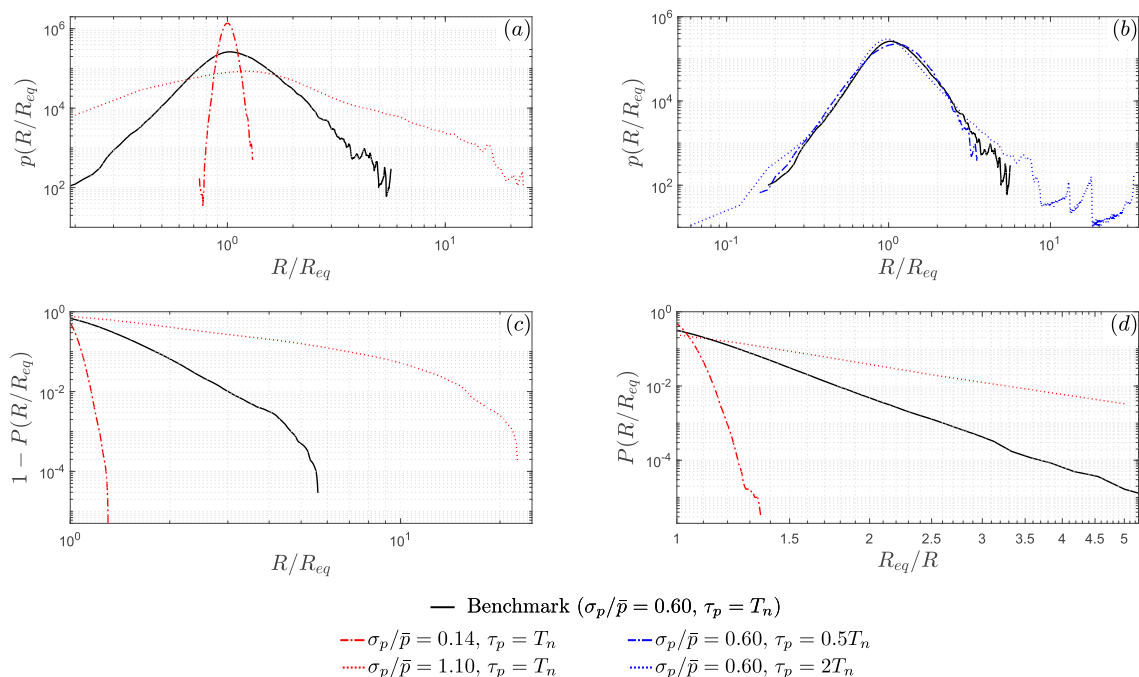


FIG. 8. (a, b) Probability density function of the metric R/R_{eq} . (c) Complementary cumulative distribution function of R/R_{eq} evaluated for $R/R_{eq} > 1$ (right tail of the distribution). (d) Cumulative distribution function of R/R_{eq} evaluated for $R/R_{eq} < 1$ (left tail of the distribution), note that the horizontal axis reports R_{eq}/R and not R/R_{eq} as in Panel (c).

776 induce the reduction of the peak height, the fattening of 785
 777 the tails, more asymmetrical pdfs, and the increment of 786
 778 the mode. Differently from σ_p/\bar{p} , changes of τ_p induce 787
 779 less relevant effects (Fig. 8b). No changes of the peak 788
 780 height, of the mode of the pdf, and of the symmetry of 789
 781 the curves are in fact observed. The only relevant effect is 790
 782 a slight expansion of the distribution range toward higher 791
 783 values of R/R_{eq} , which occurs when the correlation time 792
 784 increases (see the right tail of the dotted curve in Fig. 793

8b).
 The tails are better described by the cumulative dis-
 tribution functions. A complementary distribution is
 adopted to analyze the right tail, (see Fig. 8c). In order
 to focus on the left tail, the cumulative distribution is
 evaluated (see Fig. 8d). Increments of the noise intensity
 mainly induce a fattening of the tails and an increment
 of the range (see Panels 8c, d). In the right tail, the
 range increases from 2 to 20 when σ_p/\bar{p} increases from

0.14 to 1.10. Moreover, the frequency of occurrence of a given R/R_{eq} changes of orders of magnitude, for the same increment of σ_p/\bar{p} . The same behavior is observed in the left tail: the minimum value attained by R/R_{eq} reduces from 0.6 to 0.2, when σ_p/\bar{p} increases from 0.14 to 0.60. Interestingly, the further increment of σ_p/\bar{p} from 0.60 to 1.1 does not lead to a reduction of R/R_{eq} . The distribution does not extend beyond 0.2 ($R_{eq}/R = 5$ in Fig. 8d). However, the frequency of occurrence of this extreme value increases of more than one order of magnitude. Finally, as surmised from the analysis of Panels 8a and 8b, the pdfs of R/R_{eq} display asymmetry. In fact the right tail is always characterized by a power-law behavior (linear in the log-log diagrams of Panel 8c) for low values of R/R_{eq} followed by a cut-off. On the contrary, the left

tail is always approximately linear (Panel 8d).

Appendix C: Numerical Details

In order to evaluate the response of a gas bubble to a pressure forcing, the numerical integration of (1) is required. To this aim, the dimensional Eq. (1) is firstly made dimensionless adopting the length scale R_{eq} (i.e., the bubble radius in equilibrium conditions) and the time scale T_n (i.e., the period of bubble free oscillations, see Eq. 5). Secondly, the second-order differential dimensionless equation is transformed in the system of two first-order differential dimensionless equations

$$\begin{cases} \tilde{y}_2 = \frac{d\tilde{y}_1}{d\tilde{t}} \\ \frac{d\tilde{y}_2}{d\tilde{t}} = \frac{\frac{p_w - p(\tilde{t})}{\mathcal{P}\tilde{y}_1} + \frac{\tilde{y}_2}{\mathcal{N}\tilde{y}_1} [p_G(1 - 3k) - p(\tilde{t}) + p_v] - \frac{1}{\mathcal{N}} \frac{dp(\tilde{t})}{d\tilde{t}} - \left(1 - \frac{\text{Ma}}{3}\right) \frac{3\tilde{y}_2^2}{2\tilde{y}_1}}{1 - \text{Ma} + \frac{4\mu}{\mathcal{M}\tilde{y}_1}}, \end{cases} \quad (\text{C1})$$

where tilde denotes dimensionless quantities, $\tilde{y}_1 = R/R_{eq}$, $\text{Ma} = \tilde{y}_2 R_{eq}/cT_n$ is the Mach number, $\mathcal{P} = \rho R_{eq}^2/T_n^2$, $\mathcal{M} = c\rho/R_{eq}$, and $\mathcal{N} = \mathcal{M}/T_n$. Finally, p_w and p_G can be expressed, according to (2-4), in terms of \tilde{y}_1 and \tilde{y}_2 as

$$p_G = \left(\frac{2S}{R_{eq}} - p_v + \bar{p} \right) \left(\frac{1}{\tilde{y}_1} \right)^{3k}, \quad (\text{C2})$$

$$p_w = p_G + p_v - \frac{2S}{R_{eq}\tilde{y}_1} - \frac{4\mu\tilde{y}_2}{T_n\tilde{y}_1}. \quad (\text{C3})$$

The system of equation (C1) was numerically solved by an explicit Runge-Kutta approach by using the Dormand-Prince pair [68].

In order to select the appropriate time-step for numerical integration, a sensitivity analysis about this parameter was performed. The test case was a gas bubble with $R_{eq} = 5 \mu\text{m}$, $R(0)/R_{eq} = 2$ and $\dot{R}(0) = 0$ in a uniform pressure field. Three time steps ($\Delta t = [10^{-7}, 10^{-8}, 10^{-9}]$ s) were tested in the numerical simulations of the bubble dynamics (see Fig. 9). Panel 9a shows that $\Delta t = 10^{-9}$ s and $\Delta t = 10^{-8}$ s led to a bubble response (in terms of $R(t)$) indistinguishable, while $\Delta t = 10^{-7}$ s led to a less precise simulation of the system dynamics. To better quantify the quality of the numerical integrations, we evaluated the relative error

$$\varepsilon_R(t) = \frac{\|R(\Delta t, t) - R_{\text{REF}}(t)\|}{R_{\text{REF}}(t)}, \quad (\text{C4})$$

where $R(\Delta t, t)$ is the bubble radius at the instant t evaluated with a numerical simulation in which the time-step

Δt was adopted. The term $R_{\text{REF}}(t)$ is the “exact” reference value. In this case, we adopted $R_{\text{REF}}(t) = R(\Delta t = 10^{-9}, t)$. The time step $\Delta t = 10^{-8}$ s was found suitable for the numerical integrations, as the maximum error $\varepsilon \sim 0.02$ was attained (see Fig. 9b).

In order to guarantee that the statistical description of a stochastic process was significant, two tests were performed. The first test concerns the duration of the considered stochastic process. In particular, we studied whether the same statistical values were obtained, irrespectively of the length of the analyzed time-series. Fig. 10 reports the behavior of two statistical metrics as a function of σ_p/\bar{p} , as already discussed in Fig. 4. Each statistical index was evaluated from four time-series, $R(t)$, characterized by different durations, T . It can be observed that simulations carried out with $T > 2000T_n$ lead to curves characterized by the same behavior. The duration $T = 4000T_n$ was therefore deemed appropriate for the statistical analysis of the stochastic bubble dynamics.

The second test was to verify the independence of the results from a single realization. Namely, whether different stochastic realizations of the process lead to the same statistical indexes. Fig. 11 reports two statistical parameters of Fig. 4. Each statistical index was evaluated with seven time-series, $R(t)$, characterized by a different pressure forcing. Each pressure time-series was characterized by the same statistics (σ_p, τ_p), but a different set of random numbers (see Eq. 7) was adopted to introduce randomness. It can be observed that all simulations give curves characterized by the same behavior. Moreover, the mean value, the standard deviation and the kurtosis

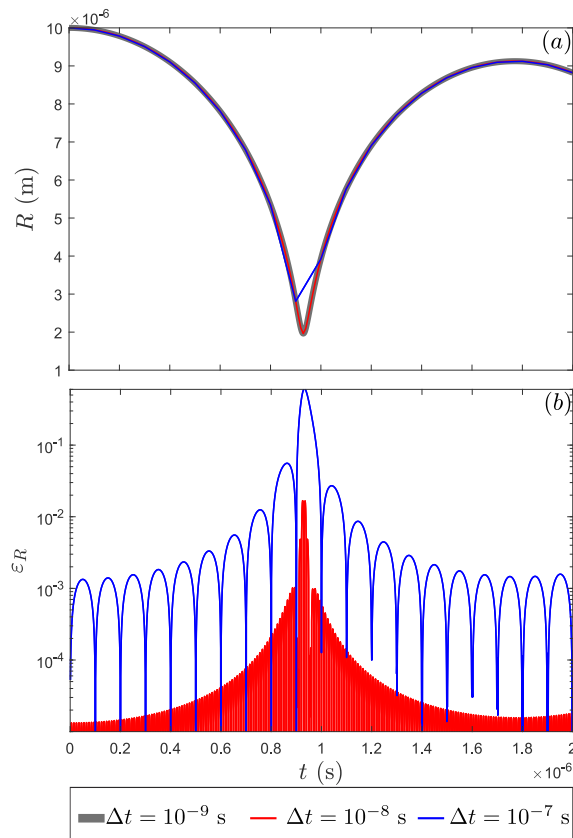


FIG. 9. (a) Example of curves $R(\Delta t, t)$ numerically computed adopting different time-steps Δt . (b) Relative error $\varepsilon_R(t)$ occurring in the numerical computation performed with different time-steps. The relative error is evaluated considering the curve computed with $\Delta t = 10^{-9}$ s the exact reference. The initial conditions are $R(0)/R_{eq} = 2$ and $\dot{R}(0) = 0$. The pressure field is uniform.

of the bubble radius were basically the same.

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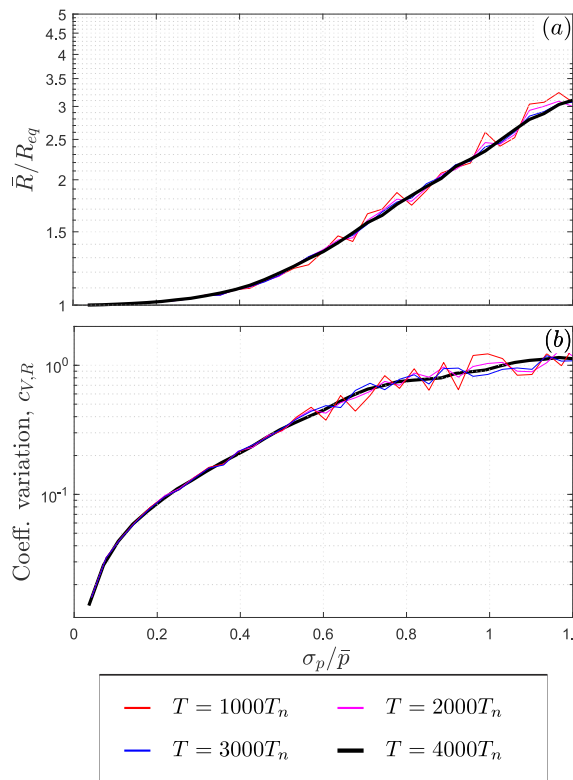


FIG. 10. Effect of the duration T of the simulation on the statistical metrics that describe the time-series $R(t)$. Similar to Fig. 4, two statistical parameters and their dependence on σ_p/\bar{p} are considered. The different curves were evaluated considering different length of the simulation. The parameter $\tau_p = T_n$ is adopted.

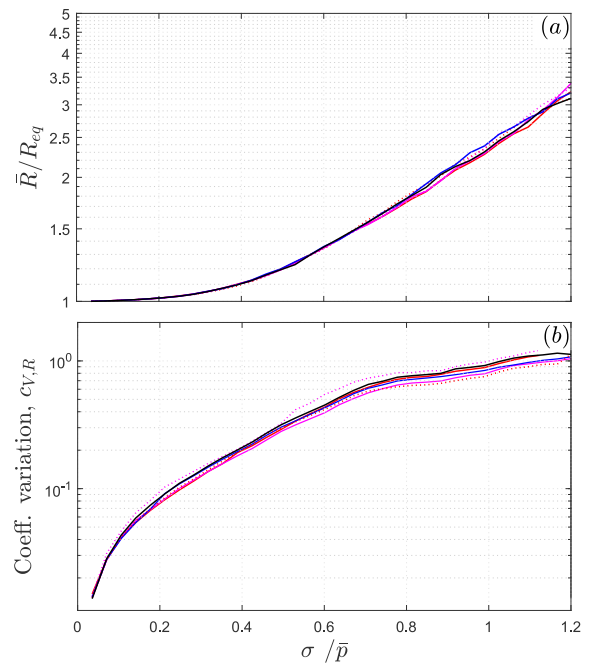


FIG. 11. Effect of different realizations of the stochastic process on the statistical parameters that describe the time-series $R(t)$. Similar to Fig. 4, two statistical parameters and their dependence on σ_p/\bar{p} are considered. The different curves were evaluated with the same noise intensities and correlation times, but with a different set of random numbers. The parameter $\tau_p = T_n$ is adopted.

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