A Riemann-Hilbert Approach to the Kissing Polynomials

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I dedicate this thesis to my mother, father, and brother, for all of their love and support, and to my grandfather, N.P. Jeganathan, for inspiring me to pursue mathematics.

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgments. More details on the originality of the work presented in this thesis is given on a chapter by chapter basis in Section 1.3.

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This thesis is the result of four years of work in Cambridge, and none of the results that follow would have been possible without the help of many people who I have met along the way.

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Abstract

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Motivated by the numerical treatment of highly oscillatory integrals, this thesis studies a family of polynomials known as the Kissing Polynomials through Riemann-Hilbert techniques. The Kissing Polynomials are a family of non-Hermitian orthogonal polynomials, which are orthogonal with respect to the complex weight function $\exp(i\omega z)$ over the interval $[-1,1]$, where $\omega > 0$. Although they have already been used to derive complex quadrature rules, there remain two main questions which this thesis addresses. The first is the existence of such polynomials; the second is the behavior of these polynomials throughout the complex plane.

The first two chapters of this thesis provide the necessary background needed for the main results presented in the later chapters. In the first chapter, the connection between the numerical integration of highly oscillatory integrals and the Kissing Polynomials is established. Furthermore, we present the theory of non-Hermitian orthogonal polynomials and provide a more detailed description of the results in this thesis. The second chapter is a review on the formulation of the Kissing Polynomials as a solution to a matrix valued Riemann-Hilbert problem. This formulation is crucial to establishing both the existence of the Kissing Polynomials and its properties throughout the complex plane. Moreover, we also provide an overview of the powerful non-commutative steepest descent technique developed by Deift and Zhou in the mid 1990s used to compute the asymptotics for oscillatory Riemann-Hilbert problems, which will be used extensively in Chapters 4 and 5.

In Chapter 3, we utilize the Riemann-Hilbert approach of Fokas, Its, and Kitaev to establish our first main result: the existence of the even degree Kissing polynomials for all values of $\omega > 0$. First, we use the Riemann-Hilbert problem to show that the Kissing Polynomials satisfy a certain linear ordinary differential equation. Then, using standard results on differential equations, along with previous results on the Kissing Polynomials found in the literature, we are able to provide the desired result.

In Chapter 4, we turn our attention to the behavior of the Kissing Polynomials as both the degree n and parameter ω become large. To achieve this, we formulate this problem in terms of varying-weight Kissing polynomials, whose asymptotics can be handled with the Deift-Zhou steepest descent analysis. Now, the weight function depends now on n, the degree of the underlying polynomial. We are able to

provide uniform asymptotics of the Kissing Polynomials as both n and ω go to infinity at a linear rate such that the ratio $\omega/n > t_c$, where t_c is a to be specified critical value.

In Chapter 5, we generalize the results of Chapter 4 and study polynomials which are orthogonal with respect to the varying, complex weight, $\exp(nsz)$, over the interval $[-1,1]$, where now $s \in \mathbb{C}$. We will see that there are curves in the s-plane, called breaking curves, which separate regions corresponding to differing asymptotic behavior of the polynomials. In this chapter, we provide the large n behavior of the recurrence coefficients associated to these polynomials. Finally, we also study the behavior of these recurrence coefficients as the parameter s approaches a breaking curve in a specified double scaling limit.

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Chapter 1

Introduction

1.1 Numerical Evaluation of Highly Oscillatory Integrals

The motivation for this thesis lies in the field of numerical analysis, in particular, the numerical treatment of highly oscillatory integrals. In this section we briefly review Gaussian quadrature, numerical steepest descent, and complex Gaussian quadrature, following the reference [33]. We are concerned with evaluating integrals of the form

$$\int_a^b g(x)w(x)\,dx, \tag{1.1}$$

where for ease of exposition, we assume w is positive, continuous, and integrable on (a,b), and g is real analytic. Above, we may potentially take $a = -\infty$ or $b = \infty$. We further assume that all moments,

$$\mu_n := \int_a^b x^n w(x)\,dx, \qquad n \in \mathbb{N}, \tag{1.2}$$

are finite.

A classical method for handling such integrals is called Gaussian quadrature, which at its heart relies on the theory of orthogonal polynomials. Historically, the field of orthogonal polynomials started in number theory with continued fractions, but grew to touch many areas of mathematics, from numerical analysis and approximation theory to probability theory and theoretical physics. Already by the turn of the twentieth century, much of the classical theory of orthogonal polynomials had been developed; standard references include [27, 85].

We say that $p_n(x)$ is the monic, degree n orthogonal polynomial with respect to the weight w if it is a monic polynomial of degree n which satisfies

$$\int_a^b p_n(x)x^k w(x)\,dx = 0, \tag{1.3a}$$

when $0 \leq k < n$, and

$$\int_a^b p_n(x) x^n w(x)\, dx = \frac{1}{\kappa_n^2} \neq 0. \tag{1.3b}$$

Sometimes, we may also consider the orthonormal polynomial $\pi_n(x) = \kappa_n p_n(x)$, which satisfies

$$\int_a^b \pi_n(x)\pi_m(x) w(x)\, dx = \delta_{n,m}. \tag{1.4}$$

Finally, if the monic polynomial of degree n with respect to a weight w exists for all $n \in \mathbb{N}$, we will refer to the sequence $\{p_n(x)\}_{n=0}^{\infty}$ as the orthogonal polynomial sequence with respect to w.

We will give more details on orthogonal polynomials in Section 1.2 below, but for now we give the following theorem on the zeros of orthogonal polynomials used in traditional numerical integration techniques.

Theorem 1.1 ([27, p. 27])**.** *Let w be positive and integrable on the interval (a,b). Then there exists a unique monic orthogonal polynomial sequence with respect to w on (a,b) whose zeros are all real, simple, and contained in (a,b).*

Some familiar examples of orthogonal polynomial sequences are the *Legendre polynomials*, which correspond to $w(x) = 1$ over the interval $(-1,1)$, and the *Laguerre polynomials*, which correspond to the weight $w(x) = e^{-x}$ over $(0,\infty)$.

Recall that we are interested in numerically evaluating (1.1). To do so, we want to choose nodes, $\{x_k\}_{k=1}^n$, and weights, $\{w_k\}_{k=1}^n$, in such a way that we may efficiently approximate the integral in question via a weighted sum as

$$\int_a^b g(x) w(x)\, dx \approx \sum_{k=1}^n w_k g(x_k). \tag{1.5}$$

The important question is how to choose both the nodes and weights, and the answer to this question is provided to us via Gaussian quadrature.

Theorem 1.2 (Gaussian Quadrature [57, p. 497])**.** *Let w be positive and integrable over the interval (a,b) and let p_n be the monic orthogonal polynomial of degree n with respect to the weight w. Moreover, assume that $g \in C^{2n}([a,b])$ and let x_k, $k = 1,\ldots,n$, be the n zeros of p_n. If we set*

$$w_k = \int_a^b \left(\prod_{\substack{i=0 \\ i\neq k}}^n \frac{x - x_i}{x_k - x_i} \right) w(x)\, dx, \tag{1.6}$$

then there exists some $\xi \in (a,b)$ such that

$$\int_a^b g(x) w(x)\, dx - \sum_{k=1}^n w_k g(x_k) = \frac{g^{(2n)}(\xi)}{(2n)!} \int_a^b p_n^2(x) w(x)\, dx. \tag{1.7}$$

The proof of Theorem 1.2 relies on the orthogonality of the polynomials in (1.3), the Euclidean algorithm for factoring polynomials, and the error incurred when approximating functions with a polynomial of degree $2n-1$. In particular, the fact that the zeros of p_n are all real and lie in the interval (a,b), while nice mathematically, is not crucial to the proof of Theorem 1.2. By virtue of (1.7), we see that Gaussian quadrature rules are exact for all functions g which are polynomials of degree less than or equal to $2n-1$. Moreover, as any rule of the form $\sum_{k=1}^{n} w_k g(y_k)$ does not integrate (1.1) exactly when $g(x) = \prod_{k=1}^{n}(x-y_k)^2$, we see that Gaussian quadrature rules are optimal in the sense that they obtain the highest polynomial order possible.

Next, we consider the case where the integrand is highly oscillatory. More precisely, consider the following integral

$$I_\omega[f] := \int_{-1}^{1} f(x)e^{i\omega x}\,dx, \qquad \omega > 0. \tag{1.8}$$

Again for ease of exposition, we take f to be an entire function which is real valued on the real line. One approach to treating this integral numerically would be to take $g_1(x) = f(x)\cos(\omega x)$ and $g_2(x) = f(x)\sin(\omega x)$. By taking real and imaginary parts, we could write $I_\omega[f] = I_1 + iI_2$, where

$$I_1 = \int_{-1}^{1} g_1(x)w(x)\,dx, \qquad I_2 = \int_{-1}^{1} g_2(x)w(x)\,dx, \tag{1.9}$$

and $w(x) = 1$. Now, both I_1 and I_2 meet the assumptions of Theorem 1.2, leading to *Gauss-Legendre quadrature*, where the nodes used in the quadrature rule are the zeros of the Legendre polynomials. However, in light of the error estimate (1.7), we see that if ω is very large, one must take n very large so as to offset the growth of $g_{1,2}^{(2n)}(\xi)$. Indeed, this has led to many developments in computing zeros of classical orthogonal polynomials of very large degree, see for instance [50].

Another approach, extensively detailed in [33], is to throw away the intuition behind classical methods such as Gaussian quadrature, and instead exploit certain structural features of the highly oscillatory integral (1.8). One such method is *numerical steepest descent*, which we briefly describe below. As f is entire, the integral (1.8) remains unchanged if we alter the path of integration from -1 to 1. Indeed, the whole idea behind numerical steepest descent is to deform the contour of integration so as transform high oscillation into exponential decay, a theme that will be prevalent throughout this thesis. As shown in [33, Section 5], provided f does not possess significant growth at infinity, we may write

$$I_\omega[f] = ie^{-i\omega}\int_0^\infty f(-1+ip)e^{-\omega p}\,dp - ie^{i\omega}\int_0^\infty f(1+ip)e^{-\omega p}\,dp. \tag{1.10}$$

Note that this change of contour has turned our original oscillatory integral into a sum of two integrals which can be treated, after a change of variable, via Theorem 1.2 with $w(x) = e^{-x}$ over the interval $(0,\infty)$. More precisely, when using numerical steepest descent, we approximate

$$I_\omega[f] \approx \frac{ie^{-i\omega}}{\omega}\sum_{k=1}^{n} w_k f\left(-1+\frac{it_k}{\omega}\right) - \frac{ie^{i\omega}}{\omega}\sum_{k=1}^{n} w_k f\left(1+\frac{it_k}{\omega}\right), \tag{1.11}$$

where the w_k and t_k are the weights and nodes, respectively, of the classical Gauss-Laguerre quadrature rule. Here, the Laguerre polynomials are orthogonal with respect to the weight $w(x) = e^{-x}$ on $(0, \infty)$. As explained in [33], this method has error which is $\mathscr{O}\left(\omega^{-2n-1}\right)$ for large ω, which is the optimal asymptotic order one can expect when using $2n$ function evaluations. It is quite counterintuitive that methods such as numerical steepest descent perform better the larger ω gets, as this is precisely where the classical methods such as Gaussian quadrature fail so spectacularly. However, it should be noted that numerical steepest descent is truly an asymptotic method. As $\omega \to 0$, we see by (1.11) that the nodes used for steepest descent tend towards infinity in the complex plane, and as such numerical steepest descent performs much worse than traditional methods in the regime of small ω.

At present, it seems that the treatment of (1.8) falls into separate and distinct camps. On the one hand, if ω is small, we can use the tried and trusted Gaussian quadrature method, which as we have seen has optimal polynomial order. On the other hand, when ω is large, we should instead opt for a method such as numerical steepest descent, which attains optimal asymptotic order.

In an attempt to address this apparent schism between the highly oscillatory and non-oscillatory regimes of the integral in (1.8), the authors of [5] proposed a quadrature rule which used Gaussian quadrature directly on the weight function $w(x) = e^{i\omega x}$ over the interval $[-1, 1]$. In particular, the nodes of the $2n$-point quadrature rule are the zeros of the monic, degree $2n$ polynomial $p_{2n}(z;\omega)$, which satisfies the following orthogonality conditions

$$\int_{-1}^{1} p_{2n}(z;\omega) z^k e^{i\omega z}\, dz = \begin{cases} 0 & 0 \le k < 2n, \\ \frac{1}{\kappa_{2n}^2(\omega)}, & k = 2n. \end{cases} \tag{1.12}$$

More information on methods such as this, which are based on orthogonal polynomials with respect to complex valued weight functions, can be found in the chapter on Complex Gaussian Quadrature in [33, Chapter 6].

For reasons which will become clear shortly, the authors of [26] called the family of polynomials $\{p_n(z;\omega)\}$ the Kissing polynomials. Note that the weight function is no longer positive, nor even real valued, and as such we no longer have any information of the location of the zeros of the Kissing polynomials. Indeed, even the existence of polynomials satisfying (1.12) can no longer be taken for granted. The zeros of the Kissing polynomials for degree $n = 4, 5$ are plotted in Figure 1.1. Figure 1.1 already highlights a difference with Theorem 1.1, in that the zeros of the Kissing polynomials appear to have non-zero imaginary part for all $\omega > 0$. Moreover, we note that the zero trajectories of the 4^{th} and 5^{th} degree Kissing polynomials coincide for a discrete set of frequencies - at these particular frequencies, the trajectories "kiss", and it is this phenomenon which gives the Kissing polynomials their name. We will see in Chapter 3 that these kissing points correspond to values of ω for which the odd degree polynomial fails to exist. This lack of existence, and seeming lack of control on the location of the zeros of the Kissing polynomials, already provides some hints at the obstacles faced when trying to study the Kissing polynomials.

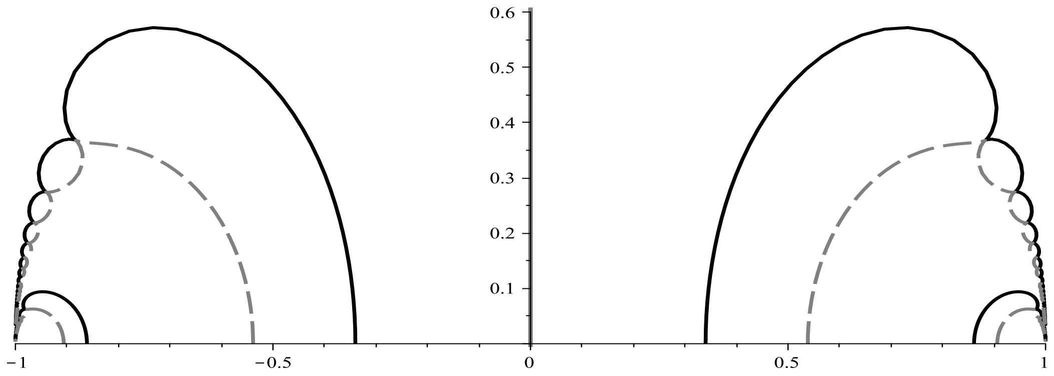


Figure 1.1: Trajectories of the zeros of p_4 (dark, solid) and p_5 (grey, dashed), as ω ranges from 0 to ∞.

Despite the fact that we are considering orthogonality with respect to complex weights, there are some strong theoretical benefits to using the complex Gaussian Quadrature method described above. First, it is easy to note that as $\omega \to 0^+$, the zeros of the Kissing polynomials converge to the zeros of the Legendre polynomials (which we recall are orthogonal with respect to the weight $w(x) = 1$ over the interval $[-1,1]$). On the other hand, the paper [5] provides some insight on the behavior of the Complex Gaussian Quadrature rule based on the Kissing polynomials. To state the theorem, let f be an analytic function and consider the Complex Gaussian Quadrature rule with $2n$ quadrature nodes based on the Kissing polynomials (provided the polynomial exists). That is, the quadrature nodes are given by the $2n$ complex zeros of $p_{2n}(z;\omega)$ defined in (1.12), which we denote $\{z_j\}_{j=1}^{2n}$. Using Theorem 4.1 of [5], if the zeros $\{z_k\}$ can be split into two groups, $\{z_j^1\}_{j=1}^n$ and $\{z_j^2\}_{j=1}^n$, where

$$z_j^1 = -1 + \mathscr{O}\left(\frac{1}{\omega}\right), \qquad z_j^2 = 1 + \mathscr{O}\left(\frac{1}{\omega}\right), \qquad \omega \to \infty, \tag{1.13}$$

then the quadrature rule possesses the following asymptotic order

$$\sum_{j=1}^{2n} w_j f(z_j) - \int_{-1}^{1} f(z) e^{i\omega z}\, dz = \mathscr{O}\left(\omega^{-2n-1}\right), \qquad \omega \to \infty. \tag{1.14}$$

Certainly, numerical experiments such as Figure 1.1 indicate that (1.13) holds. In fact, in [26] it is shown that the zeros of the Kissing polynomials can be written as

$$z_j^1 = -1 + \frac{it_j}{\omega} + \mathscr{O}\left(\frac{1}{\omega^2}\right), \qquad z_j^2 = 1 + \frac{it_j}{\omega} + \mathscr{O}\left(\frac{1}{\omega^2}\right), \qquad \omega \to \infty, \tag{1.15}$$

for $j = 1,\ldots,n$, where t_j is a zero of the Laguerre polynomial of degree n. Comparing the above to (1.11), it seems that, at least in the case of the Kissing polynomials, that numerical steepest descent is a large ω approximation to Complex Gaussian Quadrature! In this sense, the Complex Gaussian

Quadrature rule with an even number of nodes based on the Kissing polynomials nicely interpolates between steepest descent as $\omega \to \infty$ and traditional Gaussian quadrature as $\omega \to 0$.

The authors of [5] ended their work with a series of open questions, which will serve as the motivation for the present thesis. The first concerns the existence of the even degree Kissing polynomials for all $\omega > 0$, which in light of the results presented above is of crucial importance in the design of Complex Gaussian Quadrature rules based on the Kissing polynomials. The second is the behavior of the Kissing polynomials as the degree n tends to infinity. Before addressing these and other questions related to the Kissing polynomials, it is necessary to present the basic background on orthogonal polynomials.

1.2 Background on Orthogonal Polynomials

The theory of orthogonal polynomials has a long history and has been well documented in the literature. Of the many excellent works, we refer the reader to [27, 34, 85], and we will present the basic theory below.

To start with, we will consider orthogonality with respect to positive weights over the real line. As in the previous section, take a, b as the endpoints of integration and consider a positive, integrable function w defined on the interval $[a,b]$, such that all the moments (1.2) are finite. Associated with the weight function w, we consider the Hilbert space $L^2([a,b],w)$, equipped with the inner product,

$$\langle f, g\rangle = \int_a^b f(x)g(x)w(x)\,dx. \tag{1.16}$$

Using the Gram-Schmidt procedure on the basis of monomials with the norm induced from the above inner product, one may generate a basis of polynomials $\{\pi_n(x)\}$ which satisfy (1.4).

In this thesis, we prefer to work with the monic orthogonal polynomials p_n, as opposed to the orthonormal polynomials π_n, which satisfy

$$\langle p_n, p_m\rangle = \frac{\delta_{n,m}}{\kappa_n^2} \tag{1.17}$$

where κ_n is the leading coefficient of π_n.

One important consequence of the orthogonality (1.17) is that the polynomials p_n satisfy the following three-term recurrence relation

$$xp_n(x) = p_{n+1}(x) + \alpha_n p_n(x) + \beta_n p_{n-1}(x), \tag{1.18}$$

where the recurrence coefficients can be expressed as

$$\alpha_n = \frac{\langle xp_n, p_n\rangle}{\langle p_n, p_n\rangle}, \qquad \beta_n = \frac{\langle p_n, p_n\rangle}{\langle p_{n-1}, p_{n-1}\rangle}. \tag{1.19}$$

Typically, one starts with the recursion (1.18) with the conditions $p_{-1}(x) = 0$, $p_0(x) = 1$.

Orthogonal polynomials with respect to positive weights form the subject matter of many of the standard references on orthogonal polynomials, and there are many beautiful and important results on the analytic theory of these polynomials. One such result is that the zeros of these orthogonal polynomials interlace, whose proof is given in [27, Theorem 5.3]. To state this separation theorem, we first denote the k zeros of $p_k(x)$ as $\{x_{i,k}\}_{i=1}^k$, ordered so that $x_{i,k} < x_{i+1,k}$ for $i = 1,\ldots,k-1$. Then, we have the following theorem on the interlacing of the zeros of polynomials of consecutive degrees.

Theorem 1.3 (Interlacing of Zeros [27, Theorem 5.3])**.** *The zeros of $p_n(x)$ and $p_{n+1}(x)$ mutually separate each other. That is,*

$$x_{n+1,i} < x_{n,i} < x_{n+1,i+1}, \qquad i = 1,\ldots,n.$$

We now drop the assumption that w is a positive weight function and instead consider the situation where w is complex valued. For ease of exposition, we assume that w is entire, putting us in the realm of complex orthogonality as discussed in Section 1.2. Now, the bilinear form

$$\langle f,g\rangle = \int_a^b f(x)g(x)w(x)\,dx, \tag{1.20}$$

no longer constitutes an inner product. Indeed, it is possible for $\langle f,f\rangle = 0$ without $f = 0$. However, the relation $\langle f,zg\rangle = \langle zf,g\rangle$ continues to hold. Therefore, the algebraic aspects of these complex orthogonal polynomials, such as the three term recurrence relation (1.18) and Gaussian Quadrature, as in Theorem 1.2, continue to hold even though we are working in the complex setting, provided the appropriate polynomials exist. However, the analytic theory of orthogonal polynomials no longer holds, so questions of existence and results on the zeros of complex orthogonal polynomials, such as Theorems 1.1 and 1.3, can no longer be taken for granted.

In this case, the existence of the orthogonal polynomial sequence is equivalent to the non-vanishing of the related Hankel determinants. To state this precisely, we first recall that the moments of w are given by

$$\mu_n = \int_a^b x^n w(x)\,dx < \infty. \tag{1.21}$$

We form the following *Hankel matrix*,

$$H_n = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n}, \end{pmatrix} \tag{1.22}$$

and consider its determinant

$$h_n = \det H_n. \tag{1.23}$$

We then have the following theorem on the existence of the orthogonal polynomials with respect to the weight w.

Theorem 1.4 ([27, Theorem 3.1]). *A necessary and sufficient condition for the existence of an orthogonal polynomial sequence with respect to w is that*

$$h_n \neq 0, \qquad n = 0, 1, \ldots \tag{1.24}$$

We remark here that the Hankel determinants can be used to provide alternative formulas for the recurrence coefficients in (1.18). Specifically, we may write

$$\alpha_n = \frac{\tilde{h}_n}{h_n} - \frac{\tilde{h}_{n-1}}{h_{n-1}}, \qquad \beta_n = \frac{h_n h_{n-2}}{h_{n-1}^2}, \tag{1.25}$$

where

$$\tilde{h}_n = \det \begin{pmatrix} \mu_0 & \cdots & \mu_{n-1} & \mu_{n+1} \\ \mu_1 & \cdots & \mu_n & \mu_{n+2} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_n & \cdots & \mu_{2n-1} & \mu_{2n+1} \end{pmatrix}. \tag{1.26}$$

Finally, the monic orthogonal polynomials themselves may be written in terms of determinants [51, Chapter 2] as

$$p_n(x) = \frac{1}{h_{n-1}} \det \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} & 1 \\ \mu_1 & \mu_2 & \cdots & \mu_n & x \\ \vdots & \vdots & & \vdots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} & x^n \end{pmatrix}. \tag{1.27}$$

1.3 Outline of Thesis and Relation to Other Works

In this section, we outline the main findings of this thesis. Many of the results in this thesis were the result of collaboration with other mathematicians. Therefore, in addition to providing the main results of this thesis, this section also highlights the extent to which the results of this thesis came from collaboration with others. We will also emphasize in this section which chapters and results do *not* represent the original work of this author. Moreover, we also hope that this section puts the results of this thesis in context with many of the other works on the Riemann-Hilbert approach for orthogonal polynomials.

Chapter 2 is an introductory chapter and outlines both the Riemann-Hilbert problem developed by Fokas, Its, and Kitaev [46] and its method of asymptotic solution by Deift-Zhou steepest descent, originally developed in [35, 36]. We show in this chapter how one may perform the process of steepest descent provided one can construct an appropriate *modified external field*, or h-function. Both the Riemann-Hilbert problem and Deift-Zhou steepest descent have now appeared in the literature multiple times; in particular, we refer the reader to works where the authors considered exponential weights on the real line [17, 34, 39, 40, 58], real weights over a subset of the real line [65, 60, 91], and non-Hermitian orthogonality as is considered in this thesis [7, 11–13, 21, 31, 32, 59, 71]. There are even extensions of the Riemann-Hilbert formulation to various forms of orthogonality in other guises that we will not touch on in this thesis [3, 6, 14, 20, 61, 67, 90]. Although there are plentiful references on both the Fokas-Its-Kitaev Riemann-Hilbert problem and on Deift-Zhou steepest descent, this material plays a fundamental role in this thesis. As such, this chapter contains no original material and is intended as review. Furthermore, we mainly follow the guides laid out in [13, 16, 34] to present this material.

Chapter 3 constitutes the first original work of the author of this thesis. The main goal of this chapter is to prove the existence of the even degree Kissing Polynomials. The material in this chapter also forms a small section of the work [26], which was written together with Alfredo Deaño, Daan Huybrechs, and Arieh Iserles, and is to be submitted for publication shortly. It should be noted that the work [26] contains significantly more material than what is written in Chapter 2, as this author's contribution to that work was just the proof of existence as explained in this thesis. However, many of the results obtained by the three authors of [26] listed above play a fundamental role in proving the existence of the even degree Kissing polynomials, and these results are stated in Section 3.1. Where necessary proofs of these results are given, but it should be stressed that the results of Section 3.1 do not constitute original work of the author of this thesis. All material in this chapter after Section3.1, however, are original work of the author.

To prove the existence of the even degree Kissing polynomials, we first show that the polynomials satisfy a second order differential equation in the variable z. We are able to show this by using the Riemann-Hilbert problem as explained in Chapter 2 and using techniques shown in [51, Chapter 22]. After formulating this differential equation, we are able to isolate the singular points of this differential equation as functions of ω, and furthermore show that the even degree polynomials never vanish at these points. As such, we have by the standard analytic existence theorems for differential equations that the even degree Kissing polynomials do not form double zeros, which will turn out to be enough to guarantee the existence of these polynomials. The behavior of the polynomials as functions of ω plays a crucial role in this proof of existence, and one of the most important results on the deformation of the polynomials also highlights a strong connection to integrable systems. In Chapter 3, we emphasize the well known fact that the recurrence coefficients for the Kissing polynomials satisfy a complex version of the Toda Lattice equations. Moreover, this connection between the Kissing polynomials and the complex differential-difference equations satisfied by its recurrence coefficients should not be surprising; indeed, the recurrence coefficients of any family of orthogonal polynomials which are orthogonal with respect to the weight $\exp(\omega V(z))$, where V is any polynomial, will satisfy Toda-like equations. These equations are nothing more than the compatibility conditions between shifting the degree of the polynomial (via the three term recurrence relation) and differentiation in the parameter ω. For more details on this connection to integrable systems, the reader is referred to [28, 37, 45, 68, 69, 89].

Having completed the proof of existence for the even degree Kissing polynomials, the next item on the docket is for us to study the behavior of these polynomials as $n \to \infty$ or $\omega \to \infty$. It happens that both of these questions have been individually studied already in the literature. We will discuss the behavior as $\omega \to \infty$ at the beginning of Chapter 3, and the problem of studying the polynomials as $n \to \infty$ has already been addressed by Kuijlaars, McLaughlin, van Assche, and Vanlessen in [65], where the authors considered orthogonal polynomials with respect to the Jacobi-type weight $w(x) = (1-x)^{\alpha}(1+x)^{\beta}h(x)$ over the interval $[-1,1]$. Although their work was intended for the case when h is real valued on the real line, the work can be easily modified to deal with the Kissing polynomials.

Now, the remainder of this thesis focuses on the behavior of the Kissing polynomials (and its generalizations) as both n and ω tend to infinity at different rates. This question is addressed in Chapter 4, which represents joint work of the author of this thesis and Guilherme Silva which was published in *Journal of Approximation Theory* [25]. All results in this chapter are due to the collaboration of both the present author and G. Silva, unless directly specified otherwise. We address the question at hand by studying the following varying weight Kissing polynomials, which satisfy the orthogonality conditions

$$\int_{-1}^{1} p_n^N(z;t) z^k e^{-Nf(z;t)}\, dz = 0, \qquad k = 0,1,\ldots,n-1, \tag{1.28}$$

where $f(z;t) = -itz$ and $t > 0$ is fixed. We emphasize that for each $N \in \mathbb{N}$, there exists a family of polynomials $\{p_n^N\}$, and we then study these families of polynomials along the diagonal $N = n$. By doing so, we are studying the original Kissing polynomials as both n and ω tend to infinity at the rate t, or equivalently, $\omega = \omega(n) = tn$. The behavior of the polynomials for $t < t_c$, where t_c is the unique positive solution to

$$2\log\left(\frac{2+\sqrt{t^2+4}}{t}\right) - \sqrt{t^2+4} = 0, \tag{1.29}$$

has been studied already by Deaño in [31]. We will address his results in more detail in Chapter 4, but it is important to stress some of the major differences in techniques used between our work and that of Deaño [31].

As everything in the integrand of (1.28) is analytic, we may use Cauchy's Theorem to integrate over any curve connecting -1 and 1. The question of which is the correct curve to integrate over has been answered in the context of potential theory, and it is within this framework that Deaño has written [31]. In light of the asymptotic behavior of the zeros of the polynomials as $n \to \infty$, it is expected that there exists a "correct" contour over which to take the integration in (1.28). This contour should be the one on which the zeros of p_n accumulate as $n \to \infty$. The study of this intuitive notion of the "correct" curve was started by Nuttall, who conjectured that in the case where the weight function does not depend on the degree n, the correct curve should be one of minimal capacity (see also [76]). Nuttall's conjectures were then established rigorously by Stahl in [83, 82], where the correct curve was shown to satisfy a certain max-min variational problem. After Stahl's contributions, such curves became known in the literature as S-curves (where the S stands for “symmetric”) or curves which possess the S-property.

The attempt to adapt Stahl’s work to account for orthogonality with respect to varying weights, as is considered in Chapter 4, was first undertaken by Gonchar and Rakhmanov. In [49], Gonchar and Rakhmanov obtained the asymptotic zero distribution of a particular class of non-Hermitian orthogonal polynomials with varying weights, but took the existence of a curve with the S-property for granted. The question of the existence of S-curves was considered by Rakhmanov in [79], where he outlined a general max-min formulation for obtaining S-contours. In both the context of varying and non-varying weights, the probability measure which minimizes a certain energy functional on the S-curve (known as the equilibrium measure) governs the weak limit of the empirical counting measure for the zeros of the orthogonal polynomials. Indeed, the main technical differences between the subcritical case for the Kissing polynomials in [31] and the supercritical case considered in Chapter 4 is that for $t < t_c$, the equilibrium measure is supported on one analytic arc, whereas for $t > t_c$, the measure is supported on two arcs. This potential theoretic approach, known now as the Gonchar-Rakhmanov-Stahl (GRS) program, has been carried out in various scenarios, and we refer the reader to many excellent works on the subject [2, 4, 63, 70, 73, 72, 74, 75, 94], as we quickly restate the results that will be necessary in the present thesis.

To understand the GRS theory as it relates to the current situation, we first let $f(z;t)$ be a polynomial in z. We consider any analytic curve, Γ, which connects -1 to 1 in $\mathbb{C}$. Then, we seek a measure μ among all probability measures supported on Γ which minimizes the following weighted energy functional:

$$I(\mu) = \iint \log\left|\frac{1}{z-s}\right| d\mu(z)d\mu(s) + \Re\int f(s;t)\,d\mu(s). \tag{1.30}$$

From the general theory presented in [80], this problem has a unique solution, say μ_*, which we call the equilibrium measure in the presence of the external field $\Re f$ on Γ.

Furthermore, we define the logarithmic potential of a probability measure μ supported on Γ as

$$U^{\mu}(z) = \int \log\left|\frac{1}{z-s}\right| d\mu(s). \tag{1.31}$$

Again following [80], we have that the logarithmic potential of the equilibrium measure, μ_*, on Γ satisfies

$$2U^{\mu_*}(z) + \Re f(z;t) = \ell, \qquad z \in \operatorname{supp}\,\mu_*, \tag{1.32a}$$

$$2U^{\mu_*}(z) + \Re f(z;t) \geq \ell, \qquad z \in \Gamma \setminus \operatorname{supp}\,\mu_*, \tag{1.32b}$$

for some constant $\ell \in \mathbb{R}$. We then say that the contour Γ possesses the S-property in the external field $\Re f$ if for every z in the interior of the support of μ_* we have that

$$\frac{\partial}{\partial n_+}\left(2U^{\mu_*}(z)+\Re f(z;t)\right)=\frac{\partial}{\partial n_-}\left(2U^{\mu_*}(z)+\Re f(z;t)\right), \tag{1.33}$$

where $\partial n_\pm$ denote the normal derivatives on the $\pm$ side of Γ.

Another equivalent formulation of S curves comes through the guise of the g-function. Given a curve Γ, denote by μ_* the equilibrium measure in the external field $\Re f$ on Γ, and define

$$g(z)=\int_\Gamma \log(z-s)\, d\mu_*(s). \tag{1.34}$$

Above, we take the principal branch of the logarithm. Using the Cauchy-Riemann equations and that $\Re g(z)=-U^{(\mu)}(z)$, we see that the definition of the S-property is equivalent to the condition that the imaginary part of $-g_+-g_-+f$ is locally constant on the support of μ_*. Therefore, combining (1.32) and (1.33), we see that Γ possesses the S-property if its g function satisfies

$$-g_+(z)-g_-(z)+f(z;t)=\ell+i\tilde{\ell}, \qquad z\in \operatorname{supp}\mu_*, \tag{1.35a}$$

$$\Re\left(-g_+(z)-g_-(z)+f(z;t)\right)\geq \ell, \qquad z\in\Gamma\setminus\operatorname{supp}\mu_*. \tag{1.35b}$$

Above, the constant ℓ is the same in each component of the support of μ_*, whereas $\tilde{\ell}$ can differ between the individual components.

The power of the S-property comes from the GRS theory as follows. First, we introduce the normalized zero counting measure of the polynomials defined in (1.28), with $N=n$, as

$$\mu_n=\frac{1}{n}\sum_{k=1}^{n}\delta_{z_k}, \tag{1.36}$$

where z_k is a zero of the polynomial p_n^n and δ_{z_k} is the atomic mass with weight 1 at z. Then, the theorem of Gonchar and Rakhmanov [49] states that if Γ is contour with the S property, then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\delta_{z_k}\stackrel{*}{\rightharpoonup}\mu_*, \tag{1.37}$$

where μ_* is the equilibrium measure on Γ in the presence of the external field $\Re f$.

In contrast to the approach of Deaño in [31], where he explicitly constructed a curve with the S-property, in Chapter 4 we will use ad-hoc calculations inspired by the max-min approach described above to formulate an initial guess for our initial S-contour. We will then use this guess as initial input into the Riemann-Hilbert problem for computing the asymptotics of the polynomials, and verify later that this curve satisfies the S-property. In this chapter, we will also construct the appropriate h-function as described in Chapter 2. However, we will proceed with the steepest descent process via an alternate approach using a so called "symmetrized external field". The main reason for using such a symmetrized external field is so that we may more explicitly present the asymptotics for the Kissing polynomials when $t > t_c$; indeed, by using the symmetrized external field, we will be able to present the asymptotics without making reference to Theta functions, as is the case in Chapter 2. The construction of this symmetrized external field is based on the integration of precisely constructed differential forms on a certain Riemann surface, and is inspired by the work of Mo and Kuijlaars [62].

The main results of Chapter 4 are the construction of the h-function and the asymptotics of the Kissing polynomials in the supercritical regime, $t > t_c$. In particular, we will show that the normalized zero counting measure converges to a measure supported on two arcs, symmetric about the imaginary axis, and will also provide asymptotic formulas for the Kissing polynomials uniform in compact subsets of the plane. The main original findings of this chapter are the construction of the h-function and the construction of the global parametrix with the symmetrized external field.

After studying the supercritical regime for the Kissing polynomials, we turn in Chapter 5 to the study of the varying weight Kissing polynomials where the associated parameter now takes on any complex value. The results of Chapter 5 constitute joint work of the present author and Ahmad Barhoumi and Alfredo Deaño; this chapter is based on a paper which has been submitted for publication [8]. In particular, the results of this chapter are original results of the author of this thesis and his collaborators, unless directly specified otherwise. In this chapter, we study monic polynomials which satisfy

$$\int_{-1}^{1} p_n(z;s) z^k e^{-nf(z;s)}\, dz = 0, \qquad k = 0, 1, \ldots, n-1, \tag{1.38}$$

where now, $f(z;s) = sz$ and $s \in \mathbb{C}$. Clearly, the work of Deaño on the subcritical case for the Kissing polynomials [31], along with the work on supercritical case discussed in Chapter 4, fall under the umbrella of the polynomials considered in (1.38). In fact, these two works will play a vital role in the study of the orthogonal polynomials defined in (1.38). This is because we will deform the h-functions of both the sub- and supercritical regimes of the Kissing polynomials using the technique of continuation in parameter space, first developed in the context of the theory of integrable systems, see for instance [56, 86, 87]. This theory has recently been applied to orthogonal polynomials [10, 12, 13], and will allow us to deform the h-function in the parameter space. As we show in Chapter 2, given a suitable h-function, we may generically complete the process of steepest descent, and as such we will be able to perform the process of steepest descent for non-degenerate values of s by using the deformed h-function. In particular, we provide asymptotic formulas for the recurrence coefficients of the polynomials as $n \to \infty$ in various regions of the parameter space.

In the later sections of Chapter 5, we turn our attention to the behavior of the polynomials as we transition from the subcritical to the supercritical regimes, and then we focus on the behavior of the polynomials near the value $s = 2$. In both scenarios, we study the behavior of the recurrence coefficients in these regimes by considering an appropriate scaling limit. These scaling limits will be explained in greater detail in Chapter 5, but for now we note that the role of special functions, in particular a solution to Painlevé II, becomes very apparent in these sections. For more connections between orthogonal polynomials, Riemann-Hilbert problems, special functions, and Painlevé equations, we refer the reader to the works [12, 15, 30, 41, 47, 89, 93].

Chapter 2

The Riemann-Hilbert Problem and Nonlinear Steepest Descent

In its original setting, the Riemann-Hilbert problem is a question in the theory of Fuchsian differential systems, and forms the subject of Hilbert's 21^{st} problem in his famous list of problems in 1900. Over the past century, the notion of what constitutes a Riemann-Hilbert problem has evolved, and today a Riemann-Hilbert problem involves the reconstruction of an analytic function from prescribed jump conditions. The formulation of orthogonal polynomials as a solution to a 2×2 matrix valued Riemann-Hilbert problem was first given by Fokas, Its, and Kitaev in the early 1990s [46]. This formulation became even more powerful in the late 1990s due to the development of the nonlinear steepest descent method to obtain asymptotic solutions to Riemann-Hilbert problems, developed by Deift and Zhou [39, 40, 36]. There are many excellent texts on these subjects in the literature, and we follow the guide of [13, 16, 34]. For the extension of the Riemann-Hilbert formulation to deal with multiple orthogonality, see [90]. For a treatment of Riemann-Hilbert problems from a numerical perspective, the reader is referred to [88]. Finally, for a historical perspective on the Riemann-Hilbert problem and its connections to integrable systems, we recommend the survey article [52].

The purpose of this chapter is twofold. First and foremost, this chapter is meant to introduce the Fokas-Its-Kitaev Riemann-Hilbert problem and Deift-Zhou nonlinear steepest descent. Throughout this chapter, we take for granted the existence of a *modified external field*, or *h-function*, which we shall construct in later chapters. Secondly, however, this chapter will serve to reduce the length of the subsequent chapters. Indeed, we will often refer back to results presented in this chapter throughout Chapters 3-5.

2.1 The Fokas-Its-Kitaev Riemann-Hilbert Problem and The Modified External Field

In this chapter, we consider monic polynomials, $p_n^N(z;s)$, which satisfy the following orthogonality conditions

$$\int_{-1}^{1} p_n^N(z;s) z^k e^{-Nf(z;s)}\, dz = 0, \qquad k = 0,1,\ldots,n-1. \tag{2.1}$$

Here, f is an entire function, referred to as the potential, and the normalizing constant $\kappa_{n,N}$ is defined via

$$\int_{-1}^{1} (p_n^N(z;s))^2 e^{-Nf(z;s)} dz = \frac{1}{\kappa_{n,N}^2}. \tag{2.2}$$

In the case of the varying weight Kissing polynomials, we take $f(z;s) = -isz$ as explained in Chapter 1.

Now, given a smooth curve Σ connecting -1 to 1 in $\mathbb{C}$, oriented from -1 to 1, consider the following Riemann-Hilbert problem for $Y : \mathbb{C} \setminus \Sigma \to \mathbb{C}^{2\times 2}$,

$$Y_n^N(z;s) \text{ is analytic for } z \in \mathbb{C} \setminus \Sigma, \tag{2.3a}$$

$$Y_{n,+}^N(z;s) = Y_{n,-}^N(z;s) \begin{pmatrix} 1 & e^{-Nf(z;s)} \\ 0 & 1 \end{pmatrix}, \qquad z \in \Sigma, \tag{2.3b}$$

$$Y_n^N(z;s) = \left(I + \mathscr{O}\left(\frac{1}{z}\right)\right) z^{n\sigma_3}, \qquad z \to \infty, \tag{2.3c}$$

$$Y_n^N(z;s) = \mathscr{O}\begin{pmatrix} 1 & \log|z \mp 1| \\ 1 & \log|z \mp 1| \end{pmatrix}, \qquad z \to \pm 1. \tag{2.3d}$$

Above, σ_3 is the Pauli matrix given by $\sigma_3 = \operatorname{diag}(1,-1)$. The boundary values are defined for $z \in \Sigma$ as $Y_{n,\pm}^N(z;s) = \lim_{x\to z} Y_n^N(x;s)$ as x approaches z from the right (left) side of the contour Σ, where the directions are induced from the orientation on Σ. Moreover, these boundary values are assumed to exist for all interior points of the contour Σ.

One result of Fokas, Its, and Kitaev in [46] was that this Riemann-Hilbert problem has a solution which can be given explicitly in terms of orthogonal polynomials. In particular, the existence of Y is equivalent to the existence of the monic orthogonal polynomial p_n^N defined in (2.1), of degree *exactly* n; furthermore, if p_{n-1}^N also exists as a polynomial of degree $n-1$, then Y is explicitly given by

$$Y(z) = \begin{pmatrix} p_n^N(z;s) & \left(\mathscr{C} p_n^N e^{-Nf}\right)(z) \\ -2\pi i \kappa_{n-1,N}^2 p_{n-1}^N(z;s) & -2\pi i \kappa_{n-1,N}^2 \left(\mathscr{C} p_{n-1}^N e^{-Nf}\right)(z) \end{pmatrix}. \tag{2.4}$$

In (2.4), $\mathscr{C} g$ denotes the Cauchy transform of the function g along Σ, i.e.

$$\left(\mathscr{C} g\right)(z) = \frac{1}{2\pi i}\int_{\Sigma} \frac{g(u)}{u-z}\, du,$$

which is analytic in $\mathbb{C}\setminus\Sigma$.

Recall that in the present thesis, we are concerned with the asymptotic analysis along the diagonal $N=n$. For notational convenience, we drop the dependence of the Riemann-Hilbert problem and its solution on the parameters n, N, and s. It should be stressed, however, that the Riemann-Hilbert problem does indeed depend on these parameters.

In order to extract asymptotics of the orthogonal polynomials in the complex plane as $n\to\infty$, we will implement the technique of nonlinear steepest descent developed by Deift and Zhou in the mid 1990s. Key to this process is the existence of a so called *modified external field* or *h-function*. Throughout this chapter, we will assume the existence of the desired h-function, and delay proofs of its existence to the later chapters. As the Kissing polynomials are orthogonal with respect to a weight that is the exponential of a linear function, we shall only encounter "genus 0" and "genus 1" h-functions. The meaning of the term "genus" shall become clear shortly. We briefly describe properties of the genus 0 and genus 1 h-functions below, before moving on to the description of nonlinear steepest descent.

Genus 0 h-function

To state properties of the genus 0 h-function, we first define $\gamma_{c,0}^{(0)}:=(-\infty,-1]$ and $\gamma_{m,0}^{(0)}:=\Sigma$, where we recall that Σ is the contour on which the RHP for Y is posed in (2.3). The superscript (0) indicates that we are working with the genus 0 h-function. Then, defining $\Omega^{(0)}=\Omega^{(0)}(s)=\gamma_{c,0}^{(0)}\cup\gamma_{m,0}^{(0)}$, the genus 0 h-function satisfies the following scalar Riemann-Hilbert problem:

$$h(z;s) \text{ is analytic for } z\in\mathbb{C}\setminus\Omega^{(0)}, \tag{2.5a}$$

$$h_+(z;s)-h_-(z;s)=4\pi i, \qquad z\in\gamma_{c,0}^{(0)}, \tag{2.5b}$$

$$h_+(z;s)+h_-(z;s)=0, \qquad z\in\gamma_{m,0}^{(0)}, \tag{2.5c}$$

$$h(z;s)=-f(z;s)-\ell+2\log z+\mathscr{O}\left(\frac{1}{z}\right), \qquad z\to\infty \tag{2.5d}$$

$$\Re h(z;s)=\mathscr{O}\left((z\mp 1)^{1/2}\right), \qquad z\to\pm 1. \tag{2.5e}$$

Above, $\ell=\ell(s)$ is a constant which will be used in the first transformation of the steepest descent process. In addition to solving the above scalar Riemann-Hilbert problem, we also impose that the genus 0 h-function satisfies the following inequality near the arc Σ:

$$\Re h(z_0;s)>0 \text{ for } z_0 \text{ in close proximity to any interior point of } \gamma_{m,0}^{(0)}. \tag{2.6}$$

We now state properties of the genus 1 h-function.

Genus 1 h-function

As before, we define $\gamma_{c,0}^{(1)} := (-\infty,-1]$ and set $\Omega^{(1)} = \Omega^{(1)}(s) = \gamma_{c,0}^{(1)} \cup \Sigma$. We further assume that we may partition the contour Σ as $\Sigma = \gamma_{m,0}^{(1)} \cup \gamma_{c,1}^{(1)} \cup \gamma_{m,1}^{(1)}$, where the contours $\gamma_{m,0}^{(1)}$, $\gamma_{c,1}^{(1)}$, and $\gamma_{m,1}^{(1)}$ are bounded and connect -1 to λ_0, λ_0 to λ_1, and λ_1 to 1, respectively. Here $\lambda_0 = \lambda_0(s)$ and $\lambda_1 = \lambda_1(s)$ are such that $\lambda_0, \lambda_1 \in \Sigma$, and they denote the endpoints of the arc $\gamma_{c,1}^{(1)}$. These contours are chosen so that the genus 1 h-function solves the following Riemann-Hilbert problem and inequalities. The scalar Riemann-Hilbert problem for the genus 1 h-function is given as:

$$h(z;s) \text{ is analytic for } z \in \mathbb{C} \setminus \Omega^{(1)}, \tag{2.7a}$$

$$h_+(z;s) - h_-(z;s) = 4\pi i, \qquad z \in \gamma_{c,0}^{(1)}, \tag{2.7b}$$

$$h_+(z;s) + h_-(z;s) = 4\pi i \omega_0, \qquad z \in \gamma_{m,0}^{(1)}, \tag{2.7c}$$

$$h_+(z;s) - h_-(z;s) = 4\pi i \eta_1, \qquad z \in \gamma_{c,1}^{(1)}, \tag{2.7d}$$

$$h_+(z;s) + h_-(z;s) = 0, \qquad z \in \gamma_{m,1}^{(1)}, \tag{2.7e}$$

$$h(z;s) = -f(z;s) - \ell + 2\log z + \mathcal{O}\left(\frac{1}{z}\right), \qquad z \to \infty \tag{2.7f}$$

$$\Re h(z;s) = \mathcal{O}\left((z-\lambda)^{3/2}\right), \qquad z \to \lambda \in \{\lambda_0, \lambda_1\}, \tag{2.7g}$$

$$\Re h(z;s) = \mathcal{O}\left((z \mp 1)^{1/2}\right), \qquad z \to \pm 1. \tag{2.7h}$$

Above, the constants $\omega_0 \in \mathbb{R}$ and $\eta_1 \in \mathbb{R}$ are arbitrary real constants chosen so that we may construct the desired genus 1 h-function. Again, $\ell = \ell(s)$ is a constant which will be used in the first transformation of the steepest descent process. Note that given any real constants ω_0, η_1, and ℓ, there is no guarantee that a solution to the above scalar problem exists; however, if one does exist, it will be unique.

In addition to satisfying the above scalar Riemann-Hilbert problem, we also impose that the genus 1 h-function satisfies the following inequalities near the arc Σ:

$$\Re h(z) < 0 \text{ if } z \text{ is an interior point of } \gamma_{c,1}^{(1)}, \tag{2.8a}$$

$$\Re h(z_0) > 0 \text{ for } z_0 \text{ in close proximity to any interior point of } \gamma_{m,0}^{(1)} \cup \gamma_{m,1}^{(1)}. \tag{2.8b}$$

Letting L denote the genus of the h-function, we may put both scalar RHPs (2.5), (2.7) and both inequalities (2.6), (2.8), on the same notational footing as follows. First, we define the *main arcs*, $\mathfrak{M}^{(L)}$, as $\mathfrak{M}^{(L)} := \cup_{j=0}^{L} \gamma_{m,j}^{(L)}$, and the *complementary arcs*, $\mathfrak{C}^{(L)}$, as $\mathfrak{C}^{(L)} = \cup_{j=0}^{L} \gamma_{c,j}^{(L)}$. Furthermore, we denote the set of branchpoints as $\Lambda^{(0)} := \{-1,1\}$ and $\Lambda^{(1)} := \{-1,1,\lambda_0,\lambda_1\}$. We also define the constants $\eta_0 := 1$ and $\omega_1 := 0$, and drop the superscripts, (0) and (1), for notational convenience. In what follows, we will make the underlying genus explicit, so there shall be no confusion as to which h-function we are working with. With this notation in hand, the scalar RHPs (2.5) and (2.7) may be rewritten as

$$h(z;s) \text{ is analytic for } z \in \mathbb{C} \setminus (\mathfrak{C} \cup \mathfrak{M}), \tag{2.9a}$$

$$h_+(z;s) - h_-(z;s) = 4\pi i \eta_j, \qquad z \in \gamma_{c,j}, \qquad j = 0, \dots, L, \tag{2.9b}$$

$$h_+(z;s) + h_-(z;s) = 4\pi i \omega_j, \qquad z \in \gamma_{m,j}, \qquad j = 0, \dots, L, \tag{2.9c}$$

$$h(z;s) = -f(z;s) - \ell + 2\log z + \mathcal{O}\left(\frac{1}{z}\right), \qquad z \to \infty \tag{2.9d}$$

$$\Re h(z;s) = \mathcal{O}\left((z \mp 1)^{1/2}\right), \qquad z \to \pm 1, \tag{2.9e}$$

$$\Re h(z;s) = \mathcal{O}\left((z - \lambda)^{3/2}\right), \qquad z \to \lambda, \quad \lambda \in \Lambda \setminus \{\pm 1\}. \tag{2.9f}$$

Furthermore, we may rewrite the inequalities (2.6) and (2.8) as

$$\Re h(z) < 0 \text{ if } z \text{ is an interior point of any bounded complementary arc } \gamma_c \in \mathfrak{C}, \tag{2.10a}$$

$$\Re h(z_0) > 0 \text{ for } z_0 \text{ in close proximity to any interior point of a main arc } \gamma_m \in \mathfrak{M}. \tag{2.10b}$$

Finally, we call $s \in \mathbb{C}$ a *regular point* if we are able to construct either a genus 0 or genus 1 h-function, $h(z;s)$. Before moving on to discuss the method of nonlinear steepest descent when $s \in \mathbb{C}$ is a regular point, we quickly digress to discuss the terminology "genus" used extensively above.

Assume that s is a regular point, and that we are able to construct a genus L h-function. By taking derivatives in (2.9), we see that $h'(z;s)$ is analytic in $\mathbb{C} \setminus \mathfrak{M}$ and changes sign over $\mathfrak{M}$. Therefore, we may define the Riemann surface, $\mathfrak{R}$, to be the two-sheeted, genus L Riemann surface associated to the algebraic equation

$$\xi^2 = h'(z;s)^2 = M^2(z;s)R(z;s). \tag{2.11}$$

Here, R is defined by

$$R(z;s) = \frac{1}{z^2 - 1} \prod_{j=0}^{2L-1} (z - \lambda_j(s)), \tag{2.12}$$

and M is a polynomial of degree $1 - L$ in z, chosen so that h' possess the correct asymptotics at infinity, in light of (2.9d). Finally, the branchcuts of $\mathfrak{R}$ (and equivalently h') are taken along $\gamma_{m,j}$, $j = 0, \dots, L$, and the top sheet of $\mathfrak{R}$ is fixed so that

$$\xi(z) = -f'(z;s) + \mathcal{O}\left(\frac{1}{z}\right), \qquad z \to \infty_1. \tag{2.13}$$

Therefore, when we refer to the "genus" of the h-function, we are really referring to the genus of the underlying Riemann surface. For the remainder of this chapter, we take for granted the existence of an appropriate h-function, and show how this h-function can be used to complete the Deift-Zhou method of nonlinear steepest descent.

Remark 2.1. In much of the literature, the process of steepest descent uses the g function defined in (1.34). We remark here that the two approaches are equivalent. Following [13], we note that from (1.32) and (1.34), the g-function satisfies

$$g(z) \text{ is analytic for } z \in \mathbb{C} \setminus (\mathfrak{C} \cup \mathfrak{M}), \tag{2.14a}$$

$$g_+(z) - g_-(z) = 2\pi i \eta_j, \qquad z \in \gamma_{c,j}, \qquad j = 0, \ldots, L, \tag{2.14b}$$

$$g_+(z) + g_-(z) = f(z;s) + \ell + 2\pi i \omega_j, \qquad z \in \gamma_{m,j}, \qquad j = 0, \ldots, L, \tag{2.14c}$$

$$g(z) = \log z + \mathscr{O}\left(\frac{1}{z}\right), \qquad z \to \infty. \tag{2.14d}$$

Moreover, we may recover the g function from the relation that $h(z;s) = 2g(z;s) - f(z;s) - \ell$. However, in this thesis, we adopt the approach of using the h-function.

2.2 Overview of Deift-Zhou Nonlinear Steepest Descent

The power of the Fokas-Its-Kitaev Riemann-Hilbert problem for the orthogonal polynomials is that it is a useful starting point to obtain uniform asymptotics of the polynomials throughout the complex plane. Obtaining these asymptotics can be achieved by a process called nonlinear steepest descent, developed by Deift and Zhou in the mid 1990s [39, 40, 36]. We will review the steps of the steepest descent process in this section. The main idea is to transform the Riemann-Hilbert problem for Y in (2.3) via a sequence of steps into an equivalent Riemann-Hilbert problem for some R, which is of small norm. Such small norm problems can be solved asymptotically via perturbation theory, and as such we may obtain asymptotics for R as $n \to \infty$. Then, reversing the transformations that led from Y to R, we may uncover asymptotics of Y, and therefore the polynomials, as $n \to \infty$.

The transformation from Y to R is done in four steps,

$$Y \mapsto T \mapsto S \mapsto R.$$

The first transformation $Y \mapsto T$ aims to normalize the Riemann-Hilbert problem at infinity, and makes use of the h-function above. This transformation leads to the Riemann-Hilbert problem for T having highly oscillatory jump matrices over the main arcs in Σ, so the next transformation $T \mapsto S$ transforms the problem for T so that these oscillatory entries decay exponentially fast. This is done by factorizing the jump matrix for T and deforming the contour Σ to a new contour $\hat{\Sigma}$; this deformation of contours so that oscillatory entries decay exponentially fast is similar in nature to the steepest descent method for oscillatory integrals described in Chapter 1.1, and is what gives this method the name "steepest descent". After this transformation, we will see that some of the jumps for S are exponentially close to the identity. Ignoring these jumps, we obtain a model Riemann-Hilbert problem for M which we can solve exactly. The solution to the RHP for M is called the global parametrix, and it is expected that M is close to S as $n \to \infty$, as their Riemann-Hilbert problems differ by jumps which are exponentially small. However, we will see that the global parametrix is not bounded near the endpoints of the main arcs in Σ, and at these points a more refined local analysis will be needed. Therefore, in a neighborhood of each point $\lambda \in \Lambda$, which we denote D_λ, we seek a local parametrix $P^{(\lambda)}$ which solves the RHP for S *exactly* in D_λ, and agrees with the global parametrix M on ∂D_λ as $n \to \infty$. Provided we are able to solve for both the global and local parametrices, we may define $R = SM^{-1}$ away from the points $\lambda \in \Lambda$ and $R = S\left(P^{(\lambda)}\right)^{-1}$ in each neighborhood D_λ. We will find that this procedure will result in R satisfying a suitable small norm Riemann-Hilbert problem R on some new system of contours Σ_R, which we may solve by perturbation theory.

2.2.1 First Transformations

The first transformation of steepest descent aims to normalize the Riemann-Hilbert problem (2.3) at infinity. To do so, we define

$$T(z) := e^{-n\ell\sigma_3/2} Y(z) e^{-\frac{n}{2}[h(z)+f(z)]\sigma_3}, \tag{2.15}$$

where we recall that ℓ is defined by (2.9d). By making this transformation, we see that T satisfies the following Riemann-Hilbert problem:

$$T(z) \text{ is analytic for } z \in \mathbb{C} \setminus \Sigma, \tag{2.16a}$$

$$T_+(z) = T_-(z) \begin{pmatrix} e^{-\frac{n}{2}(h_+(z)-h_-(z))} & e^{\frac{n}{2}(h_+(z)+h_-(z))} \\ 0 & e^{\frac{n}{2}(h_+(z)-h_-(z))} \end{pmatrix}, \qquad z \in \Sigma, \tag{2.16b}$$

$$T(z) = I + \mathscr{O}\left(\frac{1}{z}\right), \qquad z \to \infty, \tag{2.16c}$$

$$T(z) = \mathscr{O}\begin{pmatrix} 1 & \log|z \mp 1| \\ 1 & \log|z \mp 1| \end{pmatrix}, \qquad z \to \pm 1. \tag{2.16d}$$

Equations (2.9c) and (2.10b) imply that $\Re h(z) = 0$ for $z \in \mathfrak{M}$. As $\mathfrak{M}$ is part of the zero level set of $\Re h$, the jump matrix for T has highly oscillatory diagonal entries when $z \in \mathfrak{M}$. Furthermore, if $z \in \mathfrak{C} \setminus \gamma_{c,0}$, the diagonal entries of the jump matrix will be constant and purely imaginary. Moreover, the $(1,2)$-entry of the jump matrix will decay exponentially fast to 0 by (2.10a). The next transformation of the steepest descent process deforms the jump contours so that the highly oscillatory entries of the jump matrix decay exponentially fast, and is referred to as the *opening of lenses*.

The opening of lenses relies on the following factorization of the jump matrix across a main arc

$$\begin{pmatrix} e^{-nH(z)} & e^{2\pi i n\omega_j} \\ 0 & e^{nH(z)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{n\left(H(z)-2\pi i\omega_j\right)} & 1 \end{pmatrix} \begin{pmatrix} 0 & e^{2\pi i n\omega_j} \\ -e^{-2\pi i n\omega_j} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{n\left(-H(z)-2\pi i\omega_j\right)} & 1 \end{pmatrix}, \tag{2.17}$$

where we have defined $H(z) = (h_+(z) - h_-(z))/2$. On the $+$-side ($-$-side) of each main arc, we define $\gamma_{m,j}^+$ ($\gamma_{m,j}^-$) to be an arc which starts and ends at the endpoints of $\gamma_{m,j}$ and remains entirely

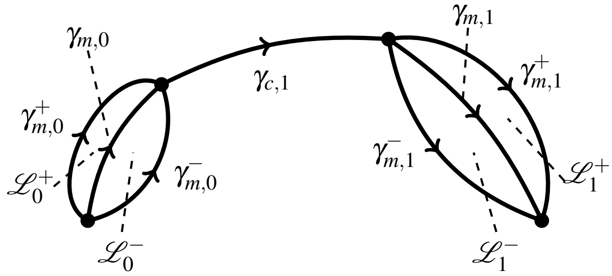


Figure 2.1: The contour $\hat{\Sigma}$ after opening lenses in the genus 1 case, $L = 1$.

on the $+(-)$ side of $\gamma_{m,j}$. For now we do not impose any restrictions on the precise description of these arcs, but we enforce that they remain in the region where $\Re h > 0$, which is possible due to (2.10b). We define $\mathscr{L}_j^{\pm}$ to be the region bounded between the arcs $\gamma_{m,j}$ and $\gamma_{m,j}^{\pm}$, respectively, and set $\hat{\Sigma} := \Sigma \cup_{j=0}^{L} \left(\gamma_{m,j}^+ \cup \gamma_{m,j}^-\right)$, as in Figure 2.1.

We can now define the third transformation of the steepest descent process as

$$S(z) := \begin{cases} T(z) \begin{pmatrix} 1 & 0 \\ \mp e^{-nh(z)} & 1 \end{pmatrix}, & z \in \mathscr{L}_j^{\pm}, \\ T(z), & \text{otherwise.} \end{cases} \tag{2.18}$$

This implies that S solves the following Riemann-Hilbert problem on $\hat{\Sigma}$:

$$S(z) \text{ is analytic for } z \in \mathbb{C} \setminus \hat{\Sigma}, \tag{2.19a}$$

$$S_+(z) = S_-(z) j_S(z), \qquad z \in \hat{\Sigma}, \tag{2.19b}$$

$$S(z) = I + \mathscr{O}\left(\frac{1}{z}\right), \qquad z \to \infty. \tag{2.19c}$$

The transformation (2.18) enforces that the endpoint behavior for S is different to that of T. In particular, we find that as $z \to \pm 1$ within the lens,

$$S(z) = \mathscr{O}\begin{pmatrix} \log|z \mp 1| & \log|z \mp 1| \\ \log|z \mp 1| & \log|z \mp 1| \end{pmatrix}. \tag{2.20}$$

We also remark that S is bounded near the endpoints λ_0 and λ_1, in the event we are working in the genus 1 case. Now, note that for $z \in \gamma_{m,j}^{\pm}$,

$$j_S(z) = \begin{pmatrix} 1 & 0 \\ e^{-nh(z)} & 1 \end{pmatrix}, \tag{2.21}$$

which decays exponentially fast to the identity as $n \to \infty$, as a consequence of (2.10b). As $S = T$ outside of the lenses, we see that there are no changes to the jump matrix across a complementary arc, so that

$$j_S(z) = \begin{pmatrix} e^{-2\pi i n \eta_j} & e^{\frac{n}{2}(h_+(z)+h_-(z))} \\ 0 & e^{2\pi i n \eta_j} \end{pmatrix}, \qquad z \in \gamma_{c,j}, \tag{2.22}$$

which again tends exponentially fast to a diagonal matrix as $n \to \infty$. Finally, we see that over $\gamma_{m,j}$, the jump matrix is given by

$$j_S(z) = \begin{pmatrix} 0 & e^{2\pi i n \omega_j} \\ -e^{-2\pi i n \omega_j} & 0 \end{pmatrix}, \qquad z \in \gamma_{m,j}, \tag{2.23}$$

which follows from the factorization (2.17).

Now consider the following model Riemann-Hilbert problem for the global parametrix, M, which is obtained by neglecting those entries in the jump matrix which are exponentially close to the identity in the Riemann-Hilbert problem for S,

$$M(z) \text{ is analytic for } z \in \mathbb{C} \setminus \Sigma, \tag{2.24a}$$

$$M_+(z) = M_-(z) \begin{pmatrix} e^{-2\pi i n \eta_j} & 0 \\ 0 & e^{2\pi i n \eta_j} \end{pmatrix}, \qquad z \in \gamma_{c,j},\ j = 1, \ldots, L, \tag{2.24b}$$

$$M_+(z) = M_-(z) \begin{pmatrix} 0 & e^{2\pi i n \omega_j} \\ -e^{-2\pi i n \omega_j} & 0 \end{pmatrix}, \qquad z \in \gamma_{m,j},\ j = 0, \ldots, L, \tag{2.24c}$$

$$M(z) = I + \mathscr{O}\left(\frac{1}{z}\right), \qquad z \to \infty. \tag{2.24d}$$

Assuming we are able to solve the model Riemann-Hilbert problem, we would like to make the final transformation by setting $R = SM^{-1}$. However, this will turn out not to be valid near the endpoints Λ. As such, we will need a more refined local analysis near these points. More precisely, we will solve the Riemann-Hilbert problem for S *exactly* near these points, and impose further that it matches with the global parametrix as $n \to \infty$.

To do so, we define $D_\lambda = D_\delta(\lambda)$ to be discs of fixed radius δ around each endpoint $\lambda \in \Lambda$. For each $\lambda \in \Lambda$, we seek a local parametrix $P^{(\lambda)}$, dependent on n, which solves:

$$P^{(\lambda)}(z) \text{ is analytic for } z \in D_\lambda \setminus \hat{\Sigma}, \tag{2.25a}$$

$$P_+^{(\lambda)}(z) = P_-^{(\lambda)}(z) j_S(z), \qquad z \in D_\lambda \cap \hat{\Sigma} \tag{2.25b}$$

$$P^{(\lambda)}(z) = M(z)\left(I + \mathscr{O}\left(\frac{1}{n}\right)\right), \qquad n \to \infty, \quad z \in \partial D_\lambda. \tag{2.25c}$$

We also require that $P^{(\lambda)}$ has a continuous extension to $\overline{D_\delta(\lambda)} \setminus \hat{\Sigma}$ and remains bounded as $z \to \lambda$. The construction of both the global and relevant local parametrices, though now standard, are included in Sections 2.3 and 2.4 of this chapter. For now, we take for granted the existence of solutions to these model Riemann-Hilbert problems, and show how to arrive at a suitable small norm Riemann-Hilbert problem and explain how to solve such a problem via perturbation techniques.

2.2.2 Small Norm Riemann-Hilbert Problems

We may complete the process of nonlinear steepest descent by defining the final transformation as

$$R(z) = \begin{cases} S(z)M(z)^{-1}, & z \in \mathbb{C} \setminus \left(\hat{\Sigma} \cup_{\lambda \in \Lambda} D_\lambda\right)) \\ S(z)P^{(\lambda)}(z)^{-1}, & z \in D_\lambda \setminus \hat{\Sigma},\, \lambda \in \Lambda. \end{cases} \tag{2.26}$$

Provided we were able to appropriately construct both the local and global parametrices, the matrix R will satisfy a "small norm" Riemann-Hilbert problem on a new contour, Σ_R, whose jumps decay to the identity in the appropriate sense. The contour Σ_R will consist of the oriented arcs forming the boundaries ∂D_λ about each $\lambda \in \Lambda$ and the portions of $\gamma^{\pm}_{m,L}$ which are not in the interior of D_λ, as illustrated in Figure 2.2 for the genus $L = 1$ case.

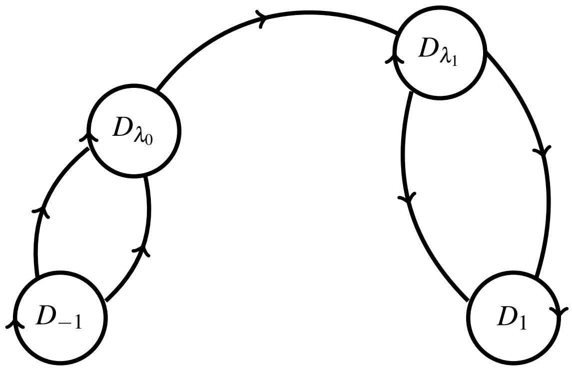


Figure 2.2: The contour Σ_R in the case $L = 1$. Note that we have chosen the contours ∂D_λ to have clockwise orientation.

Here, the jump matrix $j_R(z)$ will satisfy

$$j_R(z) = \begin{cases} I + \mathscr{O}\left(e^{-cn}\right), & z \in \Sigma_R \setminus \bigcup_{\lambda \in \Lambda} \partial D_\lambda \\ I + \mathscr{O}\left(\frac{1}{n}\right), & z \in \bigcup_{\lambda \in \Lambda} \partial D_\lambda \end{cases}, \tag{2.27}$$

for some $c > 0$ with uniform error terms. In particular, we may write the jump matrix as $j_R(z) = I + \Delta(z)$, where

$$\Delta(z) \sim \sum_{k=1}^{\infty} \frac{\Delta_k(z)}{n^k}, \qquad n \to \infty, z \in \Sigma_R. \tag{2.28}$$

By [39, Theorem 7.10], this behavior then implies that R has an asymptotic expansion of the form

$$R(z) \sim I + \sum_{k=1}^{\infty} \frac{R_k(z)}{n^k}, \qquad n \to \infty, \tag{2.29}$$

valid uniformly for $z \in \mathbb{C} \setminus \cup_{\lambda \in \Lambda}$. Above, the $R_k(z)$ are solutions to the following Riemann-Hilbert problem (c.f [39, Section 7], [65, Section 8.2]),

$$R_k(z) \text{ is analytic for } z \in \mathbb{C} \setminus \bigcup_{\lambda \in \Lambda} \partial D_\lambda \tag{2.30a}$$

$$R_{k,+}(z) = R_{k,-}(z) + \sum_{j=1}^{k-1} R_{k-j,-} \Delta_j(z), \qquad z \in \bigcup_{\lambda \in \Lambda} \partial D_\lambda \tag{2.30b}$$

$$R_k(z) = \frac{R_k^{(1)}}{z} + \frac{R_k^{(2)}}{z^2} + \mathscr{O}\left(\frac{1}{z^3}\right), \qquad z \to \infty, \tag{2.30c}$$

where the Δ_j are given by (2.28). Therefore, if we are able to determine the Δ_k in (2.28), we will be able to sequentially solve for the R_k in the expansion for R in (2.29) via the Riemann-Hilbert problem (2.30).

2.2.3 Unwinding the Transformations

The process of retracing the steps of Deift-Zhou steepest descent to obtain uniform asymptotics of the orthogonal polynomials in the plane is now standard. We outline this process below, as it will be used throughout Chapters 4 and 5 below.

First, we consider asymptotics of the polynomials for some $z \in \mathbb{C} \setminus \mathfrak{M}$. Unwinding the transformations away from the lenses, we see that

$$Y(z) = e^{n\ell\sigma_3/2} T(z) e^{\frac{n}{2}[h(z)+sz]\sigma_3} = e^{n\ell\sigma_3/2} S(z) e^{\frac{n}{2}[h(z)+sz]\sigma_3} = e^{n\ell\sigma_3/2} R(z) M(z) e^{\frac{n}{2}[h(z)+sz]\sigma_3}, \tag{2.31}$$

where $M(z)$ above is the appropriate global parametrix. By (2.4), we know the orthogonal polynomial is given by the $(1,1)$ entry of Y, so that

$$p_n(z) = e^{\frac{n}{2}[h(z)+\ell+sz]} \left(M_{11}(z) R_{11}(z) + M_{21}(z) R_{12}(z)\right), \tag{2.32}$$

where the subscript ij indicates the (i,j) entry of the relevant matrix valued function.

Similarly, we may retrace the steps of steepest descent in the upper (lower) lenses, but away from the endpoints in $\Lambda(s)$, to see that

$$Y(z) = e^{n\ell\sigma_3/2} R(z) M(z) \begin{pmatrix} 1 & 0 \\ \pm e^{-nh(z)} & 1 \end{pmatrix} e^{\frac{n}{2}[h(z)+sz]\sigma_3}, \tag{2.33}$$

where we use the $+(-)$ sign in the upper (lower) lip of the lens. Therefore, we have that

$$p_n(z) = e^{\frac{n}{2}(\ell+sz)} \left[e^{\frac{n}{2}h(z)} \left(M_{11}R_{11} + M_{21}R_{12}\right) \pm e^{-\frac{n}{2}h(z)} \left(M_{12}R_{11} + M_{22}R_{12}\right)\right] \tag{2.34}$$

in the upper (lower) lip of the lens.

Note that in both (2.32) and (2.34), we are able to express the polynomial p_n in terms of the global parametrix M and the matrix R. The knowledge that R decays to the identity, along with our explicit solution of the global parametrix, will allow us to determine the relevant asymptotics of the polynomials as $n \to \infty$. A similar procedure can be used to recover the asymptotic expansion of the recurrence coefficients, as explained below.

We recall that the three term recurrence relation is given by

$$zp_n^n(z) = p_{n+1}^n(z) + \alpha_n p_n^n(z) + \beta_n p_{n-1}^n(z).$$

To state the recurrence coefficients in terms of Y, we first note that from (2.3) we may write

$$Y(z)z^{-n\sigma_3} = I + \frac{Y_1}{z} + \frac{Y_2}{z^2} + \mathscr{O}\left(\frac{1}{z^3}\right), \qquad z \to \infty. \tag{2.35}$$

Then, we may write the recurrence coefficients (c.f. [16]) as

$$\alpha_n = \frac{[Y_2]_{12}}{[Y_1]_{12}} - [Y_1]_{22}, \qquad \beta_n = [Y_1]_{12}[Y_1]_{21}. \tag{2.36}$$

As before, we will unwind these transformations until we are able to express the recurrence coefficients in terms of the global parametrix and the matrix valued function R, defined in (2.26). We continue by writing

$$T(z) = I + \frac{T_1}{z} + \frac{T_2}{z^2} + \mathscr{O}\left(\frac{1}{z^3}\right), \qquad z \to \infty. \tag{2.37}$$

Using (2.9d), we recall that

$$h(z;s) = f(z;s) - \ell + 2\log(z) + \frac{c_1}{z} + \frac{c_2}{z^2} + \mathscr{O}\left(\frac{1}{z^3}\right), \qquad z \to \infty, \tag{2.38}$$

so that

$$e^{-\frac{n}{2}(h(z;s)+f(z;s))} = z^{-n}e^{\frac{n\ell}{2}}\left(1 - \frac{nc_1}{2z} + \frac{nc_1^2 - 4nc_2}{8z^2} + \mathscr{O}\left(\frac{1}{z^3}\right)\right), \qquad z \to \infty. \tag{2.39}$$

Next, using (2.15) we compute

$$[T_1]_{12} = e^{-n\ell}[Y_1]_{12}, \qquad [T_1]_{21} = e^{n\ell}[Y_1]_{21} \tag{2.40a}$$

$$[T_1]_{22} = [Y_1]_{22} + \frac{nc_1}{2}, \qquad [T_2]_{12} = e^{-n\ell}\left(\frac{nc_1}{2}[Y_1]_{12} + [Y_2]_{12}\right). \tag{2.40b}$$

Thus, it easily follows that (2.36) becomes

$$\alpha_n = \frac{[T_2]_{12}}{[T_1]_{12}} - [T_1]_{22}, \qquad \beta_n = [T_1]_{12}[T_1]_{21}. \tag{2.41}$$

The above equation will be the starting point of our analysis in Chapter 5, where we prove the asymptotic expansions of the recurrence coefficients.

2.3 Construction of Global Parametrices

Below, we give a detailed description on how to solve the model problem (2.24) in the genus 0 and genus 1 cases, which will be the only two regimes we see for the linear weight under consideration. The arguments below can be easily adapted to cases of higher genera corresponding to other weights, as in [13].

2.3.1 Genus 0 Global Parametrix

In the case we are working in the genus 0 regime, $\Sigma = \gamma_{m,0}(s)$, where $\gamma_{m,0}$ is chosen so that we may construct a suitable h function satisfying both (2.9) and (2.10). The model Riemann-Hilbert problem (2.24) in the genus 0 case takes the following form. We seek $M : \mathbb{C} \setminus \gamma_{m,0} \to \mathbb{C}^{2\times 2}$ such that

$$M(z) \text{ is analytic for } z \in \mathbb{C} \setminus \gamma_{m,0}, \tag{2.42a}$$

$$M_+(z) = M_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in \gamma_{m,0}, \tag{2.42b}$$

$$M(z) = I + \mathscr{O}\left(\frac{1}{z}\right), \qquad z \to \infty. \tag{2.42c}$$

This can be solved explicitly [16, 31] as

$$M(z) = \frac{1}{\sqrt{2}\,(z^2-1)^{1/4}} \begin{pmatrix} \varphi(z)^{1/2} & i\varphi(z)^{-1/2} \\ -i\varphi(z)^{-1/2} & \varphi(z)^{1/2} \end{pmatrix}, \tag{2.43}$$

where $\varphi(z) = z + (z^2-1)^{1/2}$, with branch cuts taken on $\gamma_{m,0}$ so that $\varphi(z) = 2z + \mathscr{O}(1/z)$ and $(z^2-1)^{1/4} = z^{1/2} + \mathscr{O}(z^{-3/2})$ as $z \to \infty$.

2.3.2 Genus 1 Global Parametrix

In the genus 1 regime, we have that $\Sigma = \gamma_{m,0}(s) \cup \gamma_{c,1}(s) \cup \gamma_{m,1}(s)$, and the set of branchpoints is given by $\Lambda(s) = \{-1, 1, \lambda_0(s), \lambda_1(s)\}$, where the arcs and endpoints are chosen so that we may construct a suitable h-function. Now, the model problem (2.24) takes the form

$$M(z) \text{ is analytic for } z \in \mathbb{C} \setminus \Sigma, \tag{2.44a}$$

$$M_+(z) = M_-(z) \begin{pmatrix} e^{-2\pi i n \eta_1} & 0 \\ 0 & e^{2\pi i n \eta_1} \end{pmatrix}, \qquad z \in \gamma_{c,1}, \tag{2.44b}$$

$$M_+(z) = M_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in \gamma_{m,1}, \tag{2.44c}$$

$$M_+(z) = M_-(z) \begin{pmatrix} 0 & e^{2\pi i n \omega_0} \\ -e^{-2\pi i n \omega_0} & 0 \end{pmatrix}, \qquad z \in \gamma_{m,0}, \tag{2.44d}$$

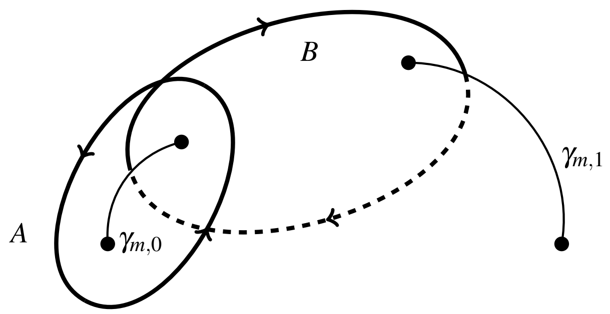


Figure 2.3: The homology basis on $\mathfrak{R}$. The bold contours are on the top sheet of $\mathfrak{R}$, and the dashed contours are on the second sheet of $\mathfrak{R}$.

$$M(z) = I + \mathscr{O}\left(\frac{1}{z}\right), \qquad z \to \infty. \tag{2.44e}$$

We follow the approach of [13, 39, 87], and solve this problem in four steps. Using the discussion preceding (2.11), $\mathfrak{R}$ is the hyperelliptic Riemann surface associated with the algebraic equation

$$\xi^2(z) = h'(z;s)^2, \tag{2.45}$$

whose branchcuts are taken along $\gamma_{m,0}$ and $\gamma_{m,1}$. To set notation, we call the top sheet $\mathfrak{R}_1$ and the bottom sheet $\mathfrak{R}_0$, so that $\mathfrak{R} = \mathfrak{R}_0 \cup \mathfrak{R}_1$. The sheet $\mathfrak{R}_1$ is fixed so that

$$\xi(z) = -f'(z;s) + \mathscr{O}\left(\frac{1}{z}\right), \tag{2.46}$$

as $z \to \infty$ on this sheet. We form a homology basis on $\mathfrak{R}$ using the A and B cycles defined in Figure 2.3.

As $\mathfrak{R}$ is of genus 1, the vector space of holomorphic differentials on $\mathfrak{R}$ has dimension 1 and is linearly generated by

$$\Omega_0 = \frac{dz}{\xi(z)\,(z^2-1)}. \tag{2.47}$$

We then define $\omega := b\Omega_0$, with b chosen to normalize ω so that

$$\oint_A \omega = 1. \tag{2.48}$$

Moreover, if we define

$$\tau := \oint_B \omega, \tag{2.49}$$

it is well known that $\Im\tau > 0$, see [43, Chapter III.2].

Step One - Removal of Jumps on Complementary Arcs

The first step aims to remove the jumps over the complementary arcs and we will follow the procedure outlined in [87]. First, we introduce the function

$$\Xi(z) = \left[\left(z^2-1\right)(z-\lambda_0)(z-\lambda_1)\right]^{1/2}, \tag{2.50}$$

with a branch cut taken on $\gamma_{m,0}$ and $\gamma_{m,1}$ and branch chosen so that $\Xi(z)/z^2 \to 1$ as $z \to \infty$. Next, define

$$\tilde{g}(z) = \Xi(z)\left[\int_{\gamma_{c,1}} \frac{\eta_1\, d\zeta}{(\zeta - z)\Xi(\zeta)} - \int_{\gamma_{m,0}} \frac{\Delta_0\, d\zeta}{(\zeta - z)\Xi_+(\zeta)}\right], \tag{2.51}$$

The constant Δ_0 is chosen so that $\tilde{g}$ is analytic at infinity. More precisely, Δ_0 is defined so that

$$\int_{\gamma_{c,1}} \frac{\eta_1\, d\zeta}{\Xi(\zeta)} - \int_{\gamma_{m,0}} \frac{\Delta_0\, d\zeta}{\Xi_+(\zeta)} = 0. \tag{2.52}$$

Note that by (2.48) and the definition of ω, it follows that $\Delta_0 = \eta_1 \tau$. Furthermore, $\tilde{g}$ is bounded near each $\lambda \in \Lambda$ and satisfies

$$\tilde{g}_+(z) - \tilde{g}_-(z) = 2\pi i \eta_1, \qquad z \in \gamma_{c,1} \tag{2.53a}$$

$$\tilde{g}_+(z) + \tilde{g}_-(z) = -2\pi i \Delta_0, \qquad z \in \gamma_{m,0}, \tag{2.53b}$$

$$\tilde{g}_+(z) + \tilde{g}_-(z) = 0, \qquad z \in \gamma_{m,1}. \tag{2.53c}$$

Next, we define

$$M_0(z) = e^{-n\tilde{g}(\infty)\sigma_3} M(z) e^{n\tilde{g}(z)\sigma_3}. \tag{2.54}$$

Then, M_0 solves the following Riemann-Hilbert problem

$$M_0(z) \text{ is analytic for } z \in \mathbb{C} \setminus \mathfrak{M}, \tag{2.55a}$$

$$M_{0,+}(z) = M_{0,-}(z) \begin{pmatrix} 0 & e^{2\pi i n(\omega_0+\Delta_0)} \\ -e^{-2\pi i n(\omega_0+\Delta_0)} & 0 \end{pmatrix}, \qquad z \in \gamma_{m,0}, \tag{2.55b}$$

$$M_{0,+}(z) = M_{0,-}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in \gamma_{m,1}, \tag{2.55c}$$

$$M_0(z) = I + \mathscr{O}\left(\frac{1}{z}\right), \qquad z \to \infty. \tag{2.55d}$$

Note that M_0 no longer has jumps over the complementary arcs. We recall above that the set of main arcs $\mathfrak{M} = \gamma_{m,0} \cup \gamma_{m,1}$.

Step Two - Solve $n = 0$

In the case that $n = 0$, the model problem for M_0 takes the form

$$M_0^{(0)}(z) \text{ is analytic for } z \in \mathbb{C} \setminus \mathfrak{M}, \tag{2.56a}$$

$$M_{0,+}^{(0)}(z) = M_{0,-}^{(0)}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in \mathfrak{M}, \tag{2.56b}$$

$$M_0^{(0)}(z) = I + \mathscr{O}\left(\frac{1}{z}\right), \qquad z \to \infty. \tag{2.56c}$$

The solution to (2.56) is well known (see for instance [16]), and is given by

$$M_0^{(0)}(z) = \frac{1}{2} \begin{pmatrix} \eta(z) + \eta(z)^{-1} & i(\eta(z) - \eta(z)^{-1}) \\ -i(\eta(z) - \eta(z)^{-1}) & \eta(z) + \eta(z)^{-1} \end{pmatrix}, \tag{2.57}$$

where

$$\eta(z) = \left(\frac{(z+1)(z-\lambda_1)}{(z-\lambda_0)(z-1)} \right)^{1/4} \tag{2.58}$$

with branch cuts on $\gamma_{m,0}$ and $\gamma_{m,1}$ and the branch of the root chosen so that

$$\lim_{z \to \infty} \eta(z) = 1. \tag{2.59}$$

It is important to understand the location of the zeros of the entries of $M_0^{(0)}(z)$, as they will play a role later in this construction. Note first that the zeros of $\eta(z) + \eta^{-1}(z)$ are the zeros of $\eta^4(z) - 1 = \left(\eta^2(z) - 1\right)\left(\eta^2(z) + 1\right)$, which is meromorphic on $\mathfrak{R}$, with a zero at ∞_1 (infinity on the sheet $\mathfrak{R}_1$) and one simple zero on each sheet of $\mathfrak{R}$. If we denote by z_1 the zero of $\eta^2(z) - 1$, then $\hat{z}_1$, which denotes the projection of z_1 onto the opposite sheet of $\mathfrak{R}$, solves $\eta^2(z) + 1$.

Step Three - Match the jumps on $\mathfrak{M}$

The next step in the solution is to match the jump conditions (2.55b) and (2.55c). We will do this by constructing two scalar functions, $\mathscr{M}_1(z,d)$ and $\mathscr{M}_2(z,d)$ which satisfy

$$\mathscr{M}_+ = \begin{cases} \mathscr{M}_- \begin{pmatrix} 0 & e^{2\pi i W} \\ e^{-2\pi i W} & 0 \end{pmatrix}, & z \in \gamma_{m,0}, \\ \mathscr{M}_- \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & z \in \gamma_{m,1}, \end{cases} \tag{2.60}$$

where

$$\mathscr{M}(z,d) = \left(\mathscr{M}_1(z,d), \mathscr{M}_2(z,d)\right), \tag{2.61}$$

$W = n(\omega_0 + \Delta_0)$, and $d \in \mathbb{C}$ is a yet to be defined constant that will be chosen to cancel the simple poles of the entries of $M_0^{(0)}$. If we can construct such functions then it is immediate from (2.56b) and (2.60) that

$$\mathscr{L}(z) := \frac{1}{2}\begin{pmatrix} \left(\eta(z)+\eta(z)^{-1}\right)\mathscr{M}_1(z,d) & i\left(\eta(z)-\eta(z)^{-1}\right)\mathscr{M}_2(z,d) \\ -i\left(\eta(z)-\eta(z)^{-1}\right)\mathscr{M}_1(z,-d) & \left(\eta(z)+\eta(z)^{-1}\right)\mathscr{M}_2(z,-d) \end{pmatrix} \tag{2.62}$$

satisfies (2.55b) and (2.55c). We can construct $\mathscr{M}_1$ and $\mathscr{M}_2$ with the help of theta functions on $\mathfrak{R}$. We define the Riemann theta function associated with τ in (2.49) in the standard way

$$\Theta(\zeta) = \sum_{m\in\mathbb{Z}} e^{2\pi i m\zeta + \pi i \tau m^2}, \qquad \zeta \in \mathbb{C}. \tag{2.63}$$

The following properties of the theta function follow immediately from (2.63):

$$\Theta \text{ is analytic in } \mathbb{C}, \tag{2.64a}$$

$$\Theta(\zeta) = \Theta(-\zeta), \tag{2.64b}$$

$$\Theta(\zeta+1) = \Theta(\zeta), \tag{2.64c}$$

$$\Theta(\zeta+\tau) = e^{-2\pi i\zeta - \pi i\tau}\Theta(\zeta). \tag{2.64d}$$

Associated with Θ is the *period lattice*, $\Lambda_\tau := \mathbb{Z} + \tau\mathbb{Z}$. If $\Theta(\zeta)$ is not identically zero, then it has a simple zero at $\zeta = \frac{1}{2} + \frac{\tau}{2} \mod \Lambda_\tau$. Next we define the Abel map as

$$u(z) = \int_1^z \omega, \qquad z \in \mathbb{C}\setminus\Sigma, \tag{2.65}$$

where we recall ω was normalized to satisfy (2.48) and the path of integration is taken on the upper sheet of $\mathfrak{R}$. By (2.48), we have that u is well defined on $\mathbb{C}\setminus\mathfrak{M}$ modulo $\mathbb{Z}$. From (2.48) and (2.49) it follows that

$$u_+(z) + u_-(z) = 0, \qquad z \in \gamma_{m,1}, \tag{2.66a}$$

$$u_+(z) + u_-(z) = \tau, \qquad z \in \gamma_{m,0}, \tag{2.66b}$$

$$u_+(z) = u_-(z), \qquad z \in \gamma_{c,1}, \tag{2.66c}$$

where again, all the equalities above are taken modulo $\mathbb{Z}$. Next we set

$$\mathscr{M}_1(z,d) := \frac{\Theta(u(z)-W+d)}{\Theta(u(z)+d)}, \qquad \mathscr{M}_2(z,d) := \frac{\Theta(-u(z)-W+d)}{\Theta(-u(z)+d)}, \tag{2.67}$$

where we recall that $W = n(\omega_0 + \Delta_0)$ and d is yet to be determined. Then, both $\mathscr{M}_1$ and $\mathscr{M}_2$ are single valued on $\mathbb{C}\setminus\mathfrak{M}$. Equations (2.64) and (2.66) immediately show that the functions $\mathscr{M}_1$ and $\mathscr{M}_2$ satisfy (2.60), as desired.

Step 4 - Choose d and normalize $\mathscr{L}$

We have now constructed $\mathscr{M}_1$ and $\mathscr{M}_2$ so that $\mathscr{L}$ defined in (2.62) satisfies (2.55b) and (2.55c). We must now choose d so that $\mathscr{L}$ is analytic in $\mathbb{C}\setminus\mathfrak{M}$ and normalize $\mathscr{L}$ so that it tends to the identity as $z\to\infty$. By the standard theory [43], for arbitrary $d\in\mathbb{C}$ the function $\Theta(u(z)-d)$ on $\mathfrak{R}$ either vanishes identically or vanishes at a single point p_1, counted with multiplicity. Recall that we have defined z_1 to be the unique finite solution to $\eta(z)^2-1=0$ and $\hat{z}_1$, its projection onto the opposite sheet of $\mathfrak{R}$, to be the unique finite solution to $\eta(z)^2+1=0$ on $\mathfrak{R}$.

We choose d so that the simple zeros of the denominators of each entry of $\mathscr{L}$ cancel the zeros of $\eta\pm\eta^{-1}$. From the remarks immediately following (2.64), this is satisfied if we set

$$d=-u(\hat{z}_1)+\frac{1}{2}+\frac{\tau}{2}\mod\Lambda_\tau, \tag{2.68}$$

as $\Theta(\zeta)=0$ when $\zeta=\frac{1}{2}+\frac{\tau}{2}\mod\Lambda_\tau$. As the Theta function is even, we have that

$$\Theta(u(\hat{z}_1)+d)=\Theta(-u(z_1)+d)=\Theta(u(z_1)-d)=0, \tag{2.69}$$

which verifies that each entry of $\mathscr{L}$ is analytic in $\mathbb{C}\setminus\mathfrak{M}$.

We must normalize $\mathscr{L}$ so that it decays to the identity as $z\to\infty$. We first note that we have an alternative formula for d,

$$d=-u(\infty_1)\mod\Lambda_\tau. \tag{2.70}$$

To see this, we note that $\eta^2(z)-1$ is meromorphic on $\mathfrak{R}$ with a zero at ∞_1, a simple zero at z_1, and poles at λ_0 and 1. By Abel's Theorem [43, Theorem III.6.3], we have that

$$u(\infty_1)+u(z_1)-u(1)-u(\lambda_0)=0\mod\Lambda_\tau.$$

Using (2.65), along with (2.48) and (2.49), we see that

$$u(1)=0,\qquad u(\lambda_0)=-\frac{1}{2}-\frac{\tau}{2}, \tag{2.71}$$

so that (2.70) follows by (2.68). As $\eta(z)-\eta(z)^{-1}\to 0$ as $z\to\infty$,

$$\det\mathscr{L}(\infty)=\mathscr{M}_1(\infty,d)\mathscr{M}_2(\infty,-d)=\frac{\Theta^2(W)}{\Theta^2(0)}. \tag{2.72}$$

As $\mathscr{L}$ has the same jumps as M_0 in (2.55b) and (2.55c), we can conclude that $\det\mathscr{L}$ is entire, and as $\mathscr{L}$ is bounded at infinity, we have that

$$\det\mathscr{L}(z)=\frac{\Theta^2(W)}{\Theta^2(0)}. \tag{2.73}$$

If $\Theta(W) \neq 0$, then

$$M_0(z) = \mathscr{L}^{-1}(\infty)\mathscr{L}(z) \tag{2.74}$$

solves (2.55). The condition $\Theta(W) \neq 0$ can be rewritten as

$$n(\omega_0 + \Delta_0) \neq \frac{1}{2} + \frac{\tau}{2} \mod \Lambda_\tau, \tag{2.75}$$

so that we have proven the following Lemma (see Theorem 2.17 of [13]).

Lemma 2.2. *The model Riemann-Hilbert problem* (2.55) *has a solution if*

$$n(\omega_0 + \Delta_0) \neq \frac{1}{2} + \frac{\tau}{2} \mod \Lambda_\tau. \tag{2.76}$$

Moreover, the solution is given by

$$M_0(z) = \mathscr{L}^{-1}(\infty)\mathscr{L}(z),$$

where $\mathscr{L}$ *is defined in* (2.62).

2.4 Construction of Local Parametrices

Recalling the discussion preceding (2.25), we will need a more detailed local analysis about the endpoints $\lambda \in \Lambda$. For non-critical situations, we will only need to use the Bessel and Airy parametrices. Although these constructions are now standard, we state them below for completeness. For complete details we refer the reader to [16, 34, 39, 65].

2.4.1 Airy Parametrix

Here, we consider the local parametrix in a neighborhood of the soft edge λ_0. Note that in light of (2.9), we are assuming we are working in the genus 1 regime, as the genus 0 regime would only require a local analysis about the endpoints ± 1.

Now, by (2.9), $\Re h(z) = c\,(z-\lambda_0)^{3/2} + \mathscr{O}\,(z-\lambda_0)^{5/2}$ as $z \to \lambda_0$ for some $c \neq 0$. We will also make use of the following function

$$h^{(\lambda)}(z) = \int_\lambda^z h'(z;s)\,ds, \tag{2.77}$$

where the path of integration emanates upwards in the complex plane from λ and does not cross $\Omega(s)$. We will also find in Chapters 4 and 5, by use of the Boutroux condition, that there exist real constants $K_\pm^\lambda$ such that

$$h_\pm^{(\lambda)}(z) = h(z) + iK_\pm^\lambda. \tag{2.78}$$

As $\lambda = \lambda_0$, the main arc $\gamma_{m,0}$ lies to the left of λ and the complementary arc $\gamma_{c,1}$ lies to the right of λ, where left and right are in reference to the orientation of $\hat{\Sigma}$.

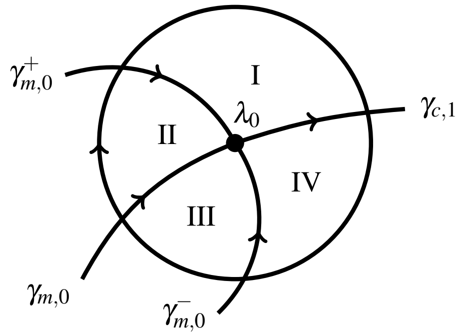


Figure 2.4: Definition of Sectors I, II, III, and IV within D_{λ_0}.

We want to solve the following Riemann-Hilbert problem in a neighborhood of λ_0, D_{λ_0},

$$P^{(\lambda_0)}(z) \text{ is analytic for } z \in D_{\lambda_0} \setminus \hat{\Sigma}, \tag{2.79a}$$

$$P_+^{(\lambda_0)}(z) = P_-^{(\lambda_0)}(z) j_S(z), \qquad z \in D_{\lambda_0} \cap \hat{\Sigma}, \tag{2.79b}$$

$$P^{(\lambda_0)}(z) = \left(I + \mathscr{O}\left(\frac{1}{n}\right)\right) M(z), \qquad n \to \infty, \quad z \in \partial D_{\lambda_0}, \tag{2.79c}$$

where we recall

$$j_S(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{-nh(z)} & 1 \end{pmatrix}, & z \in \gamma_{m,0}^{\pm}, \\ \begin{pmatrix} 0 & e^{2\pi i n \omega_0} \\ -e^{-2\pi i n \omega_0} & 0 \end{pmatrix}, & z \in \gamma_{m,0}, \\ \begin{pmatrix} e^{-2\pi i n \eta_1} & e^{\frac{n}{2}(h_+(z)+h_-(z))} \\ 0 & e^{2\pi i n \eta_1} \end{pmatrix}, & z \in \gamma_{c,1}. \end{cases} \tag{2.80}$$

We also require that $P^{(\lambda_0)}$ has a continuous extension to $\overline{D}_{\lambda_0} \setminus \hat{\Sigma}$ and remains bounded as $z \to \lambda_0$. We solve for $P^{(\lambda_0)}$ by setting

$$P^{(\lambda_0)}(z) = U^{(\lambda_0)}(z) e^{-\frac{n}{2}h(z)\sigma_3}, \tag{2.81}$$

where $U^{(\lambda_0)}$ satisfies a Riemann-Hilbert problem in D_{λ_0}, with jump $U_+^{(\lambda_0)}(z) = U_-^{(\lambda_0)}(z) j_{U^{(\lambda_0)}}(z)$ for $z \in D_{\lambda_0} \cap \hat{\Sigma}$. Here, the jump matrix $j_{U^{(\lambda_0)}}$ is given by

$$j_{U^{(\lambda_0)}}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & z \in \gamma_{m,0}^{\pm}, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \gamma_{m,0}, \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & z \in \gamma_{c,1}. \end{cases} \tag{2.82}$$

The Riemann-Hilbert problem for $U^{(\lambda_0)}$ can be constructed via Airy functions as in [16, 39]. We define

$$A(\zeta) = \begin{cases} \begin{pmatrix} y_0(\zeta) & -y_2(\zeta) \\ y_0'(\zeta) & -y_2'(\zeta) \end{pmatrix}, & \arg\zeta \in \left(0, \frac{2\pi}{3}\right), \\ \begin{pmatrix} -y_1(\zeta) & -y_2(\zeta) \\ -y_1'(\zeta) & -y_2'(\zeta) \end{pmatrix}, & \arg\zeta \in \left(\frac{2\pi}{3}, \pi\right), \\ \begin{pmatrix} -y_2(\zeta) & y_1(\zeta) \\ -y_2'(\zeta) & y_1'(\zeta) \end{pmatrix}, & \arg\zeta \in \left(-\pi, -\frac{2\pi}{3}\right), \\ \begin{pmatrix} y_0(\zeta) & y_1(\zeta) \\ y_0'(\zeta) & y_1'(\zeta) \end{pmatrix}, & \arg\zeta \in \left(-\frac{2\pi}{3}, 0\right). \end{cases} \tag{2.83}$$

where

$$y_0(\zeta) := \mathrm{Ai}(\zeta), \qquad y_1(\zeta) := \omega \mathrm{Ai}(\omega\zeta), \qquad y_2(\zeta) := \omega^2 \mathrm{Ai}(\omega^2\zeta), \tag{2.84}$$

Ai is the Airy function, and $\omega := \exp(2\pi i/3)$. We remark here, following [1, Section 10.4], that for any $\varepsilon > 0$,

$$\mathrm{Ai}(\zeta) = \frac{1}{2\sqrt{\pi}\zeta^{1/4}} e^{-\frac{2}{3}\zeta^{3/2}} \left(1 + \mathscr{O}\left(\frac{1}{\zeta^{3/2}}\right)\right), \tag{2.85a}$$

$$\mathrm{Ai}'(\zeta) = -\frac{\zeta^{1/4}}{2\sqrt{\pi}} e^{-\frac{2}{3}\zeta^{3/2}} \left(1 + \mathscr{O}\left(\frac{1}{\zeta^{3/2}}\right)\right), \tag{2.85b}$$

as $\zeta \to \infty$ with $-\pi + \varepsilon \le \arg\zeta \le \pi - \varepsilon$. We next define

$$f_{n,A}(z) = n^{2/3} f_A(z), \qquad f_A(z) = \left[-\frac{3}{4} h^{(\lambda)}(z)\right]^{2/3}, \tag{2.86}$$

so that $f_A(z)$ conformally maps a neighborhood of λ_0 to a neighborhood of 0. Recall that we still have the freedom to choose the precise description of $\gamma_{m,0}^{\pm}$, so we choose them in D_{λ_0} so they are mapped to the rays $\{z : \arg z = \pm\frac{2\pi}{3}\}$, respectively, under the map f_A.

Then, we have that

$$U^{(\lambda_0)}(z) = E_n^{(\lambda_0)}(z) A\left(f_{n,A}(z)\right), \tag{2.87}$$

where $E_n^{(\lambda_0)}$ is any analytic function, satisfies the jumps given in (2.82). The analytic prefactor $E_n^{(\lambda_0)}$ is chosen so that

$$P^{(\lambda_0)}(z) = E_n^{(\lambda_0)}(z) A\left(f_{n,A}(z)\right) e^{-\frac{n}{2}h(z)\sigma_3} \tag{2.88}$$

satisfies the matching condition (2.79c) and is given by

$$E_n^{(\lambda_0)}(z) = \begin{cases} M(z) e^{-\frac{1}{2}niK_+^{\lambda}\sigma_3} L_n^{(\lambda_0)}(z)^{-1}, & z \in \text{I}, \text{II}, \\ M(z) e^{-\frac{1}{2}niK_-^{\lambda}\sigma_3} L_n^{(\lambda_0)}(z)^{-1}, & z \in \text{III}, \text{IV}, \end{cases} \tag{2.89}$$

where Sectors I, II, III, and IV are defined in Figure 2.4, and

$$L_n^{(\lambda_0)}(z) = \frac{1}{2\sqrt{\pi}} n^{-\sigma_3/6} f_A(z)^{-\sigma_3/4} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}.$$

In the formulas above, the branch cut for $f_A^{1/4}$ is taken on $\gamma_{m,0}$ and is the principal branch.

The case where $\lambda = \lambda_1$ can be handled similarly. The main difference here is that the complementary arc leads into λ and the main arc exits λ, and in this sense the orientation of the local problem at λ_1 is the reverse of the situation depicted in Figure 2.4. However, as detailed in [34, Section 7.6], we may either use appropriate choices of Airy functions in a neighborhood of λ_1, or exploit certain symmetrical features of our problem, as in done in Chapter 4.

2.4.2 Bessel Parametrix

Now we assume that we are looking at the analysis near $z = 1$, which is a hard edge, and we recall that $\Re h(z) = \mathscr{O}\left((z-1)^{1/2}\right)$ as $z \to 1$. We will actually show in the construction of h in Chapter 4 that

$$h(z) = c\,(z-1)^{1/2} + \mathscr{O}\left((z-1)^{3/2}\right), \qquad z \to 1, \tag{2.90}$$

for some $c \neq 0$. Recall that we wish to solve

$$P^{(1)}(z) \text{ is analytic for } z \in D_1 \setminus \hat{\Sigma}, \tag{2.91a}$$

$$P_+^{(1)}(z) = P_-^{(1)}(z) j_S(z), \qquad z \in D_1 \cap \hat{\Sigma}, \tag{2.91b}$$

$$P^{(1)}(z) = \left(I + \mathscr{O}\left(\frac{1}{n}\right)\right) M(z), \qquad n \to \infty, \quad z \in \partial D_1, \tag{2.91c}$$

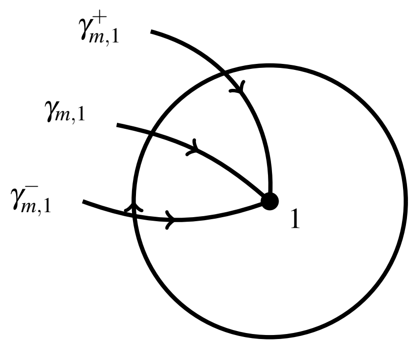


Figure 2.5: Structure of $\hat{\Sigma}$ in D_1 when $L = 1$.

where $P^{(1)}$ has a continuous extension to $\overline{D}_1 \setminus \hat{\Sigma}$ and remains bounded as $z \to 1$, and where the jump matrix j_S in D_1 is given by

$$j_S(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{-nh(z)} & 1 \end{pmatrix}, & z \in \gamma_{m,L}^{\pm}, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \gamma_{m,L}. \end{cases} \tag{2.92}$$

Analogously to the analysis in the soft edge, we define $P^{(1)}(z) = U^{(1)}(z)e^{-\frac{n}{2}h(z)\sigma_3}$, so that $U^{(1)}$ solves a new Riemann-Hilbert problem in D_1, with jump matrix given by

$$j_{U^{(1)}}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & z \in \gamma_{m,0}^{\pm}, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \gamma_{m,0}. \end{cases} \tag{2.93}$$

Now, $U^{(1)}$ can be written explicitly in terms of Bessel functions, as in [65], and we state this construction below. First set

$$b_1(\zeta) = H_0^{(1)}\left(2(-\zeta)^{1/2}\right), \qquad b_2(\zeta) = H_0^{(2)}\left(2(-\zeta)^{1/2}\right), \tag{2.94a}$$

$$b_3(\zeta) = I_0\left(2\zeta^{1/2}\right), \qquad b_4(\zeta) = K_0\left(2\zeta^{1/2}\right), \tag{2.94b}$$

where I_0 is the modified Bessel function of the first kind, K_0 is the modified Bessel function of the second kind, and $H_0^{(1)}$ and $H_0^{(2)}$ are Hankel functions of the first and second kind, respectively. With this in hand, we may define the Bessel parametrix as

$$B(\zeta) = \begin{cases} \begin{pmatrix} \frac{1}{2}b_2(\zeta) & -\frac{1}{2}b_1(\zeta) \\ -\pi z^{1/2}b_2'(\zeta) & \pi z^{1/2}b_1'(\zeta) \end{pmatrix}, & -\pi < \arg\zeta < -\frac{2\pi}{3}, \\ \begin{pmatrix} b_3(\zeta) & \frac{i}{\pi}b_4(\zeta) \\ 2\pi i z^{1/2}b_3'(\zeta) & -2z^{1/2}b_4'(\zeta) \end{pmatrix}, & |\arg\zeta| < \frac{2\pi}{3}, \\ \begin{pmatrix} \frac{1}{2}b_1(\zeta) & \frac{1}{2}b_2(\zeta) \\ \pi z^{1/2}b_1'(\zeta) & \pi\zeta^{1/2}b_2'(\zeta) \end{pmatrix}, & \frac{2\pi}{3} < \arg\zeta < \pi. \end{cases} \tag{2.95}$$

The asymptotics of the Bessel parametrix in each of the regions above are provided in [65, Section 6], where is was shown that

$$B(\zeta) = \frac{\left(2\pi\zeta^{1/2}\right)^{-\sigma_3/2}}{\sqrt{2}} \begin{pmatrix} 1+\mathscr{O}\left(\zeta^{-1/2}\right) & i+\mathscr{O}\left(\zeta^{-1/2}\right) \\ i+\mathscr{O}\left(\zeta^{-1/2}\right) & 1+\mathscr{O}\left(\zeta^{-1/2}\right) \end{pmatrix} e^{2\zeta^{1/2}\sigma_3}, \qquad \zeta\to\infty, \tag{2.96}$$

holds in each sector defined in (2.95). Using the conformal map, $f_{n,B}$, where

$$f_{n,B}(z) = n^2 f_B(z), \qquad \text{where} \qquad f_B(z) = \frac{h(z)^2}{16}, \tag{2.97}$$

the matrix $U^{(1)}$ is given by

$$U^{(1)}(z) = E_n^{(1)}(z)B(f_{n,B}(z)), \tag{2.98}$$

where $E_n^{(1)}$ is an analytic prefactor chosen to ensure the matching condition (2.91c). Therefore, we have that

$$E_n^{(1)}(z) = M(z)L_n^{(1)}(z)^{-1}, \qquad L_n^{(1)}(z) := \frac{1}{\sqrt{2}}(2\pi n)^{-\sigma_3/2} f_B(z)^{-\sigma_3/4} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \tag{2.99}$$

where all branch cuts above are again taken to be principal branches.

A similar analysis may be conducted around $z=-1$, and we state the solution to the local parametrix here is given by

$$P^{(-1)}(z) = E_n^{(-1)}(z)\tilde{B}\left(\tilde{f}_{n,B}(z)\right)e^{-\frac{n}{2}h(z)} \tag{2.100}$$

where $\tilde{B}(z) = \sigma_3 B(z)\sigma_3$,

$$\tilde{f}_{n,B}(z) = n^2\tilde{f}_B(z), \qquad \tilde{f}_B(z) = \frac{\tilde{h}(z)^2}{16}, \tag{2.101}$$

and $\tilde{h}(z) = h(z) - 2\pi i$. Similarly, we have

$$E_n^{(-1)}(z) = M(z)L_n^{(-1)}(z)^{-1}, \qquad L_n^{(-1)}(z) := \frac{1}{\sqrt{2}}(2\pi n)^{-\sigma_3/2}\tilde{f}_B(z)^{-\sigma_3/4} \begin{pmatrix} -1 & i \\ i & -1 \end{pmatrix}. \tag{2.102}$$

With the necessary background on the Fokas-Its-Kitaev Riemann-Hilbert problem and Deift-Zhou steepest descent now in hand, we move on to applying them to the Kissing polynomials.

Chapter 3

The Even Degree Kissing Polynomials

The main goal of this chapter is to provide the proof of existence for the even degree Kissing polynomials. To do this, we first use the fact that the Kissing polynomials solve the Riemann-Hilbert problem introduced in Chapter 2 to deduce that the Kissing polynomials satisfy a second order differential equation. Using results of [5] and [26], we will be able to use the basic theory of differential equations to provide the existence proof for the even degree Kissing polynomials, which incidentally are those polynomials whose zeros are used in the complex quadrature scheme introduced in Section 1.1.

3.1 Preliminary Results on the Kissing Polynomials

In this section, we recall properties of the Kissing polynomials from [5, 26] which will be used in the later sections of this chapter. As a reminder, the results of this section do not constitute original work of the author of this thesis, and are included for review purposes. We first recall the definition of the Kissing polynomial, $p_n(z,\omega)$, as a monic polynomial of degree exactly n in z which satisfies

$$\int_{-1}^{1} p_n(z;\omega) z^k e^{i\omega z}\, dz = \begin{cases} 0, & k = 0, 1, \ldots, n-1, \\ \chi_n(\omega), & k = n, \end{cases} \tag{3.1}$$

for some $\chi_n(\omega) \neq 0$. As shown in [5], the polynomials are symmetric over the imaginary axis, that is,

$$p_n(z) = (-1)^n \overline{p_n(-\bar{z})}, \qquad z \in \mathbb{C}. \tag{3.2}$$

In particular, the above equation implies that if z_1 is a zero of $p_n(z;\omega)$, then its reflection over the imaginary axis, $-\overline{z_1}$, is also a zero of $p_n(z;\omega)$.

Let

$$\mu_n(\omega) = \int_{-1}^{1} z^n e^{i\omega z}\, dz, \qquad n \in \mathbb{N} \cup \{0\} \tag{3.3}$$

be the *moments* of the weight function and set

$$H_n(\omega) = \begin{pmatrix} \mu_0(\omega) & \mu_1(\omega) & \dots & \mu_n(\omega) \\ \mu_1(\omega) & \mu_2(\omega) & \dots & \mu_{n+1}(\omega) \\ \vdots & \vdots & & \vdots \\ \mu_n(\omega) & \mu_{n+1}(\omega) & \dots & \mu_{2n}(\omega) \end{pmatrix} \tag{3.4}$$

and

$$h_n(\omega) = \det H_n(\omega), \tag{3.5}$$

where $n \in \mathbb{N}$, to be the nth *Hankel matrix* and *Hankel determinant*, respectively.

As seen in Chapter 1, the polynomial $p_n(z;\omega)$ itself can also be written in terms of these Hankel determinants as follows:

$$p_n(z;\omega) = \frac{1}{h_{n-1}} \det \begin{bmatrix} \mu_0(\omega) & \mu_1(\omega) & \cdots & \mu_{n-1}(\omega) & 1 \\ \mu_1(\omega) & \mu_2(\omega) & \cdots & \mu_n(\omega) & z \\ \vdots & \vdots & & \vdots & \vdots \\ \mu_n(\omega) & \mu_{n+1}(\omega) & \cdots & \mu_{2n-1}(\omega) & z^n \end{bmatrix}. \tag{3.6}$$

Moreover, as shown in [51, Chapter 2], both the polynomial and the Hankel determinant can be expressed as an n-fold integral. The following result, attributed to Heine, should be familiar to researchers in random matrix theory, as it expresses the Hankel determinant as a partition function. Indeed, we may write

$$h_{n-1}(\omega) = \frac{1}{n!} \int_{[-1,1]^n} \prod_{0 \le k < \ell \le n-1} (x_\ell - x_k)^2 \, e^{i\omega(x_0 + \cdots + x_{n-1})} \, dx_0 \dots dx_{n-1}, \tag{3.7a}$$

and

$$p_n(z;\omega) = \frac{1}{n! h_{n-1}} \int_{[-1,1]^n} \prod_{m=0}^{n-1} (z - x_m) \prod_{0 \le k < \ell \le n-1} (x_\ell - x_k)^2 \, e^{i\omega(x_0 + \cdots + x_{n-1})} \, dx_0 \dots dx_{n-1}. \tag{3.7b}$$

Although we will not directly use the above integral formulas in this chapter, they were of great importance in the study of the Kissing polynomials as $\omega \to \infty$. For instance, by asymptotically expanding (3.7a) and applying combinatorial arguments, the authors of [26] were able to show that h_{2n-1} does not vanish for large ω and that h_{2n} vanishes approximately at multiples of π as $\omega \to \infty$. This result already implies that the even degree Kissing polynomials exist for large enough ω, whereas the odd degree polynomials fail to exist for a discrete set of values once ω is large enough. Furthermore, by using the same asymptotic techniques on the integral in (3.7b), the authors of [26] were able to show that the zeros of $p_{2n}(z;\omega)$ could be partitioned as $\{z_j^1\}_{j=1}^n$ and $\{z_j^2\}_{j=1}^n$, where

$$z_j^1 = -1 + \frac{it_j}{\omega} + \mathcal{O}\left(\frac{1}{\omega^2}\right), \qquad z_j^2 = 1 + \frac{it_j}{\omega} + \mathcal{O}\left(\frac{1}{\omega^2}\right), \qquad \omega \to \infty, \tag{3.8}$$

for $j = 1,\ldots,n$, and t_j is a zero of the Laguerre polynomial of degree n. As shown in Chapter 1, this result was crucial in establishing the connection between numerical steepest descent and complex Gaussian quadrature.

The asymptotic techniques based on Heine's formula are very powerful when trying to establish properties of the Kissing polynomials as $\omega \to \infty$. However, in order to show that the even degree Kissing polynomials exist for all ω, and not just large enough ω, we must turn to the Riemann-Hilbert problem. As seen in Chapter 2, the Kissing polynomials can also be formulated as part of a solution to the Riemann-Hilbert problem

$$Y_n(z;\omega) \text{ is analytic for } z \in \mathbb{C}\setminus[-1,1], \tag{3.9a}$$

$$Y_{n,+}(z;\omega) = Y_{n,-}(z;\omega)\begin{pmatrix} 1 & e^{i\omega z} \\ 0 & 1 \end{pmatrix}, \qquad z \in (-1,1), \tag{3.9b}$$

$$Y_n(z;\omega) = \left(I + \mathscr{O}\left(\frac{1}{z}\right)\right) z^{n\sigma_3}, \qquad z \to \infty, \tag{3.9c}$$

$$Y_n(z;\omega) = \mathscr{O}\begin{pmatrix} 1 & \log|z\mp 1| \\ 1 & \log|z\mp 1| \end{pmatrix}, \qquad z \to \pm 1. \tag{3.9d}$$

Provided p_n and p_{n-1} both exist as monic polynomials of degree n and $n-1$, respectively, the solution to (3.9) is given uniquely by

$$Y(z) = \begin{pmatrix} p_n(z;\omega) & \frac{1}{2\pi i}\int_{-1}^{1} \frac{p_n(s,\omega)e^{i\omega s}}{s-z}\,ds \\ -2\pi i \kappa_{n-1}^2 p_{n-1}(z;\omega) & -\kappa_{n-1}^2 \int_{-1}^{1} \frac{p_{n-1}(s,\omega)e^{i\omega s}}{s-z}\,ds \end{pmatrix}, \tag{3.10}$$

where κ_{n-1} is the leading coefficient of the orthonormal polynomial, or equivalently,

$$\frac{1}{\kappa_{n-1}^2(\omega)} = \chi_{n-1}(\omega). \tag{3.11}$$

The condition (3.9c) can be rewritten

$$Y(z) = \left(I + \frac{A_n(\omega)}{z} + \frac{B_n(\omega)}{z^2} + \mathscr{O}\left(\frac{1}{z^3}\right)\right) z^{n\sigma_3}, \qquad z \to \infty, \tag{3.12}$$

where

$$A_n(\omega) = \begin{bmatrix} a_{11,n} & a_{12,n} \\ a_{21,n} & a_{22,n} \end{bmatrix}, \qquad B_n(\omega) = \begin{bmatrix} b_{11,n} & b_{12,n} \\ b_{21,n} & b_{22,n} \end{bmatrix}, \tag{3.13}$$

and the $a_{ij,n}, b_{ij,n}$ are functions of n and ω.

Provided that the polynomials $p_n(z;\omega)$ and $p_{n\pm 1}(z;\omega)$ exist for the given values of n and ω, we still have that the Kissing polynomials satisfy the three term recurrence relation (1.18),

$$p_{n+1}(z;\omega) = (z - \alpha_n(\omega))\, p_n(z;\omega) - \beta_n(\omega) p_{n-1}(z;\omega). \tag{3.14}$$

Using the fact that

$$\frac{d}{d\omega}\mu_k(\omega) = i\mu_{k+1}(\omega), \tag{3.15}$$

we may use (1.26) to write that

$$\alpha_n(\omega) = -i\left[\frac{\dot{h}_n(\omega)}{h_n(\omega)} - \frac{\dot{h}_{n-1}(\omega)}{h_{n-1}(\omega)}\right], \qquad \beta_n(\omega) = \frac{h_n(\omega)h_{n-2}(\omega)}{h_{n-1}^2(\omega)}, \tag{3.16}$$

where the $\dot{h}_n$ indicates differentiation with respect to the parameter ω.

We can calculate the relevant entries of (3.13) in terms of Hankel determinants by looking at the expansion of (3.10) at infinity, as follows:

$$a_{11,n} = i\frac{\dot{h}_{n-1}}{h_{n-1}}, \qquad a_{21,n} = -\frac{2\pi i h_{n-2}}{h_{n-1}} = -\frac{2\pi i}{\chi_{n-1}}, \qquad a_{12,n} = -\frac{h_n}{2\pi i h_{n-1}} = -\frac{\chi_n}{2\pi i}, \tag{3.17a}$$

and

$$a_{22,n} = -i\frac{\dot{h}_{n-1}}{h_{n-1}}, \qquad b_{21,n} = 2\pi\frac{\dot{h}_{n-2}}{h_{n-1}}, \qquad b_{12,n} = \frac{1}{2\pi}\frac{\dot{h}_n}{h_{n-1}}. \tag{3.17b}$$

3.1.1 Analysis of the Kissing Pattern

An important consequence of the weight function not being positive is the fact that, even when the existence of p_n is assured for some specific values of n and ω, its roots lie in the complex plane. They come in pairs of two, symmetric with respect to the imaginary axis, as a consequence of (3.2).

When $\omega = 0$, $p_n(z)$ is a multiple of the classical Legendre polynomial and its roots are real and are located in the interval $(-1,1)$. For increasing values of ω, they follow a trajectory in the upper half plane, as illustrated in Figures 3.1 and 3.2. The trajectories corresponding to polynomials of consecutive even and odd degree touch at a discrete set of frequencies ω: the zeros of the polynomials "kiss" and this phenomenon motivates their name.

Bearing in mind formula (3.6), it comes as no surprise that these critical values of ω correspond exactly to the zeros of the Hankel determinant $h_{n-1}(\omega)$. Using a different normalization we can define a polynomial that always exists, regardless of the zeros of the Hankel determinant,

$$\tilde{p}_n(z) = h_{n-1}p_n(z). \tag{3.18}$$

Observe that, unlike $p_n(z)$, this new polynomial always exists; if $h_{n-1} = 0$ for some value of ω then it has degree less than n. From the theory of quasi-orthogonal polynomials, or formal orthogonal polynomials, it is known that the degree of $\tilde{p}_n(z)$ equals the dimension of the largest leading non-singular principal submatrix of the Hankel matrix H_{n-1} [23]. This is the same as saying that the degree of $\tilde{p}_n$ is equal to the degree of the first existing polynomial p_k of lower degree $k \leq n$. The plots indicate, and we show below, that the degree of $\tilde{p}_n(z)$ at a kissing point is actually $n-1$.

We study what happens for $n \geq 1$ as ω tends to a critical value $\omega^* > 0$ such that $h_n(\omega^*) = 0$. We can rewrite the three-term recurrence relation (3.14) in terms of the new polynomials, and then using

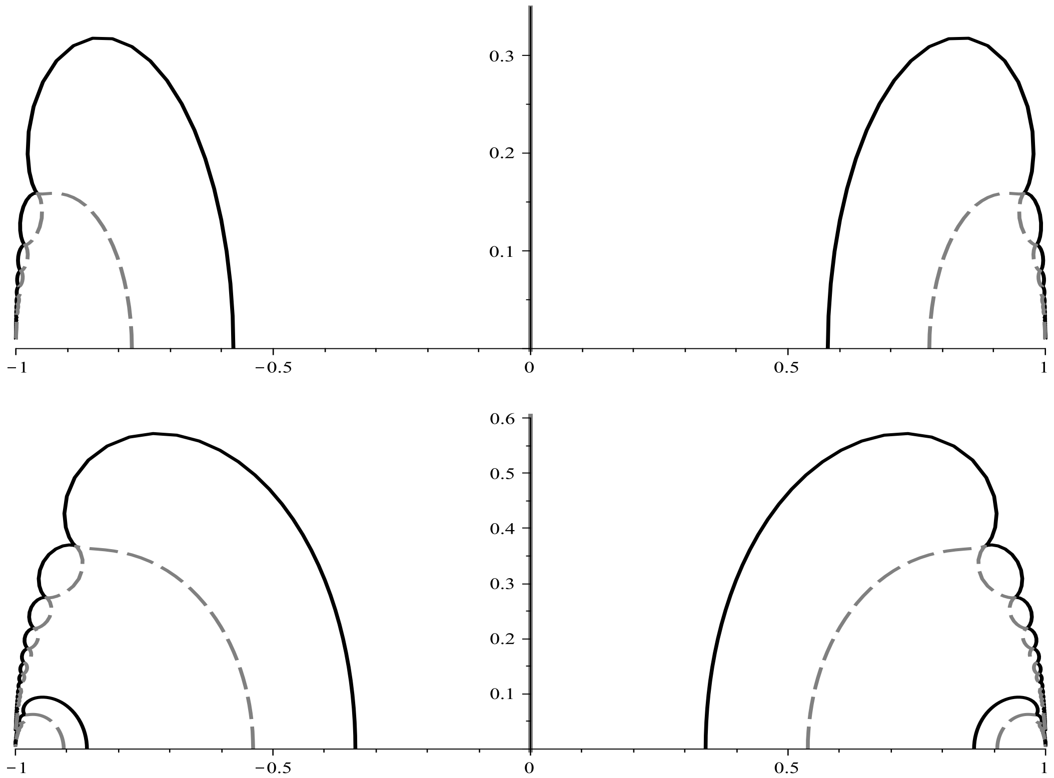


Figure 3.1: Trajectories of the zeros of p_2 (dark, solid) and p_3 (grey, dashed), at the top, and p_4 (dark, solid) and p_5 (grey, dashed), at the bottom. We note that both p_3 and p_5 always have a zero on the imaginary axis.

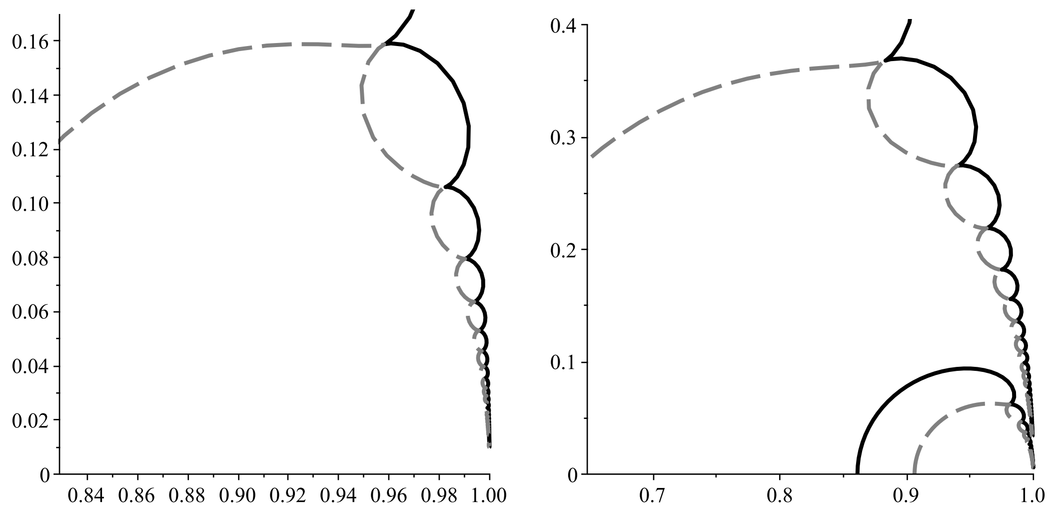


Figure 3.2: Close-ups of the kissing patterns near the right endpoint $+1$, for p_2 (dark, solid) and p_3 (grey, dashed), on the left, and for p_4 (dark, solid) and p_5 (grey, dashed), on the right.

(3.16), in terms of Hankel determinants:

$$\tilde{p}_{n+1}(z)h_{n-1}^2 = \left[h_n h_{n-1} z + i\left(\dot{h}_n h_{n-1} - \dot{h}_{n-1} h_n\right)\right]\tilde{p}_n(z) - h_n^2 \tilde{p}_{n-1}(z). \tag{3.19}$$

For critical values of ω, where h_n vanishes, this expression simplifies. Indeed, if $h_n(\omega^*) = 0$ and $\dot{h}_n(\omega^*), h_{n-1}(\omega^*) \neq 0$, it becomes

$$\tilde{p}_{n+1}(z) = i\frac{\dot{h}_n}{h_{n-1}}\tilde{p}_n(z), \tag{3.20}$$

i.e. $\tilde{p}_{n+1}(z)$ is a scalar multiple of $\tilde{p}_n(z)$. As shown below in Lemma 3.3 and its proof, $h_n(\omega^*) = 0$ guarantees that $\dot{h}_n(\omega^*), h_{n-1}(\omega^*) \neq 0$. This means that at zeros of h_n the polynomial $\tilde{p}_{n+1}$ reduces to a constant multiple of p_n of lower degree. Hence, their zeros coincide and the trajectories of both polynomials "kiss".

3.1.2 Results on the Hankel Determinants and Recurrence Coefficients

The following lemmas on the Hankel determinants and recurrence coefficients will be used extensively in this chapter. Proofs of these results can be found in [5, 26]. We first start by stating that the Hankel determinants associated to the Kissing polynomials are always real valued.

Lemma 3.1. *For any $\omega \in \mathbb{R}$ and any $n \geq 0$, the Hankel determinant $h_n(\omega)$ given by* (3.5) *is real valued. Furthermore, $h_n(\omega)$ is an even function of ω.*

The following lemma is a complex version of the Toda evolution equation, which is well known in the theory of integrable systems and in random matrix theory (cf. for instance [19, Theorem 1.4.2]), [16, Proposition 18.1], [18, Section 2], or [89, Chapter 3]).

Lemma 3.2. *It is true that*

$$\ddot{h}_n(\omega)h_n(\omega) - \dot{h}_n^2(\omega) = -h_{n-1}(\omega)h_{n+1}(\omega), \qquad n \geq 1, \tag{3.21}$$

where $\dot{}$ indicates differentiation with respect to ω. Alternatively, we may write

$$\frac{d^2}{d\omega^2}\log h_{n-1}(\omega) = -\beta_n(\omega), \tag{3.22}$$

in terms of the recurrence coefficient $\beta_n(\omega)$ in (3.14).

Similarly, as a function of the parameter ω, the recurrence coefficients themselves satisfy the differential–difference equations

$$\begin{aligned} \dot{\alpha}_n(\omega) &= i\left(\beta_{n+1}(\omega) - \beta_n(\omega)\right) \\ \dot{\beta}_n(\omega) &= i\beta_n(\omega)(\alpha_n(\omega) - \alpha_{n-1}(\omega)). \end{aligned} \tag{3.23}$$

Again, these are nothing but a complex case of the classical Toda lattice equations, written in the Flaschka variables; these equations are known to govern the deformation of the recurrence coefficients whenever the measure of orthogonality is a perturbation of a classical one with an exponential factor linear in the parameter, which is ω in our case. We refer the reader to [51, Section 2.8] for more details.

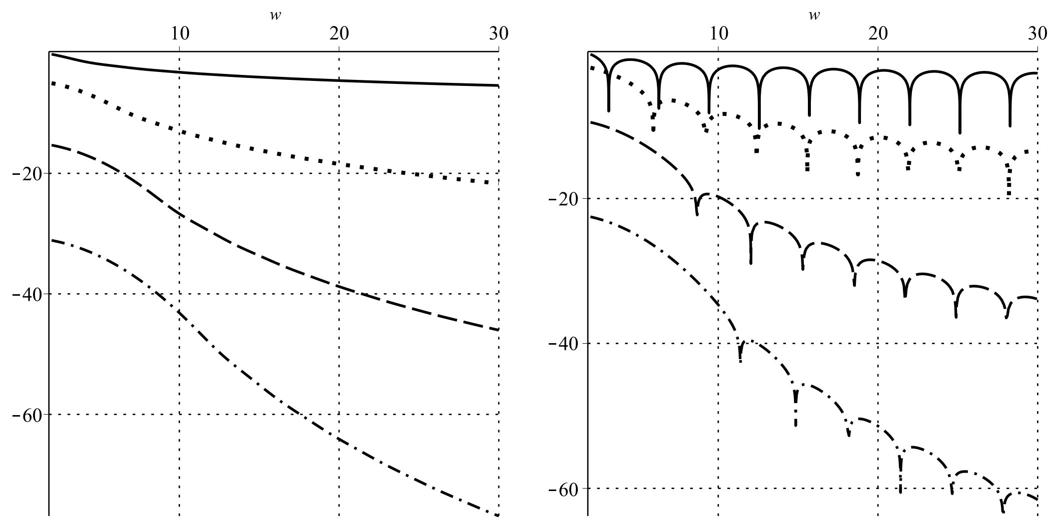


Figure 3.3: On the left, plot in log-scale of $\log|h_1(\omega)|$ (solid line), $\log|h_3(\omega)|$ (dotted), $\log|h_5(\omega)|$ (dashed) and $\log|h_7(\omega)|$ (dashed-dotted). On the right, plot in log-scale of $\log|h_2(\omega)|$ (solid line), $\log|h_4(\omega)|$ (dotted), $\log|h_6(\omega)|$ (dashed) and $\log|h_8(\omega)|$ (dashed-dotted).

In Figure 3.3 we display $\log|h_n(\omega)|$ as a function of ω for different values of n; it is apparent that there is a clear difference in behavior depending on the parity of n. Based on this figure, and similar ones that can be obtained by direct computation in MAPLE, we formulate the following two results on the zeros of the Hankel determinants. Proofs are included for convenience, but it should again be stressed that the following two lemmas are not the original work of the author of this thesis.

Lemma 3.3. *There is no $n \geq 1$ and $\omega^* > 0$ such that*

$$h_{n-1}(\omega^*) = h_n(\omega^*) = 0.$$

Proof. Assume that $h_{n-1}(\omega^*) = h_n(\omega^*) = 0$ for some $\omega = \omega^*$. Then by (3.21) we have $h_n' = 0$ and so h_n has a double root. Since both terms on the left hand side of (3.21) have a double root, so must the right hand side and this implies that either $h_{n+1} = 0$ as well, or that h_{n-1} has a double root.

In the latter case, two consecutive Hankel determinants have a double root. In the former case this happens too. Indeed, in this case we have $h_n = \dot{h}_n = h_{n-1} = h_{n+1} = 0$. We can reformulate (3.21) as

$$\ddot{h}_{n+1}(\omega)h_{n+1}(\omega) - [\dot{h}_{n+1}(\omega)]^2 = -h_n(\omega)h_{n+2}(\omega), \qquad n \geq 0.$$

It follows that $\dot{h}_{n+1}(\omega) = 0$, i.e. both $h_n(\omega)$ and $h_{n+1}(\omega)$ have a double root.

It remains to rule out two consecutive double roots. Let us assume they are h_n and h_{n+1}. In that case, the right hand side of (3.19) vanishes at $\omega = \omega^*$ but the left hand side does not, since $\tilde{p}_n$ does not vanish identically, unless also $h_{n-1} = 0$. Subsequently, we can deduce from another reformulation of (3.19) that $\dot{h}_{n-1} = 0$ too. Continuing this reasoning leads to a chain of double roots and all Hankel determinants vanishing down to $n = 0$, which is a contradiction. □

Lemma 3.4. *There is no $n \geq 0$ and $\omega^* > 0$ such that*

$$h_n(\omega^*) = h_{n+2}(\omega^*) = 0.$$

Proof. The result is true by direct computation for $n = 0$ and $n = 1$. Let us assume it is true up to $n-1$, and assume that $h_n(\omega^*) = 0$ for some $\omega^* > 0$. We intend to show that $h_{n+2}(\omega^*) \neq 0$.

We know that $h_{n-1} \neq 0$ by Lemma 3.3 and that $h_{n-2} \neq 0$ by our inductive assumption. It follows from (3.16) that α_{n-1} is analytic at ω^*. It also follows from (3.16) that $\beta_n(\omega^*) = 0$. Since $h_n' \neq 0$ and $h_{n-2} \neq 0$, this root of β_n is simple.

We reformulate the differential-difference equations (3.23) as

$$\begin{aligned} \beta_{n+1} &= -i\dot{\alpha}_n + \beta_n, \\ \alpha_{n+1} &= -i\frac{\dot{\beta}_{n+1}}{\beta_n} + \alpha_n. \end{aligned}$$

Plugging in a Taylor series of α_{n-1} and β_n around $\omega = \omega^*$ and using the above recursions, it follows after straightforward computation that α_n and α_{n+1} have a simple pole, β_{n+1} has a double pole, β_{n+2} has a simple root and α_{n+2} is analytic at ω^*. Using the expressions

$$\alpha_n = \frac{\langle zp_n, p_n \rangle}{\langle p_n, p_n \rangle} \qquad \text{and} \qquad \beta_n = \frac{\langle p_n, p_n \rangle}{\langle p_{n-1}, p_{n-1} \rangle},$$

this implies that $\langle p_{n+1}, p_{n+1} \rangle$ has a simple zero at $\omega = \omega^*$ and $\langle p_{n+2}, p_{n+2} \rangle \neq 0$. The latter in turn implies that $h_{n+2}(\omega^*) \neq 0$. □

We also remark that these ideas were used in a similar problem in the PhD thesis of N. Lejon [66]. In this setting, where Lejon considered a complex cubic potential on a union of infinite contours in $\mathbb{C}$, the analogous properties for the Hankel determinants combined with the string equations for the polynomial family under consideration led to a proof of existence for the even degree polynomials. However, in the case of the Kissing polynomials, the string equations become more involved, and such an approach does not appear to be tractable.

3.2 Differential Equations for the Kissing Polynomials

We start by using the Riemann-Hilbert problem to show that the Kissing polynomials satisfy a second order differential equation.

Lemma 3.5 (Differential Equation for the Kissing Polynomials)**.** *Let ω be such that $h_{n-1}(\omega) \neq 0$. Then the Kissing polynomials satisfy the following second order ODE:*

$$p_n''(z) + \frac{R(z;\omega)}{Q(z;\omega)} p_n'(z) + \frac{S(z;\omega)}{Q(z;\omega)} p_n(z) = 0, \tag{3.24}$$

where Q, R, S are polynomials in z, defined in (3.42)*. Moreover, if $h_n(\omega) = 0$, then the only singular points of the differential equation are at $z = \pm 1$. If $h_n(\omega) \neq 0$, the differential equation also has a regular singular point at*

$$z_*(\omega) = -\alpha_n - \frac{2n+1}{i\omega} \in i\mathbb{R}, \tag{3.25}$$

along with $z = \pm 1$.

Proof. In addition to assuming that ω is such that h_{n-1} does not vanish, we also at first assume that $h_{n-2}(\omega) \neq 0$.

We will derive the differential equation (3.24) by means of the Riemann-Hilbert problem (3.9). First, following the outline of [51, Chapter 22], we make a transformation to Y so that the resulting matrix has constant jumps over the interval $(-1,1)$. As such, we define

$$Z(z) = Y(z) \begin{pmatrix} e^{i\omega z/2} & 0 \\ 0 & e^{-i\omega z/2} \end{pmatrix}, \tag{3.26}$$

so that Z solves the following Riemann-Hilbert problem:

$$Z(z) \text{ is analytic for } z \in \mathbb{C} \setminus [-1,1], \tag{3.27a}$$

$$Z_+(z) = Z_-(z) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad z \in (-1,1), \tag{3.27b}$$

$$Z(z) = \left(I + \frac{A_n(\omega)}{z} + \frac{B_n(\omega)}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right) \right) z^{n\sigma_3} e^{i\omega z \sigma_3/2}, \qquad z \to \infty, \tag{3.27c}$$

$$Z(z) = \mathcal{O}\begin{pmatrix} 1 & \log|z \mp 1| \\ 1 & \log|z \mp 1| \end{pmatrix}, \qquad z \to \pm 1, \tag{3.27d}$$

where A_n and B_n are given by (3.13). By taking derivatives, we are also able to conclude that Z' solves the Riemann-Hilbert problem:

$$Z'(z) \text{ is analytic for } z \in \mathbb{C} \setminus [-1,1], \tag{3.28a}$$

$$Z'_+(z) = Z'_-(z) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad z \in (-1,1), \tag{3.28b}$$

$$Z'(z) = \left(\Gamma_0(\omega) + \frac{\Gamma_1(\omega)}{z} + \frac{\Gamma_2(\omega)}{z^2} + \mathscr{O}\left(\frac{1}{z^3}\right)\right) z^{n\sigma_3} e^{i\omega z\sigma_3/2}, \qquad z \to \infty, \tag{3.28c}$$

$$Z'(z) = \mathscr{O}\begin{pmatrix} 1 & |z \mp 1|^{-1} \\ 1 & |z \mp 1|^{-1} \end{pmatrix}, \qquad z \to \pm 1, \tag{3.28d}$$

where

$$\begin{aligned} \Gamma_0(\omega) &= \frac{i\omega\sigma_3}{2}, \qquad \Gamma_1(\omega) = n\sigma_3 + \frac{i\omega A_n \sigma_3}{2}, \\ \Gamma_2(\omega) &= -A_n + nA_n\sigma_3 + \frac{i\omega}{2} B_n \sigma_3. \end{aligned} \tag{3.29a}$$

By the standard technique of showing uniqueness of solutions to Riemann-Hilbert problems [34], we have that $\det Z(z) = 1$, so that Z is invertible for all $z \in \mathbb{C}$. As both Z and Z' have the same jumps over the interval $(-1,1)$, we conclude that $Z'Z^{-1}$ is analytic in $\mathbb{C} \setminus \{\pm 1\}$, where the singularities at the endpoints are at most simple poles. Therefore, $(z^2-1)Z'(z)Z^{-1}(z)$ is an entire function. We then compute

$$Z^{-1}(z) = e^{-i\omega z\sigma_3/2} z^{-n\sigma_3}\left(I + \frac{\Delta_1(\omega)}{z} + \frac{\Delta_2(\omega)}{z^2} + \mathscr{O}\left(\frac{1}{z^3}\right)\right), \qquad z \to \infty, \tag{3.30}$$

where

$$\Delta_1(\omega) = -A_n, \qquad \Delta_2(\omega) = A_n^2 - B_n, \tag{3.31}$$

and define

$$M(z) = \Gamma_0 z^2 + (\Gamma_0\Delta_1 + \Gamma_1)z + (\Gamma_1\Delta_1 + \Gamma_2 + \Gamma_0\Delta_2 - \Gamma_0). \tag{3.32}$$

By looking at the asymptotics of Z^{-1} and Z' at infinity, we conclude that

$$(z^2-1)Z'(z)Z^{-1}(z) = M(z) + \mathscr{O}\left(\frac{1}{z}\right), \qquad z \to \infty. \tag{3.33}$$

As $(z^2-1)Z'(z)Z^{-1}(z)$ is entire, an application of Liouville's Theorem then gives us that

$$(z^2-1)Z'(z) = M(z)Z(z). \tag{3.34}$$

Looking at the first column of (3.34), and using (3.10) and (3.26), we have that

$$(z^2-1)\begin{pmatrix} p_n'(z) \\ p_{n-1}'(z) \end{pmatrix} = \begin{pmatrix} N_1(z) & N_2(z) \\ N_3(z) & N_4(z) \end{pmatrix}\begin{pmatrix} p_n(z) \\ p_{n-1}(z) \end{pmatrix} \tag{3.35}$$

where

$$N_1(z) = M_{11}(z) - \frac{i\omega}{2}(z^2-1), \qquad N_2(z) = -2\pi i\kappa_{n-1}^2 M_{12}(z) \tag{3.36}$$

$$N_3(z) = -\frac{M_{21}(z)}{2\pi i\kappa_{n-1}^2}, \qquad N_4(z) = M_{22}(z) - \frac{i\omega}{2}(z^2-1). \tag{3.37}$$

Using (3.13) and (3.32), we can simplify these expressions to

$$N_1(z) = nz - i\left[\frac{\dot{h}_{n-1}}{h_{n-1}} - \omega\frac{h_n h_{n-2}}{h_{n-1}^2}\right], \tag{3.38a}$$

$$N_2(z) = -\frac{i\omega h_{n-2}h_n}{h_{n-1}^2}\left(z - z_*(\omega)\right), \tag{3.38b}$$

$$N_3(z) = i\omega\left(z - z^{(3)}(\omega)\right), \tag{3.38c}$$

and

$$N_4(z) = -i\omega(z^2-1) - nz + i\left[\frac{h'_{n-1}}{h_{n-1}} - \omega\frac{h_n h_{n-2}}{h_{n-1}^2}\right]. \tag{3.38d}$$

In (3.38b) and (3.38c), z_* and $z^{(3)}$ are given by

$$z_*(\omega) = -\alpha_n - \frac{2n+1}{i\omega}. \tag{3.39}$$

and

$$z^{(3)}(\omega) = -\alpha_{n-1} - \frac{2n-1}{i\omega}. \tag{3.40}$$

All of the N_i are well defined provided $h_{n-1} \neq 0$, with N_3 needing the additional assumption of $h_{n-2} \neq 0$ to be well defined. Combining the equations in (3.35) gives

$$\left((z^2-1)\frac{d}{dz} - N_4(z)\right)\left(\frac{(z^2-1)p'_n(z)}{N_2(z)} - \frac{N_1(z)p_n(z)}{N_2(z)}\right) = N_3(z)p_n(z).$$

Equivalently, we can write

$$p''_n(z) + \frac{R(z)}{Q(z)}p'_n(z) + \frac{S(z)}{Q(z)}p_n(z) = 0, \tag{3.41}$$

where

$$Q(z) = (z^2-1)^2 N_2(z) \tag{3.42a}$$

$$R(z) = \left(z^2-1\right)N_2(z)\left(2z - N_1(z) - N_4(z)\right) - (z^2-1)^2 N'_2(z) \tag{3.42b}$$

$$S(z) = N_2(z)\left(N_1(z)N_4(z) - (z^2-1)N'_1(z)\right) - N_2^2(z)N_3(z) + (z^2-1)N_1(z)N'_2(z) \tag{3.42c}$$

Now, using (3.38b), we see that if $h_n = 0$,

$$N_2(z) = -\frac{\omega h_{n-2}\dot{h}_n}{h_{n-1}^2} \neq 0, \tag{3.43}$$

so that the only singular points of the ODE (3.41) are at ± 1. On the other hand, if $h_n \neq 0$, N_2 has a simple, purely imaginary zero at

$$z_*(\omega) = -\alpha_n - \frac{2n+1}{i\omega} \in i\mathbb{R}, \tag{3.44}$$

which is necessarily a regular singular point of the ODE (3.41). Simplifying yields

$$\frac{R(z)}{Q(z)} = \frac{2z - N_1(z) - N_4(z)}{z^2 - 1} - \frac{1}{z - z_*(\omega)} \tag{3.45a}$$

$$\frac{S(z)}{Q(z)} = \frac{N_1(z)N_4(z) - (z^2-1)N_1'(z)}{(z^2-1)^2} - \frac{N_2(z)N_3(z)}{(z^2-1)^2} + \frac{N_1(z)}{(z^2-1)(z - z_*(\omega))} \tag{3.45b}$$

Using (3.38a) and (3.38d), we see that both N_1 and N_4 are well defined when h_{n-2} vanishes, and by (3.39) we have that z_* does not depend on h_{n-2}. Finally, if ω' is such that $h_{n-2}(\omega') = 0$, then

$$\lim_{\omega \to \omega'} N_2(z)N_3(z) = \frac{\omega'^2 h_n(\omega')\dot{h}_{n-2}(\omega')(z - z_*(\omega'))}{h_{n-1}^2(\omega')} \tag{3.46}$$

so that (3.41) holds even when $h_{n-2} = 0$, completing the proof of Lemma 3.5. □

This lemma has two immediate corollaries which will be used in the proof of existence of the even degree Kissing polynomials.

Corollary 3.6. *If $h_n(\omega) = 0$, then p_n can not have a zero of multiplicity greater than one on the imaginary axis.*

Proof. The proof is immediate, since when $h_n(\omega) = 0$, the imaginary axis consists solely of regular points of the second order differential equation (3.24). □

Corollary 3.7. *Assume $h_n \neq 0$ and $h_{n-1} \neq 0$. If p_n has a zero at $z_*(\omega)$, then it is a double zero.*

Proof. We may write

$$p_n(z) = \sum_{k=0}^{n-j} a_k (z - z_*)^{k+j}, \tag{3.47}$$

where j is yet to be determined and $a_0 \neq 0$. Using (3.45), we can expand R/Q and S/Q in a Laurent series about z_* as

$$\frac{R(z)}{Q(z)} = \sum_{k=-1}^{\infty} r_k (z - z_*)^k, \qquad \frac{S(z)}{Q(z)} = \sum_{k=-1}^{\infty} s_k (z - z_*)^k. \tag{3.48}$$

Above, we compute $r_{-1} = -1$, which follows from (3.42) in the proof of Lemma 3.5. Plugging (3.47) and (3.48) into the differential equation

$$p_n''(z) + \frac{R(z)}{Q(z)} p_n'(z) + \frac{S(z)}{Q(z)} p_n(z) = 0,$$

and looking at the coefficient of $(z-z_*)^{j-1}$ gives

$$a_0 j\,(j-2)=0. \tag{3.49}$$

As $a_0 \neq 0$, (3.49) implies that either $j=0$ or $j=2$, completing the proof. □

We now turn our attention to the behavior of the Kissing polynomials as we deform the parameter ω. The starting point of our analysis is the following relation, derived in [5, Theorem 3.2]:

$$\frac{\partial}{\partial\omega}p_n(z)=-i\beta_n p_{n-1}(z). \tag{3.50}$$

Using similar techniques to those used to derive the differential equation in the variable z, we are able to conclude that the Kissing polynomials also satisfy a second order differential equation in the parameter ω.

Lemma 3.8. *Assume that ω' is such that $h_{n-1}(\omega')\neq 0$, so that $p_n(z)$ exists as a monic polynomial of degree n in a neighborhood of ω'. Then, in this neighborhood, the Kissing polynomials satisfy*

$$\ddot{p}_n+i\,(z-\alpha_n)\,\dot{p}_n-\beta_n p_n=0, \tag{3.51}$$

where $\dot{}$ indicates differentiation with respect to the parameter ω.

Proof. Using the recurrence relation (3.14), we may transform (3.50) to

$$\frac{\partial}{\partial\omega}p_{n-1}=ip_n-i\,(z-\alpha_{n-1})\,p_{n-1}. \tag{3.52}$$

We may combine the two differential-difference equations, (3.50) and (3.52), as in the proof of Lemma 3.5 to obtain

$$\frac{i}{\beta_n}\ddot{p}_n+\left[\frac{\partial}{\partial\omega}\left(\frac{i}{\beta_n}\right)-\frac{z-\alpha_{n-1}}{\beta_n}\right]\dot{p}_n-ip_n=0. \tag{3.53}$$

Using the Toda Equations (3.23), we can simplify this to

$$\ddot{p}_n+i\,(z-\alpha_n)\,\dot{p}_n-\beta_n p_n=0,$$

completing the proof. □

Next, we write down a dynamical system governing the zeros of p_n as functions of ω, using techniques from [24] and the differential equation (3.51).

Lemma 3.9. *Assume that ω' is such that $h_{n-1}(\omega')\neq 0$, so that $p_n(z)$ exists as a monic polynomial of degree n in a neighborhood of ω'. Denote by $\{z_i(\omega)\}_{i=1}^n$ the n zeros of the polynomial p_n. Then, in* *this neighborhood of ω', the zeros evolve according to the following dynamical system:*

$$\ddot{z}_k = 2\dot{z}_i \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{\dot{z}_j}{z_k - z_j} - i\dot{z}_i\,(z_k - \alpha_n), \qquad k = 1,2,\dots,n, \tag{3.54}$$

Proof. As p_n is a monic polynomial of degree n, we write

$$p_n(z;\omega) = \prod_{i=1}^{n} (z - z_i(\omega)). \tag{3.55}$$

Differentiating with respect to ω yields

$$\frac{\partial}{\partial \omega} p_n(z)\Big|_{z=z_k(\omega)} = -\dot{z}_k(\omega) \prod_{\substack{j=1\\ j\neq k}}^{n} (z_k(\omega) - z_j(\omega)),$$

and

$$\begin{aligned}\frac{\partial^2}{\partial \omega^2} p_n(z)\Big|_{z=z_k(\omega)} = &-\ddot{z}_k(\omega) \prod_{\substack{j=1\\ j\neq k}}^{n} (z_k(\omega) - z_j(\omega)) \\ &+ 2\dot{z}_k(\omega) \sum_{\substack{\ell=1\\ \ell\neq k}}^{n} \dot{z}_\ell(\omega) \prod_{\substack{j=1\\ j\neq \ell,k}}^{n} (z_k(\omega) - z_j(\omega)).\end{aligned}$$

Evaluating (3.51) along any zero trajectory then yields the following complex n-body problem,

$$\begin{aligned}-\ddot{z}_k(\omega) \prod_{\substack{j=1\\ j\neq k}}^{n} (z_k(\omega) - z_j(\omega)) + 2\dot{z}_k(\omega) \sum_{\substack{j=1\\ j\neq k}}^{n} \dot{z}_j(\omega) \prod_{\substack{\ell=1\\ \ell\neq k,j}}^{n} (z_k(\omega) - z_\ell(\omega)) \\ - i\dot{z}_k(\omega)\,(z_k(\omega) - \alpha_n(\omega)) \prod_{\substack{j=1\\ j\neq k}}^{n} (z_k(\omega) - z_j(\omega)) = 0,\end{aligned}$$

for $k = 1,2,\dots,n$, which upon simplifying, yields (3.54). □

The above proof also lends some insight into the behavior of the zeros as soon as $\omega > 0$. It is clear that the initial positions of these zeros are the zeros of the Legendre polynomials. Let x_i, $i = 1,\dots,n$ denote the ordered zeros of the Legendre polynomials, i.e.

$$z_i(0) = x_i, \qquad i = 1,\dots,n.$$

Next, we see that

$$-i\beta_n p_{n-1}(z_i(\omega)) = \frac{\partial}{\partial \omega} p_n(z)\Big|_{z=z_i(\omega)} = -\dot{z}_i(\omega) \prod_{\substack{k=1\\ k\neq i}}^{n} (z_i(\omega) - z_k(\omega)) \tag{3.56}$$

Evaluating Equation (3.56) at $\omega = 0$, we have that

$$\dot{z}_i(0) = \frac{i\beta_{n,L} P_{n-1}(x_i)}{\prod_{\substack{k=1\\k\neq i}}^{n} (x_i - x_k)}, \qquad i = 1, \dots, n,$$

where $\beta_{n,L}$ are the recurrence coefficients for the Legendre polynomials and P_{n-1} is the monic Legendre polynomial of degree $n-1$. It is well known that

$$\beta_{n,L} = \frac{n^2}{(2n-1)(2n+1)}.$$

Therefore, we have that the zeros of the Kissing polynomials move up in the complex plane as soon as $\omega > 0$.

Another consequence of Lemma 3.9 is that the zeros are analytic functions of ω, provided the zeros are all simple and α_n is not infinite. By (3.16), we see that α_n is infinite when either h_n or h_{n-1} vanishes - that is, α_n is infinite at kissing points. This should be read in light of the discussion in Section 3.1.1, where we have shown that if h_{n-1} vanishes, p_n becomes a multiple of p_{n-1}, and Figure 3.1 and Figure 3.2 show that at these points the zero trajectories form cusp singularities.

3.3 Existence of the Even Degree Polynomials

The goal of this section is to show that the even degree Kissing Polynomials exist for all $\omega > 0$. In this proof of existence, we will make use of both the symmetry of the polynomials over the imaginary axis and the differential equation in z as stated in Lemma 3.5.

We say that p_k exists if there is a monic polynomial of degree *exactly* k which satisfies the orthogonality conditions given in (3.1). Equivalently, we have that p_k exists if the Hankel determinant h_{k-1} does not vanish. We have seen in Section 3.1 that as $\omega \to \hat{\omega}$, where $\hat{\omega}$ satisfies $h_{n-1}(\hat{\omega}) = 0$, one or more of the zeros of p_n becomes infinite. Therefore, we will prove the existence of the even degree Kissing polynomials for $\omega > 0$ by showing that their zeros do not become infinite.

We first recall from (3.2) that for each $\omega > 0$, the Kissing Polynomials satisfy

$$p_n(z;\omega) = (-1)^n \overline{p_n(-\bar{z};\omega)}. \tag{3.57}$$

This immediately implies that zeros for the even degree polynomials can become infinite in only one of two ways:

(i) Zeros tend to infinity in one or more pairs, symmetric about the imaginary axis, or

(ii) An even number of zeros form a zero of multiplicity greater than one on the imaginary axis. These zeros then split and one (or more) of the zeros travels to infinity along the imaginary axis.

We quickly rule out the first case above. We recall that the polynomials $\tilde{p}_n$, defined in (3.18) as

$$\tilde{p}_n(z) = h_{n-1} p_n(z), \tag{3.58}$$

always exists as a polynomial of degree $\leq n$.

Lemma 3.10. *If $\hat{\omega}$ is such that $h_{n-1}(\hat{\omega}) = 0$, then* $\deg(\tilde{p}_n(z;\hat{\omega})) = n-1$.

Proof. We recall (3.20) (shifted from $n \mapsto n-1$), which states that if $h_{n-1}(\hat{\omega}) = 0$,

$$\tilde{p}_n(z) = i\frac{h'_{n-1}(\hat{\omega})}{h_{n-2}(\hat{\omega})}\tilde{p}_{n-1}(z) = ih'_{n-1}(\hat{\omega})p_{n-1}(z) \tag{3.59}$$

By Lemma 3.3 and the remarks immediately following the lemma, we see that $h_{n-2}(\hat{\omega}) \neq 0$, so that p_{n-1} exists as a monic polynomial of degree $n-1$, and that $h'_{n-1}(\hat{\omega}) \neq 0$, as well. Therefore, $\deg(\tilde{p}_n) = n-1$, completing the proof. □

It immediately follows that as $\omega \to \hat{\omega}$, precisely one zero escapes to infinity, which rules out that zeros tend to infinity in one or more symmetric pairs. We therefore turn our attention to the second case, and rule out a zero of multiplicity greater than one forming on the imaginary axis. In order to accomplish this, we will need the following lemma.

Lemma 3.11. *Let $\omega \in (\omega_1, \omega_2)$ where $\omega_1 < \omega_2$ are such that $h_{2n-1}(\omega) \neq 0$ and $h_{2n}(\omega) \neq 0$ for all $\omega \in (\omega_1, \omega_2)$. Assume further that p_{2n-2} exists as a monic polynomial of degree $2n-2$ and satisfies $p_{2n-2}(z_*(\omega)) \neq 0$ for all $\omega \in (\omega_1, \omega_2)$. If $p_{2n}(z_*(\omega)) = 0$, then*

$$\frac{d}{d\omega} p_{2n}(z_*(\omega);\omega) \neq 0. \tag{3.60}$$

Proof. Using (3.50), we see that

$$\frac{d}{d\omega} p_{2n}(z_*(\omega)) = \frac{\partial}{\partial z} p_{2n}(z_*(\omega)) z'_*(\omega) - i\beta_{2n} p_{2n-1}(z_*(\omega)). \tag{3.61}$$

As both h_{2n} and h_{2n-1} are non-zero by assumption, we can use Corollary 3.7 to conclude that if p_{2n} vanishes at z_*, then its first partial derivative in z must also vanish at z_*. Therefore, if $\frac{d}{d\omega} p_{2n}(z_*(\omega)) = 0$, we would have that

$$-i\beta_{2n} p_{2n-1}(z_*(\omega)) = 0. \tag{3.62}$$

If $h_{2n-2}(\omega) \neq 0$, the three term recurrence would imply that $p_{2n-2}(z_*(\omega)) = 0$. If $h_{2n-2}(\omega) = 0$, we could use that

$$\tilde{p}_{2n-1} = i\frac{h'_{2n-2}}{h_{2n-3}}\tilde{p}_{2n-2}, \tag{3.63}$$

where $\tilde{p}_{2n-1} = h_{2n-2} p_{2n-1}$, to again conclude that $p_{2n-2}(z_*(\omega)) = 0$. In either case, we have a contradiction, completing the proof of the lemma. □

We now move to the main theorem of this chapter, proving the existence of the even degree Kissing polynomials.

Theorem 3.12. *Let* $k \in \mathbb{N} \cup \{0\}$*. Then* $p_{2k}(z;\omega)$ *exists for all* $\omega > 0$ *and does not vanish on the imaginary axis.*

Proof. The statement is clearly true for $k = 0$ and we will proceed by induction. Therefore, we assume the theorem is true for $k = 0, 1, \ldots, n-1$, and we will show that p_{2n} exists for all ω and does not vanish on the imaginary axis.

Assume for sake of contradiction that there exists an ω for which p_{2n} does not exist and let $\hat{\omega}$ be the smallest positive solution to $h_{2n-1}(\omega) = 0$. By the remarks preceding Lemma 3.10, we know there exists some $\omega_d < \hat{\omega}$ for which $p_{2n}(z;\omega_d)$ has a purely imaginary zero of multiplicity greater than one. By Lemma 3.5 and standard analytic existence theorems for ODEs, we know that any purely imaginary zero of multiplicity greater than one must be located precisely at

$$z_*(\omega_d) = -\alpha_{2n} - \frac{4n+1}{i\omega_d}.$$

We next show that $p_{2n}(z_*(\omega)) \neq 0$ for all $\omega \in (0,\hat{\omega})$, reaching a contradiction to be able to conclude that p_{2n} exists for all ω. Moreover, this in turn implies that p_{2n} does not vanish on the imaginary axis. To see this, note that when $\omega = 0$, p_{2n} is the monic Legendre Polynomial of degree $2n$, and as such is real valued and does not vanish on the imaginary axis. If there existed an ω_* for which $p_{2n}(z;\omega_*)$ vanished somewhere on the imaginary axis, the symmetry of the polynomials across the imaginary axis would imply that there exists some $\omega_d \leq \omega_*$ for which p_{2n} had a zero of even multiplicity on the imaginary axis. Therefore, showing that $p_{2n}(z_*(\omega)) \neq 0$ for all $\omega > 0$ also implies that p_{2n} does not vanish on the imaginary axis.

We want to show that $p_{2n}(z_*(\omega)) \neq 0$ for all $\omega \in (0,\hat{\omega})$. Assume first that n is odd. As n is odd, and by assumption p_{2n-2} exists for all ω and has no zeros on the imaginary axis, we have that

$$p_{2n-2}(ix) > 0, \qquad x \in \mathbb{R}, \qquad \omega > 0. \tag{3.64}$$

Next define $\hat{\Omega} := \{0,\hat{\omega}\} \cup \{\omega : h_{2n}(\omega) = 0,\ \omega < \hat{\omega}\}$, so that $|z_*(\omega)| \to \infty$ as $\omega \to \omega_* \in \hat{\Omega}$. As p_{2n} exists on the interval $(0,\hat{\omega})$, we have that $p_{2n}(z_*(\omega))$ is analytic on $(0,\hat{\omega}) \setminus \hat{\Omega}$. Observe that as n is odd,

$$p_{2n}(z_*(\omega)) \to -\infty, \qquad \omega \to \omega_* \in \hat{\Omega}. \tag{3.65}$$

Recall that the goal is to show that $p_{2n}(z_*(\omega))$ does not vanish in $(0,\hat{\omega})$. For sake of contradiction, assume there exists some $\omega_d < \hat{\omega}$ for which $p_{2n}(z_*(\omega_d)) = 0$ and define

$$\omega_0 := \sup_{\omega\in\hat{\Omega}} \{\omega : \omega < \omega_d\}, \qquad \omega_3 := \inf_{\omega\in\hat{\Omega}} \{\omega : \omega > \omega_d\}. \tag{3.66}$$

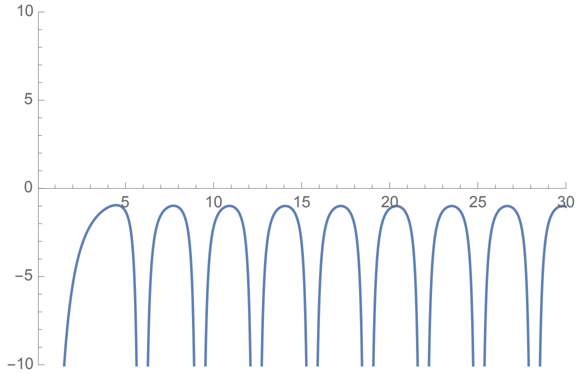
We then have that $p_{2n}(z_*(\omega))$ is analytic in (ω_0, ω_3), vanishes somewhere in this interval, and tends to $-\infty$ as we approach the endpoints of this interval.

Therefore, using Lemma 3.11, there exist $\omega_1 < \omega_2$ such that $p_{2n}(z_*(\omega_1)) = p_{2n}(z_*(\omega_2)) = 0$,

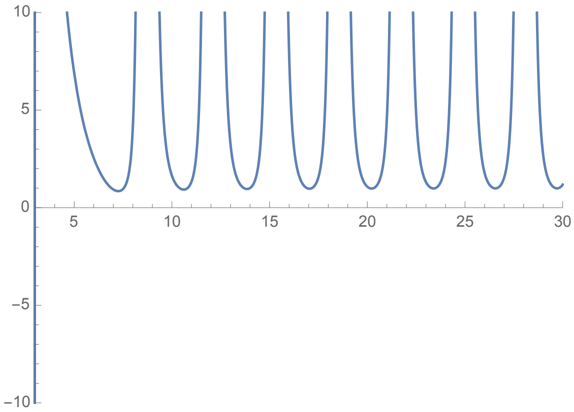
$$\frac{d}{d\omega} p_{2n}\left(z_*\left(\omega_1\right)\right) > 0, \qquad \frac{d}{d\omega} p_{2n}\left(z_*\left(\omega_2\right)\right) < 0, \tag{3.67}$$

and

$$p_{2n}(z_*(\omega)) > 0, \qquad \omega \in (\omega_1, \omega_2). \tag{3.68}$$



(a) Trajectory of $p_2(z_*(\omega))$



(b) Trajectory of $p_4(z_*(\omega))$

Figure 3.4: Trajectories of $p_{2n}(z_*(\omega)$ for $n = 1, 2$.

Next,

$$\frac{d}{d\omega} p_{2n}\left(z_*\left(\omega\right)\right) = \frac{\partial}{\partial z} p_{2n}\left(z_*(\omega)\right) z_*'(\omega) - i\beta_{2n} p_{2n-1}(z_*(\omega)). \tag{3.69}$$

As p_{2n} vanishes at $z_*(\omega_1)$ and $z_*(\omega_2)$, we have by Corollary 3.7 that

$$\frac{\partial}{\partial z} p_{2n}\left(z_*(\omega_1)\right) = \frac{\partial}{\partial z} p_{2n}\left(z_*(\omega_2)\right) = 0,$$

which using (3.67) and (3.69) gives that

$$-i\beta_{2n}\left(\omega_1\right) p_{2n-1}(z_*(\omega_1)) > 0, \tag{3.70a}$$

$$-i\beta_{2n}\left(\omega_2\right) p_{2n-1}(z_*(\omega_2)) < 0. \tag{3.70b}$$

Using the three term recurrence relation, and that $p_{2n}(z_*(\omega))$ vanishes at ω_1 and ω_2, we may write these equations as

$$f(\omega_1) p_{2n-2}(z_*(\omega_1)) > 0, \qquad f(\omega_2) p_{2n-2}(z_*(\omega_2)) < 0, \tag{3.71}$$

where

$$f(\omega) = -\frac{i\beta_{2n}(\omega)\beta_{2n-1}(\omega)}{z_*(\omega) - \alpha_{2n-1}(\omega)}. \tag{3.72}$$

Claim. *$f(\omega)$ is a well defined, continuous function on $[\omega_1, \omega_2]$ and is nonzero throughout this interval.*

Proof of Claim. Using (3.16) and (3.25), we can write

$$f(\omega) = -\frac{h_{2n}h_{2n-3}}{h_{2n-1}} \left(\frac{1}{h_{2n-2}\left(\frac{\dot{h}_{2n}}{h_{2n}} - \frac{\dot{h}_{2n-2}}{h_{2n-2}} + \frac{4n+1}{\omega}\right)} \right) \tag{3.73}$$

As h_{2n}, h_{2n-1}, and h_{2n-3} are analytic and do not vanish in $[\omega_1, \omega_2]$, we just need to focus our attention on the term in brackets. If we can show this term is never zero or infinite on $[\omega_1, \omega_2]$, the proof will be complete. The term in brackets is zero only at the poles of

$$g(\omega) := h_{2n-2}\left(\frac{\dot{h}_{2n}}{h_{2n}} - \frac{\dot{h}_{2n-2}}{h_{2n-2}} + \frac{4n+1}{\omega}\right).$$

As g only has poles at the zeros of h_{2n} and at 0, we can conclude that f is never zero in $[\omega_1, \omega_2]$. We are just left to show $g(\omega)$ does not vanish on $[\omega_1, \omega_2]$, so that f is continuous on this interval and as such does not change sign. Note that g is well defined and nonzero when $h_{2n-2} = 0$, so we must show that

$$\frac{h'_{2n}}{h_{2n}} - \frac{h'_{2n-2}}{h_{2n-2}} + \frac{4n+1}{\omega} = -i(z_* - \alpha_{2n-1})$$

does not vanish on $[\omega_1, \omega_2]$ when $h_{2n-2} \neq 0$. As $h_{2n-2} \neq 0$, we may use the recurrence relation to show that if $z_* - \alpha_{2n-1}$ vanished, then

$$p_{2n}(z_*) = -\frac{h_{2n-1}h_{2n-3}}{h_{2n-2}^2} p_{2n-2}(z_*) \tag{3.74}$$

If $z_* - \alpha_{2n-1}$ vanished at either ω_1 or ω_2, where p_{2n} also vanishes, then we would immediately have that p_{2n-2} vanished here as well. Therefore, we can conclude that $z_* - \alpha_{2n-1}$ does not vanish at the endpoints of $[\omega_1, \omega_2]$. We have by (3.68) that $p_{2n}(z_*) > 0$ for $\omega \in (\omega_1, \omega_2)$. On the other hand, as $\omega < \omega_*$, we have that h_{2n-1} is positive and by assumption h_{2n-3} is always positive as p_{2n-2} exists for all ω. We also have from (3.64) that $p_{2n-2}(z_*) > 0$ for all ω, which when combined with (3.74), yields a contradiction with $p_{2n}(z_*) > 0$. Therefore, f is continuous and can not change sign on $[\omega_1, \omega_2]$.

■

As $f(\omega)$ does not change sign on $[\omega_1, \omega_2]$, we know by (3.71) that $p_{2n-2}(z_*(\omega))$ must change sign in (ω_1, ω_2). However, this immediately gives that there is some $\omega \in (\omega_1, \omega_2)$ for which p_{2n-2} vanishes on the imaginary axis, contradicting the inductive hypothesis. Therefore, we can conclude $p_{2n}(z_*(\omega))$ does not vanish on $(0, \hat{\omega})$, as desired. The case where n is even can be handled analogously, except all the signs and inequalities in the proof above are reversed, see Figure 3.4 □

We have seen above that the key to proving the existence of the even degree Kissing Polynomials was proving that these polynomials never formed higher order zeros on the imaginary axis. Having proved that the even degree Kissing Polynomials do not have zeros of multiplicity greater than one on the imaginary axis, we may now take this a step further and show they do not have higher order zeros anywhere in the complex plane.

Lemma 3.13. *Fix $\omega > 0$ so that $h_{n-1}(\omega) \neq 0$ and $p_n(z;\omega)$ exists as a monic polynomial of degree n. Then,*

$$p_n(1;\omega) \neq 0, \qquad \textit{and} \qquad p_n(-1;\omega) \neq 0.$$

Proof. First assume further that ω is such that $h_{n-2}(\omega) \neq 0$, so that both p_n and p_{n-1} exist as monic polynomials of degree n and $n-1$, respectively. Using (3.35), we have that

$$(z^2-1)p_n'(z) = N_1(z)p_n(z) + N_2(z)p_{n-1}(z), \tag{3.75}$$

where

$$N_1(z) = nz - i\left[\frac{\dot{h}_{n-1}}{h_{n-1}} - \omega\frac{h_n h_{n-2}}{h_{n-1}^2}\right], \tag{3.76a}$$

$$N_2(z) = -\frac{i\omega h_{n-2}h_n}{h_{n-1}^2}\left(z - z_*(\omega)\right), \tag{3.76b}$$

and we recall that

$$z_*(\omega) = -\alpha_n - \frac{2n+1}{i\omega}. \tag{3.77}$$

As $h_{n-1}(\omega) \neq 0$, we may write this in terms of the polynomials $\tilde{p}_n$, which exist for all ω, as

$$\frac{z^2-1}{h_{n-1}}\tilde{p}_n'(z) = \frac{N_1(z)}{h_{n-1}}\tilde{p}_n(z) - \frac{i\omega h_n}{h_{n-1}^2}\left(z - z_*(\omega)\right)\tilde{p}_{n-1}(z). \tag{3.78}$$

As both N_1 and N_2 are well defined when h_{n-2} vanishes, (3.78) holds for any ω provided $h_{n-1}(\omega) \neq 0$, by continuity.

Now, fix ω so that $h_{n-1}(\omega) \neq 0$ and assume that $p_n(1) = 0$. Evaluating (3.78) at $z = 1$, we see that

$$\frac{i\omega h_n}{h_{n-1}^2}\left(1 - z_*(\omega)\right)\tilde{p}_{n-1}(1;\omega) = 0 \tag{3.79}$$

Note that $1 - z_*(\omega) \neq 0$ as $z_*(\omega) \in i\mathbb{R}$ for all ω. First consider the case $h_n(\omega) \neq 0$. We then immediately have that $\tilde{p}_{n-1}(1;\omega) = 0$. On the other hand, assume $\omega = \hat{\omega}$ was such that $h_n(\hat{\omega}) = 0$. Then taking the limit as $\omega \to \hat{\omega}$ in (3.78), and using (3.16) and (3.77), we see that

$$\frac{\hat{\omega}\dot{h}_n(\hat{\omega})}{h_{n-1}^2(\hat{\omega})}\tilde{p}_{n-1}^{\hat{\omega}}(1) = 0. \tag{3.80}$$

In light of Lemma 3.3 and the remarks immediately following the lemma, we see that $\dot{h}_n(\hat{\omega}) \neq 0$, so that in the case $h_n(\omega) = 0$, we still have that $\tilde{p}_{n-1}(1) = 0$. Therefore, $\tilde{p}_n(1) = 0$ implies that $\tilde{p}_{n-1}(1) = 0$.

We now show that $\tilde{p}_n(1) = 0$ implies that $\tilde{p}_{n-k}(1) = (0)$ for $k \in \{0, 1, \ldots, n\}$. As we have just shown, this statement is true for $k = 0, 1$, so we will proceed by induction and assume it holds true for $k = 0, 1, \ldots, m-1 < n$ and show that it holds true for $k = m$.

We may use the three term recurrence relation (3.19), where we shift the index $n \mapsto n - m + 1$ and use that $\tilde{p}_{n-m+1}(1) = \tilde{p}_{n-m+2}(1) = 0$ to conclude that

$$h^2_{n-m+1}(\omega)\tilde{p}_{n-m}(1) = 0. \tag{3.81}$$

If $h_{n-m+1}(\omega) \neq 0$, we immediately have $\tilde{p}_{n-m}(1) = 0$, completing the inductive step. On the other hand, assume that $\omega = \hat{\omega}$ and $h_{n-m+1}(\hat{\omega}) = 0$. Shifting $n \mapsto n - m + 1$ in (3.78) and taking limits as $\omega \to \hat{\omega}$, we arrive (in a similar fashion to (3.80)) at

$$\frac{\hat{\omega}\dot{h}_{n-m+1}(\hat{\omega})}{h^2_{n-m}(\hat{\omega})}\tilde{p}^{\hat{\omega}}_{n-m}(1) = 0.$$

By Lemma 3.3, we have that $h_{n-m}(\hat{\omega}) \neq 0$ and $\dot{h}_{n-m+1}(\hat{\omega}) \neq 0$, so that $\tilde{p}_{n-m}(1) = 0$, completing the inductive step.

In particular, this chain of reasoning implies that $\tilde{p}_0(1) = 0$. However, $\tilde{p}_0(z) \equiv 1$, so we have reached a contradiction. As such $\tilde{p}_n(1) \neq 0$, which implies that $p_n(1) \neq 0$ when $h_{n-1}(\omega) \neq 0$.

Finally, we may use the symmetry across the imaginary axis in (3.2) to conclude that $p_n(1) \neq 0$ implies that $p_n(-1) \neq 0$, completing the proof.

□

Corollary 3.14. *Assume $n \in \mathbb{N}$ and $\omega > 0$ so that $p_{2n}(z;\omega)$ exists as a polynomial of degree $2n$. Then $p_{2n}(z;\omega)$ has $2n$ simple zeros.*

Proof. For sake of contradiction, assume there existed some $\hat{z}$ so that $p^{\omega}_{2n}(\hat{z}) = 0$ and

$$\frac{\partial}{\partial z}p^{\omega}_{2n}(\hat{z}) = 0. \tag{3.82}$$

By Lemma 3.5, we know that $\hat{z} \in \{-1, 1, z_*(\omega)\}$. However, in the proof of Theorem 3.12 we showed that $p_{2n}(z_*(\omega)) \neq 0$ for all $\omega > 0$. Furthermore, Lemma 3.13 shows that $p_{2n}(1) \neq 0$ and $p_{2n}(-1) \neq 0$, which contradicts the fact that $p_{2n}(\hat{z}) = 0$, proving that the even degree Kissing Polynomials have no higher order zeros. □

Having completed the proof of existence for the even degree Kissing polynomials, we now move on to studying the behavior of the Kissing polynomials as both n and ω tend to infinity.

Chapter 4

Supercritical Regime for the Kissing Polynomials

The goal of this chapter is to study the large n asymptotics of the Kissing polynomials when the parameter ω depends on n. To make this more precise, for each $N \in \mathbb{N}$, we introduce monic polynomials of degree n, $p_n^N(z;t)$, which satisfy the following orthogonality conditions:

$$\int_{-1}^{1} p_n^N(z;t) z^k e^{-Nf(z;t)}\,dz = 0, \qquad k = 0,1,\ldots,n-1, \tag{4.1a}$$

and

$$\int_{-1}^{1} p_n^N(z;t) z^n e^{-Nf(z;t)}\,dz \neq 0, \tag{4.1b}$$

where $f(z;t) = -itz$ and $t > 0$. We then define the *varying weight Kissing polynomials* by taking $N = n$. Therefore, studying the behavior of the Kissing polynomials as ω and n tend to infinity at rate t (that is, $\omega = \omega(n) = tn$) is equivalent to studying the varying weight Kissing polynomials as $n \to \infty$. Overloading definitions, we refer to the varying weight Kissing polynomials as just Kissing polynomials for the remainder of this chapter, as the weight function under consideration, $\exp(-nf(z;t))$ will always depend on n. As we have made the decision to take asymptotics along the diagonal $N = n$, we drop the dependence of these polynomials on N, and thus study the polynomials $p_n(z;t)$.

In [31], Deaño studied the large degree asymptotics for $p_n(z;t)$, showing that for $t < t_c$ the zeros of these polynomials accumulate on a single analytic arc connecting -1 and 1. As shown in [31], the critical value t_c (numerically, $t_c \approx 1.32549$) is the unique positive solution to the equation

$$2\log\left(\frac{2+\sqrt{t^2+4}}{t}\right) - \sqrt{t^2+4} = 0. \tag{4.2}$$

Deaño also noted that for $t > t_c$, the zeros of $p_n(z;t)$ seemed to accumulate on two disjoint arcs in the complex plane, as illustrated in Figure 4.1.

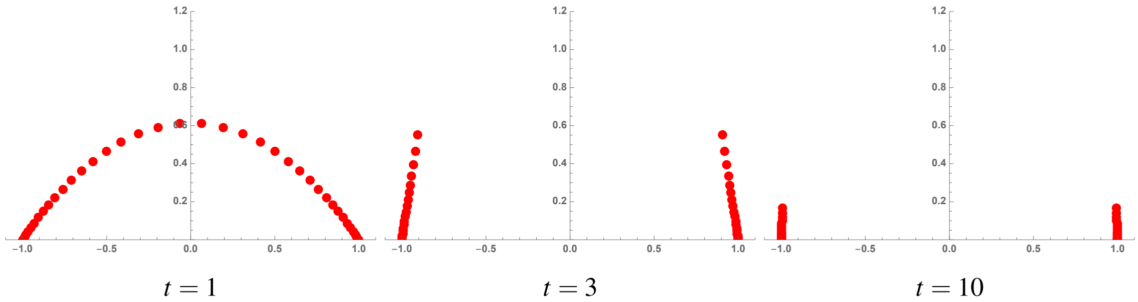


Figure 4.1: Zeros of $p_{40}(z;t)$ for $t = 1,3,10$.

The goal of this chapter is to prove that the situation depicted in Figure 4.1 is correct. Therefore, we will show that for $t > t_c$, the zeros of the varying weight Kissing polynomials do indeed accumulate on two disjoint arcs, one emanating from -1 and the other emanating from $+1$. These arcs turn out to be analytic, and we will describe them precisely. We will also provide strong asymptotic formulas for $p_n(z;t)$ in the complex plane.

4.1 Statement of Main Results

As everything in the integrand of (4.1) is analytic, we have complete freedom when choosing the path of integration connecting -1 and $+1$. On the other hand, accounting for the asymptotic behavior of $p_n(z;t)$ as $n \to \infty$, and in particular its zeros, it is expected that there exists a distinguished curve of orthogonality along which the asymptotic behavior of $p_n(z;t)$ changes depending on whether z belongs to this curve or not. This curve should be the one where the zeros of $p_n(z;t)$ asymptotically lie, as depicted in Figure 4.1. This distinguished curve should in principle possess the S-property, as described in Chapter 1.2, based on the potential theoretic arguments of the GRS program.

Following along the lines of the potential theoretic approach used in [63, 72], see also [31] for related calculations, we expect that the weak limit of the normalized zero counting measure for $p_n(z;t)$, say a probability measure μ_*, should satisfy a quadratic equation of the form

$$\left(\int \frac{d\mu_*(s)}{s-z} + \frac{f'(z;t)}{2} \right)^2 = \frac{Q(z)}{4}, \quad z \in \mathbb{C} \setminus \operatorname{supp} \mu_*, \tag{4.3}$$

where Q is a rational function to be determined, whose only singularities are simple poles at ± 1 (so as to encode that the endpoints of integration in (4.1) are ± 1). A comparison of both sides of this quadratic equation implies that

$$Q^{1/2}(z;t) = -it - \frac{2}{z} + \mathscr{O}\left(\frac{1}{z^2} \right), \qquad z \to \infty. \tag{4.4}$$

Due to the symmetry at hand in the present case, we assume that such a rational function Q takes the form

$$Q(z;t) = -\frac{t^2(z-\lambda_0(t))(z-\lambda_1(t))}{z^2-1}, \tag{4.5}$$

where

$$\lambda_0(t) = -x_*(t) + \frac{2i}{t}, \qquad \lambda_1(t) = x_*(t) + \frac{2i}{t} = -\overline{\lambda_0}, \tag{4.6}$$

and $x_*(t) \geq 0$ is to be determined. Note that the imaginary parts of λ_0 and λ_1 are determined via (4.4) and the symmetry over the imaginary axis.

For the choice $x_* = 0$ the rational function Q has a double zero at $z = 2i/t$, and the Riemann surface associated to the equation

$$\xi^2 = \frac{Q(z)}{4}, \tag{4.7}$$

has genus 0. This genus Ansatz yields the correct guess of an appropriate Q for $t < t_c$, and it is consistent with the numerical observation that for $t < t_c$, the zeros of $p_n(z;t)$ accumulate on a single analytic arc, as proven in the aforementioned work [31]. In the same work, Deaño also indicated that $x_* = 0$ should not be the correct choice for $t > t_c$. In light of the numerical outputs in Figure 4.1, which indicate that for $t > t_c$ the zeros accumulate on two disjoint arcs, we expect that the Riemann surface associated to Q as in (4.3) has genus 1. This means that Q must have simple zeros for $t > t_c$, so we must expect that $x_* > 0$.

The determination of $x_*(t)$ in the supercritical regime is our first contribution, as stated below.

Theorem 4.6. *Fix $t > t_c$ and define λ_0, λ_1 as in* (4.6). *Then there exists a unique choice $x_* = x_*(t) \in (0,1)$ for which*

$$\operatorname{Re}\int_{\lambda_1}^{1} Q^{1/2}(s;t)ds = 0. \tag{4.8}$$

Furthermore,

$$\lim_{t\to t_c^+} x_*(t) = 0, \qquad \lim_{t\to+\infty} x_*(t) = 1.$$

In the literature, the transcendental condition (4.8) goes by the name of the *Boutroux Condition* [10–12, 64]. We remark that this condition does not depend on the choice of branch of the square root, as long as it varies analytically along the contour of integration. We are implicitly assuming this fact when we write (4.8). We will shortly fix a branch of this root that will be used throughout the rest of this chapter.

Recall from Chapter 2 that one key to implementing the Deift-Zhou steepest descent procedure is the existence of an appropriate h-function. The determination of this h-function also includes the determination of the main arcs, $\mathfrak{M}$, and complementary arcs, $\mathfrak{C}$, off which the h-function is analytic. As we expect to be in the genus 1 regime, the determination of the main arcs $\mathfrak{M} = \gamma_{m,0} \cup \gamma_{m,1}$ is our next result.

Theorem 4.13. *Let $t > t_c$ and take Q, $\lambda_0 = -x_* + 2i/t$, and $\lambda_1 = -\overline{\lambda_0}$ as in* (4.5) *and* (4.6). *Here $x_* = x_*(t)$ is the one whose existence is assured via Theorem 4.6. Then there exist analytic arcs $\gamma_{m,0} = \gamma_{m,0}(t)$ and $\gamma_{m,1} = \gamma_{m,1}(t)$ with the following properties:*

(i) The arc $\gamma_{m,1}$ is in the right half plane and connects λ_1 and 1. It is the unique such arc which satisfies

$$\int_{\lambda_1}^{z} \sqrt{Q(s)}\, ds \in i\mathbb{R}, \qquad z \in \gamma_{m,1}. \tag{4.9}$$

(ii) The arc $\gamma_{m,0}$ is obtained via reflecting the arc $\gamma_{m,1}$ about the imaginary axis and satisfies

$$\int_{-1}^{z} \sqrt{Q(s)}\, ds \in i\mathbb{R}, \qquad z \in \gamma_{m,0}. \tag{4.10}$$

Having determined x_* in Theorem 4.6 and constructed the main arcs in Theorem 4.13, we may now completely determine Q by specifying its branch cuts. We first orient the arcs $\gamma_{m,0}$ and $\gamma_{m,1}$ from -1 to λ_0 and from λ_1 to 1, respectively, so that we may define $+$ and $-$ sides of this arc with respect to their orientation. The rational function Q has a well-defined analytic square root on $\mathbb{C} \setminus \mathfrak{M}$, which we choose in such a way that the asymptotic expansion (4.4) holds true. For $z \in \mathfrak{M}$, we denote by $Q_\pm^{1/2}(z)$ the boundary values of this square root as z approaches $\mathfrak{M}$ from the $\pm$-side.

Our next result assures the existence of a positive measure μ_* which is supported on $\mathfrak{M}$. This measure will turn out to be the weak limit of the normalized counting measure for the zeros of the Kissing polynomials and will additionally satisfy (4.3).

Theorem 4.14. *Let $t > t_c$ and take Q, λ_0, and λ_1 as in* (4.5) *and* (4.6). *Here $x_* = x_*(t)$ is the one whose existence is assured via Theorem 4.6. Define a complex valued measure μ_* on $\gamma_{m,0} \cup \gamma_{m,1}$ through its density with respect to the complex line element, ds, as*

$$d\mu_*(s) = \frac{1}{2\pi i} Q_+^{1/2}(s)\, ds, \qquad s \in \gamma_{m,0} \cup \gamma_{m,1}.$$

Then, μ_* is a probability measure on $\gamma_{m,0} \cup \gamma_{m,1}$. Moreover, its shifted Cauchy transform, defined below as

$$\xi(z) = C^{\mu_*}(z) - \frac{it}{2}, \qquad C^{\mu_*}(z) := \int_{\gamma_{m,0} \cup \gamma_{m,1}} \frac{d\mu_*(s)}{s - z}, \qquad z \in \mathbb{C} \setminus (\gamma_{m,0} \cup \gamma_{m,1}),$$

solves $\xi^2(z) = Q(z)/4$ for $z \in \mathbb{C} \setminus (\gamma_{m,0} \cup \gamma_{m,1})$.

We then move on to the construction of the h-function defined in Chapter 2. Although we will not use the h-function directly in this chapter, its explicit form will be important in Chapter 5. Recall that the scalar Riemann-Hilbert problem is posed on the contour $\Omega = \mathfrak{M} \cup \mathfrak{C}$. As in Chapter 2, $\mathfrak{C} = \gamma_{c,0} \cup \gamma_{c,1}$, where $\gamma_{c,0} = (-\infty, -1]$. For now, we state that $\gamma_{c,1}$ is a contour connecting λ_0 to λ_1, whose precise location will be given shortly. Recall that the h-function must satisfy all of the following conditions:

$$h(z;t) \text{ is analytic for } z \in \mathbb{C} \setminus \Omega, \tag{4.11a}$$

$$h_+(z;t) - h_-(z;t) = 4\pi i, \qquad z \in \gamma_{c,0}, \tag{4.11b}$$

$$h_+(z;t) + h_-(z;t) = 4\pi i \omega_0, \qquad z \in \gamma_{m,0}, \tag{4.11c}$$

$$h_+(z;t) - h_-(z;t) = 4\pi i \eta_1, \qquad z \in \gamma_{c,1}, \tag{4.11d}$$

$$h_+(z;t) + h_-(z;t) = 0, \qquad z \in \gamma_{m,1}, \tag{4.11e}$$

$$h(z;t) = itz - \ell + 2\log z + \mathscr{O}\left(\frac{1}{z}\right), \qquad z \to \infty, \tag{4.11f}$$

$$\Re h(z;s) = \mathscr{O}\left((z-\lambda)^{3/2}\right), \qquad z \to \lambda \in \{\lambda_0, \lambda_1\}, \tag{4.11g}$$

$$\Re h(z;s) = \mathscr{O}\left((z \mp 1)^{1/2}\right), \qquad z \to \pm 1, \tag{4.11h}$$

and the inequalities

$$\Re h(z) < 0 \text{ if } z \text{ is an interior point of } \gamma_{c,1}, \tag{4.11i}$$

$$\Re h(z_0) > 0 \text{ for } z_0 \text{ in close proximity to any interior point of } \gamma_{m,0} \cup \gamma_{m,1}. \tag{4.11j}$$

Defining

$$h(z;t) = -\int_1^z \sqrt{Q(s;t)}\, ds, \qquad z \in \mathbb{C} \setminus \Omega, \tag{4.12}$$

where the path of integration above does not cross the contour Ω, we have our next main result.

Theorem 4.15. *$h(z;t)$ defined in* (4.12) *satisfies all of the requirements listed in* (4.11).

Having constructed the appropriate h-function, we could in principle follow the guide of Chapter 2 and use Deift-Zhou steepest descent with the h-function constructed above. However, we proceed with an alternate approach, based on a "symmetrized" version of the h-function. The symmetrized h-function, which we call ϕ, is defined as

$$\phi(z) = \frac{-h(z) - i\kappa}{2}, \tag{4.13}$$

where

$$\kappa = -\pi\omega_0, \tag{4.14}$$

and ω_0 is the constant defined in (4.11c). The reason for the use of ϕ, as opposed to h, is due to the symmetry of the current problem over the imaginary axis. Indeed, by using ϕ, we will be able construct a global parametrix in a manner that does not involve the use of theta functions, as in Chapter 2.3. This in turn will allow us to present slightly more explicit formulas for the asymptotics of the orthogonal polynomials.

We note that $\kappa = \kappa(t)$ depends on the parameter t. We will show in Section 4.4 that for odd n and a function $c = c(t)$, to be defined later, the difference $2n\kappa(t) - c(t)$ takes values in $2\pi\mathbb{Z}$ for a discrete set of values of n and t. For $\varepsilon > 0$, we therefore define Θ^*_ε to be the set of pairs (n,t) for which the quantity $2n\kappa - c$ is a distance less than ε away from $2\pi\mathbb{Z}$. More details on this critical set, which intuitively plays the same role as the non-vanishing of the theta divisor in (2.76), are given in Section 4.4.5. This then leads us to our final result.

Theorem 4.26. *Fix $\varepsilon > 0$ and $t > t_c$. For n sufficiently large, and for $(n,t) \notin \Theta^*_\varepsilon$ in the case that n is odd, the Kissing polynomial $p_n(z;t)$ defined in* (4.1) *(with $N = n$) uniquely exists as a monic polynomial of degree exactly n. If we denote by $z_1,\dots,z_n$ the zeros of $p_n(z;t)$, we have the weak asymptotics,*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \delta_{z_k} \overset{*}{\rightharpoonup} \mu_*, \tag{4.15}$$

where μ_* is the measure defined in Theorem 4.14 and δ_z is the atomic measure with mass 1 *at z.*

Furthermore, as $n \to \infty$,

$$p_{2n}(z;t) = \Psi_{n,0}(z)e^{n(-i\kappa - itz + \ell - 2\phi(z))}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \tag{4.16a}$$

$$p_{2n+1}(z;t) = e^{(2n+1)\left(-\frac{i\kappa}{2} - \frac{itz}{2} + \frac{\ell}{2} - \phi(z)\right)}\left(\Psi_{n,1} + \mathcal{O}\left(\frac{1}{n}\right)\right), \tag{4.16b}$$

hold true uniformly in compact subsets of $\mathbb{C}\setminus\mathfrak{M}$ and $\mathbb{C}\setminus(\mathfrak{M}\cup\gamma_{c,0})$, respectively, where the functions $\Psi_{n,0}$ and $\Psi_{n,1}$ have the following properties:

(i) $\Psi_{n,0}$ is holomorphic in $\mathbb{C}\setminus\mathfrak{M}$, whereas $\Psi_{n,1}$ is holomorphic on $\mathbb{C}\setminus(\mathfrak{M}\cup\gamma_{c,0})$, and they remain bounded on compact subsets of their respective domains of definition as $n\to\infty$.

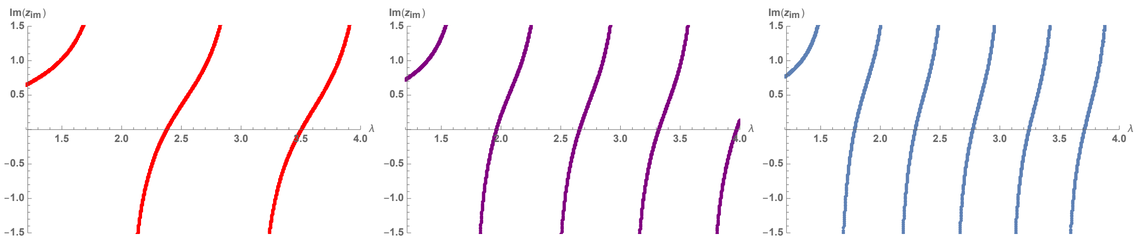
(ii) $\Psi_{n,0}$ does not have zeros.

(iii) The function $\Psi_{n,1}$ has a unique zero at a point $a_* = a_*(n,t)$, which is simple and located on the imaginary axis.

ℓ is the constant defined via (4.11f). We note that although $\Psi_{n,1}$ has a jump on a contour $\gamma_{c,0}$ which does not contain zeros of p_{2n+1}, it will turn out that $\Psi_{n,1}e^{-(2n+1)\phi}$ is analytic on $\mathbb{C}\setminus\mathfrak{M}$, so the leading term on the right-hand side of (4.16b) is in fact analytic on $\mathbb{C}\setminus\mathfrak{M}$.

The nature of the restriction to odd n in Theorem 4.26 is due to the construction of the global parametrix, whose existence can only be assured upon verifying the non-degeneracy conditions leading to the definition of Θ^*_ε above. Again, more details are provided in Section 4.4.5. The functions $\Psi_{n,0}$ and $\Psi_{n,1}$ are specific entries in this global parametrix, and as such they are initially constructed with the help of meromorphic differentials on the Riemann surface associated to $\xi^2 = Q/4$.

The appearance of a zero of $\Psi_{n,1}$ on the imaginary axis is natural in our situation, because for odd n the polynomial $p_n(z;t)$ has, by symmetry, exactly one zero on the imaginary axis. When $t < t_c$, the support of the limiting zero distribution of $p_n(z;t)$ is connected and intersects the imaginary axis, so this zero on $i\mathbb{R}$ is always encoded in the limiting distribution. However, when $t > t_c$, the support of the limiting distribution μ_* no longer touches the imaginary axis. Therefore, the purely imaginary zero of $p_{2n+1}(z;t)$ remains an outlier, and can intuitively be thought of as a "spurious" zero in the language of rational approximation.



$n=1$ $n=2$ $n=3$

Figure 4.2: Plotting the imaginary part of the purely imaginary zero of $p_{2n+1}(z;t)$ as a function of t, for $n = 1,2,3$.

As explained in Chapter 3, for any n fixed, there is always a sequence $\omega = \omega_j \to \infty$ for which $p^{\omega}_{2n+1}(z)$ never exists (as a polynomial of degree exactly $2n+1$). Having in mind the identification $\omega = \omega(n) = tn$, and leaving aside technicalities such as the uniformity of the error in (4.16b) for large t, it is therefore natural that p_{2n+1} need not exist for all values of t. As such, the restriction on odd degrees in Theorem 4.26 is to be expected.

The main original findings of this work are the construction of the h-function and the alternate construction of the global parametrix with the symmetrized h-function. Of course, the main results on the asymptotics of the polynomials themselves proceed via Deift-Zhou steepest descent, and these steps are more or less unchanged to the material as presented in Chapter 2.

4.2 Construction of the Modified External Field

We first state the following two lemmas in complex analysis will be used throughout the text.

Lemma 4.1. *Let γ_1 be an contour on the left half plane, from p to q, and γ_2 be the contour obtained from γ_1 upon reflection over the imaginary axis, oriented from $-\overline{q}$ to $-\overline{p}$.*

Suppose that a function f satisfies the symmetry relation

$$\overline{f(s)} = \delta f(-\overline{s}),$$

where $\delta \in \{+1,-1\}$. Then

$$\overline{\int_{\gamma_1} f(s)ds} = \delta \int_{\gamma_2} f(s)ds.$$

In particular,

$$\int_{\gamma_1} f(s)ds + \int_{\gamma_2} f(s)ds = \begin{cases} 2\Re \int_{\gamma_1} f(s)ds, & \text{if } \delta = 1, \\ 2i\Im \int_{\gamma_1} f(s)ds, & \text{if } \delta = -1. \end{cases}$$

Proof. We first parameterize the arcs as $\gamma_1, \gamma_2 : [0,1] \to \mathbb{C}$ so that $\gamma_1(0) = p$, $\gamma_1(1) = q$, and $\gamma_2(t) = -\overline{\gamma_1(t)}$. Then,

$$\int_{\gamma_2} \delta f(s)\, ds = \int_0^1 \delta f(\gamma_2(t))\gamma_2'(t)\, dt = \int_0^1 -\delta f\left(-\overline{\gamma_1(t)}\right)\overline{\gamma_1'(t)}\, dt$$
$$= -\overline{\int_0^1 f(\gamma_1(t))\gamma_1'(t)\, dt} = \overline{\int_{\gamma_1} f(s)\, ds},$$

completing the proof. □

Lemma 4.2. *Let $\widehat{\gamma}$ be a contour symmetric with respect to the imaginary axis and f be as in the previous lemma. Then*

$$\Im \int_{\widehat{\gamma}} f(s) ds = 0, \quad \textit{if } \delta = 1, \qquad \Re \int_{\widehat{\gamma}} f(s) ds = 0, \quad \textit{if } \delta = -1.$$

Proof. Let γ_1 be the portion of $\hat{\gamma}$ lying in the left half plane, and let γ_2 be its reflection over the imaginary axis, with both γ_1 and γ_2 inheriting their respective orientation from $\hat{\gamma}$. We may then write

$$\int_{\hat{\gamma}} f(s)\, ds = \int_{\gamma_1} f(s)\, ds + \int_{\gamma_2} f(s)\, ds.$$

Using Lemma 4.1 with $\delta = 1$, we see that the above integral has no imaginary part. Similarly, when $\delta = -1$, we se that the above integral is purely imaginary, completing the proof. □

4.2.1 The Boutroux Condition

The Boutroux Condition, introduced in [10, 11], provides one approach to determining the asymptotics of orthogonal polynomials with respect to complex weights. For a rational function R, the Boutroux condition specifies that

$$\oint_\alpha \xi\, du \in i\mathbb{R},$$

where α is any closed loop on the Riemann surface associated to the algebraic curve $\xi^2 = R(z)$.

In the present setting, for any $x > 0$, fix $t > t_c$ and consider $\lambda_1 = \lambda_1(x,t)$ as in (4.6) with $x > 0$, and the associated rational function $Q = Q(z;t)$ as in (4.5). Let L be the union of the oriented line segments connecting -1 to λ_0 and λ_1 to 1. In this section we use L as a set of branch cuts for $Q^{1/2}$, so that we fix $Q^{1/2}$ to be the square root of Q which is analytic on $\mathbb{C} \setminus L$ and that satisfies the asymptotics in (4.4). Notice that the endpoints of L (and, in loose terms, L itself) vary continuously with the parameters $x \geq 0$ and $t > 0$.

In very concrete terms, the Boutroux condition in our setting requires finding x for which

$$\Re \int_{-1}^{\lambda_0} Q_+^{1/2}(s)\, ds = \Re \int_{\lambda_0}^{\lambda_1} Q^{1/2}(s)\, ds = \Re \int_{\lambda_1}^{1} Q_+^{1/2}(s)\, ds = 0, \tag{4.17}$$

where the integration takes place along straight line segments (that is, along subarcs of L), and we recall that the subscript $+$ denotes the limiting value of $Q^{1/2}(u)$ as we approach $u \in L$ from its left-hand side w.r.t. the orientation of L.

To emphasize the dependence of Q on both x_* and t, we introduce

$$Q(z;t,x) = -\frac{t^2\left(z+x-\frac{2i}{t}\right)\left(z-x-\frac{2i}{t}\right)}{z^2-1}. \tag{4.18}$$

Note that with our assumptions that $x>0$ and $t>t_c$, we have that $Q^{1/2}$ possesses the following symmetry over the imaginary axis

$$\overline{Q^{1/2}(z;t,x)} = -Q^{1/2}\left(-\bar{z};t,x\right). \tag{4.19}$$

Lemma 4.3.

$$-\overline{\int_{\lambda_1}^{1} Q_+^{1/2}(s;t,x)\,ds} = \int_{-1}^{\lambda_0} Q_+^{1/2}(s;t,x)\,ds \quad \text{and} \quad \Re\int_{\lambda_0}^{\lambda_1} Q_+^{1/2}(s;t,x)\,ds = 0.$$

Proof. Note that the symmetry in (4.19) extends to the branch cuts, that is $\overline{Q_+^{1/2}(z;t,x)} = -Q_+^{1/2}\left(-\bar{z};t,x\right)$ for $z\in L$. The proof now immediately follows from the symmetry as described in Lemmas 4.1 and 4.2 with $\delta=-1$. □

As a result of Lemma 4.3, to determine x such that equations (4.17) are satisfied, all we have to do is to make sure that the last integral in (4.17) is purely imaginary. To do this, we consider the function

$$\psi(x) = \Re\int_{\lambda_1}^{1} Q_+^{1/2}(s;t,x)\,ds = \Re\int_{x+\frac{2i}{t}}^{1} Q_+^{1/2}(s;t,x)\,ds. \tag{4.20}$$

We emphasize that for any fixed x, the contours of integration are still straight line segments. This assures that for fixed t, the function ψ is a continuous function of $x\in[0,1]$. Our next task is to show that ψ changes sign on $[0,1]$, for any $t>t_c$. This would then immediately imply the existence of an $x_* = x_*(t)$ for which $\psi(x_*)=0$ for all $t>t_c$.

Lemma 4.4. *For all $t>t_c$, we have that*

$$\psi(0)<0.$$

Proof. By definition,

$$\psi(0) = \Re\int_{\frac{2i}{t}}^{1} Q_+^{1/2}(s;t,0)\,ds.$$

In this situation, we can write $Q^{1/2}$ explicitly as

$$Q^{1/2}(s;t,0) = \frac{-it\left(s-\frac{2i}{t}\right)}{\sqrt{s^2-1}},$$

where the branch of $\sqrt{s^2-1}$ is the principal branch, so that $\sqrt{s^2-1} \sim s$, as $s \to \infty$. Using that

$$\frac{d}{ds}\sqrt{s^2-1} = \frac{s}{\sqrt{s^2-1}} \qquad \text{and} \qquad \frac{d}{ds}\log\left(s+\sqrt{s^2-1}\right) = \frac{1}{\sqrt{s^2-1}}, \tag{4.21}$$

we calculate that

$$-it\int_{\frac{2i}{t}}^{1} \frac{\left(s-\frac{2i}{t}\right)}{\sqrt{s^2-1}}\,ds = -\sqrt{4+t^2} + 2\log\left(\frac{i\left(2+\sqrt{4+t^2}\right)}{t}\right).$$

By taking real parts, we have that

$$\psi(0) = 2\log\left(\frac{2+\sqrt{4+t^2}}{t}\right) - \sqrt{4+t^2}.$$

Note that $\psi(0) = 0$ when $t = t_c$, which follows from the definition of t_c as the only positive solution to (4.2). Furthermore,

$$\frac{d}{dt}\psi(0) = -\frac{\sqrt{4+t^2}}{t} < 0,$$

so $\psi(0) < 0$ for all $t > t_c$, as desired. □

Lemma 4.5. *For all $t > 0$, we have that*

$$\psi(1) > 0.$$

Proof. Through the linear change of variables

$$s \mapsto i(s-1),$$

we see that

$$\psi(1) = -t\int_{-\frac{2}{t}}^{0} \operatorname{Re} R_+^{1/2}(s)\,ds, \tag{4.22}$$

where

$$R(s) = \frac{(2+st)\,(2+t(2i+s))}{t^2 s\,(2i+s)}.$$

As we have fixed the branch of $Q^{1/2}$ to satisfy (4.4), the branch of $R^{1/2}$ in (4.22) behaves like

$$R^{1/2}(s) \to 1, \qquad s \to \infty.$$

Here, $R^{1/2}$ has branch cuts on the horizontal segments $\left(-\frac{2}{t},0\right)$ and $\left(-\frac{2}{t}-2i,-2i\right)$ and the integral (4.22) is computed along the first of these branch cuts. The goal now is to show that

$$\operatorname{Re} R_+^{1/2}(s) < 0, \qquad s \in \left(-\frac{2}{t},0\right),$$

which will immediately imply that $\psi(1) > 0$ for $t > 0$. To do this, first note that for $s \in \mathbb{R}$ we can split $R(s)$ into real and imaginary parts as

$$R(s) = U(s) + iV(s), \quad U(s) = \frac{(2+st)\left(2s+t\left(4+s^2\right)\right)}{t^2 s\,(4+s^2)}, \quad V(s) = -\frac{4\,(2+st)}{t^2 s\,(4+s^2)}.$$

As $s \to -\infty$,

$$R(s) = 1 + \frac{4}{ts} + \frac{4-4it}{t^2 s^2} + \mathscr{O}\left(\frac{1}{s^3}\right), \qquad s \to -\infty,$$

so that as s moves from $-\infty$ towards $-2/t$ along the negative real axis, the image $R(s)$ traces out a curve in the plane, starting at $z = 1$ and initially dropping into the lower-right hand quadrant of the complex plane. As neither the real nor imaginary parts of $R(s)$ have real zeros or poles in the interval $\left(-\infty, -\frac{2}{t}\right)$, we can conclude that $R(s)$ remains in the lower right-hand quadrant for these values of s, and consequently

$$\arg R^{1/2}(s) \in \left(-\frac{\pi}{4}, 0\right), \qquad s \in \left(-\infty, -\frac{2}{t}\right).$$

In particular, from the expansion

$$R(s) = -\frac{t^2}{2(i+t)}\left(s + \frac{2}{t}\right) + \mathscr{O}\left(\left(s + \frac{2}{t}\right)^2\right), \qquad s \to -\frac{2}{t},$$

we obtain

$$\lim_{\substack{s \to -\frac{2}{t} \\ s < -\frac{2}{t}}} \arg R^{1/2}(s) = \lim_{\substack{s \to -\frac{2}{t} \\ s < -\frac{2}{t}}} \frac{1}{2} \arctan \frac{V(s)}{U(s)} = -\frac{\arctan\left(\frac{1}{t}\right)}{2} \in \left(-\frac{\pi}{4}, 0\right).$$

Because $R^{1/2}$ vanishes as a square root at $s = -2/t$, a conformal mapping analysis implies that

$$\lim_{\substack{s \to -\frac{2}{t} \\ s > -\frac{2}{t}}} \arg R_+^{1/2}(s) = -\frac{\arctan\left(\frac{1}{t}\right)}{2} - \frac{\pi}{2} \in \left(-\frac{3\pi}{4}, -\frac{\pi}{2}\right),$$

so that as s moves from $s = -\frac{2}{t}$ towards 0 on the negative real axis, the image of $R_+^{1/2}(s)$ traces out a curve that starts at 0 and ventures into the lower left hand quadrant of the plane. Just as before, neither the real nor imaginary parts of $R(s)$ have zeros or poles in the interval $\left(-\frac{2}{t}, 0\right)$, and therefore we obtain

$$\operatorname{Re} R_+^{1/2}(s) < 0, \qquad s \in \left(-\frac{2}{t}, 0\right),$$

as desired. □

We are now ready to state and prove the first main result of this chapter.

Theorem 4.6. *Fix $t > t_c$ and define λ_0, λ_1 as in* (4.6). *Then there exists a unique choice $x_* = x_*(t) \in (0,1)$ for which*

$$\operatorname{Re} \int_{\lambda_1}^{1} Q^{1/2}(s;t)ds = 0. \tag{4.23}$$

Furthermore,

$$\lim_{t \to t_c^+} x_*(t) = 0, \qquad \lim_{t \to +\infty} x_*(t) = 1.$$

Proof. The existence of $x_* \in (0,1)$ follows from Lemmas 4.4 and 4.5 and the continuity of $\psi(x)$. The uniqueness of such x_* will follow later, in a more indirect manner. We outline this proof of uniqueness below.

First, we note that the entirety of the asymptotic analysis to be conducted later relies solely on the existence of such an $x_*(t)$, and not on its uniqueness. Indeed, we show later that for this value of x_*, there exists some measure μ_* such that the normalized zero counting measure of the orthogonal polynomials converges weakly to μ_*. Assume there is some other x_*, say $\widehat{x}_*$, which satisfies the conditions of Theorem 4.6. To this value of $\widehat{x}_*$, we will be able to construct an associated measure $\widehat{\mu}_*$ and verify the convergence of the normalized zero counting measure to $\widehat{\mu}_*$. By uniqueness of the limiting zero distribution, we would have $\widehat{\mu}_* = \mu_*$, and as such their supports would have to agree. In particular, the endpoints of the supports agree as well, thus $x_* = \widehat{x}_*$. Along the same lines, we can verify that $x_* \to 0$ as $t \to t^+$ by looking at the limiting zero distribution of p_n for the fixed choice $t = t_c$.

To show that $x_* \to 1$ as $t \to \infty$, we start with a change of variables in (4.20) to arrive at

$$\psi(x) = \operatorname{Re} \int_{-1}^{0} \left(1 - x - \frac{2i}{t}\right) Q_+^{1/2}\left(\left(1 - x - \frac{2i}{t}\right)s + 1; t, x\right) ds,$$

where we are integrating over the branch cut oriented on the real axis from -1 to 0. Another cumbersome calculation shows that

$$\left(1 - x - \frac{2i}{t}\right) Q_+^{1/2}\left(\left(1 - x - \frac{2i}{t}\right)s + 1; t, x\right) = tc_1 + c_0 + \mathscr{O}\left(\frac{1}{t}\right), \qquad t \to \infty,$$

where

$$c_1 = i(x-1)\left(\frac{(s+1)(-1-s-x+sx)}{s(-2-s+sx)}\right)_+^{1/2},$$

and

$$c_0 = -\frac{2\left(3 + 4s + s^2 + x - 4sx - 2s^2x + s^2x^2\right)}{(-2-s+sx)(-1-s-x+sx)}\left(\frac{(s+1)(-1-s-x+sx)}{s(-2-s+sx)}\right)_+^{1/2},$$

with a uniform error term for x in compact subsets of $\mathbb{R}$.

If we restrict to $x \in (0,1)$, we see that for $s \in (-1,0)$,

$$\left(\frac{(s+1)(-1-s-x+sx)}{s(-2-s+sx)}\right)_+^{1/2} \in i\mathbb{R},$$

so that $c_1 \in \mathbb{R}$ and $c_0 \in i\mathbb{R}$. Then,

$$\psi(x) = it(x-1)\int_{-1}^{0}\left(\frac{(s+1)(-1-s-x+sx)}{s(-2-s+sx)}\right)_+^{1/2} ds + \mathscr{O}\left(\frac{1}{t}\right), \qquad t \to \infty,$$

with uniform error for $x \in (0,1)$. This means that the coefficient with t in the right-hand side above is nonzero if and only if $x \neq 1$. However, we must have $x_* \in (0,1)$ and $\psi(x_*) = 0$ for every $t > t_c$, which, by virtue of the expansion above, can only happen if $x_* \to 1$ as $t \to \infty$, as desired.

□

In fact, the argument above actually shows that

$$x_*(t) = 1 + o\left(\frac{1}{t}\right), \qquad t \to \infty, \tag{4.24}$$

as the right hand side of the last identity above, when evaluated at x_*, must also be 0 in the limit $t \to \infty$. To conclude this section, it is useful to compute the first integral in (4.17).

Lemma 4.7. *For x_* given by Theorem 4.6, we have that*

$$\int_{\lambda_1}^{1} Q_+^{1/2}(s)\, ds = i\pi. \tag{4.25}$$

Proof. Using (4.23) and Lemma 4.3,

$$2\int_{\lambda_1}^{1} Q_+^{1/2}(s)\, ds = 2i\Im\int_{\lambda_1}^{1} Q_+^{1/2}(s)\, ds = \int_{\lambda_1}^{1} Q_+^{1/2}(s)\, ds + \int_{-1}^{\lambda_0} Q_+^{1/2}(s)\, ds.$$

We may write the right hand side above as a contour integral, yielding

$$2\int_{\lambda_1}^{1} Q_+^{1/2}(s)\, ds = \frac{1}{2}\int_{\alpha} Q^{1/2}(s)\, ds,$$

where α is a closed contour that encircles the whole branch cut L in the clockwise direction. To compute the latter integral, we deform α to ∞ and use the expansion (4.4) to get

$$\int_{z_*}^{1} Q_+^{1/2}(s)\, ds = \frac{1}{4}\, 2\pi i \operatorname{Res}(Q^{1/2}(s), s = \infty) = i\pi,$$

as wanted.

□

4.2.2 Construction of the Main Arcs

The goal of this section is to construct the main arcs, $\gamma_{m,0}$ and $\gamma_{m,1}$, as prescribed by the definition of the h-function. We will see that these arcs can be described as trajectories of a quadratic differential on the Riemann sphere. As such, we first describe the basic theory as needed for the present analysis.

Background on Quadratic Differentials

The theory of quadratic differentials is now standard, and as such, the material in this section is not original work of the author. The theory presented below follows [81, 84], and we refer the reader to these works for complete details. Further details can also be found in [74, Appendix B], [78, Chapter 8].

A meromorphic differential ϖ on a Riemann surface $\mathfrak{R}$ is a second order form on the Riemann surface, given locally by the expression $f(z)\,dz^2$, where f is a meromorphic function of the local coordinate z. In particular, if $z = z(\zeta)$ is a conformal change of variables,

$$\tilde{f}(\zeta)\,d\zeta^2 = f(z(\zeta))z'(\zeta)^2\,d\zeta^2 \tag{4.26}$$

represents ϖ in the local coordinate ζ. In the present context, we may always take the underlying Riemann surface to be the Riemann sphere. Of particular interest to us is the *critical graph* of a quadratic differential ϖ, which we explain below.

First, we define the *critical points* of $\varpi = f\,dz^2$ to be the zeros and poles of f. The order of the critical point, p, is the order of the zero or pole, and is denoted by $\eta(p)$. Zeros and simple poles are called *finite critical points*; all other critical points are *infinite*. Any point which is not a critical point, is a *regular point*.

In a neighborhood of any regular point p, the primitive

$$\Upsilon(z) = \int_p^z \sqrt{-\varpi} = \int_p^z \sqrt{-f(s)}\,ds \tag{4.27}$$

is well defined by specifying the branch of the root at p and analytically continuing this along the path of integration. Then, we define an arc $\gamma \subset \mathfrak{R}$ to be an *arc of trajectory* of ϖ if it is locally mapped by Υ to a vertical line. Equivalently, for any point $p \in \gamma$, there exists a neighborhood U where Υ is well defined and moreover, $\Re\Upsilon(z)$ is constant for $z \in \gamma \cap U$. A maximal arc of trajectory is called a *trajectory* of ϖ. Moreover, any trajectory which extends to a finite critical point along one of its directions is called a *critical trajectory* of ϖ. The set of critical trajectories of ϖ, along with their limit points, is defined to be the *critical graph* of ϖ.

To understand the topology of the critical graph of a quadratic differential ϖ, we must necessarily study both the local structure of trajectories near finite critical points, along with the global structure of the critical trajectories. Fortunately, the local behavior near a finite critical point is quite regular. Indeed, from a point p of order $\eta(p) = m \geq -1$ emanate $m+2$ trajectories, forming equal angles of $2\pi/(m+2)$ at p. This also includes regular points, which implies that through any regular point passes exactly one trajectory, which is locally an analytic arc. In particular, this implies that trajectories may only intersect at critical points.

The global structure of trajectories is more involved, and requires more detailed analysis. In general, a trajectory γ is either

(i) a closed curve containing no critical points,

(ii) an arc connecting two critical points (which may coincide), or

(iii) an arc that has no limit along at least one of its directions.

Trajectories satisfying (iii) are called *recurrent trajectories*, and their absence in the present work is assured by Jenkins' Three Poles Theorem [78, Theorem 8.5], formulated as Principle **P3** below.

We will also need two basic principles that follow from the general theory of trajectories of quadratic differentials. These principles are thoroughly discussed in [74, Section 4.5.1].

P1. If a critical trajectory τ emerges from a zero contained in a simply connected domain $D \subset \mathbb{C}$ that does not contain poles of $f\,dz^2$, then either τ connects to another zero inside D, or τ intersects ∂D. In a similar spirit, if $D \subset \mathbb{C}$ is simply connected and contains exactly one pole, then the trajectory emanating from this pole has to either end at a zero inside D or hit the boundary of D.

P2. If $D \subset \mathbb{C}$ is a simply connected domain whose boundary is a union of critical trajectories and it does not contain poles in its interior, then it has to contain at least one pole on its boundary.

Finally, we state Teichmüller's Lemma, which will be used throughout the construction of the main arcs. Fix a simply connected domain $D \subset \overline{\mathbb{C}}$ whose boundary is a finite union of critical trajectories. Given a critical point $p \in \partial D$, we set

$$\beta(p) = 1 - \theta(p)\frac{\eta(p)+2}{2\pi},$$

where $\theta(p) \in [0, 2\pi]$ is the inner angle of ∂D at p. The following formula, valid for any simply connected domain D as above, is known as Teichmüller's Lemma [84, Theorem 14.1],

$$\sum_{p \in \partial D} \beta(p) = 2 + \sum_{p \in D} \eta(p). \tag{4.28}$$

With the necessary background on quadratic differentials in hand, we move on to the construction of the main arcs.

The Associated Quadratic Differential and its Trajectories

In this section, we study the trajectories of $\varpi = -Q(z;t)\,dz^2$, where we recall that $Q(z;t)$ is given in (4.5) as

$$Q(z;t) = -\frac{t^2\,(z-\lambda_0(t))\,(z-\lambda_1(t))}{z^2-1},$$

with λ_0, λ_1 given by (4.6) and x_* given by Theorem 4.6. Note that $-Q\,dz^2$ has an infinite critical point at infinity of order four. To see this, we make use of the local coordinate $z = 1/\zeta$ and use (4.26) to obtain

$$-\frac{1}{\zeta^4}Q\left(\frac{1}{\zeta};t\right)d\zeta^2 = \left(\frac{t^2}{\zeta^4} + \mathscr{O}\left(\frac{1}{\zeta^3}\right)\right)d\zeta^2, \qquad \zeta \to 0. \tag{4.29}$$

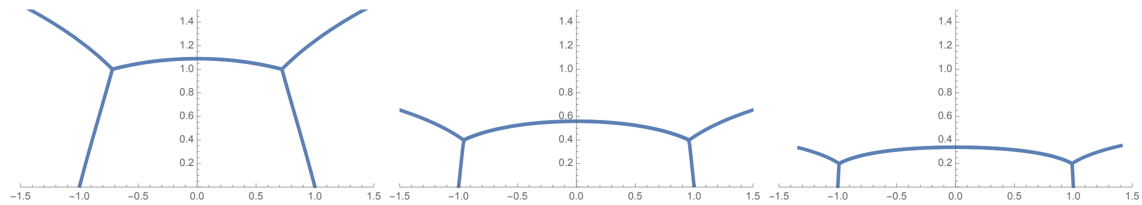


Figure 4.3: Trajectories of $-Q(z;t)\,dz^2$.

By the symmetry of Q over the imaginary axis, and its explicit form, we immediately have the following additional principles:

P3. Because $-Q\,dz^2$ has three distinct poles, any critical trajectory has to connect two critical points (possibly the same).

P4. If τ is an arc of trajectory, then its reflection $-\tau^*$ onto the imaginary axis is also an arc of trajectory. The notation τ^* indicates complex conjugation of the set τ.

The structure of the critical graph of $-Q\,dz^2$ is our main theorem in this section, and its proof will be completed via the sequence of following lemmas. The critical graph of $-Q\,dz^2$ is numerically computed in Figure 4.3 for various choices of t.

Lemma 4.8. *There is at least one trajectory emanating from λ_1 with endpoint on $\{1, \lambda_0\}$.*

Proof. As λ_1 is a simple zero of $-Q$, three trajectories emanate from λ_1. To get to a contradiction, suppose that no trajectory emanating from λ_1 meets 1 or λ_0. If a trajectory emanating from λ_1 hits $i\mathbb{R}$, then using Principle **P4** we conclude that this trajectory connects λ_1 to $-\overline{\lambda_1} = \lambda_0$. But this cannot occur by assumption, so all three trajectories emanating from λ_1 have to stay in the right half plane and none of them can end at $z = 1$.

Furthermore, none of these trajectories from λ_1 can form a closed loop in $\mathbb{C}$. Indeed, if this were the case, then Principle **P2** applied to the bounded domain D determined by this loop would guarantee that $z = 1$ is inside the loop. But then the trajectory τ emerging from $z = 1$, by Principle **P1**, would have to hit ∂D. As ∂D is a trajectory, and trajectories can only intersect at critical points, this means that τ would have to connect to the only critical point $\lambda_1 \in \partial D$, which we are assuming cannot occur.

So this discussion and Principle **P3** yield that all the trajectories from λ_1 must extend to ∞ (the only remaining critical point). Because these trajectories have to stay in the right half plane, the asymptotics (4.4) imply that they all meet at ∞ with angle 0. These three trajectories determine exactly two domains in the right half plane, whose angle at ∞ is exactly 0. At least one of these domains is pole-free, say D, and for this domain, we compute

$$\sum_{p\in D} \eta(p) = 0 \quad \text{and} \quad \sum_{p\in \partial D} \beta(p) = 1,$$

in contradiction to (4.28). The proof is complete. □

Lemma 4.9. *At most one trajectory emanating from λ_1 intersects $i\mathbb{R}$.*

Proof. Suppose that there are two trajectories which emanate from λ_1, say τ_1 and τ_2, which intersect $i\mathbb{R}$. By Principle **P4**, they must both connect to $-\overline{\lambda_1} = \lambda_0$, so their union is the boundary of a simply connected domain D. Using Principles **P2** and **P4**, we see the poles $z = \pm 1 \in D$.

Consider now the remaining trajectory τ_3 which emanates from λ_1. If τ_3 emerges within D, $z = \pm 1 \in D$ implies that no critical trajectory extends to $z = \infty$. As this can not occur, we must have that τ_3 extends to ∞. Using again Principle **P3**, we see that the trajectories from $z = \lambda_0, \lambda_1$ are fully determined as in Figure 4.4a.

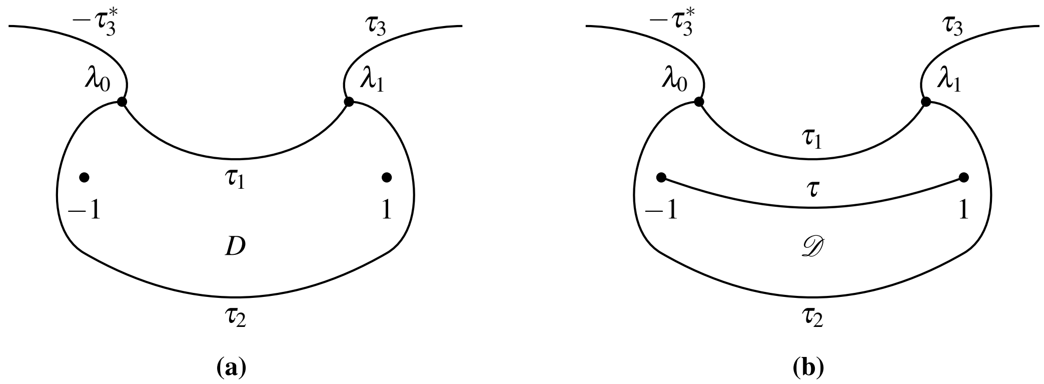


Figure 4.4: The (hypothetical) critical graph of $-Q\,dz^2$ as used in the proof of Lemma 4.9.

We now look at the trajectory τ emanating from $z = 1$. By Principle **P3**, we conclude that τ has to connect $z = 1$ and $z = -1$, so the full critical graph of $-Q\,dz^2$ is now depicted in Figure 4.4b.

Consider the domain $\mathscr{D}$, which is obtained by removing τ from the domain bounded by $\tau_1 \cup \tau_2$, see Figure 4.4b. According to the canonical decomposition of the critical graph [74, Theorem B1], the domain $\mathscr{D}$ is a ring domain, which means that for the function

$$\Upsilon(z) = \int_1^z \sqrt{Q(s)}ds$$

and some nonzero real constant c, the map $F(z) = e^{c\Upsilon(z)}$ is a conformal map from $\mathscr{D}$ to an annulus of positive radii $r < R$, and with boundary correspondence $F(\tau) = \partial D_r(0)$ and $F(\tau_2 \cup \tau_1) = \partial D_R(0)$. As $F(1) = 1$, $r = |F(1)| = 1$, and hence $R = |F(\lambda_1)| > 1$. However, using (4.23) we get also that $|F(\lambda_1)| = 1$, a contradiction, thereby concluding the proof.

□

Lemma 4.10. *If there is a trajectory connecting* 1 *and* λ_1*, then there is a trajectory connecting* λ_1 *and* λ_0.

Proof. Let τ_1 be the trajectory connecting λ_1 and $z = 1$, and τ_2 and τ_3 the remaining two trajectories emanating from λ_1. The proof will again proceed by contradiction. If we assume that there is no trajectory connecting λ_1 to λ_0, then τ_1 and τ_2 have to extend to $z = \infty$ horizontally on the right-half plane. Using the symmetry Principle **P4**, there are only two possibilities left for the critical graph of $-Qdz^2$, depending on whether or not $z = 1$ belongs or not to the domain bounded by τ_2 and τ_3 on the right half plane. These two possibilities are displayed in Figure 4.5.

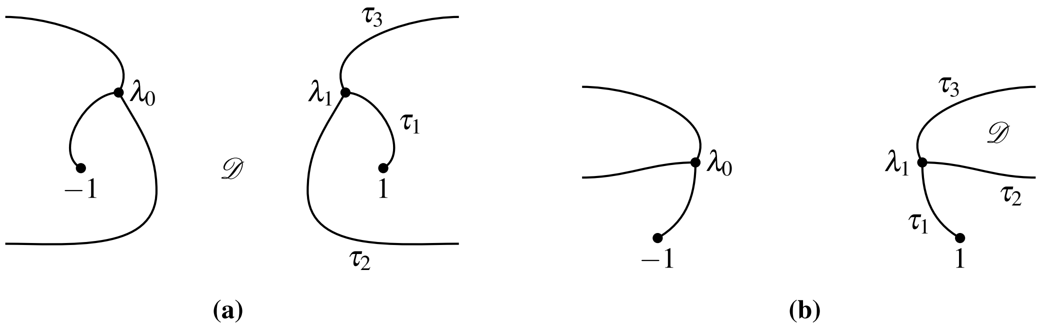


Figure 4.5: The hypothetical possibilities for the critical graph of $-Qdz^2$, as derived in the proof of Lemma 4.10.

In either of the two situations, we consider the domain $\mathscr{D}$ to be the one uniquely determined by the condition that $\partial\mathscr{D} \cap \tau_1 = \{\lambda_1\}$. This domain is also represented in Figure 4.5. Applying (4.28) to $\mathscr{D}$ as represented in Figure 4.5a, we have

$$\sum_{p\in\partial\mathscr{D}} \beta(p) = 4, \quad \sum_{p\in\mathscr{D}} \eta(p) = 0,$$

whereas in the situation depicted in Figure 4.5b ,

$$\sum_{p\in\partial\mathscr{D}} \beta(p) = 1, \quad \sum_{p\in\mathscr{D}} \eta(p) = 0.$$

Both cases violate (4.28), so we conclude the contradiction argument and the proof.

□

Lemma 4.11. *There is a trajectory connecting* 1 *and* λ_1.

Proof. Again to get to a contradiction, suppose there is no such trajectory. By Lemma 4.8, there is then a trajectory τ_1 which connects λ_1 and λ_0. Next, using Lemma 4.9 and Principle **P3**, the other two trajectories, τ_2 and τ_3, remain in the right half plane and extend to ∞ with angle 0.

These trajectories τ_2 and τ_3 determine a simply connected domain $\mathscr{H}$ in the right half plane which does not contain τ_1. Applying Teichmüller's Lemma (4.28) to this domain, we arrive at

$$\sum_{p\in\partial\mathscr{H}} \beta(p) = \beta(\lambda_1) + \beta(\infty) = 1.$$

Thus, $\mathscr{H}$ must contain the only pole in the right half plane, namely $z = 1$. This means that the critical graph is as depicted in Figure 4.6.

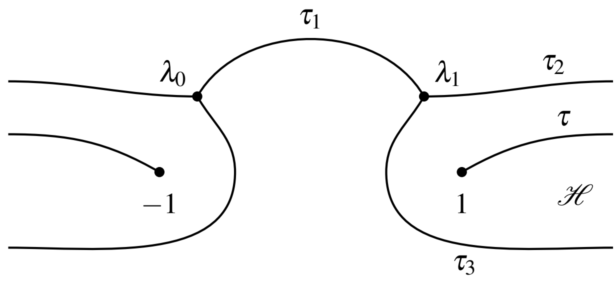


Figure 4.6: The trajectories of $-Qdz^2$ as used in the proof of Lemma 4.11.

We continue focusing on the domain $\mathscr{H}$. According to the canonical decomposition of the critical graph [74, Theorem B1], this domain $\mathscr{H}$ is a strip domain, which means that for some branch of the square root, the function

$$\Upsilon(z) = \int_{\lambda_1}^{z} \sqrt{Q(s)}ds$$

is a conformal map from $\mathscr{H}$ to a strip of the form

$$\mathscr{S} = \{z \in \mathbb{C} \mid c_1 < \operatorname{Re} z < c_2\}.$$

Moreover, Υ extends continuously to the boundary of $\mathscr{H}$, hence mapping $\tau_2 \cup \tau_3$ and τ to distinct connected components of the strip $\mathscr{S}$.

Clearly, $\Upsilon(\lambda_1) = 0$ and because of the boundary correspondence just explained we must then have $\operatorname{Re}\Upsilon(1) \neq 0$. However, using (4.23) we actually see that $\operatorname{Re}\Upsilon(1) = 0$, a contradiction. The proof is complete.

□

We now prove the following theorem on the structure of the critical graph of $-Qdz^2$, showing indeed that things are as depicted numerically in Figure 4.3.

Theorem 4.12. *The trajectories of $-Qdz^2$ are symmetric with respect to reflection over the imaginary axis. Furthermore, the critical trajectories in the right half plane are as follows:*

(i) There is a critical trajectory connecting 1 and λ_1.

(ii) There is a critical trajectory connecting λ_1 and λ_0. This is the only trajectory from λ_1 that moves to the left-half plane.

(iii) The remaining trajectory that emanates from λ_1 ends at ∞. Furthermore, $\arg z \to 0$ as $z \to \infty$ along this trajectory.

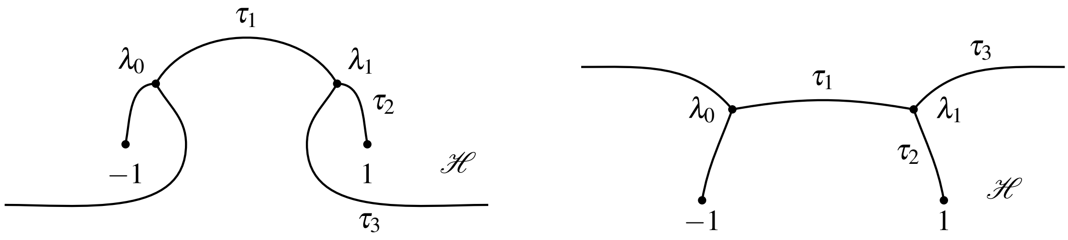
Proof. The symmetry under reflection follows immediately from Principle **P4**.

A combination of Lemmas 4.10 and 4.11 immediately yield (i) and (ii). The fact that the trajectory which connects λ_1 to λ_0 is the only one which moves to the left half plane from λ_1 follows from

Lemma 4.9. Using Lemma 4.9 again, we see that the remaining trajectory emanating from λ_1, say τ_3, must stay in the right half plane, and as such it extends to ∞ with angle 0. This concludes the proof of (iii).

□

We remark that Theorem 4.12 does not completely determine the structure of the critical graph. In principle, the trajectory in part (iii) of Theorem 4.12 could extend to ∞ going below $z = -1$, which would result on the critical graph displayed in Figure 4.7a, which we call case (i), in contrast with Figure 4.7b, which we call case (ii). We now justify why case (ii) takes place instead of case (i), which coincides with the numerical outputs in Figure 4.3.



(a) Case (i) **(b)** Case (ii)

Figure 4.7: The two possible configurations for the critical graph of $-Q\,dz^2$ after the proof of Theorem 4.12. As shown below, the correct configuration is Case (ii). We invite the reader to compare with the numerical output produced in Figure 4.3.

In either case, let $Q^{1/2}$ be the branch of the square root defined by the asymptotics (4.4), with branch cut along

$$\tau = \tau_1 \cup \tau_2 \cup \widehat{\tau}_2,$$

where $\widehat{\tau}_2$ is the reflection of τ_2 over the imaginary axis, see Figure 4.7. We orient this branch cut from -1 to 1. In either case (i) or (ii) in Figure 4.7, consider the domain $\mathscr{H}$ bounded by $\tau \cup \tau_3 \cup \widehat{\tau}_3$, with $\widehat{\tau}_3$ being the reflection of τ_3 over the imaginary axis. Informally, in case (i), $\mathscr{H}$ is the domain "above" the critical graph, and in case (ii) it is the domain "below" the critical graph.

Again using the decomposition of the critical graph as presented in [74, Theorem B.1], the domain $\mathscr{H}$ is a half-plane domain, and as such we know that the function

$$\Upsilon(z) = \int_1^z Q^{1/2}(s)\,ds, \quad z \in \mathscr{H},$$

is a conformal map from $\mathscr{H}$ to either the left or the right half plane. By looking at the asymptotics (4.4), we see that

$$\Upsilon(z) = -itz\,(1 + o(1)),$$

so when $z \to \infty$ along $\mathscr{H} \cap i\mathbb{R}$ we must either have that $\Re\Upsilon(z) \to +\infty$ (as in case (i)) or $\Re\Upsilon(z) \to -\infty$ (as in case (ii)). This means that the image $\Upsilon(\mathscr{H})$ is either the right or left half plane, corresponding to cases (i) or (ii), respectively.

By boundary correspondence, we must then have $\Upsilon_+(\tau_2) \subset i\mathbb{R}_+$ in case (i) and $\Upsilon_+(\tau_2) \subset i\mathbb{R}_-$ in case (ii), which implies that $\Im\Psi_+(\lambda_1) > 0$ and $\Im\Psi_+(\lambda_1) < 0$, respectively. But from (4.25) we know that $\Upsilon_+(\lambda_1) = -i\pi$, so that case (ii) is the correct configuration.

We are now in a position to state the existence of the main arcs used in the construction of the h-function.

Theorem 4.13. *Let $t > t_c$ and take Q, $\lambda_0 = -x_* + 2i/t$, and $\lambda_1 = -\overline{\lambda_0}$ as in* (4.5) *and* (4.6)*. Here $x_* = x_*(t)$ is the one whose existence is assured via Theorem 4.6. Then there exist analytic arcs $\gamma_{m,0} = \gamma_{m,0}(t)$ and $\gamma_{m,1} = \gamma_{m,1}(t)$ with the following properties:*

(i) The arc $\gamma_{m,1}$ is in the right half plane, and connects λ_1 and 1*. It is the unique such arc which satisfies*

$$\int_{\lambda_1}^{z} \sqrt{Q(s)}\,ds \in i\mathbb{R}, \qquad z \in \gamma_{m,1}. \tag{4.30}$$

(ii) The arc $\gamma_{m,0}$ is obtained via reflecting the arc $\gamma_{m,1}$ about the imaginary axis and satisfies

$$\int_{-1}^{z} \sqrt{Q(s)}\,ds \in i\mathbb{R}, \qquad z \in \gamma_{m,0}. \tag{4.31}$$

Proof. Set $\gamma_{m,1} = \tau_2$ as the trajectory as labeled in Figure 4.7b and $\gamma_{m,0}$ to be its reflection over the imaginary axis. That is, $\gamma_{m,1}$ is the trajectory of $-Qdz^2$ which connects λ_1 and 1, and $\gamma_{m,0}$ is the trajectory of $-Qdz^2$ which connects -1 and λ_0. Then (4.30)–(4.31) hold by the definition of a critical trajectory. The uniqueness follows by the uniqueness of the critical graph of $-Qdz^2$, and the fact that $\gamma_{m,1}$ is the only trajectory connecting 1 and λ_1.

□

4.2.3 Construction of the h-function

For the remainder of this chapter, we take the branch of $Q^{1/2}$ with cuts on $\gamma_{m,0} \cup \gamma_{m,1}$ and with branch chosen so that $Q^{1/2}$ satisfies the asymptotics (4.4). Before constructing the h-function, we posit the following theorem which guarantees the existence of a positive measure μ_* on $\gamma_{m,0} \cup \gamma_{m,1}$. This measure will turn out to be the weak limit of the normalized zero counting measure for the orthogonal polynomials, as we show in the next section.

Theorem 4.14. *Let $t > t_c$ and take Q, λ_0, and λ_1 as in* (4.5) *and* (4.6)*. Here $x_* = x_*(t)$ is the one whose existence is assured via Theorem 4.6. Define a complex valued measure μ_* on $\gamma_{m,0} \cup \gamma_{m,1}$ through its density with respect to the complex line element, ds, as*

$$d\mu_*(s) = \frac{1}{2\pi i} Q_+^{1/2}(s)\,ds, \qquad s \in \gamma_{m,0} \cup \gamma_{m,1}.$$

Then, μ_* is in fact a probability measure on $\gamma_{m,0} \cup \gamma_{m,1}$. Moreover, its shifted Cauchy transform, defined below as

$$\xi(z) = C^{\mu_*}(z) - \frac{it}{2}, \qquad C^{\mu_*}(z) := \int_{\gamma_{m,0}\cup\gamma_{m,1}} \frac{d\mu_*(s)}{s-z}, \qquad z \in \mathbb{C} \setminus (\gamma_{m,0} \cup \gamma_{m,1}),$$

solves $\xi^2(z) = Q(z)/4$ for $z \in \mathbb{C} \setminus (\gamma_{m,0} \cup \gamma_{m,1})$.

Proof. By the definition of $\gamma_{m,0}$ and $\gamma_{m,1}$ as trajectories of $-Q\,dz^2$, we know that μ_* is a real-valued measure. Also, because its density does not vanish on each of the arcs, we also know that μ_* cannot change sign in each of these arcs. From (4.25),

$$\mu_*(\gamma_{m,1}) = \frac{1}{2\pi i}\int_{\lambda_1}^{1} Q_+^{1/2}(s)ds = \frac{1}{2},$$

so μ_* has to be positive along $\gamma_{m,1}$. By symmetry $\mu_*(\gamma_{m,0}) = \mu_*(\gamma_{m,1})$ (see for instance Lemma 4.3), so μ_* has to be a probability measure.

To show that the shifted Cauchy transform satisfies the algebraic equation, we use that $Q_+^{1/2} = -Q_-^{1/2}$ along $\gamma_{m,0} \cup \gamma_{m,1}$ to write

$$C^{\mu_*}(z) = \frac{1}{4\pi i}\oint_C \frac{Q^{1/2}(s)}{s-z}ds, \quad z \in \mathbb{C} \setminus (\gamma_{m,0} \cup \gamma_{m,1}),$$

where C is a bounded contour that encircles $\gamma_{m,0} \cup \gamma_{m,1}$ in the clockwise direction and does not encircle z. Using (4.4) to compute residues,

$$C^{\mu_*}(z) = \frac{1}{2}\left(\operatorname*{Res}_{s=z} \frac{Q^{1/2}(s)}{s-z} + \operatorname*{Res}_{s=\infty} \frac{Q^{1/2}(s)}{s-z} \right) = \frac{Q^{1/2}(z)}{2} + \frac{it}{2},$$

from which the algebraic equation follows.

□

We now construct the genus 1 h-function which satisfies both the scalar Riemann-Hilbert problem and inequality constraints in (4.11). We first define $\mathscr{G}$ to be the critical graph of the quadratic differential as described by Theorem 4.12. As in Theorem 4.13, we define $\gamma_{m,0}$ and $\gamma_{m,1}$ to be the trajectories of $-Q\,dz^2$ which connect -1 to λ_0 and λ_1 to 1, respectively. Keeping with the notation of Chapter 2, we set $\gamma_{c,0} = (-\infty, -1]$. Finally, we set the arc $\gamma_{c,1}$ to be an analytic arc which connects λ_0 and λ_1. For now, the only requirement on $\gamma_{c,1}$ which we impose is that it remains entirely within the sector $\mathscr{H}_1$, which is the domain in $\mathbb{C}$ bounded by those critical trajectories in $\mathscr{G} \setminus (\gamma_{m,0} \cup \gamma_{m,1})$ shaded in gray in Figure 4.8. The domain $\mathscr{H}_2$ is given by $\mathscr{H}_2 = \mathbb{C} \setminus \overline{\mathscr{H}_1}$.

Keeping with the notation of Chapter 2, the main arcs are given by $\mathfrak{M} = \gamma_{m,0} \cup \gamma_{m,1}$, the complementary arcs are given by $\mathfrak{C} = \gamma_{c,0} \cup \gamma_{c,1}$, and the contour on which the scalar Riemann-Hilbert problem is posed is given by $\Omega = \Omega^{(1)} = \mathfrak{M} \cup \mathfrak{C}$, with orientation as depicted in Figure 4.8.

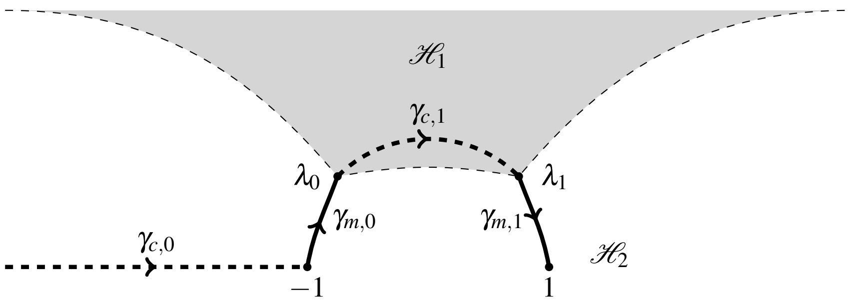


Figure 4.8: Definition of the main arcs, $\mathfrak{M}$, and complementary arcs, $\mathfrak{C}$. The main arcs are drawn in thick bold arcs and the complementary arcs are drawn in thick, dashed arcs. The remaining arcs in $\mathscr{G} \setminus (\gamma_{m,0} \cup \gamma_{m,1})$ are drawn in dashed arcs. The shaded regions corresponds to $\mathscr{H}_1$, where $\Re h(z) < 0$, and the orientation imposed on the system of arcs is indicated in the figure.

Next, we define

$$h(z;t) = -\int_1^z \sqrt{Q(s;t)}\, ds, \qquad z \in \mathbb{C} \setminus \Omega, \tag{4.32}$$

where the path of integration above does not cross the contour Ω. We recall that the branch cuts for $Q^{1/2}$ are taken on $\mathfrak{M}$ and the branch is chosen so that $Q^{1/2}$ satisfies (4.4) at infinity. We now show that the function defined in (4.32) satisfies all the requirements of the genus 1 h-function.

Theorem 4.15. *$h(z;t)$ defined in* (4.32) *satisfies all of the requirements listed in* (4.11).

Proof. First, it is clear that h defined in (4.32) is analytic in $\mathbb{C} \setminus \Omega$. Therefore, we now verify it possesses the correct jumps for $z \in \Omega$.

First we let $z \in \gamma_{c,0} = (-\infty, -1]$. We may write

$$h_+(z;t) - h_-(z;t) = -\oint_C Q^{1/2}(s;t)\, ds, \qquad z \in \gamma_{c,0},$$

where C is a loop encircling $\gamma_{m,0} \cup \gamma_{c,0} \cup \gamma_{m,1}$, oriented counterclockwise. Using (4.4), a residue calculation at infinity gives us that

$$h_+(z;t) - h_-(z;t) = 4\pi i, \qquad z \in \gamma_{c,0}, \tag{4.33}$$

proving (4.11b). Similarly, for $z \in \gamma_{c,1}$, we may write

$$h_+(z;t) - h_-(z;t) = \oint_C Q^{1/2}(s;t)\, ds, \qquad z \in \gamma_{c,1}, \tag{4.34}$$

where now C is a counter-clockwise oriented loop encircling the contour $\gamma_{m,1}$. Using (4.25), we see that

$$h_+(z;t) - h_-(z;t) = 2\pi i, \qquad z \in \gamma_{c,1}, \tag{4.35}$$

verifying (4.11d) with $\eta_1 = 1/2$. Next, using that $Q_+^{1/2} = -Q_-^{1/2}$, we immediately have that $h_+(z;t) + h_-(z;t) = 0$ for $z \in \gamma_{m,1}$, proving (4.11e). Similarly, for $z \in \gamma_{m,0}$, we have that

$$h_+(z;t) + h_-(z;t) = -2\int_{\lambda_1}^{\lambda_0} Q^{1/2}(s;t)\,ds, \qquad z \in \gamma_{m,0}. \tag{4.36}$$

By the Boutroux condition, as shown in Lemma 4.3, the integral above is purely imaginary. In particular, we may write

$$h_+(z;t) + h_-(z;t) = 4\pi i\omega_0, \qquad z \in \gamma_{m,0}, \tag{4.37}$$

where

$$\omega_0 = \frac{1}{2\pi i}\int_{\lambda_0}^{\lambda_1} Q^{1/2}(s;t)\,ds \in \mathbb{R}, \tag{4.38}$$

proving (4.11c).

From the asymptotics in (4.4), we conclude that

$$h(z;t) = itz + 2\log z - \ell + \mathscr{O}\left(\frac{1}{z}\right), \qquad z \to \infty, \tag{4.39}$$

so that h satisfies the asymptotic condition (4.11f) for some complex constant ℓ. Furthermore, the behavior near the endpoints of $\mathfrak{M}$ as described in (4.11g) and (4.11h) follows immediately from the definition of h in (4.32) and of Q in (4.5), as Q has simple zeros at $z = \lambda_0, \lambda_1$ and simple poles at $z = \pm 1$.

We now turn our attention to verifying the inequalities (4.11i) and (4.11j). To prove these inequalities we again turn to the theory of quadratic differentials, and in particular, the basic structure theorem as presented in [74, Theorem B.1]. Along a similar line of reasoning to the one presented in the discussion following Theorem 4.12, the domains $\mathscr{H}_1$ and $\mathscr{H}_2$ are half-plane domains, and as such we know that the function

$$\Upsilon(z) = \int_1^z Q^{1/2}(s)\,ds$$

is a conformal map from both $\mathscr{H}_1$ and $\mathscr{H}_2$ to either the left or the right half plane. By looking at the asymptotics (4.4), we see that

$$\Upsilon(z) = -itz\,(1 + o(1))\,.$$

As $t > 0$, when $z \to \infty$ along $\mathscr{H}_1 \cap i\mathbb{R}$ we must have that $\Re\Upsilon(z) \to +\infty$. Similarly, as $z \to \infty$ along $\mathscr{H}_2$, $\Re\Upsilon(z) \to -\infty$. This then implies that

$$\begin{aligned} h(z;t) &< 0, \qquad z \in \mathscr{H}_1, \\ h(z;t) &> 0, \qquad z \in \mathscr{H}_2. \end{aligned}$$

However, as the arc $\gamma_{c,1}$ lies entirely within $\mathscr{H}_1$, this immediately implies both inequalities (4.11i) and (4.11j), completing the proof.

□

4.3 Asymptotic Analysis with the Symmetrized h-function

Having constructed the genus 1 h-function, we could in principle follow the guide of Chapter 2 to compute the asymptotics of the polynomials. At this point, however, due to the symmetry found in the present setting, we opt to follow an alternate approach, based on a "symmetrized" version of the h-function. The symmetrized h-function, which we call ϕ, is defined as

$$\phi(z) = \frac{-h(z) - i\kappa}{2}, \tag{4.40}$$

where

$$\kappa = -\pi\omega_0, \tag{4.41}$$

and we recall that ω_0 is the constant defined in (4.11c). Using the proof of Theorem 4.15, see (4.38), we can alternatively write

$$\kappa = \frac{i}{2}\int_{\lambda_0}^{\lambda_1} Q^{1/2}(s;t)\,ds \in \mathbb{R}. \tag{4.42}$$

The following behavior for ϕ easily follows from the behavior of the h constructed in Theorem 4.15,

$$\begin{aligned} &\phi_+(z) - \phi_-(z) = -2\pi i, && z \in \gamma_{c,0}, \\ &\phi_+(z) + \phi_-(z) = (-1)^j i\kappa, && z \in \gamma_{m,j}, \\ &\phi_+(z) - \phi_-(z) = -\pi i, && z \in \gamma_{c,1}, \\ &\Re\phi(z) > 0, && z \in \gamma_{c,1}, \end{aligned} \tag{4.43}$$

where $j = 0, 1$. Moreover, the inequality

$$\Re\phi(z) < 0 \tag{4.44}$$

is valid in the immediate vicinities of any subarc of $\gamma_{m,0} \cup \gamma_{m,1}$ which does not contain their endpoints. Finally, note from (4.11f) and (4.40) that

$$\phi(z) = -\frac{itz}{2} + \frac{\ell}{2} - \frac{i\kappa}{2} - \log z + \mathscr{O}\left(\frac{1}{z}\right), \qquad z \to \infty. \tag{4.45}$$

The motivation for using ϕ in the steepest descent analysis, as opposed to h, is that we will be able to construct a global parametrix that does not require the use of theta functions. As such, we will be able to obtain more explicit information about the behavior of various quantities involved in the construction; in particular we will be able to quantify a better degeneracy condition which plays a similar role to (2.76).

We quickly recap the first steps of the steepest descent process, now using the function ϕ. For convenience, we will use the notation

$$E_{12} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad E_{21} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Defining $\Sigma = \Omega \setminus \gamma_{c,0}$, we recall from Chapter 2 that the matrix valued function

$$Y(z) = \begin{pmatrix} p_n(z;t) & \left(\mathscr{C} p_n e^{-nf}\right)(z) \\ -2\pi i \kappa_{n-1}^2 p_{n-1}(z;t) & -2\pi i \kappa_{n-1}^2 \left(\mathscr{C} p_{n-1} e^{-nf}\right)(z) \end{pmatrix} \tag{4.46}$$

solves

$$Y(z) \text{ is analytic in } \mathbb{C}\backslash\Sigma, \tag{4.47a}$$

$$Y_+(z) = Y_-(z)\left(I + e^{-nf(z;t)} E_{12}\right), \qquad z \in \Sigma, \tag{4.47b}$$

$$Y(z) = \left(I + \mathscr{O}\left(\frac{1}{z}\right)\right) z^{n\sigma_3}, \qquad z \to \infty, \tag{4.47c}$$

$$Y(z) = \mathscr{O}\begin{pmatrix} 1 & \log|z \mp 1| \\ 1 & \log|z \mp 1| \end{pmatrix}, \qquad z \to \pm 1. \tag{4.47d}$$

The first transformation, which aims to normalize Y at infinity, reads

$$T(z) = e^{\frac{n}{2}(-\ell + i\kappa)\sigma_3} Y(z) e^{n\left(\phi(z) + \frac{itz}{2}\right)\sigma_3}. \tag{4.48}$$

Note that although ϕ is analytic in $\mathbb{C} \setminus \Omega$, T is analytic in $\mathbb{C} \setminus \Sigma$, due to the jumps for ϕ over $\gamma_{c,0}$ given in (4.43). As before, T satisfies the asymptotics $T \to I$ as $z \to \infty$, and satisfies the jump $T_+ = T_- j_T$ along Σ, where the jump matrix j_T is given by

$$j_T(z) = \begin{pmatrix} e^{n(\phi_+(z) - \phi_-(z))} & e^{-n(\phi_+(z) + \phi_-(z))} \\ 0 & e^{-n(\phi_+(z) - \phi_-(z))} \end{pmatrix}, \qquad z \in \Sigma. \tag{4.49}$$

To open lenses, we define $\hat{\Sigma} := \Sigma \cup \gamma_{m,0}^{\pm} \cup \gamma_{m,1}^{\pm}$, with $\gamma_{m,j}^{\pm}$ as in Figure 4.9, while ensuring that $\hat{\Sigma}$ remains symmetric with respect to $i\mathbb{R}$. Then we open lenses $T \mapsto S$ in the standard way,

$$S(z) = \begin{cases} T(z)(I \mp e^{2n\phi(z)} E_{21}), & z \text{ in the } \pm\text{-part of the lenses}, \\ T(z), & \text{otherwise}. \end{cases} \tag{4.50}$$

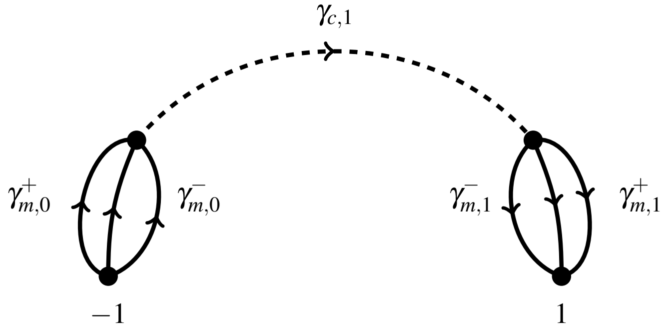


Figure 4.9: The contour $\hat{\Sigma}$.

Then, S is analytic in $\mathbb{C}\setminus\hat{\Sigma}$, tends to the identity as $z\to\infty$, and has the following jumps on $\hat{\Sigma}$:

$$j_S(z)=\begin{cases} e^{-i\kappa n}E_{12}-e^{i\kappa n}E_{21}, & z\in\gamma_{m,0},\\ e^{i\kappa n}E_{12}-e^{-i\kappa n}E_{21}, & z\in\gamma_{m,1},\\ (-1)^nI+(-1)^ne^{-2n\phi_+(z)}E_{12}, & z\in\gamma_{c,1},\\ I+e^{2n\phi(z)}E_{21}, & z\in\gamma_{m,0}^{\pm}\cup\gamma_{m,1}^{\pm}. \end{cases} \tag{4.51}$$

From the inequalities in (4.43)–(4.44), we see that many of these jumps are exponentially decaying, as expected. As before, we consider the model Riemann-Hilbert problem obtained by disregarding these jumps which decay exponentially quickly.

4.4 Construction of the Global Parametrix

Dropping the exponentially decaying terms in (4.51), we arrive at the following model Riemann-Hilbert Problem:

$$M(z) \text{ is analytic in } \mathbb{C}\backslash\Sigma, \tag{4.52a}$$

$$M_+(z)=M_-(z)j_M(z), \qquad z\in\Sigma, \tag{4.52b}$$

$$M(z)=I+\mathscr{O}\left(\frac{1}{z}\right), \qquad z\to\infty, \tag{4.52c}$$

where

$$J_M(z)=\begin{cases} e^{-i\kappa n}E_{12}-e^{i\kappa n}E_{21} & z\in\gamma_{m,0},\\ e^{i\kappa n}E_{12}-e^{-i\kappa n}E_{21} & z\in\gamma_{m,1},\\ (-1)^nI & z\in\gamma_{c,1}. \end{cases} \tag{4.53}$$

We also impose that M has at worst fourth root singularities at the endpoints of $\gamma_{m,0}\cup\gamma_{m,1}$, in addition to satisfying the model Riemann-Hilbert problem (4.52).

In contrast to the theta function approach introduced in Chapter 2, the symmetry of the problem at hand allows us to pursue an alternate approach. We discuss this construction in four steps.

The first step is standard, and consists of building a solution to the particular RHP obtained from M by setting $n = 0$. The resulting RHP is of the form one typically encounters when attempting to construct the global parametrix for orthogonal polynomials with positive weights on the real line, and yields a matrix function N.

In the second step, we start with an Ansatz on how to modify the entries of N so as to obtain the required M. This Ansatz then produces a system of scalar Riemann-Hilbert problems.

In the third step, we construct the solutions to these scalar Riemann-Hilbert problems in a somewhat explicit way, with the help of meromorphic differentials on the associated Riemann surface. This can be accomplished for generic values of t and is split into two sections. The ideas in this step are greatly inspired by the work of Kuijlaars and Mo [62].

In the fourth and final step we analyze this non-degeneracy condition on t, showing that the construction is valid provided that either n is even or n is odd and t is not in the exceptional set Θ^* mentioned in Section 4.1.

4.4.1 Step One: Construction of Simplified Parametrix

We first set $n = 0$ in (4.53) and consider the following RHP:

$$N(z) \text{ is analytic in } \mathbb{C}\setminus\mathfrak{M}, \tag{4.54a}$$

$$N_+(z) = N_-(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in \mathfrak{M}, \tag{4.54b}$$

$$N(z) = I + \mathscr{O}\left(\frac{1}{z}\right), \qquad z \to \infty. \tag{4.54c}$$

We recall above that $\mathfrak{M} = \gamma_{m,0} \cup \gamma_{m,1}$ is the set of main arcs. We recognize (4.54) as the model problem (2.57), and therefore its solution is given by (2.58) as

$$N(z) = \begin{pmatrix} \frac{\eta(z)+\eta^{-1}(z)}{2} & \frac{\eta(z)-\eta^{-1}(z)}{-2i} \\ \frac{\eta(z)-\eta^{-1}(z)}{2i} & \frac{\eta(z)+\eta^{-1}(z)}{2} \end{pmatrix}, \tag{4.55}$$

where

$$\eta(z) = \left(\frac{(z+1)(z-\lambda_1)}{(z-\lambda_0)(z-1)}\right)^{1/4} = \left(\frac{(z+1)(z-\lambda_1)}{\left(z+\overline{\lambda_1}\right)(z-1)}\right)^{1/4}, \tag{4.56}$$

with branch cuts on $\gamma_{m,0}$ and $\gamma_{m,1}$ and the branch of the root chosen so that

$$\lim_{z\to\infty} \eta(z) = 1. \tag{4.57}$$

It is important to understand the location of the zeros of $N(z)$, as they will give us an extra degree of freedom when we later modify N. Recall that $\lambda_1 = x_* + \frac{2i}{t}$, where x_* satisfies the properties of Theorem 4.6.

Lemma 4.16. *Let*

$$y_* = \frac{2i}{t(1-x_*)}. \tag{4.58}$$

Then

$$\eta\,(y_*) - \eta^{-1}\,(y_*) = 0,$$

and this is the only finite zero of $\eta\,(z) - \eta^{-1}(z)$*. Furthermore, the equation*

$$\eta\,(z) + \eta^{-1}(z) = 0,$$

has no finite solutions.

Proof. First note that the solutions to $\eta\,(z) \pm \eta^{-1}(z)$ are the solutions to

$$\eta^4(z) = \frac{(z+1)\,(z-\lambda_1)}{(z-1)\left(z+\overline{\lambda_1}\right)} = 1.$$

This equation has exactly one finite solution given by

$$z = y_* := \frac{2i}{t(1-x_*)},$$

where $x_* = \Re\lambda_1$. In other words, either $\eta^2(y_*) = 1$ or $\eta^2(y_*) = -1$, and we will now show that only the former takes place. From (4.24), we have that

$$\lim_{t\to\infty} y_* = \infty.$$

As we normalized the branch of η at infinity as in (4.57), we have that for z large

$$\eta^2(z) = 1 + \frac{1-x_*}{z} + \mathscr{O}\left(\frac{1}{z^2}\right).$$

In words, the function η^2 is a continuous function of both z and t, and we know that if t is large enough, y_* will be large enough; in this situation we have that $\eta^2(y_*)$ is close to 1. In particular, it cannot be close to -1, and as we know that $\eta^2(y_*)$ is either ± 1, we can conclude that $\eta^2(y_*) = 1$ for t large enough. However, again using continuity in t, we have that $\eta^2(y_*) = 1$ for all $t > t_c$, and not just for t large enough. Then,

$$\eta^2(y_*) = 1 \iff \eta\,(y_*) - \eta^{-1}(y_*) = 0,$$

concluding the proof. □

Recall the discussion following Theorem 4.14, where we stated that the contour $\gamma_{c,1}$ remained in the region $\mathscr{H}_1$ as depicted in Figure 4.8. We now also impose the condition that the contour $\gamma_{c,1}$ does not contain the point y_*. As the contour can be chosen arbitrarily in $\mathscr{H}_1$, it is always possible to choose it in such a way so as to ensure that it does not contain y_*.

4.4.2 Step Two: Ansatz for the Global Parametrix and Related Scalar RHPs

We now state the Ansatz for how we will solve the model problem (4.52). We seek M in the form

$$M(z) = \begin{pmatrix} c_1^{-1} & 0 \\ 0 & c_2^{-1} \end{pmatrix} \begin{pmatrix} N_{11}(z)v_1^{(1)}(z) & N_{12}(z)v_2^{(1)}(z) \\ N_{21}(z)v_1^{(2)}(z) & N_{22}(z)v_2^{(2)}(z) \end{pmatrix}, \tag{4.59}$$

where c_1 and c_2 are nonzero constants, the functions $v_j^{(k)}$ are yet to be determined, and the notation N_{ij} indicates the (i,j)-entry of the matrix N.

Comparing the RHPs for M and N, we arrive at the following desired properties for the functions $v_j^{(k)}$:

(i) If $j = k$, $v_j^{(k)}$ is analytic in $\mathbb{C}\backslash\Sigma$.

(ii) If $j \neq k$, $v_j^{(k)}$ is analytic on $\mathbb{C}\backslash(\Sigma \cup \{y_*\})$, where the singularity at y_* is a simple pole.

(iii) The $v_j^{(k)}$ have the following jumps over Σ,

$$v_{1,\pm}^{(k)}(z) = v_{2,\mp}^{(k)}(z)e^{in\kappa}, \qquad z \in \gamma_{m,0}, \qquad k = 1,2, \tag{4.60a}$$

$$v_{1,\pm}^{(k)}(z) = v_{2,\mp}^{(k)}(z)e^{-in\kappa}, \qquad z \in \gamma_{m,1}, \qquad k = 1,2, \tag{4.60b}$$

$$v_{j,+}^{(k)}(z) = (-1)^n v_{j,-}^{(k)}(z), \qquad z \in \gamma_{c,0}, \qquad k,j = 1,2. \tag{4.60c}$$

(iv) When $j = k$,

$$v_j^{(j)}(z) = c_j + \mathscr{O}\left(\frac{1}{z}\right), \quad z \to \infty, \tag{4.61}$$

for some nonzero constant c_j.

(v) If $j \neq k$, then

$$v_j^{(k)}(z) = \mathscr{O}(1), \quad z \to \infty. \tag{4.62}$$

(vi) The functions $v_j^{(k)}$ remain bounded at the endpoints of $\mathfrak{M}$.

The importance of the above requirements is made clear by the following result.

Lemma 4.17. *If there exist functions $v_j^{(k)}$ which satisfy conditions (i)–(vi) above, then the solution to the RHP* (4.52) *is given by the formula* (4.59).

Proof. The condition (i) for $v_j^{(j)}$, together with (4.54a) imply immediately that M_{11} and M_{22} are analytic in $\mathbb{C}\setminus\Sigma$. Next, condition (ii) and Lemma 4.16 give us that for $j \neq k$ the simple pole of $v_j^{(k)}$ at y_* cancels with the zero y_* of N_{kj}, so the off-diagonal entries of M are also analytic in $\mathbb{C}\setminus\Sigma$. Therfore, (4.52a) is verified.

To verify the appropriate jump conditions, note that for $z \in \gamma_{m,0}$, we use the jumps for $v_j^{(k)}$ to get

$$\begin{aligned} M_+(z) &= \begin{pmatrix} c_1^{-1} & 0 \\ 0 & c_2^{-1} \end{pmatrix} \begin{pmatrix} -N_{12,-}(z)v_{2,-}^{(1)}(z)e^{in\kappa} & N_{11,-}(z)v_{1,-}^{(1)}(z)e^{-in\kappa} \\ -N_{22,-}(z)v_{2,-}^{(2)}(z)e^{in\kappa} & N_{21,-}(z)v_{1,-}^{(2)}(z)e^{-in\kappa} \end{pmatrix} \\ &= M_-(z) \begin{pmatrix} 0 & e^{-in\kappa} \\ -e^{in\kappa} & 0 \end{pmatrix}. \end{aligned}$$

This is the same as (4.52b) along $\gamma_{m,0}$. The jump over $\gamma_{m,1}$ can be verified in the same manner. As the matrix N does not jump over $\gamma_{c,1}$, the jump of M over this contour is determined only by the jump of the $v_j^{(k)}$'s over $\gamma_{c,1}$. As such, M satisfies all the jumps given in (4.53).

The normalization of M at ∞ follows from (4.61) and (4.54c). Finally, because the $v_j^{(k)}$'s are bounded near the endpoints of $\mathfrak{M}$, the behavior of M near the endpoints is governed by that of N, and as such, M has at worst fourth-root singularities.

□

We now move on to the construction of the scalar functions, $v_j^{(k)}$.

4.4.3 Construction of the Meromorphic Differentials

Let $\mathscr{Q}$ be the genus 1 Riemann surface associated to the algebraic equation

$$\xi^2 = \frac{Q(z)}{4}, \tag{4.63}$$

where we recall from (4.5) that

$$Q(z) = -\frac{t^2(z-\lambda_1)\left(z+\overline{\lambda_1}\right)}{z^2-1}.$$

More concretely, this surface $\mathscr{Q}$ is obtained as the closure of the surface that we get when gluing the two copies

$$\mathscr{Q}_j = \overline{\mathbb{C}}\setminus\mathfrak{M}, \qquad j = 1,2,$$

along $\mathfrak{M}$ in the usual crosswise manner. On $\mathscr{Q}$, the global meromorphic solution ξ to the equation (4.63) is well defined by $\xi\Big|_{\mathscr{Q}_j} = \xi_j$, where ξ_j is the analytic branch of $\frac{1}{2}Q^{1/2}$ on $\mathscr{Q}_j$ uniquely determined by the asymptotics

$$\xi_j(z) = (-1)^j\frac{it}{2} + \mathscr{O}(z^{-1}), \quad z\to\infty, \quad j = 1,2.$$

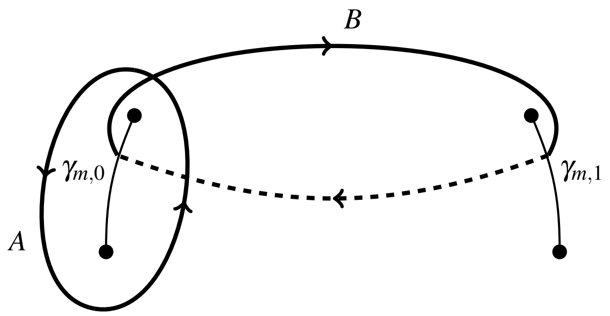


Figure 4.10: The homology basis on $\mathcal{Q}$. The bold contours are on the top sheet of $\mathcal{Q}$, and the dashed contours are on the second sheet of $\mathcal{Q}$. In particular, A is a cycle on the first sheet of $\mathcal{Q}$ that encircles $\gamma_{m,0}$ once in the counter-clockwise direction without crossing the imaginary axis. The cycle B starts on the first sheet of $\mathcal{Q}$ and passes through $\gamma_{m,0}$ and $\gamma_{m,1}$. We also impose that the cycle B is symmetric with respect to the imaginary axis.

In other words, $\xi_1 = \frac{1}{2}Q^{1/2}$, where the branch for Q is the one defined in (4.4), and $\xi_2 = -\xi_1$. For notational convenience, we denote by $a^{(j)}$ the copy of the point $a \in \mathbb{C}$ which is on the sheet $\mathcal{Q}_j$. This is well defined provided a does not belong to any of the branch cuts $\mathfrak{M}$, which is enough for our purposes. Additionally, the canonical homology basis $\{A, B\}$ on $\mathcal{Q}$ is chosen as depicted in Figure 4.10.

To construct the functions $v_k^{(j)}$ from the previous section, we will use meromorphic differentials on the Riemann surface $\mathcal{Q}$. This construction is inspired by a similar construction by Kuijlaars and Mo [62], but here we exploit the symmetry with respect to the imaginary axis in a very explicit way, which also leads to more explicit formulas when compared to those in [62].

The differential

$$\Lambda_0 = \frac{1}{\xi(z)(z^2-1)}dz$$

linearly generates the space of holomorphic differentials on $\mathcal{Q}$. For later convenience, for a positive integer j we denote

$$m_A^{(j)} = -\frac{1}{2}\oint_A z^j \Lambda_0 = \int_{\gamma_{m,0}} \frac{s^j}{Q_+^{1/2}(s)(s^2-1)}ds, \quad m_B^{(j)} = \frac{1}{2}\oint_B z^j \Lambda_0 = \int_{\lambda_0}^{\lambda_1} \frac{s^j}{Q^{1/2}(s)(s^2-1)}ds. \quad (4.64)$$

Because Λ_0 is, up to a multiplicative constant, the only holomorphic differential on $\mathcal{Q}$, the Riemann bilinear relations imply that $m_A^{(0)}$ and $m_B^{(0)}$ are nonzero. Defining

$$I(z; j) = \frac{z^j}{Q^{1/2}(z)(z^2-1)}, \quad (4.65)$$

we see using the symmetry for $Q^{1/2}$ in (4.19) that

$$I(-\bar{z}; j) = (-1)^{j+1}\overline{I(z; j)}, \quad (4.66)$$

with the above symmetry extending to the branch cuts. Then, applying the symmetry over the imaginary axis gives us that

$$\Re m_B^{(2j)} = \Im m_A^{(2j)} = 0, \qquad \Im m_B^{(2j+1)} = \Re m_A^{(2j+1)} = 0, \qquad j \geq 0. \tag{4.67}$$

Next, for $a \in i\mathbb{R}$ and $\nu = 1,2$, define a meromorphic differential $\Lambda_a^{(\nu)}$ by the formula

$$\Lambda_a^{(\nu)} = \frac{1}{2}\frac{1}{z-a}\left(1 + \frac{\xi(a^{(\nu)})(a^2-1)}{\xi(z)(z^2-1)}\right) dz$$

The only poles of $\Lambda_a^{(\nu)}$ are $a^{(\nu)}$, $\infty^{(1)}$ and $\infty^{(2)}$, which are all simple, and

$$\operatorname*{Res}_{\infty^{(1,2)}} \Lambda_a^{(\nu)} = -\frac{1}{2}, \quad \operatorname*{Res}_{a^{(\nu)}} \Lambda_a^{(\nu)} = 1. \tag{4.68}$$

Also, another calculation using symmetry gives us the following proposition

Proposition 4.18.

$$\Re \oint_B \Lambda_a^{(\nu)} = 0, \quad \Im \oint_A \Lambda_a^{(\nu)} = (-1)^\nu \frac{\pi}{2}.$$

Proof. As $a \in i\mathbb{R}$, it follows that

$$\frac{1}{-\bar{z}-a} = -\overline{\left(\frac{1}{z-a}\right)}. \tag{4.69}$$

Moreover, we also find using the symmetry for Q that

$$Q^{1/2}(-\bar{z};t)\left((-\bar{z})^2 - 1\right) = -\overline{Q^{1/2}(z)(z^2-1)}. \tag{4.70}$$

Finally, as $a \in i\mathbb{R}$, we also see that $Q^{1/2}(a)(a^2-1) \in i\mathbb{R}$. Therefore, using the symmetry from Lemma 4.2, with $\delta = -1$, we see that

$$\Re \oint_B \Lambda_a^{(\nu)} = 0. \tag{4.71}$$

Again using the symmetry with $\delta = -1$, this time with Lemma 4.1, we can write

$$2i\Im \oint_A \Lambda_a^{(\nu)} = \oint_A \Lambda_a^{(\nu)} + \oint_C \Lambda_a^{(\nu)} = -2\pi i \sum_{p \in \mathscr{Q}_1} \operatorname*{Res}_p \Lambda_a^{(\nu)}, \tag{4.72}$$

where C is a loop on the first sheet of $\mathscr{Q}$ which encircles the contour $\gamma_{m,1}$, does not cross the imaginary axis, and is oriented counter clockwise. If $\nu = 1$, there are two poles on $\mathscr{Q}_1$, whereas if $\nu = 2$, the only pole on $\mathscr{Q}_1$ is the pole at $\infty^{(1)}$. Using (4.68), we see that

$$\sum_{p \in \mathscr{Q}_1} \operatorname*{Res}_p \Lambda_a^{(\nu)} = \frac{(-1)^{\nu+1}}{2} \implies \Im \oint_A \Lambda_a^{(\nu)} = (-1)^\nu \frac{\pi}{2}. \tag{4.73}$$

□

Finally, for $a \in i\mathbb{R}$ and $b \in \mathbb{R}$, $\nu, \varsigma \in \{1,2\}$, set

$$\Omega(a,b) = \Omega(a,b;\nu,\varsigma) = \Lambda_a^{(\nu)} - \Lambda_{y_*}^{(\varsigma)} + b\Lambda_0, \tag{4.74}$$

where y_* is as in Lemma 4.16.

Lemma 4.19. *If $a = y_*$ and $\nu = \varsigma$, then $\Omega(a,b)$ is holomorphic. Otherwise, the differential $\Omega(a,b)$ has simple poles at $a^{(\nu)}$ and $y_*^{(\varsigma)}$ with residues $+1$ and -1, respectively, and no other poles. In either case*

$$\Re \oint_B \Omega(a,b) = 0, \quad \Im \oint_A \Omega(a,b) = ((-1)^\nu - (-1)^\varsigma)\frac{\pi}{2}.$$

Proof. The proof follows immediately from the properties of Λ_0, $\Lambda_a^{(\nu)}$, and Proposition 4.18. □

Moving forward, for $\tau, b \in \mathbb{R}$ we define $\Psi_A : \mathbb{R}^2 \to \mathbb{R}$ by

$$\Psi_A(\tau,b) = \Re \oint_A \Omega(i\tau,b)$$

Using (4.67) and (4.74), we see that

$$\Psi_A(\tau,b) = \Re\left[\oint_A \left(\Lambda_{i\tau}^{(\nu)} - \Lambda_{y_*}^{(\varsigma)}\right)\right] - 2bm_A^{(0)}. \tag{4.75}$$

Clearly Ψ_A is a linear function of b. Furthermore, as the A-cycle is chosen to not intersect $i\mathbb{R}$, Ψ_A is actually a real analytic function of $\tau \in \mathbb{R}$, as well. Thus, the level set determined by $\Psi_A(\tau,b) = 0$ is the graph of a real analytic function $\tau \mapsto \Psi_A(\tau, b(\tau))$, with

$$b(\tau) = \frac{1}{2m_A^{(0)}} \Re \oint_A \left(\Lambda_{i\tau}^{(\nu)} - \Lambda_{y_*}^{(\varsigma)}\right). \tag{4.76}$$

We next consider the function $\Psi_B : \mathbb{R} \to \mathbb{R}\backslash\mathbb{Z}$ given by

$$\Psi_B(\tau) = \frac{1}{2\pi i} \oint_B \Omega(i\tau, b(\tau)). \tag{4.77}$$

For the integration above, we consider the cycle B to be given by straight line segments on $\mathscr{Q}_1$ and $\mathscr{Q}_2$ with endpoints $\lambda_0 = -\overline{\lambda_1}$ and λ_1. This way, in principle, Ψ_B is well defined for $\tau \neq \Im\lambda_1$. However, because the residue of $\Omega(a,b)$ at $a^{(\nu)}$ is 1, we actually get that Ψ_B remains well defined and continuous, as a function with values on $\mathbb{R}\setminus\mathbb{Z}$, when $\tau \to \operatorname{Im}\lambda_1$. Also, when $\nu = \varsigma$, we immediately see that $b(\tau) \to 0$ as $\tau \to \Im y_*$, and a simple calculation shows that Ψ_B remains continuous with $\Psi_B(\Im y_*) = 0$, and $\Omega(i\tau, b(\tau))$ reduces to the null meromorphic differential.

In conclusion, Ψ_B is a continuous function from $\mathbb{R}$ to $\mathbb{R}\setminus\mathbb{Z}$.

Lemma 4.20. *The function Ψ_B is injective.*

Proof. Take two values τ_1 and τ_2 for which $\Psi_B(\tau_1) = \Psi_B(\tau_2)$. This implies that the imaginary part of the B-period of the difference $\widehat{\Omega} = \Omega(i\tau_1, b(\tau_1)) - \Omega(i\tau_2, b(\tau_2))$ vanishes. Using Lemma 4.19, we see that the real part of the B period of $\widehat{\Omega}$ vanishes as well, so we can conclude that

$$\oint_B \widehat{\Omega} = 0.$$

We may also write

$$\oint_A \widehat{\Omega} = \oint_A \Lambda^{(\nu)}_{i\tau_1} - \Lambda^{(\nu)}_{i\tau_2} - 2\left(b(\tau_1) - b(\tau_2)\right) m_A^{(0)}.$$

Using (4.64) and Proposition 4.18, we see that the imaginary part of the above A-period is 0. Similarly, using the definition of $b(\tau)$ in (4.76), we see that the real part of the A-period vanishes as well, as b was specifically chosen to ensure so. Therefore, we may also conclude that

$$\oint_A \widehat{\Omega} = 0.$$

Now, to get to a contradiction, assume that $\tau_1 \neq \tau_2$. Then $\widehat{\Omega}$ has residues of $+1$ and -1 at $i\tau_1^{(\nu)}$ and $i\tau_2^{(\nu)}$, respectively. Hence, for a fixed base point $P_0 \in \mathscr{Q}$, the function

$$P \mapsto \exp\left(\int_{P_0}^{P} \widehat{\Omega}\right)$$

is a well-defined meromorphic function on $\mathscr{Q}$ with only one pole, namely $i\tau_2^{(\nu)}$, having nonzero residue. As the residues of any meromorphic function have to add to 0, we have reached the desired contraction.

□

We also need to compute the limit of Ψ_B as $\tau \to \infty$. To do so, we first notice that $\Lambda^{(\varsigma)}_{y_*}$ and Λ_0 do not depend on τ. To analyze $\Lambda^{(\nu)}_{i\tau}$, we use (4.4) to derive the asymptotics

$$\frac{1}{2}\frac{1}{z-a}\frac{\xi(a^{(\nu)})}{\xi(z)}\frac{a^2-1}{z^2-1} = (-1)^{\nu-1}\left(\frac{iat}{4} + \frac{itz}{4} + \frac{1}{2}\right)\frac{1}{\xi(z)(z^2-1)} + \mathscr{O}(a^{-1}), \quad a \to \infty,$$

which is valid with uniform error term for z in compacts. With $a = i\tau$ this, in turn, implies that

$$\begin{aligned}\oint_{A,B} \Lambda^{(\nu)}_{i\tau} &= \frac{1}{2}\oint_{A,B} \frac{1}{z-a}\frac{\xi(a^{(\nu)})}{\xi(z)}\frac{a^2-1}{z^2-1}dz \\ &= \pm(-1)^{\nu-1}\left(\frac{\tau t}{2}m^{(0)}_{A,B} - m^{(0)}_{A,B} - \frac{it}{2}m^{(1)}_{A,B}\right) + \mathscr{O}(\tau^{-1}), \quad \tau \to \infty,\end{aligned}$$

where we use the $+$ $(-)$ sign for A (B). Therefore, using (4.76), we see that

$$b(\tau) = -\frac{1}{2m_A^{(0)}}\Re\oint_A \Lambda^{(\varsigma)}_{y_*} + \frac{(-1)^{\nu-1}}{4}(\tau t - 2) + \frac{(-1)^{\nu}}{4}\frac{itm_A^{(1)}}{m_A^{(0)}} + \mathscr{O}(\tau^{-1}), \quad \tau \to \infty.$$

Then,

$$\oint_B \Omega(i\tau, b(\tau)) = -\oint_B \Lambda_{y_*}^{(\varsigma)} - \frac{m_B^{(0)}}{m_A^{(0)}} \Re \oint_A \Lambda_{y_*}^{(\varsigma)} + (-1)^{\nu} \frac{it}{2} \left(\frac{m_B^{(0)} m_A^{(1)}}{m_A^{(0)}} - m_B^{(1)} \right) + \mathscr{O}(\tau^{-1}),$$

as $\tau \to \infty$. By Lemma 4.19, we know the above integral is purely imaginary. Therefore, defining $c = c(t; \nu, \varsigma)$ via

$$\lim_{\tau \to \infty} \oint_B \Omega_{i\tau}^{(\nu)} = 2\pi i \lim_{\tau \to \infty} \Psi_B(\tau) = ic(t; \nu, \varsigma),$$

we see the constant c is purely real and explicitly given by

$$c = c(t; \nu, \varsigma) := \Im \left(-\oint_B \Lambda_{y_*}^{(\varsigma)} - \frac{m_B^{(0)}}{m_A^{(0)}} \Re \oint_A \Lambda_{y_*}^{(\varsigma)} + (-1)^{\nu} \frac{it}{2} \left(\frac{m_B^{(0)} m_A^{(1)}}{m_A^{(0)}} - m_B^{(1)} \right) \right). \tag{4.78}$$

We emphasize that c depends on y_*, ς, ν and obviously t, but not on a.

As a consequence of Lemma 4.20, we see that Ψ_B is a bijection between $\mathbb{R} \cup \{\infty\}$ and an interval of the form $[q, q+1]$. As such, this means that for any $\alpha \in \mathbb{R} \setminus \{c\}$ we can always choose τ so that the B-period of $\Omega(i\tau, b(\tau))$ equals $2\pi i\alpha + 2\pi i N$ for some $N \in \mathbb{N}$. We summarize the findings of this subsection as the next result.

Lemma 4.21. *For $\mu, \varsigma \in \{1, 2\}$, $t > t_c$, and $a \in i\mathbb{R}$, $b \in \mathbb{R}$, with $a \neq y_*$ in the case $\nu = \varsigma$, the meromorphic differential $\Omega(a, b)$ in* (4.74) *has simple poles at $a^{(\nu)}$ and $y_*^{(\varsigma)}$ with residues $+1$ and -1, respectively, and is elsewhere holomorphic.*

Furthermore, if $\alpha \in \mathbb{R}$ is such that $\alpha - c \in \mathbb{R} \setminus 2\pi\mathbb{Z}$, then there exist unique $a \in i\mathbb{R}$ and $b \in \mathbb{R}$ for which

$$\oint_B \Omega(a, b) = 2\pi i\alpha, \quad \oint_A \Omega(a, b) = ((-1)^{\nu} - (-1)^{\varsigma}) \frac{\pi i}{2},$$

where these identities are understood modulo $2\pi i\mathbb{Z}$.

4.4.4 Step Three: Solving the Scalar RHP's

Note that when the conditions

$$n \in 2\mathbb{Z}, \quad \kappa \in \mathbb{Z}, \tag{4.79}$$

occur, the matrix N defined in (4.55) solves the model RHP (4.52). Therefore, for the rest of this subsection, we assume that the degeneration (4.79) does not occur.

We will now use the meromorphic differential $\Omega(a, b) = \Omega(a, b; \nu, \varsigma_n(\nu))$ as described in the previous section for a special choice of periods, whose existence will be guaranteed by Lemma 4.21. To do so, for $k = 1, 2$ and $n \in \mathbb{Z}_+$, we define $\nu, \varsigma \in \{1, 2\}$ by

$$\nu = \nu_{n,k} = n + k + 1 \mod 2, \quad \text{and} \quad \varsigma = \varsigma_k = k + 1 \mod 2. \tag{4.80}$$

The definition of ν and ς as above is chosen so as to place the poles of

$$\Omega_n^{(k)} := \Omega(a,b;\nu_{n,k},\varsigma_k) \tag{4.81}$$

in very specific locations: when n is even, the poles $a^{(\nu)}$ and $y_*^{(\nu)}$ are on the same sheet $\mathscr{Q}_{k+1}$; on the other hand, when n is odd, the pole $a^{(\nu)}$ is on the sheet $\mathscr{Q}_k$ and the pole $y_*^{(\varsigma)}$ is on the other sheet $\mathscr{Q}_{k+1}$. In the preceding comments, we identified $\mathscr{Q}_3 = \mathscr{Q}_1$.

We now choose $a = i\tau$ and b in (4.81) in very specific ways, as formulated in the next result.

Lemma 4.22. *Fix $k \in \{1,2\}$ and $n \in \mathbb{Z}_+$. Let $\nu = \nu_{n,k}$ and $\varsigma = \varsigma_k$ be as in* (4.80). *Let $c = c(n,k)$ and κ be as in* (4.41) *and* (4.78)*, respectively. Suppose that $2n\kappa - c \in \mathbb{R}\setminus 2\pi\mathbb{Z}$. Then there exists a meromorphic differential $\Omega_n^{(k)}$ with the following properties.*

(i) The only poles of $\Omega_n^{(k)}$, which are simple, are at $y_*^{(\varsigma)}$ as in (4.58) *and at another point $a_*^{(\nu)} \in i\mathbb{R}$, with*

$$\operatorname*{Res}_{a_*^{(\nu)}} \Omega_n^{(k)} = 1, \qquad \operatorname*{Res}_{y^{(\varsigma)}} \Omega_n^{(k)} = -1.$$

(ii) The periods of $\Omega_n^{(k)}$ are

$$\oint_B \Omega_n^{(k)} = 2n\kappa i, \quad \oint_A \Omega_n^{(k)} = n\pi i, \tag{4.82}$$

where these identities are understood modulo $2\pi i\mathbb{Z}$.

Proof. The proof follows from Lemma 4.19 and Lemma 4.21. We remark that because we are assuming (4.79) does not occur, the poles $y_*^{(\varsigma)}$ and $a_*^{(\nu)}$ never coincide; that is, the degeneration $a_* = y_*$ and $\nu = \varsigma$ in Lemma 4.19 and Lemma 4.21 never takes place. □

Now, define Γ as $\Gamma := (-\infty,-1)\cup\Sigma\cup(1,\infty)$. For $z \in \mathbb{C}\setminus(\Gamma\cup\{a_*,y_*\})$ define

$$u_1^{(k)}(z) = \int_1^z \Omega_n^{(k)}, \quad k = 1,2. \tag{4.83}$$

Above, the path of integration always stays on the first sheet $\mathscr{Q}_1$ and is defined as follows. For z lying above Γ, the path of integration connects 1 to z without crossing Γ, except of course at the initial point 1. For z lying below Γ, the path of integration emanates upwards from 1 and moves to the region below Γ crossing the interval $(-\infty,-1)$. The path then remains below Γ until reaching z.

Next, for $z \in \mathbb{C}\setminus(\Gamma\cup\{a_*,y_*\})$, we similarly define

$$u_2^{(k)}(z) = \int_1^z \Omega_n^{(k)} - in\kappa, \quad k = 1,2. \tag{4.84}$$

The path of integration for $u_2^{(k)}$ lies entirely in $\mathscr{Q}_2$ and specified as follows. For z lying below Γ, the path of integration connects 1 to z on the second sheet without crossing Γ, except at 1. For z lying above Γ, the path of integration emanates downwards from 1 and moves to the region above Γ across the interval $(-\infty,-1)$. The path then remains above Γ until meeting z.

Intuitively, one can think of the path of integration for $u_2^{(k)}$ as the mirror image on the other sheet of the path used for $u_1^{(k)}$, as illustrated in Figure 4.11. Also, because the residues of $\Omega_n^{(k)}$ are ± 1, these functions are well-defined analytic functions modulo $2\pi i\mathbb{Z}$.

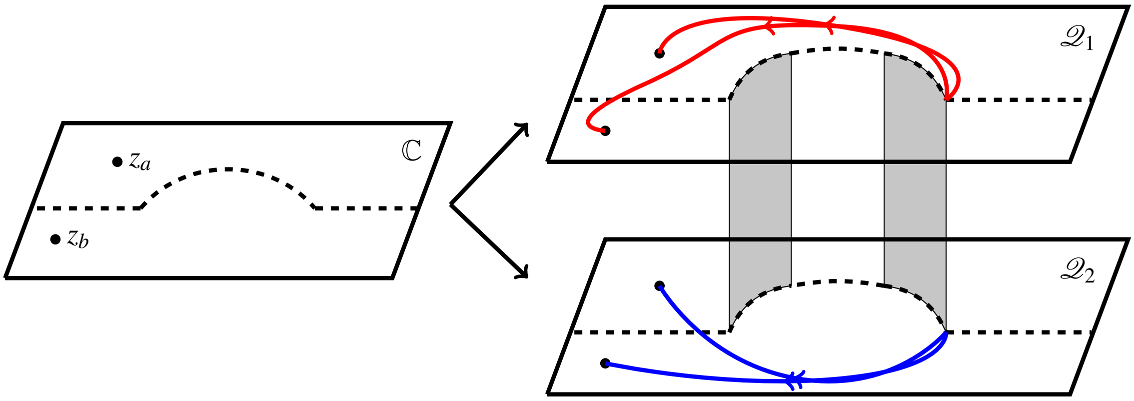


Figure 4.11: Visualization of the paths of integration for the functions $u_j^{(k)}(z)$. Here, the contour Γ is dashed and $z_a, z_b \in \mathbb{C}$ are points lying above and below Γ, respectively. The projections of $z_{a,b}$ onto $\mathscr{Q}$ are also pictured.

The main properties of $u_j^{(k)}$ are collected in the next result, where all equalities are understood modulo $2\pi i\mathbb{Z}$.

Lemma 4.23. *(a) The functions $u_j^{(k)}$, $j,k=1,2$, verify the following jumps for $z\in\Gamma$:*

(i) For $z\in(-\infty,-1)\cup(1,\infty)$,

$$u_{j,+}^{(k)}(z) = u_{j,-}^{(k)}(z). \tag{4.85}$$

(ii) For $z\in\gamma_{m,\ell}$, $\ell=0,1$,

$$u_{1,\pm}^{(k)}(z) = u_{2,\mp}^{(k)}(z) + (-1)^{\ell+1} in\kappa. \tag{4.86}$$

(iii) For $z\in\gamma_{c,1}$,

$$u_{j,+}^{(k)}(z) = u_{j,-}^{(k)}(z) + i\pi n.$$

(b) For some constants $m_{j,k}\in\mathbb{C}$ the asymptotic behavior

$$u_j^{(k)}(z) = m_{j,k} + \mathscr{O}\left(\frac{1}{z}\right), \qquad z\to\infty, \tag{4.87}$$

holds.

(c) The function $u_j^{(j)}$ is analytic at y_*, for $j=1,2$.

(d) When $j\neq k$,

$$u_j^{(k)}(z) = -\log(z-y_*) + \mathscr{O}(1), \qquad z\to y_*. \tag{4.88}$$

(e) The behavior of the $u_j^{(k)}$ *near* a_* *is as follows:*

- *For even n, the function* $u_j^{(j)}$ *is analytic at* a_**, whereas for* $j \neq k$

$$u_j^{(k)}(z) = \log(z - a_*) + \mathscr{O}(1), \qquad z \to a_*. \tag{4.89}$$

- *For odd n, when* $j \neq k$ *the function* $u_j^{(k)}$ *is analytic at* a_**, whereas*

$$u_j^{(j)}(z) = \log(z - a_*) + \mathscr{O}(1), \qquad z \to a_*. \tag{4.90}$$

Proof. First let $x \in (1, \infty)$. Then

$$u_{1,+}^{(k)}(x) - u_{1,-}^{(k)}(x) = \oint_C \Omega_n^{(k)},$$

where C is a clockwise loop on the first sheet of $\mathscr{Q}$ surrounding both $\gamma_{m,0}$ and $\gamma_{m,1}$. By transferring this loop to infinity, we have that

$$u_{1,+}^{(k)}(x) = u_{1,-}^{(k)}(x), \qquad x \in (1, \infty).$$

The deformation of C to infinity may pick up residue contributions from the poles a and y_* depending on their locations, but as all residues are ± 1, this contribution will only contribute a factor of $2\pi i$.

Now let $x \in (-\infty, 1)$. Then,

$$u_{1,+}^{(k)}(x) - u_{1,-}^{(k)}(x) = \oint_C \Omega_n^{(k)}, \qquad x \in (-\infty, 1).$$

where now C is contractible within $\mathscr{Q}$, so that (4.85) holds for $j = 1$. In a similar fashion we compute (4.85) for $j = 2$.

Next, take $z \in \gamma_{m,0}$. In this case,

$$u_{1,+}^{(k)}(z) - u_{2,-}^{(k)}(z) = \oint_C \Omega_n^{(k)} + i\kappa n,$$

where C can be deformed to the cycle $-B$, so this integral can be computed using (4.82) as

$$u_{1,+}^{(k)}(z) - u_{2,-}^{(k)}(z) = -in\kappa. \tag{4.91}$$

We may similarly compute $u_{1,-}^{(k)}(z) - u_{2,+}^{(k)}(z)$ for $z \in \gamma_{m,0}$ proving (4.86) over $\gamma_{m,0}$. The case where $z \in \gamma_{m,1}$ can be written as a contour integral over a loop which can be deformed to a point, and as such the jump in (4.86) follows over $\gamma_{m,1}$, as well.

For the final jump, take $z \in \gamma_{c,1}$. Then

$$u_{1,+}^{(k)}(z) - u_{1,-}^{(k)}(z) = \oint_A \Omega_n^{(k)} = i\pi n,$$

and this integral is computed using (4.82).

The asymptotic behavior (4.87) follows from the fact that $\Omega^{(k)}$ is regular near infinity on both sheets of $\mathscr{Q}$.

Now, consider the function $u_j^{(j)}$, which we recall is being integrated on some path on the sheet $\mathscr{Q}_j$. Using (4.80), we see that the pole of $\Omega_j^{(j)}$ at y_* is on the opposite sheet of $\mathscr{Q}$, and in particular, $\Omega_j^{(j)}$ is analytic at y_* on the sheet $\mathscr{Q}_j$. This in turn implies (c) above. On the other hand, when $j \neq k$, the simple pole at y_* is on the sheet $\mathscr{Q}_j$ and has residue -1, and as such we know that

$$\Omega_n^{(k)} = \left(-\frac{1}{z-y_*} + \mathscr{O}(1)\right) dz, \qquad z \to y_*^{(j)}.$$

Upon integration, we have (4.88). A similar argument also provides provides (4.89)–(4.90).

□

Finally, we define

$$v_j^{(k)}(z) = \exp\left(u_j(z)\right), \quad k, j = 1, 2, \tag{4.92}$$

and prove our main result of this section.

Theorem 4.24. *Let c and κ be as in* (4.78) *and* (4.41) *and suppose that $2n\kappa - c \in \mathbb{R} \setminus 2\pi\mathbb{Z}$. Then the model Riemann-Hilbert problem* (4.52) *has a unique solution $M = (M_{jk})$. Its entries satisfy the following conditions:*

(i) For n even, M_{11} and M_{22} are never zero, whereas M_{12} and M_{21} have a unique zero at $a_* \in i\mathbb{R}$.

(ii) For n odd, M_{12} and M_{21} are never zero, whereas M_{11} and M_{22} have a unique zero at $a_* \in i\mathbb{R}$.

Proof. Uniqueness of M follows in the standard way for Riemann-Hilbert problems, see for instance [34].

To prove existence, we use $v_j^{(k)}$ as in (4.92) and set M as in (4.59). By Lemma 4.17 it is enough to verify that $v_j^{(k)}$ are solutions to the RHP (1)–(6) at the start of Section 4.4.2 *et seq.*. In turn, these scalar RHP conditions follow immediately from Lemma 4.23 (a)–(d).

Finally, the properties of the zeros of $M_{j,k}$ follow from Lemma 4.23–(e). □

4.4.5 Step Four: Analysis of $2n\kappa - c$

The whole construction of the global parametrix that ended up with Theorem 4.24 relies on the assumption that $2n\kappa - c \in \mathbb{R} \setminus 2\pi\mathbb{Z}$. It turns out that we can actually remove this restriction, provided that n is even.

Indeed, recall that the meromorphic differential $\Omega_n^{(k)}$ as in (4.81) has a pole at $a^{(n+k+1)}$. Assuming that $2n\kappa - c \in 2\pi\mathbb{Z}$, when we try to match the periods of $\Omega_n^{(k)}$ with the help of the injectivity of Ψ_B in (4.77), the pole at $a = i\tau$ moves to $\infty^{(n+k+1)}$. Nevertheless, the meromorphic differential $\Omega_n^{(k)}$ still has a limit, as one can see by performing an asymptotic analysis very similar to the one which led to

(4.78). In such a case, $\Omega_n^{(k)}$ becomes the unique differential whose only singularities are simple poles at $\infty^{(n+k+1)}$ and $y_*^{(k+1)}$, with residues $+1$ and -1, respectively.

In this way, we can still define $u_j^{(k)}$ as in (4.83)–(4.84), with the paths of integration on the sheet j. Now, when n is even and $j \neq k$, as $z \to \infty$ the path of integration extends to the pole at $\infty^{(k+1)}$, and as such $u_j^{(k+1)} \approx -\log z$ as $z \to \infty$. This means that now $v_j^{(k)}(z) \to 0$ as $z \to \infty$, which is no problem at all as the behavior (4.62) is still satisfied.

On the other hand, when n is odd, the path of integration for $u_k^{(k)}$ is now the one that extends to the pole at $\infty^{(k)}$, and consequently $v_j^{(j)}$ vanishes as $z \to \infty$. In such a case, the condition (4.61) is no longer satisfied for a nonzero constant c_j.

We now verify that the condition $2n\kappa - c \in 2\pi\mathbb{Z}$ occurs only for countably many pairs (n,t), if any. To see this, first notice that $2n\kappa - c$ depends real-analytically on $t > t_c$. Likewise, c does not depend on k,n but only (real-analytically) on $t > t_c$, and we write $2n\kappa - c = 2n\kappa(t) - c(t)$. Thus, for any $\ell \in \mathbb{Z}$ and fixed $n \in \mathbb{N}$, the identity

$$2n\kappa(t) - c(t) = 2\pi\ell$$

has at most countably many solutions for t in $\mathbb{R}$, provided $2n\kappa - c$ is not constant. For fixed n, we may calculate that

$$2n\kappa(t) - c(t) = g(t)(\alpha + o(1)), \tag{4.93}$$

where $g(t) = \max\{t, t^\delta \log t\}$, and a constant $\alpha \neq 0$, so that $2n\kappa - c$ is indeed not constant. To see that this quantity is not constant, we analyze these terms as $t \to \infty$. Starting with κ, we use (4.42) to show that

$$\kappa(t) = \mathcal{O}(t),$$

which follows from

$$\lambda_0(x,t) = 1 + \frac{2i}{t} + \mathcal{O}\left(\frac{1}{t^{1+\delta}}\right), \qquad Q^{1/2}(s) = -\frac{it}{2} + \mathcal{O}(1), \quad t \to \infty.$$

Next, as

$$\frac{1}{Q^{1/2}(s)\,(s^2-1)} = \frac{2i}{(s^2-1)t} + \mathcal{O}\left(\frac{1}{t^2(s^2-1)^2}\right),$$

we have that

$$m_B^{(0)} = \frac{2i}{t}\int_{-1+\frac{2i}{t}}^{1+\frac{2i}{t}} \frac{1}{s^2-1}\,ds + \mathcal{O}\left(\frac{1}{t}\right) = -\frac{2i\log(t)}{t} + \mathcal{O}\left(\frac{1}{t}\right), \qquad t \to \infty.$$

In a similar way, we find that

$$m_B^{(1)} = -\frac{\pi}{t} + \mathcal{O}\left(\frac{\log t}{t^2}\right), \qquad t \to \infty.$$

To compute the moments over the A cycle, we can write

$$m_A^{(1)} = \frac{2i}{t}\int_{-1}^{-x} \frac{s}{\left(\left(s^2-1\right)\left(s^2-x^2\right)\right)_+^{1/2}}\,ds + \mathscr{O}\left(\frac{1}{t^2}\right) = \mathscr{O}\left(\frac{1}{t}\right), \qquad t\to\infty.$$

Similarly,

$$m_A^{(0)} = -\frac{2m_{A,\infty}^{(0)}}{t} + \mathscr{O}\left(\frac{1}{t^2}\right), \qquad t\to\infty,$$

where

$$m_{A,\infty}^{(0)} := \int_{-1}^{-x} \frac{1}{\sqrt{\left(1-s^2\right)\left(s^2-x^2\right)}} \in \mathbb{R}.$$

Combining the preceding calculations, we see that the third term in (4.78) behaves like

$$(-1)^\nu \frac{it}{2}\left(\frac{m_B^{(0)} m_A^{(1)}}{m_A^{(0)}} - m_B^{(1)}\right) = \mathscr{O}\left(\log t\right),$$

as $t\to\infty$. Next, we have that

$$-\oint_B \Lambda_{y_*}^{(\varsigma)} = \int_{\lambda_0}^{\lambda_1} \frac{(-1)^\varsigma Q^{1/2}(y_*)\left(y_*^2-1\right)}{Q^{1/2}(s)\left(s^2-1\right)\left(s-y_*\right)}.$$

From (4.58),

$$y_* = \mathscr{O}\left(t^\varepsilon\right),$$

and as $y_*\in i\mathbb{R}$, we write that

$$y_* \sim iat^\varepsilon, \qquad a\in\mathbb{R}, \qquad \lambda\to\infty.$$

Now, we have that

$$\frac{(-1)^\varsigma Q^{1/2}(y_*)\left(y_*^2-1\right)}{Q^{1/2}(s)\left(s^2-1\right)\left(s-y_*\right)} = (-1)^{\varsigma+1}\,\frac{iat^\varepsilon}{s^2-1} + \mathscr{O}\left(\frac{1}{t^{1-\varepsilon}}\right), \qquad t\to\infty, \tag{4.94}$$

so that

$$-\oint_B \Lambda_{y_*}^{(\varsigma)} = (-1)^\varsigma\, iat^\varepsilon \log t + \mathscr{O}\left(\frac{1}{t^{1-\varepsilon}}\right),$$

as $t\to\infty$. Similarly, we find that

$$\oint_A \Lambda_{y_*}^{(\varsigma)} = (-1)^{\varsigma+1}\, at^\varepsilon m_{A,\infty}^{(0)} + \mathscr{O}\left(\frac{1}{t^{1-\varepsilon}}\right), \qquad t\to\infty.$$

Therefore,

$$-\frac{m_A^{(0)}}{m_B^{(0)}}\Re\oint_A \Lambda_{y_*}^{(\varsigma)} \sim (-1)^\varsigma\, iat^\varepsilon \log t, \qquad t\to\infty.$$

Hence, both the first and middle terms in (4.78) tend to infinity at a rate $\mathscr{O}(t^{\varepsilon}\log t)$ and moreover do not cancel as $t\to\infty$. The final term tends to infinity like $\mathscr{O}(\log t)$, while $2\kappa(t)=\mathscr{O}(t)$ as $t\to\infty$. Thus the function $2\kappa(t)-c(t)$ does indeed tend to ∞ as $t\to\infty$. In fact, numerical experiments suggest that $\varepsilon=1$, implying that $2\kappa-c$ tends to infinity at a rate $\mathscr{O}(t\log t)$. In any case, we have shown that for large t the function $2\kappa(t)-c(t)$ is not a constant and tends to infinity as $t\to\infty$, verifying (4.93).

Now, we fix $\varepsilon>0$ and define the set Θ_{ε}^{*} as

$$\Theta_{\varepsilon}^{*}:=\{(n,t):\operatorname{dist}(2n\kappa(t)-c(t),2\pi\mathbb{Z})<\varepsilon\}. \tag{4.95}$$

By virtue of this discussion, Theorem 4.24 is improved to the following form.

Theorem 4.25. *Fix $\varepsilon>0$. For n even with $t>t_c$ or n odd with $t>t_c$ and $(n,t)\notin\Theta_{\varepsilon}^{*}$, the model Riemann-Hilbert problem* (4.52) *has a unique solution $M=(M_{jk})$. Its diagonal entries satisfy the following:*

(i) For n even, M_{11} and M_{22} are never zero.

(ii) For n odd, M_{11} and M_{22} have a unique zero at $a_*\in i\mathbb{R}$ which is simple.

Furthermore, the entries of M remain bounded on compacts as $n\to\infty$ with n even or n odd with $(n,t)\notin\Theta_{\varepsilon}$.

4.5 Asymptotics for the Kissing Polynomials in the Supercritical Regime

We complete the process of Deift-Zhou steepest descent and use it to extract asymptotics of the Kissing polynomials in the supercritical regime. First, we must turn our attention to the local parametrices.

4.5.1 Local Parametrices

Having constructed the global parametrix, we now quickly turn to the construction of the local parametrices. Much of the following is the same as presented in Chapter 2, albeit we now use the ϕ function, as opposed to the h-function.

Hard Edges

Let $D_1:=D_{\delta}(1)$ be a disc centered at 1 of small radius $\delta>0$. We seek a local parametrix, $P^{(1)}(z)$, defined in D_1, which solves the following Riemann-Hilbert problem

$$P^{(1)}(z) \text{ is analytic in } D_1\setminus\hat{\Sigma}, \tag{4.96a}$$

$$P_{+}^{(1)}(z)=P_{-}^{(1)}(z)j_S(z), \qquad z\in D_1\cap\hat{\Sigma}, \tag{4.96b}$$

$$P^{(1)}(z)=\left(I+\mathscr{O}\left(\frac{1}{n}\right)\right)M(z), \qquad \text{uniformly on } \partial D_1 \text{ as } n\to\infty. \tag{4.96c}$$

We also require that $P^{(1)}$ has a continuous extension to $\overline{D_1}\setminus\hat{\Sigma}$ and remains bounded as $z\to 1$. We seek $P^{(1)}$ in the form

$$P^{(1)}(z)=U^{(1)}(z)e^{n\phi(z)\sigma_3}, \tag{4.97}$$

where $U^{(1)}$ has the following jumps over $\hat{\Sigma}\cap D_1$

$$U_+^{(1)}(z)=U_-^{(1)}(z)\begin{cases}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, & z\in\gamma_{m,1}^{\pm}\cap D_1,\\ \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, & z\in\gamma_{m,1}\cap D_1.\end{cases} \tag{4.98}$$

Let $f_{n,B}$ be the map

$$f_{n,B}(z)=n^2 f_B(z), \qquad \text{where} \qquad f_B(z)=\frac{1}{4}\left(-\phi(z)-\frac{i\kappa}{2}\right)^2. \tag{4.99}$$

Using (4.43), as $\phi_+(z)+\phi_-(z)=-i\kappa$ for $z\in\gamma_{m,1}$, we have that

$$\left(-\phi(z)-\frac{i\kappa}{2}\right)_+=-\left(-\phi(z)-\frac{i\kappa}{2}\right)_-, \qquad z\in\gamma_{m,1}. \tag{4.100}$$

Therefore, $f_{n,B}$ is analytic in $D_1\setminus\{1\}$, as it has no jumps within the disc D_1. Moreover, since $h(z)\to 0$ as $z\to 1$, and as $-\phi(z)-i\kappa/2=h(z)/2$, we see that $f_{n,B}\to 0$ as $z\to 1$, so that the apparent singularity at $z=1$ is in fact removable, and as such $f_{n,B}$ is analytic in D_1.

Using (4.32) and (4.40), we see that

$$-\phi(z)-\frac{i\kappa}{2}=\frac{1}{2}h(z)=-\frac{1}{2}\int_1^z\sqrt{Q(s;t)}\,ds, \tag{4.101}$$

where we recall that

$$Q(z;t)=-\frac{t^2(z-\lambda_0)(z-\lambda_1)}{z^2-1}, \qquad \lambda_0=-x_*+\frac{2i}{t}, \qquad \lambda_1=x_*+\frac{2i}{t}. \tag{4.102}$$

Then, as $z\to 1$, we have

$$-\phi(z)-\frac{i\kappa}{2}=-\sqrt{\frac{4+t\,(4i+t\,(x^2-1))}{2}}\sqrt{z-1}+\mathscr{O}\left((z-1)^{3/2}\right), \qquad z\to 1, \tag{4.103}$$

so that

$$f_B(z)=\frac{4+t\left(4i+t\left(x^2-1\right)\right)}{8}(z-1)+\mathscr{O}\left((z-1)^2\right), \qquad z\to 1. \tag{4.104}$$

As $f'_{n,B}(1)\neq 0$, $f_{n,B}$ is then a conformal mapping from a neighborhood of 1 to a neighborhood of 0. Further note that as $h(z)$ is purely imaginary on $\gamma_{m,1}$, the map $f_{n,B}$ takes the main arc $\gamma_{m,1}$ to the negative real axis. Moreover, we now choose the definitions of the arcs $\gamma_{m,1}^{\pm}$, so that they are mapped by $f_{n,B}$ to the rays $\{z : \arg z = \pm\frac{2\pi}{3}\}$, respectively.

Then, using the Bessel parametrix defined in (2.95), we see that

$$U^{(1)}(z) := E^{(1)}(z)B(f_{n,B}(z)), \tag{4.105}$$

where $E^{(1)}$ is analytic and satisfies the jumps (4.98). Therefore, $P^{(1)}$ defined in (4.97) satisfies both (4.96a) and (4.96b). We now choose $E^{(1)}$ so that $P^{(1)}$ matches M on the boundary of D_1, as specified by (4.96c). This can be done by individually considering the asymptotics of the proposed $P^{(1)}$ in each sector. We first consider the case where $|\arg f_{n,B}| < \frac{2\pi}{3}$. As shown in [65, Section 6],

$$B(\zeta) = \left(2\pi\zeta^{1/2}\right)^{-\sigma_3/2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1+\mathscr{O}\left(\frac{1}{\zeta^{1/2}}\right) & i+\mathscr{O}\left(\frac{1}{\zeta^{1/2}}\right) \\ i+\mathscr{O}\left(\frac{1}{\zeta^{1/2}}\right) & 1+\mathscr{O}\left(\frac{1}{\zeta^{1/2}}\right) \end{pmatrix} e^{2\zeta^{1/2}\sigma_3}, \qquad \zeta\to\infty, \tag{4.106}$$

for $|\arg\zeta| < \frac{2\pi}{3}$. In fact, (4.106) holds in all sectors of the disc D_1, as shown in [65].

Using the definitions of $P^{(1)}$ and $U^{(1)}$, we find that

$$P^{(1)}(z) = E^{(1)}(z)\left(2\pi n f_B^{1/2}(z)\right)^{-\sigma_3/2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\left(I+\mathscr{O}\left(\frac{1}{n}\right)\right)e^{-\frac{ni\kappa}{2}\sigma_3}, \tag{4.107}$$

as $n\to\infty$. By setting

$$E^{(1)}(z) = M(z)e^{\frac{in\kappa}{2}\sigma_3}L^{(1)}(z)^{-1}, \quad L^{(1)}(z) := \frac{1}{\sqrt{2}}(2\pi n)^{-\sigma_3/2} f_B(z)^{-\sigma_3/4}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \tag{4.108}$$

we see that $P^{(1)}(z) = M(z)\left(I+\mathscr{O}\left(n^{-1}\right)\right)$ as $n\to\infty$ on ∂D_1. We are now just left to verify that the matrix valued $E^{(1)}(z)$ is in fact analytic within D_1, as claimed.

Recall from (4.53) that M is analytic on $D_1\setminus\gamma_{m,1}$ and satisfies

$$M_+(z) = M_-(z)\begin{pmatrix} 0 & e^{i\kappa n} \\ -e^{-i\kappa n} & 0 \end{pmatrix}, \qquad z\in\gamma_{m,1}. \tag{4.109}$$

Furthermore, the matrix $L^{(1)}(z)$ is also analytic on $D_1\setminus\gamma_{m,1}$, with jumps for $L^{(1)}$ coming from the branch of the quarter root of f_B, which satisfies

$$f_{B,+}(z)^{\sigma_3/4} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} f_{B,-}(z)^{\sigma_3/4}, \qquad z\in\gamma_{m,1}. \tag{4.110}$$

Then, we see that first factor of $E^{(1)}$ satisfies

$$M_+(z)e^{\frac{in\kappa}{2}\sigma_3} = M_-(z)e^{\frac{in\kappa}{2}\sigma_3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in \gamma_{m,1}. \tag{4.111}$$

Next, we also see that

$$L_+^{(1)}(z) = L_-^{(1)}(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{4.112}$$

so that $M\exp(\frac{i}{2}n\kappa\sigma_3)$ and $L^{(1)}$ have the same jumps over $\gamma_{m,1}$. Therefore, $E^{(1)}$ has no jumps in D_1 and is analytic in $D_1 \setminus \{1\}$. As shown further in [65], the entries of $E^{(1)}$ have at most square root singularities at $z = 1$, so that the apparent singularity is removable and in fact $E^{(1)}$ is analytic within D_1, as desired.

The parametrix $P^{(-1)}$ in a small neighborhood D_{-1} of the hard edge -1 can be constructed by exploring the symmetry with respect to the imaginary axis, which leads to

$$P^{(-1)}(z) := \overline{P^{(1)}(-\bar{z})}, \quad z \in D_{-1}.$$

Soft Edges

Let $D_{\lambda_0} := D_\delta(\lambda_0)$ be a small disc centered at λ_0 of radius $\delta > 0$. We seek a local parametrix, $P^{(\lambda_0)}(z)$, defined on D_{λ_0}, which is the solution to the following Riemann-Hilbert problem:

$$P^{(\lambda_0)}(z) \text{ is analytic in } D_{\lambda_0}\backslash\hat{\Sigma}, \tag{4.113a}$$

$$P_+^{(\lambda_0)}(z) = P_-^{(\lambda_0)}(z)\, j_S(z), \qquad z \in D_{\lambda_0} \cap \hat{\Sigma}, \tag{4.113b}$$

$$P^{(\lambda_0)}(z) = \left(I + \mathscr{O}\left(\frac{1}{n}\right)\right)M(z), \qquad \text{uniformly on } \partial D_{\lambda_0} \text{ as } n \to \infty. \tag{4.113c}$$

We also require that $P^{(\lambda_0)}(z)$ has a continuous extension to $\overline{D_{\lambda_0}}\backslash\hat{\Sigma}$ and remains bounded as $z \to \lambda_0$. First, we introduce

$$\phi^{(\lambda_0)}(z) := \frac{1}{2}\int_{\lambda_0}^{z} Q^{1/2}(s)\,ds, \tag{4.114}$$

where the path of integration emanates upwards from λ_0 and does not cross Ω. Note by using (4.32) and (4.40), we may express

$$\phi^{(\lambda_0)}(z) = \phi(z) + \frac{i\kappa}{2} - \frac{1}{2}\int_1^{\lambda_0} Q_+^{1/2}(s)\,ds. \tag{4.115}$$

Then using (4.25) and (4.42), we may simplify this further to

$$\phi^{(\lambda_0)}(z) = \phi(z) + \frac{i\pi}{2} - \frac{i\kappa}{2}. \tag{4.116}$$

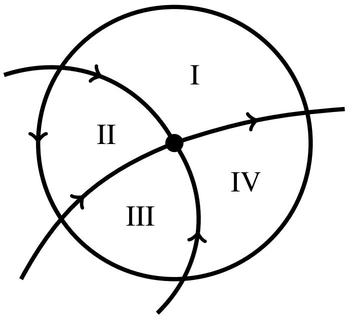


Figure 4.12: Definition of Sectors I, II, III, and IV within D_{λ_0}.

As before, we seek a solution of the form

$$P^{(\lambda_0)}(z) = U^{(\lambda_0)}(z)e^{n\phi(z)\sigma_3}, \tag{4.117}$$

so that $U^{(\lambda)}(z)$ has the following jumps within D_{λ_0}:

$$U_+^{(\lambda_0)}(z) = U_-^{(\lambda_0)}(z) \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & z \in \gamma_{m,0}^{\pm} \cap D_{\lambda_0}, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \gamma_{m,0} \cap D_{\lambda_0}, \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & z \in \gamma_{c,1} \cap D_{\lambda_0}. \end{cases} \tag{4.118}$$

We may match the jumps for $U^{(\lambda_0)}$ using the Airy parametrix defined in (2.83). Then $U^{(\lambda_0)}$ takes the form

$$U^{(\lambda_0)}(z) = E^{(\lambda_0)}(z)A(f_{n,A}(z)), \tag{4.119}$$

where A is the Airy parametrix, $E^{(\lambda_0)}$ is analytic, and $f_{n,A}$ is the conformal map given by

$$f_{n,A}(z) = n^{2/3} f_A(z), \qquad f_A(z) = \left[\frac{3}{2}\left(\phi^{(\lambda_0)}(z)\right)\right]^{2/3}. \tag{4.120}$$

From (4.114), we can write for z in a neighborhood λ_0,

$$\phi^{(\lambda_0)}(z) = \frac{2}{3}(z-\lambda_0)^{3/2} g(z), \tag{4.121}$$

where the cut for $(z-\lambda_0)^{3/2}$ is taken on $\gamma_{m,0}$ and $g(z)$ is analytic in a neighborhood of λ_0, with $g(\lambda_0) \neq 0$.

From (4.121), we see that f_A has no jumps within D_{λ_0}, and since $f_A(\lambda_0) = 0$, we can conclude that the apparent singularity at λ_0 is again removable. Since $f_A'(\lambda_0) = g(\lambda_0) \neq 0$, we see that f_A is indeed a conformal mapping from a neighborhood of λ_0 onto a neighborhood of 0. Therefore, we are just left to choose the prefactor $E^{(\lambda_0)}$ so that it is analytic within D_{λ_0} and that the matching condition (4.113c) is satisfied. As before, this can be accomplished by studying the asymptotics of the Airy parametrix.

As seen in [16], we may study the asymptotics in each sector of D_{λ_0} to find that

$$A(\zeta) = \frac{1}{2\sqrt{\pi}}\zeta^{-\sigma_3/4}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}\left(I + \mathscr{O}\left(\frac{1}{\zeta^{3/2}}\right)\right)e^{-\frac{2}{3}\zeta^{3/2}\sigma_3}, \qquad \zeta \to \infty. \tag{4.122}$$

Using (4.116) and (4.121), we set

$$E^{(\lambda_0)}(z) = \begin{cases} M(z)e^{n\left[\frac{i\pi}{2}-\frac{i\kappa}{2}\right]\sigma_3}L^{(\lambda_0)}(z)^{-1}, & z \in \mathrm{I,II}, \\ M(z)e^{n\left[-\frac{i\pi}{2}-\frac{i\kappa}{2}\right]\sigma_3}L^{(\lambda_0)}(z)^{-1}, & z \in \mathrm{III,IV}, \end{cases} \tag{4.123}$$

where Sectors I, II, III, and IV are defined in Figure 4.12, and

$$L^{(\lambda_0)}(z) = \frac{1}{2\sqrt{\pi}}n^{-\sigma_3/6}f_A(z)^{-\sigma_3/4}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}.$$

In this way, the matching condition (4.113c) is satisfied. In the formulas above, all the roots are taken to be the principal branches.

Again, we are just left to verify that $E^{(\lambda_0)}$ is analytic within D_{λ_0}. As $f_A(z)$ is positive on $\gamma_{c,1}$ and negative on $\gamma_{m,0}$, we find that $f_A^{-1/4}$ only has a jump within D_{λ_0} for $z \in \gamma_{m,0} \cap D_1$, which satisfies

$$f_{A,+}(z)^{\sigma_3/4} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} f_{A,-}(z)^{\sigma_3/4}, \qquad z \in \gamma_{m,0}. \tag{4.124}$$

Therefore, we see that

$$L_+^{(\lambda_0)}(z) = L_-^{(\lambda_0)}(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in \gamma_{m,0}. \tag{4.125}$$

Next we define,

$$\tilde{M}(z) = \begin{cases} M(z)e^{n\left[\frac{i\pi}{2}-\frac{i\kappa}{2}\right]\sigma_3}, & z \in \mathrm{I,II}, \\ M(z)e^{n\left[-\frac{i\pi}{2}-\frac{i\kappa}{2}\right]\sigma_3}, & z \in \mathrm{III,IV}, \end{cases} \tag{4.126}$$

so that $E^{(\lambda_0)}(z) = \tilde{M}(z)L^{(\lambda_0)}(z)$. Our goal is to show that $\tilde{M}$ has the same jumps as $L^{(\lambda_0)}$ within D_{λ_0}, just as in the Bessel parametrix case.

First, we consider $z \in \gamma_{c,1}$. Then we see that

$$\tilde{M}_+(z) = \tilde{M}_-(z)e^{-n\left[-\frac{i\pi}{2}-\frac{i\kappa}{2}\right]\sigma_3}(-1)^n I e^{n\left[\frac{i\pi}{2}-\frac{i\kappa}{2}\right]\sigma_3} = \tilde{M}_-(z), \tag{4.127}$$

so that $\tilde{M}$ has no jump over $\gamma_{c,1}$. For $z \in \gamma_{m,0}$, we find that

$$\tilde{M}_+(z) = \tilde{M}_-(z) e^{-n\left[-\frac{i\pi}{2}-\frac{i\kappa}{2}\right]\sigma_3} \begin{pmatrix} 0 & e^{-i\kappa n} \\ -e^{i\kappa n} & 0 \end{pmatrix} e^{n\left[\frac{i\pi}{2}-\frac{i\kappa}{2}\right]\sigma_3} = \tilde{M}_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad (4.128)$$

so that we may conclude that $E^{(\lambda_0)}(z)$ is analytic for $z \in D_{\lambda_0} \setminus \{\lambda_0\}$. However, we again see that the singularity at λ_0 is removable, so that $E^{(\lambda_0)}$ is analytic within the entire disc, and as such, the construction of the local parametrix is complete.

In a similar fashion to the hard edge scenario, we compute the local parametrix $P^{(\lambda_1)}$ in a small neighborhood, D_{λ_1}, of λ_1 via symmetry, which leads to

$$P^{(\lambda_1)}(z) := \overline{P^{(\lambda_0)}(-\bar{z})}, \quad z \in D_{\lambda_1}.$$

4.5.2 Final Transformation and Asymptotics

Recall that we have defined the set of branchpoints, using the notation of Chapter 2, as $\Lambda = \{-1, 1, \lambda_0, \lambda_1\}$. We now define the final transformation of the steepest descent method as

$$R(z) = \begin{cases} S(z)M(z)^{-1}, & z \in \mathbb{C} \setminus \left(\bigcup\limits_{\lambda \in \Lambda} D_\lambda \cup \hat{\Sigma} \right), \\ S(z)P^{(\lambda)}(z)^{-1}, & z \in D_\lambda \setminus \hat{\Sigma}, \qquad \lambda \in \Lambda. \end{cases} \qquad (4.129)$$

Then R solves a Riemann-Hilbert problem of the form,

$$R(z) \text{ is analytic in } \mathbb{C} \setminus \Sigma_R, \qquad (4.130a)$$

$$R_+(z) = R_-(z) j_R(z), \qquad z \in \Sigma_R, \qquad (4.130b)$$

$$R(z) = I + \mathcal{O}\left(\frac{1}{z}\right), \qquad z \to \infty, \qquad (4.130c)$$

where the contour Σ_R is depicted in Figure 4.13, and the jump matrix j_R that satisfies

$$j_R(z) = \begin{cases} I + \mathcal{O}\left(e^{-cn}\right), & z \in \Sigma_R \setminus \left(\bigcup\limits_{\lambda \in \Lambda} \partial D_\lambda \right), \\ I + \mathcal{O}\left(\frac{1}{n}\right), & z \in \bigcup\limits_{\lambda \in \Lambda} \partial D_\lambda, \end{cases}$$

with uniform error terms. The definition of M, and of R itself, only requires that n is even or $2n\kappa(t) - c(t) \notin 2\pi\mathbb{Z}$. The condition that $(n,t) \notin \Theta_\varepsilon$ is used to ensure that the entries of M remain bounded as $n \to \infty$, see Theorem 4.25 above, and consequently ensure that j_R indeed decays to the identity as claimed.

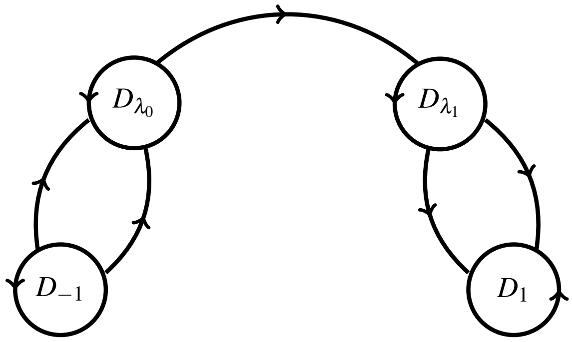


Figure 4.13: The contour Σ_R.

As seen in Chapter 2.2.2 and the references therein, the above conditions are enough to assure that

$$R(z) = I + \mathscr{O}\left(\frac{1}{n}\right), \qquad n \to \infty, \tag{4.131}$$

uniformly for $z \in \mathbb{C} \backslash \Sigma_R$ as $n \to \infty$ with n even or with $(n,t) \notin \Theta_\varepsilon$. We are now left to retrace our steps in the transformations of nonlinear steepest descent to obtain the uniform asymptotic formulas for the Kissing Polynomials in the supercritical regime.

Using the same procedures that led to (2.32) and (2.34), we arrive at the following theorem.

Theorem 4.26. *Fix $\varepsilon > 0$ and $t > t_c$. For n sufficiently large, and for $(n,t) \notin \Theta_\varepsilon^*$ in the case that n is odd, the Kissing polynomial $p_n(z;t)$ defined in* (4.1) *(with $N = n$) uniquely exists as a monic polynomial of degree exactly n. If we denote by $z_1, \ldots, z_n$ the zeros of $p_n(z;t)$, we have the weak asymptotics,*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \delta_{z_k} \xrightarrow{*} \mu_*, \tag{4.132}$$

where μ_* is the measure defined in Theorem 4.14 and δ_z is the atomic measure with mass 1 *at z.*

Furthermore, as $n \to \infty$,

$$p_{2n}(z;t) = \Psi_{n,0}(z) e^{n(-i\kappa - itz + \ell - 2\phi(z))} \left(1 + \mathscr{O}\left(\frac{1}{n}\right)\right), \tag{4.133a}$$

$$p_{2n+1}(z;t) = e^{(2n+1)\left(-\frac{i\kappa}{2} - \frac{itz}{2} + \frac{\ell}{2} - \phi(z)\right)} \left(\Psi_{n,1}(z) + \mathscr{O}\left(\frac{1}{n}\right)\right), \tag{4.133b}$$

hold true uniformly in compact subsets of $\mathbb{C} \setminus \mathfrak{M}$ and $\mathbb{C} \setminus (\mathfrak{M} \cup \gamma_{c,0})$, respectively, where the functions $\Psi_{n,0}$ and $\Psi_{n,1}$ have the following properties:

(i) $\Psi_{n,0}$ is holomorphic in $\mathbb{C} \setminus \mathfrak{M}$, whereas $\Psi_{n,1}$ is holomorphic on $\mathbb{C} \setminus (\mathfrak{M} \cup \gamma_{c,0})$, and they remain bounded on compact subsets of their respective domains of definition as $n \to \infty$.

(ii) $\Psi_{n,0}$ does not have zeros.

(iii) The function $\Psi_{n,1}$ has a unique zero at a point $a_* = a_*(n,t)$, which is simple and located on the imaginary axis.

Proof. For the weak convergence of the counting measure, we turn back to Remark 2.1, where we have that $h = 2g - f - \ell$. Using (4.11), we may write that

$$h_+(z) + h_-(z) = 4\pi i \omega_j, \qquad z \in \mathfrak{M}, \tag{4.134a}$$

$$\Re h(z) < 0, \qquad z \in \gamma_{c,1}. \tag{4.134b}$$

Using that supp $\mu_* = \gamma_{m,0} \cup \gamma_{m,1}$, we may translate the above properties into

$$-g_+(z;t) - g_-(z;t) + f(z;t) = -\ell - 2\pi i \omega_j, \qquad z \in \text{supp}\, \mu_*, \tag{4.135a}$$

$$\Re\left(-g_+(z;t) - g_-(z;t) + f(z;t)\right) > -\ell, \qquad z \in \Sigma \setminus \text{supp}\, \mu_*, \tag{4.135b}$$

which are precisely the conditions on Σ given in (1.35) to ensure that it possesses the S-property. By the theorem of Gonchar and Rakhmanov [49] (see also the discussion around (1.36)-(1.37)), the convergence in (4.132) is assured.

We now turn to the asymptotic formulas for the polynomials in compact subsets of $\mathbb{C} \setminus \Sigma$. We follow the same outline as presented in Section 2.2.3, the only difference again being that we use the ϕ function as opposed to the h function. Unwinding the transformations starting from (4.48), we find that for $z \in \mathbb{C} \setminus \Sigma$

$$Y(z) = e^{-\frac{n}{2}(-\ell + i\kappa)\sigma_3} R(z) M(z) e^{-n\left(\phi(z) + \frac{itz}{2}\right)\sigma_3}. \tag{4.136}$$

By looking at the $(1,1)$-entry of Y, as in (4.46), we find that we may write the Kissing polynomial as

$$p_n(z;t) = e^{\frac{n}{2}(-i\kappa + \ell - itz - 2\phi(z))}\left(M_{11}(z) R_{11}(z) + M_{21}(z) R_{12}(z)\right). \tag{4.137}$$

We now use that $R = I + \mathscr{O}\left(\frac{1}{n}\right)$ and Theorem 4.25 on the global parametrix to complete the proof of the theorem. We first consider the case when the polynomial is of even degree. We have

$$p_{2n}(z;t) = e^{n(-i\kappa - itz + \ell - 2\phi(z))}\left(M_{11}(z)\left(1 + \mathscr{O}\left(\frac{1}{n}\right)\right) + M_{21}(z)\mathscr{O}\left(\frac{1}{n}\right)\right), \qquad n \to \infty. \tag{4.138}$$

By Theorem 4.25, we have that M_{11} is never zero and that M_{21} is bounded, so that we recover (4.133a) with

$$\Psi_{n,0}(z) = M_{11}(z). \tag{4.139}$$

The fact that $\Psi_{n,0}$ is analytic on $\mathbb{C} \setminus \mathfrak{M}$ follows from (4.52) and the fact that the model Riemann-Hilbert problem does not jump over $\gamma_{c,0}$ in the even case.

Similarly, in the case that we are working with the polynomial of degree $2n+1$, (4.137) becomes

$$p_{2n+1}(z;t) = e^{(2n+1)\left(-\frac{i\kappa}{2} - \frac{itz}{2} + \frac{\ell}{2} - \phi(z)\right)}\left(M_{11}(z) + \mathscr{O}\left(\frac{1}{n}\right)\right). \tag{4.140}$$

This again gives (4.133b), with $\Psi_{n,1}(z) = M_{11}(z)$, but it is noted that we are not able to factor this term out, as it has a zero on the imaginary axis. $\square$

Chapter 5

Global Phase Portrait and Asymptotic Regimes for the Kissing Polynomials

The main goal of this chapter is to determine the asymptotic behavior of the recurrence coefficients for polynomials satisfying the following non-Hermitian, degree dependent, orthogonality conditions:

$$\int_{-1}^{1} p_n(z;s) z^k e^{-nf(z;s)}\, dz = 0, \qquad k = 0, 1, \ldots, n-1, \tag{5.1a}$$

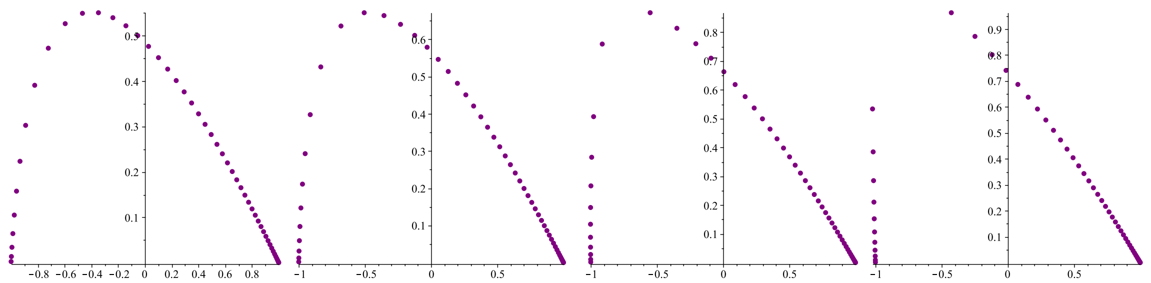
and

$$\int_{-1}^{1} p_n(z;s) z^n e^{-nf(z;s)}\, dz \neq 0, \tag{5.1b}$$

where p_n is a monic polynomial of degree n in the variable z, $f(z;s) = sz$, and $s \in \mathbb{C}$ is arbitrary. By doing so, we generalize the results of the previous chapter, by letting our parameter s take values in $\mathbb{C}$, as opposed to the imaginary axis. However, the results from the previous chapter on the supercritical regime, along with the results of Deaño on the subcritical regime [31], will be crucial in this chapter. Indeed, in this chapter, we adopt a viewpoint based on deformation techniques born from advances in the theory of random matrices and integrable systems. We will make heavy use of the technique known as *continuation in parameter space*, first developed in the context of integrable systems (c.f. [56, 86, 87]), but which has only recently been applied in the field of orthogonal polynomials [10, 12, 13]. We will see that we may deform the genus 1 h-function of Chapter 4 and the genus 0 h-function of [31] away from the imaginary axis, provided we do not cross certain curves in the parameter space, called *breaking curves*.

5.1 Statement of Main Results

Recall from Chapter 2, that we may complete the process of Deift-Zhou steepest descent provided we are able to construct an appropriate h-function. This h-function must solve a certain scalar Riemann-Hilbert problem on both the main and complementary arcs, while also satisfying certain inequality constraints. We quickly note that as the weight function we consider, $\exp(-nf(z;s))$, depends on the parameter s, the scalar Riemann-Hilbert problem also depends on the parameter s. Importantly, the number of arcs over which this Riemann-Hilbert problem is posed, or equivalently the genus of the underlying Riemann surface, is also to be determined. Indeed, we will see that h-functions corresponding to Riemann surfaces of different genus lead to asymptotic expansions which possess markedly different behavior as $n \to \infty$. This difference is analogous to the difference in asymptotic behavior of the polynomials (and their recurrence coefficients) in the one cut (subcritical) and two cut (supercritical) cases. However, once one proves that for a specified genus and corresponding $s \in \mathbb{C}$ the scalar problem for the h-function has a solution, one may continue with the process of steepest descent as outlined in Chapter 2.



(a) $s = -1 - 0.85i$. **(b)** $s = -1 - 0.95i$. **(c)** $s = -1 - 1.05i$. **(d)** $s = -1 - 1.15i$.

Figure 5.1: Zeros of $p_{50}(z;s)$ defined in (5.1) as s moves from $s = -1 - 0.85i \in \mathfrak{S}_0$ to $s = -1 - 1.15i \in \mathfrak{S}_1^-$.

The partitioning of the parameter space into genus 0 and genus 1 regions constitutes our first main result. We recall from Chapter 2 that the h-function must satisfy the following conditions:

$$h(z;s) \text{ is analytic for } z \in \mathbb{C} \setminus (\mathfrak{C} \cup \mathfrak{M}), \tag{5.2a}$$

$$h_+(z;s) - h_-(z;s) = 4\pi i \eta_j, \qquad z \in \gamma_{c,j}, \qquad j = 0, \ldots, L, \tag{5.2b}$$

$$h_+(z;s) + h_-(z;s) = 4\pi i \omega_j, \qquad z \in \gamma_{m,j}, \qquad j = 0, \ldots, L, \tag{5.2c}$$

$$h(z;s) = -f(z;s) - \ell + 2\log z + \mathcal{O}\left(\frac{1}{z}\right), \qquad z \to \infty \tag{5.2d}$$

$$\Re h(z;s) = \mathcal{O}\left((z \mp 1)^{1/2}\right), \qquad z \to \pm 1, \tag{5.2e}$$

$$\Re h(z;s) = \mathcal{O}\left((z - \lambda)^{3/2}\right), \qquad z \to \lambda, \quad \lambda \in \Lambda \setminus \{\pm 1\}. \tag{5.2f}$$

Furthermore, we may rewrite the inequalities (2.6) and (2.8) as

$$\Re h(z) < 0 \text{ if } z \text{ is an interior point of any bounded complementary arc } \gamma_c \in \mathfrak{C}, \tag{5.3a}$$

$$\Re h(z_0) > 0 \text{ for } z_0 \text{ in close proximity to any interior point of a main arc } \gamma_m \in \mathfrak{M}. \tag{5.3b}$$

We will build solutions to the above problem for all $s \in \mathbb{C}$ by deforming the h functions of [31] and the one constructed in Chapter 4 away from the imaginary axis by the process of continuation in parameter space mentioned above. During this deformation process, we will encounter curves in the parameter space which separate regions corresponding to different genera. We call such curves *breaking curves*, and the set of breaking curves, along with their endpoints, is denoted $\mathfrak{B}$. The description of the set $\mathfrak{B}$ is our first main result.

Theorem 5.11. *There are two critical breaking points at $s = \pm 2$ and $\mathfrak{B} = \mathfrak{b}_{-\infty} \cup \mathfrak{b}_{\infty} \cup \mathfrak{b}_{+} \cup \mathfrak{b}_{-} \cup \{\pm 2\}$. Here, $\mathfrak{b}_{-\infty} = (-\infty, -2)$ and $\mathfrak{b}_{\infty} = (2, \infty)$. The breaking curve $\mathfrak{b}_{+}$ connects -2 and 2 while remaining in the upper half plane, and the breaking curve $\mathfrak{b}_{-}$ is obtained by reflecting $\mathfrak{b}_{+}$ about the real axis.*

As seen in Figure 5.2, the set $\mathfrak{B}$ divides the parameter space into three simply connected components: $\mathfrak{G}_0$ and $\mathfrak{G}_1^{\pm}$. The notion of critical breaking point, as introduced in the theorem above, will be defined in due course. Yet, it is not surprising given Figure 5.2 that the behavior of the polynomials near $s = \pm 2$ will be qualitatively different from the behavior near other points on the breaking curve. We will see that the region $\mathfrak{G}_0$ corresponds to the genus 0 region, whereas the regions $\mathfrak{G}_1^{\pm}$ correspond to genus 1 regions.

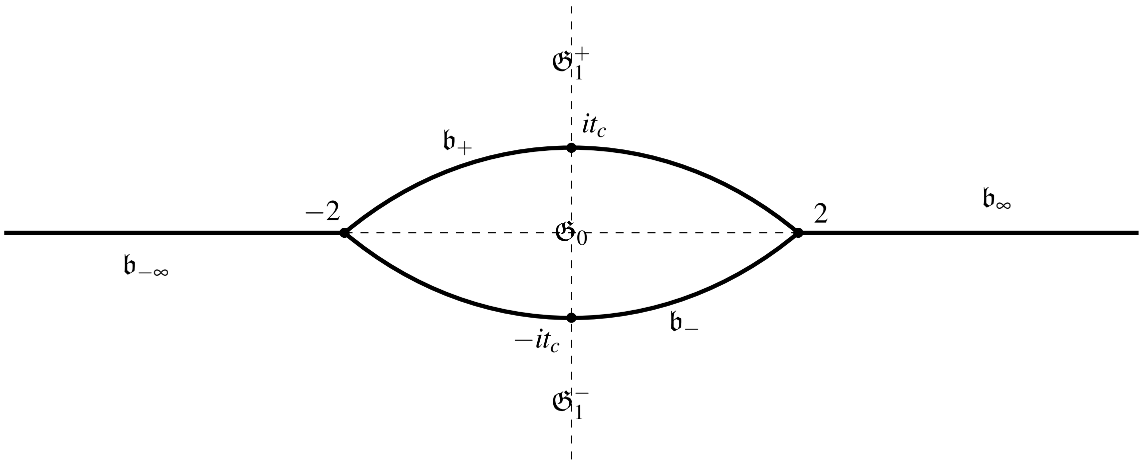


Figure 5.2: Definitions of the regions $\mathfrak{G}_0$ and $\mathfrak{G}_1^{\pm}$ in the s-plane. The set $\mathfrak{B}$ is drawn in bold. The regular breaking points $\pm it_c$ are indicated on the breaking curves $\mathfrak{b}^{\pm}$.

Having determined the description of the set $\mathfrak{B}$, we will be able to deduce asymptotic formulas for the recurrence coefficients for the orthogonal polynomials defined in (5.1) for all $s \in \mathbb{C} \setminus \mathfrak{B}$ via deformation techniques. We quickly digress to discuss notation before stating these results. We first introduce monic polynomials, $p_n^N(z;s)$ which satisfy the following orthogonality conditions.

$$\int_{-1}^{1} p_n^N(z;s) z^k e^{-Nf(z;s)}\, dz = 0, \qquad k = 0, 1, \ldots, n-1, \tag{5.4}$$

where N is a fixed integer. Note that for each $N \in \mathbb{N}$, we have a family of polynomials $\{p_n^N(z;s)\}_{n=0}^{\infty}$. The polynomials that we consider in (5.1) are given by $p_n(z;s) = p_n^n(z;s)$; that is, we consider the polynomials along the diagonal where $N = n$. Provided the polynomials exist for the appropriate values for n, N, and s, they satisfy the following three term recurrence relation

$$zp_n^N(z;s) = p_{n+1}^N(z;s) + \alpha_n^N(s)p_n^N(z;s) + \beta_n^N(s)p_{n-1}^N(z;s). \tag{5.5}$$

In the present work, we concern ourselves with the situation $N = n$, and for sake of notation we set $\alpha_n := \alpha_n^n$ and $\beta_n := \beta_n^n$. It should be stressed that the polynomials p_{n-1}, p_n and p_{n+1} do *not* satisfy the recurrence relation (5.5). We now state our second result, on the asymptotics of the recurrence coefficients in the region $\mathfrak{G}_0$.

Theorem 5.12. *Let $s \in \mathfrak{G}_0$. Then the recurrence coefficients α_n and β_n exist for large enough n, and they satisfy*

$$\alpha_n(s) = \frac{2s}{(s^2-4)^2}\frac{1}{n^2} + \mathscr{O}\left(\frac{1}{n^3}\right), \tag{5.6a}$$

and

$$\beta_n(s) = \frac{1}{4} + \frac{s^2+4}{4(s^2-4)^2}\frac{1}{n^2} + \mathscr{O}\left(\frac{1}{n^4}\right), \tag{5.6b}$$

as $n \to \infty$.

As mentioned above, for $s \in \mathfrak{G}_1^{\pm}$, the underlying Riemann surface has genus 1. Indeed, the Riemann surface corresponds to the algebraic equation $\xi^2 = Q(z;s)$, where Q is a rational function, and we take the branch cuts for the Riemann surface on two arcs - one connecting -1 to $\lambda_0(s)$, labeled $\gamma_{m,0}$, and the other connecting $\lambda_1(s)$ to 1, labeled $\gamma_{m,1}$, where λ_0 and λ_1 will be determined. Moreover, for $s \in \mathfrak{G}_1^{\pm}$, the asymptotics of the recurrence coefficients will depend on theta functions on our Riemann surface. As shown in Chapter 2, these theta functions will be used to construct functions $\mathscr{M}_1(z,k)$ and $\mathscr{M}_2(z,k)$, along with a constant d, whose precise descriptions are provided in Section 2.3.2. Of particular importance to us at the moment is (2.72), which implies that we want both $\mathscr{M}_1(\infty,d)$ and $\mathscr{M}_2(\infty,-d)$ to be separated from 0.

Therefore, keeping in mind (2.72), for $N = n$ and given $\varepsilon > 0$, we define $\mathbb{N}(s,\varepsilon)$ to be a sequence of indices, n, such that $\mathscr{M}_1(z,d), \mathscr{M}_2(z,d)$ are non-vanishing in $\{z : |z| > 1/\varepsilon\}$. In particular,

$$\mathbb{N}(s,\varepsilon) := \left\{n \in \mathbb{Z} : |\mathscr{M}_i(\infty,(-1)^{i-1}d)| > C_i(s,\varepsilon), \quad i = 1,2\right\},$$

for some constants C_i. To make use of these functions in our asymptotic analysis, we need to know that the cardinality of the set $\mathbb{N}(s,\varepsilon)$ is not finite. This is asserted in the following result,

Lemma 5.13. *For all $n \geq 1$ and $\varepsilon > 0$ small enough, if $n \notin \mathbb{N}(s,\varepsilon)$, then $n+1 \in \mathbb{N}(s,\varepsilon)$.*

Intuitively, taking limits within the set $\mathbb{N}(s,\varepsilon)$ in this chapter plays a similar role to taking limits away from the set Θ_ε^* in Theorem 4.26 in the previous chapter. Indeed, in both situations, these restrictions are imposed to ensure the construction of a suitable global parametrix. After proving the above Lemma, our next result concerns the asymptotics of the recurrence coefficients for $s \in \mathfrak{G}_1^{\pm}$.

Theorem 5.15. *Let $s \in \mathfrak{G}_1^{\pm}$ and $n \in \mathbb{N}(s,\varepsilon)$. Then the recurrence coefficients α_n and β_n exist for large enough n, and they satisfy*

$$\alpha_n(s) = \frac{\lambda_0^2(s) - \lambda_1^2(s)}{4 + 2\lambda_0(s) - 2\lambda_1(s)} + \frac{d}{dz}\left[\log \mathscr{M}_1(1/z,d) - \log \mathscr{M}_1(1/z,-d)\right]\Big|_{z=0} + \mathscr{O}_\varepsilon\left(\frac{1}{n}\right), \tag{5.7}$$

and

$$\beta_n(s) = \frac{(2+\lambda_0(s)-\lambda_1(s))^2}{16}\frac{\mathscr{M}_1(\infty,-d)\mathscr{M}_2(\infty,d)}{\mathscr{M}_1(\infty,d)\mathscr{M}_2(\infty,-d)} + \mathscr{O}_\varepsilon\left(\frac{1}{n}\right), \tag{5.8}$$

as $n \to \infty$.

Above, the notation $f(n) = \mathscr{O}_\varepsilon(1/n)$ indicates that there exists a constant which depends only on ε, $M = M(\varepsilon)$, such that $|f(n)| \leq M/n$ for large enough n. We recall that the parameter ε is used to define the set of valid indices, $\mathbb{N}(s,\varepsilon)$, along which we take limits. Having determined the asymptotics of the recurrence coefficients of the polynomials in (5.1) when $s \in \mathbb{C} \setminus \mathfrak{B}$, our final two results recover these asymptotics when $s \in \mathfrak{B}$.

As seen in Theorem 5.11, the breaking curves $\mathfrak{b}_{-\infty}$ and $\mathfrak{b}_\infty$ are the intervals $(-\infty,-2)$ and $(2,\infty)$, respectively. The theory of orthogonal polynomials with respect to real weights, varying or otherwise, has been written about extensively in the literature, most notably from the viewpoint of potential theory. In particular, the results of Deift, Kriecherbauer, and McLaughlin in [38] can be applied in conjunction with the GRS program to show that the empirical zero counting measure of the polynomials in (5.1) converge to a continuous measure supported on the interval $[-1,1]$ as $n \to \infty$, when $s \in \mathbb{R}$ and $|s| < 2$. The results of [38] can also be used to show that the corresponding limit measure is supported on $[-1,a)$ for some $a < 1$ when $s > 2$. Similarly, one also has that this measure is supported on $(b,1]$ for some $b > -1$ when $s \in \mathbb{R}$ is such that $s < -2$. The difference in the support of the limiting measure when $|s| > 2$ and $|s| < 2$ is also of interest in random matrix theory, and occurs when the soft edge meets the hard edge (see the work of Claeys and Kuijlaars [29]). From the viewpoint of the present work, these transitions at $s = \pm 2$ can be seen to come from the fact these are critical breaking points.

As the case where $s \in \mathbb{R} \cap \mathfrak{B}$ has been extensively studied, we next consider the asymptotic behavior of the recurrence coefficients as we approach a regular breaking point which is not on the real line. More precisely, we let s_* be a regular breaking point in $\mathfrak{B} \setminus ((-\infty,2] \cup [2,\infty))$ and we let s approach s_* as

$$s = s_* + \frac{L_1}{n}, \tag{5.9}$$

where $L_1 \in \mathbb{C}$ is such that $s = s(n) \in \mathfrak{G}_0$ for large enough n. The scaling limit (5.9) is referred to as the double scaling limit, as it describes the behavior of the polynomials as both $n \to \infty$ and $s \to s_*$. This formulation then leads to the following description of the recurrence coefficients in the aforementioned double scaling limit.

Theorem 5.19. *Let $s_* \in \mathfrak{B} \setminus ((-\infty, 2] \cup [2, \infty))$ and let $s \to s_*$ as described in* (5.9). *Then the recurrence coefficients exist for large enough n, and they satisfy*

$$\alpha_n(s) = \frac{\delta_n \left(s^2 + 2s\left(\frac{4}{s^2} - 1\right)^{1/2} - 4\right)}{\sqrt{\pi} s^3} \frac{1}{n^{1/2}} + \frac{2\delta_n^2 \left(s^2 + 4s\left(\frac{4}{s^2} - 1\right)^{1/2} - 8\right)}{\pi s^5} \frac{1}{n} + \mathscr{O}\left(\frac{1}{n^{3/2}}\right), \quad (5.10a)$$

and

$$\beta_n(s) = \frac{1}{4} + \frac{\delta_n}{2\sqrt{\pi} s}\left(\frac{4}{s^2} - 1\right)^{1/2} \frac{1}{n^{1/2}} - \frac{\delta_n^2}{2\pi s^2} \frac{1}{n} + \mathscr{O}\left(\frac{1}{n^{3/2}}\right), \quad (5.10b)$$

as $n \to \infty$, where

$$\delta_n = \delta_n(L_1) = e^{-in\kappa} \exp\left(L_1 \left(\frac{4}{s_*^2} - 1\right)^{1/2}\right), \qquad \kappa \in \mathbb{R}. \quad (5.11)$$

The constant $\kappa \in \mathbb{R}$ will be determined in Section 5.3. Note that

$$|\delta_n| = \exp\left(\Re\left[L_1 \left(\frac{4}{s_*^2} - 1\right)^{1/2}\right]\right), \quad (5.12)$$

as $\kappa \in \mathbb{R}$ and that the recurrence coefficients decay at a rate of $n^{1/2}$. In particular, the modulus of δ_n does not depend on n.

Now, we are just left with investigating the behavior of the recurrence coefficients for s near the critical breaking points $s = \pm 2$. For brevity, we focus just on the case $s = 2$. To state our results, we consider the Painlevé II equation [77, Chapter 32]:

$$q''(x) = xq(x) + 2q^3(x) - \alpha, \qquad \alpha \in \mathbb{C}. \quad (5.13)$$

Next, let $q = q(w)$ be the generalized Hastings-McLeod solution to Painlevé II with the parameter $\alpha = 1/2$, which is characterized by the following asymptotic behavior

$$q(x) = \begin{cases} \sqrt{-\frac{x}{2}} + \mathscr{O}\left(\frac{1}{x}\right), & x \to -\infty \\ \frac{1}{2x} + \mathscr{O}\left(\frac{1}{x^4}\right) & x \to \infty. \end{cases} \quad (5.14)$$

In order to study the asymptotics of the recurrence coefficients as $s \to 2$, we take s in a double scaling regime near this critical point as

$$s = 2 + \frac{L_2}{n^{2/3}}, \quad (5.15)$$

where we impose that $L_2 < 0$. This leads us to our final main finding.

Theorem 5.22. *Let $s \to 2$ as described in* (5.15). *Then the recurrence coefficients exist for large enough n, and they satisfy*

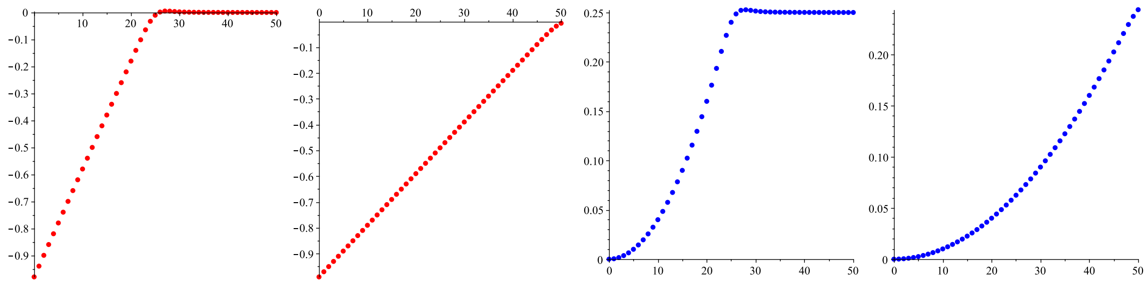
$$\alpha_n(s) = -\frac{q^2(-L_2) + q'(-L_2)}{n^{2/3}} + \mathscr{O}\left(\frac{1}{n}\right), \tag{5.16a}$$

and

$$\beta_n(s) = \frac{1}{4} - \frac{q^2(-L_2) + q'(-L_2)}{2} \frac{1}{n^{2/3}} + \mathscr{O}\left(\frac{1}{n}\right), \tag{5.16b}$$

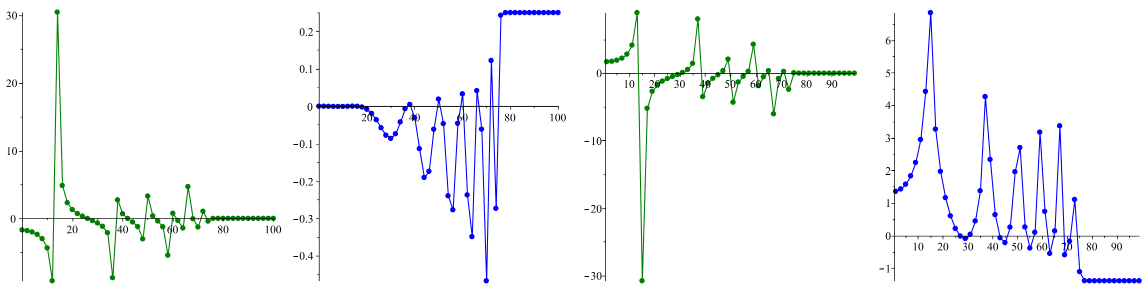
as $n \to \infty$, where q is the generalized Hastings-McLeod solution to Painlevé II with parameter $\alpha = 1/2$. Furthermore, the function $U(w) = q^2(w) + q'(w)$ is free of poles for $w \in \mathbb{R}$.

Plots of the recurrence coefficients are given in Figures 5.3 and 5.4, and should be compared with Theorems 5.12 and 5.22.



(a) $\alpha_n(1)$. **(b)** $\alpha_n(2)$. **(c)** $\beta_n(1)$. **(d)** $\beta_n(2)$.

Figure 5.3: Plots of $\alpha_n(s)$ and $\beta_n(s)$ for $n = 0, \ldots, 50$, with $s = 1, 2$.



(a) $\Im \alpha_{2n}(i)$. **(b)** $\Re \beta_{2n}(i)$. **(c)** $\Im \alpha_{2n+1}(i)$. **(d)** $\log \Re \beta_{2n+1}(i)$.

Figure 5.4: Plots of $\Im \alpha_n(s)$ and $\Re \beta_n(s)$ for $n = 0, \ldots, 100$, with $s = i$.

Figures 5.1, 5.3, and 5.4 have been computed using the nonlinear discrete string equations for the recurrence coefficients presented in [9, Theorem 2, Theorem 4], see also [89, §5.2]. In Figure 5.4, we have used from [26] that $\beta_n(s) \in \mathbb{R}$ and $\alpha_n(s) \in i\mathbb{R}$ when $s \in i\mathbb{R}$. Moreover, it was also shown in [26] that for fixed n, $\alpha_n(it)$ and $\beta_{2n+1}(it)$ will have poles (as a function of t) for $t \in \mathbb{R}$. As such, we have plotted $\Re\beta_{2n+1}$ on a log scale in Figure 5.4d. Once the recurrence coefficients $\alpha_n(s)$ and $\beta_n(s)$ have been computed, we assemble the Jacobi matrix for the orthogonal polynomials and calculate its eigenvalues, which correspond to the zeros of $p_{50}(z;s)$, as explained for instance in [27]. Calculations have been done in MAPLE, using an extended precision of 100 digits.

5.2 The Global Phase Portrait - Continuation in Parameter Space

Below, we will first define breaking points and breaking curves. The set of breaking curves along with their endpoints will be denoted as $\mathfrak{B}$ and we will show that the inequalities (5.3) can only break down as we cross a breaking curve. We then construct the breaking curves for the weight function under consideration in this chapter.

5.2.1 Breaking Curves

Following [13], we define a breaking point as follows: $s_b \in \mathbb{C}$ is a breaking point if there exists a saddle point $z_0 \in \Omega(s)$ such that

$$h'(z_0;s_b) = 0, \qquad \text{and} \qquad \Re h(z_0;s_b) = 0. \tag{5.17}$$

Above, we also impose that the zero of h' is of at least order 1. We call a breaking point *critical* if either:

(i) The saddle point in (5.17) coincides with a branchpoint in $\Lambda(s)$, or

(ii) The order of the zero at the saddle point is greater than one or there are at least two saddle points of h on Ω counted with multiplicity.

If a breaking point s is not a critical breaking point, it is a *regular breaking point*.

Remark 5.1. Note that h' is analytic in $\mathbb{C}\setminus\mathfrak{M}(s)$. In the above definition of breaking point, if $z_0 \in \mathfrak{M}(s)$, we mean $h'(z_0) = 0$ in the following sense. We note that $h'_+(z)$ and $h'_-(z)$ have analytic extensions to a neighborhood of $z_0 \in \mathfrak{M}(s)$. Moreover, in this neighborhood, the two extensions are related via $h'_+(z) = -h'_-(z)$. Therefore, if z_0 is such that $h'_+(z_0) = 0$ (where here we are referring to the extension so this is well defined), then $h'_-(z_0) = 0$, so we say $h'(z_0) = 0$.

We have the following lemma from [13, Lemma 4.3], and we include the proof for convenience.

Lemma 5.2. *Let $s = s_1 + is_2$ where $s_1, s_2 \in \mathbb{R}$ and let s_b be a regular breaking point. If $\partial_{s_k} h(z_0;s_b) \neq 0$ for at least one of $k = 1,2$, then there a exists smooth curve passing through s_b consisting of breaking points.*

Proof. Writing $z = u + iv$ and $s = s_1 + is_2$, we may consider (5.17) to be a system of 3 real equations in 4 real unknowns in the form $G(u, v, s_1, s_2) = 0$. We may choose either $j = 1$ or $j = 2$ so that $\Re\partial_{s_j} h(z_0; s_b) \neq 0$. Then, as $h'(z_0; s_b) = 0$, we may calculate the Jacobian as

$$\det\left(\frac{\partial G}{\partial(u, v, s_j)}\right) = i^{j-1}\Re h_{s_j}(z_0; s_b)\begin{vmatrix} \frac{\partial}{\partial u}\Re h'(z_0; s_b) & \frac{\partial}{\partial v}\Re h'(z_0; s_b) \\ \frac{\partial}{\partial u}\Im h'(z_0; s_b) & \frac{\partial}{\partial v}\Im h'(z_0; s_b) \end{vmatrix}$$
$$= i^{j-1}\Re h_{s_j}(z_0; s_b)\left|h''(z_0; s_b)\right|^2,$$

where we have used the Cauchy-Riemann equations for the second equality above. As $h'' \neq 0$, as s_b is a regular breaking point, the Implicit Function Theorem completes the proof.

□

The curves in Lemma 5.2 are defined to be *breaking curves*. We will see that the breaking curves partition the parameter space so as to separate regions of different genus of h function, as they are precisely where the inequalities on h break down.

Lemma 5.3. *Let $s(t)$ for $t \in [0, 1]$ be a smooth curve in the parameter space starting from $s_0 = s(0)$ and ending at $s_1 = s(1)$. Assume further that $s(t)$ is a regular point for all $0 \leq t < 1$. Then, the inequalities* (5.3) *do not hold at s_1 if and only if s_1 is a breaking point.*

Proof. To see this, first consider the case that the inequality (5.3b) breaks down in a vicinity of z_0, where z_0 is an interior point of a main arc. By definition, $\Re h(z) = 0$ for all interior points z of a main arc, so clearly we must have that $\Re h(z_0; s_1) = 0$. To show that s_1 is a breaking point, we must just show that $h'(z_0; s_1) = 0$. To get a contradiction, assume that $h'(z_0; s_1) \neq 0$. As h_+ is analytic at z_0 and its derivative doesn't vanish, we may write that $h'_+(z) = c + (z - z_0)a(z)$, where a is analytic in a neighborhood of z_0 and does not vanish in this neighborhood and c is a purely imaginary constant. Therefore, $\Re h_+(z)$ does not change sign in close proximity to z_0 on the $+$-side of the cut, and as $h = h_+$ here, the real part of h does not change on the $+$ side of the cut in close proximity of z_0. A similar argument applied to h_- shows that the real part of h does not change on the $-$-side of the cut in close proximity of z_0, either. As $\Re h(z; s(t)) > 0$ for all z in close proximity of a main arc for $t < 1$, we have that by continuity in s and by the constant sign of $\Re h(z; s_1)$ in close proximity to z_0 that $\Re h(z; s_1) > 0$ for all z in close proximity to z_0. This is precisely the inequality which we have assumed to have broken down, so we have reached the desired contradiction. As such, $h'(z_0; s_1) = 0$, and s_1 is a breaking point. Going the other way, we have that the real part of h_+ must change sign above/below the cut if $h'_\pm(z_0) = 0$, which clearly violates inequality (5.3b).

Next, assume that inequality (5.3a) breaks down at z_0, where z_0 is an interior point of a complementary arc, γ_c. Given that $\Re h(z; s(t)) < 0$ for all interior points of a complementary arc, we have by continuity that if the inequality breaks down for s_1 at some point z_0, we must have that $\Re h(z_0; s_1) = 0$. We are now left to show that $h'(z_0) = 0$. To get a contradiction, assume that $h'(z_0) \neq 0$. Then, there is a zero level curve of $\Re h(z)$ passing through z_0 which looks locally like an analytic arc (that is, no intersections). Furthermore, the sign of $\Re h(z)$ is constant on either side of γ_c in close proximity to z_0. By continuity, we have that $\Re h(z;s_1) < 0$ for all interior points $z \in \gamma_c \setminus \{z_0\}$. Therefore, we are able to deform the complementary arc back into the region where $\Re h(z) < 0$ for all $z \in \gamma_c$, contradicting the assumption the inequality was violated. Therefore, we must have that $h'(z_0;s_1) = 0$, and as such s_1 is a breaking point. On the other hand, assume that s_1 is a breaking point. Then as $\Re h(z_0;s_1) = 0$, we clearly have that the strict inequality (5.3a) is violated at z_0. Moreover, the condition that $h'(z_0) = 0$ enforces that we can not deform the complementary arc so as to fix the inequality.

□

5.2.2 The Genus 0 and 1 h-functions

In this section, we review the previous work in the literature for polynomials of the form (5.1) where $s \in i\mathbb{R}$.

Genus 0

The case where $s = -it$ and $0 < t < t_c$ was studied in [31]. We recall that t_c was defined as the unique positive solution to

$$2\log\left(\frac{2+\sqrt{t^2+4}}{t}\right) - \sqrt{t^2+4} = 0. \tag{5.18}$$

We want to show that we may extend the results of [31], by using the technique of continuation in parameter space discussed above, to construct a genus 0 h-function which satisfies both (5.2) and (5.3). In order to state some of the results from [31], we first define

$$h'(z;s) = \frac{2-sz}{(z^2-1)^{1/2}}. \tag{5.19}$$

Next, we consider the quadratic differential $\varpi_s := -h'(z;s)^2\,dz^2$. The following is a restatement of [31, Theorem 2.1].

Lemma 5.4. *Let $s = -it$ where $0 < t < t_c$. There exists a smooth curve $\gamma_{m,0}(s)$ connecting -1 and 1 which is a trajectory of the quadratic differential ϖ_s.*

With this lemma in hand, we take the branch cut of (5.19) on $\gamma_{m,0}(s)$, with the branch chosen so that

$$h'(z;s) = -s + \frac{2}{z} + \mathcal{O}\left(\frac{1}{z^2}\right), \qquad z \to \infty. \tag{5.20}$$

The critical graph of ϖ_s is depicted in Figure 5.5. We see that there are four trajectories emanating from the double zero at $z = 2i/t = 2/s$, two of which form a loop surrounding the endpoints -1 and 1. We may now extend this critical graph from the subset of the imaginary axis to all $s \in \mathfrak{G}_0$.

Lemma 5.5. *For all $s \in \mathfrak{G}_0$, there exists a smooth curve $\gamma_{m,0}(s)$ connecting -1 and 1 which is a trajectory of the quadratic differential ϖ_s.*

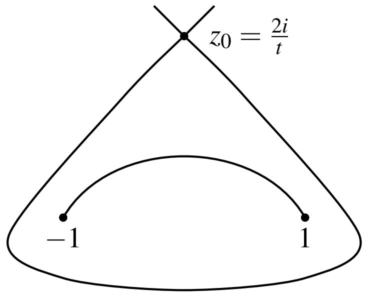


Figure 5.5: Critical Graph of $-h'^2\,dz^2$ for h' defined in (5.19) and $s = -it$ with $0 < t < t_c$.

Proof. Fix some $s_0 = -it$ with $0 < t < t_c$ and some $s_1 \in \mathfrak{G}_0$. The goal is to show that there exists a trajectory of ϖ_{s_1} which connects -1 to 1. As $\mathfrak{G}_0$ is simply connected, we may connect s_0 to s_1 with a curve that lies completely within $\mathfrak{G}_0$, which we call ρ. As we deform s along ρ towards s_1, we note that the topology of the critical graph of ϖ_s will only change if a trajectory emanating from $2/s$ ever meets $\gamma_{m,0}(s)$. Assume for sake of contradiction, there existed some $s_* \in \rho$ for which this occurred. We would then have $\Re h(z;s_*) = 0$ for $z \in \gamma_{m,0}(s)$, as it is a trajectory of the quadratic differential ϖ_s. Moreover, we would also have that $h'(2/s_*;s_*) = 0$ as $2/s_*$ is a zero of $h'(z;s_*)$. In other words, s_* is a breaking point. However, this contradicts the fact that ρ lies completely within $\mathfrak{G}_0$, which by definition contains no breaking points in its interior. As such, the topology of the critical graph at s_1 is the same as it was at s_0, and we conclude that there exists a trajectory of ϖ_{s_1} connecting -1 and 1. □

In light of the lemma above, we keep the notation of $\gamma_{m,0}(s)$ to be the trajectory of ϖ_s which connects -1 and 1. We then have $\Omega(s) := \gamma_{c,0} \cup \gamma_{m,0}(s)$, where we recall $\gamma_{c,0} = (-\infty, -1]$. Now, consider the function

$$h(z;s) = \int_1^z h'(u;s)\,du, \tag{5.21}$$

where the path of integration is taken in $\mathbb{C} \setminus \Omega(s)$.

Lemma 5.6. *Let $s \in \mathfrak{G}_0$. Then, $h(z;s)$ defined in* (5.21) *solves the Riemann-Hilbert problem* (5.2) *and satisfies the inequalities* (5.3).

Proof. It is clear that h is analytic in $\mathbb{C} \setminus \Omega(s)$. Next, note that $\Re h(z;s) \to 0$ as $z \to 1$ and $\Re h(z;s)$ is constant along $\gamma_{m,0}(s)$, as it is a trajectory of ϖ_s. Therefore, we have that $\Re h(z;s) = 0$ for $z \in \gamma_{m,0}(s)$. As $h'_+ = -h'_-$ on $\gamma_{m,0}$, we have that $h_+(z) + h_-(z) = 0$ for $z \in \gamma_{m,0}$, so that h satisfies the appropriate jump over $\gamma_{m,0}$. Next, a residue calculation gives us that $h_+(z) - h_-(z) = 4\pi i$ for $z \in \gamma_{c,0}$.

We can integrate (5.21) directly to find,

$$h(z;s) = 2\log\left(z + \left(z^2-1\right)^{1/2}\right) - s\left(z^2-1\right)^{1/2}. \tag{5.22}$$

From this, we can deduce that

$$h(z;s) = -sz + 2\log 2 + 2\log z + \mathcal{O}\left(\frac{1}{z}\right), \qquad z \to \infty, \tag{5.23}$$

so that h satisfies (5.2d). Finally, we have from (5.19) that $\Re h(z) = \mathcal{O}\left(\sqrt{z \mp 1}\right)$ as $z \to \pm 1$, so that the h constructed above satisfies all of the requirements of (5.2).

To see that $h(z;s)$ satisfies (5.3), we note that the inequalities were proven directly in [31] for $s = -it$ with $0 < t < t_c$. By using Lemma 5.3, we see that the inequalities will hold for all $s \in \mathfrak{G}_0$, completing the proof. □

With the genus 0 h-function constructed explicitly for all $s \in \mathfrak{G}_0$, we now turn to the genus 1 case.

Genus 1

The genus 1 case is slightly more involved, but as before, we will deform the existing solution on the imaginary axis to all other values of s. Therefore, we start with defining

$$h'(z;s) = -s\left(\frac{(z-\lambda_0(s))(z-\lambda_1(s))}{z^2-1}\right)^{1/2}, \tag{5.24}$$

and we now set $\varpi_s := -h'(z;s)^2\,dz^2$, where h' is defined in (5.24). It was shown in Chapter 4, in particular Theorem 4.15, that for $s = -it$ where $t > t_c$, there exist trajectories of the quadratic differential ϖ_s connecting -1 to λ_0 and λ_1 to 1. Here, λ_0 and λ_1 satisfy

$$\lambda_0 + \lambda_1 = \frac{4}{s}, \qquad \Re \oint_C h'(z)\,dz = 0, \tag{5.25}$$

and where C is any loop on the Riemann surface $\mathfrak{R}$ associated to the algebraic equation $y^2 = (h')^2$.

Note that the first condition in (5.25) ensures that

$$h'(z) = -s + \frac{2}{z} + \mathcal{O}\left(\frac{1}{z^2}\right), \qquad z \to \infty. \tag{5.26}$$

The second condition of (5.25) is the Boutroux Condition, as discussed in Section 4.2.1. The critical graph of ϖ_s for $s \in i\mathbb{R} \cap \mathfrak{G}_1^-$ as proven in Chapter 4 is displayed in Figure 5.6. In this case, the critical graph is symmetric with respect to the imaginary axis, and there exists a trajectory connecting -1 to λ_0 and one connecting $\lambda_1 = -\overline{\lambda_0}$ to 1.

We consider the case $s \in \mathfrak{G}_1^-$. As in the proof of Lemma 5.5, we note that for any $s \in \mathfrak{G}_1^-$, there will exist trajectories connecting -1 to λ_0 and λ_1 to 1, which we define to be $\gamma_{m,0}(s)$ and $\gamma_{m,1}(s)$, respectively. Further, we define $\gamma_{c,1}$ to be a curve connecting λ_0 to λ_1 along which $\Re h(z) < 0$, whose existence is guaranteed by the definition of $\mathfrak{G}_1^-$. Finally, assume for now that we may deform s within $\mathfrak{G}_1^-$ so as to preserve the conditions (5.25). Then, for $s \in \mathfrak{G}_1^-$ we have $\Omega(s) = \gamma_{c,0} \cup \gamma_{m,0} \cup \gamma_{c,1} \cup \gamma_{m,1}$

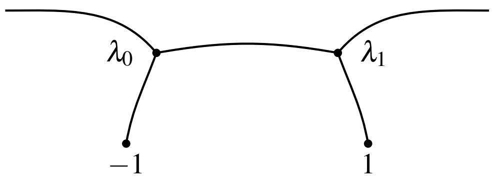


Figure 5.6: Critical Graph of $-h'^2\,dz^2$ for h' defined in (5.24) and $s \in i\mathbb{R}$ with $\Im s < -t_c$.

and we define

$$h(z;s) = \int_1^z h'(u;s)\,du, \tag{5.27}$$

where the path of integration is taken in $\mathbb{C} \setminus \Omega(s)$ and h' is given in (5.24). We have the following Lemma, which shows that the so-constructed h function is the correct one needed for genus 1 asymptotics.

Lemma 5.7. *Let $s \in \mathfrak{S}_1^-$. Then, $h(z;s)$ defined in* (5.27) *solves the Riemann-Hilbert problem* (5.2) *and satisfies the inequalities* (5.3).

Proof. Again, it is immediate that h is analytic in $\mathbb{C} \setminus \Omega(s)$ and has the appropriate endpoint behavior near all endpoints in Λ. Moreover, from the first condition of (5.25), we ensure that h has the correct asymptotics at infinity. The Boutroux condition ensures that we have a purely imaginary jump over $\gamma_{c,1}$ and the same residue calculation as in the genus 0 case yields that $h_+(z) - h_-(z) = 4\pi i$ for $z \in \gamma_{c,0}$. Finally, as $\Re h(z) = 0$ for $z \in \mathfrak{M}$, along with $h'_+(z) + h'_-(z) = 0$ for $z \in \mathfrak{M}$ and the Boutroux condition, we have that $h_+ + h'_-$ is purely imaginary on the main arcs $\gamma_{m,0}$ and $\gamma_{m,1}$.

As before, the inequalities (5.3) were established in Theorem 4.15 directly for $s \in i\mathbb{R}$ with $\Im s < -t_c$, so we may again use Lemma 5.3 to show that the inequalities continue to hold for all $s \in \mathfrak{S}_1^-$. □

Therefore, it is left to show that we may deform s while preserving (5.25). Denoting $\lambda_0 = u_0 + iv_0$ and $\lambda_1 = u_1 + iv_1$, we may write the conditions (5.25) as $F(s;u_0,v_0,u_1,v_1) = 0$, where $F = (f_1, f_2, f_3, f_4)$ and

$$f_1 = u_0 + u_1 - \Re\frac{4}{s}, \tag{5.28a}$$

$$f_2 = v_0 + v_1 - \Im\frac{4}{s}, \tag{5.28b}$$

$$f_3 = \Re\oint_A h'(z)\,dz, \tag{5.28c}$$

$$f_4 = \Re\oint_B h'(z)\,dz. \tag{5.28d}$$

Note that f_3 and f_4 are equivalent to the Boutroux condition, as any loop on $\mathfrak{R}$ may be written as a combination of the A and B cycle on $\mathfrak{R}$. Taking the Jacobian of the above conditions with respect to the endpoints yields

$$\nabla F = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \Re \oint_A h'_{\lambda_0}\, dz & \Im \oint_A h'_{\lambda_0}\, dz & \Re \oint_A h'_{\lambda_1}\, dz & \Im \oint_A h'_{\lambda_1}\, dz \\ \Re \oint_B h'_{\lambda_0}\, dz & \Im \oint_B h'_{\lambda_0}\, dz & \Re \oint_B h'_{\lambda_1}\, dz & \Im \oint_B h'_{\lambda_1}\, dz \end{pmatrix}, \tag{5.29}$$

where

$$h'_{\lambda_j}(z) = \frac{-1}{2(z-\lambda_j)} h'(z), \qquad j = 1, 2. \tag{5.30}$$

As $\lambda_0 \neq \lambda_1$, since we are at a regular point, note that

$$\left(h'_{\lambda_1}(z) - h'_{\lambda_0}(z)\right) dz \tag{5.31}$$

is the unique (up to multiplicative constant) holomorphic differential on $\mathfrak{R}$. Subtracting the first and second columns from the third and fourth columns, we find

$$\det \nabla F = \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \Re \oint_A h'_{\lambda_0}\, dz & \Im \oint_A h'_{\lambda_0}\, dz & \Re \mathscr{A} & \Im \mathscr{A} \\ \Re \oint_B h'_{\lambda_0}\, dz & \Im \oint_B h'_{\lambda_0}\, dz & \Re \mathscr{B} & \Im \mathscr{B} \end{pmatrix}, \tag{5.32}$$

where

$$\mathscr{A} = \oint_A \left(h'_{\lambda_1}(z) - h'_{\lambda_0}(z)\right) dz, \qquad \mathscr{B} = \oint_B \left(h'_{\lambda_1}(z) - h'_{\lambda_0}(z)\right) dz. \tag{5.33}$$

That is, $\mathscr{A}$ and $\mathscr{B}$ are the A and B periods of a holomorphic differential on $\mathfrak{R}$, and the determinant is given by

$$\det \nabla F = \Im(\overline{A}B) > 0, \tag{5.34}$$

which follows from Riemann's Bilinear inequality (c.f. [22, Section 1.4] or [42, Theorem 1.4]). As this determinant is non-zero, we can deform the endpoints continuously in s so as to preserve (5.25), verifying that for all $s \in \mathfrak{G}_1^-$, we may construct a genus 1 h-function.

The case $s \in \mathfrak{G}_1^+$ may be easily obtained via reflection. To see this, note that if $s \in \mathfrak{G}_1^+$, then $-s \in \mathfrak{G}_1^-$. Take $\lambda_0(s) = -\lambda_0(-s)$ and $\lambda_1(s) = -\lambda_1(-s)$, so that $h'(z;s) = -h'(-z;-s)$, and we may use the results for $-s \in \mathfrak{G}_1^-$ to construct the appropriate genus 1 h-function.

5.2.3 Proof of Theorem 5.11

We recall that the aim of Theorem 5.11 is to verify that Figure 5.2 is the accurate picture of the set of breaking curves in the parameter space.

As the genus of $\mathfrak{R}(s)$ is either 0 or 1, we have that the genus must be 0 along a breaking curve. That is, $\Omega(s) = \gamma_{c,0} \cup \gamma_{m,0}$. We have seen in (5.22) that the regular genus 0 h-function is given by

$$h(z;s) = 2\log\left(z + \left(z^2-1\right)^{1/2}\right) - s\left(z^2-1\right)^{1/2}. \tag{5.35}$$

Remark 5.8. Note that there is one other genus zero h function which occurs when $s \in \mathbb{R}$ and $|s| > 2$. Here, we have that

$$h'(z) = \sqrt{\frac{z-\lambda_1(s)}{z-1}}, \qquad \text{or} \qquad h'(z) = \sqrt{\frac{z-\lambda_2(s)}{z+1}},$$

with a cut taken on the real line connecting λ_1 and 1 or λ_2 and -1, depending on the situation. However, neither of these h-functions admit saddle points, so they do not need to be considered when looking for breaking points.

By looking at (5.35), we see that the only saddle point is at $z_0 = 2/s$. As this is a simple zero of h', it follows that the only critical breaking points occur when the saddle point coincides with the branchpoints in $\Lambda(s)$. That is, the only critical breaking points are $s = \pm 2$. We now have the following simple calculation.

Proposition 5.9. *If s_b is a regular breaking point, then*

$$\frac{d}{ds}h\left(\frac{2}{s_b}, s_b\right) \neq 0. \tag{5.36}$$

Proof. We write

$$h\left(\frac{2}{s}, s\right) = 2\log\left(\frac{2}{s} + \left(\frac{4}{s^2}-1\right)^{1/2}\right) - s\left(\frac{4}{s^2}-1\right)^{1/2}, \tag{5.37}$$

so that

$$h'\left(\frac{2}{s}, s\right) = -\left(-1+\frac{4}{s^2}\right)^{1/2}. \tag{5.38}$$

Note that this vanishes only for $s = \pm 2$, which are critical breaking points, so that the proposition above is true for all regular breaking points. □

By Lemma 5.2, the above proposition immediately implies the following, just as in [13, Corollary 6.2].

Corollary 5.10. *Breaking curves are smooth, simple curves consisting of regular breaking points (except possibly the endpoints). They do not intersect each other except perhaps at critical breaking points or at infinity. They can originate and end only at critical breaking points and at infinity.*

Now, we can indeed verify that the global phase portrait depicted in Figure 5.2 is the correct picture.

Theorem 5.11. *There are two critical breaking points at* $s = \pm 2$ *and* $\mathfrak{B} = \mathfrak{b}_{-\infty} \cup \mathfrak{b}_{\infty} \cup \mathfrak{b}_{+} \cup \mathfrak{b}_{-} \cup \{\pm 2\}$. *Here,* $\mathfrak{b}_{-\infty} = (-\infty, -2)$ *and* $\mathfrak{b}_{\infty} = (2, \infty)$. *The breaking curve* $\mathfrak{b}_{+}$ *connects* -2 *and* 2 *while remaining in the upper half plane, and the breaking curve* $\mathfrak{b}_{-}$ *is obtained by reflecting* $\mathfrak{b}_{+}$ *about the real axis.*

Proof. To find the breaking curves, we recall that the only saddle point occurs at

$$z_0(s) = \frac{2}{s}, \tag{5.39}$$

so that the breaking curves are part of the zero level set

$$\Re\left(2\log\left(\frac{2}{s} + \left(\frac{4}{s^2} - 1\right)^{1/2}\right) - s\left(\frac{4}{s^2} - 1\right)^{1/2}\right) = 0. \tag{5.40}$$

Recall also, that the only critical breaking points are $s = \pm 2$, at which the saddle point collides with the hard edge at ± 1, respectively. As $h(2/s, s) = \mathscr{O}\left((s \mp 2)^{3/2}\right)$ as $s \to \pm 2$, we note that 3 breaking curves emanate from each of ± 2.

Now, if $s \in \mathbb{R}$ and $|s| > 2$, then

$$-s\left(\frac{4}{s^2} - 1\right)^{1/2} \in i\mathbb{R},$$

where we have taken the branch cut to be the interval $[-1, 1]$. Furthermore, recall that the map $z \to z + \left(z^2 - 1\right)^{1/2}$ sends the interval $(-1, 1)$ to the unit circle. As such, we also have that

$$2\log\left(\frac{2}{s} + \left(\frac{4}{s^2} - 1\right)^{1/2}\right) \in i\mathbb{R}$$

when $s \in \mathbb{R}$ and $|s| > 2$. Therefore, the rays $(2, \infty)$ and $(-\infty, -2)$ are both breaking curves. Finally, note that

$$h\left(\frac{2}{s}, s\right) = -is + i\pi + \mathscr{O}\left(\frac{1}{s}\right), \qquad s \to \infty, \tag{5.41}$$

so that the two rays emanating from ± 2 towards infinity along the real axis are the only two portions of the breaking curve which intersect at infinity.

According to Corollary 5.10, the remaining breaking curves either emanate from ± 2 or form closed loops in the s-plane consisting of only regular breaking points. As $h(2/s; s)$ has non-zero real part for $s \in (-2, 2)$, we conclude that the remaining breaking curves do not intersect the real axis. Next, note that $\Re h(2/s; s)$ is harmonic for s off the real axis, so that off the real axis, there are no closed loops along which $\Re h(2/s, s) = 0$. Therefore, the remaining breaking curves begin and end at ± 2. Finally, as

$$h\left(\frac{2}{\bar{s}}, \bar{s}\right) = \overline{h\left(\frac{2}{s}, s\right)}, \tag{5.42}$$

we see that the breaking curves which connect -2 and 2 are symmetric about the real axis. $\square$

5.2.4 Proof of Theorem 5.12

Having successfully verified the global phase portrait is as depicted in Figure 5.2, with $\mathfrak{G}_0$ corresponding to the genus 0 region and $\mathfrak{G}_1^{\pm}$ corresponding to the genus 1 regions, we may now use the techniques illustrated in Section 2.2.3 to obtain asymptotics of the recurrence coefficients for $s \in \mathbb{C} \setminus \mathfrak{B}$.

With the h function for $s \in \mathfrak{G}_0$ in hand, we now follow the procedure described in Chapter 2 to both construct the global parametrix M and unwind the transformations to arrive at a small norm Riemann-Hilbert problem for R.

For $s \in \mathfrak{G}_0$, we are in the genus 0 region and as such we will use the global parametrix defined in Subsection 2.3.1. We recall that the global parametrix is given in (2.43) as

$$M(z) = \frac{1}{\sqrt{2}\,(z^2-1)^{1/4}} \begin{pmatrix} \varphi(z)^{1/2} & i\varphi(z)^{-1/2} \\ -i\varphi(z)^{-1/2} & \varphi(z)^{1/2} \end{pmatrix}, \tag{5.43}$$

where $\varphi(z) = z + (z^2-1)^{1/2}$, with the branch cut taken on $\gamma_{m,0}$ so that $\varphi(z) = 2z + \mathscr{O}(1/z)$, $(z^2-1)^{1/4} = z^{1/2} + \mathscr{O}(z^{-3/2})$ as $z \to \infty$. From this, we immediately see that

$$M(z) = I + \frac{M_1}{z} + \frac{M_2}{z^2} + \mathscr{O}\left(\frac{1}{z^3}\right), \qquad z \to \infty, \tag{5.44}$$

where

$$M_1 = \begin{pmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} \frac{1}{8} & 0 \\ 0 & \frac{1}{8} \end{pmatrix}. \tag{5.45}$$

Recall from (2.41),

$$\alpha_n = \frac{[T_2]_{12}}{[T_1]_{12}} - [T_1]_{22}, \qquad \beta_n = [T_1]_{12}\,[T_1]_{21}, \tag{5.46}$$

where T_1 and T_2 are defined as the following terms of the asymptotic expansion of T at infinity,

$$T(z) = I + \frac{T_1}{z} + \frac{T_2}{z^2} + \mathscr{O}\left(\frac{1}{z^3}\right), \qquad z \to \infty.$$

In Section 2.2.2, we stated that R has an asymptotic expansion of the form

$$R(z) = I + \sum_{k=1}^{\infty} \frac{R_k(z)}{n^k}, \qquad n \to \infty, \tag{5.47}$$

which is valid uniformly near infinity and each R_k satisfies

$$R_k(z) = \frac{R_k^{(1)}}{z} + \frac{R_k^{(2)}}{z^2} + \mathscr{O}\left(\frac{1}{z^3}\right), \qquad z \to \infty. \tag{5.48}$$

Recalling that $T(z) = R(z)M(z)$ for large enough z, we may write

$$T_1 = M_1 + \frac{R_1^{(1)}}{n} + \frac{R_2^{(1)}}{n^2} + \mathscr{O}\left(\frac{1}{n^3}\right), \qquad n \to \infty, \tag{5.49a}$$

and

$$T_2 = M_2 + \frac{R_1^{(1)}M_1 + R_1^{(2)}}{n} + \frac{R_2^{(1)}M_1 + R_2^{(2)}}{n^2} + \mathscr{O}\left(\frac{1}{n^3}\right), \qquad n \to \infty, \tag{5.49b}$$

and as such we turn our attention to determining R_1 and R_2.

We recall the discussion in Section 2.2.2, where we wrote $j_R(z) = I + \Delta(z)$, where Δ admits an asymptotic expansion in inverse powers of n as

$$\Delta(z) \sim \sum_{k=1}^{\infty} \frac{\Delta_k(z)}{n^k}, \qquad n \to \infty. \tag{5.50}$$

As $\Delta(z)$ decays exponentially quickly for $z \in \Sigma_R \setminus \cup_{\lambda\in\Lambda} \partial D_\lambda$, we have that

$$\Delta_k(z) = 0, \qquad z \in \Sigma_R \setminus \bigcup_{\lambda\in\Lambda} \partial D_\lambda. \tag{5.51}$$

The behavior of $\Delta_k(z)$ for $z \in \partial D_\lambda$ can be determined in terms of the appropriate local parametrix used at the particular $\lambda \in \Lambda$.

We give an explicit formula for $\Delta_k(z)$ for $z \in \partial D_1$ following [65, Section 8]. We compute that the Bessel parametrix defined in (2.95) satisfies

$$B(\zeta) = \frac{1}{\sqrt{2}}(2\pi)^{-\sigma_3/2}\zeta^{-\sigma_3/4}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\left(I + \sum_{k=1}^{\infty} \frac{B_k}{\zeta^{k/2}}\right) e^{2\zeta^{1/2}\sigma_3} \tag{5.52}$$

uniformly as $\zeta \to \infty$, where the matrices B_k are defined as

$$B_k := \frac{(-1)^{k-1}\prod_{j=1}^{k-1}(2j-1)^2}{4^{2k-1}(k-1)!}\begin{pmatrix} \frac{(-1)^k}{k}\left(\frac{k}{2}-\frac{1}{4}\right) & -i\left(k-\frac{1}{2}\right) \\ (-1)^k i\left(k-\frac{1}{2}\right) & \frac{1}{k}\left(\frac{k}{2}-\frac{1}{4}\right) \end{pmatrix}. \tag{5.53}$$

As $j_R(z) = P^{(1)}(z)M^{-1}(z) - I$ for $z \in \partial D_1$, we may use (2.91c)-(2.97) to see that

$$\Delta(z) = P^{(1)}(z)M^{-1}(z) - I = M(z)\left[\sum_{k=1}^{\infty} \frac{4^k B_k}{n^k h(z)^k}\right] M^{-1}(z), \qquad n \to \infty, \tag{5.54}$$

so that we have by direct inspection,

$$\Delta_k(z) = \frac{(-1)^{k-1}\prod_{j=1}^{k-1}(2j-1)^2}{4^{k-1}(k-1)!h(z)^k} M(z)\begin{pmatrix} \frac{(-1)^k}{k}\left(\frac{k}{2}-\frac{1}{4}\right) & -i\left(k-\frac{1}{2}\right) \\ (-1)^k i\left(k-\frac{1}{2}\right) & \frac{1}{k}\left(\frac{k}{2}-\frac{1}{4}\right) \end{pmatrix} M^{-1}(z), \tag{5.55}$$

for $z \in \partial D_1$. Defining $\tilde{h}(z) = h(z) - 2\pi i$, we are able to similarly compute that

$$\Delta_k(z) = \frac{(-1)^{k-1}\prod_{j=1}^{k-1}(2j-1)^2}{4^{k-1}(k-1)!\tilde{h}(z)^k} M(z) \begin{pmatrix} \frac{(-1)^k}{k}\left(\frac{k}{2}-\frac{1}{4}\right) & i\left(k-\frac{1}{2}\right) \\ (-1)^{k+1} i\left(k-\frac{1}{2}\right) & \frac{1}{k}\left(\frac{k}{2}-\frac{1}{4}\right) \end{pmatrix} M^{-1}(z), \tag{5.56}$$

when $z \in \partial D_{-1}$. It was also shown in [65, Section 8] that we may write

$$\Delta_1(z) = \begin{cases} \dfrac{A^{(1)}}{z-1} + \mathscr{O}(1), & z \to 1, \\ \dfrac{B^{(1)}}{z+1} + \mathscr{O}(1), & z \to -1, \end{cases} \tag{5.57}$$

for some constant matrices $A^{(1)}$ and $B^{(1)}$. Using the behavior of h defined in (5.22) and φ near ± 1, we find that

$$A^{(1)} = \frac{1}{8(s-2)} \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix}, \qquad B^{(1)} = \frac{1}{8(s+2)} \begin{pmatrix} -1 & -i \\ -i & 1 \end{pmatrix}. \tag{5.58}$$

We recall from Section 2.2.2 that the Δ_k may be used to solve for the R_k via the following Riemann-Hilbert problem:

$$R_k(z) \text{ is analytic for } z \in \mathbb{C} \setminus (\partial D_1 \cup \partial D_{-1}), \tag{5.59a}$$

$$R_{k,+}(z) = R_{k,-}(z) + \sum_{j=1}^{k-1} R_{k-j,-}\Delta_j(z), \qquad z \in \partial D_1 \cup \partial D_{-1}, \tag{5.59b}$$

$$R_k(z) = \mathscr{O}\left(\frac{1}{z}\right), \qquad z \to \infty. \tag{5.59c}$$

Having determined the $\Delta_k(z)$ for $z \in \partial D_{\pm 1}$, we may solve for the R_k directly. By inspection, we see that

$$R_1(z) = \begin{cases} \dfrac{A^{(1)}}{z-1} + \dfrac{B^{(1)}}{z+1}, & z \in \mathbb{C} \setminus (D_1 \cup D_{-1}), \\ \dfrac{A^{(1)}}{z-1} + \dfrac{B^{(1)}}{z+1} - \Delta_1(z), & z \in D_1 \cup D_{-1}, \end{cases} \tag{5.60}$$

solves the Riemann-Hilbert problem (5.59) for R_1.

To determine R_2, we again follow [65] where it was shown

$$R_1(z)\Delta_1(z) + \Delta_2(z) = \begin{cases} \dfrac{A^{(2)}}{z-1} + \mathscr{O}(1), & z \to 1, \\ \dfrac{B^{(2)}}{z+1} + \mathscr{O}(1), & z \to -1, \end{cases} \tag{5.61}$$

for some constant matrices $A^{(2)}$ and $B^{(2)}$. As we now have explicit formulas for R_1, Δ_1, and Δ_2, we may use the properties of h and φ to obtain

$$A^{(2)} = \frac{1}{16(s-2)^2(s+2)} \begin{pmatrix} \frac{s-2}{4} & i(2s+5) \\ -i(2s+5) & \frac{s-2}{4} \end{pmatrix} \tag{5.62a}$$

and

$$B^{(2)} = \frac{1}{16(s-2)(s+2)^2} \begin{pmatrix} -\frac{s+2}{4} & i(2s-5) \\ -i(2s-5) & -\frac{s+2}{4} \end{pmatrix}. \tag{5.62b}$$

Having determined the $A^{(2)}$ and $B^{(2)}$, we may again solve the Riemann-Hilbert problem for R_2 by inspection as

$$R_2(z) = \begin{cases} \dfrac{A^{(2)}}{z-1} + \dfrac{B^{(2)}}{z+1}, & z \in \mathbb{C} \setminus (D_1 \cup D_{-1}), \\ \dfrac{A^{(2)}}{z-1} + \dfrac{B^{(2)}}{z+1} - R_1(z)\Delta_1(z) - \Delta_2(z), & z \in D_1 \cup D_{-1}. \end{cases} \tag{5.63}$$

Now, we may expand the R_k at infinity to determine the appropriate terms in (5.49). As $R_k(z) = A^{(k)}/(z-1) + B^{(k)}/(z+1)$ for $k = 1, 2$ and $z \in \mathbb{C} \setminus (D_1 \cup D_{-1})$, we have that

$$R_k(z) = \frac{A^{(k)} + B^{(k)}}{z} + \frac{A^{(k)} - B^{(k)}}{z^2} + \mathscr{O}\left(\frac{1}{z^3}\right), \qquad z \to \infty. \tag{5.64}$$

Using the explicit formula for the $A^{(k)}$ and $B^{(k)}$, we obtain

$$R_1^{(1)} = \frac{1}{4\,(4-s^2)} \begin{pmatrix} s & -2i \\ -2i & -s \end{pmatrix}, \qquad R_1^{(2)} = \frac{1}{4\,(4-s^2)} \begin{pmatrix} 2 & -is \\ -is & -2 \end{pmatrix} \tag{5.65a}$$

$$R_2^{(1)} = \frac{i(s^2+5)}{4\,(s^2-4)^2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad R_2^{(2)} = \frac{1}{32\,(s^2-4)^2} \begin{pmatrix} s^2-4 & 36is \\ -36is & s^2-4 \end{pmatrix} \tag{5.65b}$$

Finally, using (5.45) and (5.65) in (5.46) and (5.49), we see that we have successfully proven the theorem below.

Theorem 5.12. *Let $s \in \mathfrak{S}_0$. Then the recurrence coefficients α_n and β_n exist for large enough n, and they satisfy*

$$\alpha_n(s) = \frac{2s}{(s^2-4)^2} \frac{1}{n^2} + \mathscr{O}\left(\frac{1}{n^3}\right), \tag{5.66a}$$

and

$$\beta_n(s) = \frac{1}{4} + \frac{s^2+4}{4(s^2-4)^2} \frac{1}{n^2} + \mathscr{O}\left(\frac{1}{n^4}\right), \tag{5.66b}$$

as $n \to \infty$.

5.2.5 Proof of Theorem 5.15

For $s \in \mathfrak{G}_1^{\pm}$, the h-function is of genus 1, and we must use the global parametrix constructed in Section 2.3.2. We recall Lemma 2.2, which states that we may construct a global parametrix provided

$$n(\omega_0 + \Delta_0) \neq \frac{1}{2} + \frac{\tau}{2} \mod \Lambda_\tau, \tag{5.67}$$

where ω_0 is the constant from (5.2c), τ is given by (2.49), $\Delta_0 = \eta_1 \tau$ (where η_1 is defined in 5.2b) and Λ_τ is the period lattice associated with τ.

The above degeneracy condition can be qualified via the following Lemma. We recall that for fixed $\varepsilon > 0$, the set $\mathbb{N}(s,\varepsilon)$ is the set of all indices, n, for which there exist constants $C_i(s,\varepsilon) > 0,\ i = 1,2$ such that

$$|\mathscr{M}_i(\infty,(-1)^{i-1}d)| > C_i(s,\varepsilon), \quad i = 1,2.$$

Lemma 5.13. *For all $n \geq 1$ and $\varepsilon > 0$ small enough, if $n \notin \mathbb{N}(s,\varepsilon)$, then $n+1 \in \mathbb{N}(s,\varepsilon)$.*

Proof. First, we remark that for notational convenience that we have dropped the dependence of the functions $\mathscr{M}_i$ on n. However, this distinction will play a role in this proof, so we now use the notation $\mathscr{M}_{1,n}(z,d)$.

Denote by z_n the zero of $\mathscr{M}_{1,n}(z,d)$. As $\frac{1}{2} + \frac{\tau}{2}$ is the only zero of the theta function (which is also simple), it follows from (2.67) and (2.70) that the zero of $\mathscr{M}_{1,n}(z,d)$ is defined by the equation

$$u(z_n) - n(\Delta_0 + \omega_0) - u(\infty_1) = \frac{1}{2} + \frac{\tau}{2} \mod \Lambda_\tau. \tag{5.68}$$

Next, note that when the degeneracy (5.67) takes place, that is when

$$n(\omega_0 + \Delta_0) = \frac{1}{2} + \frac{\tau}{2} \mod \Lambda_\tau, \tag{5.69}$$

we have that $z_n = \infty_1$. Moreover, (5.68) applied to both n and $n+1$ yields

$$u(z_{n+1}) - u(z_n) = \Delta_0 + \omega_0 \mod \Lambda_\tau.$$

Now, let $\varepsilon_0 > 0$ be such that for all $\varepsilon < \varepsilon_0$, $n \notin \mathbb{N}(s,\varepsilon)$. For the sake of a contradiction, assume $n+1 \notin \mathbb{N}(s,\varepsilon)$. Then, taking $\varepsilon_0 \to 0$, the above equation immediately yields that $0 = \Delta_0 + \omega_0 \mod \Lambda_\tau$. However, by deforming the contour and using the expansion (5.26), one can check (as shown in [31] and Chapter 4) that

$$\frac{1}{2\pi i} h'_+(z;s)\, dz,\ z \in \mathfrak{M}$$

is a probability measure on $\mathfrak{M}$ and that $\Delta_0 = \tau\eta_1$, where η_1 is the measure of $\gamma_{m,1}$. Hence, $\eta_1 \in (0,1)$. As the period lattice is given by $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$, and as $\omega_0 \in \mathbb{R}$ and $\operatorname{Im}\tau > 0$, we have $\Delta_0 + \omega_0 \not\equiv 0 \mod \Lambda_\tau$, reaching a contradiction. Similar considerations can be given to $\mathscr{M}_{2,n}(z,s)$. □

Throughout this section, we are working with the assumption that $n \in \mathbb{N}(s,\varepsilon)$, so that the global parametrix exists by Lemma 2.2. Recall again (2.41) , which states that

$$\alpha_n = \frac{[T_2]_{12}}{[T_1]_{12}} - [T_1]_{22}, \qquad \beta_n = [T_1]_{12}[T_1]_{21},$$

where T_1 and T_2 are defined via the asymptotic expansion of T at infinity,

$$T(z) = I + \frac{T_1}{z} + \frac{T_2}{z^2} + \mathscr{O}\left(\frac{1}{z^3}\right), \qquad z \to \infty.$$

We see by (2.26), that for for large z, $T(z) = R(z)M(z)$. Therefore, $T_k = M_k + \mathscr{O}\left(\frac{1}{n}\right)$ as $n \to \infty$ for $k = 1,2$, so we have that

$$\alpha_n = \frac{[M_2]_{12}}{[M_1]_{12}} - [M_1]_{22} + \mathscr{O}\left(\frac{1}{n}\right), \qquad \beta_n = [M_1]_{12}[M_1]_{21} + \mathscr{O}\left(\frac{1}{n}\right), \tag{5.70}$$

as $n \to \infty$.

Remark 5.14. In order to compute higher order terms in the expansion of the recurrence coefficients in the genus 1 regime, one would again need to write the jump matrix for R as a perturbation of the identity. This would involve writing the jump matrix on ∂D_λ in terms of the appropriate local parametrix used at λ. As the expansion of the Bessel parametrix was given in (5.52), we give the expansion of the Airy parametrix below. Using (2.83) and [1, Section 10.4], we have that

$$A(\zeta) = \frac{1}{2\sqrt{\pi}}\zeta^{-\frac{1}{4}\sigma_3}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}\left(I + \sum_{k=1}^{\infty}\frac{A_k}{\zeta^{\frac{3k}{2}}}\right)e^{-\frac{2}{3}\zeta^{\frac{3}{2}}\sigma_3}, \tag{5.71}$$

uniformly as $\zeta \to \infty$. In the above,

$$A_k = \begin{pmatrix} \left(-\frac{2}{3}\right)^k \frac{c_k+d_k}{2} & \left(\frac{2}{3}\right)^k \frac{d_k-c_k}{2i} \\ \left(-\frac{2}{3}\right)^k \frac{c_k-d_k}{2i} & \left(\frac{2}{3}\right)^k \frac{c_k+d_k}{2} \end{pmatrix} \tag{5.72}$$

and

$$c_k = \frac{\Gamma\left(3k+\frac{1}{2}\right)}{54^k k!\,\Gamma\left(k+\frac{1}{2}\right)}, \qquad d_k = -\frac{6k-1}{6k+1}c_k, \qquad k = 1,2,\ldots \tag{5.73}$$

One could again carry out the process detailed in Section 5.2.4 to obtain higher order terms in the genus 1 regime, but we just concern ourselves with the leading term.

By Lemma 2.2, as $n \in \mathbb{N}(s,\varepsilon)$, the global parametrix is defined as

$$M(z) = e^{n\tilde{g}(\infty)\sigma_3}\mathscr{L}^{-1}(\infty)\mathscr{L}(z)e^{-n\tilde{g}(z)\sigma_3}, \tag{5.74}$$

where we recall from (2.51) and (2.62) that

$$\mathscr{L}(z) := \frac{1}{2}\begin{pmatrix} \left(\eta(z)+\eta(z)^{-1}\right)\mathscr{M}_1(z,d) & i\left(\eta(z)-\eta(z)^{-1}\right)\mathscr{M}_2(z,d) \\ -i\left(\eta(z)-\eta(z)^{-1}\right)\mathscr{M}_1(z,-d) & \left(\eta(z)+\eta(z)^{-1}\right)\mathscr{M}_2(z,-d) \end{pmatrix}, \tag{5.75}$$

and

$$\tilde{g}(z) = \Xi(z)\left[\int_{\gamma_{c,1}} \frac{\eta_1\, d\zeta}{(\zeta-z)\Xi(\zeta)} - \int_{\gamma_{m,0}} \frac{\Delta_0\, d\zeta}{(\zeta-z)\Xi_+(\zeta)}\right]. \tag{5.76}$$

Above, η is defined in (2.58) as

$$\eta(z) = \left(\frac{(z+1)(z-\lambda_1)}{(z-\lambda_0)(z-1)}\right)^{1/4}, \tag{5.77}$$

with branch cuts on $\gamma_{m,0}$ and $\gamma_{m,1}$ and the branch of the root chosen so that $\eta(\infty)=1$ and the constant Δ_0 was chosen to satisfy

$$\int_{\gamma_{c,1}} \frac{\eta_1\, d\zeta}{\Xi(\zeta)} - \int_{\gamma_{m,0}} \frac{\Delta_0\, d\zeta}{\Xi_+(\zeta)} = 0. \tag{5.78}$$

We see that

$$\tilde{g}(z) = \tilde{g}(\infty) + \frac{\tilde{g}_1}{z} + \frac{\tilde{g}_2}{z^2} + \mathscr{O}\left(\frac{1}{z^3}\right), \qquad z\to\infty, \tag{5.79}$$

where

$$\tilde{g}(\infty) = \delta_1, \tag{5.80a}$$

$$\tilde{g}_1 = \delta_2 - \frac{\delta_1(\lambda_0+\lambda_1)}{2}, \tag{5.80b}$$

$$\tilde{g}_2 = \delta_3 - \frac{\delta_2(\lambda_0+\lambda_1)}{2} - \frac{\delta_1\left(4+(\lambda_0-\lambda_1)^2\right)}{8}, \tag{5.80c}$$

and

$$\delta_k := \int_{\gamma_{c,1}} \frac{\zeta^k \eta_1\, d\zeta}{\Xi(\zeta)} - \int_{\gamma_{m,0}} \frac{\zeta^k \Delta_0\, d\zeta}{\Xi_+(\zeta)}. \tag{5.80d}$$

Therefore,

$$e^{-n\tilde{g}(z)\sigma_3} = \left[I - \frac{ng_1\sigma_3}{z} + \frac{n^2\tilde{g}_1^2 I - 2n\tilde{g}_2\sigma_3}{2z^2} + \mathscr{O}\left(\frac{1}{z^3}\right)\right] e^{-n\tilde{g}(\infty)\sigma_3}, \qquad z\to\infty. \tag{5.81}$$

Next we turn to the expansion of the matrix $\mathscr{L}$. We have

$$\mathscr{L}(z) = \mathscr{L}(\infty) + \frac{\mathscr{L}_1}{z} + \frac{\mathscr{L}_2}{z^2} + \mathscr{O}\left(\frac{1}{z^3}\right), \qquad z\to\infty. \tag{5.82}$$

To calculate $\mathscr{L}_1$ and $\mathscr{L}_2$, we first see that by (2.58)

$$\eta(z) = 1 + \frac{n_1}{z} + \frac{n_2}{z^2} + \mathscr{O}\left(\frac{1}{z^3}\right), \qquad z\to\infty, \tag{5.83}$$

where

$$n_1 = \frac{2+\lambda_0-\lambda_1}{4}, \tag{5.84a}$$

and

$$n_2 = \frac{4+4\lambda_0+5\lambda_0^2-4\lambda_1-2\lambda_0\lambda_1-3\lambda_1^2}{32}. \tag{5.84b}$$

This then gives us that

$$\eta(z)+\eta(z)^{-1} = 2+\frac{n_1^2}{z^2}+\mathscr{O}\left(\frac{1}{z^3}\right), \qquad z\to\infty, \tag{5.85a}$$

and

$$\eta(z)-\eta(z)^{-1} = \frac{2n_1}{z}+\frac{2n_2-n_1^2}{z^2}+\mathscr{O}\left(\frac{1}{z^3}\right), \qquad z\to\infty, \tag{5.85b}$$

which implies

$$\mathscr{L}_1 = \begin{pmatrix} \frac{d}{dz}\mathscr{M}_1\left(\frac{1}{z},d\right)\Big|_{z=0} & in_1\mathscr{M}_2(\infty,d) \\ -in_1\mathscr{M}_1(\infty,-d) & \frac{d}{dz}\mathscr{M}_2\left(\frac{1}{z},-d\right)\Big|_{z=0} \end{pmatrix}, \tag{5.86a}$$

and

$$\mathscr{L}_2 = \begin{pmatrix} \frac{1}{2}\mathscr{M}_1(\infty,d)\,n_1^2+\frac{d^2}{dz^2}\mathscr{M}_1\left(\frac{1}{z},d\right)\Big|_{z=0} & \frac{n_1^2-2n_2}{2i}\mathscr{M}_2(\infty,d)+i\eta_1\frac{d}{dz}\mathscr{M}_2\left(\frac{1}{z},d\right)\Big|_{z=0} \\ \frac{2n_2-n_1^2}{2i}\mathscr{M}_1(\infty,-d)-in_1\frac{d}{dz}\mathscr{M}_1\left(\frac{1}{z},-d\right)\Big|_{z=0} & \frac{1}{2}\mathscr{M}_2(\infty,-d)\,n_1^2+\frac{d^2}{dz^2}\mathscr{M}_2\left(\frac{1}{z},-d\right)\Big|_{z=0} \end{pmatrix}. \tag{5.86b}$$

Putting this all together yields

$$M_1 = e^{n\tilde{g}(\infty)\sigma_3}\left[\mathscr{L}^{-1}(\infty)\mathscr{L}_1-ng_1\sigma_3\right]e^{-n\tilde{g}(\infty)\sigma_3}, \tag{5.87a}$$

and

$$M_2 = e^{n\tilde{g}(\infty)\sigma_3}\left[\frac{n^2\tilde{g}_1^2\sigma_3^2-2n\tilde{g}_2\sigma_3}{2}-n\tilde{g}_1\mathscr{L}^{-1}(\infty)\mathscr{L}_1\sigma_3+\mathscr{L}^{-1}(\infty)\mathscr{L}_2\right]e^{-n\tilde{g}(\infty)\sigma_3}. \tag{5.87b}$$

Using this in (5.70), we find that

$$\beta_n = \frac{\mathscr{M}_1(\infty,-d)\mathscr{M}_2(\infty,d)}{\mathscr{M}_1(\infty,d)\mathscr{M}_2(\infty,-d)}n_1^2+\mathscr{O}\left(\frac{1}{n}\right), \qquad n\to\infty, \tag{5.88}$$

and

$$\alpha_n = \frac{n_2}{n_1}-\frac{n_1}{2}+\frac{d}{dz}\left[\log\mathscr{M}_1(1/z,d)-\log\mathscr{M}_1(1/z,-d)\right]\Big|_{z=0}+\mathscr{O}\left(\frac{1}{n}\right). \tag{5.89}$$

Using (5.84), we arrive at the theorem below.

Theorem 5.15. *Let $s \in \mathfrak{G}_1^{\pm}$ and $n \in \mathbb{N}(s,\varepsilon)$. Then the recurrence coefficients α_n and β_n exist for large enough n, and they satisfy*

$$\alpha_n(s) = \frac{\lambda_0^2(s) - \lambda_1^2(s)}{4 + 2\lambda_0(s) - 2\lambda_1(s)} + \frac{d}{dz}\left[\log \mathscr{M}_1(1/z, d) - \log \mathscr{M}_1(1/z, -d)\right]\Big|_{z=0} + \mathscr{O}_\varepsilon\left(\frac{1}{n}\right), \tag{5.90}$$

and

$$\beta_n(s) = \frac{(2 + \lambda_0(s) - \lambda_1(s))^2}{16} \frac{\mathscr{M}_1(\infty, -d)\mathscr{M}_2(\infty, d)}{\mathscr{M}_1(\infty, d)\mathscr{M}_2(\infty, -d)} + \mathscr{O}_\varepsilon\left(\frac{1}{n}\right), \tag{5.91}$$

as $n \to \infty$.

5.3 Double Scaling Limit near Regular Breaking Points

Having determined the behavior of the recurrence coefficients as $n \to \infty$ with $s \in \mathfrak{G}_0 \cup \mathfrak{G}_1^{\pm}$, we turn our attention to the behavior of these coefficients for critical values of $s_* \in \mathfrak{B}$ where $s_* \notin \mathbb{R}$. Below, the double scaling limit describes the asymptotics of the recurrence coefficients as both $n \to \infty$ and $s \to s_*$ simultaneously at an appropriate scaling rate.

5.3.1 Definition of the Double Scaling Limit

In the remainder of this section, we will assume that s approaches s_* within the region $\mathfrak{G}_0$. In particular, we fix $s_* \in \mathfrak{B} \setminus ((-\infty, -2] \cup [2, \infty))$ and take

$$s = s_* + \frac{L_1}{n}, \qquad L_1 \in \mathbb{C}, \tag{5.92}$$

where the constant L_1 is chosen so that $s \in \mathfrak{G}_0$ for all n large enough. Furthermore, we impose that $\Im s_* < 0$, so that $\Im \frac{2}{s_*} > 0$; this requirement is for ease of exposition, and the case where $\Im s_* > 0$ can be handled similarly. As $s \to s_*$ within $\mathfrak{G}_0$, we have that $\Omega(s) = \gamma_{c,0} \cup \gamma_{m,0}(s)$. Furthermore, there exists a genus 0 h-function which satisfies the following:

$$h \text{ is analytic in } \mathbb{C} \setminus \Omega(s), \tag{5.93a}$$

$$h_+(z) - h_-(z) = 4\pi i, \qquad z \in \gamma_{c,0}, \tag{5.93b}$$

$$h_+(z) + h_-(z) = 0, \qquad z \in \gamma_{m,0}, \tag{5.93c}$$

$$h(z) = -sz - l + 2\log z + \mathscr{O}\left(\frac{1}{z}\right), \qquad z \to \infty, \tag{5.93d}$$

$$\Re h(z) = \mathscr{O}\left((z \mp 1)^{1/2}\right), \qquad z \to \pm 1, \tag{5.93e}$$

where $l \in \mathbb{R}$. As s_* is a regular breaking point, we now have that $\Re(h(2/s_*; s_*)) = 0$, by definition, and a more detailed local analysis will be needed in the vicinity of this point.

As the first transformation is the same as the first transformation in Chapter 2, we briefly restate it below. We recall that Y defined in (2.4) solves the Riemann-Hilbert problem (2.3). By setting

$$T(z) := e^{-n\ell\sigma_3/2}Y(z)e^{-\frac{n}{2}[h(z)+f(z)]\sigma_3}, \tag{5.94}$$

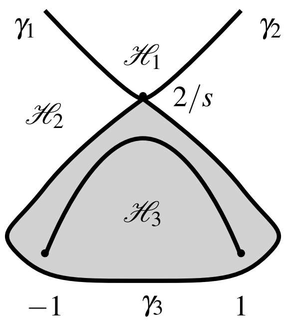
we then have that T defined above solves the Riemann-Hilbert problem (2.16).

5.3.2 Opening of the Lenses

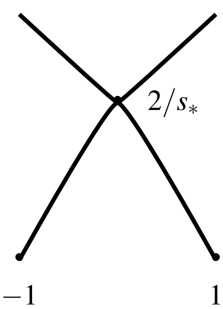
In order to address some of the more technical issues which arise when attempting to open lenses, we turn again to the theory of quadratic differentials. Recall that $\gamma_{m,0}(s)$ is defined to be the trajectory of the quadratic differential

$$\varpi_s = -\frac{(2-sz)^2}{z^2-1}dz^2 \tag{5.95}$$

which connects -1 and 1, whose existence is assured due to Lemma 5.5. Moreover, we also have that four trajectories ϖ_s emanate from $z = 2/s$ at equal angles of $\pi/2$, as described in Section 4.2.2. Finally, an application of Teichmüller's Lemma (c.f. [84, Theorem 14.1]) shows that the trajectories define two infinite sectors and one finite sector whose boundary is formed by a closed trajectory from $z = 2/s$ which encircles both ± 1. Moreover, at the critical value s_*, we have that two trajectories go to infinity from $z = 2/s_*$, and the other two connect $z = 2/s_*$ with ± 1. Another application of Teichmüller's Lemma shows that the two infinite trajectories tend to infinity in opposite directions. The depictions of these critical graphs are given in Figure 5.7; for more details on the precise structure of the the critical graph we refer the reader to [31, Section 3.2].



(a) The critical graph of ϖ_s when $s = -it \in \mathfrak{G}_0$ with $t > 0$. The figure depicts the situation when s is close to s_*. The shaded region is $\mathscr{H}_3$.



(b) The critical graph of ϖ_s when $s = s_*$ where $s_* \in \mathfrak{B} \cap i\mathbb{R}_-$.

Figure 5.7: The critical graphs of ϖ_s for s close to s_* and for $s = s_*$

Recall that the key to the opening of lenses is that the jump matrices decay exponentially fast to the identity along the lips of the lens. In the sections above, this immediately followed from the inequality (5.3b) which stated that sign of the real part of h was greater than zero. However, at the critical value of s_*, this will no longer be true above the critical point $2/s_*$, and a more detailed local analysis will be needed. We label the trajectories emanating from $z = 2/s$ as γ_i, $i = 1,2,3$, and the regions bounded by these trajectories as $\mathscr{H}_j$, $j = 1,2,3$, as in Figure 5.7.

To understand the sign of the real part of h, consider the function

$$\Upsilon(z;s) = \int_{2/s}^{z} \frac{2-su}{(u^2-1)^{1/2}}\,du, \tag{5.96}$$

with the branch cut taken on $\gamma_{m,0}(s)$ and branch chosen so that $\Upsilon(z;s) = -sz + \mathscr{O}(1)$ as $z \to \infty$. In terms of the h-function, we may write

$$h(z;s) = h(2/s;s) + \Upsilon(z;s). \tag{5.97}$$

We may now state the following lemma.

Lemma 5.16. *Fix $s \in \mathfrak{S}_0$ so that $\Im s < 0$. Then,*

$$\Re h\left(\frac{2}{s};s\right) > 0, \tag{i}$$

$$\Re h(z;s) > 0, \qquad z \in \mathscr{H}_2 \cup \mathscr{H}_3. \tag{ii}$$

Proof. By the basic theory (c.f. [74, Appendix B], [55, Chapter 3]) the domains $\mathscr{H}_1$ and $\mathscr{H}_2$ are half plane domains which are conformally mapped by Υ to either the left or right half planes. As $\Im s < 0$, there exists some $t_0 > 0$ so that $z = -it \in \mathscr{H}_2$ for all $t > t_0$. Recalling that

$$\Upsilon(z;s) = -sz + \mathscr{O}(1), \qquad z \to \infty,$$

we may use that $\Im s < 0$ to conclude that $\Re\Upsilon(z;s) > 0$ for $z = -it$, where $t > t_0$. Therefore, we must have that Υ conformally maps $\mathscr{H}_2$ to the right half plane and as such

$$\Re\Upsilon(z;s) > 0, \qquad z \in \mathscr{H}_2. \tag{5.98}$$

Similarly, as Υ is analytic around $z = 2/s$ and has a double zero at $z = 2/s$, we can conclude that $\Re\Upsilon(z;s) < 0$ for z in $\mathscr{H}_1 \cup \mathscr{H}_3$ in close proximity to $z = 2/s$. As $\mathscr{H}_1$ is a half plane domain, we immediately have that

$$\Re\Upsilon(z;s) < 0, \qquad z \in \mathscr{H}_1. \tag{5.99}$$

Again following the theory laid out in [74, Appendix B], it follows that $\mathscr{H}_3$ is a ring domain. Therefore there exists some $c > 0$ so that the function $z \mapsto \exp(c\Upsilon(z;s))$ maps $\mathscr{H}_1$ conformally to an annulus

$$R = \{w \in \mathbb{C} : r_1 < |w| < 1\}. \tag{5.100}$$

In particular we have that

$$0 > \Re\Upsilon(z;s) > \Re\Upsilon(1,s), \qquad z \in \mathscr{H}_3 \tag{5.101}$$

As $\Upsilon(1;s) = -h(2/s;s)$, we have proven (i). Furthermore, (ii) now follows directly from (5.97), (5.98), and (5.101). $\square$

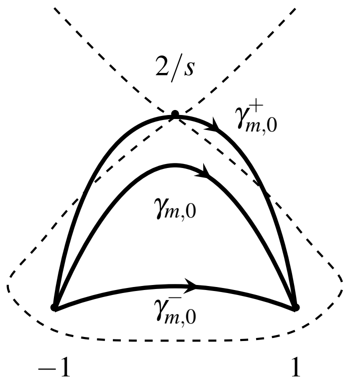


Figure 5.8: Opening of lenses in the double scaling regime near a regular breaking point. The trajectories of ϖ_s are indicated by dashed lines.

We now open lenses as depicted in Figure 5.8. Note that the upper lip of the lens, $\gamma_{m,0}^{+}$ passes through $z = 2/s$ and both $\gamma_{m,0}^{\pm}$ remain entirely within $\mathscr{H}_2 \cup \mathscr{H}_3$. As before, we define $\mathscr{L}_j^{\pm}$ to be the region bounded between the arcs $\gamma_{m,j}$ and $\gamma_{m,j}^{\pm}$, respectively, and set $\hat{\Sigma} := \Sigma \cup_{j=0}^{L} \left(\gamma_{m,j}^{+} \cup \gamma_{m,j}^{-}\right)$. We can now define the third transformation of the steepest descent process as

$$S(z) := \begin{cases} T(z) \begin{pmatrix} 1 & 0 \\ \mp e^{-nh(z)} & 1 \end{pmatrix}, & z \in \mathscr{L}_j^{\pm}, \\ T(z), & \text{otherwise}. \end{cases} \tag{5.102}$$

We then consider the model Riemann-Hilbert problem formed by disregarding the jumps on $\gamma_{m,0}^{\pm}$. In particular, we seek M such that

$$M(z) \text{ is analytic for } z \in \mathbb{C} \setminus \gamma_{m,0}(s), \tag{5.103a}$$

$$M_+(z) = M_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in \gamma_{m,0}, \tag{5.103b}$$

$$M(z) = I + \mathscr{O}\left(\frac{1}{z}\right), \qquad z \to \infty. \tag{5.103c}$$

The solution to this Riemann-Hilbert problem was provided in Chapter 2, see (2.43).

Note that the jump on $\gamma_{m,0}^{+}(s)$ is no longer exponentially decaying to the identity as $s \to s_*$ in a neighborhood of $z = 2/s$. Moreover, the matrix M is not bounded near the endpoints $z = \pm 1$. Therefore, we define $D_c := D_\delta(2/s)$, $D_{-1} := D_\delta(-1)$, and $D_1 := D_\delta(1)$ to be discs of radius δ centered at $z = 2/s, -1$, and 1, respectively. We take δ small enough so that $D_c \cap \gamma_{m,0}^{-} = \emptyset$. Note that for s near s_*, the trajectory $\gamma_{m,0}(s)$ is close to $2/s_*$, so that for n large enough we must have that

$D_c \cap \gamma_{m,0}(s) \neq \emptyset$. In each D_k, $k \in \{c, -1, 1\}$, we seek a local parametrix $P^{(k)}$ such that

$$P^{(k)}(z) \text{ is analytic for } z \in D_\lambda \setminus \hat{\Sigma}, \tag{5.104a}$$

$$P_+^{(k)}(z) = P_-^{(k)}(z) j_S(z), \qquad z \in D_k \cap \hat{\Sigma}, \tag{5.104b}$$

$$P^{(\lambda)}(z) = M(z)\left(I + o(1)\right), \qquad n \to \infty, \quad z \in \partial D_k. \tag{5.104c}$$

As shown in Section 2.4, $P^{(1)}$ and $P^{(-1)}$ are given by

$$\begin{aligned} P^{(1)}(z) &= E_n^{(1)}(z) B\left(f_{n,B}(z)\right) e^{-\frac{n}{2} h(z)\sigma_3}, \\ P^{(-1)}(z) &= E_n^{(-1)}(z) \tilde{B}\left(\tilde{f}_{n,B}(z)\right) e^{-\frac{n}{2} \tilde{h}(z)}, \end{aligned} \tag{5.105a}$$

where $\tilde{h}(z) = h(z) - 2\pi i$, B is the Bessel parametrix defined in (2.95), and $\tilde{B}(z) = \sigma_3 B(z) \sigma_3$. Above,

$$f_{n,B}(z) = \frac{h(z)^2}{16}, \qquad \tilde{f}_{n,B}(z) = \frac{\tilde{h}(z)^2}{16}, \tag{5.106a}$$

$$E_n^{(1)}(z) = M(z) L_n^{(1)}(z)^{-1}, \qquad L_n^{(1)}(z) := \frac{1}{\sqrt{2}} (2\pi n)^{-\sigma_3/2} f_B(z)^{-\sigma_3/4} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \tag{5.106b}$$

and

$$E_n^{(-1)}(z) = M(z) L_n^{(-1)}(z)^{-1}, \qquad L_n^{(-1)}(z) := \frac{1}{\sqrt{2}} (2\pi n)^{-\sigma_3/2} \tilde{f}_B(z)^{-\sigma_3/4} \begin{pmatrix} -1 & i \\ i & -1 \end{pmatrix}. \tag{5.106c}$$

We will now move on to the construction of the local parametrix $P^{(c)}$ within D_c.

5.3.3 Parametrix around the Critical Point

We consider a disc D_c around $z = 2/s$ of small radius δ. We partition D_c into D_c^+ and D_c^- as in Figure 5.9, so that D_c^+ is the region within D_c that lies to the left of $\gamma_{m,0}$ and D_c^- is the region which lies to the right. We define the following function in D_c^+:

$$\tilde{h}_c(z; s) = \int_{2/s_*}^{z} \frac{2 - su}{\left(u^2 - 1\right)^{1/2}} \, du, \qquad z \in D_c^+, \tag{5.107}$$

where the path of integration does not cross $\gamma_{m,0}(s)$. Note that $\tilde{h}_c(z;s)$ is analytic within D_c^+. Next, denote by h_c the analytic continuation of $\tilde{h}_c$ into D_c^-.

In terms of the h function, we may write

$$h_c(z; s) = \begin{cases} h(z; s) - h\left(\frac{2}{s_*}; s\right), & z \in D_c^+, \\ -h(z; s) - h\left(\frac{2}{s_*}; s\right), & z \in D_c^-. \end{cases} \tag{5.108}$$

We have the following lemma, following the lines laid out in [13, Proposition 4.5].

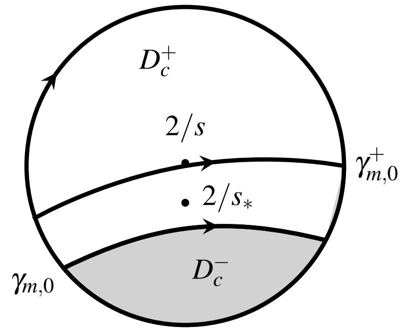


Figure 5.9: Definitions of the regions $D_c^\pm$ within D_c. The region D_c^- is shaded in the figure.

Lemma 5.17. *There exists a jointly analytic function $\zeta(z;s)$ which is univalent in a fixed neighborhood of $z = 2/s_*$, with s in a neighborhood of s_*, and an analytic function $K(s)$ near $s = s_*$ so that*

$$h_c(z;s) = \frac{1}{2}\zeta^2(z;s) + K(s)\zeta(z;s), \tag{5.109}$$

where $K(2/s_*) = 0$ and

$$\zeta\left(\frac{2}{s_*}, s\right) \equiv 0 \tag{5.110}$$

for s in a neighborhood of s_*.

Proof. Define $h_{cr}(s) := h_c(2/s;s)$. Then, we have that

$$h_{cr}(s) = \frac{2}{s_*^3\left(\frac{4}{s_*^2}-1\right)^{1/2}}(s-s_*)^2\left[1+\mathscr{O}(s-s_*)\right]. \tag{5.111}$$

Therefore, we may write

$$h_{cr}(s) = -\frac{1}{2}K^2(s), \tag{5.112}$$

where $K(s)$ is analytic near $s = s_*$ and satisfies

$$K(s) = k_1(s-s_*) + \mathscr{O}(s-s_*)^2, \tag{5.113}$$

with

$$k_1 = \frac{2i}{s_*^{3/2}}\left(\frac{4}{s_*^2}-1\right)^{-1/4}. \tag{5.114}$$

Moreover, we can calculate that

$$h_c(z;s) - h_{cr}(s) = -\frac{s}{2}\left(\frac{4}{s^2}-1\right)^{-1/2}\left(z-\frac{2}{s}\right)^2\left[1+\mathscr{O}\left(z-\frac{2}{s}\right)\right]. \tag{5.115}$$

Next define

$$\frac{\zeta(z;s)}{\sqrt{2}} := \sqrt{h_c(z;s)+\frac{K^2(s)}{2}} - \frac{K(s)}{\sqrt{2}}. \tag{5.116}$$

We immediately have that ζ satisfies (5.109), is conformal map in a neighborhood of $z = 2/s$ and satisfies $\zeta(2/s_*, s) \equiv 0$. □

We now specify that the size of the disc D_c is chosen to be small enough so that $\zeta(z;s) + K(s)$ is conformal for n large enough (or equivalently, when s is close to s_*), which is possible via the lemma above. Moreover, we also impose that the arc $\gamma_{m,0}^+$ is mapped to the real line via $\zeta(z;s) + K(s)$ within D_c.

From the proof of Lemma 5.17, we see that

$$K(s) = \frac{2i}{s_*^{3/2}} \left(\frac{4}{s_*^2} - 1 \right)^{-1/4} (s - s_*) + \mathscr{O}(s - s_*)^2. \tag{5.117}$$

Therefore, we note that the double scaling limit (5.92) can be equivalently stated by taking $n \to \infty$ and $s \to s_*$, so that

$$\lim_{n\to\infty,\, s\to s_*} nK(s) = \frac{2iL_1}{s_*^{3/2}} \left(\frac{4}{s_*^2} - 1 \right)^{-1/4} = L_1 k_1, \tag{5.118}$$

where k_1 is given in (5.114). We may obtain the local parametrix about $z = 2/s$ by solving the following Riemann-Hilbert problem:

$$P^{(c)}(z) \text{ is analytic for } z \in D_c \setminus \hat{\Sigma}, \tag{5.119a}$$

$$P_+^{(c)}(z) = P_-^{(c)}(z) j_S(z), \qquad z \in D_c \cap \hat{\Sigma}, \tag{5.119b}$$

$$P^{(c)}(z) = (I + o(1)) M(z), \qquad n \to \infty, \quad z \in \partial D_c. \tag{5.119c}$$

We recall that the jumps in (5.119b) are given by

$$P_+^{(c)}(z) = P_-^{(c)}(z) \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{-nh(z;s)} & 1 \end{pmatrix}, & z \in D_c \cap \gamma_{m,0}^+(s), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in D_c \cap \gamma_{m,0}(s). \end{cases} \tag{5.120}$$

We solve for $P^{(c)}$ by first defining $U^{(c)}$ so that

$$P^{(c)}(z) = U^{(c)}(z) e^{-\frac{n}{2} h(z) \sigma_3}. \tag{5.121}$$

Then, $U^{(c)}$ is also analytic for $z \in D_c \setminus \hat{\Sigma}$ and satisfies the following jump conditions within D_c:

$$U_+^{(c)}(z) = U_-^{(c)}(z) \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & z \in D_c \cap \gamma_{m,0}^+(s), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in D_c \cap \gamma_{m,0}(s). \end{cases} \tag{5.122}$$

We may solve for $U^{(c)}$ using the error function parametrix presented in [21, Section 7.5]. We introduce

$$C(\zeta) := \begin{pmatrix} e^{\zeta^2} & 0 \\ b(\zeta) & e^{-\zeta^2} \end{pmatrix}, \tag{5.123}$$

where

$$b(\zeta) := \frac{1}{2} e^{-\zeta^2} \begin{cases} \operatorname{erfc}\left(-i\sqrt{2}\zeta\right), & \Im\zeta > 0, \\ -\operatorname{erfc}\left(i\sqrt{2}\zeta\right), & \Im\zeta < 0. \end{cases} \tag{5.124}$$

Then, $C(\zeta)$ is analytic for $\zeta \in \mathbb{C} \setminus \mathbb{R}$ and satisfies

$$C_+(\zeta) = C_-(\zeta) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \zeta \in \mathbb{R}. \tag{5.125}$$

Moreover, it possesses the following asymptotic expansion, uniform in the upper and lower half planes:

$$C(\zeta) = \left(I + \sum_{k=0}^{\infty} \begin{pmatrix} 0 & 0 \\ b_k & 0 \end{pmatrix} \zeta^{-2k-1} \right) e^{\zeta^2 \sigma_3}, \qquad \zeta \to \infty, \tag{5.126}$$

where

$$b_k = \frac{i}{\sqrt{2\pi}} \frac{\Gamma\left(k + \frac{1}{2}\right)}{2^{k+1}\Gamma\left(\frac{1}{2}\right)}. \tag{5.127}$$

Next define,

$$f_{n,C}(z;s) = \left(\frac{n}{2}\right)^{1/2} f_C(z;s), \qquad f_C(z;s) = \frac{1}{\sqrt{2}} \left(\zeta(z;s) + K(s)\right), \tag{5.128}$$

where ζ and K are as defined via Lemma 5.17. Using the proof of Lemma 5.17, we see that $f_C(z;s)$ conformally maps a neighborhood of $z = 2/s$ to a neighborhood of $z = 0$. If we define

$$J(z) = \begin{cases} I, & z \in D_c^+, \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & z \in D_c^-, \end{cases} \tag{5.129}$$

we see that

$$P^{(c)}(z) = E_n^{(c)}(z)C\left(f_{n,C}(z)\right)J(z)e^{-\frac{n}{2}h(z)\sigma_3}, \tag{5.130}$$

where $E_n^{(c)}$ is any matrix which is analytic throughout D_c, solves (5.119a) and (5.119b). We now choose $E_n^{(c)}$ so that $P^{(c)}$ satisfies (5.119c). As $n \to \infty$ for $z \in D_c^+$, we have

$$P^{(c)}(z) = E_n^{(c)}(z)\left(I + \sum_{k=0}^{\infty}\begin{pmatrix}0 & 0\\ b_k & 0\end{pmatrix}\left(\frac{2}{n}\right)^{k+1/2}(f_C(z;s))^{-2k-1}\right)e^{\frac{n}{2}\left[f_C^2(z;s)-h(z;s)\right]\sigma_3}. \tag{5.131}$$

Similarly, we have that as $n \to \infty$ for $z \in D_c^-$,

$$P^{(c)}(z) = E_n^{(c)}(z)\left(I + \sum_{k=0}^{\infty}\begin{pmatrix}0 & 0\\ b_k & 0\end{pmatrix}\left(\frac{2}{n}\right)^{k+1/2}(f_C(z;s))^{-2k-1}\right)e^{\frac{n}{2}\left[f_C^2(z;s)+h(z;s)\right]\sigma_3}J(z). \tag{5.132}$$

Therefore, if we set

$$E_n^{(c)}(z) = M(z)J^{-1}(z)e^{-\frac{n}{2}\left[K^2(s)/2-h(2/s_*;s)\right]\sigma_3} \qquad z \in D_c, \tag{5.133}$$

we see that $P_n^{(c)}(z)$ satisfies the matching condition (5.119c). It is easy enough to see that $E_n^{(c)}$ is analytic within D_c as both M and J have the same jumps over $\gamma_{m,0}$ and are bounded within D_c. Moreover, we see that

$$P^{(c)}(z) = \left(I + n^{-1/2}\sum_{k=0}^{\infty}\frac{P_{k,n}(z;s)}{n^k}\right)M(z), \qquad n \to \infty, \tag{5.134}$$

where

$$P_{k,n}(z;s) = \frac{2^{k+1/2}}{f_C(z;s)^{2k+1}}e^{\frac{n}{2}\left(K^2(s)-2h(2/s_*;s\right)}\begin{cases}\begin{pmatrix}0 & 0\\ b_k & 0\end{pmatrix}, & z \in D_c^+,\\ \begin{pmatrix}0 & -b_k\\ 0 & 0\end{pmatrix}, & z \in D_c^-.\end{cases} \tag{5.135}$$

Now, as $s \to s_*$,

$$K^2(s) - 2h(2/s_*;s) = -2h(2/s_*;s_*) + 2\left(\frac{4}{s_*^2}-1\right)^{1/2}(s-s_*) + k_1^2(s-s_*)^2 + \mathscr{O}(s-s*)^3 \tag{5.136a}$$

$$= -2h(2/s_*;s_*) + 2L_1\left(\frac{4}{s_*^2}-1\right)^{1/2}\frac{1}{n} + \frac{L_1^2k_1^2}{n^2} + \mathscr{O}\left(\frac{1}{n^3}\right). \tag{5.136b}$$

Moreover, as s_* is a regular breaking point, we have that $h(2/s_*;s_*) = i\kappa$, where $\kappa \in \mathbb{R}$. Then, as $n \to \infty$ (and as such $s \to s_*$),

$$e^{\frac{n}{2}\left(K^2(s)-2h(2/s_*;s)\right)} = e^{-in\kappa}\exp\left(L_1\left(\frac{4}{s_*^2}-1\right)^{1/2}\right)\left(1+\frac{L_1^2k_1^2}{2n}+\mathscr{O}\left(\frac{1}{n^2}\right)\right). \tag{5.137}$$

We then have that

$$P^{(c)}(z) = \left(I+n^{-1/2}\sum_{k=0}^{\infty}\frac{P_k(z;s)}{n^k}\right)M(z), \qquad n\to\infty, \tag{5.138}$$

where P_0 is given by

$$P_0(z;s) = \frac{\sqrt{2}\delta_n(L_1)}{f_C(z;s)}\begin{cases}\begin{pmatrix}0 & 0\\ \frac{i}{2\sqrt{2\pi}} & 0\end{pmatrix}, & z\in D_c^+,\\ \begin{pmatrix}0 & -\frac{i}{2\sqrt{2\pi}}\\ 0 & 0\end{pmatrix}, & z\in D_c^-,\end{cases} \tag{5.139}$$

where for ease of notation we have defined

$$\delta_n(L_1) := e^{-in\kappa}\exp\left(L_1\left(\frac{4}{s_*^2}-1\right)^{1/2}\right). \tag{5.140}$$

Note above that $\left|e^{-in\kappa}\right| = 1$ as

$$\kappa = \Im h(2/s_*;s_*). \tag{5.141}$$

5.3.4 Proof of Theorem 5.19

The final transformation is

$$R(z) = S(z)\begin{cases}M(z)^{-1}, & z\in\mathbb{C}\setminus\overline{(D_{-1}\cup D_1\cup D_c)},\\ P^{(-1)}(z)^{-1}, & z\in D_{-1},\\ P^{(1)}(z)^{-1}, & z\in D_1,\\ P^{(c)}(z)^{-1}, & z\in D_c.\end{cases} \tag{5.142}$$

We write the jump matrix $j_R(z) = I+\Delta(z)$, where

$$\Delta(z) = \sum_{k=1}^{\infty}\frac{\Delta_{k/2}(z)}{n^{k/2}}. \tag{5.143}$$

As before, we have that $\Delta_k(z) = 0$ for $z\in\Sigma_R\setminus(\partial D_{-1}\cup\partial D_1\cup\partial D_c)$, as the jump matrix decays exponentially fast to the identity off of the boundaries of the discs D_{-1}, D_1, and D_c. From (5.55), (5.56), and (5.138), we have for $k \in \mathbb{N}$ that

$$\Delta_k(z) = \begin{cases} \dfrac{(-1)^{k-1}\prod_{j=1}^{k-1}(2j-1)^2}{4^{k-1}(k-1)!\tilde{h}(z)^k} M(z) \begin{pmatrix} \frac{(-1)^k}{k}\left(\frac{k}{2}-\frac{1}{4}\right) & i\left(k-\frac{1}{2}\right) \\ (-1)^{k+1} i\left(k-\frac{1}{2}\right) & \frac{1}{k}\left(\frac{k}{2}-\frac{1}{4}\right) \end{pmatrix} M^{-1}(z), & z \in D_{-1}, \\ \dfrac{(-1)^{k-1}\prod_{j=1}^{k-1}(2j-1)^2}{4^{k-1}(k-1)!h(z)^k} M(z) \begin{pmatrix} \frac{(-1)^k}{k}\left(\frac{k}{2}-\frac{1}{4}\right) & -i\left(k-\frac{1}{2}\right) \\ (-1)^{k} i\left(k-\frac{1}{2}\right) & \frac{1}{k}\left(\frac{k}{2}-\frac{1}{4}\right) \end{pmatrix} M^{-1}(z), & z \in D_{1}, \\ 0, & z \in D_c, \end{cases} \tag{5.144a}$$

and

$$\Delta_{k+\frac{1}{2}}(z) = \begin{cases} 0 & z \in D_1 \cup D_{-1}, \\ M(z)P_k(z;s)M^{-1}(z), & z \in D_c, \end{cases} \tag{5.144b}$$

where we have used (5.138). As $\Delta(z)$ possesses the expansion (5.143), we may again use the arguments presented in [39, Section 7] and [65, Section 8] to conclude that R has an asymptotic expansion in inverse powers of $n^{1/2}$ of the form

$$R(z) = \sum_{k=0}^{\infty} \frac{R_{k/2}(z)}{n^{k/2}}, \qquad n \to \infty, \tag{5.145}$$

where each $R_{k/2}$ solves the following Riemann-Hilbert problem:

$$R_{k/2}(z) \text{ is analytic for } z \in \mathbb{C} \setminus (\partial D_{-1} \cup \partial D_{-1} \cup \partial D_c), \tag{5.146a}$$

$$R_{k/2,+}(z) = R_{k/2,-}(z) + \sum_{j=1}^{k-1} R_{(k-j)/2,-}\Delta_{j/2}(z), \qquad z \in \partial D_{-1} \cup \partial D_{-1} \cup \partial D_c, \tag{5.146b}$$

$$R_{k/2}(z) = \frac{R_{k/2}^{(1)}}{z} + \frac{R_{k/2}^{(2)}}{z^2} + \mathscr{O}\left(\frac{1}{z}\right), \qquad z \to \infty. \tag{5.146c}$$

Above, we have $R_0(z) \equiv I$. Following [65], we have the following lemma.

Lemma 5.18.

(i) The restriction of Δ_1 to ∂D_{-1} has a meromorphic continuation to a neighborhood of D_{-1}. This continuation is analytic, except at -1, where Δ_1 has a pole of order 1.

(ii) The restriction of Δ_1 to ∂D_1 has a meromorphic continuation to a neighborhood of D_1. This continuation is analytic, except at 1*, where Δ_1 has a pole of order at most* 1.

(iii) The restriction of $\Delta_{1/2}$ to ∂D_c has a meromorphic continuation to a neighborhood of D_c. This continuation is analytic, except at $2/s$, where $\Delta_{1/2}$ has a pole of order at most 1.

Proof. (i) and (ii) are given in [65, Lemma 8.2], so we prove (iii). As both M and $P_k(z;s)$ are analytic within $D_c^\pm$, we have that $\Delta_{1/2}(z)$ is analytic in both $D_c^\pm$. Furthermore, it is straightforward to check using (5.139) and (5.103b) that

$$\Delta_{1/2,+}(z) = \Delta_{1/2,-}(z), \qquad z \in \gamma_{m,0}, \tag{5.147}$$

so that $\Delta_{1/2}(z)$ is analytic in $D_c \setminus \{2/s\}$. As $f_C(z;s) = \mathscr{O}(z-2/s)$ as $z \to 2/s$, we have by (5.135) that the isolated singularity is pole of order 1. □

Next we recall (2.41)

$$\alpha_n = \frac{[T_2]_{12}}{[T_1]_{12}} - [T_1]_{22}, \qquad \beta_n = [T_1]_{12}[T_1]_{21}, \tag{5.148}$$

where T_1 and T_2 are defined via the expansion of T at infinity,

$$T(z) = I + \frac{T_1}{z} + \frac{T_2}{z^2} + \mathscr{O}\left(\frac{1}{z^3}\right). \tag{5.149}$$

By (5.102) and (5.142) we have that $T(z) = R(z)M(z)$ for z outside of the lens. Using (5.145), we then have that

$$T_1 = M_1 + \frac{R_{1/2}^{(1)}}{n^{1/2}} + \frac{R_1^{(1)}}{n} + \mathscr{O}\left(\frac{1}{n^{3/2}}\right), \qquad n \to \infty, \tag{5.150a}$$

$$T_2 = M_2 + \frac{R_{1/2}^{(1)}M_1 + R_{1/2}^{(2)}}{n^{1/2}} + \frac{R_1^{(1)}M_1 + R_1^{(2)}}{n} + \mathscr{O}\left(\frac{1}{n^{3/2}}\right), \qquad n \to \infty, \tag{5.150b}$$

where M_1 and M_2 were calculated in (5.45) as

$$M_1 = \begin{pmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} \frac{1}{8} & 0 \\ 0 & \frac{1}{8} \end{pmatrix}. \tag{5.151}$$

We first solve for $R_{1/2}(z)$. Using Lemma 5.18, we may write

$$\Delta_{1/2}(z) = \frac{C^{(1/2)}}{z-2/s}, \qquad z \to 2/s, \tag{5.152}$$

for some constant matrix $C^{(1/2)}$. Using the explicit expression (5.144b) for $\Delta_{1/2}$, we can compute $C^{(1/2)}$ as

$$C^{(1/2)} = \frac{\delta_n(L_1)}{2s\sqrt{\pi}} \begin{pmatrix} 1 & -\frac{s\left(\frac{4}{s^2}-1\right)^{1/2}-2}{is} \\ \frac{s\left(\frac{4}{s^2}-1\right)^{1/2}+2}{is} & -1 \end{pmatrix} \tag{5.153}$$

where we have used (5.116) to calculate that

$$f_C(z;s) = -\frac{s}{2}\left(\frac{4}{s^2}-1\right)^{-1/2}\left(z-\frac{2}{s}\right)+\mathcal{O}\left(z-\frac{2}{s}\right)^2. \tag{5.154}$$

Then

$$R_{1/2}(z) := \begin{cases} \dfrac{C^{(1/2)}}{z-2/s}, & z\in\mathbb{C}\setminus D_c, \\ \dfrac{C^{(1/2)}}{z-2/s}-\Delta_{1/2}(z), & z\in D_c, \end{cases} \tag{5.155}$$

solves (5.146) with $k=1$. Next, as shown in (5.57) and (5.58),

$$\Delta_1(z) = \begin{cases} \dfrac{A^{(1)}}{z-1}+\mathcal{O}(1), & z\to 1, \\ \dfrac{B^{(1)}}{z+1}+\mathcal{O}(1), & z\to -1, \end{cases} \tag{5.156}$$

where

$$A^{(1)} = \frac{1}{8(s-2)}\begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix}, \qquad B^{(1)} = \frac{1}{8(s+2)}\begin{pmatrix} -1 & -i \\ -i & 1 \end{pmatrix}. \tag{5.157}$$

We can then compute that

$$R_{1/2}(z)\Delta_{1/2}(z)+\Delta_1(z) = \begin{cases} \dfrac{A^{(1)}}{z-1}+\mathcal{O}(1), & z\to 1, \\ \dfrac{B^{(1)}}{z+1}+\mathcal{O}(1), & z\to -1, \\ \dfrac{C^{(1)}}{z-2/s}+\mathcal{O}(1), & z\to 2/s, \end{cases} \tag{5.158}$$

where

$$C^{(1)} = -\frac{\delta_n^2(L_1)}{4\pi s^2\left(\frac{4}{s^2}-1\right)^{1/2}}\begin{pmatrix} 1 & -\frac{s\left(\frac{4}{s^2}-1\right)^{1/2}-2}{is} \\ \frac{s\left(\frac{4}{s^2}-1\right)^{1/2}+2}{is} & -1 \end{pmatrix}. \tag{5.159}$$

Then,

$$R_1(z) = \begin{cases} \dfrac{A^{(1)}}{z-1}+\dfrac{B^{(1)}}{z+1}+\dfrac{C^{(1)}}{z-2/s}, & z\in\mathbb{C}\setminus(D_{-1}\cup D_1\cup D_c), \\ \dfrac{A^{(1)}}{z-1}+\dfrac{B^{(1)}}{z+1}+\dfrac{C^{(1)}}{z-2/s}-R_{1/2}(z)\Delta_{1/2}(z)-\Delta_1(z), & z\in D_{-1}\cup D_1\cup D_c, \end{cases} \tag{5.160}$$

solves the Riemann-Hilbert problem (5.146) with $k=2$. As we now have explicit expressions for $R_{1/2}$ and R_1, we may expand at infinity to get

$$R_{1/2}^{(1)} = C^{(1/2)}, \qquad R_{1/2}^{(2)} = \frac{2}{s}C^{(1/2)}, \tag{5.161a}$$

$$R_1^{(1)} = A^{(1)} + B^{(1)} + C^{(1)}, \qquad R_1^{(2)} = A^{(1)} - B^{(1)} + \frac{2}{s}C^{(1)}. \tag{5.161b}$$

Using (5.148) and (5.150), we arrive at the following theorem.

Theorem 5.19. *Let $s_* \in \mathfrak{B} \setminus ((-\infty, 2] \cup [2, \infty))$ and let $s \to s_*$ as described in* (5.92)*. Then the recurrence coefficients exist for large enough n, and they satisfy*

$$\alpha_n(s) = \frac{\delta_n \left(s^2 + 2s\left(\frac{4}{s^2}-1\right)^{1/2} - 4\right)}{\sqrt{\pi}s^3 n^{1/2}} + \frac{2\delta_n^2 \left(s^2 + 4s\left(\frac{4}{s^2}-1\right)^{1/2} - 8\right)}{\pi s^5 n} + \mathscr{O}\left(\frac{1}{n^{3/2}}\right), \tag{5.162a}$$

and

$$\beta_n(s) = \frac{1}{4} + \frac{\delta_n}{2\sqrt{\pi}s}\left(\frac{4}{s^2}-1\right)^{1/2}\frac{1}{n^{1/2}} - \frac{\delta_n^2}{2\pi s^2}\frac{1}{n} + \mathscr{O}\left(\frac{1}{n^{3/2}}\right), \tag{5.162b}$$

as $n \to \infty$, where

$$\delta_n = \delta_n(L_1) = e^{-in\kappa}\exp\left(L_1\left(\frac{4}{s_*^2}-1\right)^{1/2}\right), \qquad \kappa \in \mathbb{R}. \tag{5.163}$$

5.4 Double Scaling Limit near a Critical Breaking Point

We now take s in a double scaling regime near the critical point $s=2$ as

$$s = 2 + \frac{L_2}{n^{2/3}}, \tag{5.164}$$

where $L_2 < 0$. Note that as $L_2 < 0$, we have that $s \in \mathfrak{G}_0$ for large enough n.

5.4.1 Outline of Steepest Descent

Although we are now considering the case where s depends on n via the double scaling limit (5.164), the first two transformations of steepest descent remain unchanged to the previous analysis, and as such, we summarize the steps briefly and refer the reader to Chapter 2.

As $s \in \mathfrak{G}_0$ for n large enough, we have immediately that there is a genus 0 h-function and a contour $\gamma_{m,0} = [-1,1]$ such that the following conditions are valid:

$$h(z;s) \text{ is analytic for } z \in \mathbb{C} \setminus \Omega, \tag{5.165a}$$

$$h_+(z;s) - h_-(z;s) = 4\pi i, \qquad z \in \gamma_{c,0}, \tag{5.165b}$$

$$h_+(z;s) + h_-(z;s) = 0, \qquad z \in \gamma_{m,0}, \tag{5.165c}$$

$$h(z;s) = -sz - \ell + 2\log z + \mathscr{O}\left(\frac{1}{z}\right), \qquad z \to \infty \tag{5.165d}$$

$$\Re h(z;s) = \mathscr{O}\left((z \mp 1)^{1/2}\right), \qquad z \to \pm 1, \tag{5.165e}$$

where in the formulas above, $\Omega = \gamma_{c,0} \cup \gamma_{m,0}$. Moreover, this h function satisfies both inequalities in (2.10) for large enough n. Finally, we remark that as we are in the genus 0 regime, we have an explicit formula for the h function, given in (5.22) as

$$h(z;s) = 2\log\left(z + \left(z^2 - 1\right)^{1/2}\right) - s\left(z^2 - 1\right)^{1/2}. \tag{5.166}$$

We recall that Y defined in (2.4) solves the Riemann-Hilbert problem (2.3). By setting

$$T(z) := e^{-n\ell\sigma_3/2} Y(z) e^{-\frac{n}{2}[h(z)+f(z)]\sigma_3}, \tag{5.167}$$

we then have that T defined above solves the Riemann-Hilbert problem (2.16). We then open lenses by defining $\gamma_{m,0}^{\pm}$ to be the arcs connecting -1 and 1 which remain entirely within the region $\Re h(z) > 0$ such that $\gamma_{m,0}^{\pm}$ remains on the $+(-)$ side of $\gamma_{m,0}$, respectively. We then define $\mathscr{L}_0^{\pm}$ to be the region bounded between the arcs $\gamma_{m,0}$ and $\gamma_{m,0}^{\pm}$, respectively, and set $\hat{\Sigma} := \gamma_{m,0} \cup \gamma_{m,0}^{+} \cup \gamma_{m,0}^{-}$, as before.

We can now define the third transformation of the steepest descent process as

$$S(z) := \begin{cases} T(z)\begin{pmatrix} 1 & 0 \\ \mp e^{-nh(z)} & 1 \end{pmatrix}, & z \in \mathscr{L}_0^{\pm}, \\ T(z), & \text{otherwise}, \end{cases} \tag{5.168}$$

so that S solves the following Riemann-Hilbert problem on $\hat{\Sigma}$:

$$S(z) \text{ is analytic for } z \in \mathbb{C} \setminus \hat{\Sigma}, \tag{5.169a}$$

$$S_+(z) = S_-(z) j_S(z), \qquad z \in \hat{\Sigma}, \tag{5.169b}$$

$$S(z) = I + \mathscr{O}\left(\frac{1}{z}\right), \qquad z \to \infty, \tag{5.169c}$$

where

$$j_S(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{-nh(z)} & 1 \end{pmatrix}, & z \in \gamma_{m,0}^{\pm}, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \gamma_{m,0}. \end{cases} \tag{5.170}$$

To complete the process of nonlinear steepest descent, we must find suitable global and local parametrices, M and $P^{(\pm 1)}$, which are suitably close to the solution S. We define $D_{\pm 1}$ to be discs of fixed radius δ about $z=\pm 1$, respectively, and we seek parametrices which solve the following Riemann-Hilbert problems:

$$M(z) \text{ is analytic for } z\in\mathbb{C}\setminus\gamma_{m,0}(s), \tag{5.171a}$$
$$M_+(z)=M_-(z)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \qquad z\in\gamma_{m,0}, \tag{5.171b}$$
$$M(z)=I+\mathscr{O}\left(\frac{1}{z}\right), \qquad z\to\infty, \tag{5.171c}$$

and

$$P^{(\pm 1)}(z) \text{ is analytic for } z\in D_{\pm 1}\setminus\hat{\Sigma}, \tag{5.172a}$$
$$P_+^{(\pm 1)}(z)=P_-^{(\pm 1)}(z)j_S(z), \qquad z\in D_{\pm 1}\cap\hat{\Sigma}, \tag{5.172b}$$
$$P^{(\pm 1)}(z)=M(z)\left(I+o(1)\right), \qquad n\to\infty,\quad z\in\partial D_{\pm 1}. \tag{5.172c}$$

We have seen in Section 2.3 that M is given by (2.43).

Moreover, we have seen in Section 2.4 that the local parametrix $P^{(-1)}$ is given by

$$P^{(-1)}(z)=E_n^{(-1)}(z)\tilde{B}\left(\tilde{f}_{n,B}(z)\right)e^{-\frac{n}{2}h(z)}, \tag{5.173}$$

where $\tilde{B}(z)=\sigma_3 B(z)\sigma_3$ and B is the Bessel parametrix constructed in (2.95). In the above formulas,

$$\tilde{f}_{n,B}(z)=n^2\tilde{f}_B(z), \qquad \tilde{f}_B(z)=\frac{\tilde{h}(z)^2}{16}, \tag{5.174}$$

$\tilde{h}(z)=h(z)-2\pi i$, and

$$E_n^{(-1)}(z)=M(z)L_n^{(-1)}(z)^{-1}, \quad L_n^{(-1)}(z):=\frac{1}{\sqrt{2}}(2\pi n)^{-\sigma_3/2}\tilde{f}_B(z)^{-\sigma_3/4}\begin{pmatrix}-1 & i\\ i & -1\end{pmatrix}. \tag{5.175}$$

The main difference between the case of regular points and the critical breaking point at $s=2$ comes in the analysis about $z=1$. Note that the map

$$f_{n,B}(z;s)=\frac{h(z;s)^2}{16} \tag{5.176}$$

defined in (2.97) is no longer conformal when $s=2$. Indeed,

$$f_{n,B}(z;s)=\frac{(s-2)^2}{8}(z-1)+\frac{(s-2)(3s+2)}{48}(z-1)^2+\mathscr{O}\left((z-1)^3\right), \qquad z\to 1, \tag{5.177}$$

so that $f_{n,B}(z,2)=\mathscr{O}\left((z-1)^3\right)$ as $z\to 1$. Therefore, a different analysis will be needed in D_1 in the double scaling limit (5.164).

5.4.2 Local parametrix at $z = 1$.

We consider a disc, D_1, around $z = 1$ of fixed radius $\delta > 0$. The local parametrix about $z = 1$ solves the following Riemann-Hilbert problem:

$$P^{(1)}(z) \text{ is analytic for } z \in D_1 \setminus \hat{\Sigma}, \tag{5.178a}$$

$$P^{(1)}_+(z) = P^{(1)}_-(z) j_S(z), \qquad z \in D_1 \cap \hat{\Sigma}, \tag{5.178b}$$

$$P^{(1)}(z) = (I + o(1))M(z), \qquad n \to \infty, \quad z \in \partial D_1. \tag{5.178c}$$

We will solve for $P^{(1)}$ by setting

$$P^{(1)}(z) = U^{(1)}(z) e^{-\frac{n}{2}h(z)\sigma_3}, \tag{5.179}$$

where $U^{(1)}$ has the following jumps over $\hat{\Sigma}$ within D_1:

$$U^{(1)}_+(z) = U^{(1)}_-(z) \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & z \in D_1 \cap \left(\gamma^+_{m,0} \cup \gamma^-_{m,0}\right), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in D_1 \cap \gamma_{m,0}. \end{cases} \tag{5.180}$$

We will solve this local problem using a parametrix related to the Painlevé II and Painlevé XXXIV differential equations. Finally, we remark that as we transition from $s < 2$ (the situation we consider) to $s > 2$, the hard edge becomes a soft edge.

The Painlevé XXXIV Parametrix

Let $q = q(w)$ be a solution of the Painlevé II equation

$$q'' = wq + 2q^3 - \alpha, \qquad \alpha \in \mathbb{C}. \tag{5.181}$$

We define the following function $D = D(w)$, which is closely related to the Hamiltonian function for Painlevé II:

$$D = (q')^2 - q^4 - wq^2 + 2\alpha q. \tag{5.182}$$

Next, we consider the following Riemann–Hilbert problem, which appears in [53, 54, 93, 92]. This problem appears in works related to orthogonal polynomials on the real line and Hermitian random matrix ensembles with a Fisher–Hartwig singularity or with critical behavior at the edge of the spectrum.

Let $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, where $\Gamma_1 = \left\{\arg\zeta = -\frac{2\pi}{3}\right\}$, $\Gamma_2 = \{\arg\zeta = 0\}$, $\Gamma_3 = \left\{\arg\zeta = \frac{2\pi}{3}\right\}$, and $\Gamma_4 = \{\arg\zeta = \pi\}$, with orientation as in Figure 5.10, and define the sectors Ω_j as in Figure 5.10.

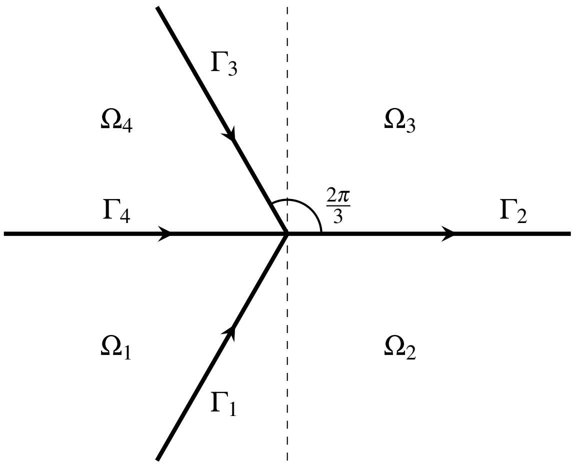


Figure 5.10: Contour for the RH problem for $\Psi_\alpha(\zeta;w)$.

Consider the following Riemann-Hilbert problem for $\Psi(\zeta,w)$ posed on Γ.

$$\Psi(\zeta,w) \text{ is analytic for } \zeta\in\mathbb{C}\setminus(\Gamma_1\cup\Gamma_3\cup\Gamma_4)\,, \tag{5.183a}$$

$$\Psi_+(\zeta,w)=\Psi_-(\zeta,w)\begin{cases}\begin{pmatrix}1&0\\1&1\end{pmatrix}, & \zeta\in\Gamma_1\cup\Gamma_3,\\ \begin{pmatrix}1&a_2\\0&1\end{pmatrix}, & \zeta\in\Gamma_2,\\ \begin{pmatrix}0&1\\-1&0\end{pmatrix}, & \zeta\in\Gamma_4,\end{cases} \tag{5.183b}$$

$$\Psi(\zeta,w)=\left(1+\frac{\Psi_1(w)}{\zeta}+\mathscr{O}\left(\frac{1}{\zeta^2}\right)\right)\zeta^{-\sigma_3/4}\left(\frac{I+i\sigma_1}{\sqrt{2}}\right)e^{-\left(\frac{4}{3}\zeta^{3/2}-w\zeta^{1/2}\right)\sigma_3}, \qquad \zeta\to\infty, \tag{5.183c}$$

$$\Psi(\zeta,w)=\begin{cases}\mathscr{O}\begin{pmatrix}1&\log|\zeta|\\1&\log|\zeta|\end{pmatrix}, & \zeta\in\Omega_2\cup\Omega_3,\\ \mathscr{O}\begin{pmatrix}\log|\zeta|&\log|\zeta|\\ \log|\zeta|&\log|\zeta|\end{pmatrix}, & \zeta\in\Omega_1\cup\Omega_4,\end{cases} \tag{5.183d}$$

where

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}, \qquad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{5.184}$$

In [93, Section 2], it is shown[1], via a vanishing lemma (Lemma 1), that this Riemann–Hilbert problem has a unique solution for all real values of w if $a_2\in\mathbb{C}\setminus(-\infty,0)$. In the present case, we are taking $a_2=0$ (therefore, no jump on Σ_2), so the result applies. We note that the vanishing lemma for

[1]Our Ψ function corresponds to Ψ_0 in their notation.

Painlevé II, along with the vanishing lemma for Painlevé IV, was first proven by Fokas and Zhou in [44, Lemma 4.1, Lemma 4.2]. The existence result also follows from [53, Proposition 2.3], identifying $\Psi(\zeta,w)$ with the function $\Psi^{(spec)}(\zeta,s)$ in their notation.

In order to calculate the entries of the matrix $\Psi_1(w)$ in (5.183c), that will be needed later to obtain the asymptotics of the recurrence coefficients, we use the fact that this Riemann–Hilbert problem originates from a folding procedure of Flaschka–Newell for Painlevé II. Applying formulas (25) and (37) in [93], we have

$$\Psi(\zeta,w)=\begin{pmatrix}1 & 0\\ -\dfrac{D+q}{2i} & 1\end{pmatrix}\zeta^{-\frac{\sigma_3}{4}}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\Phi(i\zeta^{\frac{1}{2}},w), \tag{5.185}$$

where $\Phi(\lambda,w)$ solves a Riemann–Hilbert problem corresponding to Painlevé II, see [93, Section 2] and also [47, Theorem 5.1 and (5.0.51)]. Here $q=q(w)$ solves Painlevé II and $D=D(w)$ is given by (5.182). Furthermore, we observe that the solution $\Psi(\zeta,w)$ that we study corresponds to the Stokes multipliers $b_1=0$ and $b_2=b_4=1$, in the notation used in [54, §1.3], and therefore $a_2=0$ and $a_1=a_3=-i$ in terms of the Stokes multipliers for Painlevé II, see [54, (A.10)]. This is in fact the generalized Hastings–McLeod solution to Painlevé II, with parameter $\alpha=1/2$, which is characterized by the following asymptotic behavior:

$$\begin{aligned} q_{\mathrm{HM}}(x)&=\sqrt{-\frac{x}{2}}+\mathcal{O}(x^{-1}), \qquad x\to-\infty,\\ q_{\mathrm{HM}}(x)&=\frac{\alpha}{x}+\mathcal{O}(x^{-4})=\frac{1}{2x}+\mathcal{O}(x^{-4}), \qquad x\to+\infty. \end{aligned} \tag{5.186}$$

Further properties of the Painlevé functions associated to $\Psi(\zeta,w)$ are proved in [53, Lemma 3.5].

As $\lambda\to\infty$, we have the expansion

$$\Phi(\lambda,w)=\left(I+\frac{m_1(w)}{\lambda}+\frac{m_2(w)}{\lambda^2}+\mathcal{O}(\lambda^{-3})\right)e^{-i\left(\frac{4}{3}\lambda^3+w\lambda\right)\sigma_3}, \tag{5.187}$$

where the entries of the matrices $m_1(w)$ and $m_2(w)$ are given explicitly in formula (21) in [93], see also [47, (5.0.7)], again in terms of u, u' and D (we omit the dependence on w for brevity):

$$m_1(w)=\frac{1}{2}\begin{pmatrix}-iD & q\\ q & iD\end{pmatrix}, \qquad m_2(w)=\frac{1}{8}\begin{pmatrix}q^2-D^2 & 2i(qD+q')\\ -2i(qD+q') & q^2-D^2\end{pmatrix}. \tag{5.188}$$

Combining (5.185), (5.187) and (5.188), we arrive at the following formulas for the entries of the matrix $\Psi_1(w)$ in (5.183c):

$$\begin{aligned} \Psi_{1,11} &= \frac{D^2 - q^2}{8} - \frac{qD + q'}{4}, \\ \Psi_{1,22} &= -\frac{D^2 - q^2}{8} + \frac{qD + q'}{4}, \\ \Psi_{1,12} &= \frac{i}{2}(D - q). \end{aligned} \tag{5.189}$$

Construction of the Local Parametrix

We now continue to build the local parametrix in the disc D_1. First, we have the following lemma, following the ideas laid out in [15, Proposition 4.5].

Lemma 5.20. *There exists a function $\zeta(z;s)$ which is conformal in a fixed neighborhood of $z = 1$, with s close to 2, and an analytic function $A(s)$, such that*

$$-\frac{h(z)}{2} = \frac{4}{3}\zeta(z;s)^{3/2} - A(s)\zeta(z;s)^{1/2}, \tag{5.190}$$

and

$$\zeta(1,s) \equiv 0, \qquad A(2) = 0. \tag{5.191}$$

Proof. As h has a critical point at $z = \frac{2}{s}$, we write

$$h_{cr}(s) = h\left(\frac{2}{s}, s\right) = 2\log\left(\frac{2}{s} + \left(\frac{4}{s^2} - 1\right)^{1/2}\right) - s\left(\frac{4}{s^2} - 1\right)^{1/2}. \tag{5.192}$$

Near $s = 2$, we see that $h_{cr}(s) = \mathscr{O}\left((s-2)^{3/2}\right)$, so that

$$h_{cr}(s) = \frac{2}{3}A^{3/2}(s), \tag{5.193}$$

for some $A(s)$ analytic in a neighborhood of $s = 2$ satisfying

$$A(s) = a_1(s-2) + \mathscr{O}\left((s-2)^2\right), \qquad s \to 2, \tag{5.194}$$

where

$$a_1 = -1. \tag{5.195}$$

Next, define

$$\xi(z;s) = -3h(z;s) + \left(-4A^3(s) + 9h^2(z;s)\right)^{1/2}, \tag{5.196}$$

and set

$$u(z;s) = u_1(z;s) + u_2(z;s), \tag{5.197}$$

where

$$u_1(z;s) = \frac{A(s)}{2^{2/3}\xi^{1/3}(z;s)}, \qquad u_2(z;s) = \frac{\xi^{1/3}(z;s)}{2^{4/3}}. \tag{5.198}$$

Then, u solves the equation

$$\frac{4}{3}u^3(z;s) - A(s)u(z;s) = -\frac{h(z;s)}{2}. \tag{5.199}$$

Now, for s in a neighborhood of 2, we have that for z in a neighborhood of 1,

$$\begin{aligned}\xi(z;s) = {} & 2(-A(s))^{3/2} + 3\sqrt{2}(s-2)(z-1)^{\frac{1}{2}} + \frac{9(s-2)^2}{2(-A(s))^{3/2}}(z-1)\\ & + \frac{2+3s}{2\sqrt{2}}(z-1)^{\frac{3}{2}} + \mathscr{O}\left((z-1)^2\right).\end{aligned} \tag{5.200}$$

From this, we then have that

$$\begin{aligned}u_1(z) = {} & -\frac{(-A(s))^{1/2}}{\sqrt{2}} - \frac{s-2}{2\sqrt{2}A(s)}(z-1)^{\frac{1}{2}} - \frac{(s-2)^2}{8(-A(s))^{5/2}}(z-1)\\ & - \frac{A^3(s)(3s+2)+8(s-2)^3}{24\sqrt{2}A^4(s)}(z-1)^{\frac{3}{2}}\\ & + \frac{(s-2)\left(35(s-2)^3+4A^3(s)(3s+2)\right)}{192(-A(s))^{\frac{11}{2}}}(z-1)^2 + \mathscr{O}\left((z-1)^{\frac{5}{2}}\right),\end{aligned} \tag{5.201}$$

and

$$\begin{aligned}u_2(z) = {} & \frac{(-A(s))^{1/2}}{\sqrt{2}} - \frac{s-2}{2\sqrt{2}A(s)}(z-1)^{\frac{1}{2}} + \frac{(s-2)^2}{8(-A(s))^{5/2}}(z-1)\\ & - \frac{A^3(s)(3s+2)+8(s-2)^3}{24\sqrt{2}A^4(s)}(z-1)^{\frac{3}{2}}\\ & - \frac{(s-2)\left(35(s-2)^3+4A^3(s)(3s+2)\right)}{192(-A(s))^{\frac{11}{2}}}(z-1)^2 + \mathscr{O}\left((z-1)^{\frac{5}{2}}\right).\end{aligned} \tag{5.202}$$

Combining these two, we have that

$$u(z;s) = -\frac{(s-2)}{\sqrt{2}A(s)}(z-1)^{1/2} - \frac{A^3(s)(3s+2)+8(s-2)^3}{12\sqrt{2}A^4(s)}(z-1)^{3/2} + \mathscr{O}\left((z-1)^{\frac{5}{2}}\right). \tag{5.203}$$

Making the change of variables $u^2 \mapsto \zeta$, we have that

$$\zeta(z;s) = \frac{(s-2)^2}{2A^2(s)}(z-1) + \mathscr{O}\left((z-1)^2\right), \tag{5.204}$$

so that ζ is a conformal map in a neighborhood of $z=1$ when s is in a neighborhood of 2. Note that when $s=2$, we have that

$$\zeta(z,2) = \frac{1}{2}(z-1) + \mathscr{O}\left((z-1)^2\right), \tag{5.205}$$

where we have used (5.194) and (5.195), so that ζ is still conformal when $s=2$. Finally, it is immediate from (5.199) that ζ solves (5.192), which completes the proof. □

Using (5.194) and (5.204), we may compute

$$\zeta(z,s) = \zeta_1(s)(z-1) + \mathscr{O}\left((z-1)^2\right), \qquad z \to 1, \tag{5.206}$$

where

$$\zeta_1(s) = \frac{1}{2} + \mathscr{O}(s-2), \qquad s \to 2. \tag{5.207}$$

As $s \in \mathbb{R}$, we see that $\gamma_{m,0}$ is mapped to the ray Γ_4 by the conformal map ζ. Moreover, we now choose the lips of the lens, $\gamma_{m,0}^{\pm}$, within the disc so that they are mapped by ζ to the rays Γ_3 and Γ_1, respectively.

Next, we set

$$E_n^{(1)}(z) = M(z)\left(\frac{I+i\sigma_1}{\sqrt{2}}\right)^{-1}\left(n^{2/3}\zeta(z;s)\right)^{\sigma_3/4}, \tag{5.208}$$

where the branch cut for $\zeta^{1/4}$ is taken on $\gamma_{m,0}(s)$. As

$$M_+(z) = M_-(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \zeta_+^{1/4}(z,s) = i\zeta_-^{1/4}(z,s), \qquad z \in \gamma_{m,0}(s), \tag{5.209}$$

we see that $E_n^{(1)}(z)$ has no jumps within D_1. By (2.43) each entry of M is $\mathscr{O}(z-1)^{1/4}$ as $z \to 1$, so the singularity of $E_n^{(1)}$ at $z=1$ is removable. Therefore, we see that $E_n^{(1)}(z)$ is analytic in D_1. We may then conclude that

$$P^{(1)}(z) = E_n^{(1)}(z)\Psi\left(n^{2/3}\zeta(z;s), n^{2/3}A(s)\right)e^{-\frac{n}{2}h(z)\sigma_3} \tag{5.210}$$

solves (5.178). Indeed, as $\zeta(z;s)$ maps $\gamma_{m,0}$, $\gamma_{m,0}^{+}$, and $\gamma_{m,0}^{-}$ to Γ_4, Γ_3, and Γ_1, respectively, we see that $P^{(1)}$ is analytic in $D_1 \setminus \hat{\Sigma}$. Next, using Lemma 5.20 and (5.183c), we see that $P^{(1)}$ satisfies (5.178c). Finally, we note that as $P^{(1)}$ and S have the same jumps within D_1, the combination $S(z)P^{(1)}(z)^{-1}$ is analytic on $D_1 \setminus \{1\}$. Also note that the behavior of S and $P^{(1)}$ are the same as $z \to 1$, so that the singularity is removable.

5.4.3 Proof of Theorem 5.22

The final transformation is

$$R(z) = S(z)\begin{cases} M(z)^{-1}, & z \in \mathbb{C} \setminus \overline{(D_{-1} \cup D_1)}, \\ P^{(-1)}(z)^{-1}, & z \in D_{-1}, \\ P^{(1)}(z)^{-1}, & z \in D_1. \end{cases} \tag{5.211}$$

As before, we want to write the jump matrix as $I+\Delta(z)$, where $\Delta(z)$ has an expansion in inverse powers of n^{α}, for some α to be determined. We recall (5.56), where we showed that

$$\Delta(z)=\sum_{k=1}^{\infty}\frac{\Delta_k(z)}{n^k},\qquad n\to\infty,\quad z\in D_{-1}, \tag{5.212}$$

with

$$\Delta_k(z)=\frac{(-1)^{k-1}\prod_{j=1}^{k-1}(2j-1)^2}{4^{k-1}(k-1)!\tilde{h}(z)^k}M(z)\begin{pmatrix}\frac{(-1)^k}{k}\left(\frac{k}{2}-\frac{1}{4}\right) & i\left(k-\frac{1}{2}\right)\\ (-1)^{k+1}i\left(k-\frac{1}{2}\right) & \frac{1}{k}\left(\frac{k}{2}-\frac{1}{4}\right)\end{pmatrix}M^{-1}(z), \tag{5.213}$$

and $\tilde{h}(z)=h(z)-2\pi i$.

To compute the jumps over ∂D_1, we first recall that

$$\Psi(\zeta,w)=\left(1+\frac{\Psi_1(w)}{\zeta}+\mathscr{O}\left(\frac{1}{\zeta^2}\right)\right)\zeta^{-\sigma_3/4}\left(\frac{I+i\sigma_1}{\sqrt{2}}\right)e^{-\left(\frac{4}{3}\zeta^{3/2}-w\zeta^{2/3}\right)\sigma_3},\qquad \zeta\to\infty. \tag{5.214}$$

We may then use (5.183c), (5.208), and (5.210) to see that

$$P^{(1)}(z)M^{-1}(z)=M(z)\left(I+\frac{\tilde{\Psi}_{1/3}(z,s)}{n^{1/3}}+\frac{\tilde{\Psi}_{2/3}(z,s)}{n^{2/3}}+\mathscr{O}\left(\frac{1}{n}\right)\right)M^{-1}(z),\qquad n\to\infty, \tag{5.215}$$

where

$$\tilde{\Psi}_{1/3}(z,s)=\frac{\Psi_{1,12}(w)}{2\zeta^{1/2}(z,s)}\begin{pmatrix} i & 1\\ 1 & -i\end{pmatrix}, \tag{5.216a}$$

and

$$\tilde{\Psi}_{2/3}(z,s)=\frac{1}{2\zeta(z,s)}\begin{pmatrix}\Psi_{1,11}(w)+\Psi_{1,22}(w) & i\left(\Psi_{1,11}(w)-\Psi_{1,22}(w)\right)\\ -i\left(\Psi_{1,11}(w)-\Psi_{1,22}(w)\right) & \Psi_{1,11}(w)+\Psi_{1,22}(w)\end{pmatrix}, \tag{5.216b}$$

where $\Psi_{1,ij}$ refers to the (i,j) entry of the matrix Ψ_1. Moreover, above we have defined

$$w=w(s)=n^{2/3}A(s), \tag{5.217}$$

where A is the analytic function given in Lemma 5.17. By the double scaling limit (5.164) and (5.194), we also have that

$$w=-L_2+\mathscr{O}\left(\frac{1}{n^{2/3}}\right),\qquad n\to\infty. \tag{5.218}$$

It is now straightforward to see that Δ can be written in inverse powers of $n^{1/3}$ as

$$\Delta(z)=\sum_{k=1}^{\infty}\frac{\Delta_{k/3}(z)}{n^{1/3}},\qquad n\to\infty, z\in\Sigma_R, \tag{5.219}$$

where $\Delta_{k/3}(z) \equiv 0$ for $z \in \Sigma_R \setminus (\partial D_1 \cup \partial D_{-1})$,

$$\Delta_{k/3}(z) = \begin{cases} 0, & \frac{k}{3} \notin \mathbb{N}, \\ \dfrac{(-1)^{k-1}\prod_{j=1}^{k-1}(2j-1)^2}{4^{k-1}(k-1)!\tilde{h}(z)^k} M(z) \begin{pmatrix} \frac{(-1)^k}{k}\left(\frac{k}{2}-\frac{1}{4}\right) & i\left(k-\frac{1}{2}\right) \\ (-1)^{k+1} i\left(k-\frac{1}{2}\right) & \frac{1}{k}\left(\frac{k}{2}-\frac{1}{4}\right) \end{pmatrix} M^{-1}(z), & \frac{k}{3} \in \mathbb{N}, \end{cases}$$

for $z \in \partial D_1$, and

$$\Delta_{k/3}(z) = M(z)\tilde{\Psi}_{k/3}(z,s)M^{-1}(z), \qquad z \in \partial D_1, \tag{5.220}$$

where the $\tilde{\Psi}_{k/3}$ can be computed using the expansion of Ψ in (5.183c) along with the definitions of the conformal maps and analytic prefactor given in Lemma 5.17 and (5.208), respectively. We recall that both $\tilde{\Psi}_{1/3}$ and $\tilde{\Psi}_{2/3}$ are given in (5.216).

Now, we may again use the arguments presented in [39, Section 7] and [65, Section 8] to conclude that R has an asymptotic expansion in inverse powers of $n^{1/3}$ of the form

$$R(z) = \sum_{k=0}^{\infty} \frac{R_{k/3}(z)}{n^{k/3}}, \qquad n \to \infty, \tag{5.221}$$

where each $R_{k/3}$ solves the following Riemann-Hilbert problem:

$$R_{k/3}(z) \text{ is analytic for } z \in \mathbb{C} \setminus (\partial D_{-1} \cup \partial D_1), \tag{5.222a}$$

$$R_{k/3,+}(z) = R_{k/3,-}(z) + \sum_{j=1}^{k-1} R_{(k-j)/3,-}\Delta_{j/3}(z), \qquad z \in \partial D_{-1} \cup \partial D_1, \tag{5.222b}$$

$$R_{k/3}(z) = \frac{R^{(1)}_{k/3}}{z} + \frac{R^{(2)}_{k/3}}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right), \qquad z \to \infty. \tag{5.222c}$$

Next we recall (2.41)

$$\alpha_n = \frac{[T_2]_{12}}{[T_1]_{12}} - [T_1]_{22}, \qquad \beta_n = [T_1]_{12}[T_1]_{21}, \tag{5.223}$$

where T_1 and T_2 appear in the expansion of T at infinity,

$$T(z) = I + \frac{T_1}{z} + \frac{T_2}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right). \tag{5.224}$$

By (5.168) and (5.211) we have that $T(z) = R(z)M(z)$ for z outside of the lens. Using (5.221), we then have that

$$T_1 = M_1 + \frac{R^{(1)}_{1/3}}{n^{1/3}} + \frac{R^{(1)}_{2/3}}{n^{2/3}} + \mathcal{O}\left(\frac{1}{n}\right), \qquad n \to \infty, \tag{5.225a}$$

$$T_2 = M_2 + \frac{R^{(1)}_{1/3}M_1 + R^{(2)}_{1/3}}{n^{1/3}} + \frac{R^{(1)}_{2/3}M_1 + R^{(2)}_{2/3}}{n^{2/3}} + \mathcal{O}\left(\frac{1}{n}\right), \qquad n \to \infty, \tag{5.225b}$$

where M_1 and M_2 were calculated in (5.45) as

$$M_1 = \begin{pmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} \frac{1}{8} & 0 \\ 0 & \frac{1}{8} \end{pmatrix}. \tag{5.226}$$

We therefore turn our attention to computing the first few terms of the expansions of both $R_{1/3}$ and $R_{2/3}$. Before doing so, we first present the following lemma.

Lemma 5.21. *The restrictions of $\Delta_{1/3}$ and $\Delta_{2/3}$ to ∂D_1 have meromorphic continuations to a neighborhood of D_1. These continuations are analytic, except at* 1*, where they have poles of order* 1.

Proof. We first consider $\Delta_{1/3}$, defined as

$$\Delta_{1/3}(z) = M(z)\tilde{\Psi}_{1/3}(z,s)M^{-1}(z), \tag{5.227}$$

where

$$\tilde{\Psi}_{1/3}(z,s) = \frac{\Psi_{1,12}(w)}{2\zeta^{1/2}(z,s)} \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix},$$

where the branch cut of $\zeta^{1/2}$ is taken to be $\gamma_{m,0}(s)$. Next, since

$$M_+(z) = M_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \zeta_+^{1/2}(z,s) = -\zeta_-^{1/2}(z,s), \qquad z \in \gamma_{m,0}(s), \tag{5.228}$$

we see that $\Delta_{1/3,+}(z) = \Delta_{1/3,-}(z)$ for $z \in \gamma_{m,0}$ so that $\Delta_{1/3}$ is analytic in $D_1 \setminus \{1\}$. Since

$$M(z) \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} M^{-1}(z) = \sqrt{2} \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} \frac{1}{(z-1)^{1/2}} + \mathscr{O}\left((z-1)^{1/2}\right), \qquad z \to 1, \tag{5.229}$$

and $\zeta(z,s) = \zeta_1(s)(z-1) + \mathscr{O}(z-1)^2$, where $\zeta_1(s) \neq 0$ as ζ is a conformal mapping from 1 to 0, we see that the isolated singularity at $z = 1$ is a simple pole.

In the case, of $\Delta_{2/3}$, we note that

$$M(z)\tilde{\Psi}_{2/3}(z,s)M^{-1}(z) = \tilde{\Psi}_{2/3}(z,s), \tag{5.230}$$

so that the lemma follows immediately from (5.216b). □

In light of the lemma above, we may write that

$$\Delta_{1/3}(z) = \frac{C^{(1/3)}}{z-1}, \qquad z \to 1. \tag{5.231}$$

Using that $\zeta(z,s) = \zeta_1(s)(z-1) + \mathscr{O}(z-1)^2$ as $z \to 1$, we compute that

$$C^{(1/3)} = \frac{\Psi_{1,12}(w)}{\sqrt{2}\zeta_1^{1/2}(s)} \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}. \tag{5.232}$$

By direct inspection we see that

$$R_{1/3}(z) = \begin{cases} \dfrac{C^{(1/3)}}{z-1}, & z \in \mathbb{C} \setminus D_1, \\ \dfrac{C^{(1/3)}}{z-1} - \Delta_{1/3}(z), & z \in D_1, \end{cases} \tag{5.233}$$

solves the Riemann-Hilbert problem (5.222) when $k = 1$, so that

$$R_{1/3}^{(1)} = R_{1/3}^{(2)} = C^{(1/3)}. \tag{5.234}$$

We analogously solve for the terms in the expansion of $R_{2/3}$ by writing

$$R_{1/3}(z)\Delta_{1/3}(z) + \Delta_{2/3}(z) = \frac{C^{(2/3)}}{z-1}, \tag{5.235}$$

where we may compute that

$$C^{(2/3)} = \frac{1}{2\zeta_1(s)} \begin{pmatrix} \Psi_{1,11}(w) + \Psi_{1,22}(w) & i(\Psi_{1,11}(w) - \Psi_{1,22}(w)) \\ -i(\Psi_{1,11}(w) - \Psi_{1,22}(w)) & \Psi_{1,11}(w) + \Psi_{1,22}(w) \end{pmatrix}. \tag{5.236}$$

Then,

$$R_{1/3}(z) = \begin{cases} \dfrac{C^{(2/3)}}{z-1}, & z \in \mathbb{C} \setminus D_1, \\ \dfrac{C^{(2/3)}}{z-1} - R_{1/3}(z)\Delta_{1/3}(z) - \Delta_{2/3}(z), & z \in D_1, \end{cases} \tag{5.237}$$

solves (5.222), and we may compute that the terms in the large z expansion of $R_{2/3}$ are given by

$$R_{2/3}^{(1)} = R_{2/3}^{(2)} = C^{(2/3)}. \tag{5.238}$$

Combining the previous equations (in particular (5.223), (5.225), (5.226), (5.234), and (5.238)), we have

$$\alpha_n(s) = \frac{\Psi_{1,11}(w) - \Psi_{1,22}(w) + \Psi_{1,12}^2(w)}{\zeta_1(s)} \frac{1}{n^{2/3}} + \mathscr{O}\left(\frac{1}{n}\right), \qquad n \to \infty, \tag{5.239a}$$

and

$$\beta_n(s) = \frac{1}{4} + \frac{\Psi_{1,11}(w) - \Psi_{1,22}(w) + \Psi_{1,12}^2(w)}{2\zeta_1(s)} \frac{1}{n^{2/3}} + \mathscr{O}\left(\frac{1}{n^{4/3}}\right), \qquad n \to \infty. \tag{5.239b}$$

Next, using (5.207) and the double scaling limit (5.164), along with the formula for w in (5.217), we have that

$$\alpha_n(s) = 2\left(\Psi_{1,11}(w) - \Psi_{1,22}(w) + \Psi_{1,12}^2(w)\right) \frac{1}{n^{2/3}} + \mathscr{O}\left(\frac{1}{n}\right), \qquad n \to \infty, \tag{5.240a}$$

and

$$\beta_n(s) = \frac{1}{4} + \left(\Psi_{1,11}(w) - \Psi_{1,22}(w) + \Psi_{1,12}^2(w)\right) \frac{1}{n^{2/3}} + \mathscr{O}\left(\frac{1}{n}\right), \qquad n \to \infty. \tag{5.240b}$$

Using (5.189), we can simplify the previous combination of entries of $\Psi_1(w)$:

$$\Psi_{1,11}(w) - \Psi_{1,22}(w) + \Psi_{1,12}^2(w) = -\frac{1}{2}(q^2(w) + q'(w)),$$

so that by using (5.218) we have the following theorem.

Theorem 5.22. *Let $s \to 2$ as described in* (5.15). *Then the recurrence coefficients exist for large enough n, and they satisfy*

$$\alpha_n(s) = -\frac{q^2(-L_2) + q'(-L_2)}{n^{2/3}} + \mathscr{O}\left(\frac{1}{n}\right), \tag{5.241a}$$

and

$$\beta_n(s) = \frac{1}{4} - \frac{q^2(-L_2) + q'(-L_2)}{2} \frac{1}{n^{2/3}} + \mathscr{O}\left(\frac{1}{n}\right), \tag{5.241b}$$

as $n \to \infty$, where q is the generalized Hastings-McLeod solution to Painlevé II with parameter $\alpha = 1/2$. Furthermore, the function $U(w) = q^2(w) + q'(w)$ is free of poles for $w \in \mathbb{R}$.

The fact that the function $U(w) = q^2(w) + q'(w)$ is free of poles for $w \in \mathbb{R}$ follows from [53, Lemma 3.5], as well as from [93, Lemma 1, Corollary 1]; in this last reference, the theorem is a consequence of the vanishing lemma applied to the Painlevé XXXIV Riemann–Hilbert problem, and then translating the result to solutions of Painlevé II. This completes the proof of Theorem 5.22.

Chapter 6

Conclusion and Outlook

In this thesis, we have seen how to apply the Riemann-Hilbert approach to answer certain questions dealing with the Kissing polynomials. There are many natural extensions of this work, and we will provide some details on these below.

As seen in Chapter 1, the original motivation for studying the Kissing polynomials comes from their use in complex Gaussian quadrature rules as explained in [5, 33]. Despite the many theoretical benefits that come from quadrature rules based on the Kissing polynomials, these methods are often not used in practice. Indeed, as shown in Chapter 1, when the oscillatory parameter is small we may use regular Gaussian quadrature; when it is large, we may opt instead to use numerical steepest descent. Both of these methods, when applied to the oscillatory integral

$$I_\omega[f] = \int_{-1}^{1} f(x) e^{i\omega x}\, dx,$$

require the calculation of zeros of classical families of orthogonal polynomials. In particular, when ω is small, we may use Gaussian quadrature with nodes at the zeros of the Legendre polynomials and when ω is large we may use nodes which are a simple rescaling of the zeros of the Laguerre polynomials, as seen in (1.11). In both cases, we may compute these zeros in a stable and quick manner (c.f. [48]).

In order for complex Gaussian quadrature rules based on the Kissing polynomials to become widely used, we must first be able to efficiently compute the zeros of the Kissing polynomials. One approach is to consider the Jacobi matrix formed with the recurrence coefficients, as explained in [48, Section 4]. We recall that the monic Kissing polynomials, provided the corresponding polynomials exist, satisfy a recurrence relation of the form

$$z p_n(z) = p_{n+1}(z) + \alpha_n p_n(z) + \beta_n p_{n-1}(z). \tag{6.1}$$

With the recurrence coefficients, we form the infinite, tridiagonal Jacobi matrix

$$J_\infty = \begin{pmatrix} \alpha_0 & \beta_1 & & & & 0 \\ 1 & \alpha_1 & \beta_2 & & & \\ & 1 & \alpha_2 & \beta_3 & & \\ & & \ddots & \ddots & \ddots & \\ 0 & & & & & \end{pmatrix}. \tag{6.2}$$

The Gaussian Quadrature nodes and weights can then be expressed in terms of the eigenvalues and eigenvectors of the principal minors of this matrix. By setting

$$J_{2n} = (J_\infty)_{2n\times 2n} \tag{6.3}$$

to be the $2n \times 2n$ principal minor of J_∞, we let λ_ν and u_ν be the the corresponding eigenvalues and normalized eigenvectors of J_{2n}. By [48, Theorem 4], the quadrature method

$$\int_{-1}^{1} f(x)e^{i\omega x}\,dx \approx \sum_{\nu=1}^{2n} w_\nu f(\lambda_\nu), \tag{6.4}$$

where

$$w_\nu = \frac{2\sin\omega}{\omega} u_{\nu,1}^2 \tag{6.5}$$

and $u_{\nu,1}$ is the first entry of the corresponding eigenvector, is equivalent to the complex Gaussian quadrature scheme which motivated this thesis. Therefore, in order to quickly and efficiently implement complex Gaussian quadrature, we must quickly and efficiently compute the principal minors of this Jacobi matrix.

One way to approach this problem is to leverage the fact that the recurrence coefficients satisfy complex versions of the Toda equations and study the deformation of the Jacobi matrix in time. Such an approach, in relation to Hermitian orthogonality, has been considered in [37], where the authors considered the evolution of a real valued Jacobi matrix under the Toda Flow. There are many interesting connections here with the theory of Lie Groups and this is an avenue of research that should be explored. For more details on this approach, see also the report [28].

Another approach to computing recurrence coefficients could come from the so called *string equations*, which are nonlinear difference equations that are typically satisfied by the recurrence coefficients of orthogonal polynomials. In fact, in [26], we show that the recurrence coefficients of the Kissing polynomials satisfy

$$-2(n+1)\alpha_n + 2\sigma_{n,n-1} + i\omega\left(1-\beta_{n+1}-\beta_n-\alpha_n^2\right) = 0, \tag{6.6}$$

where $\sigma_{n,n-1}$ is the subleading coefficient of the monic Kissing polynomial. In the recent book [89], see also [45] , many examples of such string equations were given and many of these were shown to be discrete Painlevé equations. The study of the equation (6.6) from the viewpoint of discrete integrable systems is another promising, yet unexplored, area of research.

Finally, we note that the Kissing polynomials are just the tip of the iceberg. There are many other families of non-Hermitian orthogonality that one could study which lead to many interesting questions. This theory is still being developed and is filled with many exciting areas to explore.

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