Probabilistic inferences from conjoined to iterated conditionals ☆

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Abstract

There is wide support in logic, philosophy, and psychology for the hypothesis that the probability of the indicative conditional of natural language, $P(if\ A\ then\ B)$, is the conditional probability of B given A, $P(B|A)$. We identify a conditional which is such that $P(if\ A\ then\ B) = P(B|A)$ with de Finetti's conditional event, $B|A$. An objection to making this identification in the past was that it appeared unclear how to form compounds and iterations of conditional events. In this paper, we illustrate how to overcome this objection with a probabilistic analysis, based on coherence, of these compounds and iterations. We interpret the compounds and iterations as conditional random quantities which, given some logical dependencies, may reduce to conditional events. We show how the inference to $B|A$ from A and B can be extended to compounds and iterations of both conditional events and biconditional events. Moreover, we determine the respective uncertainty propagation rules. Finally, we make some comments on extending our analysis to counterfactuals.

Keywords: Centering, Coherence, p-entailment, Compound conditionals, Iterated conditionals, Counterfactuals

1. Introduction

There is wide agreement in logic and philosophy that the indicative conditional of natural language, *if A then B*, cannot be adequately represented as the material conditional of binary logic, logically equivalent to $\bar{A} \vee B$ (*not-A or B*) [26]. Psychological studies have also shown that ordinary people do not judge the probability of *if A then B*, $P(if\ A\ then\ B)$, to be the probability of the material conditional, $P(\bar{A} \vee B)$, but rather tend to assess it as the conditional probability of B given A, $P(B|A)$, or at least to converge on this assessment [5, 28, 30, 55, 70, 71, 83]. These psychological results have been taken to imply [5, 29, 41, 64, 71, 72], that *if A then B* is best represented, either as the probability conditional of Adams [3], or as the *conditional event B|A* of de Finetti [21, 22], the probability of which is $P(B|A)$. We will adopt the latter view in the present paper and base our analysis on conditional events and coherence (for related analyses, specifically on categorical syllogisms, squares of opposition under coherence and on generalized argument forms see [42, 77, 76, 82]). One possible objection to holding that $P(if\ A\ then\ B) = P(B|A)$ is that it is supposedly unclear how this relation extends to compounds of conditionals and makes sense of them [24, 26, 87]. Yet consider:

$$\overbrace{She\ will\ be\ angry}^{a}\ if\ \overbrace{her\ son\ gets\ a\ B}^{b}\ and\ \overbrace{she\ will\ be\ furious}^{f}\ if\ \overbrace{he\ gets\ a\ C}^{c}. \qquad (1)$$

The above conjunction appears to make sense, as does the following seemingly even more complex conditional construction [24]:

If *she will be angry* if *her son gets a* B, then *she will be furious* if *he gets a* C. (2)

We will show below, in reply to the objection, how to give sense to (1) and (2) in terms of *compound conditionals*. Specifically, we will interpret (1) as a *conjunction* of two conditionals ($a|b$ and $f|c$) and (2) in terms of a *conditional* whose antecedent ($a|b$) and consequent ($f|c$) are both conditionals (if $a|b$, then $f|c$). But we note first that the *iterated conditional* (2) validly follows from the conjunction (1) by the form of inference we will call *centering* which, as we will show, can be extended to the compounds of conditionals (see Section 3 below). We point out that our framework is quantitative rather than a logical one. Indeed in our approach, syntactically conjoined and iterated conditionals in natural language are analyzed as conditional random quantities, which can sometimes reduce to conditional events, given logical dependencies ([47, 50]). For instance, the biconditional event $A||B$, which we will define by $(B|A) \wedge (A|B)$, reduces to the conditional $(A \wedge B)|(A \vee B)$. Moreover, the notion of *biconditional centering* will be given.

The outline of the paper is as follows. In Section 2 we give some preliminaries on the notions of coherence and p-entailment for conditional random quantities, which assume values in $[0, 1]$. In Section 3, after recalling the notions of conjoined conditional and iterated conditional, we study the p-validity of centering in the case where the basic events are replaced by conditionals. In Section 4 we give some results on coherence, by determining the lower and upper bounds for the conclusion of two-premise centering; we also examine the classical case by obtaining the same lower and upper bounds. In Section 5, after recalling the classical biconditional introduction rule, we present an analogue in terms of conditional events (*biconditional AND rule*); we also obtain one-premise and two-premise biconditional centering. In Section 6 we determine the lower and upper bounds for the conclusion of two-premise biconditional centering. In Section 7 we investigate reversed inferences (i.e., inferences from the conclusion to its premises), by determining the lower and upper bounds for the premises of the biconditional AND rule. Section 8 sketches how to apply results of this paper to study selected counterfactuals, and remark that the Import-Export Principle is not valid in our approach which allows us to avoid Lewis' notorious triviality results. Section 9 concludes with some remarks on future work. Further details which expand Section 2 are given in Appendix A.

2. Some preliminaries

The coherence-based approach to probability and to other uncertain measures has been adopted by many authors (see, e.g., [7, 8, 12, 13, 14, 15, 16, 17, 18, 38, 49, 71, 88]); we recall below some basic aspects on the notions of coherence and of p-entailment. In Appendix A we will give further details on coherence of probability and prevision assessments.

2.1. Events and constituents

In our approach events represent uncertain facts described by (non ambiguous) logical propositions. An event A is a two-valued logical entity which is either true (T), or false (F). The indicator of an event A is a two-valued numerical quantity which is 1, or 0, according to whether A is true, or false, respectively, and we use the same symbol to refer to an event and its indicator. We denote by Ω the sure event and by $\varnothing$ the impossible one (notice that, when necessary, the symbol $\varnothing$ will denote the empty set). Given two events A and B, we denote by $A \wedge B$ the logical intersection, or conjunction, of A and B; moreover, we denote by $A \vee B$ the logical union, or disjunction, of A and B. To simplify notations, in many cases we denote the conjunction of A and B (and its indicator) as AB; of course, AB coincides with the product of A and B. We denote by $\bar{A}$ the negation of A. Of course, the truth values for conjunctions, disjunctions and negations are obtained by applying the propositional logic. Given any events A and B, we simply write $A \subseteq B$ to denote that A logically implies B, that is $A\bar{B} = \varnothing$, which means that A and $\bar{B}$ cannot both be true.

Given n events $A_1, \ldots, A_n$, as $A_i \vee \bar{A}_i = \Omega$, $i = 1, \ldots, n$, by expanding the expression $\bigwedge_{i=1}^{n}(A_i \vee \bar{A}_i)$, we obtain

$$\Omega = \bigwedge_{i=1}^{n}(A_i \vee \bar{A}_i) = (A_1 \cdots A_n) \vee (A_1 \cdots A_{n-1}\bar{A}_n) \vee \cdots \vee (\bar{A}_1 \cdots \bar{A}_n);$$

that is the sure event Ω is represented as the disjunction of 2^n logical conjunctions. By discarding the conjunctions which are impossible (if any), the remaining ones are the *constituents* generated by $A_1, \ldots, A_n$. Of course, the constituents are pairwise logically incompatible; then, they are a partition of Ω. We recall that $A_1, \ldots, A_n$ are logically independent when the number of constituents generated by them is 2^n. Of course, in case of some logical dependencies among $A_1, \ldots, A_n$, the number of constituents is less than 2^n. For instance, given two events A, B, with $A \subseteq B$, the constituents are: $AB, \bar{A}B, \bar{A}\bar{B}$. If not stated otherwise, we assume logical independence throughout the paper.

2.2. Conditional events and coherent probability assessments

Given two events E, H, with $H \neq \emptyset$, the conditional event $E|H$ is defined as a three-valued logical entity which is true (T), or false (F), or void (V), according to whether EH is true, or $\bar{E}H$ is true, or $\bar{H}$ is true, respectively. The notion of logical inclusion among events has been generalized to conditional events by Goodman and Nguyen in [52] (see also [49]). Given two conditional events $E_1|H_1$ and $E_2|H_2$, we say that $E_1|H_1$ implies $E_2|H_2$, denoted by $E_1|H_1 \subseteq E_2|H_2$, iff E_1H_1 *true* implies E_2H_2 *true* and $\bar{E}_2H_2$ *true* implies $\bar{E}_1H_1$ *true*; i.e., iff $E_1H_1 \subseteq E_2H_2$ and $\bar{E}_2H_2 \subseteq \bar{E}_1H_1$.

We recall that, agreeing to the betting metaphor, if you assess $P(E|H) = p$, then, for every real number s, you are willing to pay an amount ps and to receive s, or 0, or ps, according to whether EH is true, or $\bar{E}H$ is true, or $\bar{H}$ is true (bet called off), respectively. Then, the random gain associated with the assessment $P(E|H) = p$ is $G = sH(E - p)$.

Given a real function $P : \mathcal{K} \rightarrow \mathcal{R}$, where $\mathcal{K}$ is an arbitrary family of conditional events, let us consider a subfamily $\mathcal{F}_n = \{E_1|H_1, \ldots, E_n|H_n\}$ of $\mathcal{K}$, and the vector $\mathcal{P}_n = (p_1, \ldots, p_n)$, where $p_i = P(E_i|H_i)$, $i = 1, \ldots, n$. We denote by $\mathcal{H}_n$ the disjunction $H_1 \vee \cdots \vee H_n$. As $E_iH_i \vee \bar{E}_iH_i \vee \bar{H}_i = \Omega$, $i = 1, \ldots, n$, by expanding the expression $\bigwedge_{i=1}^{n}(E_iH_i \vee \bar{E}_iH_i \vee \bar{H}_i)$ we can represent Ω as the disjunction of 3^n logical conjunctions, some of which may be impossible. The remaining ones are the constituents generated by $\mathcal{F}_n$ and, of course, are a partition of Ω. We denote by $C_1, \ldots, C_m$ the constituents which logically imply $\mathcal{H}_n$ and (if $\mathcal{H}_n \neq \Omega$) by C_0 the remaining constituent $\bar{\mathcal{H}}_n = \bar{H}_1 \cdots \bar{H}_n$, so that

$$\mathcal{H}_n = C_1 \vee \cdots \vee C_m, \quad \Omega = \bar{\mathcal{H}}_n \vee \mathcal{H}_n = C_0 \vee C_1 \vee \cdots \vee C_m, \quad m + 1 \leqslant 3^n.$$

In the context of betting scheme, with the pair $(\mathcal{F}_n, \mathcal{P}_n)$ we associate the random gain $G = \sum_{i=1}^{n} s_iH_i(E_i - p_i)$, where $s_1, \ldots, s_n$ are n arbitrary real numbers. We observe that G is the difference between the amount that you receive, $\sum_{i=1}^{n} s_i(E_iH_i + p_i\bar{H}_i)$, and the amount that you pay, $\sum_{i=1}^{n} s_ip_i$, and represents the net gain from engaging each transaction $H_i(E_i - p_i)$, the scaling and meaning (buy or sell) of the transaction being specified by the magnitude and the sign of s_i respectively. Let g_h be the value of G when C_h is true; then $G \in \mathcal{D} = \{g_0, g_1, \ldots, g_m\}$. Of course, $g_0 = 0$. We denote by $\mathcal{D}_{\mathcal{H}_n}$ the set of values of G restricted to $\mathcal{H}_n$, that is $\mathcal{D}_{\mathcal{H}_n} = \{g_1, \ldots, g_m\}$.

Definition 1. The function P defined on $\mathcal{K}$ is said to be *coherent* if and only if, for every integer n, for every finite subfamily $\mathcal{F}_n$ of $\mathcal{K}$ and for every real numbers $s_1, \ldots, s_n$, one has: $\min \mathcal{D}_{\mathcal{H}_n} \leqslant 0 \leqslant \max \mathcal{D}_{\mathcal{H}_n}$.

Notice that the condition $\min \mathcal{D}_{\mathcal{H}_n} \leqslant 0 \leqslant \max \mathcal{D}_{\mathcal{H}_n}$ can be written in two equivalent ways: $\min \mathcal{D}_{\mathcal{H}_n} \leqslant 0$, or $\max \mathcal{D}_{\mathcal{H}_n} \geqslant 0$. As shown by Definition 1, a probability assessment is coherent if and only if, for any finite combination of n bets, it does not happen that the values $g_1, \ldots, g_m$ are all positive, or all negative (*no Dutch Book*). Further technical details on coherence of probability assessments on conditional events and on conditional random quantities are given in Appendix A.

2.3. Conditional random quantities and the notions of p-consistency and p-entailment

In what follows, if not specified otherwise, we will consider conditional random quantities which take values in a finite subset of $[0, 1]$. Based on the notions of p-consistency and p-entailment of Adams ([1]), which were formulated for conditional events in the setting of coherence (see, e.g., [43, 45, 48]), we will generalize these notions to these conditional random quantities. Let $X|H$ be a finite conditional random quantity and let $\{x_1, \ldots, x_r\}$ denote the set of possible values for the restriction of X to H. Then, $X|H \in [0, 1]$ if and only if $x_j \in [0, 1]$ for each $j = 1, \ldots, r$; indeed in this case coherence requires that $\mathbb{P}(X|H) \in [0, 1]$ (see, e.g., [50]).

Definition 2. Let $\mathcal{F}_n = \{X_i|H_i, \ i = 1, \ldots, n\}$ be a family of n conditional random quantities which take values in a finite subset of $[0, 1]$. Then, $\mathcal{F}_n$ is *p-consistent* if and only if, the (prevision) assessment $(\mu_1, \mu_2, \ldots, \mu_n) = (1, 1, \ldots, 1)$ on $\mathcal{F}_n$ is coherent.

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Definition 3. A p-consistent family $\mathcal{F}_n = \{X_i | H_i, \ i = 1, \ldots, n\}$ *p-entails* a conditional random quantity $X|H$ which takes values in a finite subset of $[0, 1]$, denoted by $\mathcal{F}_n \models_p X|H$, if and only if for any coherent (prevision) assessment $(\mu_1, \ldots, \mu_n, z)$ on $\mathcal{F}_n \cup \{X|H\}$ it holds that: if $\mu_1 = \cdots = \mu_n = 1$, then $z = 1$.

Of course, when $\mathcal{F}_n$ p-entails $X|H$, there may be coherent assessments $(\mu_1, \ldots, \mu_n, z)$ with $z \neq 1$, but in such case $\mu_i \neq 1$ for at least one index i. We say that the inference from $\mathcal{F}_n$ to $X|H$ is *p-valid* if and only if $\mathcal{F}_n \models_p X|H$.

Now, we generalize the notion of p-entailment between two finite families of conditional random quantities $\mathcal{F}$ and $\mathcal{F}'$.

Definition 4. Given two p-consistent finite families of conditional random quantities $\mathcal{F}$ and $\mathcal{F}'$, we say that $\mathcal{F}$ p-entails $\mathcal{F}'$ if and only if $\mathcal{F}$ p-entails $X|H$, for every $X|H \in \mathcal{F}'$.

Transitivity property of p-entailment: Of course, p-entailment is transitive; that is, given three p-consistent families of conditional random quantities $\mathcal{F}, \mathcal{F}', \mathcal{F}''$, if $\mathcal{F} \models_p \mathcal{F}'$ and $\mathcal{F}' \models_p \mathcal{F}''$, then $\mathcal{F} \models_p \mathcal{F}''$.

Remark 1. Notice that, from Definition 3, we trivially have that $\mathcal{F}$ p-entails $X|H$, for every $X|H \in \mathcal{F}$; then, by Definition 4, it immediately follows

$$\mathcal{F} \models_p \mathcal{F}', \ \ \forall \mathcal{F}' \subseteq \mathcal{F}, \ \mathcal{F}' \neq \varnothing. \tag{3}$$

Remark 2. Notice that, if we consider conditional events instead of conditional random quantities, we recover the usual notions of p-consistency, p-entailment, and p-validity.

3. Centering

Given a conditional event $B|A$, if you assess $P(B|A) = x$, then for the indicator of $B|A$ we have $B|A = AB + x\bar{A}$ (see Appendix Appendix A.3). Thus, when the conditioning event A is true then $B|A$ has the same value as B and as AB, while, when the conditioning event A is false then $B|A$ coincides with $x = P(B|A)$. This aspect seems related to the notion of (strong) centering used in Lewis' logic ([61]) in order to assign truth values to counterfactuals. In Remark 3 of this section we will show that $(B|A) \wedge A = AB$, that is $(B|A) \wedge A$ and AB are the same object; then, by the compound probability theorem, it holds that $P[(B|A) \wedge A] = P(AB) = P(B|A)P(A)$ and then $P[(B|A) \wedge A] = P(AB) \leqslant P(B|A)$. This inequality also follows by the Goodman & Nguyen inclusion relation $AB \subseteq B|A$ ([49, Theorem 6]). We recall that the equality $P[(B|A) \wedge A] = P(B|A)P(A)$ in [54] is named "the probabilistic version of centering" and it has been usually looked at as a probabilistic independence of the conditional *if A then B* from its premise A. We also recall that in [54, footnote 5] Hajek and Hall observe that "*centering is a slight misnomer, since this name usually refers to a property of the nearness relation used to give the truth conditions for the conditional (each world is the nearest world to itself)*"[2]. Interestingly, in [2] Adams has shown that the Lewis theory of nearest possible worlds can be interpreted as a theory of worlds nearest in probability; in other words, according to Adams' viewpoint, the Lewis logic may be considered as "the logic not of truth, but of high probability". By the previous remarks and in agreement with [54] (see also [53, p. 442]), we simply use the term centering also for the kind of inferences which we study in this paper.

There is *one-premise centering*: inferring *if A then B* from the single premise AB. And *two-premise centering*: inferring *if A then B* from the two separate premises A and B. Centering is valid for quite a wide range of conditionals ([19, 20, 66]). It is clearly valid for the material conditional, since *not-A or B* must be true when A *and* B is true. It is also valid for Lewis conditional *if A then B* ([61]), which holds, roughly, when B is true in the closest world in which A is true. In [61] Lewis has a semantic condition of centering, which states that the actual world is the closest world to itself. The characteristic axiom for this semantic condition is what we are also calling centering. It is probabilistically valid, *p-valid*, for the conditional event, i.e. AB p-entails $B|A$ and $\{A, B\}$ p-entails $B|A$. Centering is, however, not valid for inferentialist accounts of conditionals, where an inferential relation between antecedent and consequent is presupposed (see, e.g., [23]).

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[2] Lewis distinguished between "centering" and "weak centering" ([61]). We use "centering" simply for the equivalent of the former, "strong" notion on centering. There is psychological evidence that ordinary people conform to this notion of centering for indicatives ([20, 80]).

A (p-consistent) set of premises p-entails a conclusion if and only if the conclusion must have probability one when all the premises have probability one [48]. Clearly, one-premise centering is p-valid, indeed the p-entailment of $B|A$ from AB follows by observing that $P(AB) = P(A)P(B|A)$ and so $P(AB) \leqslant P(B|A)$: if $P(AB) = 1$, then $P(B|A) = 1$. Two-premise centering is also clearly p-valid, as it is p-valid to infer AB from A and B, and then one-premise centering can be used to infer $B|A$: if $P(A) = x$ and $P(B) = y$, coherence requires that $P(AB)$ has to be in the interval $[\max\{x + y - 1, 0\}, \min\{x, y\}]$, with $P(AB) \leqslant P(B|A)$. Therefore, if $P(A) = P(B) = 1$, it follows $P(AB) = P(B|A) = 1$ and then $\{A, B\}$ p-entails $B|A$.

We will study the p-validity of generalized versions of one-premise and two-premise centering, where the unconditional events A and B are replaced by the conditional events $A|H$ and $B|K$, respectively. These kinds of centering involve the notions of conjunction and of iterated conditioning for conditional events. Conjunction and iteration among conditionals have been studied from the viewpoint of random variables by many authors (see, e.g. [57, 58, 59, 85]); for an overview on conditionals, see, e.g., [3, 25, 27]. In our approach we exploit recent results obtained in the setting of coherence for conditional random quantities (see, e.g. [46, 47, 50, 51]).

3.1. Conjunction of two conditional events

We recall and discuss the notion of conjunction of two conditional events. Note that, in numerical terms, two conditional events $A|H$ and $B|K$, with $P(A|H) = x$ and $P(B|K) = y$, coincide with the random quantities $AH + x\bar{H}$ and $BK + y\bar{K}$, respectively. Then, $\min\{A|H, B|K\} = \min\{AH + x\bar{H}, BK + y\bar{K}\}$.

Definition 5 (Conjunction). Given any pair of conditional events $A|H$ and $B|K$, with $P(A|H) = x, P(B|K) = y$, we define their conjunction as the conditional random quantity

$$(A|H) \wedge (B|K) = \min\{A|H, B|K\} \,|\, (H \vee K) = \min\{AH + x\bar{H}, BK + y\bar{K}\} \,|\, (H \vee K).$$

Then, defining $z = \mathbb{P}[(A|H) \wedge (B|K)]$, we have

$$(A|H) \wedge (B|K) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } \bar{A}H \vee \bar{B}K \text{ is true,} \\ x, & \text{if } \bar{H}BK \text{ is true,} \\ y, & \text{if } AH\bar{K} \text{ is true,} \\ z, & \text{if } \bar{H}\bar{K} \text{ is true.} \end{cases} \tag{4}$$

From (4), the conjunction $(A|H) \wedge (B|K)$ is the following random quantity

$$(A|H) \wedge (B|K) = 1 \cdot AHBK + x \cdot \bar{H}BK + y \cdot AH\bar{K} + z \cdot \bar{H}\bar{K}. \tag{5}$$

Notice that the quantity $z = \mathbb{P}[(A|H) \wedge (B|K)]$ represents the value that you assess, with the proviso that, for each real number s, you will pay the amount sz by receiving the random quantity $s[(A|H) \wedge (B|K)]$. In particular, if $s = 1$, then you agree to pay z with the proviso that you will receive: 1, if both conditional events are true; 0, if at least one of the conditional events is false; x, if $A|H$ is void and $B|K$ is true; y, if $B|K$ is void and $A|H$ is true; z, if both conditional events are void. Notice that this notion of conjunction, with positive probabilities for the conditioning events, has been already proposed in [63].

Remark 3. We remark that in particular, given two events A and H, with $H \neq \emptyset$, $P(A|H) = x$, $P(H) = y$, $\mathbb{P}[(A|H) \wedge H] = z$, by (5) it holds that

$$(A|H) \wedge H = (A|H) \wedge (H|\Omega) = AHH\Omega + x \cdot \bar{H}H\Omega + y \cdot AH\emptyset + z \cdot \bar{H}\emptyset = AH. \tag{6}$$

Then, the conjunction $(A|H) \wedge H$ is equivalent to the unconditional event AH and $\mathbb{P}[(A|H) \wedge H] = P(AH) = P(A|H)P(H)$.

Notice that the notion of conjunction given in Definition 5, with *positive* probabilities for the *conditioning* events, has been already proposed in the context of betting scheme in [63]. By linearity of prevision it holds that

$$z = P(AHBK) + xP(\bar{H}BK) + yP(AH\bar{K}) + zP(\bar{H}\bar{K});$$

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in particular, if $P(H \vee K) > 0$ we obtain the following result given in [59, 63]:

$$\mathbb{P}[(A|H) \wedge (B|K)] = \frac{P(AHBK) + P(A|H)P(\bar{H}BK) + P(B|K)P(AH\bar{K})}{P(H \vee K)}.$$

We recall that a well-known notion of conjunction among conditional events, which plays an important role in non-monotonic reasoning, is the quasi conjunction [1, 6, 48], i.e., the following conditional event:

$$QC(A|H, B|K) = (AH \vee \bar{H}) \wedge (BK \vee \bar{K})|(H \vee K),$$

or in numerical terms, since $AH \vee \bar{H} = AH + \bar{H}$ and $BK \vee \bar{K} = BK + \bar{K}$:

$$QC(A|H, B|K) = \min\{AH + \bar{H}, BK + \bar{K}\} \,|\, (H \vee K).$$

The event $AH \vee \bar{H}$ is the material conditional associated with the conditional "if H then A". Then, the quasi conjunction is defined by taking the minimum of the material conditionals given $H \vee K$. However, we define the conjunction by taking the minimum of the conditional events given $H \vee K$. Our conjunction is (in general) a conditional random quantity, whereas the quasi conjunction is a conditional event. In some particular cases conjunction and quasi conjunction coincide; two cases examined in [47] are: (i) $x = y = 1$; (ii) $K = AH$ (or symmetrically $H = BK$). Moreover, classical results concerning lower and upper bounds for the conjunction of unconditional events, which do not hold for the upper bound of the quasi conjunction ([39, 49]), still hold for our notion of conjunction. This is shown in the next result ([50]).

Theorem 1. Given any coherent assessment (x, y) on $\{A|H, B|K\}$, with A, H, B, K logically independent, and with $H \neq \varnothing, K \neq \varnothing$, the extension $z = \mathbb{P}[(A|H) \wedge (B|K)]$ is coherent if and only if the Fréchet-Hoeffding bounds are satisfied:

$$max\{x + y - 1, 0\} = z' \leqslant z \leqslant z'' = min\{x, y\}. \tag{7}$$

Remark 4. We recall that, by logical independence of A, H, B, K, the assessment (x, y) is coherent for every $(x, y) \in [0, 1]^2$. From Theorem 1, the set Π of all coherent assessment (x, y, z) on $\mathcal{F} = \{A|H, B|K, (A|H) \wedge (B|K)\}$ is $\Pi = \{(x, y, z) : (x, y) \in [0, 1]^2, max\{x + y - 1, 0\} \leqslant z \leqslant min\{x, y\}\}$. Then, $z \in [0, 1]$ and $(A|H) \wedge (B|K) \in [0, 1]$. Moreover, as $(1, 1, 1) \in \Pi$, the family $\mathcal{F}$ (and so each subfamily of $\mathcal{F}$) is p-consistent. In particular, if $x = 1, y = 1$, then z must be equal to 1. Then, by Definition 3, $\{A|H, B|K\}$ p-entails $(A|H) \wedge (B|K)$, i.e.,

$$\{A|H, B|K\} \models_p (A|H) \wedge (B|K). \tag{8}$$

We call this inference rule "*AND* rule for conditional events". We also notice that the assessment $(x, y, 1) \in \Pi$ if and only if $x = 1$ and $y = 1$. Then, both $x = 1$ and $y = 1$ follow from $z = 1$, i.e. $(A|H) \wedge (B|K) \models_p \{A|H, B|K\}$, which is the converse of (8).

Remark 5. Assuming $HK = \varnothing$, it holds that the conjunction $(A|H) \wedge (B|K)$ coincides with the product $(A|H) \cdot (B|K)$; moreover

$$\mathbb{P}[(A|H) \wedge (B|K)] = \mathbb{P}[(A|H) \cdot (B|K)] = P(A|H)P(B|K),$$

which states that the random quantities $A|H$ and $B|K$ are *uncorrelated*; more details are given in ([50]).

3.2. Iterated conditioning

We recall and discuss the notion of iterated conditioning.

Definition 6 (Iterated conditioning). Given any pair of conditional events $A|H$ and $B|K$, the iterated conditional $(B|K)|(A|H)$ is defined as the conditional random quantity $(B|K)|(A|H) = (B|K) \wedge (A|H) + \mu\bar{A}|H$, where $\mu = \mathbb{P}[(B|K)|(A|H)]$.

Notice that, in the context of betting scheme, μ represents the amount you agree to pay, with the proviso that you will receive the quantity

$$(B|K)|(A|H) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } AH\bar{B}K \text{ is true,} \\ y, & \text{if } AH\bar{K} \text{ is true,} \\ \mu, & \text{if } \bar{A}H \text{ is true,} \\ x + \mu(1-x), & \text{if } \bar{H}BK \text{ is true,} \\ \mu(1-x), & \text{if } \bar{H}\,\bar{B}K \text{ is true,} \\ z + \mu(1-x), & \text{if } \bar{H}\,\bar{K} \text{ is true.} \end{cases} \tag{9}$$

We recall the following product formula ([47]).

Theorem 2 (Product formula). Given any assessment $x = P(A|H), \mu = \mathbb{P}[(B|K)|(A|H)], z = \mathbb{P}[(B|K) \wedge (A|H)]$, if (x, μ, z) is coherent, then $z = \mu \cdot x$, i.e.,

$$\mathbb{P}[(B|K) \wedge (A|H)] = \mathbb{P}[(B|K)|(A|H)]P(A|H). \tag{10}$$

As $z = \mu x$, it follows that $z + \mu(1-x) = \mu$. Then, from (9), $(B|K)|(A|H)$ coincides with

$$AHBK + yAH\bar{K} + (x + \mu(1-x))\bar{H}BK + \mu(1-x)\bar{H}\bar{B}K + \mu(\bar{A}H \vee \bar{H}\bar{K}).$$

Remark 6. As $x \geqslant z$, for $x > 0$ one has $\mu = \frac{z}{x} \in [0,1]$; moreover $x + \mu(1-x)$ is a linear convex combination of the values μ and 1, then $x + \mu(1-x) \in [\mu, 1]$. Therefore, for $x > 0$, $(B|K)|(A|H) \in [0,1]$. As shown in Theorem 4, $\mu \in [0,1]$ also for $x = 0$. Thus, $(B|K)|(A|H) \in [0,1]$ in all cases.

3.3. One-premise and two-premise centering: p-validity

The *one-premise centering* involving conditional events is represented by the following inference rule: *from* $(A|H) \wedge (B|K)$ *infer* $(B|K)|(A|H)$. Likewise, *two-premise centering* involving conditional events is represented by: *from* $\{A|H, B|K\}$ *infer* $(B|K)|(A|H)$. Are these inference rules p-valid?

One-premise centering is p-valid; indeed, from (10) it holds that

$$\mathbb{P}[(B|K) \wedge (A|H)] \leqslant \mathbb{P}[(B|K)|(A|H)], \tag{11}$$

then $\mathbb{P}[(B|K) \wedge (A|H)] = 1$ implies $\mathbb{P}[(B|K)|(A|H)] = 1$, i.e.,

$$(B|K) \wedge (A|H) \models_p (B|K)|(A|H). \tag{12}$$

Two-premise centering is also p-valid; indeed, from (8) and (12), by transitivity,

$$\{(A|H), (B|K)\} \models_p (B|K)|(A|H), \tag{13}$$

that is, if $P(A|H) = 1$ and $P(B|K) = 1$, then $\mathbb{P}[(B|K)|(A|H)] = 1$.

4. Lower and upper bounds for two-premise centering

In this section we give a probabilistic analysis of two-premise centering by determining the coherent lower and upper bounds for the conclusion. We first consider the general case: *from* $\{A|H, B|K\}$ *infer* $(B|K)|(A|H)$. Then, we consider two-premise centering with unconditional events in the premise set: *from* $\{A, B\}$ *infer* $B|A$, which is a particular case where $H = K = \Omega$.

4.1. The general case: "from $\{A|H, B|K\}$ infer $(B|K)|(A|H)$"

We start by computing the set of all coherent assessments on the elements of centering.

Theorem 3. Let A, B, H, K be any logically independent events. The set Π of all coherent assessments (x, y, z, μ) on the family $\mathcal{F} = \{A|H, B|K, (A|H) \wedge (B|K), (B|K)|(A|H)\}$ is $\Pi = \Pi' \cup \Pi''$, where

$$\Pi' = \{(x, y, z, \mu) : x \in (0, 1], y \in [0, 1], z \in [z', z''], \mu = \tfrac{z}{x}\},$$
$$\text{with } z' = \max\{x + y - 1, 0\}, z'' = \min\{x, y\}, \text{ and} \qquad (14)$$
$$\Pi'' = \{(0, y, 0, \mu) : (y, \mu) \in [0, 1]^2\}.$$

Proof. We recall that the assessment (x, y) on $\{A|H, B|K\}$ is coherent for every $(x, y) \in [0, 1]^2$. By Theorem 1, the assessment $z = \mathbb{P}[(A|H) \wedge (B|K)]$ is a coherent extension of (x, y) if and only if $z \in [z', z'']$, where $z' = \max\{x + y - 1, 0\}$ and $z'' = \min\{x, y\}$. Moreover, assuming $x > 0$, by Theorem 2 it holds that $\mu = \tfrac{z}{x}$. Then, every $(x, y, z, \mu) \in \Pi'$ is coherent, that is $\Pi' \subseteq \Pi$. Of course, if $x > 0$ and $(x, y, z, \mu) \notin \Pi'$, then (x, y, z, μ) is not coherent. Now, we assume $x = 0$, so that $z' = z'' = 0$. Then, we show that the assessment $(0, y, 0, \mu)$ is coherent if and only if $(y, \mu) \in [0, 1]^2$, that is $(0, y, 0, \mu) \in \Pi''$. As $x = 0$, it holds that $A|H = AH + x\bar{H} = AH$. Then, $(B|K)|(A|H) = (B|K)|AH = (BK + y\bar{K})|AH$ and $\mathcal{F} = \{A|H, B|K, (A|H) \wedge (B|K), (BK + y\bar{K})|AH\}$. The constituents C_h's and the points Q_h's associated with $(\mathcal{F}, \mathcal{M})$, where $\mathcal{M} = (0, y, 0, \mu)$, are given in Table 1. Denoting by $\mathcal{I}$ be the convex hull generated by $Q_1, Q_2, \dots, Q_8$,

	C_h	Q_h	
C_1	$AHBK$	$(1, 1, 1, 1)$	Q_1
C_2	$AH\bar{B}K$	$(1, 0, 0, 0)$	Q_2
C_3	$AH\bar{K}$	$(1, y, y, y)$	Q_3
C_4	$\bar{A}HBK$	$(0, 1, 0, \mu)$	Q_4
C_5	$\bar{A}H\bar{B}K$	$(0, 0, 0, \mu)$	Q_5
C_6	$\bar{A}H\bar{K}$	$(0, y, 0, \mu)$	Q_6
C_7	$\bar{H}BK$	$(0, 1, 0, \mu)$	Q_7
C_8	$\bar{H}\bar{B}K$	$(0, 0, 0, \mu)$	Q_8
C_0	$\bar{H}\bar{K}$	$(0, y, 0, \mu)$	$Q_0 = \mathcal{M}$

Table 1: Constituents C_h's and points Q_h's associated with the prevision assessment $\mathcal{M} = (0, y, 0, \mu)$ on $\mathcal{F} = \{A|H, B|K, (A|H) \wedge (B|K), (BK + y\bar{K})|AH\}$.

the coherence of the prevision assessment $\mathcal{M}$ on $\mathcal{F}$ requires that the condition $\mathcal{P} \in \mathcal{I}$ be satisfied; this amounts to the solvability of the following system

$$\mathcal{M} = \sum_{h=1}^{8} \lambda_h Q_h, \quad \sum_{h=1}^{8} \lambda_h = 1, \quad \lambda_h \geqslant 0, \quad h = 1, \dots, 8. \qquad (15)$$

As $\mathcal{M} = yQ_4 + (1 - y)Q_5$, the vector $(\lambda_1, \dots, \lambda_8) = (0, 0, 0, y, 1 - y, 0, 0, 0)$ is a solution of system (15) such that $\sum_{h:C_h \subseteq HK} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 = \lambda_4 + \lambda_5 = 1 > 0$, so that $\sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \dots + \lambda_6 = 1 > 0$ and $\sum_{h:C_h \subseteq K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_7 + \lambda_8 = 1 > 0$; while, $\sum_{h:C_h \subseteq AH} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 = 0$. Then, by (A.2), $\mathcal{I}_0 \subseteq \{4\}$ and $\mathcal{F}_0 \subseteq \{(BK + y\bar{K})|AH\}$. Thus, from Theorem 13, for checking coherence of $\mathcal{M}$ on $\mathcal{F}$ it is sufficient to study the coherence of $\mu = \mathbb{P}(\{(BK + y\bar{K})|AH\})$. The random gain for the assessment μ is

$$G = sAH(BK + y\bar{K} - \mu), \quad s \in \mathbb{R}.$$

Without loss of generality, we can assume $s = 1$. The constituents contained in AH are: $C_1 = AHBK, C_2 = AH\bar{B}K, C_3 = AH\bar{K}$. The corresponding values for the random gain G are: $g_1 = (1 - \mu)$, $g_2 = -\mu$, $g_3 = y - \mu$. Then, the set of values of G restricted to AH is $\mathcal{D}_{AH} = \{g_1, g_2, g_3\}$. As it can be verified,

$$\min \mathcal{D}_{AH} > 0 \iff \mu < 0, \quad \max \mathcal{D}_{AH} < 0 \iff \mu > 1.$$

Therefore, the condition of coherence on μ, that is $\min \mathcal{D}_{AH} \cdot \max \mathcal{D}_{AH} \leqslant 0$, is satisfied if and only if $\mu \in [0, 1]$. Thus, every assessment $(0, y, 0, \mu)$ is coherent if and only if $(0, y, 0, \mu) \in \Pi'' = \{(0, y, 0, \mu) : (y, \mu) \in [0, 1]^2\}$. Therefore $\Pi = \Pi' \cup \Pi''$. $\qquad \square$

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Based on Theorem 3, we obtain the following prevision propagation rule for two-premise centering:

Theorem 4. Let A, B, H, K be any logically independent events. Given a coherent assessment (x, y) on $\{A|H, B|K\}$, for the iterated conditional $(B|K)|(A|H)$ the extension $\mu = \mathbb{P}((B|K)|(A|H))$ is coherent if and only if $\mu \in [\mu', \mu'']$, where

$$\mu' = \begin{cases} \max\left\{0, \frac{x+y-1}{x}\right\}, & \text{if } x > 0; \\ 0, & \text{if } x = 0; \end{cases} \quad \mu'' = \begin{cases} \min\left\{1, \frac{y}{x}\right\}, & \text{if } x > 0; \\ 1, & \text{if } x = 0. \end{cases} \quad (16)$$

Proof. Assume that $x = 0$. From Theorem 3 it follows that the set of all coherent assessments (x, y, z, μ) on $\mathcal{F} = \{A|H, B|K, (A|H) \wedge (B|K), (B|K)|(A|H)\}$ is $\Pi'' = \{(0, y, 0, \mu) : (y, \mu) \in [0, 1]^2\}$. Then, μ is a coherent extension of (x, y) if and only if $\mu \in [\mu', \mu'']$, where $\mu' = 0$ and $\mu'' = 1$.

Assume that $x > 0$. From Theorem 3 it follows that the set of all coherent assessments (x, y, z, μ) on $\mathcal{F}$ is $\Pi' = \{(x, y, z, \mu) : 0 < x \leqslant 1, 0 \leqslant y \leqslant 1, z' \leqslant z \leqslant z'', \mu = \frac{z}{x}\}$, where $z' = \max\{x + y - 1, 0\}$ and $z'' = \min\{x, y\}$. Then, μ is a coherent extension of (x, y) if and only if $\mu \in [\mu', \mu'']$, where $\mu' = \frac{z'}{x} = \max\left\{\frac{x+y-1}{x}, 0\right\}$ and $\mu'' = \frac{z''}{x} = \min\left\{\frac{y}{x}, 1\right\}$. □

Remark 7. The p-validity of two-premise centering given in (13) directly follows as an instantiation of Theorem 4 with $x = 1$ and $y = 1$.

4.2. The case $H = K = \Omega$

In case of logical dependencies among events, as we know, the set of all coherent assessments may be smaller than the set given in Theorem 3. We examine the case $H = K = \Omega$, by showing that the set Π of all coherent assessments on $\mathcal{F} = \{A, B, AB, B|A\}$ is still the same as in Theorem 3.

Theorem 5. Let A, B be any logically independent events. The set Π of all coherent assessments (x, y, z, μ) on the family $\mathcal{F} = \{A, B, AB, B|A\}$ is $\Pi = \Pi' \cup \Pi''$, with Π' and Π'' as defined in formula (14).

Proof. We recall that the assessment (x, y) on $\{A, B\}$ is coherent for every $(x, y) \in [0, 1]^2$. The assessment $z = P(AB)$ is a coherent extension of (x, y) if and only if $z \in [z', z'']$, where $z' = \max\{x + y - 1, 0\}$ and $z'' = \min\{x, y\}$. Moreover, assuming $x > 0$, by compound probability theorem it holds that $\mu = \frac{z}{x}$. Then, every $(x, y, z, \mu) \in \Pi'$ is coherent, that is $\Pi' \subseteq \Pi$. Of course, if $x > 0$ and $(x, y, z, \mu) \notin \Pi'$, then (x, y, z, μ) is not coherent. Now, we assume $x = 0$, so that $z' = z'' = 0$. Then, we show that the assessment $(0, y, 0, \mu)$ is coherent if and only if $(y, \mu) \in [0, 1]^2$, that is $(0, y, 0, \mu) \in \Pi''$. The constituents C_h's and the points Q_h's associated with $(\mathcal{F}, \mathcal{P})$, where $\mathcal{P} = (0, y, 0, \mu)$, are given in Table 2. Denoting by $\mathcal{I}$ the convex hull generated by Q_1, Q_2, Q_3, Q_4, the coherence of the prevision assessment $\mathcal{P}$

	C_h	Q_h	
C_1	AB	$(1, 1, 1, 1)$	Q_1
C_2	$A\bar{B}$	$(1, 0, 0, 0)$	Q_2
C_3	$\bar{A}B$	$(0, 1, 0, \mu)$	Q_3
C_4	$\bar{A}\bar{B}$	$(0, 0, 0, \mu)$	Q_4

Table 2: Constituents C_h's and points Q_h's associated with the probability assessment $\mathcal{P} = (0, y, 0, \mu)$ on $\mathcal{F} = \{A, B, AB, B|A\}$.

on $\mathcal{F}$ requires that the condition $\mathcal{P} \in \mathcal{I}$ be satisfied; this amounts to the solvability of the following system

$$\mathcal{P} = \sum_{h=1}^{4} \lambda_h Q_h, \quad \sum_{h=1}^{4} \lambda_h = 1, \quad \lambda_h \geqslant 0, \quad h = 1, \ldots, 4. \quad (17)$$

As $\mathcal{P} = yQ_3 + (1 - y)Q_4$, the vector $(\lambda_1, \ldots, \lambda_4) = (0, 0, y, 1 - y)$ is a solution of system (17), with $\sum_{h:C_h \subseteq A} \lambda_h = 0$. Then, by (A.2), $\mathcal{I}_0 \subseteq \{4\}$ and $\mathcal{F}_0 \subseteq \{B|A\}$. Thus, from Theorem 13, for checking coherence of $\mathcal{P}$ on $\mathcal{F}$ it is sufficient to study the coherence of $\mu = P(B|A)$. Of course, $\mu = P(B|A)$ is coherent if and only if $\mu \in [0, 1]$. Thus, every assessment $(0, y, 0, \mu)$ is coherent if and only if $(0, y, 0, \mu) \in \Pi'' = \{(0, y, 0, \mu) : (y, \mu) \in [0, 1]^2\}$. Therefore $\Pi = \Pi' \cup \Pi''$. □

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Based on Theorem 5, we obtain the following prevision propagation rule for two-premise centering with unconditional events in the premise set:

Theorem 6. Let A, B be any logically independent events. Given a coherent assessment (x, y) on $\{A, B\}$, for the conditional event $B|A$ the extension $\mu = P(B|A)$ is coherent if and only if $\mu \in [\mu', \mu'']$, where

$$\mu' = \begin{cases} \max\left\{\frac{x+y-1}{x}, 0\right\}, & \text{if } x > 0; \\ 0, & \text{if } x = 0; \end{cases} \qquad \mu'' = \begin{cases} \min\left\{\frac{y}{x}, 1\right\}, & \text{if } x > 0; \\ 1, & \text{if } x = 0. \end{cases}$$

Proof. The proof is the same of Theorem 4, with $\mathcal{F} = \{A, B, AB, B|A\}$ and with Theorem 3 replaced by Theorem 5. $\qquad \square$

Remark 8. As shown by theorems 4 and 6, the lower and upper bounds on the conclusion of two-premise centering involving iterated conditionals coincide with the respective bounds on the conclusion of the (non-iterated) two-premise centering.

5. Biconditional centering

In classical logic the biconditional $A \leftrightarrow B$ (defined by $(\overline{A \vee B}) \vee (AB)$) can be represented by the conjunction of the two material conditionals $\bar{A} \vee B$ and $\bar{B} \vee A$. Therefore, $\{\bar{A} \vee B, \bar{B} \vee A\} \models A \leftrightarrow B$[3], which is called *biconditional introduction rule*. With the material conditional interpretation of a conditional, the biconditional $A \leftrightarrow B$ represents the conjunction of the two conditionals *if A then B* and *if B then A*. In this section, we present an analogue in terms of conditional events, by also giving a meaning to the conjunction of two conditional events $A|B$ and $B|A$.

From centering it follows that $\{A, B\} \models_p B|A$ and $\{A, B\} \models_p A|B$. Then, from $P(A) = P(B) = 1$ it follows that $P(B|A) = P(A|B) = 1$, which we denote by: $\{A, B\} \models_p \{A|B, B|A\}$. Thus, by applying (8) with $H = B$ and $K = A$, we obtain $\{A|B, B|A\} \models_p (A|B) \wedge (B|A)$ (which we call *biconditional introduction* rule, or *biconditional AND* rule). Then, by transitivity

$$\{A, B\} \models_p (A|B) \wedge (B|A). \tag{18}$$

In a similar way, we can prove that

$$AB \models_p (A|B) \wedge (B|A). \tag{19}$$

We recall that the conditional event $(AB)\,|\,(A \vee B)$, denoted by $A||B$, captures the notion of the *biconditional event*, which has been seen as the conjunction of two conditionals with the same truth table as the "defective" biconditional discussed in [32]; see also [30]. We have

Theorem 7. Given two events A and B it holds that: $(A|B) \wedge (B|A) = (AB)|(A \vee B) = A||B$.

Proof. We note that $(A|B) \wedge (B|A) = \min(A|B, B|A)|(A \vee B) = AB + \mu \cdot \bar{A}\bar{B}$, where $\mu = \mathbb{P}[(A|B) \wedge (B|A)]$; we also observe that $(AB)|(A \vee B) = AB + p \cdot \bar{A}\bar{B}$, where $p = P[(AB)|(A \vee B)]$. Then, under the assumption that "$(A \vee B)$ is true", the two random quantities $(A|B) \wedge (B|A)$ and $(AB)|(A \vee B)$ coincide. By coherence (see [50, Theorem 4]) it follows that these two random quantities coincide also under the assumption that "$(A \vee B)$ is false", that is μ and p coincide. Therefore, $(A|B) \wedge (B|A) = (AB)|(A \vee B)$. $\qquad \square$

Based on Theorem 7, we can now really interpret the biconditional event $A||B$ as the conjunction of the two conditionals $(B|A)$ and $(A|B)$. Moreover, equations (18) and (19) represent what we call *two-premise biconditional centering* and *one-premise biconditional centering* respectively, that is $\{A, B\} \models_p A||B$ and $AB \models_p A||B$.

Though in classical logic $\{\bar{A}, \bar{B}\} \models (A \leftrightarrow B)$, the analogue does not hold in our approach, since we do not have p-entailment of $A||B$ from $\bar{A}, \bar{B}$, indeed if $P(\bar{A}) = P(\bar{B}) = 1$, then $P(A \vee B) = 0$ and therefore $P(A||B) = P((AB)|(A \vee B)) \in [0, 1]$ (see Theorem 8 below). The biconditional event $A||B$ is of interest to psychologists because there is evidence that children go through a developmental stage in which they judge that $P(\textit{if A then B}) = P[(AB)|(A \vee B)]$,

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[3] We recall that the symbol $\models$ denotes the relation of logical entailment. In our case, if both events $\bar{A} \vee B$ and $\bar{B} \vee A$ are true, then the biconditional $A \leftrightarrow B$ is true.

with this judgment being replaced by *P(if A then B) = P(B|A)* as they grow older ([32]). We recall that, given two conditional events $A|H$ and $B|K$, their quasi conjunction is defined as the conditional event $Q(A|H, B|K) = [(AH \vee \bar{H}) \wedge (BK \vee \bar{K})]|(H \vee K)$. Quasi conjunction is a basic notion in the work of Adams ([1]) and plays a role in characterizing entailment from a conditional knowledge base (see also [6]). We recall that in [49] $A||B$ was interpreted by the quasi conjunction of $A|B$ and $B|A$, by obtaining $A||B = Q(A|B, B|A) = (AB)|(A \vee B)$.

6. Lower and upper bounds for two-premise biconditional centering

In this section we determine the lower and upper bounds for the conclusion of two premise biconditional centering.[4]

Theorem 8. Let A, B be any logically independent events. Given any (coherent) assessment $(x, y) \in [0, 1]^2$ on $\{A, B\}$, for the biconditional event $A||B$ the extension $z = P(A||B)$ is coherent if and only if $z \in [z', z'']$, where

$$z' = \max\{x + y - 1, 0\}, \quad z'' = \begin{cases} \frac{\min\{x,y\}}{\max\{x,y\}}, & \text{if } x > 0 \text{ or } y > 0, \\ 1, & \text{if } x = 0 \text{ and } y = 0. \end{cases} \tag{20}$$

Proof. We consider two cases: (i) $x > 0$ or $y > 0$; (ii) $x = 0$ and $y = 0$.

Case (i). As $P(A \vee B) \geqslant \max\{x, y\}$, it follows that $P(A \vee B) > 0$. Then, defining $v = P(AB)$, one has $P(A||B) = \frac{P(AB)}{P(A \vee B)} = \frac{v}{x+y-v}$. We recall that v is a coherent extension (x, y) if and only if $v \in [v', v'']$, where $v' = \max\{x + y - 1, 0\}$ and $v'' = \min\{x, y\}$. By observing that $f(v) = \frac{v}{x+y-v}$ is an increasing function of v, it follows that the assessment $z = P(A||B)$ is a coherent extension of (x, y) if and only if $z \in [z', z'']$, where $z' = \frac{v'}{x+y-v'} = \frac{\max\{x+y-1,0\}}{x+y-\max\{x+y-1,0\}} = \frac{\max\{x+y-1,0\}}{\min\{x+y,1\}} = \max\{x + y - 1, 0\}$, and $z'' = \frac{v''}{x+y-v''} = \frac{\min\{x,y\}}{\max\{x,y\}}$. We observe that if $x = 0$ or $y = 0$, then $z'' = z' = 0$; if $x = y = 1$, then $z' = z'' = 1$. Moreover, if $x > 0$ and $y > 0$, then $z'' = \min\{\frac{x}{y}, \frac{y}{x}\}$.

Case (ii) $(x = y = 0)$. The constituents C_h's, $h = 1, 2, 3, 4$, associated with the assessment $(0, 0, z)$ on $\{A, B, AB|(A \vee B)\}$, and the corresponding points Q_h's are $C_1 = AB, C_2 = A\bar{B}, C_3 = \bar{A}B, C_4 = \bar{A}\ \bar{B}$, and $Q_1 = (1, 1, 1), Q_2 = (1, 0, 0), Q_3 = (0, 1, 0), Q_4 = (0, 0, z)$, respectively. As the prevision point $(0, 0, z)$ coincides with Q_4, then it belongs to the convex hull of points $Q_1, \dots, Q_4$, that is the associated system (Σ), as defined in Section 2.2, is solvable. As $P(A \vee B) \leqslant \min\{x + y, 1\} = 0$, each solution $(\lambda_1, \dots, \lambda_4)$ of (Σ) is such that $\sum_{h: C_h \subseteq A \vee B} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 = 0$, so that $\mathcal{I}_0 = \{3\}$. By Theorem 13, $(0, 0, z)$ is coherent if and only if z is coherent, which amounts to $z \in [0, 1]$. $\square$

7. Reversed inferences and bounds on biconditional AND rule

In this section we first recall the lower and upper bounds on the conclusion of the biconditional AND rule. Then, we study the reverse inferences from the prevision assessment on the conclusion $A||B = (A|B) \wedge (B|A)$ to the premises $\{A|B, B|A\}$. That is, starting with a given assessment $z \in [0, 1]$ on $A||B$, we determine the set D_z of all coherent extensions (x, y), where $x = P(A|B)$ and $y = P(B|A)$. We recall the following probabilistic propagation rule ([49]). Let (x, y) be any coherent assessment on $\{A|B, B|A\}$; then, the probability assessment $z = P(A||B)$ is a coherent extension of (x, y) if and only if

$$z = T_0^H(x, y) = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0, \\ \frac{xy}{x+y-xy} = \frac{1}{\frac{1-x}{x} + \frac{1-y}{y} + 1}, & \text{if } 0 < x \leqslant 1 \text{ and } 0 < y \leqslant 1, \end{cases} \tag{21}$$

where $T_0^H(x, y)$ is the Hamacher t-norm, with parameter $\lambda = 0$. We obtain

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[4]Coherence of probability assessments on conditional events can be checked, for example, by the CkC-package [4].

Theorem 9. Let A, B be any logically independent events. Given any assessment $z \in [0, 1]$ on $A||B$, the extension (x, y) on $\{A|B, B|A\}$ is coherent if and only if $(x, y) \in D_z$, where

$$D_z = \begin{cases} \{(x, y) \in [0, 1]^2 : x = 0 \text{ or } y = 0\}, & \text{if } z = 0, \\ \{(x, y) \in [0, 1]^2 : z \leqslant x \leqslant 1, y = \frac{xz}{x-z+xz}\}, & \text{if } 0 < z \leqslant 1. \end{cases}$$

Proof. From (21) the set Π of all coherent assessments (x, y, z) on $\{A|B, B|A, A||B\}$ is

$$\Pi = \{(x, y, z) : (x, y) \in [0, 1]^2, z = T_0^H(x, y)\}. \tag{22}$$

Assume that $z = 0$. We notice that the assessment $(x, y, 0) \in \Pi$ if and only if $x = 0$ or $y = 0$, with $(x, y) \in [0, 1]^2$. Then, $D_0 = \{(x, y) \in [0, 1]^2 : x = 0 \text{ or } y = 0\}$. Assume that $0 < z \leqslant 1$. By Goodman and Nguyen inclusion relation among conditional events, as $AB|(A \vee B) \subseteq A|B$ and $AB|(A \vee B) \subseteq B|A$, coherence requires that (see, e.g., [49, Theorem 6]) $x \geqslant z > 0$ and $y \geqslant z > 0$; thus $xy > 0$, $x + y - xy > 0$, and $x - z + xz > 0$. Then, from (21) and (22) it holds that $z = \frac{xy}{x+y-xy}$, so that $y = \frac{xz}{x-z+xz}$. Therefore, $D_z = \{(x, y) : z \leqslant x \leqslant 1, y = \frac{xz}{x-z+xz}\}$. $\square$

Remark 9. Based on Theorem 9, the set Π of all coherent assessments (x, y, z) on $\{A|B, B|A, A||B\}$ can also be written as

$$\Pi = \{(x, y, z) : z \in [0, 1], (x, y) \in D_z\}.$$

Moreover, by symmetry, we observe that, if $z > 0$, the set D_z in Theorem 9 can also be written as $D_z = \{(x, y) \in [0, 1]^2 : z \leqslant y \leqslant 1, x = \frac{yz}{y-z+yz}\}$.

8. Two-premise centering with logical relations and counterfactuals

In this section we consider an instance of two premise-centering, with a logical dependency, that can be used to study some counterfactuals. Specifically, we consider the inference: $\{B|\Omega, C|A\}$ p-entails $(C|A)|(B|\Omega)$, with $AB = \varnothing$. As $B|\Omega = B$, this inference can be simply written as

$$\{B, C|A\} \models_p (C|A)|B, \quad \text{with } AB = \varnothing. \tag{23}$$

We first show that, assuming $P(B) > 0$, the prevision of the conclusion $(C|A)|B$ coincides just with $P(C|A)$, i.e., $\mathbb{P}[(C|A)|B] = P(C|A)$. By (5) the conjunction of B and $C|A$ reduces to the random quantity $(C|A) \wedge B = yB$, where $y = P(C|A)$. Then, by linearity of the prevision, $\mathbb{P}[(C|A) \wedge B] = P(C|A)P(B)$. Moreover, by (10), it holds that $\mathbb{P}[(C|A) \wedge B] = \mathbb{P}[(C|A)|B]P(B)$ and then, by assuming $P(B) > 0$, we obtain

$$\mathbb{P}[(C|A)|B] = \frac{\mathbb{P}[(C|A) \wedge B]}{P(B)} = \frac{P(C|A)P(B)}{P(B)} = P(C|A). \tag{24}$$

Now we show that (24) holds in general, even if $P(B) = 0$, by also showing that the iterated conditional $(C|A)|B$ is constant and coincides with $P(C|A)$, when $AB = \varnothing$. As $(C|A) \wedge B = yB$, by Definition 6, $(C|A)|B = yB + \mu\bar{B}$. Moreover, as $B \subseteq \bar{A}$, *conditionally on B being true*, it holds that: $C|A = AC + y\bar{A} = y$; that is, when B is true, $C|A$ is constant and equal to y. Then, by coherence, $\mu = \mathbb{P}[(C|A)|B] = \mathbb{P}[(AC + y\bar{A})|B] = \mathbb{P}(y|B) = y$ (see [46, Remark 1]). Therefore, when $AB = \varnothing$ it holds that $(C|A)|B = yB + y\bar{B} = y$, i.e., the iterated conditional $(C|A)|B$ is constant and equal to $P(C|A)$. Then, trivially, when $AB = \varnothing$ it holds that $\mathbb{P}[(C|A)|B] = P(C|A)$, i.e. the prevision of the iterated conditional "if B then (if A then C)" coincides with the probability of "(if A then C)". Therefore, the probability of B does not play a role in propagating the uncertainty from the premise set $\{B, C|A\}$ to the conclusion $(C|A)|B$. In particular, if $B = \bar{A}$, then $(C|A)|\bar{A} = P(C|A)$.

This result can be used as a model for some instances of counterfactuals. *Counterfactuals* are conditionals in the subjunctive mood, which people usually use when they believe that the antecedents are false. For example, the assertion of "If the glass had fallen from the table, then it would have broken" conversationally implies that the speaker believes that the glass did not fall. Counterfactuals are important for causal reasoning and for hypothetical thinking in general. There is experimental evidence that people judge the probability of a counterfactual, "If A were the case, then C would be the case", as the conditional probability, $P(C|A)$ [68, 78, 80]. Moreover, when presented with causal,

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e.g., "If a patient were to take certain drug, the symptoms would diminish", or non-causal task material, e.g., "If the card were to show a square, it would be black", people judge the negations of the antecedents to be irrelevant to the evaluation of the counterfactuals [78, 80]. These negations state the actual facts, e.g., "The patient does not take the drug", or "The side does not show a square", respectively. This speaks for the psychological plausibility of our basic intuition, which also underlies Stalnaker's extension of the Ramsey test to counterfactuals [26, 29, 84]: when we evaluate the counterfactual "If A were the case, C would be the case", we hypothetically remove, or set aside, our information that A is false from our beliefs and assess C under the assumption that A is true. This matches the psychological data [78, 80]. One starting point of modeling such situations is given by the aforementioned iterated conditional $(C|A)|B$, with $B \subseteq \bar{A}$, where B represents the factual statement which provides evidence that $\bar{A}$.

We remark that, contrary to [63], in general the iterated conditional $(C|A)|B$, when A, B, C are logically independent, does not coincide with the conditional event $C|AB$. Indeed, by setting $\mathbb{P}[(C|A)|B] = \mu$ and $P(C|A) = y$, from Definition 6 we obtain

$$(C|A)|B = (C|A) \wedge B + \mu\bar{B} = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } AB\bar{C} \text{ is true,} \\ y, & \text{if } \bar{A}B \text{ is true,} \\ \mu, & \text{if } \bar{B} \text{ is true,} \end{cases}$$

while, assuming $AB \neq \varnothing$ and $P(C|AB) = z$, it holds that

$$C|AB = ABC + z\overline{AB} = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } AB\bar{C} \text{ is true,} \\ z, & \text{if } \overline{AB} \text{ is true;} \end{cases}$$

thus: $(C|A)|B \neq C|AB$. Moreover, as $(C|A)|B = (AC + y\bar{A})|B = AC|B + y\bar{A}|B$, by linearity of prevision and product formula

$$\mathbb{P}[(C|A)|B] = P(C|AB)P(A|B) + P(C|A)P(\bar{A}|B). \tag{25}$$

Therefore, like in [1, 59], the Import-Export Principle is not valid in our approach. Then, as proved in [50], we avoid the counter-intuitive consequences related to Lewis' well-known first triviality result ([62]). Moreover, if the Import-Export Principle were added as an axiom to our theory, assuming $AB = \varnothing, A \neq \varnothing, B \neq \varnothing$, we would have on one hand $(C|A)|B = P(C|A)$; on the other hand it would be $(C|A)|B = C|AB = C|\varnothing$; thus, we would obtain an inconsistency. We also recall that, following de Finetti, objects like $C|\varnothing$ are not considered in our approach. Finally, we point out that we are able to manage counterfactuals; indeed, in our approach the counterfactual $C|A$ when A is believed to be false is not $C|\varnothing$, but $(C|A)|\bar{A}$, which coincides with $P(C|A)$.

9. Conclusions

We have presented a probabilistic analysis of the conjunction and iteration of conditional events, and of the centering inference for these conjunctions and iterations. In our approach conjoined conditionals and iterated conditionals are conditional random quantities defined in the setting of coherence. By this approach we can overcome some objections made in the past to the conditional probability hypothesis for natural language conditionals, that $P(if\ A\ then\ B) = P(B|A)$. This hypothesis is fundamental for the new Bayesian and probabilistic approaches in the psychology of reasoning and has been confirmed in many papers ([65, 67, 73, 74, 75, 79, 81]). This identity is also central to our analysis of both indicative and counterfactual conditionals as conditional events.

We have proved the p-validity of one-premise and two-premise centering when basic events are replaced by conditional events. We have determined the lower and upper bounds for the conclusion of two-premise centering; we have also studied the classical case and have obtained the same lower and upper bounds. We have proved the p-validity of an analogue of the classical biconditional introduction rule for conditional events (biconditional AND rule). We have verified the p-validity of one-premise and two-premise biconditional centering, and have given the lower and upper bounds for the conclusion of two-premise biconditional centering. We have investigated reversed inferences, by determining the lower and upper bounds for the premises of the biconditional AND rule. We have briefly indicated how to apply our results to the study of selected counterfactuals.

It is often argued that there are deep differences between indicative and counterfactual conditionals. For example, the indicative conditional, "If Oswald did not kill Kennedy then someone else did", is not equivalent to the counterfactual conditional, "If Oswald had not killed Kennedy then someone else would have" ([26]). We will explore in future work the more detailed similarities and differences between these two forms of the conditional.

Acknowledgement

Appendix A. Some technical aspects on coherence

In this Appendix, which expands Section 2.2, we illustrate some technical aspects which concern coherence of probability and prevision assessments on conditional events and conditional random quantities.

Appendix A.1. Coherence Checking

Let be given a family $\mathcal{F}_n = \{E_1|H_1, \ldots, E_n|H_n\}$ and a probability assessment $\mathcal{P}_n = (p_1, \ldots, p_n)$ on $\mathcal{F}_n$, where $p_i = P(E_i|H_i)$, $i = 1, \ldots, n$. We set $J_n = \{1, 2, \ldots, n\}$. We recall that the constituents $C_0, C_1, \ldots, C_m$ generated by $\mathcal{F}_n$, where $C_0 = \overline{\mathcal{H}}_n = \overline{H}_1\overline{H}_2\cdots\overline{H}_n$, are obtained by expanding the expression $\bigwedge_{i=1}^{n}(E_iH_i \vee \overline{E}_iH_i \vee \overline{H}_i)$. For each $h \in J_m$, with the constituent C_h we associate a point $Q_h = (q_{h1}, \ldots, q_{hn})$, where for each $j \in J_n$, $q_{hj} = 1$, or 0, or p_j, according to whether $C_h \subseteq E_jH_j$, or $C_h \subseteq \overline{E}_jH_j$, or $C_h \subseteq \overline{H}_j$. Denoting by $\mathcal{I}$ the convex hull of $Q_1, \ldots, Q_m$, by a suitable alternative theorem ([31, Theorem 2.9]), the condition $\mathcal{P}_n \in \mathcal{I}$ is equivalent to the condition $\min \mathcal{D}_{\mathcal{H}_n} \leqslant 0 \leqslant \max \mathcal{D}_{\mathcal{H}_n}$ given in Definition 1 (see, e.g., [37, 49]). Moreover, the condition $\mathcal{P}_n \in \mathcal{I}$ amounts to the solvability of the following system (Σ) in the unknowns $\lambda_1, \ldots, \lambda_m$

$$(\Sigma): \qquad \sum_{h=1}^{m} q_{hj}\lambda_h = p_j, \quad j \in J_n; \quad \sum_{h=1}^{m} \lambda_h = 1; \quad \lambda_h \geqslant 0, \ h \in J_m.$$

We say that system (Σ) is associated with the pair $(\mathcal{F}_n, \mathcal{P}_n)$. Hence, the following result provides a characterization of the notion of coherence given in Definition 1 ([33, Theorem 4.4], see also [34, 44, 49])

Theorem 10. Let $\mathcal{K}$ be an arbitrary family of conditional events and let P be a probability function defined on $\mathcal{K}$. The function P is coherent if and only if, for every finite subfamily $\mathcal{F}_n = \{E_1|H_1, \ldots, E_n|H_n\}$ of $\mathcal{K}$, denoting by $\mathcal{P}_n$ the vector $(p_1, \ldots, p_n)$, where $p_j = P(E_j|H_j)$, $j = 1, 2, \ldots, n$, the system (Σ) associated with the pair $(\mathcal{F}_n, \mathcal{P}_n)$ is solvable.

We recall now some results on the coherence checking of a probability assessment on a finite family of conditional events. Given a probability assessment $\mathcal{P}_n = (p_1, \ldots, p_n)$ on a finite family of conditional events $\mathcal{F}_n = \{E_1|H_1, \ldots, E_n|H_n\}$, let S be the set of solutions $\Lambda = (\lambda_1, \ldots, \lambda_m)$ of the system (Σ). Then, assuming $S \neq \emptyset$, we define

$$\Phi_j(\Lambda) = \Phi_j(\lambda_1, \ldots, \lambda_m) = \sum_{r:C_r \subseteq H_j} \lambda_r, \quad j \in J_n; \ \Lambda \in S;$$
$$M_j = \max_{\Lambda \in S} \Phi_j(\Lambda), \quad j \in J_n; \quad I_0 = \{j : M_j = 0\}.$$

Of course, if $S \neq \emptyset$, then S is a closed bounded set and the maximum M_j of the linear function $\Phi_j(\Lambda) = \sum_{r:C_r \subseteq H_j} \lambda_r$ there exists for every $j \in J_n$. We observe that, assuming $\mathcal{P}_n$ coherent, each solution $\Lambda = (\lambda_1, \ldots, \lambda_m)$ of system (Σ) is a coherent extension of the assessment $\mathcal{P}_n$ on $\mathcal{F}_n$ to the family $\{C_1|\mathcal{H}_n, C_2|\mathcal{H}_n, \ldots, C_m|\mathcal{H}_n\}$. Then, by the additivity property, the quantity $\Phi_j(\Lambda)$ is the conditional probability $P(H_j|\mathcal{H}_n)$ and the quantity M_j is the upper probability $P^*(H_j|\mathcal{H}_n)$ over all the solutions Λ of system (Σ). Of course, $j \in I_0$ if and only if $P^*(H_j|\mathcal{H}_n) = 0$. Notice that I_0 is a strict subset of J_n. We denote by $(\mathcal{F}_0, \mathcal{P}_0)$ the pair associated with I_0. Given the pair $(\mathcal{F}_n, \mathcal{P}_n)$ and a (nonempty) strict

subset J of J_n, we denote by $(\mathcal{F}_J, \mathcal{P}_J)$ the pair associated with J and by (Σ_J) the corresponding system. We observe that (Σ_J) is solvable if and only if $\mathcal{P}_J \in \mathcal{I}_J$, where $\mathcal{I}_J$ is the convex hull associated with the pair $(\mathcal{F}_J, \mathcal{P}_J)$. Then, we have ([35, Theorem 3.2]; see also [9, 36])

Theorem 11. Given a probability assessment $\mathcal{P}_n$ on the family $\mathcal{F}_n$, if the system (Σ) associated with $(\mathcal{F}_n, \mathcal{P}_n)$ is solvable, then for every $J \subset J_n$, such that $J \backslash I_0 \neq \emptyset$, the system (Σ_J) associated with $(\mathcal{F}_J, \mathcal{P}_J)$ is solvable too.

The previous result says that the condition $\mathcal{P}_n \in \mathcal{I}$ implies $\mathcal{P}_J \in \mathcal{I}_J$ when $J \backslash I_0 \neq \emptyset$. We observe that, if $\mathcal{P}_n \in \mathcal{I}$, then for every nonempty subset J of $J_n \backslash I_0$ it holds that $J \backslash I_0 = J \neq \emptyset$; hence, by Theorem 10, the subassessment $\mathcal{P}_{J_n \backslash I_0}$ on the subfamily $\mathcal{F}_{J_n \backslash I_0}$ is coherent. In particular, when I_0 is empty, coherence of $\mathcal{P}_n$ amounts to solvability of system (Σ), that is to condition $\mathcal{P}_n \in \mathcal{I}$. When I_0 is not empty, coherence of $\mathcal{P}_n$ amounts to the validity of both conditions $\mathcal{P}_n \in \mathcal{I}$ and $\mathcal{P}_0$ coherent, as shown below ([35, Theorem 3.3]).

Theorem 12. The assessment $\mathcal{P}_n$ on $\mathcal{F}_n$ is coherent if and only if the following conditions are satisfied: (i) $\mathcal{P}_n \in \mathcal{I}$; (ii) if $I_0 \neq \emptyset$, then $\mathcal{P}_0$ is coherent.

Appendix A.2. Coherent conditional prevision assessments

We denote by X a random quantity, that is (following de Finetti, see also [60]) an uncertain real quantity, which has a well determined but unknown value. We recall that in the axiomatic approach to probability, usually X is defined as a random variable. We assume that X is a finite random quantity, that is X has a finite set of possible values. In particular, (the indicator of) any given event A is a two-valued random quantity, with $A \in \{0, 1\}$. Given an event $H \neq \emptyset$ and a finite random quantity X, let $\{x_1, x_2, \ldots, x_r\}$ be the set of possible values of X restricted to H, which means that if H is true then $X \in \{x_1, x_2, \ldots, x_r\}$. As an example, given an event $H \neq \emptyset$ and a finite random quantity X with a set of possible values $\{x_1, x_2, \ldots, x_n\}$. Let be $A_i = (X = x_i)$, $i = 1, 2, \ldots, n$. Assume that $A_i H \neq \emptyset$ for $i = i_1, i_2, \ldots, i_r$ and with $A_i H = \emptyset$ otherwise. Then, the set of possible values of X restricted to H is $\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\}$. Notice that the set $\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\}$ is nonempty because $\bigvee_{i=1}^n A_i H = (\bigvee_{i=1}^n A_i) H = \Omega H = H \neq \emptyset$. Indeed, if it were $A_i H = \emptyset$ for $i = 1, \ldots, n$, then it would follows that $H = \bigvee_{i=1}^n A_i H = \emptyset$ (which is a contradiction).

Agreeing to the betting metaphor, by assessing the prevision of "*X conditional on H*" (also named "*X given H*"), $\mathbb{P}(X|H)$, as the amount μ, then for any given real number s you are willing to pay an amount μs and to receive Xs, or μs, according to whether H is true, or false (bet called off), respectively. Then, the random gain associated with the assessment $\mathbb{P}(X|H) = \mu$ is $G = sH(X - \mu)$. We remark that, differently from the notion of expected value, in the subjective approach of de Finetti the prevision of a random quantity is a primitive notion and its value can be assessed in a direct way. In particular, when X is (the indicator of) an event A, then $\mathbb{P}(X|H) = P(A|H)$. We recall the notion of coherence for conditional prevision assessments ([10, 11, 50, 69]). Given a function $\mathbb{P}$ defined on an arbitrary family $\mathcal{K}$ of finite conditional random quantities, consider a finite subfamily $\mathcal{F}_n = \{X_i|H_i, i \in J_n\} \subseteq \mathcal{K}$ and the vector $\mathcal{M}_n = (\mu_i, i \in J_n)$, where $\mu_i = \mathbb{P}(X_i|H_i)$, $i \in J_n$. With the pair $(\mathcal{F}_n, \mathcal{M}_n)$ we associate the random gain $G = \sum_{i \in J_n} s_i H_i(X_i - \mu_i)$, where $s_1, s_2, \ldots, s_n$ are arbitrary real numbers; moreover, as made for the conditional events, we denote by $\mathcal{D}$ the set of values of G and by $\mathcal{D}_{\mathcal{H}_n}$, where $\mathcal{H}_n = H_1 \vee \cdots \vee H_n$, the set of values of G restricted to $\mathcal{H}_n$. Then, using the *betting scheme* of de Finetti, we have

Definition 7. The function $\mathbb{P}$ defined on $\mathcal{K}$ is coherent if and only if, $\forall n \geqslant 1$, $\forall \mathcal{F}_n \subseteq \mathcal{K}$, $\forall s_1, \ldots, s_n \in \mathbb{R}$, it holds that: $\min \mathcal{D}_{\mathcal{H}_n} \leqslant 0 \leqslant \max \mathcal{D}_{\mathcal{H}_n}$.

Given a family $\mathcal{F}_n = \{X_1|H_1, \ldots, X_n|H_n\}$, for each $i \in J_n$ we denote by $\{x_{i1}, \ldots, x_{ir_i}\}$ the set of possible values for the restriction of X_i to H_i; then, for each $i \in J_n$ and $j = 1, \ldots, r_i$, we set $A_{ij} = (X_i = x_{ij})$. Of course, for each $i \in J_n$, the family $\{\bar{H}_i, A_{ij} H_i, j = 1, \ldots, r_i\}$ is a partition of the sure event Ω, with $A_{ij} H_i = A_{ij}$, $\bigvee_{j=1}^{r_i} A_{ij} = H_i$. Then, the constituents generated by the family $\mathcal{F}_n$ are (the elements of the partition of Ω) obtained by expanding the expression $\bigwedge_{i \in J_n}(A_{i1} \vee \cdots \vee A_{ir_i} \vee \bar{H}_i)$. We set $C_0 = \bar{H}_1 \cdots \bar{H}_n$ (C_0 may be equal to $\emptyset$); moreover, we denote by $C_1, \ldots, C_m$ the constituents contained in $\mathcal{H}_n = H_1 \vee \cdots \vee H_n$. Hence $\bigwedge_{i \in J_n}(A_{i1} \vee \cdots \vee A_{ir_i} \vee \bar{H}_i) = \bigvee_{h=0}^m C_h$. With each C_h, $h \in J_m$, we associate a vector $Q_h = (q_{h1}, \ldots, q_{hn})$, where $q_{hi} = x_{ij}$ if $C_h \subseteq A_{ij}$, $j = 1, \ldots, r_i$, while $q_{hi} = \mu_i$ if $C_h \subseteq \bar{H}_i$; the vector associated with C_0 is $Q_0 = \mathcal{M}_n = (\mu_1, \ldots, \mu_n)$. Denoting by $\mathcal{I}$ the convex hull of $Q_1, \ldots, Q_m$, the condition $\mathcal{M}_n \in \mathcal{I}$ amounts to the existence of a vector $(\lambda_1, \ldots, \lambda_m)$ such that: $\sum_{h \in J_m} \lambda_h Q_h = \mathcal{M}_n$, $\sum_{h \in J_m} \lambda_h = 1$, $\lambda_h \geqslant 0$, $\forall h$; in other words, $\mathcal{M}_n \in \mathcal{I}$ is equivalent to the solvability of the following system, associated with $(\mathcal{F}_n, \mathcal{M}_n)$,

$$\sum_{h \in J_m} \lambda_h q_{hi} = \mu_i, \ i \in J_n; \ \sum_{h \in J_m} \lambda_h = 1; \ \lambda_h \geqslant 0, \ h \in J_m. \tag{A.1}$$

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Given the assessment $\mathcal{M}_n = (\mu_1, \ldots, \mu_n)$ on $\mathcal{F}_n = \{X_1|H_1, \ldots, X_n|H_n\}$, let S be the set of solutions $\Lambda = (\lambda_1, \ldots, \lambda_m)$ of system (A.1). Then, assuming the system (A.1) solvable, that is $S \neq \varnothing$, we define:

$$I_0 = \{i : \max_{\Lambda \in S} \sum_{h:C_h \subseteq H_i} \lambda_h = 0\}, \ \mathcal{F}_0 = \{X_i|H_i, i \in I_0\}, \ \mathcal{M}_0 = (\mu_i, i \in I_0) . \tag{A.2}$$

Then, the following theorem can be proved ([10, Theorem 3]):

Theorem 13. [*Operative characterization of coherence*] A conditional prevision assessment $\mathcal{M}_n = (\mu_1, \ldots, \mu_n)$ on the family $\mathcal{F}_n = \{X_1|H_1, \ldots, X_n|H_n\}$ is coherent if and only if the following conditions are satisfied:
(i) the system (A.1) is solvable;
(ii) if $I_0 \neq \varnothing$, then $\mathcal{M}_0$ on $\mathcal{F}_0$ is coherent.

Appendix A.3. Conditional previsions as previsions of conditional random quantities

We recall that usually in the literature a conditional random quantity $X|H$ is understood as the restriction of X to H, with $X|H$ undefined when H is false. By the betting scheme, if you assess $\mathbb{P}(X|H) = \mu$, the random quantity that you receive by paying μ is $XH + \mu\bar{H}$. From coherence, it holds that $\mathbb{P}(XH + \mu\bar{H}) = \mu$; indeed, if you would assess $\mathbb{P}(XH + \mu\bar{H}) = \mu^* \neq \mu$, then the random gain associated with the assessment (μ, μ^*) on $\{X|H, XH + \mu\bar{H}\}$ would be

$$G = s_1 H(X - \mu) + s_2(XH + \mu\bar{H} - \mu^*) .$$

Then, by choosing $s_1 = 1$ and $s_2 = -1$, the random gain G would be equal to the nonzero constant: $\mu^* - \mu$ (a Dutch book). In what follows, by the symbol $X|H$ we denote the random quantity $XH + \mu\bar{H}$, where $\mu = \mathbb{P}(X|H)$. This random quantity, which extends the restriction of X to H, coincides with X when H is true and is equal to μ when H is false ([50]; see also [46, 47, 60]). As shown before the conditional prevision $\mathbb{P}(X|H)$ is the prevision of the conditional random quantity $X|H$. In this way, based on the betting scheme $X|H$ is the amount that you receive in a bet on X conditional on H, if you agree to pay $\mathbb{P}(X|H)$. Moreover, denoting by $\{x_1, x_2, \ldots, x_r\}$ the set of possible values of X when H is true, and defining $A_i = (X = x_i)$, $i = 1, 2, \ldots, r$, the family $\{A_1 H, \ldots, A_r H, \bar{H}\}$ is a partition of the sure event Ω and we have

$$X|H = XH + \mu\bar{H} = x_1 A_1 H + \cdots + x_r A_r H + \mu\bar{H} \in \{x_1, x_2, \ldots, x_r, \mu\} .$$

In particular, when X is (the indicator of) an event A, the prevision of $X|H$ is the probability of the conditional event $A|H$ and, if you assess $P(A|H) = p$, then for the indicator of $A|H$ we have $A|H = AH + p\bar{H} \in \{1, 0, p\}$. We observe that the choice of p as the value of $A|H$ when H is false has been also considered in some previous works ([17, 33, 57, 60, 63, 85, 86]).

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