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Generalized logical operations among conditional events

Angelo Gilio[1] · Giuseppe Sanfilippo[2]

Abstract

We generalize, by a progressive procedure, the notions of conjunction and disjunction of two conditional events to the case of n conditional events. In our coherence-based approach, conjunctions and disjunctions are suitable conditional random quantities. We define the notion of negation, by verifying De Morgan's Laws. We also show that conjunction and disjunction satisfy the associative and commutative properties, and a monotonicity property. Then, we give some results on coherence of prevision assessments for some families of compounded conditionals; in particular we examine the Fréchet-Hoeffding bounds. Moreover, we study the reverse probabilistic inference from the conjunction $\mathcal{C}_{n+1}$ of $n+1$ conditional events to the family $\{\mathcal{C}_n, E_{n+1}|H_{n+1}\}$. We consider the relation with the notion of quasi-conjunction and we examine in detail the coherence of the prevision assessments related with the conjunction of three conditional events. Based on conjunction, we also give a characterization of p-consistency and of p-entailment, with applications to several inference rules in probabilistic nonmonotonic reasoning. Finally, we examine some non p-valid inference rules; then, we illustrate by an example two methods which allow to suitably modify non p-valid inference rules in order to get inferences which are p-valid.

Keywords Conditional events · Conditional random quantities · Conjunction · Disjunction · Negation · Fréchet-Hoeffding bounds · Coherent prevision assessments · Coherent extensions · Quasi conjunction · Probabilistic reasoning · p-entailment · Inference rules

1 Introduction

The research on combining logic and probability has a long history (see, e.g., [2, 8, 13, 16, 33]). In this paper we use a coherence-based approach to probability, which allows to introduce probability assessments on arbitrary families of conditional events, by properly managing conditioning events of zero probability (see, e.g., [4, 5, 13, 18, 19, 25, 26, 28, 42]). In probability theory and in probability logic a

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relevant problem, largely discussed by many authors (see, e.g., [3, 14, 15, 32]), is that of suitably defining logical operations among conditional events. In a pioneering paper, written in 1935 by de Finetti [16], it was proposed a three-valued logic for conditional events coinciding with that one of Lukasiewicz. A survey of the many contributions by different authors (such as Adams, Belnap, Calabrese, de Finetti, Dubois, van Fraassen, McGee, Goodmann, Lewis, Nguyen, Prade, Schay) to research on three-valued logics and compounds of conditionals has been given in [39]; conditionals have also been extensively studied in [17, 38]. In the literature, the conjunction and disjunction have been usually defined as suitable conditionals; see e.g. [2, 9, 11, 32]. A theory for the compounds of conditionals has been proposed in [34, 38]. A related theory has been developed in the setting of coherence in [22, 23, 26]; in these papers, conjunction and disjunction of two conditional events are not defined as conditional events, but as suitable *conditional random quantities*, with values in the interval [0, 1]. In the present paper we generalize the notions of conjunction and disjunction of two conditional events to the case of n conditional events; we also give the notion of negation. Then, we examine a monotonicity

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property for conjunction and disjunction. Moreover, we give some results on coherence of prevision assessments for some families of compounded conditionals; in particular we examine the Fréchet-Hoeffding bounds. Finally, we examine in detail the coherence of prevision assessments related with the conjunction of three conditional events. The paper is organized as described below. In Section 2 we recall some preliminary notions and results which concern coherence, quasi conjunction, conjunction, disjunction, and Fréchet-Hoeffding bounds. In Section 3 we introduce, in a progressive way, the notions of conjunction and disjunction for n conditional events; then, we define the notion of negation and we show that De Morgan's Laws are satisfied. We define the notion of conjunction (resp., disjunction) for the conjunctions (resp., disjunctions) associated with two families of conditional events, by showing then the validity of commutative and associative properties. In Section 4, after a preliminary result concerning the inequality $X|H \leq Y|K$ between two conditional random quantities, we show that the conjunction $\mathcal{C}_{n+1}$ of $n + 1$ conditional events is a conditional random quantity less than or equal to any conjunction $\mathcal{C}_n$ of a subfamily of n conditional events. Likewise, we show that the disjunction $\mathcal{D}_{n+1}$ of $n + 1$ conditional events is greater than or equal to any disjunction $\mathcal{D}_n$ of a subfamily of n conditional events. We also show that $\mathcal{C}_n$ and $\mathcal{D}_n$ belong to the interval $[0, 1]$. Moreover, we derive some inequalities from the monotony property. In Section 5, based on a geometrical approach, we characterize by an iterative procedure the set of coherent assessments on the family $\{C_n, E_{n+1}|H_{n+1}, \mathcal{C}_{n+1}\}$. In Section 6 we study the (reverse) inference from $\mathcal{C}_{n+1}$ to $\{\mathcal{C}_n, E_{n+1}|H_{n+1}\}$, by determining the set of coherent extensions (μ_n, x_{n+1}) of any coherent assessment μ_{n+1}, where $\mu_n = \mathbb{P}(\mathcal{C}_n)$, $x_{n+1} = P(E_{n+1}|H_{n+1})$, and $\mu_{n+1} = \mathbb{P}(\mathcal{C}_{n+1})$. In Section 7 we show that the prevision of the conjunction $\mathcal{C}_n$ satisfies the Fréchet-Hoeffding bounds. Then, by exploiting De Morgan's Laws, we give the dual result for the disjunction $\mathcal{D}_n$. In Section 8 we examine in detail the conjunction for a family of three conditional events $E_1|H_1, E_2|H_2, E_3|H_3$. We also consider the relation with the notion of quasi-conjunction studied in [2]; see also [24, 25]. We also determine the set of coherent prevision assessments on the whole family $\{E_1|H_1, E_2|H_2, E_3|H_3, (E_1|H_1) \wedge (E_2|H_2), (E_1|H_1) \wedge (E_3|H_3), (E_2|H_2) \wedge (E_3|H_3), (E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)\}$. Moreover, we consider the particular case where $H_1 = H_2 = H_3 = H$. In Section 9, by applying our notion of conjunction, we give a characterization of p-consistency and p-entailment and we examine some p-valid inference rules in probabilistic nonmonotonic reasoning. Moreover, based on a suitable notion of iterated conditioning, we briefly describe a characterization of p-entailment in the case of two premises. In Section 10, after examining some non p-valid inference rules, we illustrate two methods

which allow to construct p-valid inferences. Finally, in Section 11 we give a summary of results. Notice that for almost all (new) results of this paper the proofs are given in Appendix A.

2 Some preliminaries

In this section we recall some basic notions and results on coherence (see, e.g., [5, 7, 10, 13, 40]). In our approach an event A represents an uncertain fact described by a (non ambiguous) logical proposition; hence A is a two-valued logical entity which can be *true*, or *false*. The indicator of A, denoted by the same symbol, is 1, or 0, according to whether A is true, or false. The sure event is denoted by Ω and the impossible event is denoted by $\emptyset$. Moreover, we denote by $A \wedge B$, or simply AB, (resp., $A \vee B$) the logical conjunction (resp., logical disjunction). The negation of A is denoted $\overline{A}$. Given any events A and B, we simply write $A \subseteq B$ to denote that A logically implies B, that is $A\overline{B}$ is the impossible event $\emptyset$. We recall that n events are logically independent when the number m of constituents, or possible worlds, generated by them is 2^n (in general $m \leq 2^n$).

2.1 Conditional events and coherent probability assessments

Given two events E, H, with $H \neq \emptyset$, the conditional event $E|H$ is defined as a three-valued logical entity which is *true*, or *false*, or *void*, according to whether EH is true, or $\overline{E}H$ is true, or $\overline{H}$ is true, respectively. We recall that, agreeing to the betting metaphor, if you assess $P(E|H) = p$, then, for every real number s, you are willing to pay an amount ps and to receive s, or 0, or ps, according to whether EH is true, or $\overline{E}H$ is true, or $\overline{H}$ is true (bet called off), respectively. Then, the random gain associated with the assessment $P(E|H) = p$ is $G = sHE + ps\overline{H} - ps = sH(E - p)$. Given a real function $P : \mathcal{K} \to \mathcal{R}$, where $\mathcal{K}$ is an arbitrary family of conditional events, let us consider a subfamily $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\}$ of $\mathcal{K}$ and the vector $\mathcal{P} = (p_1, \ldots, p_n)$, where $p_i = P(E_i|H_i)$, $i \in J_n = \{1, \ldots, n\}$. We denote by $\mathcal{H}_n$ the disjunction $H_1 \vee \cdots \vee H_n$. As $E_i H_i \vee \overline{E}_i H_i \vee \overline{H}_i = \Omega$, $i \in J_n$, by expanding the expression $\bigwedge_{i \in J_n} (E_i H_i \vee \overline{E}_i H_i \vee \overline{H}_i)$ we can represent Ω as the disjunction of 3^n logical conjunctions, some of which may be impossible. The remaining ones are the constituents generated by $\mathcal{F}$ and, of course, are a partition of Ω. We denote by $C_1, \ldots, C_m$ the constituents which logically imply $\mathcal{H}_n$ and (if $\mathcal{H}_n \neq \Omega$) by C_0 the remaining constituent $\overline{\mathcal{H}}_n = \overline{H}_1 \cdots \overline{H}_n$, so that

$$\mathcal{H}_n = C_1 \vee \cdots \vee C_m,$$
$$\Omega = \overline{\mathcal{H}}_n \vee \mathcal{H}_n = C_0 \vee C_1 \vee \cdots \vee C_m, \quad m + 1 \leq 3^n.$$

In the context of betting scheme, with the pair $(\mathcal{F}, \mathcal{P})$ we associate the random gain $G = \sum_{i \in J_n} s_i H_i(E_i - p_i)$, where $s_1, \ldots, s_n$ are n arbitrary real numbers. We observe that G is the difference between the amount that you receive, $\sum_{i \in J_n} s_i(E_i H_i + p_i \overline{H}_i)$, and the amount that you pay, $\sum_{i \in J_n} s_i p_i$, and represents the net gain from engaging each transaction $H_i(E_i - p_i)$, the scaling and meaning (buy or sell) of the transaction being specified by the magnitude and the sign of s_i, respectively. Let g_h be the value of G when C_h is true; then $G \in \{g_0, g_1, \ldots, g_m\}$. Of course, $g_0 = 0$. We denote by $\mathcal{G}_{\mathcal{H}_n}$ the set of values of G restricted to $\mathcal{H}_n$, that is $\mathcal{G}_{\mathcal{H}_n} = \{g_1, \ldots, g_m\}$. Then, based on the *betting scheme* of de Finetti, we have

Definition 1 The function P defined on $\mathcal{K}$ is said to be *coherent* if and only if, for every integer n, for every finite subfamily $\mathcal{F}$ of $\mathcal{K}$ and for every real numbers $s_1, \ldots, s_n$, one has: $\min \mathcal{G}_{\mathcal{H}_n} \leq 0 \leq \max \mathcal{G}_{\mathcal{H}_n}$.

Notice that the condition $\min \mathcal{G}_{\mathcal{H}_n} \leq 0 \leq \max \mathcal{G}_{\mathcal{H}_n}$ can be written in two equivalent ways: $\min \mathcal{G}_{\mathcal{H}_n} \leq 0$, or $\max \mathcal{G}_{\mathcal{H}_n} \geq 0$. As shown by Definition 1, a probability assessment is coherent if and only if, in any finite combination of n bets, it does not happen that the values $g_1, \ldots, g_m$ are all positive, or all negative (*no Dutch Book*).

2.2 Coherent conditional prevision assessments

Given a prevision function $\mathbb{P}$ defined on an arbitrary family $\mathcal{K}$ of finite conditional random quantities, consider a finite subfamily $\mathcal{F} = \{X_1|H_1, \ldots, X_n|H_n\} \subseteq \mathcal{K}$ and the vector $\mathcal{M} = (\mu_1, \ldots, \mu_n)$, where $\mu_i = \mathbb{P}(X_i|H_i)$ is the assessed prevision for the conditional random quantity $X_i|H_i$, $i \in J_n$. With the pair $(\mathcal{F}, \mathcal{M})$ we associate the random gain $G = \sum_{i \in J_n} s_i H_i(X_i - \mu_i)$; moreover, we denote by $\mathcal{G}_{\mathcal{H}_n}$ the set of values of G restricted to $\mathcal{H}_n = H_1 \vee \cdots \vee H_n$. Then, by the *betting scheme*, we have

Definition 2 The function $\mathbb{P}$ defined on $\mathcal{K}$ is coherent if and only if, $\forall n \geq 1, \forall \mathcal{F} \subseteq \mathcal{K}, \forall s_1, \ldots, s_n \in \mathbb{R}$, it holds that: $\min \mathcal{G}_{\mathcal{H}_n} \leq 0 \leq \max \mathcal{G}_{\mathcal{H}_n}$.

Given a family $\mathcal{F} = \{X_1|H_1, \ldots, X_n|H_n\}$, for each $i \in J_n$ we denote by $\{x_{i1}, \ldots, x_{ir_i}\}$ the set of possible values for the restriction of X_i to H_i; then, for each $i \in J_n$ and $j = 1, \ldots, r_i$, we set $A_{ij} = (X_i = x_{ij})$. Of course, for each $i \in J_n$, the family $\{\overline{H}_i, A_{ij} H_i, \ j = 1, \ldots, r_i\}$ is a partition of the sure event Ω, with $A_{ij} H_i = A_{ij}$, $\bigvee_{j=1}^{r_i} A_{ij} = H_i$. Then, the constituents generated by the family $\mathcal{F}$ are (the elements of the partition of Ω) obtained by expanding the expression $\bigwedge_{i \in J_n}(A_{i1} \vee \cdots \vee A_{ir_i} \vee \overline{H}_i)$. We set $C_0 = \overline{H}_1 \cdots \overline{H}_n$ (it may be $C_0 = \emptyset$); moreover, we denote by $C_1, \ldots, C_m$ the constituents contained in $\mathcal{H}_n$. Hence $\bigwedge_{i \in J_n}(A_{i1} \vee \cdots \vee$

$A_{ir_i} \vee \overline{H}_i) = \bigvee_{h=0}^m C_h$. With each C_h, $h \in J_m$, we associate a vector $Q_h = (q_{h1}, \ldots, q_{hn})$, where $q_{hi} = x_{ij}$ if $C_h \subseteq A_{ij}$, $j = 1, \ldots, r_i$, while $q_{hi} = \mu_i$ if $C_h \subseteq \overline{H}_i$; with C_0 it is associated $Q_0 = \mathcal{M} = (\mu_1, \ldots, \mu_n)$. Denoting by $\mathcal{I}$ the convex hull of $Q_1, \ldots, Q_m$, the condition $\mathcal{M} \in \mathcal{I}$ amounts to the existence of a vector $(\lambda_1, \ldots, \lambda_m)$ such that: $\sum_{h \in J_m} \lambda_h Q_h = \mathcal{M}$, $\sum_{h \in J_m} \lambda_h = 1$, $\lambda_h \geq 0$, $\forall h$; in other words, $\mathcal{M} \in \mathcal{I}$ is equivalent to the solvability of the system (Σ), associated with $(\mathcal{F}, \mathcal{M})$,

$$(\Sigma) \quad \sum_{h \in J_m} \lambda_h q_{hi} = \mu_i, \ i \in J_n;$$
$$\sum_{h \in J_m} \lambda_h = 1; \lambda_h \geq 0, \ h \in J_m. \qquad (1)$$

Given the assessment $\mathcal{M} = (\mu_1, \ldots, \mu_n)$ on $\mathcal{F} = \{X_1|H_1, \ldots, X_n|H_n\}$, let S be the set of solutions $\Lambda = (\lambda_1, \ldots, \lambda_m)$ of system (Σ) defined in (1). Then, the following characterization theorem for coherent assessments on finite families of conditional events can be proved [6]

Theorem 1 (*Characterization of coherence*) *Given a family of n conditional random quantities $\mathcal{F} = \{X_1|H_1, \ldots, X_n|H_n\}$, with finite sets of possible values, and a vector $\mathcal{M} = (\mu_1, \ldots, \mu_n)$, the conditional prevision assessment $\mathbb{P}(X_1|H_1) = \mu_1, \ldots, \mathbb{P}(X_n|H_n) = \mu_n$ is coherent if and only if, for every subset $J \subseteq J_n$, defining $\mathcal{F}_J = \{X_i|H_i, i \in J\}, \mathcal{M}_J = (\mu_i, i \in J)$, the system (Σ_J) associated with the pair $(\mathcal{F}_J, \mathcal{M}_J)$ is solvable.*

We point out that the solvability of system (Σ) (i.e., the condition $\mathcal{M} \in \mathcal{I}$) is a necessary (but not sufficient) condition for coherence of $\mathcal{M}$ on $\mathcal{F}$. Moreover, assuming the system (Σ) solvable, that is $S \neq \emptyset$, we define:

$$I_0 = \left\{ i : \max_{\Lambda \in S} \sum_{h:C_h \subseteq H_i} \lambda_h = 0 \right\}, \ \mathcal{F}_0 = \{X_i|H_i, i \in I_0\},$$
$$\mathcal{M}_0 = (\mu_i, i \in I_0). \quad (2)$$

Then, the following theorem can be proved [6, Theorem 3]

Theorem 2 (*Operative characterization of coherence*) *A conditional prevision assessment $\mathcal{M} = (\mu_1, \ldots, \mu_n)$ on the family $\mathcal{F} = \{X_1|H_1, \ldots, X_n|H_n\}$ is coherent if and only if the following conditions are satisfied:*

(i) *the system (Σ) defined in (1) is solvable;*
(ii) *if $I_0 \neq \emptyset$, then $\mathcal{M}_0$ is coherent.*

By Theorem 2, the following algorithm checks in a finite number of steps the coherence of the prevision assessment $\mathcal{M}$ on $\mathcal{F}$.

Algorithm 1 Let be given the pair $(\mathcal{F}, \mathcal{M})$.

1. Construct the system (Σ) defined in (1) and check its solvability;

2. If the system (Σ) is not solvable then $\mathcal{M}$ is not coherent and the procedure stops, otherwise compute the set I_0;
3. If $I_0 = \emptyset$ then $\mathcal{M}$ is coherent and the procedure stops, otherwise set $(\mathcal{F}, \mathcal{M}) = (\mathcal{F}_0, M_0)$ and repeat steps 1–3.

By following the approach given in [12, 22, 23, 26, 29, 35] a conditional random quantity $X|H$ can be seen as the random quantity $XH + \mu\overline{H}$, where $\mu = \mathbb{P}(X|H)$. In particular, in numerical terms, $A|H$ is the random quantity $AH + x\overline{H}$, where $x = P(A|H)$. Then, when $H \subseteq A$, coherence requires that $P(A|H) = 1$ and hence $A|H = H + \overline{H} = 1$. Notice that, as $\overline{H}|H = 0$ and $XH|H = X|H$, it holds that: $(XH + \mu\overline{H})|H = XH|H + \mu\overline{H}|H = X|H$, where $\mu = \mathbb{P}(X|H)$. Moreover, the negation of $A|H$ is defined as $\overline{A|H} = 1 - A|H = \overline{A}|H$. Coherence can be characterized in terms of proper scoring rules [7, 21], which can be related to the notion of entropy in information theory [36, 37].

We recall a result (see [26, Theorem 4]) which shows that, given two conditional random quantities $X|H, Y|K$, if $X|H = Y|K$ when $H \vee K$ is true, then $X|H = Y|K$ also when $H \vee K$ is false, so that $X|H = Y|K$.

Theorem 3 *Given any events $H \neq \emptyset$, $K \neq \emptyset$, and any r.q.'s X, Y, let Π be the set of the coherent prevision assessments $\mathbb{P}(X|H) = \mu, \mathbb{P}(Y|K) = \nu$.*

(i) *Assume that, for every $(\mu, \nu) \in \Pi$, $X|H = Y|K$ when $H \vee K$ is true; then $\mu = \nu$ for every $(\mu, \nu) \in \Pi$.*
(ii) *For every $(\mu, \nu) \in \Pi$, $X|H = Y|K$ when $H \vee K$ is true if and only if $X|H = Y|K$.*

2.3 Quasi conjunction, conjunction, and disjunction of two conditional events

The notion of quasi conjunction plays an important role in nonmonotonic reasoning. In particular for two conditional events $A|H, B|K$ the quasi conjunction $QC(A|H, B|K)$ is the conditional event $(\overline{H} \vee A) \wedge (\overline{K} \vee B) \mid (H \vee K)$. Note that: $QC(A|H, B|K)$ is true, when a conditional event is true and the other one is not false; $QC(A|H, B|K)$ is false, when a conditional event is false; $QC(A|H, B|K)$ is void, when $H \vee K$ is false. In other words, the quasi conjunction is the conjunction of the two material conditionals $\overline{H} \vee A$, $\overline{K} \vee B$ given the disjunction of the conditioning events H, K. In numerical terms one has

$$QC(A|H, B|K) = \min\{\overline{H} \vee A, \overline{K} \vee B\} \mid (H \vee K) \qquad (3)$$

and, if we replace the material conditionals $\overline{H} \vee A$, $\overline{K} \vee B$ by the conditional events $A|H, B|K$, from formula (3) we obtain the definition below [23].

Definition 3 Given any pair of conditional events $A|H$ and $B|K$, with $P(A|H) = x$, $P(B|K) = y$, we define their conjunction as the conditional random quantity $(A|H) \wedge (B|K) = Z \mid (H \vee K)$, where $Z = \min\{A|H, B|K\}$.

Then, defining $z = \mathbb{P}[(A|H) \wedge (B|K)]$, we have

$$(A|H) \wedge (B|K) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } \overline{A}H \vee \overline{B}K \text{ is true,} \\ x, & \text{if } \overline{H}BK \text{ is true,} \\ y, & \text{if } AH\overline{K} \text{ is true,} \\ z, & \text{if } \overline{H}\,\overline{K} \text{ is true.} \end{cases} \qquad (4)$$

Remark 1 We recall that $A|H = AH + x\overline{H}$, where $x = P(A|H)$. Then, by Definition 3, it holds that

$$(A|H) \wedge (A|H) = (A|H)|H = (AH + x\overline{H})|H = AH|H = A|H.$$

From (4), the conjunction $(A|H) \wedge (B|K)$ is the following random quantity

$$(A|H) \wedge (B|K) = 1 \cdot AHBK + x \cdot \overline{H}BK + y \cdot AH\overline{K} + z \cdot \overline{H}\,\overline{K}. \qquad (5)$$

For the quasi conjunction it holds that

$$QC(A|H, B|K) = AHBK + \overline{H}BK + AH\overline{K} + \nu \cdot \overline{H}\,\overline{K}, \qquad (6)$$

where $\nu = P(QC(A|H, B|K))$. We recall that, if $P(A|H) = P(B|K) = 1$, then $\nu = 1$ (see, e.g., [25, Section 3]). We also recall a result which shows that Fréchet-Hoeffding bounds still hold for the conjunction of conditional events [26, Theorem 7].

Theorem 4 *Given any coherent assessment (x, y) on $\{A|H, B|K\}$, with A, H, B, K logically independent, $H \neq \emptyset$, $K \neq \emptyset$, the extension $z = \mathbb{P}[(A|H) \wedge (B|K)]$ is coherent if and only if the following Fréchet-Hoeffding bounds are satisfied:* $\max\{x + y - 1, 0\} = z' \leq z \leq z'' = \min\{x, y\}$.

Remark 2 We observe that, if $x = y = 1$, then coherence requires that $z = \nu = 1$ and then by (5) and (6) it follows that $(A|H) \wedge (B|K) = QC(A|H, B|K)$.

We recall now the notion of disjunction of two conditional events.

Definition 4 Given any pair of conditional events $A|H$ and $B|K$, with $P(A|H) = x$, $P(B|K) = y$, we define their disjunction as $(A|H) \vee (B|K) = W \mid (H \vee K)$, where $W = \max\{A|H, B|K\}$.

Then, defining $w = \mathbb{P}[(A|H) \vee (B|K)]$, we have

$$(A|H) \vee (B|K) = \begin{cases} 1, & \text{if } AH \vee BK \text{ is true,} \\ 0, & \text{if } \overline{A}H\overline{B}K \text{ is true,} \\ x, & \text{if } \overline{H}\,\overline{B}K \text{ is true,} \\ y, & \text{if } \overline{A}H\overline{K} \text{ is true,} \\ w, & \text{if } \overline{H}\,\overline{K}\text{ is true.} \end{cases} \quad (7)$$

Remark 3 We recall that $A|H = AH + x\overline{H}$, where $x = P(A|H)$. Then, by Definition 4, it holds that

$$(A|H) \vee (A|H) = (A|H)|H = (AH + x\overline{H})|H = AH|H = A|H.$$

From (7), the disjunction $(A|H) \vee (B|K)$ is the following random quantity

$$(A|H) \vee (B|K) = 1 \cdot AH \vee BK + x \cdot \overline{H}\,\overline{B}K + y \cdot \overline{A}H\overline{K} + w \cdot \overline{H}\,\overline{K}. \quad (8)$$

3 Conjunction, disjunction, and negation

We now define the conjunction and the disjunction of n conditional events in a progressive way by specifying the possible values of the corresponding conditional random quantities. Given a family of n conditional events $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\}$, we denote by $C_0, C_1, \ldots, C_m$, with $m + 1 \leq 3^n$, the constituents associated with $\mathcal{F}$, where $C_0 = \overline{H}_1\overline{H}_2 \cdots \overline{H}_n$. With each C_h, $h = 0, 1, \ldots, m$, we associate a tripartition (S'_h, S''_h, S'''_h) of the set $\{1, \ldots, n\}$, such that, for each $i \in \{1, \ldots, n\}$ it holds that: $i \in S'_h$, or $i \in S''_h$, or $i \in S'''_h$, according to whether $C_h \subseteq E_iH_i$, or $C_h \subseteq \overline{E}_iH_i$, or $C_h \subseteq \overline{H}_i$. In other words, for each $h = 0, 1, \ldots, m$, we have

$$S'_h = \{i : C_h \subseteq E_iH_i\}, \quad S''_h = \{i : C_h \subseteq \overline{E}_iH_i\},$$
$$S'''_h = \{i : C_h \subseteq \overline{H}_i\}. \quad (9)$$

Definition 5 (**Conjunction of n conditionals**) Let be given a family of n conditional events $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\}$. For each non-empty subset S of $\{1, \ldots, n\}$, let x_S be a prevision assessment on $\bigwedge_{i \in S}(E_i|H_i)$. Then, the conjunction $\mathcal{C}(\mathcal{F}) = (E_1|H_1) \wedge \cdots \wedge (E_n|H_n)$ is defined as

$$Z_n|(H_1 \vee \cdots \vee H_n) = \sum_{h=0}^{m} z_hC_h,$$

$$\text{where } z_h = \begin{cases} 1, & \text{if } S'_h = \{1, \ldots, n\}, \\ 0, & \text{if } S''_h \neq \emptyset, \\ x_{S'''_h}, & \text{if } S''_h = \emptyset \text{ and } S'''_h \neq \emptyset. \end{cases} \quad (10)$$

Remark 4 As shown by (10), the conjunction $(E_1|H_1) \wedge \cdots \wedge (E_n|H_n)$ assumes one of the following possible values: 1, when every conditional event is true; 0, when at least one conditional event is false; x_S, when the conditional event $E_i|H_i$ is void, for every $i \in S$, and is true for every $i \notin S$. In the case $S = \{i\}$, we simply set $x_S = x_i$.

Notice that the notion of conjunction given in (10) has been already proposed, with positive probabilities for the conditioning events, in [38]. But, our approach is developed in the setting of coherence, where conditional probabilities and conditional previsions are primitive notions. Moreover, coherence allows to properly manage zero probabilities for conditioning events.

Remark 5 We observe that to introduce the random quantity defined by formula (10) we need to specify in a coherent way the set of prevision assessments $\{x_S : S \subseteq \{1, 2, \ldots, n\}\}$. In particular, when the conditioning events $H_1, \ldots, H_n$ are all false, i.e. C_0 is true, the associated tripartition is $(S'_0, S''_0, S'''_0) = (\emptyset, \emptyset, \{1, 2, \ldots, n\})$ and the value of the conjunction $\mathcal{C}(\mathcal{F})$ is its prevision $x_{S'''_0} = \mathbb{P}[\mathcal{C}(\mathcal{F})]$. Moreover, we observe that the set of the constituents $\{C_0, \ldots, C_m\}$ associated with $\mathcal{F}$ is invariant with respect to any permutation of the conditional events in $\mathcal{F}$. Then, the operation of conjunction introduced by Definition 5 is invariant with respect to any permutation of the conditional events in $\mathcal{F}$.

Definition 6 Given two finite families of conditional events $\mathcal{F}'$ and $\mathcal{F}''$, based on Definition 5, we set $\mathcal{C}(\mathcal{F}') \wedge \mathcal{C}(\mathcal{F}'') = \mathcal{C}(\mathcal{F}' \cup \mathcal{F}'')$.

Proposition 1 *The operation of conjunction is associative and commutative.*

Proof Concerning the commutative property, let be given two finite families of conditional events $\mathcal{F}'$ and $\mathcal{F}''$. As $\mathcal{F}'' \cup \mathcal{F}' = \mathcal{F}' \cup \mathcal{F}''$, it holds that $\mathcal{C}(\mathcal{F}'') \wedge \mathcal{C}(\mathcal{F}') = \mathcal{C}(\mathcal{F}'' \cup \mathcal{F}') = \mathcal{C}(\mathcal{F}' \cup \mathcal{F}'') = \mathcal{C}(\mathcal{F}') \wedge \mathcal{C}(\mathcal{F}'')$. Concerning the associative property, let be given three finite families of conditional events $\mathcal{F}'$, $\mathcal{F}''$ and $\mathcal{F}'''$. We have

$$[\mathcal{C}(\mathcal{F}') \wedge \mathcal{C}(\mathcal{F}'')] \wedge \mathcal{C}(\mathcal{F}''') = \mathcal{C}(\mathcal{F}' \cup \mathcal{F}'') \wedge \mathcal{C}(\mathcal{F}''')$$
$$= \mathcal{C}(\mathcal{F}' \cup \mathcal{F}'' \cup \mathcal{F}''') = \mathcal{C}(\mathcal{F}') \wedge \mathcal{C}(\mathcal{F}'' \cup \mathcal{F}''')$$
$$= \mathcal{C}(\mathcal{F}') \wedge [\mathcal{C}(\mathcal{F}'') \wedge \mathcal{C}(\mathcal{F}''')] = \mathcal{C}(\mathcal{F}') \wedge \mathcal{C}(\mathcal{F}'') \wedge \mathcal{C}(\mathcal{F}''').$$

$\square$

Definition 7 (**Disjunction of n conditionals**) Let be given a family of n conditional events $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\}$. Morever, for each non-empty subset S of $\{1, \ldots, n\}$, let y_S be a prevision assessment on $\bigvee_{i \in S}(E_i|H_i)$.

Then, the disjunction $\mathcal{D}(\mathcal{F}) = (E_1|H_1) \vee \cdots \vee (E_n|H_n)$ is defined as the following conditional random quantity

$$W_n|(H_1 \vee \cdots \vee H_n) = \sum_{h=0}^{m} w_h C_h,$$

$$\text{where } w_h = \begin{cases} 1, & \text{if } S_h' \neq \emptyset, \\ 0, & \text{if } S_h'' = \{1, 2, \ldots, n\}, \\ y_{S_h'''}, & \text{if } S_h' = \emptyset \text{ and } S_h''' \neq \emptyset. \end{cases} \quad (11)$$

We recall that $S_0''' = \{1, 2, \ldots, n\}$; thus $y_{S_0'''} = \mathbb{P}[\bigvee_{i=1}^{n} (E_i|H_i)] = \mathbb{P}[\mathcal{D}(\mathcal{F})]$. As shown by (11), the disjunction $\mathcal{D}(\mathcal{F})$ assumes one of the following possible values: 1, when at least one conditional event is true; 0, when every conditional event is false; y_S, when the conditional event $E_i|H_i$ is void, for every $i \in S$, and is false for every $i \notin S$.

Definition 8 Given two finite families of conditional events $\mathcal{F}'$ and $\mathcal{F}''$, based on Definition 7, we set $\mathcal{D}(\mathcal{F}') \vee \mathcal{D}(\mathcal{F}'') = \mathcal{D}(\mathcal{F}' \cup \mathcal{F}'')$.

Proposition 2 *The operation of disjunction is associative and commutative.*

Proof The proof is analogous to that of Proposition 1. □

We give below the notion of negation for the conjunction and the disjunction of a family of conditional events.

Definition 9 Given a family of conditional events $\mathcal{F}$, the negations for the conjunction $\mathcal{C}(\mathcal{F})$ and the disjunction $\mathcal{D}(\mathcal{F})$ are defined as $\overline{\mathcal{C}(\mathcal{F})} = 1 - \mathcal{C}(\mathcal{F})$ and $\overline{\mathcal{D}(\mathcal{F})} = 1 - \mathcal{D}(\mathcal{F})$, respectively.

Given a family of n conditional events $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\}$, we denote by $\overline{\mathcal{F}}$ the family $\{\overline{E}_1|H_1, \ldots, \overline{E}_n|H_n\}$. Of course $\overline{\overline{\mathcal{F}}} = \mathcal{F}$. In the next result we show that De Morgan's Laws are satisfied.

Theorem 5 *Given a family of n conditional events $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\}$, it holds that:*

(i) $\overline{\mathcal{D}(\mathcal{F})} = \mathcal{C}(\overline{\mathcal{F}})$, *that is* $\mathcal{D}(\mathcal{F}) = \overline{\mathcal{C}(\overline{\mathcal{F}})}$;
(ii) $\overline{\mathcal{C}(\mathcal{F})} = \mathcal{D}(\overline{\mathcal{F}})$, *that is* $\mathcal{C}(\mathcal{F}) = \overline{\mathcal{D}(\overline{\mathcal{F}})}$.

Proof See Appendix A. □

4 Monotonicity property

For any given n conditional events $E_1|H_1, \ldots, E_n|H_n$, we set $\mathcal{C}_n = \bigwedge_{i=1}^{n}(E_i|H_i)$ and $\mathcal{D}_n = \bigvee_{i=1}^{n}(E_i|H_i)$.

Moreover, for every non empty subset S of $\{1, 2, \ldots, n\}$ we set

$$\mathcal{C}_S = \bigwedge_{i \in S}(E_i|H_i), \quad \mathcal{D}_S = \bigvee_{i \in S}(E_i|H_i).$$

In this section, among other results, we will show the monotonicity property of conjunction and disjunction, that is $\mathcal{C}_{n+1} \leq \mathcal{C}_n$ and $\mathcal{D}_{n+1} \geq \mathcal{D}_n$, for every $n \geq 1$.

We first prove a preliminary result, which in particular shows that, given two conditional random quantities $X|H$, $Y|K$, if $X|H \leq Y|K$ when $H \vee K$ is true, then $X|H \leq Y|K$ also when $H \vee K$ is false, so that $X|H \leq Y|K$. This result generalizes Theorem 3, as the symbol $=$ is replaced by $\leq$, and it will be used in Theorem 7.

Theorem 6 *Given any events $H \neq \emptyset$, $K \neq \emptyset$, and any r.q.'s X, Y, let Π be the set of the coherent prevision assessments $\mathbb{P}(X|H) = \mu$, $\mathbb{P}(Y|K) = \nu$.*

(i) *Assume that, for every $(\mu, \nu) \in \Pi$, $X|H \leq Y|K$ when $H \vee K$ is true; then $\mu \leq \nu$ for every $(\mu, \nu) \in \Pi$.*
(ii) *For every $(\mu, \nu) \in \Pi$, $X|H \leq Y|K$ when $H \vee K$ is true if and only if $X|H \leq Y|K$.*

Proof See Appendix A. □

The next two results illustrate the monotonicity property of conjunction and disjunction.

Theorem 7 *Given $n + 1$ arbitrary conditional events $E_1|H_1, \ldots, E_{n+1}|H_{n+1}$, with $n \geq 1$, for the conjunctions $\mathcal{C}_n$ and $\mathcal{C}_{n+1}$ it holds that $\mathcal{C}_{n+1} \leq \mathcal{C}_n$.*

Proof See Appendix A. □

Theorem 8 *Given $n + 1$ arbitrary conditional events $E_1|H_1, \ldots, E_{n+1}|H_{n+1}$, with $n \geq 1$, for the disjunctions $\mathcal{D}_n$ and $\mathcal{D}_{n+1}$ it holds that $\mathcal{D}_{n+1} \geq \mathcal{D}_n$.*

Proof Defining $\mathcal{F}_n = \{E_1|H_1, \ldots, E_n|H_n\}$ and $\mathcal{F}_{n+1} = \mathcal{F}_n \cup \{E_{n+1}|H_{n+1}\}$, by Theorems 5 and 7 it holds that

$$\mathcal{D}_{n+1} = \mathcal{D}(\mathcal{F}_{n+1}) = \overline{\mathcal{C}(\overline{\mathcal{F}}_{n+1})} = 1 - \mathcal{C}(\overline{\mathcal{F}}_{n+1})$$
$$\geq 1 - \mathcal{C}(\overline{\mathcal{F}}_n) = \overline{\mathcal{C}(\overline{\mathcal{F}}_n)} = \mathcal{D}_n. \quad □$$

The next result shows that the conjunction and the disjunction are random quantities with values in the interval [0, 1].

Theorem 9 *Given n arbitrary conditional events $E_1|H_1, \ldots, E_n|H_n$, it holds that: (i) $\mathcal{C}_n \in [0, 1]$; (ii) $\mathcal{D}_n \in [0, 1]$.*

Proof See Appendix A. □

Remark 6 From Theorem 7, it holds that $\mathcal{C}_n \leq \mathcal{C}_{n-1} \leq \ldots \leq \mathcal{C}_1$; in particular $\mathbb{P}(\mathcal{C}_n) \leq \mathbb{P}(\mathcal{C}_k)$, $k = 1, 2, \ldots, n - 1$. More generally, for every non empty subset S of $\{1, \ldots, n\}$, it holds that $\mathbb{P}(\mathcal{C}_n) \leq \mathbb{P}(\mathcal{C}_S)$. In particular, $\mathbb{P}(\mathcal{C}_n) \leq P(E_n|H_n)$. Then, $\mathbb{P}(\mathcal{C}_k) \leq \min\{\mathbb{P}(\mathcal{C}_{k-1}), P(E_k|H_k)\}$, $k = 2, 3, \ldots, n$, and by iterating it follows

$$\mathbb{P}(\mathcal{C}_n) \leq \min\{P(E_1|H_1), \ldots, P(E_n|H_n)\}. \tag{12}$$

Likewise, by Theorem 8,

$$\mathbb{P}(\mathcal{D}_k) \geq \max\{\mathbb{P}(\mathcal{D}_{k-1}), (E_k|H_k)\}, k = 2, 3, \ldots, n,$$
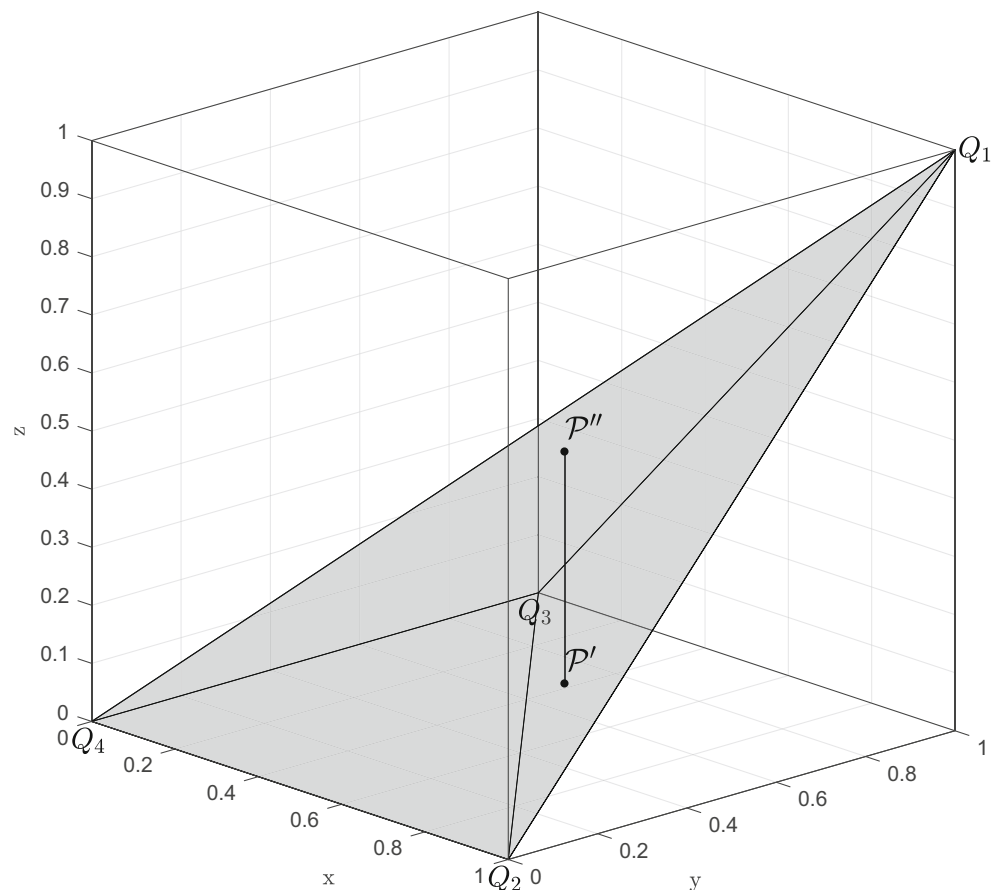
and by iterating it follows

$$\mathbb{P}(\mathcal{D}_n) \geq \max\{P(E_1|H_1), \ldots, P(E_n|H_n)\}. \tag{13}$$

5 Coherent assessments on $\{\mathcal{C}_n, E_{n+1}|H_{n+1}, \mathcal{C}_{n+1}\}$

Given any $n + 1$ arbitrary conditional events $E_1|H_1, \ldots, E_{n+1}|H_{n+1}$, let us consider the conjunctions $\mathcal{C}_n = (E_1|H_1) \wedge \cdots \wedge (E_n|H_n)$ and $\mathcal{C}_{n+1} = (E_1|H_1) \wedge \cdots \wedge (E_{n+1}|H_{n+1})$. We set $\mathbb{P}(\mathcal{C}_n) = \mu_n$, $\mathbb{P}(\mathcal{C}_{n+1}) = \mu_{n+1}$ and $P(E_{n+1}|H_{n+1}) = x_{n+1}$.

Remark 7 Let us consider the points

$$Q_1 = (1, 1, 1), \quad Q_2 = (1, 0, 0), \quad Q_3 = (0, 1, 0), \quad Q_4 = (0, 0, 0).$$

We observe that the equations of the three planes containing the points Q_1, Q_2, Q_3, or Q_1, Q_2, Q_4, or Q_1, Q_3, Q_4, are $z = x + y - 1$, or $z = x$, or $z = y$, respectively. It can be shown that a point (x, y, z) belongs to the convex hull $\mathcal{I}$ of Q_1, Q_2, Q_3, Q_4 if and only if

$$(x, y) \in [0, 1]^2, \quad \max\{x + y - 1, 0\} \leq z \leq \min\{x, y\}. \tag{14}$$

The convex hull $\mathcal{I}$, which is a tetrahedron with vertices Q_1, Q_2, Q_3, Q_4, is depicted in Fig. 1.

We observe that the lower and upper bounds in (14) are the Fréchet-Hoeffding bounds, which characterize the next result.

Theorem 10 *Assume that the events $E_1, \ldots, E_{n+1}, H_1, \ldots, H_{n+1}$ are logically independent. Let $\mathcal{I}$ be the convex hull of the points $Q_1 = (1, 1, 1)$, $Q_2 = (1, 0, 0)$, $Q_3 = (0, 1, 0)$, $Q_4 = (0, 0, 0)$. Then, the assessment*

Fig. 1 Convex hull of the points Q_1, Q_2, Q_3, Q_4. $\mathcal{P}' = (x, y, z')$, $\mathcal{P}'' = (x, y, z'')$, where $(x, y) \in [0, 1]^2$, $z' = \max\{x + y - 1, 0\}$, $z'' = \min\{x, y\}$. In the figure the numerical values are: $x = 0.6$, $y = 0.5$, $z' = 0.1$, and $z'' = 0.5$

$\mathcal{M} = (\mu_n, x_{n+1}, \mu_{n+1})$ *on the family* $\{\mathcal{C}_n, E_{n+1}|H_{n+1}, \mathcal{C}_{n+1}\}$ *is coherent if and only if* $\mathcal{M} \in \mathcal{I}$, *that is if and only if*

$$(\mu_n, x_{n+1}) \in [0,1]^2, \quad \mu'_{n+1} \leq \mu_{n+1} \leq \mu''_{n+1},$$

where $\mu'_{n+1} = \max\{\mu_n + x_{n+1} - 1, 0\}$ *and* $\mu''_{n+1} = \min\{\mu_n, x_{n+1}\}$.

Proof See Appendix A. □

Remark 8 We observe that the representation of each coherent assessment $\mathcal{M} = (\mu_n, x_{n+1}, \mu_{n+1})$ as a linear convex combination $\lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3 + \lambda_4 Q_4$ (where $\sum_{h=1}^4 \lambda_h = 1, \lambda_h \geq 0, h = 1, 2, 3, 4$) is unique, with

$$\begin{cases} \lambda_1 = \mu_{n+1} = \mathbb{P}(\mathcal{C}_{n+1}) \geq 0, \\ \lambda_2 = \mu_n - \mu_{n+1} = \mathbb{P}(\mathcal{C}_n) - \mathbb{P}(\mathcal{C}_{n+1}) \geq 0, \\ \lambda_3 = x_{n+1} - \mu_{n+1} = P(E_{n+1}|H_{n+1}) - \mathbb{P}(\mathcal{C}_{n+1}) \geq 0, \\ \lambda_4 = 1 - \mu_n - x_{n+1} + \mu_{n+1} \\ \quad = 1 - \mathbb{P}(\mathcal{C}_n) - P(E_{n+1}|H_{n+1}) + \mathbb{P}(\mathcal{C}_{n+1}) \geq 0. \end{cases}$$

In particular, concerning the extreme cases $\mu_{n+1} = \mu'_{n+1}$, or $\mu_{n+1} = \mu''_{n+1}$, we can examine four cases: 1) $\mu'_{n+1} = \mu_n + x_{n+1} - 1 > 0$; 2) $\mu'_{n+1} = 0$; 3) $\mu''_{n+1} = \mu_n$ and 4) $\mu''_{n+1} = x_{n+1}$.

In the case 1 the point $\mathcal{M} = (\mu_n, x_{n+1}, \mu_{n+1})$ is a linear convex combination $\lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3 + \lambda_4 Q_4$, with $\lambda_1 = \mu'_{n+1} = \mu_n + x_{n+1} - 1, \lambda_2 = 1 - x_{n+1}, \lambda_3 = 1 - \mu_n, \lambda_4 = 0$.

In the case 2 it holds that $\lambda_1 = \mu'_{n+1} = 0, \lambda_2 = \mu_n, \lambda_3 = x_{n+1}, \lambda_4 = 1 - \mu_n + x_{n+1}$.

In the case 3 it holds that $\lambda_1 = \mu''_{n+1} = \mu_n, \lambda_2 = 0, \lambda_3 = x_{n+1} - \mu_n, \lambda_4 = 1 - x_{n+1}$.

In the case 4 it holds that $\lambda_1 = \mu''_{n+1} = x_{n+1}, \lambda_2 = \mu_n - x_{n+1}, \lambda_3 = 0, \lambda_4 = 1 - \mu_n$.

6 Probabilistic inference from $\mathcal{C}_{n+1}$ to $\{\mathcal{C}_n, E_{n+1}|H_{n+1}\}$

In this section, given any coherent prevision assessment μ_{n+1} on $\mathcal{C}_{n+1}$, we find the set of coherent extensions (μ_n, x_{n+1}) on $\{\mathcal{C}_n, E_{n+1}|H_{n+1}\}$. As we will see, it is enough to illustrate the case $n = 1$, by finding the set of coherent extensions (x, y) on $\{E_1|H_1, E_2|H_2\}$ of any assessment $z = \mathbb{P}[(E_1|H_1) \wedge (E_2|H_2)] \in [0, 1]$.

Theorem 11 *Given any prevision assessment* z *on* $(E_1|H_1) \wedge (E_2|H_2)$, *with* $z \in [0, 1]$, *with* E_1, H_1, E_2, H_2 *logically independent, with* $H_1 \neq \emptyset$ *and* $H_2 \neq \emptyset$, *the extension* $x = P(E_1|H_1)$, $y = P(E_2|H_2)$ *is coherent if and only*

if (x, y) *belongs to the set* $T_z = \{(x, y) : x \in [z, 1], y \in [z, 1 + z - x]\}$.

Proof We recall that, by logical independence of E_1, H_1, E_2, H_2, the assessment (x, y) is coherent for every $(x, y) \in [0, 1]^2$. From Theorem 4, the set Π of all coherent assessment (x, y, z) on $\{E_1|H_1, E_2|H_2, (E_1|H_1) \wedge (E_2|H_2)\}$ is $\Pi = \{(x, y, z) : (x, y) \in [0, 1]^2, \max\{x + y - 1, 0\} \leq z \leq \min\{x, y\}\}$. We note that

$$\Pi = \{(x, y, z) : z \in [0, 1], x \in [z, 1], y \in [z, z + 1 - x]\} = $$
$$ = \{(x, y, z) : z \in [0, 1], (x, y) \in T_z\}.$$

Then, (x, y) is a coherent extension of z if and only if $(x, y) \in T_z$. □

Remark 9 We observe that, given any $z \in [0, 1]$ and defining $\Pi_z = \{(x, y, z) : (x, y) \in T_z\}$, it holds that $\Pi = \bigcup_{z \in [0,1]} \Pi_z$ (see Fig. 2). The set Π is the tetrahedron depicted in Fig. 1. Hence, contrarily to the general case, for the family $\{E_1|H_1, E_2|H_2, (E_1|H_1) \wedge (E_2|H_2)\}$ the set of coherent prevision assessments Π is convex. Indeed, Π is also the (convex) set of coherent probability assessment (x, y, z) on the family of unconditional events $\{E_1, E_2, E_1 E_2\}$. We recall that, assuming $H_1 \wedge H_2 = \emptyset$, the set of coherent prevision assessments (x, y, z) on $\{E_1|H_1, E_2|H_2, (E_1|H_1) \wedge (E_2|H_2)\}$ is the surface $\{(x, y, z) : (x, y) \in [0, 1]^2, z = xy\}$, which is a strict non-convex subset of Π (see [23, Section 5]).

Theorem 12 *Given any prevision assessment* $\mu_{n+1} = \mathbb{P}(\mathcal{C}_{n+1}) \in [0, 1]$, *with* $\mu_{n+1} \in [0, 1]$, *the extension* $\mu_n = \mathbb{P}(\mathcal{C}_n)$, $x_{n+1} = P(E_{n+1}|H_{n+1})$ *is coherent if and only if*
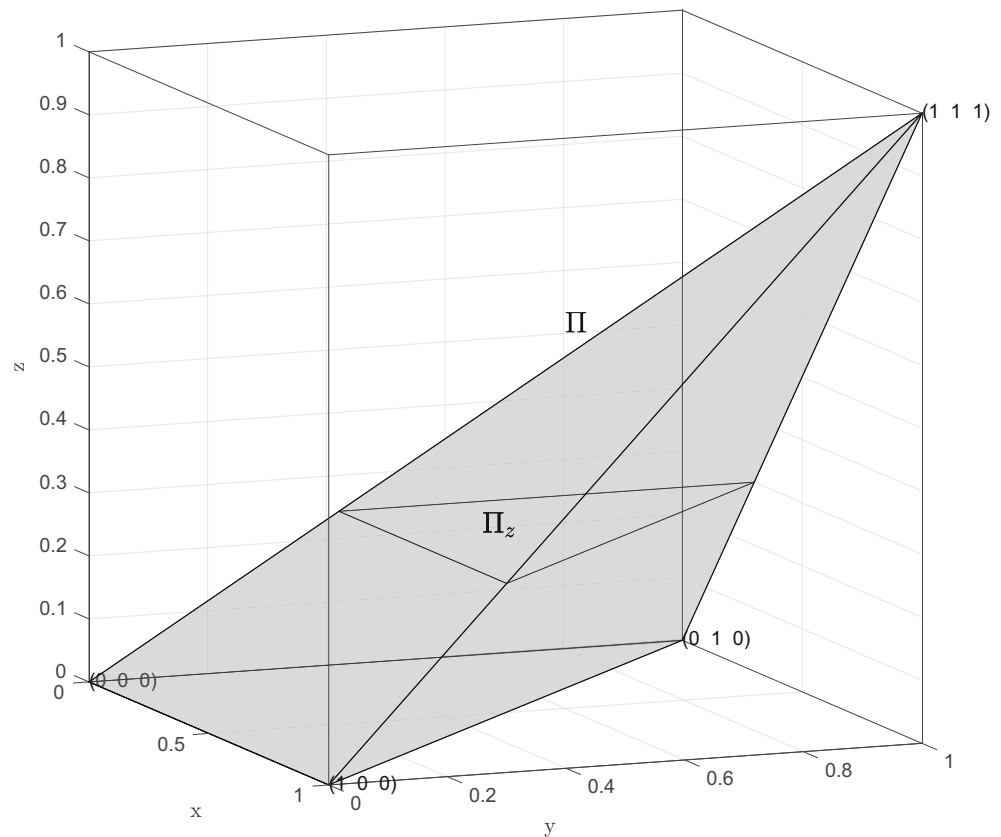
$$(\mu_n, x_{n+1}) \in \{(\mu_n, x_{n+1}) : \mu_n \in [\mu_{n+1}, 1],$$
$$x_{n+1} \in [\mu_{n+1}, 1 + \mu_{n+1} - \mu_n]\}.$$

Proof From Theorem 10, the set Π of all coherent assessment $(\mu_n, x_{n+1}, \mu_{n+1})$ on $\{\mathcal{C}_n, E_{n+1}|H_{n+1}, \mathcal{C}_{n+1}\}$ is $\Pi = \{(\mu_n, x_{n+1}, \mu_{n+1}) : (\mu_n, x_{n+1}) \in [0, 1]^2, \max\{\mu_n + x_{n+1} - 1, 0\} \leq \mu_{n+1} \leq \min\{\mu_n, x_{n+1}\}\}$. Moreover, as observed in the proof of Theorem 11, the set Π coincides with the set

$$\{(\mu_n, x_{n+1}, \mu_{n+1}) : \mu_{n+1} \in [0, 1], \mu_n \in [\mu_{n+1}, 1],$$
$$x_{n+1} \in [\mu_{n+1}, 1 + \mu_{n+1} - \mu_n]\}.$$

Then, (μ_n, x_{n+1}) is a coherent extension of μ_{n+1} if and only if (μ_n, x_{n+1}) belongs to the set $\{(\mu_n, x_{n+1}) : \mu_n \in [\mu_{n+1}, 1], x_{n+1} \in [\mu_{n+1}, 1 + \mu_{n+1} - \mu_n]\}$. □

7 Fréchet-Hoeffding bounds

In the next result we show that the prevision of the conjunction $C_n = E_1|H_1 \wedge \cdots \wedge E_n|H_n$ satisfies the Fréchet-Hoeffding bounds.

Theorem 13 *Let be given n conditional events $E_1|H_1$, $E_2|H_2, \ldots, E_n|H_n$, with $x_i = P(E_i|H_i)$, $i = 1, 2, \ldots, n$, and with $\mathbb{P}(C_n) = \mu_n$. Then*

$$\max\{x_1 + \cdots + x_n - (n-1), 0\} \leq \mu_n \leq \min\{x_1, \ldots, x_n\}. \quad (15)$$

Proof From Theorem 10, it holds that

$$\mu_n \geq \mu_{n-1} + x_n - 1 \geq \mu_{n-2} + x_{n-1} + x_n - 2 \geq \cdots \geq$$
$$x_1 + \cdots + x_n - (n-1).$$

Then, by inequality (12) and by Theorem 9 it holds that the inequalities in (15) are satisfied. $\square$

Likewise, the following result holds for the prevision η_n of the disjunction $\mathcal{D}_n = E_1|H_1 \vee E_2|H_2 \vee \cdots \vee E_n|H_n$.

Theorem 14 *Let be given n conditional events $E_1|H_1$, $E_2|H_2, \ldots, E_n|H_n$, with $x_i = P(E_i|H_i)$, $i = 1, 2, \ldots, n$, and with $\mathbb{P}(\mathcal{D}_n) = \eta_n$. Then*

$$\max\{x_1, \ldots, x_n\} \leq \eta_n \leq \min\{x_1 + \cdots + x_n, 1\}. \quad (16)$$

Proof By Definition 9, Theorems 5 and 13, defining $\overline{\mathcal{F}}_n = \{\overline{E}_1|H_1, \overline{E}_2|H_2, \ldots, \overline{E}_n|H_n\}$ it holds that

$$\mathbb{P}(\mathcal{D}_n) = 1 - \mathbb{P}(\overline{\mathcal{D}}_n) = 1 - \mathbb{P}(\mathcal{C}(\overline{\mathcal{F}}_n)) = 1 - \mathbb{P}\left(\bigwedge_{i=1}^{n}(\overline{E}_i|H_i)\right)$$
$$\leq 1 - [(1 - x_1) + \cdots + (1 - x_n) - (n-1)]$$
$$= x_1 + \cdots + x_n.$$

Then, by (13) and by Theorem 9, the inequalities in (16) are satisfied. $\square$

8 Conjunction of three conditional events

Given a family of three conditional events $\mathcal{F} = \{E_1|H_1, E_2|H_2, E_3|H_3\}$, we set $P(E_i|H_i) = x_i$, $i = 1, 2, 3$,

$\mathbb{P}[(E_i|H_i) \wedge (E_j|H_j)] = x_{ij} = x_{ji}, i \neq j$, and $x_{123} = \mathbb{P}[(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)]$. Then, by Definition 5, the conjunction of $E_1|H_1, E_2|H_2, E_3|H_3$ is the conditional random quantity

$$
\mathcal{C}(\mathcal{F}) = (E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)
$$

$$
= \begin{cases}
1, & \text{if } E_1 H_1 E_2 H_2 E_3 H_3 \text{ is true}, \\
0, & \text{if } \overline{E}_1 H_1 \vee \overline{E}_2 H_2 \vee \overline{E}_3 H_3 \text{ is true}, \\
x_1, & \text{if } \overline{H}_1 E_2 H_2 E_3 H_3 \text{ is true}, \\
x_2, & \text{if } \overline{H}_2 E_1 H_1 E_3 H_3 \text{ is true}, \\
x_3, & \text{if } \overline{H}_3 E_1 H_1 E_2 H_2 \text{ is true}, \\
x_{12}, & \text{if } \overline{H}_1 \overline{H}_2 E_3 H_3 \text{ is true}, \\
x_{13}, & \text{if } \overline{H}_1 \overline{H}_3 E_2 H_2 \text{ is true}, \\
x_{23}, & \text{if } \overline{H}_2 \overline{H}_3 E_1 H_1 \text{ is true}, \\
x_{123}, & \text{if } \overline{H}_1 \overline{H}_2 \overline{H}_3 \text{ is true}.
\end{cases} \tag{17}
$$

Remark 10 Notice that in the betting scheme x_{123} is the quantity to be paid in order to receive $\mathcal{C}(\mathcal{F})$. Assuming that the assessment $(x_1, x_2, x_3, x_{12}, x_{13}, x_{23})$ on $\{E_1|H_1, E_2|H_2, E_3|H_3, (E_1|H_1) \wedge (E_2|H_2), (E_1|H_1) \wedge (E_3|H_3), (E_2|H_2) \wedge (E_3|H_3)\}$ is coherent, we are interested in finding the values x_{123} which are a coherent extension of $(x_1, x_2, x_3, x_{12}, x_{13}, x_{23})$. Of course, as $x_i \in [0, 1]$, $i = 1, 2, 3$, and $x_{ij} \in [0, 1]$, $i \neq j$, a necessary condition for coherence is $x_{123} \in [0, 1]$.

From Remark 5 and Proposition 1 the conjunction $\mathcal{C}(\mathcal{F})$ is invariant with respect to any given permutation (i_1, i_2, i_3) of $(1, 2, 3)$; that is $\mathcal{C}(\mathcal{F}) = (E_{i_1}|H_{i_1}) \wedge (E_{i_2}|H_{i_2})) \wedge (E_{i_3}|H_{i_3})$, for any permutation (i_1, i_2, i_3) of $(1, 2, 3)$.

8.1 Study of coherence

Notice that in general, if we assess the values $x_S = \mathbb{P}(\mathcal{C}_S)$ for some $S \subset \{1, 2 \ldots, n\}$, then the study of coherence may be very complex. In this section we study coherence in the case $n = 3$ when we assess the prevision $x_S = \mathbb{P}(\mathcal{C}_S)$ for every $S \subseteq \{1, 2, 3\}$. In the next result we determine the set of coherent assessments $\mathcal{M} = (x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})$ on the family $\mathcal{F} = \{E_1|H_1, E_2|H_2, E_3|H_3, (E_1|H_1) \wedge (E_2|H_2), (E_1|H_1) \wedge (E_3|H_3), (E_2|H_2) \wedge (E_3|H_3), (E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)\} = \{\mathcal{C}_S : \emptyset \neq S \subseteq \{1, 2, 3\}\}$.

Theorem 15 *Assume that the events $E_1, E_2, E_3, H_1, H_2, H_3$ are logically independent, with $H_1 \neq \emptyset, H_2 \neq \emptyset, H_3 \neq \emptyset$. Then, the set Π of all coherent assessments $\mathcal{M} = (x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})$ on $\mathcal{F} = \{E_1|H_1, E_2|H_2, E_3|H_3, (E_1|H_1)| \wedge (E_2|H_2), (E_1|H_1) \wedge (E_3|H_3), (E_2|H_2) \wedge (E_3|H_3), (E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)\}$ is the set of*

points $(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})$ which satisfy the following conditions

$$
\begin{cases}
(x_1, x_2, x_3) \in [0, 1]^3, \\
\max\{x_1 + x_2 - 1, x_{13} + x_{23} - x_3, 0\} \leq x_{12} \leq \min\{x_1, x_2\}, \\
\max\{x_1 + x_3 - 1, x_{12} + x_{23} - x_2, 0\} \leq x_{13} \leq \min\{x_1, x_3\}, \\
\max\{x_2 + x_3 - 1, x_{12} + x_{13} - x_1, 0\} \leq x_{23} \leq \min\{x_2, x_3\}, \\
1 - x_1 - x_2 - x_3 + x_{12} + x_{13} + x_{23} \geq 0, \\
x_{123} \geq \max\{0, x_{12} + x_{13} - x_1, x_{12} + x_{23} - x_2, x_{13} + x_{23} - x_3\}, \\
x_{123} \leq \min\{x_{12}, x_{13}, x_{23}, 1 - x_1 - x_2 - x_3 + x_{12} + x_{13} + x_{23}\}.
\end{cases} \tag{18}
$$

Proof See Appendix A. □

We observe that, from (18) it follows that the coherence of $(x_1, x_2, x_3, x_{12}, x_{13}, x_{23})$ amounts to the inequality

$$
\min\{x_{12}, x_{13}, x_{23}, 1 - x_1 - x_2 - x_3 + x_{12} + x_{13} + x_{23}\}
$$
$$
\leq \max\{0, x_{12} + x_{13} - x_1, x_{12} + x_{23} - x_2, x_{13} + x_{23} - x_3\}.
$$

Then, by Theorem 15 it follows

Corollary 1 *For any coherent assessment $(x_1, x_2, x_3, x_{12}, x_{13}, x_{23})$ on*

$$
\{E_1|H_1, E_2|H_2, E_3|H_3, (E_1|H_1) \wedge (E_2|H_2),
$$
$$
(E_1|H_1) \wedge (E_3|H_3), (E_2|H_2) \wedge (E_3|H_3)\}
$$

the extension x_{123} on $(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)$ is coherent if and only if $x_{123} \in [x'_{123}, x''_{123}]$, where

$$
x'_{123} = \max\{0, x_{12} + x_{13} - x_1, x_{12} + x_{23} - x_2, x_{13}, + x_{23} - x_3\}
$$

$$
x''_{123} = \min\{x_{12}, x_{13}, x_{23}, 1 - x_1 - x_2 - x_3 + x_{12} + x_{13} + x_{23}\}. \tag{19}
$$

Proof As shown in (18), (see also (46) in the Appendix A), the coherence of $(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})$ amounts to the condition

$$
\min\{x_{12}, x_{13}, x_{23}, 1 - x_1 - x_2 - x_3 + x_{12} + x_{13} + x_{23}\} \leq x_{123}
$$
$$
\leq \max\{0, x_{12} + x_{13} - x_1, x_{12} + x_{23} - x_2, x_{13} + x_{23} - x_3\}.
$$

Then, in particular, the extension x_{123} on $(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)$ is coherent if and only if $x_{123} \in [x'_{123}, x''_{123}]$, where

$$
x'_{123} = \max\{0, x_{12} + x_{13} - x_1, x_{12} + x_{23} - x_2, x_{13} + x_{23} - x_3\},
$$

$$
x''_{123} = \min\{x_{12}, x_{13}, x_{23}, 1 - x_1 - x_2 - x_3 + x_{12} + x_{13} + x_{23}\}. □
$$

8.2 The Case $H_1 = H_2 = H_3$

We recall that in case of logical dependencies, the set of all coherent assessments may be smaller than that one associated with the case of logical independence. However, in this section we show that the results of Theorem 15 and Corollary 1 still hold when the conditioning events H_1, H_2, and H_3 coincide.

Theorem 16 *Let be given any logically independent events E_1, E_2, E_3, H, with $H \neq \emptyset$. Then, the set Π of all coherent assessments $\mathcal{M} = (x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})$ on $\mathcal{F} = \{E_1|H, E_2|H, E_3|H, (E_1|H) \wedge (E_2|H), (E_1|H) \wedge (E_3|H), (E_2|H) \wedge (E_3|H), (E_1|H) \wedge (E_2|H) \wedge (E_3|H)\}$ is the set of points $(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})$ which satisfy the conditions in formula* (18).

Proof See Appendix A. □

Corollary 2 *For any coherent assessment $(x_1, x_2, x_3, x_{12}, x_{13}, x_{23})$ on*

$$\{E_1|H, E_2|H, E_3|H, (E_1E_2)|H, (E_1E_3)|H, (E_2E_3)|H\}$$

the extension x_{123} on $(E_1E_2E_3)|H$ is coherent if and only if $x_{123} \in [x'_{123}, x''_{123}]$, where x'_{123} and x''_{123} are defined in (19).

Proof The proof is the same as for Corollary 1. □

Of course, the results of Theorem 16 and Corollary 2 still hold in the unconditional case where $H = \Omega$.

Remark 11 As shown in this section, a consistent management of conjunctions (and/or disjunctions) defined on a given family of conditional events $\mathcal{F}$ essentially requires an (iterative) coherence checking and propagation of probability and prevision assessments on compounded conditionals, for each subfamily of $\mathcal{F}$. Then, an analysis of complexity in our context would be of the same kind of the exhaustive complexity analysis given in [5] for probabilistic reasoning under coherence.

9 Characterization of p-consistency and p-entailment with applications to nonmonotonic reasoning

In this section we apply our notion of conjunction to characterize the notions of p-consistency and p-entailment. Then, we examine some inference rules related with probabilistic nonmonotonic reasoning. We also briefly describe a characterization of p-entailment by a notion of iterated conditioning, in the case of two premises. We recall below the notions of p-consistency and p-entailment of Adams [2] as formulated for conditional events in the setting of coherence (see, e.g., [5, 20, 24]).

Definition 10 Let $\mathcal{F} = \{E_i|H_i, i = 1, \ldots, n\}$ be a family of n conditional events. Then, $\mathcal{F}$ is *p-consistent* if and only if the probability assessment $(p_1, p_2, \ldots, p_n) = (1, 1, \ldots, 1)$ on $\mathcal{F}$ is coherent.

Definition 11 A p-consistent family $\mathcal{F} = \{E_i|H_i, i = 1, \ldots, n\}$ *p-entails* a conditional event $E_{n+1}|H_{n+1}$ if and only if for any coherent probability assessment $(p_1, \ldots, p_n, p_{n+1})$ on $\mathcal{F} \cup \{E_{n+1}|H_{n+1}\}$ it holds that: if $p_1 = \cdots = p_n = 1$, then $p_{n+1} = 1$.

We recall below the notion of logical implication [31] between two conditional events.

Definition 12 Given two conditional events $A|H$ and $B|K$ we say that $A|H$ *logically implies* $B|K$, which we denote by $A|H \subseteq B|K$, if and only if AH true implies BK true and $\bar{B}K$ true implies $\bar{A}H$ true; that is: $AH \subseteq BK$ and $\bar{B}K \subseteq \bar{A}H$.

A suitable conditional event, named "implication", which could be related with the relation of logical implication was introduced by de Finetti in [16]. We observe that, by coherence, it holds that (see, e.g., [25, Theorem 7]).

$$A|H \subseteq B|K \implies P(A|H) \leq P(B|K). \tag{20}$$

We also recall the notion of quasi conjunction for a general family of n conditional events.

Definition 13 Given a family $\mathcal{F} = \{E_i|H_i, i = 1, \ldots, n\}$ of n conditional events, the *quasi conjunction* $QC(\mathcal{F})$ of the conditional events in $\mathcal{F}$ is defined as the following conditional event

$$QC(\mathcal{F}) = \bigwedge_{i=1}^{n} (\bar{H}_i \vee E_i H_i) \left| \left(\bigvee_{i=1}^{n} H_i \right) \right. .$$

Remark 12 We observe that, by Definition 13, based on (9) the quasi conjunction can be represented as

$$QC(\mathcal{F}) = \sum_{h=0}^{m} v_h C_h,$$

$$\text{where } v_h = \begin{cases} 1, & \text{if } S'_h \neq \emptyset \text{ and } S''_h = \emptyset, \\ 0, & \text{if } S''_h \neq \emptyset \\ v, & \text{if } S'''_h = \{1, \ldots, n\}, \end{cases} \tag{21}$$

where $v = P(QC(\mathcal{F}))$. Therefore, by (10), (21), and by also recalling Theorem 6, it holds that $z_h \leq v_h$, $h = 0, 1, \ldots, m$; thus

$$\mathcal{C}(\mathcal{F}) \leq QC(\mathcal{F}). \tag{22}$$

In particular, if $\mathcal{F}$ is p-consistent and $P(E_i|H_i) = 1$, $i = 1, \ldots, n$, then from (15) it holds that $x_S = \mathbb{P}(\mathcal{C}(\mathcal{F}_S)) = 1$ for every $S \subseteq \{1, 2, \ldots, n\}$, where $\mathcal{F}_S = \{E_i|H_i \in \mathcal{F} : i \in S\}$; then $z_h = v_h$, $h = 0, 1, \ldots, m$, and $\mathcal{C}(\mathcal{F}) = QC(\mathcal{F})$.

9.1 Characterization of p-consistency and p-entailment

We illustrate below a characterization of p-consistency of a family $\mathcal{F}$ in terms of the coherence of the prevision assessment $\mathbb{P}[\mathcal{C}(\mathcal{F})] = 1$.

Theorem 17 *A family of n conditional events $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\}$ is p-consistent if and only if the prevision assessment $\mathbb{P}[\mathcal{C}(\mathcal{F})] = 1$ is coherent.*

Proof ($\Rightarrow$) By Definition 10, as $\mathcal{F}$ is p-consistent, the probability assessment $(x_1, x_2, \ldots, x_n) = (1, 1, \ldots, 1)$ on $\mathcal{F}$ is coherent. Then, by (15) the extension $\mathbb{P}[\mathcal{C}(\mathcal{F})] = 1$ is unique and of course coherent.

($\Leftarrow$) By (15) it holds that $\mathbb{P}[\mathcal{C}(\mathcal{F})] \leq \min\{x_1, \ldots, x_n\}$ and hence $\mathbb{P}[\mathcal{C}(\mathcal{F})] = 1$ implies $(x_1, x_2, \ldots, x_n) = (1, 1, \ldots, 1)$ on $\mathcal{F}$. Moreover, the coherence of $\mathbb{P}[\mathcal{C}(\mathcal{F})] = 1$ requires that the (unique) extension $(1, 1, \ldots, 1)$ on $\mathcal{F}$ be coherent. Thus, $\mathcal{F}$ is p-consistent. $\square$

We observe that, in the case where $H_1 = \ldots = H_n = H$, the assessment $P(E_1|H) = \ldots P(E_n|H) = 1$ is coherent (that is, $\mathcal{F}$ is p-consistent) if and only if $P[(E_1 \cdots E_n)|H] = 1$ is coherent.

The next theorem gives a characterization of p-entailment in terms of a result which involves suitable conjunctions associated with the premise set and the conclusion of the given inference rule.

Theorem 18 *Let be given a p-consistent family of n conditional events $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\}$ and a further conditional event $E_{n+1}|H_{n+1}$. Then, the following assertions are equivalent:*

(i) *$\mathcal{F}$ p-entails $E_{n+1}|H_{n+1}$;*
(ii) *the conjunction $\mathcal{C}_{n+1} = (E_1|H_1) \wedge \cdots \wedge (E_n|H_n) \wedge (E_{n+1}|H_{n+1})$ coincides with the conjunction $\mathcal{C}_n = (E_1|H_1) \wedge \cdots \wedge (E_n|H_n)$;*
(iii) *the inequality $\mathcal{C}_n \leq (E_{n+1}|H_{n+1})$ is satisfied.*

Proof See Appendix A. $\square$

As a first simple application of Theorem 18 we observe that, given two conditional events $A|H$, with $AH \neq \emptyset$, and $B|K$, the p-entailment of $B|K$ from $A|H$ amounts to the condition (ii), i.e., $A|H \wedge B|K = A|H$, or equivalently condition (iii), i.e., $A|H \leq B|K$. In particular, (ii) and (iii) are both satisfied when $A|H \subseteq B|K$.

9.2 Applications to some p-valid inference rules

We recall that an inference from a p-consistent family $\mathcal{F}$ to $E|H$ is *p-valid* if and only if $\mathcal{F}$ p-entails $E|H$. We will examine some p-valid inference rules by verifying that conditions (ii) and (iii) in Theorem 18 are satisfied. In particular we consider the following inference rules of System P: *And*, *Cut*, *CM*, and *Or*. In what follows, if not specified otherwise, the basic events are assumed to be logically independent.

And rule The family $\{B|A, C|A\}$ p-entails $BC|A$. It holds that $(B|A) \wedge (C|A) = BC|A = (B|A) \wedge (C|A) \wedge (BC|A)$ and $(B|A) \wedge (C|A) = BC|A \leq BC|A$; that is, conditions (ii) and (iii) are satisfied.

Cut rule The family $\{C|AB, B|A\}$ p-entails $C|A$. By (5), as $\overline{A}\overline{B}AB = \overline{A}ABC = \emptyset$, it holds that

$$(C|AB) \wedge (B|A) = ABC + z\overline{A},$$

where $z = \mathbb{P}[(C|AB) \wedge (B|A)]$. Moreover, $BC|A = ABC + x\overline{A}$, where $x = P(BC|A)$. As $(C|AB) \wedge (B|A)$ and $BC|A$ coincide conditionally on A being true, by Theorem 3, it follows that $(C|AB) \wedge (B|A) = BC|A$. Then, condition (ii) is satisfied, that is $(C|AB) \wedge (B|A) \wedge (C|A) = (BC|A) \wedge (C|A) = BC|A = (C|AB) \wedge (B|A)$. Moreover, $C|AB \wedge B|A = BC|A \leq C|A$, that is condition (iii) is satisfied too.

Remark 13 As shown in the analysis of Cut rule, it holds that $C|AB \wedge B|A = BC|A$. Then, the family $\{C|AB, B|A\}$ p-entails $BC|A$. This p-valid rule is called CCT (Conjunctive Cumulative Transitivity); see, e.g., [43].

CM rule The family $\{C|A, B|A\}$ p-entails $C|AB$. It holds that $(C|A) \wedge (B|A) = BC|A$. Moreover, $(C|A) \wedge (B|A) \wedge (C|AB) = (BC|A) \wedge (C|AB)$. By (5), it holds that

$$(BC|A) \wedge (C|AB) = ABC + z\overline{A},$$

where $z = \mathbb{P}[(BC|A) \wedge (C|AB)]$. Moreover, $BC|A = ABC + x\overline{A}$, where $x = P(BC|A)$. As $(BC|A) \wedge (C|AB)$ and $BC|A$ coincide conditionally on A being true, by Theorem 3 it follows that $(BC|A) \wedge (C|AB) = BC|A$; so that $(C|A) \wedge (B|A) \wedge (C|AB) = BC|A = (C|A) \wedge (B|A)$, so that condition (ii) is satisfied. Moreover, based on Definition 12, it holds that $(C|A) \wedge (B|A) = BC|A \subseteq$

$C|AB$, then $(C|A) \wedge (B|A) \leq C|AB$, so that condition (iii) is satisfied too.

Or rule The family $\{C|A, C|B\}$ p-entails $C|(A \vee B)$. We set $P(C|A) = x$, $P(C|B) = y$, and $\mathbb{P}((C|A) \wedge (C|B)) = z$; then, by observing that the family $\{ABC, \overline{A}BC, A\overline{B}C, (A \vee B)\overline{C}, \overline{A}\,\overline{B}\}$ is a partition of the sure event, we obtain

$$(C|A) \wedge (C|B) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } (A \vee B)\overline{C} \text{ is true,} \\ x, & \text{if } \overline{A}BC \text{ is true,} \\ y, & \text{if } A\overline{B}C \text{ is true,} \\ z, & \text{if } \overline{A}\,\overline{B} \text{ is true.} \end{cases} \tag{23}$$

Moreover, by defining $\mathbb{P}[(C|A) \wedge (C|B) \wedge (C|(A \vee B))] = t$, we obtain

$$(C|A) \wedge (C|B) \wedge (C|(A \vee B)) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } (A \vee B)\overline{C} \text{ is true,} \\ x, & \text{if } \overline{A}BC \text{ is true,} \\ y, & \text{if } A\overline{B}C \text{ is true,} \\ t, & \text{if } \overline{A}\,\overline{B} \text{ is true.} \end{cases}$$

As we can see, $(C|A) \wedge (C|B) \wedge (C|(A \vee B))$ and $(C|A) \wedge (C|B)$ coincide when $A \vee B$ is true; then, by Theorem 3 it holds that $t = z$, so that

$$(C|A) \wedge (C|B) \wedge (C|(A \vee B)) = (C|A) \wedge (C|B),$$

that is condition (ii) is satisfied. Moreover, defining $P(C|(A \vee B)) = w$, we have

$$C|(A \vee B) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } (A \vee B)\overline{C} \text{ is true,} \\ 1, & \text{if } \overline{A}BC \text{ is true,} \\ 1, & \text{if } A\overline{B}C \text{ is true,} \\ w, & \text{if } \overline{A}\,\overline{B} \text{ is true.} \end{cases} \tag{24}$$

Based on (23) and (24), it holds that $(C|A) \wedge (C|B) \leq C|(A \vee B)$ conditionally on $A \vee B$ being true. Then, from Theorem 6 it holds that $\mathbb{P}((C|A) \wedge (C|B)) = t \leq w = P(C|(A \vee B))$; thus $(C|A) \wedge (C|B) \leq C|(A \vee B)$, that is condition (iii) is satisfied.

An inference rule related to Or rule [1, Rule 5, p. 189] In this inference rule the premise set is $\{C|(A \vee B), \overline{C}|A\}$ and the conclusion is $C|B$. We first observe that the premise set $\mathcal{F} = \{C|(A \vee B), \overline{C}|A\}$ is p-consistent because the assessment $P(C|(A \vee B)) = P(\overline{C}|A) = 1$ is coherent. Indeed, by applying Algorithm 1 to the pair $(\mathcal{F}, \mathcal{M}) = (\{C|(A \vee B), \overline{C}|A\}, (1, 1))$, it holds that the starting system (Σ) is solvable, with $\mathcal{F}_0 = \{\overline{C}|A\}$. Then, by repeating the steps of the algorithm, the assessment $P(\overline{C}|A) = 1$ is coherent. Thus, the assessment $(1, 1)$ on $\mathcal{F}$ is coherent and hence $\mathcal{F}$ is p-consistent. We also note that, defining $P(C|(A \vee B)) = x$, $P(\overline{C}|A) = y$, and $\mathbb{P}((C|(A \vee B)) \wedge (\overline{C}|A)) = z$, the coherence of $(x, y) = (1, 1)$ from

(15) amounts to coherence of $z = 1$, which by Theorem 17 is another characterization for the p-consistency of $\mathcal{F}$. Concerning p-entailment, we observe that

$$(C|A \vee B) \wedge (\overline{C}|A) = \begin{cases} 0, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } AB\overline{C} \text{ is true,} \\ 0, & \text{if } A\overline{B}C \text{ is true,} \\ 0, & \text{if } A\overline{B}\,\overline{C} \text{ is true,} \\ y, & \text{if } \overline{A}BC \text{ is true,} \\ 0, & \text{if } \overline{A}B\overline{C} \text{ is true,} \\ z, & \text{if } \overline{A}\,\overline{B} \text{ is true,} \end{cases}$$

$$= \begin{cases} 0, & \text{if } A \vee \overline{A}B\overline{C} \text{ is true,} \\ y, & \text{if } \overline{A}BC \text{ is true,} \\ z, & \text{if } \overline{A}\,\overline{B} \text{ is true.} \end{cases} \tag{25}$$

Moreover, by defining $\mathbb{P}[(C|(A \vee B)) \wedge (\overline{C}|A) \wedge (C|B)] = t$, we obtain

$$(C|(A \vee B)) \wedge (\overline{C}|A) \wedge (C|B) = \begin{cases} 0, & \text{if } A \vee \overline{A}B\overline{C} \text{ is true,} \\ y, & \text{if } \overline{A}BC \text{ is true,} \\ t, & \text{if } \overline{A}\,\overline{B} \text{ is true.} \end{cases} \tag{26}$$

As we can see from (25) and (26), the two quantities $(C|(A \vee B)) \wedge (\overline{C}|A) \wedge (C|B)$ and $(C|(A \vee B)) \wedge (\overline{C}|A)$ coincide when $A \vee B$ is true; then, by Theorem 3 it holds that $t = z$, so that

$$(C|(A \vee B)) \wedge (\overline{C}|A) \wedge (C|B) = (C|(A \vee B)) \wedge (\overline{C}|A),$$

that is condition (ii) is satisfied. Moreover, defining $P(C|B) = w$, we have

$$C|B = \begin{cases} 1, & \text{if } BC \text{ is true,} \\ 0, & \text{if } B\overline{C} \text{ is true,} \\ w, & \text{if } \overline{B} \text{ is true.} \end{cases} = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } AB\overline{C} \text{ is true,} \\ w, & \text{if } A\overline{B}C \text{ is true,} \\ w, & \text{if } A\overline{B}\,\overline{C} \text{ is true,} \\ 1, & \text{if } \overline{A}BC \text{ is true,} \\ 0, & \text{if } \overline{A}B\overline{C} \text{ is true,} \\ w, & \text{if } \overline{A}\,\overline{B} \text{ is true,} \end{cases} \tag{27}$$

Based on (25) and (27), it holds that $(C|(A \vee B)) \wedge (\overline{C}|A) \leq C|B$ conditionally on $A \vee B$ being true. Then, from Theorem 6 it holds that $\mathbb{P}((C|(A \vee B)) \wedge (\overline{C}|A)) = t \leq w = P(C|B)$; thus $(C|(A \vee B)) \wedge (\overline{C}|A) \leq C|B$, that is condition (iii) is satisfied. Thus, this inference rule is p-valid. Notice that the p-validity of the rule could be also derived by using the lower and upper bounds given for Or rule in [18]. Indeed, using Or rule, when $P(C|A) = 0$ and $P(C|B) = y$ it holds that $z = \underline{P}(C|A \vee B) \in [0, y]$, so that $\underline{P}(C|(A \vee B)) \leq P(C|B)$. Then, $P(C|(A \vee B)) = 1$ and $P(\overline{C}|A) = 1$ implies $P(C|B) = 1$, that is $\{C|(A \vee B), \overline{C}|A\}$ p-entails $C|B$.

Generalized Or rule In this p-valid rule, studied in [19] (see also [25]), the p-consistent premise set is $\{C|A_1, C|A_2, \ldots, C|A_n\}$ and the conclusion is $C|(A_1 \vee A_2 \vee \cdots \vee$

A_n). For each nonempty subset $S \subset \{1, 2, \ldots, n\}$, we define $\mathbb{P}[\bigwedge_{i \in S} (C|A_i)] = x_S$; moreover, we set $\mathbb{P}[\bigwedge_{i=1}^{n} (C|A_i)] = z$. Then,

$$(C|A_1) \wedge \cdots \wedge (C|A_n) = \begin{cases} 1, & \text{if } A_1 A_2 \cdots A_n C \text{ is true,} \\ 0, & \text{if } (A_1 \vee A_2 \vee \cdots \vee A_n)\overline{C} \text{ is true,} \\ x_S, & \text{if } \bigwedge_{i \in S} \overline{A}_i \bigwedge_{j \notin S} A_j C \text{ is true,} \\ z, & \text{if } \overline{A}_1 \overline{A}_2 \cdots \overline{A}_n \text{ is true.} \end{cases}$$

(28)

Moreover, by defining $\mathbb{P}[(C|A_1) \wedge \cdots \wedge (C|A_n) \wedge (C|(A_1 \vee A_2 \vee \cdots \vee A_n))] = t$, we obtain

$$\begin{aligned} &(C|A_1) \wedge \cdots \wedge (C|A_n) \wedge (C|(A_1 \vee A_2 \vee \cdots \vee A_n)) \\ &= \begin{cases} 1, & \text{if } A_1 A_2 \cdots A_n C \text{ is true,} \\ 0, & \text{if } (A_1 \vee A_2 \vee \cdots \vee A_n)\overline{C} \text{ is true,} \\ x_S, & \text{if } \bigwedge_{i \in S} \overline{A}_i \bigwedge_{j \notin S} A_j C \text{ is true,} \\ t, & \text{if } \overline{A}_1 \overline{A}_2 \cdots \overline{A}_n \text{ is true.} \end{cases} \end{aligned}$$

(29)

As we can see from (28) and (29), $(C|A_1) \wedge \cdots \wedge (C|A_n) \wedge (C|(A_1 \vee A_2 \vee \cdots \vee A_n))$ and $(C|A_1) \wedge \cdots \wedge (C|A_n)$ coincide when $A_1 \vee \cdots \vee A_n$ is true; then, by Theorem 3 it holds that $t = z$, so that

$$\begin{aligned} &(C|A_1) \wedge \cdots \wedge (C|A_n) \wedge (C|(A_1 \vee A_2 \vee \cdots \vee A_n)) \\ &= (C|A_1) \wedge \cdots \wedge (C|A_n), \end{aligned}$$

that is condition (ii) is satisfied. Moreover,

$$C|(A_1 \vee A_2 \vee \cdots \vee A_n) = \begin{cases} 1, & \text{if } A_1 A_2 \cdots A_n C \text{ is true,} \\ 0, & \text{if } (A_1 \vee A_2 \vee \cdots \vee A_n)\overline{C} \text{ is true,} \\ 1, & \text{if } \bigwedge_{i \in S} \overline{A}_i \bigwedge_{j \notin S} A_j C \text{ is true,} \\ w, & \text{if } \overline{A}_1 \overline{A}_2 \cdots \overline{A}_n \text{ is true,} \end{cases}$$

(30)

where $w = P(C|(A_1 \vee A_2 \vee \cdots \vee A_n))$. Based on (28) and (30), it holds that $(C|A_1) \wedge \cdots \wedge (C|A_n) \leq C|(A_1 \vee A_2 \vee \cdots \vee A_n)$ conditionally on $A_1 \vee \cdots \vee A_n$ being true. Then, from Theorem 6 it holds that $t \leq w$; thus $(C|A_1) \wedge \cdots \wedge (C|A_n) \leq C|(A_1 \vee A_2 \vee \cdots \vee A_n)$, that is condition (iii) is satisfied.

9.3 Iterated conditioning and p-entailment

We now briefly describe a characterization of p-entailment of a conditional event $E_3|H_3$ from a p-consistent family $\{E_1|H_1, E_2|H_2\}$, which exploits a suitable notion of iterated conditioning.

Definition 14 Let be given $n + 1$ conditional events $E_1|H_1, \ldots, E_{n+1}|H_{n+1}$, with $(E_1|H_1) \wedge \cdots \wedge (E_n|H_n) \neq 0$. We denote by $(E_{n+1}|H_{n+1})|((E_1|H_1) \wedge \cdots \wedge (E_n|H_n)) = (E_{n+1}|H_{n+1})|\mathcal{C}_n$ the random quantity

$$(E_1|H_1) \wedge \cdots \wedge (E_{n+1}|H_{n+1}) + \mu(1 - (E_1|H_1) \wedge \cdots \wedge (E_n|H_n)) = \mathcal{C}_{n+1} + \mu(1 - \mathcal{C}_n),$$

where $\mu = \mathbb{P}[(E_{n+1}|H_{n+1})|\mathcal{C}_n]$.

We observe that, based on the betting metaphor, the quantity μ is the amount to be paid in order to receive the amount $\mathcal{C}_{n+1} + \mu(1 - \mathcal{C}_n)$. Definition 14 generalizes the notion of iterated conditional $(E_2|H_2)|(E_1|H_1)$ given in previous papers (see, e.g., [22, 23, 26]). We also observe that, defining $\mathbb{P}(\mathcal{C}_n) = z_n$ and $\mathbb{P}(\mathcal{C}_{n+1}) = z_{n+1}$, by the linearity of prevision it holds that $\mu = z_{n+1} + \mu(1 - z_n)$; then, $z_{n+1} = \mu z_n$, that is $\mathbb{P}(\mathcal{C}_{n+1}) = \mathbb{P}[(E_{n+1}|H_{n+1})|\mathcal{C}_n]\mathbb{P}(\mathcal{C}_n)$, which is the compound prevision theorem.

By applying Definition 14 with $n = 2$, given a p-consistent family $\{E_1|H_1, E_2|H_2\}$ and a further event $E_3|H_3$, it can be proved that [30]

$$\{E_1|H_1, E_2|H_2\} \text{ p-entails } E_3|H_3 \iff$$
$$(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) = 1,$$

that is: $\{E_1|H_1, E_2|H_2\}$ p-entails $E_3|H_3$ if and only if the iterated conditional $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2))$ is constant and equal to 1.

10 From non p-valid to p-valid inference rules

In this section we first examine some non p-valid inference rules, by showing that conditions (ii) and (iii) of Theorem 18 are not satisfied. Then, we illustrate by an example two different methods which allow to get p-valid inference rules starting by non p-valid ones.

10.1 Some non p-valid inference rules

We start by showing that Transitivity is not p-valid.

Transitivity In this rule the p-consistent premise set is $\{C|B, B|A\}$ and the conclusion is $C|A$. The rule is not p-valid [28], indeed we can show that

$$(C|B) \wedge (B|A) \wedge (C|A) \neq (C|B) \wedge (B|A) \text{ and}$$
$$(C|B) \wedge (B|A) \nleq C|A.$$

Defining $P(B|A) = x$, $P(BC|A) = y$, $P(C|A) = t$, $\mathbb{P}((C|B) \wedge (B|A) \wedge (C|A)) = \mu$, $\mathbb{P}((C|B) \wedge (B|A)) = z$, we have

$$\begin{aligned} &(C|B) \wedge (B|A) \wedge (C|A) = (C|B) \wedge (BC|A) \\ &= \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } AB\overline{C} \text{ is true,} \\ 0, & \text{if } A\overline{B}C \text{ is true,} \\ 0, & \text{if } A\overline{B}\,\overline{C} \text{ is true,} \\ y, & \text{if } \overline{A}BC \text{ is true,} \\ 0, & \text{if } \overline{A}B\overline{C} \text{ is true,} \\ \mu, & \text{if } \overline{A}\,\overline{B} \text{ is true,} \end{cases} \end{aligned}$$

(31)

and

$$(C|B) \wedge (B|A) = \begin{cases} 1, & \text{if } ABC \text{ is true}, \\ 0, & \text{if } AB\overline{C} \text{ is true}, \\ 0, & \text{if } A\overline{B}C \text{ is true}, \\ 0, & \text{if } A\overline{B}\,\overline{C} \text{ is true}, \\ x, & \text{if } \overline{A}BC \text{ is true}, \\ 0, & \text{if } \overline{A}B\overline{C} \text{ is true}, \\ z, & \text{if } \overline{A}\,\overline{B} \text{ is true}. \end{cases} \quad (32)$$

Then, as (in general) $x \neq y$, it holds that $(C|B) \wedge (B|A) \wedge (C|A) \neq (C|B) \wedge (B|A)$, so that condition (ii) is not satisfied. Moreover,

$$C|A = \begin{cases} 1, & \text{if } ABC \text{ is true}, \\ 0, & \text{if } AB\overline{C} \text{ is true}, \\ 0, & \text{if } A\overline{B}C \text{ is true}, \\ 0, & \text{if } A\overline{B}\,\overline{C} \text{ is true}, \\ t, & \text{if } \overline{A}BC \text{ is true}, \\ t, & \text{if } \overline{A}B\overline{C} \text{ is true}, \\ t, & \text{if } \overline{A}\,\overline{B} \text{ is true}. \end{cases} \quad (33)$$

Then, by observing that (in general) $x \not\leq t$ it follows that $(C|B) \wedge (B|A) \not\leq C|A$, so that condition (iii) is not satisfied. Therefore, Transitivity rule is not p-valid.

Denial of the antecedent We consider the rule where the premise set is $\{\overline{A}, C|A\}$ and the conclusion is $\overline{C}$. The premise set $\{\overline{A}, C|A\}$ is p-consistent because, by applying the Algorithm 1, the assessment $P(\overline{A}) = P(C|A) = 1$ is coherent. We verify that $\overline{A} \wedge (C|A) \wedge \overline{C} \neq \overline{A} \wedge (C|A)$ and that $\overline{A} \wedge (C|A) \not\leq \overline{C}$, that is the Denial of the antecedent is not p-valid. We set $P(C|A) = y$, then

$$\overline{A} \wedge (C|A) \wedge \overline{C} = \begin{cases} 0, & \text{if } A \text{ is true}, \\ 0, & \text{if } \overline{A}C \text{ is true}, \\ y, & \text{if } \overline{A}\,\overline{C} \text{ is true}, \end{cases}$$

and

$$\overline{A} \wedge (C|A) = \begin{cases} 0, & \text{if } A \text{ is true}, \\ y, & \text{if } \overline{A}C \text{ is true}, \\ y, & \text{if } \overline{A}\,\overline{C} \text{ is true}. \end{cases}$$

Assuming $y > 0$, when $\overline{A}C$ is true it holds that

$$\overline{A} \wedge (C|A) \wedge \overline{C} = 0 < y = \overline{A} \wedge (C|A), \quad \overline{A} \wedge (C|A) = y > 0 = \overline{C},$$

thus: $\overline{A} \wedge (C|A) \wedge \overline{C} \neq \overline{A} \wedge (C|A)$ and $\overline{A} \wedge (C|A) \not\leq \overline{C}$, that is conditions (ii) and (iii) are not satisfied.

Affirmation of the consequent We consider the rule where the (p-consistent) premise set is $\{C, C|A\}$ and the conclusion is A. We verify that $C \wedge (C|A) \wedge A \neq C \wedge (C|A)$ and $C \wedge (C|A) \not\leq A$, that is the *Affirmation of the consequent* rule is not p-valid. We set $P(C|A) = y$, then

$$C \wedge (C|A) \wedge A = AC = \begin{cases} 1, & \text{if } AC \text{ is true}, \\ 0, & \text{if } \overline{AC} \text{ is true}, \end{cases}$$

and

$$C \wedge (C|A) = \begin{cases} 1, & \text{if } AC \text{ is true}, \\ 0, & \text{if } \overline{C} \text{ is true}, \\ y, & \text{if } \overline{A}C \text{ is true}. \end{cases} \quad (34)$$

Assuming $y > 0$, when $\overline{A}C$ is true it holds that

$$C \wedge (C|A) \wedge A = 0 < y = C \wedge (C|A),$$
$$C \wedge (C|A) = y > 0 = A,$$

thus: $C \wedge (C|A) \wedge A \neq C \wedge (C|A)$ and $C \wedge (C|A) \not\leq A$, that is conditions (ii) and (iii) are not satisfied.

Remark 14 We now will make a comparison between the two objects $C \wedge (C|A)$ and $C|(A \vee \overline{C})$, by showing they do not coincide. Defining $P(C|(A \vee \overline{C})) = t$, it holds that

$$C|(A \vee \overline{C}) = \begin{cases} 1, & \text{if } AC \text{ is true}, \\ 0, & \text{if } \overline{C} \text{ is true}, \\ t, & \text{if } \overline{A}C \text{ is true}. \end{cases} \quad (35)$$

It could seem, from (34) and (35), that y and t should be equal and then $C \wedge (C|A)$ and $C|(A \vee \overline{C})$ should coincide. However, in this case the conditioning event for $C \wedge (C|A)$ is $\Omega \vee A = \Omega$, so that the disjunction of the conditioning events is $\Omega \vee (A \vee \overline{C}) = \Omega$; the two objects $C \wedge (C|A)$ and $C|(A \vee \overline{C})$ do not coincide conditionally on Ω; then $C \wedge (C|A)$ and $C|(A \vee \overline{C})$ do not coincide (condition (i) of Theorem 3 is not satisfied). We also observe that, defining $\mathbb{P}(C \wedge (C|A)) = \mu$, (in general) μ does not belong to the set $\{1, 0, y\}$ of possible values of $C \wedge (C|A)$, because μ is a linear convex combination of the values $\{1, 0, y\}$. As a further aspect, we verify below that $t \leq \mu \leq y$. The constituents generated by $\{A, C\}$ are: $AC, A\overline{C}, \overline{A}C, \overline{A}\,\overline{C}$; then, the associated values for the random vector $(C|(A \vee \overline{C}), C \wedge (C|A), C|A)$ are

$$(1, 1, 1), \quad (0, 0, 0), \quad (t, y, y), \quad (0, 0, y). \quad (36)$$

Based on Theorem 6, we observe that:

- $C|(A \vee \overline{C}) \leq C|A$ conditionally on $A \vee \overline{C} \vee A = A \vee \overline{C}$, hence $P(C|(A \vee \overline{C})) = t \leq y = P(C|A)$;
- $C \wedge (C|A) \leq C|A$ conditionally on $\Omega \vee A = \Omega$, hence $\mathbb{P}(C \wedge (C|A)) = \mu \leq y = P(C|A)$;
- $C|(A \vee \overline{C}) \leq C \wedge (C|A)$ conditionally on $A \vee \overline{C} \vee \Omega = \Omega$, hence $P(C|(A \vee \overline{C})) = t \leq \mu = \mathbb{P}(C \wedge (C|A))$.

In other words: $t \leq \mu \leq y$. We observe that these inequalities also follow because coherence requires that the prevision point (t, μ, y) must be a linear convex combination of points in (36).

On combining evidence: an example from Boole We now examine an example studied in [8, p. 632] (see also [33, Theorem 5.45]), where p-entailment does not hold. Indeed, it can be proved that the extension $w = P(C|AB)$ of any (coherent) assessment (x, y) on $\{C|A, C|B\}$ is

coherent for every $w \in [0, 1]$. Using conditions (ii) and (iii) of Theorem 18, we show that the p-consistent family $\{C|A, C|B\}$ does not p-entail $C|AB$. We set $P(C|A) = x$, $P(C|B) = y$, and $\mathbb{P}((C|A) \wedge (C|B)) = z$; then,

$$(C|A) \wedge (C|B) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } AB\overline{C} \text{ is true,} \\ 0, & \text{if } A\overline{B}\,\overline{C} \text{ is true,} \\ 0, & \text{if } \overline{A}B\overline{C} \text{ is true,} \\ x, & \text{if } \overline{A}BC \text{ is true,} \\ y, & \text{if } A\overline{B}C \text{ is true,} \\ z, & \text{if } \overline{A}\,\overline{B} \text{ is true.} \end{cases} = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } (A \vee B)\overline{C} \text{ is true,} \\ x, & \text{if } \overline{A}BC \text{ is true,} \\ y, & \text{if } A\overline{B}C \text{ is true,} \\ z, & \text{if } \overline{A}\,\overline{B} \text{ is true.} \end{cases}$$

$$(37)$$

Moreover, by defining $\mathbb{P}[(C|A) \wedge (C|AB)] = u$, $\mathbb{P}[(C|B) \wedge (C|AB)] = v$ and $\mathbb{P}[(C|A) \wedge (C|B) \wedge (C|AB)] = t$, we obtain

$$(C|A) \wedge (C|B) \wedge (C|AB) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } (A \vee B)\overline{C} \text{ is true,} \\ u, & \text{if } \overline{A}BC \text{ is true,} \\ v, & \text{if } A\overline{B}C \text{ is true,} \\ t, & \text{if } \overline{A}\overline{B} \text{ is true.} \end{cases}$$

As in general $x \neq u$ and $y \neq v$, then $(C|A) \wedge (C|B) \wedge (C|AB)$ and $(C|A) \wedge (C|B)$ do not coincide, so that condition (ii) is not satisfied. Moreover,

$$C|AB = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } AB\overline{C} \text{ is true,} \\ w, & \text{if } A\overline{B}\,\overline{C} \text{ is true,} \\ w, & \text{if } \overline{A}B\overline{C} \text{ is true,} \\ w, & \text{if } \overline{A}BC \text{ is true,} \\ w, & \text{if } A\overline{B}C \text{ is true,} \\ w, & \text{if } \overline{A}\,\overline{B} \text{ is true,} \end{cases} = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } AB\overline{C} \text{ is true,} \\ w, & \text{if } \overline{AB} \text{ is true.} \end{cases}$$

$$(38)$$

Based on (37) and (38), we can see that $(C|A) \wedge (C|B) \not\leq C|(AB)$, so that condition (iii) is not satisfied. Thus, the inference from $\{C|A, C|B\}$ to $C|AB$ is not p-valid.

10.2 Two methods for constructing p-valid inference rules

We now illustrate by an example two different methods by means of which, starting by a non p-valid inference rule, we get p-valid inference rules: a) to add a suitable premise; b) to add a suitable logical constraint. The further premise, or logical constraint, (must preserve p-consistency and) is determined by analyzing the possible values of conjunctions.

Weak Transitivity In our example we start by the (non p-valid) Transitivity rule where the premise set is $\{C|B, B|A\}$ and the conclusion is $C|A$.

Method a). We add the premise $A|(A \vee B)$, so that the premise set is $\{C|B, B|A, A|(A \vee B)\}$, while the conclusion is still $C|A$. The premise set $\{C|B, B|A, A|(A \vee B)\}$ is p-consistent; indeed as $ABC \neq \emptyset$, by evaluating $P(ABC) = 1$ we get $P(C|B) = P(B|A) = P(A|(A \vee B)) = 1$. We show that $(C|B) \wedge (B|A) \wedge (A|(A \vee B)) \wedge (C|A) = (C|B) \wedge (B|A) \wedge (A|(A \vee B))$ and $(C|B) \wedge (B|A) \wedge (A|(A \vee B)) \leq C|A$.

Defining $\mathbb{P}[(C|B) \wedge (B|A) \wedge (A|(A \vee B)) \wedge (C|A)] = \mu$, we have

$$(C|B) \wedge (B|A) \wedge (A|(A \vee B)) \wedge (C|A) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } AB\overline{C} \text{ is true,} \\ 0, & \text{if } A\overline{B}C \text{ is true,} \\ 0, & \text{if } A\overline{B}\,\overline{C} \text{ is true,} \\ 0, & \text{if } \overline{A}BC \text{ is true,} \\ 0, & \text{if } \overline{A}B\overline{C} \text{ is true,} \\ \mu, & \text{if } \overline{A}\,\overline{B} \text{ is true.} \end{cases}$$

Moreover, defining $\mathbb{P}[(C|B) \wedge (B|A) \wedge (A|(A \vee B))] = z$, we have

$$(C|B) \wedge (B|A) \wedge (A|(A \vee B)) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } AB\overline{C} \text{ is true,} \\ 0, & \text{if } A\overline{B}C \text{ is true,} \\ 0, & \text{if } A\overline{B}\,\overline{C} \text{ is true,} \\ 0, & \text{if } \overline{A}BC \text{ is true,} \\ 0, & \text{if } \overline{A}B\overline{C} \text{ is true,} \\ z, & \text{if } \overline{A}\,\overline{B} \text{ is true.} \end{cases}$$

Conditionally on $A \vee B$ being true it holds that $(C|B) \wedge (B|A) \wedge (A|(A \vee B)) \wedge (C|A) = (C|B) \wedge (B|A) \wedge (A|(A \vee B)) = ABC|(A \vee B)$. Then, by Theorem 3 we have $(C|B) \wedge (B|A) \wedge (A|(A \vee B)) \wedge (C|A) = (C|B) \wedge (B|A) \wedge (A|(A \vee B)) = ABC|(A \vee B)$, so that condition (ii) is satisfied. Finally, as $ABC|(A \vee B) \subseteq C|A$, it holds that $(C|B) \wedge (B|A) \wedge (A|(A \vee B)) = ABC|(A \vee B) \leq C|A$, so that condition (iii) is satisfied. Therefore this Weak Transitivity rule is p-valid. We observe that another p-valid version of Weak Transitivity would be obtained by adding the premise $A|B$ instead of $A|(A \vee B)$.

Method b). We add the logical constraint $\overline{A}BC = \emptyset$, that is $BC \subseteq A$. The p-consistency of the premise set $\{C|B, B|A\}$ is preserved because, as before $ABC \neq \emptyset$ and by evaluating $P(ABC) = 1$ we get $P(C|B) = P(B|A) = 1$. Based on (31), (32), (33) it holds that

$$(C|B) \wedge (B|A) \wedge (C|A) = (C|B) \wedge (BC|A)$$

$$= \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } AB\overline{C} \text{ is true,} \\ 0, & \text{if } A\overline{B}C \text{ is true,} \\ 0, & \text{if } A\overline{B}\,\overline{C} \text{ is true,} \\ 0, & \text{if } \overline{A}B\overline{C} \text{ is true,} \\ \mu, & \text{if } \overline{A}\,\overline{B} \text{ is true,} \end{cases}$$

and

$$(C|B) \wedge (B|A) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } AB\overline{C} \text{ is true,} \\ 0, & \text{if } A\overline{B}C \text{ is true,} \\ 0, & \text{if } A\overline{B}\,\overline{C} \text{ is true,} \\ 0, & \text{if } \overline{A}BC \text{ is true,} \\ z, & \text{if } \overline{A}\,\overline{B} \text{ is true.} \end{cases}$$

As we can see $(C|B) \wedge (B|A) \wedge (C|A) = (C|B) \wedge (B|A)$ conditionally on $A \vee B$ being true. Then, by Theorem 3 condition (ii) is satisfied. Moreover,

$$C|A = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } AB\overline{C} \text{ is true,} \\ 0, & \text{if } A\overline{B}C \text{ is true,} \\ 0, & \text{if } A\overline{B}\,\overline{C} \text{ is true,} \\ t, & \text{if } \overline{A}BC \text{ is true,} \\ t, & \text{if } \overline{A}\,\overline{B} \text{ is true.} \end{cases}$$

Then, $(C|B) \wedge (B|A) \leq C|A$ conditionally on $A \vee B$ being true. Thus, by Theorem 6 condition (iii) is satisfied too. Therefore, under the logical constraint $\overline{A}BC = \emptyset$, the family $\{C|B, B|A\}$ p-entails $C|A$, which is another p-valid version of Weak Transitivity.

We observe that in [28, Theorem 5] it has been shown that another p-valid version of Weak Transitivity is obtained by adding the probabilistic constraint $P(A|(A \vee B)) > 0$, that is

$$P(C|B) = 1, \ P(B|A) = 1, \ P(A|(A \vee B)) > 0$$
$$\implies P(C|A) = 1.$$

11 Conclusions

We generalized the notions of conjunction and disjunction of two conditional events to the case of n conditional events. We introduced the notion of negation and we showed that De Morgan's Laws still hold. We also verified that the associative and commutative properties are satisfied. We studied the monotonicity property, by proving that $\mathcal{C}_{n+1} \leq \mathcal{C}_n$ and $\mathcal{D}_{n+1} \geq \mathcal{D}_n$ for every n. We computed the set of all coherent assessments on the family $\{\mathcal{C}_n, E_{n+1}|H_{n+1}, \mathcal{C}_{n+1}\}$, by showing that Fréchet-Hoeffding bounds still hold in this case; then, we examined the (reverse) probabilistic inference from $\mathcal{C}_{n+1}$ to the family $\{\mathcal{C}_n, E_{n+1}|H_{n+1}\}$. Moreover, given a family $\mathcal{F} = \{E_1|H_1, E_2|H_2, E_3|H_3\}$ of three conditional events, with $E_1, E_2, E_3, H_1, H_2, H_3$ logically independent, we determined the set Π of all coherent prevision assessments for the set of conjunctions $\{\mathcal{C}_S : \emptyset \neq S \subseteq \{1, 2, 3\}\}$. In particular, we verified that the set Π is the same in the case where $H_1 = H_2 = H_3$ and we also considered the relation between conjunction and quasi-conjunction. By using conjunction we also characterized

p-consistency and p-entailment; then, we examined several examples of p-valid inference rules. We briefly described a characterization of p-entailment, in the case of two premises, by using a suitable notion of iterated conditioning. Then, after examining some non p-valid inference rules, we illustrated by an example two methods for constructing p-valid inference rules. In particular, we applied these methods to Transitivity by obtaining p-valid versions of the rule (Weak Transitivity). Future work could concern the extension of the results of this paper to more complex cases, with possible applications to the psychology of cognitive reasoning under uncertainty. This work should lead, for instance, to further developments of the results given in [41, 42].

Appendix A

Proof of Theorem 5 We observe that (ii) follows by (i), by replacing $\mathcal{F}$ by $\overline{\mathcal{F}}$; indeed, by (i) it holds that $\overline{\mathcal{D}(\overline{\mathcal{F}})} = \mathcal{C}(\overline{\overline{\mathcal{F}}}) = \mathcal{C}(\mathcal{F})$. Then, it is enough to proof the assertion (i). We will prove the assertion by induction.

Step 1: $n = 1$, $\mathcal{F} = \{E_1|H_1\}$.
We have $\overline{\mathcal{D}(\mathcal{F})} = \overline{E_1|H_1} = 1 - E_1|H_1 = \overline{E}_1|H_1 = \mathcal{C}(\overline{\mathcal{F}})$.
Thus the assertion holds when $n = 1$.

Step 2: $n = 2$, $\mathcal{F} = \{E_1|H_1, E_2|H_2\}$.
We set

$$P(E_1|H_1) = x, \ P(E_2|H_2) = y, \ \mathbb{P}[(E_1|H_1)$$
$$\vee (E_2|H_2)] = w, \ \mathbb{P}[(\overline{E}_1|H_1) \wedge (\overline{E}_2|H_2)] = t.$$

We observe that the family $\{E_1H_1 \vee E_2H_2, \overline{E}_1H\overline{E}_2H_2, \overline{H}_1\overline{E}_2H_2, \overline{E}_1H_1\overline{H}_2, \overline{H}_1\overline{H}_2\}$ is a partition of the sure event Ω. Moreover, by Definitions 3 and 4 we have

$$\overline{\mathcal{D}(\mathcal{F})} = 1 - (E_1|H_1) \vee (E_2|H_2)$$
$$= \begin{cases} 0, & \text{if } E_1H_1 \vee E_2H_2 \text{ is true,} \\ 1, & \text{if } \overline{E}_1H_1\overline{E}_2H_2 \text{ is true,} \\ 1-x, & \text{if } \overline{H}_1\overline{E}_2H_2 \text{ is true,} \\ 1-y, & \text{if } \overline{E}_1H_1\overline{H}_2 \text{ is true,} \\ 1-w, & \text{if } \overline{H}_1\overline{H}_2 \text{ is true.} \end{cases} \quad (39)$$

and

$$\mathcal{C}(\overline{\mathcal{F}}) = (\overline{E}_1|H_1) \wedge (\overline{E}_2|H_2)$$
$$= \begin{cases} 0, & \text{if } E_1H_1 \vee E_2H_2 \text{ is true,} \\ 1, & \text{if } \overline{E}_1H\overline{E}_2H_2 \text{ is true,} \\ 1-x, & \text{if } \overline{H}_1\overline{E}_2H_2 \text{ is true,} \\ 1-y, & \text{if } \overline{E}_1H_1\overline{H}_2 \text{ is true,} \\ t, & \text{if } \overline{H}_1\overline{H}_2 \text{ is true.} \end{cases} \quad (40)$$

We observe that $\overline{\mathcal{D}(\mathcal{F})}$ and $\mathcal{C}(\overline{\mathcal{F}})$ coincide when $H_1 \vee H_2$ is true. Thus, by Theorem 3, $\mathbb{P}(\overline{\mathcal{D}(\mathcal{F})}) = \mathbb{P}(\mathcal{C}(\overline{\mathcal{F}}))$ and hence $1 - w = t$. Therefore $\overline{\mathcal{D}(\mathcal{F})}$ still coincides with $\mathcal{C}(\overline{\mathcal{F}})$ when $H_1 \vee H_2$ is false, so that $\overline{\mathcal{D}(\mathcal{F})} = \mathcal{C}(\overline{\mathcal{F}})$.

Step 3: $\mathcal{F} = \{E_1|H_1, E_2|H_2, \ldots, E_n|H_n\}$.

(*Inductive Hypothesis*) Let us assume that for any (strict) subset $S \subset \{1, \ldots, n\}$, by defining $\mathcal{F}_S = \{E_i|H_i, i \in S\}$, it holds that $\overline{\mathcal{D}(\mathcal{F}_S)} = \mathcal{C}(\overline{\mathcal{F}_S})$. Now we will prove that $\overline{\mathcal{D}(\mathcal{F}_S)} = \mathcal{C}(\overline{\mathcal{F}_S})$ when $S = \{1, \ldots, n\}$, in which case $\mathcal{F}_S = \mathcal{F}$. By Definition 7 we have

$$\overline{\mathcal{D}(\mathcal{F})} = \sum_{h=0}^{m} w_h C_h, \text{ where}$$

$$w_h = \begin{cases} 0, & \text{if } S_h' \neq \emptyset, \\ 1, & \text{if } S_h'' = \{1, 2, \ldots, n\}, \\ 1 - y_{S_h'''}, & \text{if } S_h' = \emptyset \text{ and } S_h''' \neq \emptyset. \end{cases} \tag{41}$$

We continue to use the subsets S_h', S_h'', S_h''' as defined in formula (9) also with the family $\overline{\mathcal{F}}$; moreover we set $t_S = \mathbb{P}[\bigwedge_{i \in S}(\overline{E}_i|H_i)] = \mathbb{P}[\mathcal{C}(\overline{\mathcal{F}}_S)]$. Based on Definition 5, we have

$$\mathcal{C}(\overline{\mathcal{F}}) = \sum_{h=0}^{m} z_h C_h, \text{ where}$$

$$z_h = \begin{cases} 0, & \text{if } S_h' \neq \emptyset, \\ 1, & \text{if } S_h'' = \{1, 2, \ldots, n\}, \\ t_{S_h'''}, & \text{if } S_h' = \emptyset \text{ and } S_h''' \neq \emptyset. \end{cases} \tag{42}$$

Then, $\overline{\mathcal{D}(\mathcal{F})} - \mathcal{C}(\overline{\mathcal{F}}) = \sum_{h=0}^{m}(w_h - z_h)C_h$, where

$$\begin{aligned}&w_h - z_h \\ &= \begin{cases} 0, & \text{if } S_h' \neq \emptyset, \\ 0, & \text{if } S_h'' = \{1, 2, \ldots, n\}, \\ 1 - y_{S_h'''} - t_{S_h'''}, & \text{if } S_h' = \emptyset \text{ and } S_h''' \neq \emptyset. \end{cases}\end{aligned} \tag{43}$$

By the inductive hypothesis, it holds that $1 - y_{S_h'''} = \mathbb{P}[\overline{\mathcal{D}(\mathcal{F}_{S_h'''})}] = \mathbb{P}[\mathcal{C}(\overline{\mathcal{F}}_{S_h'''})] = t_{S_h'''}$ for $h = 1, \ldots, m$, because $S_h''' \subset \{1, 2, \ldots, n\}$. Then, $\overline{\mathcal{D}(\mathcal{F})} - \mathcal{C}(\overline{\mathcal{F}}) = \sum_{h=0}^{m}(w_h - z_h)C_h$, where

$$w_h - z_h = \begin{cases} 0, & h = 1, \ldots, m, \\ 1 - y_{S_0'''} - t_{S_0'''}, & h = 0. \end{cases} \tag{44}$$

By recalling that $S_0''' = \{1, 2, \ldots, n\}$, $\overline{\mathcal{D}(\mathcal{F})}$ and $\mathcal{C}(\overline{\mathcal{F}})$ coincide when $H_1 \vee H_2 \vee \cdots \vee H_n$ is true. Thus, by Theorem 3, $\mathbb{P}[\overline{\mathcal{D}(\mathcal{F})}] = \mathbb{P}[\mathcal{C}(\overline{\mathcal{F}})]$, that is $1 - y_{S_0'''} = t_{S_0'''}$. Therefore $\overline{\mathcal{D}(\mathcal{F})}$ still coincides with $\mathcal{C}(\overline{\mathcal{F}})$ when $H_1 \vee H_2 \vee \cdots \vee H_n$ is false, so that $\overline{\mathcal{D}(\mathcal{F})} = \mathcal{C}(\overline{\mathcal{F}})$.   □

Proof of Theorem 6   (i) Assume that, for every $(\mu, \nu) \in \Pi$, the values of $X|H$ and $Y|K$ associated with the constituent C_h are such that $X|H \leq Y|K$, for each C_h contained in $H \vee K$; then for each given coherent assessment (μ, ν), by choosing $s_1 = 1$, $s_2 = -1$ in the random gain, we have

$$\begin{aligned} G &= H(X - \mu) - K(Y - \nu) = (X|H - \mu) - (Y|K - \nu) \\ &= (X|H - Y|K) + (\nu - \mu). \end{aligned}$$

Then, by the hypothesis, $\mathcal{G}_{H \vee K} \leq (\nu - \mu)$ and by coherence $0 = \mathbb{P}(\mathcal{G}_{H \vee K}) \leq \nu - \mu$. Then $\mu \leq \nu$, $\forall(\mu, \nu) \in \Pi$.

(ii) By hypothesis, it holds that $(XH + \mu H^c)(H \vee K) \leq (YK + \nu K^c)(H \vee K)$; moreover, from condition (i), $\mu \leq \nu$ for every $(\mu, \nu) \in \Pi$; then

$$\begin{aligned} X|H &= XH + \mu \overline{H} = (XH + \mu \overline{H})(H \vee K) \\ &\quad + (XH + \mu \overline{H})\overline{H}\,\overline{K} = (XH + \mu \overline{H})(H \vee K) \\ &\quad + \mu \overline{H}\,\overline{K} \leqslant (YK + \nu \overline{K})(H \vee K) + \nu \overline{H}\,\overline{K} \\ &= (YK + \nu \overline{K})(H \vee K) + (YK + \nu \overline{K})\overline{H}\,\overline{K} \\ &= YK + \nu \overline{K} = Y|K. \end{aligned}$$

Vice versa, $X|H \leq Y|K$ trivially implies $X|H \leq Y|K$ when $H \vee K$ is true.   □

Proof of Theorem 7 We distinguish three cases: (*a*) the value of $\mathcal{C}_n$ is 0, with some $E_i|H_i$ false, $i \leq n$; (*b*) the value of $\mathcal{C}_n$ is 1, with $E_i|H_i$ true, $i = 1, \ldots, n$; (*c*) the value of $\mathcal{C}_n$ is $\mathbb{P}[\bigwedge_{i \in S}(E_i|H_i)] = \mathbb{P}(\mathcal{C}_S) = x_S$, for some subset $S \subseteq \{1, 2, \ldots, n\}$.

Case (*a*).   It holds that $\mathcal{C}_{n+1} = 0 = \mathcal{C}_n$.

Case (*b*).   The value of $\mathcal{C}_{n+1}$ is 1, or 0, or x_{n+1}, according to whether $E_{n+1}|H_{n+1}$ is true, or false, or void; thus $\mathcal{C}_{n+1} \leq \mathcal{C}_n$.

Case (*c*).   We distinguish three cases: (*i*) $E_{n+1}|H_{n+1}$ is true; (*ii*) $E_{n+1}|H_{n+1}$ is false; (*iii*) $E_{n+1}|H_{n+1}$ is void. In the case (*i*) the value of $\mathcal{C}_{n+1}$ is x_S, thus $\mathcal{C}_{n+1} = \mathcal{C}_n$. In the case (*ii*) the value of $\mathcal{C}_{n+1}$ is 0, thus $\mathcal{C}_{n+1} \leq \mathcal{C}_n$. In the case (*iii*) the value of $\mathcal{C}_{n+1}$ is $x_{S \cup \{n+1\}} = \mathbb{P}[\bigwedge_{i \in S \cup \{n+1\}}(E_i|H_i)]$; then, in order to prove that $\mathcal{C}_{n+1} \leq \mathcal{C}_n$, we need to prove that $x_{S \cup \{n+1\}} \leq x_S$. We proceed by induction on the cardinality of S, denoted by s. Let be $s = 1$, with $\mathcal{C}_S = E_i|H_i$, for some $i \in \{1, \ldots, n\}$. We note that $x_S = \mathbb{P}(E_i|H_i) = x_i$, $x_{S \cup \{n+1\}} = \mathbb{P}((E_i|H_i) \wedge (E_{n+1}|H_{n+1})) = x_{\{i, n+1\}}$ and by Theorem 4 it holds that $x_{S \cup \{n+1\}} = x_{\{i, n+1\}} \leq x_i = x_S$. Now, let be $s \geq 2$ and $x_{S \cup \{n+1\}} \leq x_S$ for every $s < n$, so that, based on Definition 5, $\mathcal{C}_{n+1} \leq \mathcal{C}_n$ when S is a strict subset of $\{1, 2, \ldots, n\}$. If $S = \{1, 2, \ldots, n\}$, as $E_i|H_i$ is

void for all $i = 1, \ldots, n + 1$, it holds that $\mathcal{C}_n = \mathbb{P}(\mathcal{C}_n) = x_{\{1,\ldots,n\}}$ and $\mathcal{C}_{n+1} = x_{\{1,\ldots,n+1\}} = \mathbb{P}(\mathcal{C}_{n+1})$ and, in order to prove that $\mathcal{C}_{n+1} \leq \mathcal{C}_n$, it remains to prove that $\mathbb{P}(\mathcal{C}_{n+1}) \leq \mathbb{P}(\mathcal{C}_n)$. By applying Theorem 6, with $X|H = \mathcal{C}_{n+1} = Z_{n+1}|(H_1 \vee \cdots \vee H_{n+1})$ and $Y|K = \mathcal{C}_n = Z_n|(H_1 \vee \cdots \vee H_n)$, as $\mathcal{C}_{n+1} \leq \mathcal{C}_n$ when $H_1 \vee \cdots \vee H_{n+1}$ is true (i.e., $s < n$), it follows that $\mathbb{P}(\mathcal{C}_{n+1}) \leq \mathbb{P}(\mathcal{C}_n)$; therefore $\mathcal{C}_{n+1} \leq \mathcal{C}_n$. $\qquad\square$

Proof of Theorem 9.

Case (i). We proceed by induction. The property is satisfied for $n = 1$; indeed, if $\mathcal{C}_1 = E_1|H_1 \in \{1, 0, x_1\}$, where $x_1 = \mathbb{P}(E_1|H_1) \in [0, 1]$, then $\mathcal{C}_1 \in [0, 1]$. Let us assume that the property holds for $k < n$, that is $\mathcal{C}_k \in [0, 1]$, for every $k < n$. Based on Definition 5 we distinguish three cases: (a) the value of $\mathcal{C}_n$ is 0; (b) the value of $\mathcal{C}_n$ is 1; (c) the value of $\mathcal{C}_n$ is $\mathbb{P}[\bigwedge_{i \in S}(E_i|H_i)] = x_S$, for some subset $S \subseteq \{1, 2, \ldots, n\}$. In the cases (a) and (b), $\mathcal{C}_n \in [0, 1]$. In the case (c), if $S = \{i_1, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$, then $\mathcal{C}_n \in [0, 1]$, because $x_S = \mathbb{P}(\bigwedge_{j=1}^{k}(E_{i_j}|H_{i_j}))$ is a possible value of $\mathcal{C}_k = \bigwedge_{j=1}^{k}(E_{i_j}|H_{i_j})$, with $k < n$. Finally, if $S = \{1, 2, \ldots, n\}$ (that is the conditioning events $H_1, \ldots, H_n$ are all false), then $\mathcal{C}_n = \mathbb{P}(\mathcal{C}_n)$ and $\mathbb{P}(\mathcal{C}_n) \in [0, 1]$ because the values of $\mathcal{C}_n$ restricted to $H_1 \vee \cdots \vee H_n$ all belong to $[0, 1]$. Therefore $\mathcal{C}_n \in [0, 1]$. By a similar reasoning, based on Definition 7 we can prove that $\mathcal{D}_n \in [0, 1]$. $\qquad\square$

Proof of Theorem 10 Let $C_0, \ldots, C_m$, with $m = 3^n - 1$ be the constituents associated with $\mathcal{F}_{n+1} = \{E_1|H_1, \ldots, E_{n+1}|H_{n+1}\}$, where $C_0 = \overline{H}_1 \cdots \overline{H}_{n+1}$. With each C_h, $h = 1, \ldots, m$, we associate the point $Q_h = (q_{h1}, q_{h2}, q_{h3})$, which represents the value of the random vector $(\mathcal{C}_n, E_{n+1}|H_{n+1}, \mathcal{C}_{n+1})$ when C_h is true, where q_{h1} is the value of $\mathcal{C}_n$, q_{h2} is the value of $E_{n+1}|H_{n+1}$, and q_{h3} is the value of $\mathcal{C}_{n+1}$. With C_0 it is associated the point $Q_0 = (\mu_n, x_{n+1}, \mu_{n+1}) = \mathcal{M}$. We observe that the set of points $\{Q_h, h = 1, \ldots, m\}$ contains in particular the points

$$Q_1 = (1, 1, 1), \quad Q_2 = (1, 0, 0), \quad Q_3 = (0, 1, 0), \quad Q_4 = (0, 0, 0),$$

which are respectively associated with the following constituents or logical disjunction of constituents

$$E_1 H_1 \cdots E_n H_n E_{n+1} H_{n+1}, \quad E_1 H_1 \cdots E_n H_n \overline{E}_{n+1} H_{n+1},$$
$$(\overline{E}_1 H_1 \vee \cdots \vee \overline{E}_n H_n) \wedge E_{n+1} H_{n+1},$$
$$(\overline{E}_1 H_1 \vee \cdots \vee \overline{E}_n H_n) \wedge \overline{E}_{n+1} H_{n+1}.$$

Based on Remark 7, we need to prove that the set of coherent assessments Π on $\{\mathcal{C}_n, E_{n+1}|H_{n+1}, \mathcal{C}_{n+1}\}$ coincides with the convex hull $\mathcal{I}$ of Q_1, Q_2, Q_3, Q_4. We recall that coherence of $(\mu_n, x_{n+1}, \mu_{n+1})$ implies coherence of all the sub-assessments on the associated subfamilies of $\{\mathcal{C}_n, E_{n+1}|H_{n+1}, \mathcal{C}_{n+1}\}$. The coherence of the single

assessments μ_n on $\mathcal{C}_n$, or x_{n+1} on $E_{n+1}|H_{n+1}$, or μ_{n+1} on $\mathcal{C}_{n+1}$, simply amounts to conditions

$$\mu_n \in [0, 1], \ x_{n+1} \in [0, 1], \ \mu_{n+1} \in [0, 1],$$

respectively. Then, by the hypothesis of logical independence, the sub-assessment (μ_n, x_{n+1}) is coherent, for every $(\mu_n, x_{n+1}) \in [0, 1]^2$. By Remark 6, the coherence of the sub-assessments (μ_n, μ_{n+1}) and (x_{n+1}, μ_{n+1}) amounts to the conditions $0 \leq \mu_{n+1} \leq \mu_n \leq 1$ and $0 \leq \mu_{n+1} \leq x_{n+1} \leq 1$. Finally, assuming that the above conditions are satisfied, to prove coherence of $(\mu_n, x_{n+1}, \mu_{n+1})$, by Theorem 1, it is enough to show that the point $(\mu_n, x_{n+1}, \mu_{n+1})$ belongs to the convex hull of the points $Q_1, \ldots, Q_m$. Moreover, in order $\mathcal{M}$ belongs to the convex hull of $Q_1, \ldots, Q_m$ the following system (Σ) must solvable

$$\mathcal{M} = \sum_{h=1}^{m} \lambda_h Q_h, \ \sum_{h=1}^{m} \lambda_h = 1, \ \lambda_h \geq 0, \ \forall h. \tag{45}$$

We show that the convex hull of the points $Q_1, \ldots, Q_m$ coincides with the convex hull $\mathcal{I}$ of the points Q_1, Q_2, Q_3, Q_4, described in Remark 7, because all the other points $Q_5, \ldots, Q_m$, are linear convex combinations of Q_1, Q_2, Q_3, Q_4, that is $Q_h \in \mathcal{I}$ for each $h = 5, \ldots, m$.

We examine the following different cases which depend on the logical value of $E_{n+1}|H_{n+1}$: a) $E_{n+1}|H_{n+1}$ is true; b) $E_{n+1}|H_{n+1}$ is false; c) $E_{n+1}|H_{n+1}$ is void.

a) In this case

$$Q_h = (q_{h1}, 1, q_{h1}) = q_{h1}(1, 1, 1) + (1 - q_{h1})(0, 1, 0)$$
$$= q_{h1}Q_1 + (1 - q_{h1})Q_3.$$

b) In this case

$$Q_h = (q_{h1}, 0, 0) = q_{h1}(1, 0, 0) + (1 - q_{h1})(0, 0, 0)$$
$$= q_{h1}Q_2 + (1 - q_{h1})Q_4.$$

c) In this case $Q_h = (q_{h1}, x_{n+1}, q_{h3})$ and we distinguish the following subcases: (i) $\bigwedge_{i=1}^{n} E_i H_i$ true, so that $Q_h = (1, x_{n+1}, x_{n+1})$; (ii) $\bigvee_{i=1}^{n} \overline{E}_i H_i$ true, so that $Q_h = (0, x_{n+1}, 0)$; (iii) $E_i|H_i$ void, for every $i \in S$ and $E_i|H_i$ true for every $i \in \{1, 2, \ldots, n\} \setminus S$, for some $\emptyset \neq S \subset \{1, 2, \ldots, n\}$, so that $Q_h = (x_S, x_{n+1}, x_{S \cup \{n+1\}})$. In subcase (i) it holds that

$$Q_h = (1, x_{n+1}, x_{n+1}) = x_{n+1}(1, 1, 1) + (1 - x_{n+1})(1, 0, 0)$$
$$= x_{n+1}Q_1 + (1 - x_{n+1})Q_2.$$

In subcase (ii) it holds that

$$Q_h = (0, x_{n+1}, 0) = x_{n+1}(0, 1, 0) + (1 - x_{n+1})(0, 0, 0)$$
$$= x_{n+1}Q_3 + (1 - x_{n+1})Q_4.$$

In subcase (iii), it can be verified by a finite iterative procedure that the point $Q_h = (x_S, x_{n+1}, x_{S \cup \{n+1\}}) \in$

$\mathcal{I}$. We examine the different cases on the cardinality s of S. We recall that $\bigwedge_{i \in S}(E_i|H_i)$ is denoted by $\mathcal{C}_S$.

Step 1. $s = 1$. Without loss of generality we assume $S = \{1\}$, so that $Q_h = (x_S, x_{n+1}, x_{S \cup \{n+1\}}) = (x_1, x_{n+1}, x_{\{1,n+1\}})$, where $x_1 = P(E_1|H_1)$, $x_{\{1,n+1\}} = \mathbb{P}[(E_1|H_1) \wedge (E_{n+1}|H_{n+1})]$. By Theorem 4 it holds that $\max\{x_S + x_{n+1} - 1, 0\} \leq x_{S \cup \{n+1\}} \leq \min\{x_S, x_{n+1}\}$, with $(x_S, x_{n+1}) \in [0, 1]^2$. In other words, $Q_h = (x_S, x_{n+1}, x_{S \cup \{n+1\}}) \in \mathcal{I}$. The reasoning is the same for $S = \{i\}$, $i = 2, \ldots, n$.

Step 2. $s = 2$. Without loss of generality we assume $S = \{1, 2\}$, so that $x_S = \mathbb{P}[(E_1|H_1) \wedge (E_2|H_2)]$, $x_{S \cup \{n+1\}} = \mathbb{P}[\mathcal{C}_{S \cup \{n+1\}}] = \mathbb{P}[(E_1|H_1) \wedge (E_2|H_2) \wedge (E_{n+1}|H_{n+1})]$. We denote by $C_0^*, C_1^*, \ldots, C_{m^*}^*$, the constituents associated with $\{E_i|H_i, i \in S \cup \{n+1\}\}$, where $C_0^* = \bigwedge_{i \in S \cup \{n+1\}} \overline{H}_i$. Moreover, with C_h^*, $h = 0, 1, \ldots, m^*$, we associate the point $Q_h^* = (q_{h1}^*, q_{h2}^*, q_{h3}^*)$ which represents the value of the random vector $\{\mathcal{C}_S, E_{n+1}|H_{n+1}, \mathcal{C}_{S \cup \{n+1\}}\}$ when C_h^* is true. We observe that $Q_0^* = (x_S, x_{n+1}, x_{S \cup \{n+1\}})$ and that Q_1, Q_2, Q_3, Q_4 still belongs to the set of points $\{Q_h^*, h = 1, \ldots, m^*\}$. In order that the assessment $(x_S, x_{n+1}, x_{S \cup \{n+1\}})$ on $\{\mathcal{C}_S, E_{n+1}|H_{n+1}, \mathcal{C}_{S \cup \{n+1\}}\}$ be coherent, the point $Q_0^* = (x_S, x_{n+1}, x_{S \cup \{n+1\}})$ must belong to the convex hull of points $Q_1^*, Q_2^*, \ldots, Q_m^*$. We show that for each point $Q_h^* \neq Q_i$, $i = 1, 2, 3, 4$, it holds that $Q_h^* \in \mathcal{I}$. By repeating the previous reasoning we only need to analyze the subcase (iii) of case $c)$. We have to show that, for every nonempty subset $S' \subset S$, the point $Q_h^* = (x_{S'}, x_{n+1}, x_{S' \cup \{n+1\}})$ belongs to the convex hull $\mathcal{I}$ of $Q_1, \ldots, Q_4$. As $S = \{1, 2\}$, it holds that $S' = \{1\}$, or $S' = \{2\}$, so that $Q_h^* = (x_{S'}, x_{n+1}, x_{S' \cup \{n+1\}}) = (x_1, x_{n+1}, x_{\{1,n+1\}})$, or $Q_h^* = (x_2, x_{n+1}, x_{\{2,n+1\}})$. By *Step 1*, in both cases $Q_h^* \in \mathcal{I}$. Thus $Q_h = (x_S, x_{n+1}, x_{S \cup \{n+1\}}) \in \mathcal{I}$. In other words, $\max\{x_S + x_{n+1} - 1, 0\} \leq x_{S \cup \{n+1\}} \leq \min\{x_S, x_{n+1}\}$, with $(x_S, x_{n+1}) \in [0, 1]^2$. The reasoning is the same for every $S = \{i, j\} \subset \{1, 2, \ldots, n\}$.

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Step k + 1. $s = k + 1$, $2 < k + 1 < n$. By induction, assume that $(x_{S'}, x_{n+1}, x_{S' \cup \{n+1\}}) \in \mathcal{I}$ for every $S' = \{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$. Then, by the previous reasoning, it follows that $Q_h = (x_S, x_{n+1}, x_{S \cup \{n+1\}}) \in \mathcal{I}$ for every $S = \{i_1, i_2, \ldots, i_{k+1}\}$. In other words, $\max\{x_S + x_{n+1} - 1, 0\} \leq x_{S \cup \{n+1\}} \leq \min\{x_S, x_{n+1}\}$, with $(x_S, x_{n+1}) \in [0, 1]^2$, for every $S = \{i_1, i_2, \ldots, i_{k+1}\}$.

Thus, by this iterative procedure, also in the subcase (iii) of case $c)$ it holds that $Q_h \in \mathcal{I}$. Then, $Q_h \in \mathcal{I}$, $h = 5, \ldots, m$. Finally, the condition (45) is equivalent to $\mathcal{M} \in \mathcal{I}$, so that the assessment $\mathcal{M}$ is coherent if and only if

$$(\mu_n, x_{n+1}) \in [0, 1]^2, \quad \max\{\mu_n + x_{n+1} - 1, 0\} \leq \mu_{n+1}$$
$$\leq \min\{\mu_n, x_{n+1}\}. \qquad \square$$

Proof of Theorem 15 The computation of the set Π is based on Section 2.2. The constituents C_h's and the points Q_h's associated with $(\mathcal{F}, \mathcal{M})$ are illustrated in Table 1.

We recall that $Q_h = (q_{h1}, \ldots, q_{h7})$ represents the value associated with C_h of the random vector $(E_1|H_1, E_2|H_2, E_3|H_3, (E_1|H_1) \wedge (E_2|H_2), (E_1|H_1) \wedge (E_3|H_3), (E_2|H_2) \wedge (E_3|H_3), (E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3))$, $h = 1, \ldots, 26$. With $C_0 = \overline{H}_1 \overline{H}_2 \overline{H}_3$ it is associated $Q_0 = \mathcal{M}$. Denoting by $\mathcal{I}$ the convex hull generated by $Q_1, Q_2, \ldots, Q_{26}$, the coherence of the prevision assessment $\mathcal{M}$ on $\mathcal{F}$ requires that the condition $\mathcal{M} \in \mathcal{I}$ be satisfied; this amounts to the solvability of the following system

$$(\Sigma) \qquad \mathcal{M} = \sum_{h=1}^{26} \lambda_h Q_h, \quad \sum_{h=1}^{26} \lambda_h = 1,$$
$$\lambda_h \geq 0, h = 1, \ldots, 26.$$

We observe that

$$Q_3 = x_3 Q_1 + (1 - x_3)Q_2, \quad Q_6 = x_3 Q_4 + (1 - x_3)Q_5,$$
$$Q_7 = x_2 Q_1 + (1 - x_2)Q_4, \quad Q_8 = x_2 Q_2 + (1 - x_2)Q_5,$$
$$Q_9 = x_{23} Q_1 + (x_2 - x_{23})Q_2 + (x_3 - x_{23})Q_4$$
$$+ (x_{23} - x_2 - x_3 + 1)Q_5,$$
$$Q_{12} = x_3 Q_{10} + (1 - x_3)Q_{11},$$
$$Q_{15} = x_3 Q_{13} + (1 - x_3)Q_{14},$$
$$Q_{16} = x_2 Q_{10} + (1 - x_2)Q_{13},$$
$$Q_{17} = x_2 Q_{11} + (1 - x_2)Q_{14},$$
$$Q_{18} = x_{23} Q_{10} + (x_2 - x_{23})Q_{11} + (x_3 - x_{23})Q_{13}$$
$$+ (x_{23} - x_2 - x_3 + 1)Q_{14},$$
$$Q_{19} = x_1 Q_1 + (1 - x_1)Q_{10}, \quad Q_{20} = x_1 Q_2 + (1 - x_1)Q_{11},$$
$$Q_{21} = x_{13} Q_1 + (x_1 - x_{13})Q_2 + (x_3 - x_{13})Q_{10}$$
$$+ (x_{13} - x_1 - x_3 + 1)Q_{11},$$
$$Q_{22} = x_1 Q_4 + (1 - x_1)Q_{13}, \quad Q_{23} = x_1 Q_5 + (1 - x_1)Q_{14},$$
$$Q_{24} = x_{13} Q_4 + (x_1 - x_{13})Q_5 + (x_3 - x_{13})Q_{13}$$
$$+ (x_{13} - x_1 - x_3 + 1)Q_{14},$$
$$Q_{25} = x_{12} Q_1 + (x_1 - x_{12})Q_4 + (x_2 - x_{12})Q_{10}$$
$$+ (x_{12} - x_1 - x_2 + 1)Q_{13},$$

$$Q_{26} = x_{12}Q_2 + (x_1 - x_{12})Q_5 + (x_2 - x_{12})Q_{11}$$
$$+ (x_{12} - x_1 - x_2 + 1)Q_{14}.$$

Thus, $\mathcal{I}$ coincides with the convex hull of the points $Q_1, Q_2,$ $Q_4, Q_5, Q_{10}, Q_{11}, Q_{13}, Q_{14}$. For the sake of simplicity, we set: $Q_1' = Q_1, Q_2' = Q_2, Q_3' = Q_4, Q_4' = Q_5, Q_5' = Q_{10},$ $Q_6' = Q_{11}, Q_7' = Q_{13}, Q_8' = Q_{14}$. Then, the condition $\mathcal{M} \in \mathcal{I}$ amounts to the solvability of the following system

$$(\Sigma') \qquad \mathcal{M} = \sum_{h=1}^{8} \lambda_h' Q_h', \quad \sum_{h=1}^{8} \lambda_h' = 1,$$
$$\lambda_h' \geq 0, \ h = 1, \dots, 8,$$

that is

$$(\Sigma') \begin{cases} \lambda_1' + \lambda_2' + \lambda_3' + \lambda_4' = x_1, \ \lambda_1' + \lambda_2' + \lambda_5' + \lambda_6' = x_2, \\ \lambda_1' + \lambda_3' + \lambda_5' + \lambda_7' = x_3, \ \lambda_1' + \lambda_2' = x_{12}, \\ \lambda_1' + \lambda_3' = x_{13}, \ \lambda_1' + \lambda_5' = x_{23}, \ \lambda_1' = x_{123}, \\ \sum_{h=1}^{8} \lambda_h' = 1, \ \lambda_h' \geq 0, \ h = 1, 2, \dots, 8. \end{cases}$$

System (Σ') can be written as

$$(\Sigma') \begin{cases} \lambda_1' = x_{123}, \ \lambda_2' = x_{12} - x_{123}, \\ \lambda_3' = x_{13} - x_{123}, \ \lambda_4' = x_1 - x_{12} - x_{13} + x_{123}, \\ \lambda_5' = x_{23} - x_{123}, \ \lambda_6' = x_2 - x_{12} - x_{23} + x_{123}, \\ \lambda_7' = x_3 - x_{13} - x_{23} + x_{123}, \\ \lambda_8' = 1 - x_1 - x_2 - x_3 + x_{12} + x_{13} + x_{23} - x_{123}, \\ \lambda_h' \geq 0, \ h = 1, 2, \dots, 8. \end{cases}$$

As it can be verified, by non-negativity of $\lambda_1', \dots, \lambda_8'$ it follows that (Σ') is solvable (with a unique solution) if and only if

$$\begin{cases} x_{123} \geq \max\{0, x_{12} + x_{13} - x_1, x_{12} + x_{23} - x_2, x_{13} + x_{23} - x_3\}, \\ x_{123} \leq \min\{x_{12}, x_{13}, x_{23}, 1 - x_1 - x_2 - x_3 + x_{12} + x_{13} + x_{23}\}, \end{cases} \tag{46}$$

or, in a more explicit way, if and only if the following conditions are satisfied

$$\begin{cases} (x_1, x_2, x_3) \in [0, 1]^3, \\ \max\{x_1 + x_2 - 1, x_{13} + x_{23} - x_3, 0\} \leq x_{12} \leq \min\{x_1, x_2\}, \\ \max\{x_1 + x_3 - 1, x_{12} + x_{23} - x_2, 0\} \leq x_{13} \leq \min\{x_1, x_3\}, \\ \max\{x_2 + x_3 - 1, x_{12} + x_{13} - x_1, 0\} \leq x_{23} \leq \min\{x_2, x_3\}, \\ 1 - x_1 - x_2 - x_3 + x_{12} + x_{13} + x_{23} \geq 0, \\ x_{123} \geq \max\{0, x_{12} + x_{13} - x_1, x_{12} + x_{23} - x_2, x_{13} + x_{23} - x_3\}, \\ x_{123} \leq \min\{x_{12}, x_{13}, x_{23}, 1 - x_1 - x_2 - x_3 + x_{12} + x_{13} + x_{23}\}. \end{cases} \tag{47}$$

Notice that the conditions in (47) coincide with that ones in (18). Moreover, assuming (Σ') solvable, with the solution $(\lambda_1', \dots, \lambda_8')$, we associate the vector $(\lambda_1, \lambda_2, \dots, \lambda_{26})$, with $\lambda_1 = \lambda_1', \ \lambda_2 = \lambda_2', \ \lambda_4 = \lambda_3', \ \lambda_5 = \lambda_4', \ \lambda_{10} =$

$\lambda_5', \ \lambda_{11} = \lambda_6', \ \lambda_{13} = \lambda_7', \ \lambda_{14} = \lambda_8', \ \lambda_h = 0, h \notin \{1, 2, 4, 5, 10, 11, 13, 14\}$, which is a solution of (Σ). Moreover, defining $\mathcal{J} = \{1, 2, 4, 5, 10, 11, 13, 14\}$, it holds that $\bigvee_{h \in \mathcal{J}} C_h = H_1 \wedge H_2 \wedge H_3$. Therefore, $\sum_{h \in \mathcal{J}} \lambda_h = \sum_{h: C_h \subseteq H_1 H_2 H_3} \lambda_h = 1$ and hence $\sum_{h: C_h \subseteq H_i} \lambda_h = 1, \ i = 1, 2, 3, \ \sum_{h: C_h \subseteq H_i \vee H_j} \lambda_h = 1, \ i \neq j,$ $\sum_{h: C_h \subseteq H_1 \vee H_2 \vee H_3} \lambda_h = 1$; thus, by (2), $I_0 = \emptyset$. Then, by Theorem 2, the solvability of (Σ) is also sufficient for the coherence of $\mathcal{M}$. Finally, Π is the set of conditional prevision assessments $(x_1, x_2, x_3, x_{12}, x_{13,23}, x_{123})$ which satisfy the conditions in (18). $\qquad \square$

Proof of Theorem 16 Notice that, $(E_i | H) \wedge (E_j | H) = (E_i E_j) | H$, for every $\{i, j\} \subset \{1, 2, 3\}$, and $(E_1 | H) \wedge (E_2 | H) \wedge (E_3 | H) = (E_1 E_2 E_3) | H$. Then $\mathcal{F} = \{E_1 | H, E_2 | H, E_3 | H, (E_1 E_2) | H, (E_1 E_3) | H, (E_2 E_3) | H, (E_1 E_2 E_3) | H\}$. The computation of the set Π is based on Section 2.2. The constituents C_h's and the points Q_h's associated with $(\mathcal{F}, \mathcal{M})$ are illustrated in Table 2.

We recall that $Q_h = (q_{h1}, \dots, q_{h7})$ represents the value associated with C_h of the random vector $(E_1 | H, E_2 | H, E_3 | H, (E_1 E_2) | H, (E_1 E_3) | H, (E_2 E_3) | H, (E_1 E_2 E_3) | H), h = 1, \dots, 8$. With $C_0 = \overline{H}$ it is associated $Q_0 = \mathcal{M}$. Denoting by $\mathcal{I}$ the convex hull generated by $Q_1, Q_2, \dots, Q_8$, as all the conditioning events coincide with H the assessment $\mathcal{M}$ on $\mathcal{F}$ is coherent if and only if $\mathcal{M} \in \mathcal{I}$; that is, if and only if the following system is solvable

$$\mathcal{M} = \sum_{h=1}^{8} \lambda_h Q_h, \quad \sum_{h=1}^{8} \lambda_h = 1, \quad \lambda_h \geq 0, \ h = 1, \dots, 8. \tag{48}$$

The points $Q_1, Q_2, \dots, Q_8$ coincide with the points $Q_1', Q_2', \dots, Q_8'$ in the proof of Theorem 15, respectively. Then, system (48) coincides with system $(\Sigma)'$ in the proof of Theorem 15. Therefore, it is solvable if and only if the conditions in (18) are satisfied. In other words, the set Π of all coherent assessments $\mathcal{M}$ on $\mathcal{F}$ coincides with the set of points $(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})$ which satisfy the conditions in (18). $\qquad \square$

Proof of of Theorem 18 In order to prove the theorem it is enough to prove the following implications: a) $(i) \Rightarrow (ii)$; b) $(ii) \Rightarrow (iii)$; c) $(iii) \Rightarrow (i)$.

a) $(i) \Rightarrow (ii)$. We recall that $\mathcal{F}$ p-entails $E_{n+1} | H_{n+1}$ if and only if either $H_{n+1} \subseteq E_{n+1}$, or there exists a nonempty $\mathcal{F}_\Gamma \subseteq \mathcal{F}$, where $\Gamma \subseteq \{1, \dots, n\}$, such that $QC(\mathcal{F}_\Gamma)$ implies $E_{n+1} | H_{n+1}$ (see, e.g. [24, Theorem 6]). Let us first consider the case where $H_{n+1} \subseteq E_{n+1}$. In this case $P(E_{n+1} | H_{n+1}) = 1$ and $E_{n+1} | H_{n+1} = H_{n+1} + \overline{H}_{n+1} = 1$. We have $\mathcal{C}_{n+1} = \mathcal{C}_n \wedge (E_{n+1} | H_{n+1})$, with $E_{n+1} | H_{n+1} = 1$. We distinguish two cases:

Table 1 Constituents C_h's and corresponding points Q_h's associated with $(\mathcal{F}, \mathcal{M})$, where $\mathcal{M} = (x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})$ is a prevision assessment on $\mathcal{F} = \{E_1|H_1, E_2|H_2, E_3|H_3, (E_1|H_1) \wedge (E_2|H_2), (E_1|H_1) \wedge (E_3|H_3), (E_2|H_2) \wedge (E_3|H_3), (E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)\}$

	C_h	Q_h							
C_1	$E_1 H_1 E_2 H_2 E_3 H_3$	1	1	1	1	1	1	1	Q_1
C_2	$E_1 H_1 E_2 H_2 \overline{E}_3 H_3$	1	1	0	1	0	0	0	Q_2
C_3	$E_1 H_1 E_2 H_2 \overline{H}_3$	1	1	x_3	1	x_3	x_3	x_3	Q_3
C_4	$E_1 H_1 \overline{E}_2 H_2 E_3 H_3$	1	0	1	0	1	0	0	Q_4
C_5	$E_1 H_1 \overline{E}_2 H_2 \overline{E}_3 H_3$	1	0	0	0	0	0	0	Q_5
C_6	$E_1 H_1 \overline{E}_2 H_2 \overline{H}_3$	1	0	x_3	0	x_3	0	0	Q_6
C_7	$E_1 H_1 \overline{H}_2 E_3 H_3$	1	x_2	1	x_2	1	x_2	x_2	Q_7
C_8	$E_1 H_1 \overline{H}_2 \overline{E}_3 H_3$	1	x_2	0	x_2	0	0	0	Q_8
C_9	$E_1 H_1 \overline{H}_2 \overline{H}_3$	1	x_2	x_3	x_2	x_3	x_{23}	x_{23}	Q_9
C_{10}	$\overline{E}_1 H_1 E_2 H_2 E_3 H_3$	0	1	1	0	0	1	0	Q_{10}
C_{11}	$\overline{E}_1 H_1 E_2 H_2 \overline{E}_3 H_3$	0	1	0	0	0	0	0	Q_{11}
C_{12}	$\overline{E}_1 H_1 E_2 H_2 \overline{H}_3$	0	1	x_3	0	0	x_3	0	Q_{12}
C_{13}	$\overline{E}_1 H_1 \overline{E}_2 H_2 E_3 H_3$	0	0	1	0	0	0	0	Q_{13}
C_{14}	$\overline{E}_1 H_1 \overline{E}_2 H_2 \overline{E}_3 H_3$	0	0	0	0	0	0	0	Q_{14}
C_{15}	$\overline{E}_1 H_1 \overline{E}_2 H_2 \overline{H}_3$	0	0	x_3	0	0	0	0	Q_{15}
C_{16}	$\overline{E}_1 H_1 \overline{H}_2 E_3 H_3$	0	x_2	1	0	0	x_2	0	Q_{16}
C_{17}	$\overline{E}_1 H_1 \overline{H}_2 \overline{E}_3 H_3$	0	x_2	0	0	0	0	0	Q_{17}
C_{18}	$\overline{E}_1 H_1 \overline{H}_2 \overline{H}_3$	0	x_2	x_3	0	0	x_{23}	0	Q_{18}
C_{19}	$\overline{H}_1 E_2 H_2 E_3 H_3$	x_1	1	1	x_1	x_1	1	x_1	Q_{19}
C_{20}	$\overline{H}_1 E_2 H_2 \overline{E}_3 H_3$	x_1	1	0	x_1	0	0	0	Q_{20}
C_{21}	$\overline{H}_1 E_2 H_2 \overline{H}_3$	x_1	1	x_3	x_1	x_{13}	x_3	x_{13}	Q_{21}
C_{22}	$\overline{H}_1 \overline{E}_2 H_2 E_3 H_3$	x_1	0	1	0	x_1	0	0	Q_{22}
C_{23}	$\overline{H}_1 \overline{E}_2 H_2 \overline{E}_3 H_3$	x_1	0	0	0	0	0	0	Q_{23}
C_{24}	$\overline{H}_1 \overline{E}_2 H_2 \overline{H}_3$	x_1	0	x_3	0	x_{13}	0	0	Q_{24}
C_{25}	$\overline{H}_1 \overline{H}_2 E_3 H_3$	x_1	x_2	1	x_{12}	x_1	x_2	x_{12}	Q_{25}
C_{26}	$\overline{H}_1 \overline{H}_2 \overline{E}_3 H_3$	x_1	x_2	0	x_{12}	0	0	0	Q_{26}
C_0	$\overline{H}_1 \overline{H}_2 \overline{H}_3$	x_1	x_2	x_3	x_{12}	x_{13}	x_{23}	x_{123}	Q_0

(α) H_{n+1} is true; (β) H_{n+1} is false. In case (α), by Definition 5 and Remark 4, as $E_{n+1}|H_{n+1}$ is true it follows that the values of $\mathcal{C}_{n+1}$ and of $\mathcal{C}_n$ coincide. In case (β), let $C_0, \ldots, C_m$ be the constituents associated with $\mathcal{F}$, where $C_0 = \overline{H}_1 \cdots \overline{H}_n$. Then, the constituents $C_0', \ldots, C_m'$ associated with $\mathcal{F} \cup \{E_{n+1}|H_{n+1}\}$ and

Table 2 Constituents C_h's and corresponding points Q_h's associated with $(\mathcal{F}, \mathcal{M})$, where $\mathcal{M} = (x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})$ is a prevision assessment on $\mathcal{F} = \{E_1|H, E_2|H, E_3|H, (E_1 E_2)|H, (E_1 E_3)|H, (E_2 E_3)|H, (E_1 E_2 E_3)|H\}$

	C_h	Q_h							
C_1	$E_1 E_2 E_3 H$	1	1	1	1	1	1	1	Q_1
C_2	$E_1 E_2 \overline{E}_3 H$	1	1	0	1	0	0	0	Q_2
C_3	$E_1 \overline{E}_2 E_3 H$	1	0	1	0	1	0	0	Q_3
C_4	$E_1 \overline{E}_2 \overline{E}_3 H$	1	0	0	0	0	0	0	Q_4
C_5	$\overline{E}_1 E_2 E_3 H$	0	1	1	0	0	1	0	Q_5
C_6	$\overline{E}_1 E_2 \overline{E}_3 H$	0	1	0	0	0	0	0	Q_6
C_7	$\overline{E}_1 \overline{E}_2 E_3 H$	0	0	1	0	0	0	0	Q_7
C_8	$\overline{E}_1 \overline{E}_2 \overline{E}_3 H$	0	0	0	0	0	0	0	Q_8
C_0	$\overline{H}$	x_1	x_2	x_3	x_{12}	x_{13}	x_{23}	x_{123}	Q_0

contained in $\overline{H}_{n+1}$ are $C'_0 = C_0\overline{H}_{n+1}, \ldots, C'_m = C_m\overline{H}_{n+1}$. For each constituent C'_h, $h = 1, \ldots, m$, by formula (10) the corresponding value of $\mathcal{C}_n$ is $z_h \in \{1, 0, x_{S'''_h}\}$. We denote by z'_h the value of $\mathcal{C}_{n+1}$ associated with z_h and we recall that $C'_h \subseteq \overline{H}_{n+1}$, $h = 0, 1, \ldots, m$. For each index h, if $z_h = 1$, then $z'_h = 1$; if $z_h = 0$, then $z'_h = 0$; if $z_h = x_{S'''_h}$, then $z'_h = x_{S'''_h \cup \{n+1\}}$. We set $P(E_{n+1}|H_{n+1}) = x_{n+1}$; in our case $x_{n+1} = 1$. Moreover, by Theorem 10

$$\max\left\{x_{S'''_h} + x_{n+1} - 1, 0\right\} \leq x_{S'''_h \cup \{n+1\}} \leq \min\left\{x_{S'''_h}, x_{n+1}\right\};$$

therefore $x_{S'''_h \cup \{n+1\}} = x_{S'''_h}$. Then, the values of $\mathcal{C}_{n+1}$ and of $\mathcal{C}_n$ coincide for every C'_h. Thus, $\mathcal{C}_{n+1} = \mathcal{C}_n$ when $H_{n+1} \subseteq E_{n+1}$.

We consider now the case where there exists $\mathcal{F}_\Gamma \subseteq \mathcal{F}$, $\mathcal{F}_\Gamma \neq \emptyset$, such that $QC(\mathcal{F}_\Gamma) \subseteq E_{n+1}|H_{n+1}$. First of all we prove that $\mathcal{C}(\mathcal{F}_\Gamma \cup \{E_{n+1}|H_{n+1}\}) = \mathcal{C}(\mathcal{F}_\Gamma)$. For the sake of simplicity, we set $\mathcal{C}(\mathcal{F}_\Gamma) = \mathcal{C}_\Gamma$ and $\mathcal{C}(\mathcal{F}_\Gamma \cup \{E_{n+1}|H_{n+1}\}) = \mathcal{C}_{\Gamma \cup \{n+1\}}$.

If the value of $\mathcal{C}_\Gamma$ is 1 (because all the conditional events in $\mathcal{F}_\Gamma$ are true), then $QC(\mathcal{F}_\Gamma)$ is true and hence $E_{n+1}|H_{n+1}$ is also true; thus $\mathcal{C}_{\Gamma \cup \{n+1\}} = 1$, so that $\mathcal{C}_{\Gamma \cup \{n+1\}} = \mathcal{C}_\Gamma$.

If the value of $\mathcal{C}_\Gamma$ is 0 (because some conditional event in $\mathcal{F}_\Gamma$ is false), then $\mathcal{C}_{\Gamma \cup \{n+1\}}$ is 0 too, so that $\mathcal{C}_{\Gamma \cup \{n+1\}} = \mathcal{C}_\Gamma$.

If $\mathcal{C}_\Gamma$ is x_S for some nonempty subset $\mathcal{S} \subset \Gamma$ (that is, all the conditional events in $\mathcal{F}_S$ are void and the other ones in $\mathcal{F}_{\Gamma \setminus S}$ are true), then $QC(\mathcal{F}_\Gamma)$ is true and and hence $E_{n+1}|H_{n+1}$ is also true; thus $\mathcal{C}_{\Gamma \cup \{n+1\}} = x_S$, so that $\mathcal{C}_{\Gamma \cup \{n+1\}} = \mathcal{C}_\Gamma$.

If $\mathcal{C}_\Gamma$ is x_Γ because all the conditional events in $\mathcal{F}_\Gamma$ are void, then $QC(\mathcal{F}_\Gamma)$ is void and for $E_{n+1}|H_{n+1}$ there are two cases: 1) $E_{n+1}|H_{n+1}$ true; 2) $E_{n+1}|H_{n+1}$ void. In case 1), by also recalling Remark 4, it holds that $\mathcal{C}_{\Gamma \cup \{n+1\}} = x_\Gamma$ so that $\mathcal{C}_{\Gamma \cup \{n+1\}} = \mathcal{C}_\Gamma$.

In case 2) it holds that $\mathcal{C}_{\Gamma \cup \{n+1\}} = x_{\Gamma \cup \{n+1\}}$, where $x_{\Gamma \cup \{n+1\}} = \mathbb{P}(\mathcal{C}_{\Gamma \cup \{n+1\}})$. Now, we observe that the random quantities $\mathcal{C}_\Gamma$ and $\mathcal{C}_{\Gamma \cup \{n+1\}}$ coincide conditionally on $\bigvee_{i \in \Gamma \cup \{n+1\}} H_i$ being true; then by Theorem 3 it holds that $\mathbb{P}(\mathcal{C}_\Gamma) = \mathbb{P}(\mathcal{C}_{\Gamma \cup \{n+1\}})$, that is $x_\Gamma = x_{\Gamma \cup \{n+1\}}$; thus $\mathcal{C}_{\Gamma \cup \{n+1\}} = \mathcal{C}_\Gamma$.

Finally, denoting by Γ^c the set $\{1, \ldots, n\} \setminus \Gamma$, by the associative property of conjunction we obtain

$$\mathcal{C}_{n+1} = \mathcal{C}_n \wedge E_{n+1}|H_{n+1} = \mathcal{C}_{\Gamma^c} \wedge \mathcal{C}_\Gamma \wedge E_{n+1}|H_{n+1}$$
$$= \mathcal{C}_{\Gamma^c} \wedge \mathcal{C}_\Gamma = \mathcal{C}_n.$$

b) $(ii) \Rightarrow (iii)$. By monotonicity property of conjunction it holds that $\mathcal{C}_{n+1} \leq E_{n+1}|H_{n+1}$. Then, by assuming $\mathcal{C}_n = \mathcal{C}_{n+1}$, it follows $\mathcal{C}_n \leq E_{n+1}|H_{n+1}$.

c) $(iii) \Rightarrow (i)$. Let us assume that $\mathcal{C}_n \leq E_{n+1}|H_{n+1}$, so that $\mathbb{P}(\mathcal{C}_n) \leq P(E_{n+1}|H_{n+1})$. Moreover, by assuming

that $P(E_i|H_i) = 1$, $i = 1, \ldots, n$, from (15) it follows $\mathbb{P}(\mathcal{C}_n) = 1$ and hence $P(E_{n+1}|H_{n+1}) = 1$, that is $\mathcal{F}$ p-entails $E_{n+1}|H_{n+1}$. $\qquad\square$

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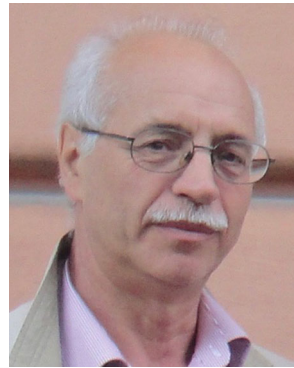
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