Croatian Journal of Philosophy
Vol. XX, No. 58, 2020

The Effectiveness of Representations in Mathematics

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This article focuses on particular ways in which visual representations contribute to the development of mathematical knowledge. I give examples of diagrammatic representations that enable one to observe new properties and cases where representations contribute to classification. I propose that fruitful representations in mathematics are iconic representations that involve conventional or symbolic elements, that is, iconic metaphors. In the last part of the article, I explain what these are and how they apply in the considered examples.

Keywords: Visual representations, discovery, iconic metaphors, manipulation, Peirce.

1. *Introduction*

Many scholars have commented on the advantages for mathematics of choosing appropriate notations. Euler, for example, expressed that Leibniz's notation for the differential was superior to Newton's:

> It might be uncivil to argue with the English about the use of words and a definition, and we might easily be defeated in a judgment about the purity of Latin and the adequacy of expression, but there is no doubt that we have won the prize from the English when it is a question of notation. For example, the tenth differential, or fluxion, is very inconveniently represented with ten dots, while our notation, $d^{10}y$, is very easily understood. (Euler 2000: 64)

Other mathematicians comment on the potential "fruitfulness" of a good choice of notation:

> It only becomes possible at all after the mathematical notation has, as a result of genuine thought, been so developed that it does the thinking for us, so to speak. (Frege 1953: xvi)

Another concern is the choice of representations in mathematics. A recent such interest is the role of visual representations, or diagrams. The aim here is to show particular ways in which visual representations contribute to the development of mathematical knowledge. One focus will be to illustrate how these representations enable you to *see* or observe certain patterns which leads to the formulation of new hypotheses. A puzzle that I will address—but only partially solve—concerns the question of *how* and *why* certain representations contribute to the development of mathematics. One part of the answer (see Carter 2019) is that it is often fruitful to have available *iconic* representations that are possible to *manipulate*. Taking as a starting point Peirce's characterisation of an icon, I will first propose that icons used in mathematics are best understood as iconic metaphors and explain what this means. In this context, I will note that iconic representations that can be manipulated play a key role in Peirce's characterisation of mathematical reasoning. Second, I will indicate that we still lack an account of how to find a useful representation or notation in mathematics.

The use of visual representations and notations has contributed to the development of mathematics in various ways. Sometimes the choice of a particular notation enables one to *see* that there is a problem of a certain type. As an example, I could mention Descartes' convention of writing a^2 instead of $a \cdot a$, a^3 instead of $a \cdot a \cdot a$ and so on. This made him able to write, for the first time, a quadratic equation (almost) as we do today, for example as $ax^2 + bx = c$. This convention made it possible to formulate a general n-degree equation and formulate the Fundamental Theorem of Algebra. As is noted by Manders this invention also suggested to Descartes why the classical problems of duplicating a cube and trisecting an angle by ruler and compass were impossible to solve:

> First, its degree, algebraically the key feature. Descartes guesses that the degree determines by what means solutions may be constructed, e.g., because angle trisection problem gives an irreducible third-degree equation, it cannot be done by ruler and compass. But there is no direct way to predict the degree of its equation from the appearance of a geometrical figure. (Manders 1989: 558).

Descartes was able to translate, for example, the problem of duplicating a cube into the cubic equation $z^3 = 2b^3$. Given a cube with side b and volume b^3, z corresponds to the side of the cube that has two times this volume. Having found that roots of quadratic equations *could* be constructed by ruler and compass, Descartes formed a hypothesis that this could not be the case for irreducible cubic equations. He also formed what he thought was a proof of this. But it turned out not to be correct. See (Lützen 2010) for details. Descartes did not yet have the required algebraic tools, for example, field extensions and formulated a geometric proof.[1]

[1] Lützen (2010) remarks that it is not strange that Descartes formulated a geometric proof: There was a long tradition of giving geometrical proofs at the time, combined with the fact that algebra was still in its infancy and so not considered as trustworthy.

Another example concerns how a particular choice of representation of a problem contributes to *classification:* a particular representation may help one to formulate—and solve—all problems of a particular type in a systematic way. The Arabic mathematician Al-Khwarizmi (c. 780–850) formulated quadratic equations in terms of the "three types of numbers" roots (the unknown), squares and numbers.[2] One of these types of equations is 'Square and roots is equal to a number'. Perhaps these expressions and their geometrical representations, when demonstrating their solution, helped him to formulate all types of quadratic equations. In any case, one usually attributes to the Arabic mathematicians the first systematic solution of quadratic equations. Other examples of representations contributing to a classification of a type of objects can be found in (Eckes and Giardino 2018).

In the next section we shall see that these two roles of representations also occur in contemporary mathematics. That is, one finds examples of representations that enable one to *see* certain properties and cases where representations contribute to *classification.* In both examples the representations consist of diagrams.

2. *Visual representations in contemporary mathematics*

Representations in free probability theory—seeing

It is possible to find examples from contemporary mathematics where a specific form of representation has contributed to the formulation of new hypotheses. One such example is presented in Carter (2010). This example illustrates how the visual appearance of a particular representation may lead to the formulation of a new concept. The example has to do with permutations on the set $\{1,2,3,\dots,p\}$ which appear in a certain combinatorial expression in free probability theory. By representing these permutations in a certain way, certain properties of them became visible. Similar representations further contribute to make visible that these properties have an effect on the value of the expression.

The expression and its value is $\mathbb{E} \circ Tr_n\left[B_1^* B_{\pi(1)} \cdot\dots\cdot B_p^* B_{\pi(p)}\right] = m^{e(\hat{\pi})} \cdot n^{o(\hat{\pi})}$. The B_i's in the expression stand for $m \times n$ matrices and their entries are Gaussian random variables. After taking the trace of the multiplied matrices, it therefore makes sense to take the expectation, '$\mathbb{E}$'. The indices contain 'π' which denotes a permutation on the set $\{1,2,3,\dots,p\}$. A permutation is a 1–1 and onto function on a set to itself. The numbers, $o(\hat{\pi})$ and $e(\hat{\pi})$, in the above formula refer to the number of odd and even numbers, respectively, of certain equivalence classes on the set $\{1,2,3,\dots,2p\}$. The total number of equivalence classes turns out to depend on properties of the permutation. I will show the representation of permutations that revealed this property. Representing a permuta-

[2] Al-Khwarizmi formulates and solves six different problems, for example, the problem 'square and roots identical to number' and 'square and number identical to roots', see (Berggren 1986).

tion by certain diagrams gives rise to the concept of a ‘non-crossing permutation’. See (Carter 2010) or (Haagerup and Thorbjørnsen 1999) for further details about the case.

Below are two examples of representing a permutation on the set $\{1, 2, 3, 4, 5, 6\}$. In the diagram on the left in figure 1, you may observe that the lines do not cross, whereas they do in the right-hand diagram. This gives rise to the notion of a non-crossing and a crossing permutation.

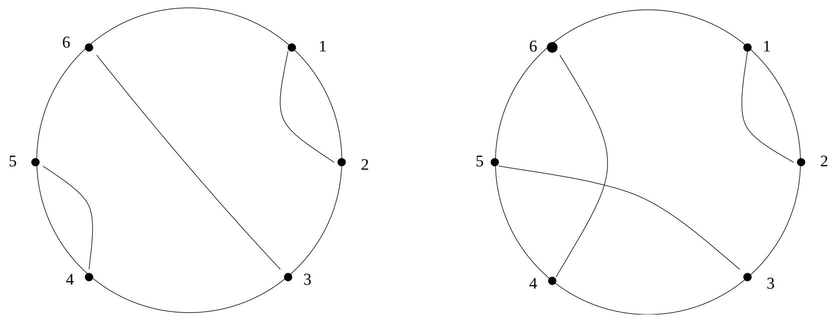

Figure 1. *The left diagram is a representation of a non-crossing permutation. In two-cycles, the permutation can be written as (12)(36)(45). The diagram on the right shows a crossing permutation. The represented permutation in this case is (12)(35)(46).*

It turns out that the above-mentioned result depends on whether lines cross or not, that is, whether the permutation is crossing or not. To see this, the mathematicians visualised, or represented, equivalence classes of an equivalence relation formed on the set $\{1,2,3,\dots,2p\}$. (First the permutation is rewritten, taking into account that there are $2p$ matrices in the expression. The new permutation is denoted $\hat{\pi}$.) The relation is $i \sim \hat{\pi}(i) + 1$ (mod 2p). Representations of such equivalence classes can be seen below in figure 2.

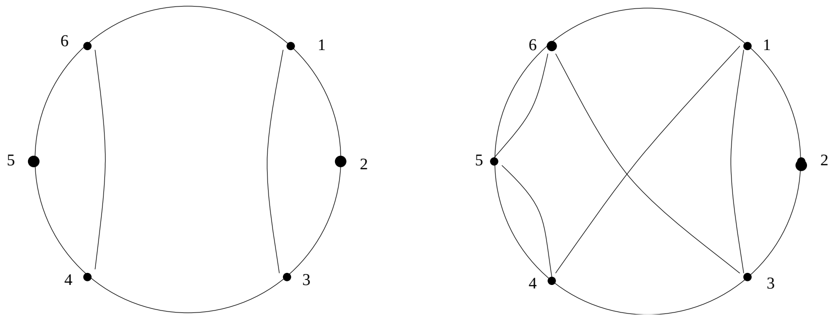

Figure 2. *Numbers that are in the same equivalence class are joined by lines. I have identified the equivalence classes of the two permutations shown in figure 1. In the left figure one sees that the number 1 is related to* $\hat{\pi}(1) + 1 =$ *3.* $\hat{\pi}(1)$ *is seen to be 2 in the left-hand diagram in figure 1. Similarly,* $\hat{\pi}(3) + 1 = 6$ *+1 = 1 (mod 6), so {1,3} form one equivalence class. It can be seen that 2 is related to itself, so there is only one number in this equivalence class (marked by a filled circle). It is seen that there are 4 equivalence classes in the left-hand diagram, whereas there are only 2 in the right-hand diagram. Recall that this corresponds to the crossing permutation.*

By drawing a number of such diagrams, varying the permutation, it is possible to detect a pattern. If $p = 3$ and so $\hat{\pi}: \{1, 2, \ldots 6\} \to \{1, 2, \ldots, 6\}$ one will see that whenever the permutation is non-crossing, there are 4 equivalence classes. If the lines cross, there will be fewer. In general, the mathematicians were able to formulate the hypothesis, that the total number of equivalence classes depends on whether the permutation is crossing or non-crossing: If it is non-crossing, the number of equivalence classes is *p+1*. If it is crossing, this number will be strictly less.

In the published papers presenting this result, there are no diagrams. In order to formulate these propositions and proofs of them, the property of being a crossing permutation therefore had to be reformulated. The formal definition of a crossing permutation is as follows: A permutation $\pi: \{1, 2, \ldots, p\} \to \{1, 2, \ldots p\}$ has a crossing, if for some $a<b<c<d$ in $\{1,2, \ldots, p\}$ it is the case that $\pi(a) = c$ and $\pi(b) = d$. If it has no crossings, it is said to be a non-crossing permutation.

One point is that there is a difference in how we *perceive* these definitions. In the diagrams the properties are *shown*. One can actually perceive the lines crossing. In the formal mathematical language, we cannot see this directly. The definitions of these properties are only *described*. (See Carter 2019 for an elaboration of this point.) Note also that this example illustrates Manders' point; that a different representation may reveal new properties or explanations. Whereas Manders discusses an algebraic representation of geometrical figures, the example presented here conversely considers a representation of a formal expression.

Representations in analysis—classification

In analysis, one field studied concerns C^*-algebras and their classification. That is, having defined C^*-algebras, one wishes to figure out the different types of such objects there are up to isomorphism. A tool to do that is to define so-called invariants. The mathematician George Elliott has formulated a program where the hope is that K-theory could provide such a tool: That two C^*-algebras are isomorphic if and only if their corresponding K-groups are pairwise isomorphic. This turned out only to be true in simple cases. The study of their K-groups, however, is still an important field of study. For C^*-algebras it is possible to define two such groups, denoted K_0 and K_1. It is generally quite complicated to calculate these groups from their original definitions. Recently a much easier way to calculate them has been found. The trick is first to represent the algebras in a different way, as directed graphs. From this representation, it is possible to find a different way to access these groups. I give a few details of these concepts here before coming to the main (philosophical) points: That certain diagrammatic representations are used as tools for classifying C^*-algebras and that these diagrams can be manipulated.

A directed graph is defined by a four-tuple, $E = (E^0, E^1, r, s)$. Here E^0 consists of the vertices of the graph and E^1 consists of the edges. That E is a *directed* graph means that edges have a direction, which is expressed by a *range* and a *source* function. For each edge, these functions say where it ends and starts: $r, s\colon E^1 \to E^0$. An example of a (finite) graph is given in figure 3. This graph has three vertices, named v_1, v_2 and v_3, and three edges, e_1, e_2 and e_3. The arrows indicate their source and range. The source of the first two is v_1, the source of e_3 is the vertex v_2. The ranges are given as follows: $r(e_1) = v_1, r(e_2) = v_2$ and $r(e_3) = v_3$.

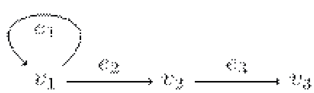

Figure 3. *A directed graph, E, with three vertices.*

A directed graph gives rise to certain generators and relations that the generators must fulfil, which then generate a C^*-algebra. The C^*-algebra generated by the graph, E, is denoted $C^*(E)$. For details of how such algebras are constructed, see Szymanski (2002). Read in a different way, a graph gives rise to a linear map, $\Delta\colon \mathbb{Z}V \to \mathbb{Z}E^0$, where V is the set of vertices that emit edges. It turns out that the two K-groups can easily be calculated from this map. First, the linear map is defined on vertices, v, that emit edges as

$$\Delta_E(v) = (\sum_{s(e)=v} r(e)) - v\cdot$$

The two groups K_0 and K_1 can be calculated as the cokernel and kernel of this map:

$$K_0(C^*(E)) \cong coker\ (\Delta_E)$$

and

$$K_1(C^*(E)) \cong ker\ (\Delta_E).$$

In the case of quadratic equations, I suggested that the geometric representation of them contributed to the formulation of, and solution to, all types of such equations. In other words: a classification of quadratic equations. The directed graphs can be used as *tools* for classification. But they are not themselves objects of such a classification in the sense that two different graphs correspond to two different types of C^*-algebras. To a particular directed graph corresponds a linear map from which the proposed invariants, K_0 and K_1 can be obtained. Furthermore, two different graphs will give rise to different linear maps. But unfortunately, the information obtained from the K-groups is not always sufficient to tell whether the corresponding C^*-algebras are isomorphic or not. The graphs are epistemic tools in the sense that they have made calculations of the K-groups easier (Carter 2018).

Another point is that the directed graphs can be manipulated. In order to illustrate this point, we consider a result from (Szymanski 2002). It is proven that a large class of C^* −algebras can be generated by directed graphs—and so that their *K*-groups can easily be calculated. This result has been found by manipulating directed graphs. The result states that, given two specific groups, K_0 and K_1, it is possible to construct a directed graph, *E,* such that the C^* −algebra it generates has these two as its K_0 and K_1-groups, that is, $K_i(C^*(E)) \cong K_i$ for i=0 and 1. The proof—and the way this result was found—starts by considering a particular graph that gets the result partially. That is, the first graph has the right K_0-group but the other group is zero. After that a number of subgraphs are added, so one gradually gets closer to the sought for graph. One adds vertices and edges and along the way calculates how these changes alter the K-groups. Manipulating graphs, i.e., adding and removing edges and vertices, therefore contributed to the result in question.

Manipulating iconic representations

We now address the observed similarities of the two case studies. In both cases certain objects are represented by diagrams. In the first case, the objects represented are permutations and, in the second, C^*-algebras. In the first case the visual representation contributed with a new concept (that of a crossing permutation). The second example is slightly different—the representation has made progress possible because calculations of K-groups turned out to be much easier. In both cases particular instances of concepts, that is particular examples of permutations and C^*-algebras, can be represented by diagrams. One reason that these representations contribute to new knowledge, is the fact that they can be manipulated. In this way they become tools for experimentation. By, for example, producing a number of examples of permutations and their equivalence classes one is able to detect a general pattern: that this number depends on the visual appearance of the lines in the diagram.

Another key feature of a fruitful representation is that it shares relevant “structure” with the problem, it represents. In C.S. Peirce’s semiotics such representations are referred to as *icons.* An icon is the particular type of sign that is able to represent its object because it is like this object in some respect. This also entails that an iconic sign should hold the capacity to reveal more information about the object it represents, than is required to identify it as a representation of that object. Stjernfelt (2007) refers to this feature as the ‘operational account’ of similarity, and so of an icon. Simple examples of iconic representations consist of images and, in mathematics, of geometrical figures. These representations visually resemble what they represent. According to Peirce, icons play a key role in mathematics in general. But mathematical icons are rarely simply pictures of what they stand for. This means that the likeness must consist of something else besides visual resemblance. When Peirce characterises icons, he sometimes refers to them as having conventional (i.e. symbolic) features or that they have a purpose:

> For example, a geometrical figure drawn on paper may be an icon of a triangle or other geometrical form. If one meets a man whose language one does not know and resorts to imitative sounds and gestures, these approach the character of an icon. The reason they are not pure icons is that the purpose of them is emphasized. A pure icon is independent of any purpose. It serves as a sign solely and simply by exhibiting the quality it serves to signify. (Peirce 1998: 309)

Note that, according to Peirce, not even a drawn geometrical figure is a pure icon. I therefore propose that icons used in mathematics contain conventional, or symbolic elements—and so cannot be pure icons. They are what he refers to as *iconic metaphors* (*Collected Papers* 2.277). A related point is that, according to Peirce, a sign must be interpreted as a sign in order to function as such. To identify in which respect a sign stands for another mathematical object is therefore part of the role of the interpretant of a sign. The conventional element of an iconic sign or, in other words, the information given so that one may identify how a given sign stands for another object, I will refer to as formulating *the underlying convention or rule for interpretation.* I propose that it is a combination of (what follows from) the underlying conventions and properties of the representation that contributes to the successful use of iconic representations in mathematics.

To give a simple example of an iconic metaphor, I return to the second example mentioned in the introduction. The particular example concerns the geometric representations of quadratic equations. One of the problems formulated by Al-Khwarizmi was ‘A square and 10 of its roots is 39’. Using contemporary notation, we can also write: $x^2 + 10x = 39$. When forming a geometric representation of this problem we could formulate the following conventions: (1) both sides of the equality sign denote (the area of) geometrical figures, (2) ‘x’ and 10 refer to (the length of) line segments, (3) addition means that the geometrical figures are joined, (4) multiplication of two line segments gives a rectangle (or a square). These lead to a representation of the equation as shown in figure 4. This geometric figure can be manipulated to determine the line segment, x. I speculate that it is easier to obtain the solution of the equation by these manipulations than manipulating the corresponding expression or equation. It appears at least to be the way that the solution was originally found: Al Khwarizmi is said to have been inspired by Babylonian mathematicians. According to (Høyrup 2002) they solved such equations geometrically. The steps are shown in figure 5. One first cuts off half of the rectangle and moves it below the figure as shown in figure 5. In the next step, the “square is completed”: one adds a square with area $5 \cdot 5 = 25$, so that the area is now 64. The side of the square is then 8 and the sought for line segment is 8–5=3.

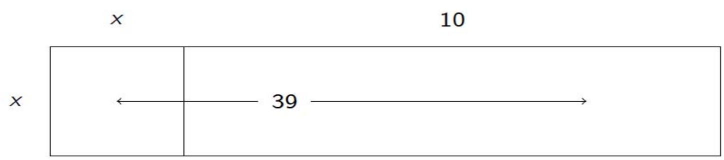

Figure 4. *A geometric representation of* $x^2 + 10x = 39$.

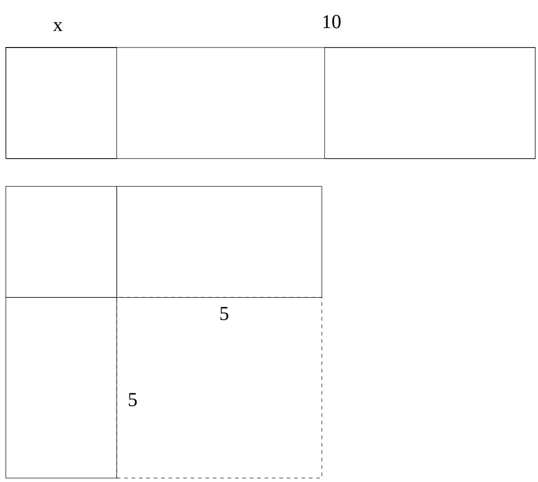

Figure 5. *Illustrating the geometric solution of* $x^2 + 10x = 39$.

Once the solution has been found geometrically, it is possible to formulate the manipulations of the figures in Figure 5 in the original language: The top figure expresses that $x^2 + 10x = 39$ is the same as $x^2 + 5x + 5x = 39$. The stippled lines in the bottom figure state that: $x^2 + 5x + 5x + 5 \cdot 5 = 39 + 5 \cdot 5$. The figure further shows that this is a square with side $5 + x$, that is, $(5 + x)^2 = 64$. Finally, one takes the square root and subtracts 5 to obtain $x = 3$.

Given this terminology, we can say that a C^*-algebra is an iconic metaphor of a directed graph. There are specific rules that define how to read a particular graph. Similarly, other definitions say how to read the graph in a different way and so obtain the linear map. This means that the linear map is also a metaphorical representation of the directed graph. Intricate mathematical arguments are needed in order to determine the relation between this map and the K-groups referred to above.

It is much easier to comprehend how diagrams represent permutations as was shown in the first case study. The employed convention is simply to place numbers on a circle and to draw a line between the numbers i and $\pi(i)$ of a given permutation, π. By using this convention, one may consider these diagrams as iconic representations of permutations. After manipulating such diagrams, the discovered property of being a crossing permutation can be reformulated in the original vocabulary of the permutation as a mapping.

The examples shown illustrate that the manipulation of iconic representations is a fruitful practice in mathematics. This brings me to the final point of this paper: that both these features play a central role in Peirce's characterisation of mathematical reasoning. In 'On the algebra of logic. A contribution to the philosophy of notation' Peirce writes the following about reasoning, mentioning the role of icons and our ability to manipulate them:

> The truth, however, appears to be that all deductive reasoning, even simple syllogism, involves an element of observation; namely, deduction consists in constructing an icon or diagram the relations of whose parts shall present a complete analogy with those of the parts of the object of reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts. ... As for algebra, the very idea of the art is that it presents formulae which can be manipulated, and that by observing the effects of such manipulation we find properties not to be otherwise discerned. (Peirce in *Collected Papers* 3.363)

In this paper I have emphasised the role of visual representations, or diagrams. But it is clear from the above quote, that also other types of representations, that is, general mathematical expressions, are examples of iconic representations that can be manipulated—and so contribute to the development of mathematics.

3. *Conclusion*

I have shown various examples illustrating the effectiveness of visual representations in contemporary mathematics. In the first example a particular diagrammatic representation revealed new properties of a permutation. In the second example, a diagrammatic representation has contributed with tools that potentially make classification of C^*-algebras simpler.

I have also noted that fruitful representations in mathematics are iconic metaphors that can be manipulated. Furthermore, such representations need not be visual or diagrammatic. Finally, I should say that what has been formulated here is only a proposal of what kinds of representations are effective. The question of how they can be found remains.

References

Berggren, J. L. 1986. *Episodes in the mathematics of medieval Islam*. New York: Springer.

Carter, J. 2010. "Diagrams and Proofs in Analysis." *International Studies in the Philosophy of Science* 24: 1–14.

Carter, J. 2018. "Graph-algebras—faithful representations and mediating objects in mathematics." *Endeavour* 42: 180–188.

Carter, J. 2019. "Exploring the fruitfulness of diagrams in mathematics." *Synthese* 196: 4011–4032.

Eckes, C. and Giardino, V. 2018. "The Classificatory Function of Diagrams: Two Examples from Mathematics." *In Diagrammatic Representation and Inference—0th International Conference, Diagrams 2018*. New York: Springer: 120–136.

Euler, L. 2000. *Foundations of Differential Calculus*. Translated by John D. Blanton. New York: Springer-Verlag.

Frege, G. 1953. *The Foundations of Arithmetic*. Translated by J.L. Austin. Harper Torchbooks. New York: Harper and Brothers.

Haagerup, U. and Thorbjørnsen, S. 1999. "Random matrices and K-theory for exact C^*-algebras." *Documenta Mathematica* 4: 341–450.

Høyrup, J. 2002. *Lengths, Widths, Surfaces: An Examination of Old Babylonian Algebra and Its Kin*. New York: Springer.

Kumjian, A. et al. 1997. "Graphs, groupoids, and Cuntz-Krieger algebras." *Journal of Functional Analysis* 144: 505–541.

Lützen, J. 2010. "The algebra of geometric impossibility: Descartes and Montucla on the impossibility of the duplication of the cube and the trisection of the angle." *Centaurus* 52 (1): 4–37.

Manders, K. 1989. "Domain extensions and the philosophy of mathematics." *The Journal of Philosophy* 86 (10): 553–562.

Peirce, C. S. (1931–1960). *Collected Papers of Charles Sanders Peirce*. Vol I–IV. Edited by Charles Hartshorne and Paul Weiss. Cambridge: The Belknap Press of Harvard University Press.

Peirce, C. S. 1998. *The Essential Peirce. Selected Philosophical Writings*. Volume 2 (1893–1913). Edited by the Peirce Edition Project. Bloomington: Indiana University Press.

Stjernfelt, F. 2007. *Diagrammatology. An Investigation on the Borderlines of Phenomenology, Ontology and Semiotics.* Synthese Library (336). Dordrecht: Springer.

Szymanski, W. 2002. "The range of *K*-invariants for *C**-algebras of infinite graphs." *Indiana University Mathematics Journal* 51 (1): 239–249.