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CHARACTERIZATIONS OF INEQUALITY ORDERINGS BY MEANS OF DISPERSIVE ORDERINGS

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The generalized Lorenz order and the absolute Lorenz order are used in economics to compare income distributions in terms of social welfare. In Section 2, we show that these orders are equivalent to two stochastic orders, the concave order and the dilation order, which are used to compare the dispersion of probability distributions. In Section 3, a sufficient condition for the absolute Lorenz order, which is often easy to verify in practice, is presented. This condition is applied in Section 4 to the ordering of generalized gamma distributions with different parameters.

Keywords: Generalized Lorenz order, absolute Lorenz order, concave order, dilation order

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1. INTRODUCTION

The basic concepts of inequality and dispersion arise in many and diverse fields, so it is difficult to give brief definitions that will command universal acceptance. Loosely speaking, inequality in income distributions is seen as a particular aspect of variability when the variables considered are non-negative and represent quantities that can be transferred from one unit to another. In economics, inequality is usually used in connection with concepts such as injustice or social welfare. Champernowne and Cowell (1998) provide a convenient reference on this topic.

Several authors have approached the problem of ranking income distributions by seeking a dominance relationship between concentration curves. In this context, the generalized Lorenz curve and the absolute Lorenz curve have been used to compare two income distributions, in terms of social welfare and inequality. One of the purposes of this paper is to show that the partial orderings of income distributions induced by such curves are equivalent to two other stochastic orders used to compare probability distributions in terms of dispersion: the concave order and the dilation order, respectively (the definition of these orders is given below). These results are stated in Section 2, and some aspects of economic inequality and the usual concept of dispersion are connected.

The definition of the generalized Lorenz curve $GL_X(p)$ corresponding to the non-negative random variable X, which represents the income of a society or community, with distribution function $F(x)$ is (Shorrocks, 1983):

$$GL_X(p) = \int_0^p F_X^{-1}(t)\,dt, \;\; p \in [0,1],$$

where F_X^{-1} denotes the inverse of F_X :

$$F_X^{-1}(a) = \inf\{x : F_X(x) \geq a\},\; a \in [0,1].$$

The generalized Lorenz curve can be used to define a partial ordering (denoted $\leq_{gl}$) on the class of non-negative random variables as follows:

$$X \leq_{gl} Y \;\text{ if and only if }\; GL_X(p) \geq GL_Y(p) \;\text{ for all } p \in [0,1].$$

Then, we say that X exhibits more welfare in the generalized Lorenz sense than Y. Generalized Lorenz ordering reflects a desire for both greater equality and higher incomes. Some recent results on this ordering can be found in Ramos *et al.* (2000).

The welfare judgements embodied in the generalized Lorenz ordering are not universally accepted (see, e.g., Kolm, 1976). If one assumes an alternative concept of «efficiency preference», which corresponds to a preference for higher incomes, while keeping the same absolute differences between incomes, then absolute Lorenz ordering obtains (Moyes, 1987). For a non-negative random variable X with finite mean μ_X, let $X-\mu_X$ be the mean-centred distribution obtained from X, and denote $F_{X-\mu_X}$ as its distribution function. The absolute Lorenz curve corresponding to X (Moyes, 1987) is defined as:

$$LA_X(p) = \int_0^p F^{-1}_{X-\mu_X}(t)\,dt, \quad p \in [0,1]. \tag{1}$$

$LA_X(p)$ represents the average income short-fall of the $100p\%$ poorest individuals, i.e., the average income that would be necessary in order to provide to anyone of them the society's mean income. The absolute Lorenz curve induces a partial ordering (denoted $\leq_{la}$) on the class of non-negative random variables, as follows:

$$X \leq_{la} Y \text{ if and only if } LA_X(p) \geq LA_Y(p) \text{ for all } p \in [0,1].$$

If $X \leq_{la} Y$, then X is said to exhibit less inequality in the absolute Lorenz sense than Y.

Although, for any finite population, there is no problem evaluating the absolute and generalized Lorenz curves, for a continuous distribution an analytic expression for these curves is rarely available. Ramos *et al.* (2000) gave a sufficient condition for the generalized Lorenz order that does not involve the explicit form of the generalized Lorenz curve. In Section 3, we complete the study they began by obtaining sufficient conditions for the absolute Lorenz ordering of random variables. These do not require a direct comparison of the involved absolute Lorenz curves. These results are applied in Section 4 to ordering of generalized gamma distributions with different parameters.

In the literature there are many partial orderings of probability distributions (e.g. Lewis and Thompson, 1981; Stoyan, 1983; Hickey, 1986). Some of them are defined by requiring

$$E[\Phi(X)] \leq E[\Phi(Y)] \tag{2}$$

to hold for all functions Φ in some class of functions. The concave order (Stoyan, 1983) is defined by saying that X is smaller than Y in the sense of the concave order (denoted by $X \leq_{cv} Y$) if (2) holds for all non-decreasing and concave functions Φ for which these expectations exist. The concave order has both properties of ordering by size and/or variability: if $X \leq_{cv} Y$ then X is both smaller and/or more variable than Y in some stochastic sense (see Chapter 3 in Shaked and Shanthikumar, 1994). It is known (see Stoyan, 1983) that

$$X \leq_{cv} Y \iff \int_{-\infty}^{x} F_X(t)\,dt \geq \int_{-\infty}^{x} F_Y(t)\,dt \text{ for all } x \in \mathbb{R} \tag{3}$$

provided the integrals exist.

Now let X and Y be random variables with finite means μ_X and μ_Y, respectively. Following Hickey (1986), we say that the random variable X is smaller than Y, in the dilation sense, (denoted by $X \leq_{dil} Y$) if

$$E\left[\Phi\left(X-\mu_X\right)\right] \leq E\left[\Phi\left(Y-\mu_Y\right)\right] \tag{4}$$

for all convex functions Φ, provided these expectations exist. Clearly, dilation generalizes the use of variance to compare distributions in terms of dispersion.

For non-negative random variables, we show in Section 2 that X exhibits less welfare (*inequality*) than Y in the generalized (*absolute*) Lorenz sense if and only if X is smaller than Y in the concave (*dilation*) sense.

Several authors (Shorrocks, 1983; Lambert, 1993; Yitzhaki, 1999) have studied connections between generalized Lorenz order and increasing concave functions. Actually, in these papers, the authors restricted themselves to discrete distributions (Shorrocks, 1983) or absolutely continuous distributions having finite support $[a,b]$, $0 \leq a < b < \infty$ (Lambert, 1993; Yitzhaki, 1999). Nevertheless, from the result of Section 2 it follows that these restrictions are not needed. Our measure-theoretic approach permit us to handle, in one framework, both discrete and continuous distibutions, as well as combinations thereof.

The characterizations are obtained as an easy consequence of the theory of submajorization, as applied to decreasing rearrangements of functions. This notion has been discussed by several authors (see, e.g., Hardy *et al.*, 1929; Ryff, 1963; Chong, 1974). We first recall some definitions and results. Denote by $M(\Omega,\mu)$ the set of all extended real-valued measurable functions on a measure space (Ω,Λ,μ). For each $f \in M(\Omega,\mu)$ consider the function $D_f : \overline{\mathbb{R}} \longrightarrow [0,\mu(\Omega)]$ defined by

$$D_f(t) = \mu\left(\{x : f(x) > t\}\right),\ t \in \overline{\mathbb{R}}, \tag{5}$$

with $\overline{\mathbb{R}}$ denoting the extended real line. The decreasing rearrangement of f is defined by

$$\delta_f(t) = \inf\left\{s \in \mathbb{R} : D_f(s) \leq t\right\},\ t \in [0,\mu(\Omega)].$$

Let $f,g \in M(\Omega,\mu) \cup M(\Omega',\mu')$, where $\mu(\Omega) = \mu'(\Omega') = a < \infty$ and denote by m the Lebesgue measure on $\mathbb{R}$. Then, we write $f \ll g$ whenever

$$\int_0^t \delta_f dm \leq \int_0^t \delta_g dm \text{ for all } t \in [0,a]$$

and $f \prec g$ whenever $f \ll g$ and

$$\int_0^a \delta_f dm = \int_0^a \delta_g dm.$$

If a is infinite, then the order relations $\prec$ and $\ll$ are defined for non-negative integrable functions $f,g \in L^1(\Omega,\mu) \cup L^1(\Omega',\mu')$ analogously.

We have the following results from Chong (1974).

Theorem 1. *$f \ll g$ if and only if $\int_u^{+\infty} D_f dm \le \int_u^{+\infty} D_g dm$ for all $u \in \mathbb{R}$.*

Theorem 2. *If $f \in L^1(\Omega,\mu)$, $g \in L^1(\Omega',\mu')$ where $\mu(\Omega)$ and $\mu'(\Omega')$ are finite and equal, then $f \prec g$ if and only if*

$$\int_{\Omega} \Phi(f)d\mu \le \int_{\Omega'} \Phi(g)d\mu'$$

for all convex functions $\Phi : \mathbb{R} \longmapsto \mathbb{R}$.

2. THE CHARACTERIZATIONS

Theorem 3. *Let X and Y be non-negative random variables with finite means μ_X and μ_Y, respectively. Then*

(i) $X \ge_{gl} Y$ if and only if $X \le_{cv} Y$

(ii) $X \le_{la} Y$ if and only if $X \le_{dil} Y$.

Proof

(i) Let F_X and F_Y be the distribution functions of X and Y, respectively, and let $(\boldsymbol{\Omega}, \mathcal{B}, m)$ be the measure space defined by $\Omega = \mathbb{R}^+$, $\mathcal{B}$ the Borel algebra of Ω and m the Lebesgue measure. Define $a(t) = 1 - F_X(t)$ and $b(t) = 1 - F_Y(t)$ for all $t \in \mathbb{R}^+$. Then $a(t)$ and $b(t)$ are non-increasing integrable functions and are equal to their respective decreasing rearrangements. It is easy to see that

$$D_a(t) = m\left\{x \in \mathbb{R}^+ : a(x) > t\right\} = F_X^{-1}(1-t), \text{ for all } t \in [0,1]$$

and, analogously,

$$D_b(t) = F_Y^{-1}(1-t) \text{ for all } t \in [0,1]\,.$$

From Theorem 1 it follows that

$$\int_0^x (1 - F_X(t))\,dt \le \int_0^x (1 - F_Y(t))\,dt \;\text{ for all } x > 0 \tag{6}$$

if and only if

$$\int_u^1 F_X^{-1}(1-t)\,dt \le \int_u^1 F_Y^{-1}(1-t)\,dt \;\text{ for all } u \in [0,1]\,. \tag{7}$$

Clearly, (6) holds if and only if

$$\int_0^x F_X(t)\,dt \geq \int_0^x F_Y(t)\,dt \text{ for all } x > 0,$$

which is $X \leq_{cv} Y$ from (3). Now, by a change of variable it is seen that (7) is equivalent to

$$\int_0^u F_X^{-1}(t)\,dt \leq \int_0^u F_Y^{-1}(t)\,dt \text{ for all } u \in [0,1],$$

which is $X \geq_{gl} Y$ and the result holds.

(ii) Now let $(\mathbf{\Omega}_X, \mathcal{B}_X, P_X)$ and $(\mathbf{\Omega}_Y, \mathcal{B}_Y, P_Y)$ be the probability spaces on which X and Y, respectively, are defined. Define $a(\omega) = X(\omega) - \mu_X$ for all $\omega \in \mathbf{\Omega}_X$ and $b(\omega) = Y(\omega) - \mu_Y$ for all $\omega \in \mathbf{\Omega}_Y$. Then

$$D_a(t) = P_X\{\omega \in \Omega_X : a(\omega) > t\} = 1 - F_{X-\mu_X}(t), \text{ for all } t \in \mathbb{R},$$

and, analogously,

$$D_b(t) = 1 - F_{Y-\mu_Y}(t), \text{ for all } t \in \mathbb{R}.$$

The decreasing rearrangements of a and b are given, respectively, by $\delta_a(t) = F_{X-\mu_X}^{-1}(1-t)$ and $\delta_b(t) = F_{Y-\mu_Y}^{-1}(1-t)$, for all $t \in [0,1]$. From Theorem 2 it follows that

$$\text{(8)} \qquad \int_0^u F_{X-\mu_X}^{-1}(1-t)\,dt \leq \int_0^u F_{Y-\mu_Y}^{-1}(1-t)\,dt \text{ for all } u \in [0,1]$$

if and only if (4) holds for all convex functions $\Phi : \mathbb{R} \longmapsto \mathbb{R}$, which is $X \leq_{dil} Y$. Since $X - \mu_X$ and $Y - \mu_Y$ have the same mean, it is not hard to see that (8) can be written as

$$\text{(9)} \qquad \int_0^{1-u} F_{X-\mu_X}^{-1}(t)\,dt \geq \int_0^{1-u} F_{Y-\mu_Y}^{-1}(t)\,dt \text{ for all } u \in [0,1]$$

which is $X \leq_{la} Y$. Hence, the proof is complete. □

3. SUFFICIENT CONDITIONS FOR ABSOLUTE LORENZ ORDERING

Let X and Y be non-negative random variables with respective means μ_X and μ_Y, having continuous distribution functions F and G, respectively. The following theorem provides a sufficient condition for X and Y to be ordered in the absolute Lorenz sense, by means of a «single-crossing property» on the distribution functions of the random variables $X - \mu_X$ and $Y - \mu_Y$.

Theorem 4. *Suppose that $F(x + \mu_X) - G(x + \mu_Y)$ has at most one sign change (from $-$ to $+$) on $\mathbb{R}$. Then $X \leq_{la} Y$.*

Proof

Let $F_{X-\mu_X}$ and $G_{Y-\mu_Y}$ be the distribution functions of $X-\mu_X$ and $Y-\mu_Y$, respectively, defined by

$$F_{X-\mu_X}(x) = F\left(x+\mu_X\right) \text{ and } G_{Y-\mu_Y}(x) = G\left(x+\mu_Y\right).$$

Since the difference $F_{X-\mu_X} - G_{Y-\mu_Y}$ changes sign at most once with sequence $-,+$, by the assumptions on F and G, then the difference between the corresponding inverse distribution functions $F^{-1}_{X-\mu_X} - G^{-1}_{Y-\mu_Y}$ changes sign at most once with sequence $+,-$. Since $X-\mu_X$ and $Y-\mu_Y$ have the same mean, we have that

$$\int_0^1 F^{-1}_{X-\mu_X}(t)dt = \int_0^1 G^{-1}_{Y-\mu_Y}(t)dt$$

and it follows that

$$\int_0^p \left[F^{-1}_{X-\mu_X}(t) - G^{-1}_{Y-\mu_Y}(t)\right] dt \geq \int_0^1 \left[F^{-1}_{X-\mu_X}(t) - G^{-1}_{Y-\mu_Y}(t)\right] dt = 0$$

for all p in $[0,1]$. Hence, $LA_X(p) \geq LA_Y(p)$ holds for all p in $[0,1]$ and, consequently, $X \leq_{la} Y$. □

Suppose now that X and Y are absolutely continuous random variables with density functions f and g, respectively. Let $f_{X-\mu_X}$ and $g_{Y-\mu_Y}$ respectively denote the density functions of the random variables $X-\mu_X$ and $Y-\mu_Y$. The next result provides a convenient sufficient condition for the absolute Lorenz comparison of two random variables.

Corollary 1. *Assume that* $\operatorname{supp}(X-\mu_X) \subseteq \operatorname{supp}(Y-\mu_Y)$. *If* $f(x+\mu_X)/g(x+\mu_Y)$ *is unimodal for x restricted to* $\operatorname{supp}(Y-\mu_Y)$, *where the mode is a supremum, then* $X \leq_{la} Y$.

Proof

Let $S(h)$ be the number of sign changes of the function $h(t)$. Since

$$f_{X-\mu_X}(x) = f\left(\mu_X + x\right), \quad g_{Y-\mu_Y}(x) = g\left(\mu_Y + x\right),$$

and $f(x+\mu_X)/g(x+\mu_Y)$ is unimodal on $\operatorname{supp}(Y-\mu_Y)$, so is $f_{X-\mu_X}(x)/g_{Y-\mu_Y}(x)$, with the mode yielding a supremum. Hence

$$S\left(f_{X-\mu_X} - g_{Y-\mu_Y}\right) = S\left(\frac{f_{X-\mu_X}}{g_{Y-\mu_Y}} - 1\right) \leq 2$$

and the sign sequence is $-,+,-$, in the case of equality. This condition implies that the difference $F_{X-\mu_X} - G_{Y-\mu_Y}$ changes sign at most once with sequence $-,+$. From Theorem 4 it follows that $X \leq_{la} Y$. □

4. ABSOLUTE LORENZ ORDERING OF GENERALIZED GAMMA DISTRIBUTIONS

Let $GG(p,\beta,\gamma,a)$ be the four-parameter generalized gamma distribution with density

$$f(x) = \frac{a(x-\gamma)^{ap-1}\exp\left[-\left(\frac{x-\gamma}{\beta}\right)^{a}\right]}{\beta^{ap}\Gamma(p)},\quad x\geq\gamma,\ a>0,\ \beta>0,\ p>0, \tag{10}$$

where $\Gamma(\cdot)$ denotes the complete gamma function. If $\gamma=0$, we have the three-parameter generalized gamma distribution considered, among others, by McDonald (1989), as descriptive model for the distribution of income. The four-parameter generalized gamma distribution includes Weibull distributions $(p=1)$, half-normal distributions $(p=1/2,\ a=2,\ \gamma=0)$ and, of course, ordinary gamma distributions $(a=1)$. The moment of order r about γ of $GG(p,\beta,\gamma,a)$ can be found in Johnson *et al.* (1994):

$$E\left[(X-\gamma)^{r}\right] = \frac{\beta^{r}\Gamma\left(p+\frac{r}{a}\right)}{\Gamma(p)}. \tag{11}$$

McDonald (1989) fitted the gamma generalized distribution to U.S.A. family nominal income for 1970-1980 and obtained estimates of a, β and p for 1970, 1975 and 1980. The results of these estimations show large variations on β for the period under consideration, whereas variations on a and p are very small. This suggests to consider the impact upon absolute Lorenz curve of variations on β, when the other parameters are fixed. The next result characterizes the parameter space of the four-parameter generalized gamma distribution in terms of absolute Lorenz ordering, when a and p are fixed.

Corollary 2. *Let* $X_1\sim GG(p,\beta_1,\gamma_1,a)$ *and* $X_2\sim GG(p,\beta_2,\gamma_2,a)$. *If* $(a-1)\cdot(ap-1)\geq 0$ *then*

$$X_1\leq_{la}X_2 \Longleftrightarrow \beta_1\leq\beta_2.$$

Proof

It is clear from (1) that the absolute Lorenz curve is invariant under location changes, so the location parameters γ_1 and γ_2 can be set null, without loss of generality.

$(\Rightarrow)$ From Theorem 3 (ii) it follows that if $X_1\leq_{la}X_2$, then

$$E\left[\Phi(X_1-\mu_1)\right]\leq E\left[\Phi(X_2-\mu_2)\right]$$

for all convex functions Φ, provided these expectations exist. In particular, by taking $\Phi(x)=x^2$, we obtain that $V[X_1]\leq V[X_2]$. From (11) we have that

$$V[X_i] = \beta_i^2\frac{\Gamma(p)\Gamma\left(p+\frac{2}{a}\right)-\left[\Gamma\left(p+\frac{1}{a}\right)\right]^2}{[\Gamma(p)]^2},\quad (i=1,2).$$

Therefore $\beta_1 \leq \beta_2$.

$(\Leftarrow)$ Suppose $\beta_1 < \beta_2$ (the case $\beta_1 = \beta_2$ is trivial). Let f_1 and f_2 be the density functions of the random variables X_1 and X_2, respectively. From (11), it follows that

$$E\left[X_i\right] = \frac{\beta_i \Gamma\left(p+\frac{1}{a}\right)}{\Gamma(p)} = \beta_i t, \quad (i = 1,2),$$

where $t = \Gamma\left(p+\frac{1}{a}\right) / \Gamma(p)$. Since $\beta_1 < \beta_2$, by Corollary 1, we will show that the ratio $f_1(x+\beta_1 t)/f_2(x+\beta_2 t)$ is unimodal for $x \geq -\beta_2 t$. It is clear that $f_1(x+\beta_1 t)/f_2(x+\beta_2 t) = 0$ for $-\beta_2 t \leq x \leq -\beta_1 t$. Now suppose $x > -\beta_1 t$ and denote

$$h(x) = \frac{f_1(x+\beta_1 t)}{f_2(x+\beta_2 t)} = \left(\frac{\beta_2}{\beta_1}\right)^{ap} \cdot \left(\frac{x+\beta_1 t}{x+\beta_2 t}\right)^{ap-1} \cdot \exp\left\{\left(\frac{x+\beta_2 t}{\beta_2}\right)^{a} - \left(\frac{x+\beta_1 t}{\beta_1}\right)^{a}\right\}.$$

Since $h(x) > 0$ for all $x > -\beta_1 t$ and $\lim\limits_{x \to -\beta_1 t} h(x) = 0$, in order to prove the unimodality of $h(x)$, it is sufficient to show that $h'(x)$ has at most one real root on $(-\beta_1 t, \infty)$. Since

$$h'(x) = M(x) \cdot \left[\frac{(ap-1)(\beta_2-\beta_1)t}{(x+\beta_2 t)(x+\beta_1 t)} + \frac{a(x+\beta_2 t)^{a-1}}{\beta_2^a} - \frac{a(x+\beta_1 t)^{a-1}}{\beta_1^a}\right],$$

where

$$M(x) = \left(\frac{\beta_2}{\beta_1}\right)^{ap} \cdot \exp\left\{\left(\frac{x+\beta_2 t}{\beta_2}\right)^{a} - \left(\frac{x+\beta_1 t}{\beta_1}\right)^{a}\right\} \cdot \frac{(x+\beta_1 t)^{ap+a-2}}{(x+\beta_2 t)^{ap-1}} \geq 0$$

for $x \geq -\beta_1 t$, it follows that $h'(x) = 0$ if and only if

$$\frac{(ap-1)(\beta_2-\beta_1)t}{(x+\beta_2 t)(x+\beta_1 t)^a} + \frac{a}{\beta_2^a}\left(\frac{x+\beta_2 t}{x+\beta_1 t}\right)^{a-1} = \frac{a}{\beta_1^a}. \tag{12}$$

By defining

$$R(x) = \frac{(ap-1)(\beta_2-\beta_1)t}{(x+\beta_2 t)(x+\beta_1 t)^a}, \; S(x) = \frac{a}{\beta_2^a}\left(\frac{x+\beta_2 t}{x+\beta_1 t}\right)^{a-1},$$

we have that $h'(x_0) = 0$ if and only if

$$R(x_0) + S(x_0) = \frac{a}{\beta_1^a}. \tag{13}$$

When $ap > 1$ and $a > 1$, both functions $R(x)$ and $S(x)$ are strictly decreasing, for $x \geq -\beta_1 t$. Hence, $R(x)+S(x)$ is strictly decreasing and (13) has at most one solution on $(-\beta_1 t, \infty)$. Similarly, if $ap < 1$ and $a < 1$, it follows that $R(x)+S(x)$ is strictly increasing for $x \geq -\beta_1 t$, attaining the value a/β_1^a at most once. If $a = 1$ or $ap = 1$, it is easy to see that (13) has at most one solution on $(-\beta_1 t, \infty)$. $\square$

Salem and Mount (1974) used ordinary gamma distributions for fitting empirical income data. The ordinary gamma distribution $G(p, \beta, \gamma)$ is obtained by setting $a = 1$ in (10). The next result characterizes, as a particular case of Corollary 2, the parameter space of this family, in terms of absolute Lorenz ordering when the shape parameter p is fixed.

Corollary 3. *Let* $X_1 \sim G(p, \beta_1, \gamma_1)$ *and* $X_2 \sim G(p, \beta_2, \gamma_2)$. *Then*

$$X_1 \leq_{la} X_2 \Longleftrightarrow \beta_1 \leq \beta_2.$$

Corollary 1 can also be used to prove that the ordinary gamma distribution can be ordered by the parameter p when the scale parameter β is fixed. The following result can be proven as Corollary 2.

Corollary 4. *Let* $X_1 \sim G(p_1, \beta, \gamma_1)$ *and* $X_2 \sim G(p_2, \beta, \gamma_2)$. *Then*

$$X_1 \leq_{la} X_2 \Longleftrightarrow p_1 \leq p_2.$$

5. REMARKS AND RELATED TOPICS

The generalized Lorenz order and the absolute Lorenz order are closely connected with the usual Lorenz ordering (see Arnold, 1987). In general, the rankings produced by these orderings do not coincide unless distributions have common means. In fact, if $\mu_X = \mu_Y$, Theorem 3.2 of Arnold (1987) is obtained as a particular case of Theorem 3 (i). The Lorenz and generalized Lorenz orderings within the gamma parametric family have been considered in Wilfling (1996) and Ramos *et al.* (2000), respectively.

The concave order is of interest in reliability theory. Stoyan (1983) states that X is smaller in mean used life than Y if $X \leq_{cv} Y$ holds. The origins of this ordering may be found in Marshall and Proschan (1970), where it appeared as a particular relationship useful for obtaining bounds for the mean lifetime of series systems in reliability theory. The concave ordering is a counterpart of the so-called convex ordering (Stoyan, 1983). The convex order is defined by requiring (2) to hold for all non-decreasing and convex functions Φ for which the expectations exist. In the case of equal means, the convex and the dilation orders are equivalent.

The relationships between some notions that are common to reliability theory and economics have been studied by several authors. Chandra and Singpurwalla (1981) proved that the Lorenz curve can be used to characterize DFR (Decreasing Failure Rate) random variables. Kochar (1989) showed that Lorenz ordering is the same as HNBUE-ordering (Harmonic New Better than Used in Expectation), which is a well-known concept for the comparison of the aging properties of two life distributions.

It is well known that the concave order is preserved under an increasing concave transformation (see Theorem 3.A.5 of Shaked and Shanthikumar, 1994). Now, from Theorem 3 (i), it follows that this property holds with $\leq_{gl}$ replacing $\leq_{cv}$. Thus, the following result generalizes the sufficient condition of Theorem 3.1, given by Moyes (1989) for a finite population.

Corollary 5. *Let X and Y be two non-negative random variables. If $X \leq_{gl} Y$ and g is any increasing and concave function, then $g(X) \leq_{gl} g(Y)$.*

In economics, the preservation of the ranking of distributions has obvious implications when one views the transformation g as a taxation scheme (see Moyes, 1988, for further discussion).

The relationship between the absolute Lorenz curve and the dilation order suggests a similar characterization between the so-called dispersion function and the dilation order. The *dispersion function* of a random variable X with a finite mean is given by (Muñoz-Pérez and Sánchez-Gómez, 1990)

$$D_X(u) = E[|X-u|] \quad \text{for all } u \in \mathbb{R}.$$

This function characterizes the distribution function and measures directly the dispersion of X about each point $u \in \mathbb{R}$. The dispersion function provides a partial ordering between random variables with respect to the dispersion in the following sense: we say that Y is at least as dispersed as X if $D_{X-\mu_X}(u) \leq D_{Y-\mu_Y}(u)$ for all $u \in \mathbb{R}$. Now, from Theorem 1 of Muñoz-Pérez and Sánchez-Gómez (1990) and Theorem 3 (ii), it follows that the ordering induced by such a function is equivalent to the ordering induced by the absolute Lorenz curve. This is stated formally in the following result.

Corollary 6. *Let X and Y be two non-negative random variables. Then*

$$X \leq_{la} Y \text{ if and only if } D_{X-\mu_X}(u) \leq D_{Y-\mu_Y}(u) \text{ for all } u \in \mathbb{R}.$$

This result suggests that the dispersion function and the absolute Lorenz curve play a similar role in statistics.

On the other hand, the functional form of inequality indices has been studied by several authors (see, for example, Kolm, 1976; Atkinson, 1970; Nygard and Sandström, 1981; Champernowne and Cowell, 1998). Moyes (1987) showed that the absolute Lorenz ordering is consistent and is implied by the unanimous partial order generated by the class of *absolute inequality indices*, introduced by Kolm (1976). Theorem 3 (ii) suggests that a reasonable summary measure of absolute inequality is provided by an index of the form $E\left[\Phi\left(X-\mu_X\right)\right]$ for any convex functions Φ. In particular, the choices $\Phi(x)=x^2$ and $\Phi(x)=|x|$ lead, respectively, to the variance and the absolute mean deviation of X. These results show that, in some way, the concepts of *absolute inequality* and *dispersion* could be considered as *allotropic forms* of the same primary concept.

Some other general results and examples relating dispersion and inequality can be found in Frosini (1984).

Finally, the «single-crossing property» has often been used to compare distributions under orderings related to the absolute Lorenz order. Theorem 6.4 of Arnold (1987) and Theorem 2.1 of Ramos *et al.* (2000) provide sufficient conditions for the Lorenz and the generalized Lorenz orders, respectively, based on this property. Theorem 1.5.1 of Stoyan (1983) provides a sufficient condition for the convex order based on the «cut criterion», which is a property very similar to the «single-crossing property».

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REFERENCES

Arnold, B. C. (1987). *Majorization and the Lorenz order: A brief Introduction*. Springer, New York.

Atkinson, A. B. (1970). «On the Measurement of Inequality». *Journal of Economic Theory*, 2, 244-263.

Chandra, M. and Singpurwalla, N. D. (1981). «Relationships between some notions which are common to reliability theory and economics». *Mathematics of Operations Research*, 6, 113-121.

Champernowne, D. G. and Cowell, F. A. (1998). *Economic inequality income distribution*. Cambridge University Press, United Kingdom.

Chong, K. M. (1974). «Some extensions of a theorem of Hardy, Littlewood and Polya and their applications». *Canadian Journal of Mathematics*, 26, 1321-1340.

Frosini, B. V. (1984). «Concentration, dispersion and spread: an insight into their relationship». *Statistica*, 3, 373-393.

Johnson, N. L., Kotz, S. and Balakrishnan, N. (1994). *Continuous univariate distributions* (2.[a] ed.). John Wiley & Sons, inc. New York.

Hardy, G. H., Littlewood, J. E. and Pólya, G. (1929). «Some simple inequalities satisfied by convex functions». *Messenger of Mathematics*, 58, 145-152.

Hickey, R. J. (1986). «Concepts of dispersion in distributions: a comparative note». *Journal of Applied Probability*, 23, 914-921.

Kochar, S. C. (1989). «On extensions of DMRL and related partial orderings of life distributions». *Communications in Statistics-Stochastic Models*, 5, 235-245.

Kolm, S.-C. (1976). «Unequal inequalities 1». *Journal of Economic Theory*, 12, 416-422.

Lambert, P. (1993). *The distribution and redistribution of income*. Manchester: Manchester University Press.

Lewis, T. and Thompson, J. W. (1981). «Dispersive distributions and the connection between dispersivity and strong unimodality». *Journal of Applied Probability*, 18, 76-90.

Marshall, A. W. and Proschan, F. (1970). «Mean life of series and parallel systems». *Journal of Applied Probability*, 7, 165-174.

McDonald, J. B. (1984). «Some generalized functions for the size distribution of income». *Econometrica*, 52, 647-663.

Moyes, P. (1987). «A new concept of Lorenz domination». *Economic Letters*, 23, 203-207.

Moyes, P. (1988). «A note on minimally progressive taxation and absolute income inequality». *Social Choice and Welfare*, 5, 227-234.

Moyes, P. (1989). «Some Classes of Functions that Preserve the Inequality and Welfare Orderings of Income Distributions». *Journal of Economic Theory*, 49, 347-359.

Muñoz-Pérez, J. and Sánchez-Gómez, A. (1990). «Dispersive ordering by dilation». *Journal of Applied Probability*, 27, 440-444.

Nygard, F. and Sandström, A. (1981). *Measuring income inequality*. Almqvist & Wiksell International, Stockholm.

Ramos, H. M., Ollero, J. and Sordo, M. A. (2000). «A sufficient condition for generalized Lorenz order». *Journal of Economic Theory*, 90, 286-292.

Ryff, J. V. (1963). «On the representation of doubly stochastic operators». *Pacific Journal of Mathematics*, 13, 1379-1386.

Salem, A. B. Z. and Mount, T. D. (1974). «A convenient descriptive model of income distribution». *Econometrica*, 42, 1115-1127.

Shaked, M. and Shanthikumar, J. G. (1994). *Stochastic orders and their applications*. Academic Press, London.

Shorrocks, A. F. (1983). «Ranking Income Distributions». *Economica*, 50, 3-17.

Stoyan, D. (1983). *Comparison Methods for Queues and Other Stochastic Models*. John Wiley, New York.

Wilfling, B. (1996). «A sufficient condition for Lorenz ordering». *Sankhyã*, Series B 58, 62-69.

Yitzhaki, S. (1999). «Neccesary and sufficient conditions for dominace using generalized Lorenz curves». In: *Advances in Econometrics, Income distribution and methodology of science: essays in honor of Camilo Dagum*. Springer.