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PERIODIC SOLUTIONS OF CONTINUOUS THIRD–ORDER DIFFERENTIAL EQUATIONS WITH PIECEWISE POLYNOMIAL NONLINEARITIES

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Abstract. We consider third–order autonomous continuous piecewise differential equations in the variable x. For such differential equations with nonlinearities of the form x^m, we investigate their periodic solutions using the averaging theory.

1. Introduction and Statement of the Main Result

For studying some electrical circuits Sprott and Sun [9, 10, 11] considered the third order differential equation $\dddot{x} = -\dot{x} - a\ddot{x} + g(x)$, where g is an elemental piecewise function. He showed that some of these equations exhibit chaos.

In this paper we are interested in studying the third order of differential equations of the form

$$\dddot{x} = -\dot{x} + \varepsilon|\ddot{x}| - \varepsilon a x^m, \tag{1}$$

where a and ε are parameters and ε is small. But our interest is in studying how their periodic solutions depend on the parameter a and on the exponent m.

We can write the third order differential equations (1) as the following differential system of first order

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -y + \varepsilon|z| - \varepsilon a x^m, \end{aligned} \tag{2}$$

where the dot denotes derivative with respect an independent variable t, usually the time.

We remark that the differential system (2) is only continuous due to the existence of the term $|z|$, so we cannot apply to it the classical averaging

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theory for studying their periodic solutions because that theory needs that the differential system be of class C^2. We shall apply recent extensions of the averaging theory to continuous differential systems, see section 2.

In what follows we state our main result. Recall that $n!!$ denotes the double factorial of n.

Theorem 1. *Let $|\varepsilon| \neq 0$ be a sufficiently small parameter. Then the following statements hold.*

(a) *If m is even and $a > 0$, then system* (2) *has a periodic solution of the form*

$$x(t,\varepsilon) = -r_0^* + \mathcal{O}(\varepsilon), \quad y(t,\varepsilon) = r_0^* \sin t + \mathcal{O}(\varepsilon), \quad z(t,\varepsilon) = r_0^* \cos t + \mathcal{O}(\varepsilon),$$

where

$$r_0^* = \left(\frac{2(m)!!}{a\pi(m-1)!!} \right)^{\frac{1}{m-1}},$$

bifurcating from the periodic solutions of system (2) *with $\varepsilon = 0$ obtained by using the averaging theory of first order.*

(b) *If either m is odd and $a \neq 0$, or m is even and $a < 0$, or $a = 0$, then the averaging theory of fist order does not provide any information on the periodic solutions of system* (2).

Theorem 1 is proved in section 3. As we said we shall prove it using the averaging theory of first order for continuous differential systems that we summarize in section 2. Finally in section 4 we do an application of Theorem 1 to a particular continuous differential system (2).

2. The averaging theory of first order for continuous differential systems

Now we summarize the results on the averaging theory that we need for proving Theorem 1.

We work with a system of the form

$$\dot{\mathrm{x}} = F_0(t,\mathrm{x}) + \varepsilon F_1(t,\mathrm{x}) + \mathcal{O}(\varepsilon^2), \tag{3}$$

where $\varepsilon \neq 0$ is a small parameter and the functions $F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $T-$periodic in the first variable, F_0 is C^1, DF_0, F_1 and R are locally Lipschitz in the variable x, and $\Omega \subseteq \mathbb{R}^n$ is an open subset.

We assume that there exists a submanifold of dimension n formed by periodic solutions of the unperturbed system

$$\dot{\mathrm{x}} = F_0(t,\mathrm{x}). \tag{4}$$

Consider $\mathrm{Cl}(U)$ the closure of U. We suppose that there exists an open set U with $\mathrm{Cl}(U) \subset \Omega$ satisfying that for each $z \in \mathrm{Cl}(U)$ the solution $\mathrm{x}(t,\mathrm{z},0)$

of the unperturbed system (4) is $T-$periodic. We note that $\mathrm{Cl}(U)$ is an isochronous set for system (4), because all its periodic orbits have period T.

Consider $x(t,z,\varepsilon)$ the solution of system (4) satisfying the initial condition $\mathrm{x}(0,\mathrm{z},\varepsilon)=z$. The linearization of system (4) in a periodic solution $\mathrm{x}(t,\mathrm{z},0)$ is given by

$$\dot{\mathrm{y}} = D_{\mathrm{x}}F_0(t,\mathrm{x}(t,\mathrm{z},0))\mathrm{y}, \tag{5}$$

where y is an $n\times n$ matrix. Let $M_{\mathrm{z}}(t)$ be the fundamental matrix of the linearized system (5) such that $M_{\mathrm{z}}(0)$ is the identity matrix.

The next result of the averaging theory allows to obtain $T-$periodic solutions of the differential system (3) with $\varepsilon\neq 0$ sufficiently small bifurcating from the periodic solutions $\mathrm{x}(t,\mathrm{z},0)$ contained in $\mathrm{Cl}(U)$ of the unperturbed differential system (4). The first version of this result is due to Malkin [6] and Roseau [7] for C^2 differential systems, a more clear and shorter proof of this result was done by Buică et al. in [2]. The extension of this result to continuous differential systems has been done by Llibre et al. in [4]. For a general introduction to the averaging theory see for instance the book of Sanders et al. [8].

Theorem 2. *Consider a continuous differential system* (3). *We suppose that there exists an open and bounded set U with $\mathrm{Cl}(U)\subset\Omega$ such that for each $\mathrm{z}\in\mathrm{Cl}(U)$, the solution $\mathrm{x}(t,\mathrm{z},0)$ of system* (4) *is $T-$periodic. Let $f:\mathrm{Cl}(U)\to\mathbb{R}^n$ be the function*

$$f(\mathrm{z})=\frac{1}{T}\int_0^T M_{\mathrm{z}}^{-1}(t)F_1(t,\mathrm{x}(t,\mathrm{z},0))dt,$$

called the averaging function of first order. Then for each $\mathbf{a}\in U$ satisfying $f(\mathbf{a})=0$ there exists a neighborhood V of $\mathbf{a}$ such that $f(z)\neq 0$ for all $z\in\overline{V}\setminus\{\mathbf{a}\}$ and the Brouwer degree of f at $\mathbf{a}$ is not zero, i.e. $d_B(f,V,0)\neq 0$. Then there exists a $T-$periodic solution $\mathrm{x}(t,\varepsilon)$ of system (3) *such that $\mathrm{x}(0,\varepsilon)\to\mathbf{a}$ when $\varepsilon\to 0$.*

For details on the Brouwer degree see for instante the paper of Browder [1]. We must note that if the averaging function $f(\mathrm{z})$ is of class C^1 then Brouwer degree $d_B(f,V,0)\neq 0$ if the Jacobian $\det(Df(\mathbf{a}))\neq 0$, see for instance [5].

3. Proof of Theorem 1

First we consider the case $m=2n$ with n a positive integer. We write the differential system (2) in the cylindrical coordinates (x,r,θ) defined by

$x = x$, $y = r\sin\theta$, $z = r\cos\theta$, and we obtain the differential system

$$\begin{aligned} \dot{x} &= r\sin\theta, \\ \dot{r} &= \varepsilon\cos\theta\left(|r\cos\theta| - ax^{2n}\right), \\ \dot{\theta} &= 1 + \frac{\varepsilon}{r}\sin\theta\left(ax^{2n} - |r\cos\theta|\right). \end{aligned} \tag{6}$$

Taking as new independent variable the variable θ, system (6) becomes

$$\begin{aligned} \frac{dx}{d\theta} &= x' = r\sin\theta + \varepsilon\sin^2\theta\left(|r\cos\theta| - ax^{2n}\right) + \mathcal{O}(\varepsilon^2), \\ \frac{dr}{d\theta} &= r' = \varepsilon\cos\theta\left(|r\cos\theta| - ax^{2n}\right) + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{7}$$

We note that this differential system is now in the normal form (3) for applying the averaging theory described in section 2.

The unperturbed system (4) is now

$$\begin{aligned} x' &= r\sin\theta, \\ y' &= 0. \end{aligned} \tag{8}$$

The solution of system (8) with initial condition (x_0, r_0) is $\varphi(\theta, (x_0, r_0)) = (x_0 + r_0(1-\cos\theta), r_0)$. So all the solutions with $r_0 > 0$ of the unperturbed system (8) are periodic with the same period 2π.

The fundamental matrix $M_{(x_0,r_0)}(\theta) = M(\theta)$ of the variational differential system (5) associated to system (8) evaluated on the periodic solution $(x_0 + r_0(1-\cos\theta), r_0)$ is

$$M(\theta) = \begin{pmatrix} 1 & 1-\cos\theta \\ 0 & 1 \end{pmatrix}. \tag{9}$$

Note that $M(0)$ is the identity matrix.

According with the averaging theory of section 2 for studying the periodic solutions of the continuous differential system (7) we must study the zeros of the averaging function

$$f(x_0, r_0) = \frac{1}{2\pi}\int_0^{2\pi} M(\theta)^{-1} F_1(\theta, \varphi(\theta, (x_0, r_0)))d\theta,$$

where

$$F_1(\theta, (x, r)) = \left(\sin^2\theta(|r\cos\theta| - ax^{2n}), \cos\theta(|r\cos\theta| - ax^{2n})\right).$$

Thus we have

$$\begin{aligned} f(x_0, r_0) = & \frac{1}{2\pi}\int_0^{2\pi} \begin{pmatrix} (1-\cos\theta)\left(r_0|\cos\theta| - a(r_0 + x_0 - r_0\cos\theta)^{2n}\right) \\ \cos\theta\left(r_0|\cos\theta| - a(r_0 + x_0 - r_0\cos\theta)^{2n}\right) \end{pmatrix} d\theta \\ = & \begin{pmatrix} f_1(x_0, r_0) \\ f_2(x_0, r_0) \end{pmatrix}. \end{aligned}$$

A simple computation shows

$$\int_0^{2\pi} r_0 \cos\theta |\cos\theta| d\theta = 0. \tag{10}$$

Therefore since $r_0 > 0$ we have that

$$\begin{aligned}
f_2(x_0, r_0) &= -\frac{a}{2\pi}\int_0^{2\pi} \cos\theta (r_0 + x_0 - r_0\cos\theta)^{2n} d\theta \\
&= -\frac{a}{2\pi}\int_0^{2\pi} r_0^{2n} \cos\theta \left(\frac{r_0 + x_0}{r_0} - \cos\theta\right)^{2n} d\theta \\
&= -\frac{a}{2\pi}\int_0^{2\pi} r_0^{2n} \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \left(\frac{r_0 + x_0}{r_0}\right)^{2n-k} \cos^{k+1}\theta \, d\theta.
\end{aligned}$$

If k is even then $\displaystyle\int_0^{2\pi} \cos^{k+1}\theta \, d\theta = 0$. So we can rewrite $f_2 = f_2(x_0, r_0)$ as

$$\begin{aligned}
f_2 &= \frac{a}{2\pi}\sum_{k=0}^{n-1} \binom{2n}{2k+1} r_0^{2n} \left(\frac{r_0 + x_0}{r_0}\right)^{2n-2k-1} \int_0^{2\pi} \cos^{2k+2}\theta \, d\theta \\
&= \frac{a}{2\pi}\sum_{k=0}^{n-1} \binom{2n}{2k+1} r_0^{2k+1} (r_0 + x_0)^{2n-2k-1} \int_0^{2\pi} \cos^{2k+2}\theta \, d\theta \\
&= \frac{a}{2\pi} r_0 (r_0 + x_0) \sum_{k=0}^{n-1} \binom{2n}{2k+1} r_0^{2k} (r_0 + x_0)^{2n-2k-2} \int_0^{2\pi} \cos^{2k+2}\theta \, d\theta.
\end{aligned}$$

Analyzing the last right hand side of the previous equation it is clear that the function $f_2(x_0, r_0)$ only vanishes at $x_0 = -r_0$. Replacing $x_0 = -r_0$ in the integrant of the definition of the function $f_1(x_0, r_0)$ it becomes

$$B(\theta) = -r_0 \cos\theta |\cos\theta| + r_0 |\cos\theta| + a r_0^{2n} (\cos^{2n+1}\theta - \cos^{2n}\theta).$$

By using (10) and Formula Bl(68)(8) of [3] (p. 405, with $a = 0$ and $b = 1$) we obtain that

$$f_1(-r_0, r_0) = \frac{1}{\pi}\left(2r_0 - \pi a \, r_0^{2n} \frac{(2n-1)!!}{(2n)!!}\right).$$

Solving $f_1(-r_0, r_0) = 0$ in the variable r_0, we get the unique real solution

$$r_0^* = \left(\frac{2(2n)!!}{a\pi (2n-1)!!}\right)^{\frac{1}{2n-1}}.$$

Note that $a > 0$ ensures the existence of the solution $r_0^* > 0$, and if $a < 0$ then the function $f_1(-r_0, r_0)$ has no positive real solutions for r_0, and consequently when m is even and $a < 0$ the averaging theory does not provide any information on the periodic solutions of the differential system (7).

In summary, if m is even and $a > 0$ the averaging function $f(x_0, r_0)$ has the unique zero $(-r_0^*, r_0^*)$ with $r_0^* > 0$. Now in order to verify the assumptions of Theorem 2 we must prove that the Jacobian

$$\det(Df(-r_0^*, r_0^*)) = \left.\frac{\partial(f_1, f_2)}{\partial(x_0, r_0)}\right|_{(x_0,r_0)=(-r_0,r_0)} \neq 0. \tag{11}$$

From the expression of $f_2(x_0, r_0)$ we obtain that

$$\begin{aligned} \frac{\partial f_2}{\partial x_0}(-r_0, r_0) &= \frac{a}{2\pi} \begin{pmatrix} 2n \\ 2n-1 \end{pmatrix} r_0^{2n-1} 2\pi \frac{(2n-1)!!}{(2n)!!} \\ &= \frac{2a\, n\, r_0^{2n-1}(2n-1)!!}{(2n)!!}, \end{aligned}$$

and

$$\frac{\partial f_2}{\partial r_0}(-r_0, r_0) = \frac{\partial f_2}{\partial x_0}(-r_0, r_0).$$

Analogously writing f_1 as a binomial expression we get

$$\begin{aligned} \frac{\partial f_1}{\partial x_0}(-r_0, r_0) &= -\frac{\partial f_2}{\partial x_0}(-r_0, r_0), \\ \frac{\partial f_1}{\partial r_0}(-r_0, r_0) &= 2/\pi - 2\frac{\partial f_2}{\partial x_0}(-r_0, r_0). \end{aligned}$$

Now replacing r_0 by r_0^* in the previous derivatives we have the Jacobian matrix

$$Df(-r_0^*, r_0^*) = \begin{pmatrix} -\dfrac{4n}{\pi} & \dfrac{2-8n}{\pi} \\ \dfrac{4n}{\pi} & \dfrac{4n}{\pi} \end{pmatrix},$$

and its determinant is $\dfrac{8n(2n-1)}{\pi^2} \neq 0$ for all positive integer n. Hence (11) holds and we can apply Theorem 2 to the continuous differential system (7).

From Theorem 2 it follows that system (7) has the periodic solution

$$x(\theta, \varepsilon) = -r_0^* + \mathcal{O}(\varepsilon), \quad r(\theta, \varepsilon) = r_0^* + \mathcal{O}(\varepsilon).$$

Going back through the change of cylindrical coordinates system (2) has the periodic solution

$$x(t, \varepsilon) = -r_0^* + \mathcal{O}(\varepsilon), \quad y(t, \varepsilon) = r_0^* \sin t + \mathcal{O}(\varepsilon), \quad z(t, \varepsilon) = r_0^* \cos t + \mathcal{O}(\varepsilon).$$

This completes the proof of statement (a) of Theorem 1.

Now suppose that $m = 2n+1$. Then we have that

$$
\begin{aligned}
f_2(x_0, r_0) = & -\frac{a}{2\pi}\int_0^{2\pi} \cos\theta (r_0 + x_0 - r_0\cos\theta)^{2n+1} d\theta \\
= & -\frac{a}{2\pi}\int_0^{2\pi} r_0^{2n+1}\cos\theta \left(\frac{r_0 + x_0}{r_0} - \cos\theta\right)^{2n+1} d\theta \\
= & -\frac{a}{2\pi}\int_0^{2\pi} r_0^{2n+1}\sum_{k=0}^{2n+1}(-1)^k \binom{2n+1}{k}\left(\frac{r_0+x_0}{r_0}\right)^{2n+1-k}\cos^{k+1}\theta\, d\theta.
\end{aligned}
$$

If k is odd then $\displaystyle\int_0^{2\pi}\cos^k\theta\, d\theta = 0$. So we can rewrite the function $f_2(x_0, r_0)$ as follows

$$
\begin{aligned}
f_2(x_0, r_0) = & \frac{a}{2\pi}\sum_{k=0}^{n}\binom{2n+1}{2k+1} r_0^{2n+1}\left(\frac{r_0+x_0}{r_0}\right)^{2n-2k}\int_0^{2\pi}\cos^{2k+2}\theta\, d\theta \\
= & \frac{a}{2\pi} r_0 \sum_{k=0}^{n-1}\binom{2n+1}{2k+1} r_0^{2k}(r_0+x_0)^{2n-2k}\int_0^{2\pi}\cos^{2k+2}\theta\, d\theta.
\end{aligned}
$$

Therefore since $r_0 > 0$ the function $f_2(x_0, r_0)$ does not vanish, and the averaging function $f(x_0, r_0)$ has no real zeros, and consequently if m is odd the averaging theory of first order does not provide any information on the periodic solutions of the differential system (2).

The case $a = 0$ is the most simple, because its averaging function is

$$
f(x_0, r_0) = \left(\frac{2\, r_0}{\pi}, 0\right),
$$

which again has no zeros when $r_0 > 0$, and so the averaging theory of first order does not provide any information on the periodic solutions of the differential system (2).

In short, taking into account that inside the proof of statement (a) of Theorem 1 we have shown that if m is even and $a < 0$ the averaging theory of first order does not provide any information on the periodic solutions of the differential system (2), the proof of statement (b) of Theorem 1 is completed.

4. An application of Theorem 1

In this section we provide an application of Theorem 1. Consider the perturbed system

$$
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= -y + \varepsilon\left(|z| - \frac{4}{\pi}x^2\right),
\end{aligned}
\tag{12}
$$

where ε is a small parameter.

Writing system (12) in the previous cylindrical coordinates we obtain the system

$$\begin{aligned}\dot{x} &= r\sin\theta,\\ \dot{r} &= \varepsilon\cos\theta\left(r|\cos\theta| - \tfrac{4x^2}{\pi}\right),\\ \dot{\theta} &= 1 + \tfrac{\varepsilon\sin\theta\left(4x^2-\pi r|\cos\theta|\right)}{\pi r},\end{aligned}$$

or equivalent the following system with the independent variable θ

$$\begin{aligned}x' &= r\sin\theta + \varepsilon\sin^2\theta\left(r|\cos\theta| - \frac{4x^2}{\pi}\right),\\ r' &= \varepsilon\cos\theta\left(r|\cos\theta| - \frac{4x^2}{\pi}\right).\end{aligned}$$

The unperturbed system and the fundamental matrix associated to it are given respectively in (4) and (9). For system (12) we have that the function

$$F_1(\theta,(x,r)) = \left(\sin^2\theta\left(r|\cos\theta| - \frac{4x^2}{\pi}\right), \cos\theta\left(r|\cos\theta| - \frac{4x^2}{\pi}\right)\right).$$

So the averaging function

$$\begin{aligned}f(x_0,r_0) &= \frac{1}{2\pi}\int_0^{2\pi} M(\theta)^{-1}F_1(\theta,\varphi(\theta,(x_0,r_0)))d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi}\begin{pmatrix}(1-\cos\theta)(r_0|\cos\theta| - A(\theta))\\ \cos\theta(r_0|\cos\theta| - A(\theta))\end{pmatrix}d\theta,\end{aligned}$$

where

$$A(\theta) = \frac{4(r_0 + x_0 - r_0\cos\theta)^2}{\pi}.$$

Computing the previous integral we obtain

$$f(x_0,r_0) = -\frac{2}{\pi}\left(5r_0^2 + 6r_0x_0 - r_0 + 2x_0^2, -2r_0(r_0+x_0)\right).$$

The second component of the function $f(x_0,r_0)$ vanishes at $x_0 = -r_0$. Replacing $x_0 = -r_0$ in the first component of the function $f(x_0,r_0)$, we get

$$f_1(x_0,r_0) = 2r_0(r_0-1).$$

Since $r_0 > 0$ it is clear that $f_1(x_0,r_0)$ only vanishes when $r_0 = 1$. So the unique zero of the function $f(x_0,r_0)$ is $(x_0,r_0) = (-1,1)$. Moreover the Jacobian matrix of $f(x_0,r_0)$ evaluated at the zero $(-1,1)$ is

$$\begin{pmatrix}-\frac{4}{\pi} & -\frac{6}{\pi}\\ \frac{4}{\pi} & \frac{4}{\pi}\end{pmatrix}$$

and its determinant is $8/\pi^2 \neq 0$. Thus, from Theorem 1 system (12) has the periodic solution

$$x(t,\varepsilon) = -1 + \mathcal{O}(\varepsilon), \quad y(t,\varepsilon) = \sin t + \mathcal{O}(\varepsilon), \quad z(t,\varepsilon) = \cos t + \mathcal{O}(\varepsilon).$$

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References

[1] F. Browder, *Fixed point theory and nonlinear problems*, Bull. Amer. Math. Soc. **9** (1983), 1–39.
[2] A. Buică, J.P. Françoise and J. Llibre, *Periodic solutions of nonlinear periodic differential systems with a small parameter*, Commun. Pure Appl. Anal. **6** (2007), 103–111.
[3] I.S. Gradshteyn and I.M. Ryzhik, *Table of integrals, series, and products*, Elsevier, Oxford, 2007.
[4] J. Llibre, D. D. Novaes and M. A. Teixeira, *Higher order averaging theory for finding periodic solutions via Brouwer degree*, Nonlinearity, **27** (2014), 563–583.
[5] N.G. Lloyd, *Degree Theory*, Cambridge University Press, 1978.
[6] I.G. Malkin, *On Poincaré's theory of periodic solutions*, Akad. Nauk SSSR. Prikl. Mat. Meh. **13** (1949), 633–646 (in Russian).
[7] M. Roseau, *Vibrations non linéaires et théorie de la stabilité*, (French) Springer Tracts in Natural Philosophy, Vol.**8** Springer–Verlag, Berlin–New York, 1966.
[8] J.A. Sanders F. Verhulst and J. Murdock, *Averaging Methods in Nonlinear Dynamical Systems*, Second edition, Applied Mathematical Sciences **59**, Springer, New York, 2007. 91–172.
[9] J.C. Sprott, *Simplest chaotic systems and circuits*, Am. J. Phys. **68** (2000), 758–763.
[10] J.C. Sprott, *A new class of chaotic circuit*, Phys. Lett. A **266** (2000) 19–23.
[11] K.H. Sun and J.C. Sprott, *A simple Jerk system with piecewise exponential nonlinearity*, Int. J. of Nonlinear Sciences & Numerical Simulation **10** (2009), 1443–1450.

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