This is a preprint of: "The zero-Hopf bifurcations in the Kolmogorov systems of degree 3 in R^3", Erika Diz Pita, Jaume Llibre, M. Victoria Otero Espinar, Clàudia Valls, *Commun. Nonlinear Sci. Numer. Simul.*, vol. 95, 105621, 2021.
DOI: [10.1016/j.cnsns.2020.105621]

The zero-Hopf bifurcations in the Kolmogorov systems of degree 3 in $\mathbb{R}^3$

Érika Diz-Pita[a], Jaume Llibre[b], M. Victoria Otero-Espinar[a], Claudia Valls[c]

[a] *Departamento de Estatística, Análise Matemática e Optimización, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain*
[b] *Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain*
[c] *Center for Mathematical Analysis, Geometry and Dynamical Systems, Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, 1049-001, Lisboa, Portugal*

Abstract

In this work we study the periodic orbits which bifurcate from all zero-Hopf bifurcations that an arbitrary Kolmogorov system of degree 3 in $\mathbb{R}^3$ can exhibit. The main tool used is the averaging theory.

Keywords: Lotka–Volterra system, Kolmogorov systems, phase portraits, Hopf bifurcation, zero-Hopf bifurcation, limit cycle

1. Introduction and statement of the main results

Lotka-Volterra systems were initially proposed, independently, by Alfred J. Lotka in 1925 [1] and Vito Volterra in 1926 [2], both in the context of competing species. These Lotka-Volterra systems are polynomial differential systems of the form

$$\dot{x} = xP(x,y), \qquad \dot{y} = yQ(x,y),$$

where P and Q are polynomials of degree 1. Later on the *Lotka-Volterra systems* were generalized and considered on arbitrary dimension $n \geq 2$, i.e.

$$\dot{x}_i = x_i P_i(x_1, \ldots, x_n),$$

where P_i are polynomials of degree 1. Finally in 1936 Andrei Kolmogorov [3] extended those systems to arbitrary degree, i.e. the polynomials P_i can have any degree. These last systems are now called *Kolmogorov systems*.

The Lotka-Volterra and Kolmogorov systems have been used for modelling many natural phenomena, such as the time evolution of conflicting species in biology [4], chemical reactions [5], plasma physics [6], hydrodynamics [7], and many other phenomena as social science and economics [8]. Recently limit cycles for differential systems in $\mathbb{R}^3$ also are studied for discontinuous differential systems see for instance [9] and the references quoted therein.

We want to study the limit cycles of the Kolmogorov systems of degree 3 in $\mathbb{R}^3$ which bifurcate in the zero-Hopf bifurcations of the singular points (a,b,c) which are not on the invariant planes $x=0$, $y=0$ and $z=0$ of the Kolmogorov system

$$\dot{x} = xP(x,y,z), \quad \dot{y} = yQ(x,y,z), \quad \dot{z} = zR(x,y,z),$$

with P, Q and R polynomials of degree 2. Doing the scaling $(x,y,z) \to (x/a, y/b, z/c)$ we can assume without loss of generality that $(a,b,c) = (1,1,1)$. Therefore it is sufficient to study the limit cycles which can bifurcate

Email addresses: erikadiz.pita@usc.es (Érika Diz-Pita), jllibre@mat.uab.cat (Jaume Llibre), mvictoria.otero@usc.es (M. Victoria Otero-Espinar), cvalls@mat.tecnico.ulisboa.pt (Claudia Valls)

from the singular point $(1,1,1)$ of the system

$$
\begin{aligned}
\dot{x} &= x\left(a_1(x-1)+a_2(y-1)+a_3(z-1)+a_4(x-1)^2+a_5(x-1)(y-1)\right.\\
&\quad \left.+\,a_6(x-1)(z-1)+a_7(y-1)^2+a_8(y-1)(z-1)+a_9(z-1)^2\right),\\
\dot{y} &= y\left(b_1(x-1)+b_2(y-1)+b_3(z-1)+b_4(x-1)^2+b_5(x-1)(y-1)\right.\\
&\quad \left.+\,b_6(x-1)(z-1)+b_7(y-1)^2+b_8(y-1)(z-1)+b_9(z-1)^2\right),\\
\dot{z} &= z\left(c_1(x-1)+c_2(y-1)+c_3(z-1)+c_4(x-1)^2+c_5(x-1)(y-1)\right.\\
&\quad \left.+\,c_6(x-1)(z-1)+c_7(y-1)^2+c_8(y-1)(z-1)+c_9(z-1)^2\right),
\end{aligned}
\tag{1.1}
$$

when this singular point is a *zero-Hopf equilibrium*, i.e. when the eigenvalues of the linear part of the system at $(1,1,1)$ are of the form 0 and $\pm\beta i$ with $\beta>0$. Here the dot denotes derivative with respect to the time t.

In the next result we characterize when the singular point $(1,1,1)$ is zero-Hopf.

Proposition 1.1. *The singular point $(1,1,1)$ of system (1.1) is zero-Hopf if and only if one of the following sets of conditions hold.*

(i) $\gamma = a_3b_3(b_2-a_1)-a_2b_3^2+a_3^2b_1 \neq 0,\ \ c_3=-a_1-b_2,$

$$c_1 = \frac{1}{\gamma}\left(a_1^3b_3-a_1^2a_3b_1-a_1\left(a_3b_1b_2-b_3\left(2a_2b_1+\beta^2\right)\right)-b_1(a_2(a_3b_1-b_2b_3)+a_3\left(\beta^2+b_2^2\right))\right) \text{ and}$$

$$c_2 = \frac{1}{\gamma}\left(a_1^2a_2b_3+a_1a_2(b_2b_3-a_3b_1)+a_2^2b_1b_3-a_3b_2\left(\beta^2+b_2^2\right)+a_2\left(b_3\left(\beta^2+b_2^2\right)-2a_3b_1b_2\right)\right).$$

(ii) $a_3b_3\neq 0,\ \ a_2=\dfrac{a_3b_2}{b_3},\ \ b_1=\dfrac{a_1b_3}{a_3},\ \ c_3=-a_1-b_2$ *and* $c_2=-\dfrac{(a_1+b_2)^2+a_3c_1+\beta^2}{b_3}.$

(iii) $b_3\neq 0,\ \ a_1=a_2=a_3=0,\ \ c_2=-\dfrac{b_2^2+\beta^2}{b_3}$ *and* $c_3=-b_2.$

(iv) $a_3\neq 0,\ \ b_1=b_2=b_3=0,\ \ c_1=-\dfrac{a_1^2+\beta^2}{a_3}$ *and* $c_3=-a_1.$

(v) $b_1\neq 0,\ \ a_2=-\dfrac{a_1^2+\beta^2}{b_1},\ \ a_3=b_3=c_3=0$ *and* $b_2=-a_1.$

Proposition 1.1 is proved in section 2.

Using the averaging theory of first order in Theorem 3 of [10] are provided sufficient conditions in order that the Kolmogorov systems (1.1) under conditions (i) exhibit a zero-Hopf bifurcation from which two limit cycles bifurcate, the kind of stability or inestability of these limit cycles is also provided. We include the result here for completeness. The expressions of c, d, e, and f, are the ones given in [10], and we denote $k=(2cf+dS_2)^2-8cdfS_1$, $S_2=(c_{31}+2c\beta)/\beta$ and $S_1=(dS_2+cf)/f$.

Theorem 1.2. *If* $\gamma\neq 0$, $c_1=(a_1^3b_3-a_1^2a_3b_1-a_1\left(a_3b_1b_2-b_3\left(2a_2b_1+\beta^2\right)\right)-b_1(a_2(a_3b_1-b_2b_3)+a_3\left(\beta^2+b_2^2\right)))/\gamma$, $c_2=(a_1^2a_2b_3+a_1a_2(b_2b_3-a_3b_1)+a_2^2b_1b_3+a_2\left(b_3\left(\beta^2+b_2^2\right)-2a_3b_1b_2\right)-a_3b_2(\beta^2+b_2^2))/\gamma$, $c_3=-a_1-b_2$, $a_2b_1-a_1b_2\neq 0$, $c\neq 0$, $d\neq 0$, $e\neq 0$, $f\neq 0$, $c_{31}+2c\beta\neq 0$ *and* $ec(c_{31}d+c\beta(2d+f))<0$, *then the Kolmogorov system has two limit cycles bifurcating from the zero-Hopf equilibrium point $(1,1,1)$. Moreover the following statements hold.*

(a) If $S_2S_1<0$, $k>0$ and $|-2cf-dS_2|<\sqrt{k}$, then the two limit cycles are unstable and have a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders.

(b) If $S_2<0$, $S_1<0$ and

- *either $k>0$, $d(-2cf-dS_2+\sqrt{k})>0$ and $d(-2cf-dS_2-\sqrt{k})>0$,*
- *or $k\leq 0$ and $d(-2cf-dS_2)>0$;*

or if $S_2>0$, $S_1>0$ and

- *either* $k > 0$, $d(-2cf - dS_2 + \sqrt{k}) < 0$ *and* $d(-2cf - dS_2 - \sqrt{k}) < 0$,
- *or* $k \leq 0$ *and* $d(-2cf - dS_2) < 0$;

then one limit cycle is a local repeller, and the other is a local attractor.

(c) *If* $S_2 > 0$, $S_1 > 0$, $k > 0$ *and* $|-2cf - dS_2| < \sqrt{k}$; *or if* $S_2 S_1 < 0$ *and*

- *either* $k > 0$, $d(-2cf - dS_2 + \sqrt{k}) > 0$ *and* $d(-2cf - dS_2 - \sqrt{k}) > 0$,
- *or* $k \leq 0$ *and* $d(-2cf - dS_2) > 0$; *then both limit cycles are unstable. One limit cycle is a local repeller, and the other has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders.*

(d) *If* $S_2 < 0$, $S_1 < 0$, $k > 0$ *and* $|-2cf - dS_2| < \sqrt{k}$; *or if* $S_2 S_1 < 0$ *and*

- *either* $k > 0$, $d(-2cf - dS_2 + \sqrt{k}) < 0$ *and* $d(-2cf - dS_2 - \sqrt{k}) < 0$,
- *or* $k \leq 0$ *and* $d(-2cf - dS_2) < 0$;

then one limit cycle is a local attractor, and the other is unstable and has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders.

(e) *If* $S_2 S_1 < 0$ *and* $k < 0$ *and* $-2cf = dS_2$, *then one limit cycle is unstable and has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders and we cannot decide about the stability of the other.*

Here we provide examples showing that all the sets of conditions given in this theorem are non-empty, see section 3. Such examples were not given in [10].

In this paper we use the averaging theory of first order for studying the limit cycles bifurcating from the zero-Hopf bifurcations of the Kolmogorov systems (1.1) under conditions (ii) to (v).

Our main result concerning the Kolmogorov systems (1.1) under the conditions (ii) is the following. The expressions of A_i, with $i = 0, ..., 4$, K_1 and N are defined in Appendix B.

Theorem 1.3. *If* $a_3 b_3 \neq 0$, $N \neq 0$, $a_2 = a_3 b_2 / b_3$, $b_1 = a_1 b_3 / a_3$, $c_3 = -a_1 - b_2$, $c_2 = -((a_1 + b_2)^2 + a_3 c_1 + \beta^2)/b_3$, $A_1 \neq 0$, $A_2 \neq 0$, $A_3 \neq 0$,*and* $A_0 A_4 (A_1 A_2 - A_0 A_3) > 0$, *then the Kolmogorov system* (1.1) *has two limit cycles bifurcating from the zero-Hopf equilibrium point* $(1, 1, 1)$. *Moreover the following statements hold.*

(a) *If* $K_1 > 0$, $A_2 A_3 (A_0 A_3 - A_1 A_2) N < 0$ *and* $|2A_0 A_3 - A_1 A_2| < \sqrt{K_1}$, *then the two limit cycles have a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders.*

(b) *If* $b_3 A_2 N > 0$, $b_3 A_3 (A_0 A_3 - A_1 A_2) > 0$ *and*

- *either* $K_1 > 0$, $b_3 A_1 N (2A_0 A_3 - A_1 A_2 - \sqrt{K_1}) < 0$ *and* $b_3 A_1 N (2A_0 A_3 - A_1 A_2 + \sqrt{K_1}) < 0$,
- *or* $K_1 \leq 0$ *and* $b_3 A_1 N (2A_0 A_3 - A_1 A_2) < 0$;

or if $b_3 A_2 N < 0$, $b_3 A_3 (A_0 A_3 - A_1 A_2) < 0$ *and*

- *either* $K_1 > 0$, $b_3 A_1 N (2A_0 A_3 - A_1 A_2 - \sqrt{K_1}) > 0$ *and* $b_3 A_1 N (2A_0 A_3 - A_1 A_2 + \sqrt{K_1}) > 0$,
- *or* $K_1 \leq 0$ *and* $b_3 A_1 N (2A_0 A_3 - A_1 A_2) > 0$;

then one limit cycle is local repeller, and the other is a local attractor.

(c) *If* $b_3 A_2 N > 0$, $b_3 A_3 (A_0 A_3 - A_1 A_2) > 0$, $K_1 > 0$ *and* $|2A_0 A_3 - A_1 A_2| < \sqrt{K_1}$; *or if* $A_2 A_3 (A_0 A_3 - A_1 A_2) N < 0$ *and*

- *either* $K_1 > 0$, $b_3 A_1 N (2A_0 A_3 - A_1 A_2 - \sqrt{K_1}) > 0$ *and* $b_3 A_1 N (2A_0 A_3 - A_1 A_2 + \sqrt{K_1}) > 0$,
- *or* $K_1 \leq 0$ *and* $b_3 A_1 (2A_0 A_3 - A_1 A_2) N > 0$;

then both limit cycles are unstable. One limit cycle is a local repeller, and the other has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders.

(d) *If* $b_3A_2N < 0$, $b_3A_3(A_0A_3 - A_1A_2) < 0$, $K_1 > 0$ *and* $|2A_0A_3 - A_1A_2| < \sqrt{K_1}$*; or if* $A_2A_3(A_0A_3 - A_1A_2)N < 0$ *and*

- *either* $K_1 > 0$, $b_3A_1N(2A_0A_3 - A_1A_2 - \sqrt{K_1}) < 0$ *and* $b_3A_1N(2A_0A_3 - A_1A_2 + \sqrt{K_1}) < 0$,
- *or* $K_1 \leq 0$ *and* $b_3A_1(2A_0A_3 - A_1A_2)N < 0$*;*

then one limit cycle is a local attractor, and the other is unstable and has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders.

(e) *If* $K_1 < 0$, $A_2A_3(A_0A_3 - A_1A_2)N < 0$ *and* $2A_0A_3 = A_1A_2$*; then one limit cycle is unstable and has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders and we cannot decide about the stability of the other.*

The main result concerning the Kolmogorov systems (1.1) under the conditions (iii) is the following. The expressions of B_i with $i = 0, ..., 4$, and K_2 are given in Appendix B.

Theorem 1.4. *If* $b_3 \neq 0$, $a_1 = a_2 = a_3 = 0$, $c_2 = -(b_2^2 + \beta^2)/b_3$, $c_3 = -b_2$, $B_1 \neq 0$, $B_2 \neq 0$, $B_3 \neq 0$ *and* $B_0B_4(B_1B_2 - B_0B_3) > 0$*, then the Kolmogorov system* (1.1) *has two limit cycles bifurcating from the zero-Hopf equilibrium point* (1, 1, 1)*. Moreover the following statements hold.*

(a) *If* $K_2 > 0$, $B_2B_3(B_0B_3 - B_1B_2) > 0$ *and* $|B_1B_2 - 2B_0B_3| < \sqrt{K_2}$*; then the two limit cycles are unstable and have a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders.*

(b) *If* $B_2 > 0$, $B_3(B_0B_3 - B_1B_2) < 0$ *and*

- *either* $K_2 > 0$, $B_1 > 0$ *and* $B_1B_2 - 2B_0B_3 < -\sqrt{K_2}$,
- *or* $K_2 \leq 0$ *and* $B_1(B_1B_2 - 2B_0B_3) < 0$*;*

then the two limit cycles are local attractors.

(c) *If* $B_2 < 0$, $B_3(B_0B_3 - B_1B_2) > 0$, $K_2 > 0$ *and* $|B_1B_2 - 2B_0B_3| < \sqrt{K_2}$*; or if* $B_2B_3(B_0B_3 - B_1B_2) > 0$, $K_2 > 0$, $B_1 < 0$ *and* $B_1B_2 - 2B_0B_3 < -\sqrt{K_2}$*; then both limit cycles are unstable. One limit cycle is a local repeller, and the other has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders.*

(d) *If* $B_2 < 0$, $B_3(B_0B_3 - B_1B_2) > 0$ *and*

- *either* $K_2 > 0$, $B_1(B_1B_2 - 2B_0B_3 - \sqrt{K_2}) < 0$ *and* $B_1(B_1B_2 - 2B_0B_3 + \sqrt{K_2}) < 0$,
- *or* $K_2 \leq 0$ *and* $B_1(B_1B_2 - 2B_0B_3) < 0$*;*

or if $B_2 > 0$, $B_3(B_0B_3 - B_1B_2) < 0$, $K_2 > 0$, $B_1 < 0$ *and* $B_1B_2 - 2B_0B_3 < -\sqrt{K_2}$*; then one limit cycle is a local attractor and the other limit cycle is a local repeller.*

(e) *If* $B_2 > 0$, $B_3(B_0B_3 - B_1B_2) < 0$, $K_2 > 0$ *and* $|B_1B_2 - 2B_0B_3| < \sqrt{K_2}$*; or if* $B_2B_3(B_0B_3 - B_1B_2) > 0$ *and*

- *either* $K_2 > 0$, $B_1(B_1B_2 - 2B_0B_3 - \sqrt{K_2}) < 0$ *and* $B_1(B_1B_2 - 2B_0B_3 + \sqrt{K_2}) < 0$,
- *or* $K_2 \leq 0$ *and* $B_1(B_1B_2 - 2B_0B_3) < 0$,

then one limit cycle is a local attractor and the other limit cycle is unstable and has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders.

(f) $B_2B_0 < 0$, $K_2 < 0$ *and* $B_1B_2 = 2B_0B_3$*; then one limit cycle is unstable and has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders and we cannot decide about the stability of the other.*

Theorems 1.3 and 1.4 are proved in section 2. Examples showing that the conditions provided by both theorems are non-empty are given in section 3.

Kolmogorov systems (1.1) under conditions (iv) are the same as under conditions (iii) but interchanging the variables x and y, so if we change the conditions $b_3 \neq 0$, $a_1 = a_2 = a_3 = 0$, $c_2 = -(b_2^2 + \beta^2)/b_3$, $c_3 = -b_2$ into $a_3 \neq 0$, $b_1 = b_2 = b_3 = 0$, $c_1 = -(a_1^2 + \beta^2)/a_3$, $c_3 = -a_1$, and redefine the constants B_i for $i = 0, ..., 4$ as it is indicated in Appendix B the same Theorem 1.4 holds.

At last our main result concerning the Kolmogorov systems (1.1) under the conditions (v) is the following, with the expressions of D_i, for $i = 0, ..., 4$, and K_4 given in Appendix B.

Theorem 1.5. *If $b_1 \neq 0$, $a_3 = b_3 = c_3 = 0$, $a_2 = -(a_1^2 + \beta^2)/b_1$, $b_2 = -a_1$, $D_1 \neq 0$, $D_2 \neq 0$, $D_3 \neq 0$ and $D_0D_4(D_1D_2 - D_0DB_3) > 0$, then the Kolmogorov system (1.1) has two limit cycles bifurcating from the zero-Hopf equilibrium point $(1, 1, 1)$. Moreover the following statements hold.*

(a) If $K_4 > 0$, $D_2D_3(a_1c_1 + b_1c_2)(D_0D_3 - D_1D_2) > 0$ and $|D_1D_2 - 2D_0D_3| < \sqrt{K_4}$; then the two limit cycles are unstable and have a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders.

(b) If $b_1D_2(a_1c_1 + b_1c_2) < 0$, $b_1D_3(D_0D_3 - D_1D_2) > 0$ and

- *either $K_4 > 0$, $b_1D_1(a_1c_1 + b_1c_2)(D_1D_2 - 2D_0D_2 - \sqrt{K_4}) < 0$ and $b_1D_1(a_1c_1 + b_1c_2)(D_1D_2 - 2D_0D_2 + \sqrt{K_4}) < 0$,*
- *or $K_4 \leq 0$ and $b_1D_1(a_1c_1 + b_1c_2)(D_1D_2 - 2D_0D_3) < 0$;*

or if $b_1D_2(a_1c_1 + b_1c_2) > 0$, $b_1D_3(D_0D_3 - D_1D_2) < 0$ and

- *either $K_4 > 0$, $b_1D_1(a_1c_1 + b_1c_2)(D_1D_2 - 2D_0D_2 - \sqrt{K_4}) > 0$ and $b_1D_1(a_1c_1 + b_1c_2)(D_1D_2 - 2D_0D_2 + \sqrt{K_4}) > 0$,*
- *or $K_4 \leq 0$ and $b_1D_1(a_1c_1 + b_1c_2)(D_1D_2 - 2D_0D_3) > 0$;*

then one limit cycle is a local repeller, and the other is a local attractor.

(c) If $b_1D_2(a_1c_1 + b_1c_2) < 0$, $b_1D_3(D_0D_3 - D_1D_2) > 0$, $K_4 > 0$ and $|D_1D_2 - 2D_0D_3| < \sqrt{K_4}$; or if $D_2D_3(a_1c_1 + b_1c_2)(D_0D_3 - D_1D_2) > 0$ and

- *either $K_4 > 0$, $b_1D_1(a_1c_1 + b_1c_2)(D_1D_2 - 2D_0D_2 - \sqrt{K_4}) > 0$ and $b_1D_1(a_1c_1 + b_1c_2)(D_1D_2 - 2D_0D_2 + \sqrt{K_4}) > 0$,*
- *or $K_4 \leq 0$ and $b_1D_1(a_1c_1 + b_1c_2)(D_1D_2 - 2D_0D_3) > 0$;*

then both limit cycles are unstable. One limit cycle is a local repeller, and the other has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders.

(d) If $b_1D_2(a_1c_1 + b_1c_2) > 0$, $b_1D_3(D_0D_3 - D_1D_2) < 0$ $K_4 > 0$ and $|D_1D_2 - 2D_0D_3| < \sqrt{K_4}$; or if $D_2D_3(a_1c_1 + b_1c_2)(D_0D_3 - D_1D_2) > 0$ and

- *either $K_4 > 0$, $b_1D_1(a_1c_1 + b_1c_2)(D_1D_2 - 2D_0D_2 - \sqrt{K_4}) < 0$ and $b_1D_1(a_1c_1 + b_1c_2)(D_1D_2 - 2D_0D_2 + \sqrt{K_4}) < 0$,*
- *or $K_4 \leq 0$ and $b_1D_1(a_1c_1 + b_1c_2)(D_1D_2 - 2D_0D_3) < 0$;*

then one limit cycle is a local attractor, and the other is unstable and has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders.

(e) If $D_2D_3(a_1c_1 + b_1c_2)(D_0D_2 - D_1D_2) > 0$, $K_4 < 0$ and $D_1D_2 = 2D_0D_3$; then one limit cycle is unstable and has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders and we cannot decide about the stability of the other.

Theorem 1.5 is proved in section 2. In section 3 can be found examples showing that the conditions provided by this theorem are non-empty.

2. Proof of results

Proof of Proposition 1.1. We want to characterize when the singular point $(1,1,1)$ of system (1.1) is a zero-Hopf equilibrium. At first, through the change of variables $(x,y,z) \to (x+1,y+1,z+1)$, we translate the point $(1,1,1)$ to the origin of coordinates, obtaining the system:

$$\begin{aligned}
\dot{x} &= (1+x)(a_1 z + a_2 y + a_3 z + a_4 x^2 + a_5 xy + a_6 xz + a_7 y^2 + a_8 yz + a_9 z^2),\\
\dot{y} &= (1+y)(b_1 x + b_2 y + b_3 z + b_4 x^2 + b_5 xy + b_6 xz + b_7 y^2 + b_8 yz + b_9 z^2),\\
\dot{z} &= (1+z)(c_1 x + c_2 y + c_3 z + c_4 x^2 + c_5 xy + c_6 xz + c_7 y^2 + c_8 yz + c_9 z^2).
\end{aligned} \tag{2.1}$$

In order that the origin of system (2.1) can exhibit a zero-Hopf bifurcation we must require that the eigenvalues of the linear part of the system at the origin be of the form 0 and $\pm\beta i$ with $\beta > 0$. We compute the characteristic polynomial and require that it has the form $\lambda(\lambda^2+\beta^2)$. Solving the resultant equation we get the five solutions given in (i)–(v). □

Proof of Theorem 1.3. We consider system (1.1) under conditions (ii) of Proposition 1.1, and we proceed to study the limit cycles bifurcating from the zero-Hopf equilibrium point, applying the averaging theory of first order, summarized in Theorem A.1 of Appendix A. To do so we perturb the parameters a_2, b_1, c_2 and c_3 which define the zero-Hopf equilibrium under the assumption (ii) as follows

$$a_2 = \frac{a_3 b_2}{b_3} + \varepsilon a_{21}, \qquad b_1 = \frac{a_1 b_3}{a_3} + \varepsilon b_{11}, c_2 = -\frac{(a_1+b_2)^2 + a_3 c_1 + \beta^2}{b_3} + \varepsilon c_{21}, \qquad c_3 = -a_1 - b_2 + \varepsilon c_{31},$$

where ε is a small parameter.

We write the lineal part of system (2.1) at the origin in its real Jordan normal form, and the associated system becames system (1) in file ss[[2]].

Now we want to write the system in such a way that conditions of Theorem A.1 are satisfied. For this we write the system in cylindrical coordinates by means of the change of variables $(X,Y,Z) \to (r\cos\theta, r\sin\theta, Z)$ obtaining system (2) of file ss[[2]].

In order to study the periodic solutions in a neigborhood of the origin, i.e. in a neigborhood of the zero-Hopf equilibrium, we do the scaling $(r,Z) \to (\varepsilon R, \varepsilon Z)$, where $\varepsilon > 0$ is the same parameter used before. We obtain system (3) of file ss[[2]].

We take the variable θ as the new independent variable and so we obtain the system

$$R' = \varepsilon F_{11} + O(\varepsilon^2), \qquad Z' = \varepsilon F_{12} + O(\varepsilon^2), \tag{2.2}$$

with coefficients F_{11} and F_{12} given in the file ss[[2]].

Note that system (2.2) is in the normal form (A.1), so we can apply the averaging theory with $T = 2\pi$, $x = (R,Z)$, $t = \theta$ and $\varepsilon R(\theta,x,\varepsilon) = O(\varepsilon^2)$. The functions F_{11}, F_{12} and R are $\mathcal{C}^2$ in x and 2π-periodic in θ. Applying Theorem A.1 we compute the averaging function of first order $f_1 = (f_{11}(R,Z), f_{12}(R,Z))$, and we obtain

$$f_{11} = \frac{\pi R(A_0 + A_1 Z)}{a_3^2 b_3 \beta^5}, \qquad f_{12} = -\frac{\pi(A_2 Z + A_3 Z^2 + A_4 R^2)}{a_3^2 b_3 \beta^5 N},$$

and A_i, for $i = 0,...,4$, K_1 and N are given in Appendix B.

We look for the isolated solutions of the equation $(f_{11}(R,Z), f_{12}(R,Z)) = (0,0)$, and we obtain, appart from the origin, $(R_1,Z_1) = (0, -A_2/A_3)$ and $(R_2,Z_2) = \left(\pm\sqrt{A_0(A_1A_2 - A_0A_3)}/(A_1\sqrt{A_4}), -A_0/A_1\right)$. We consider always the positive expression of R_2, i.e. we consider the positive sign if $A_1 > 0$ and the negative sign if $A_1 < 0$.

We compute the Jacobian matrix of f_1, which is

$$\begin{pmatrix}
\dfrac{\pi(A_0 + A_1 Z)}{a_3^2 b_2 \beta^5} & \dfrac{\pi R A_1}{a_3^2 b_3 \beta^5}\\[2ex]
-\dfrac{2\pi R A_4}{a_3^2 b_3 \beta^5 N} & -\dfrac{\pi(A_2 + 2A_3 Z)}{a_3^2 b_3 \beta^5 N}
\end{pmatrix},$$

and its determinant is $\pi^2(-2A_1A_4R^2+(A_0+A_1Z)(A_2+2A_3Z))/(a_3^4b_3^2\beta^{10}N)$. Evaluating the determinant at the solution (R_1,Z_1) we get that it is equal to $\pi^2A_2(A_0A_3-A_1A_2)/(a_3^4b_3^2A_3\beta^{10}N)$, and at the solutions (R_2,Z_2) we get that it is equal to $2\pi^2A0(A_1A_2-A_0A_3)/(a_3^4b_3^2A_1\beta^{10}N)$.

From the hypothesis considered these determinants are nonzero, therefore it follows from Theorem A.1 that for ε sufficiently small system (2.2) has two 2π-periodic solutions $(R_1(\theta,\varepsilon),Z_1(\theta,\varepsilon))$ and $(R_2(\theta,\varepsilon),Z_2(\theta,\varepsilon))$ such that $(R_j(\theta,\varepsilon),Z_j(\theta,\varepsilon))\to(R_j,Z_j)$ for $j=1,2$ when $\varepsilon\to 0$.

Moreover the Jacobian matrix evaluated at the solution (R_1,W_1) has eigenvalues equal to $\pi A_2/(a_3^2b_3\beta^5N)$ and $\pi(A_0A_3-A_1A_2)/(a_3^2b_3A_3\beta^5)$. Since the eigenvalues of the Jacobian matrix evaluated at the solutions provide the stability of the fixed point corresponding to the Poincaré map defined in a neighborhood of the solution, if $A_2b_3N>0$ and $A_3b_3(A_0A_3-A_1A_2)>0$, then the fixed point of the Poincaré map has an unstable manifold of dimension two, and the corresponding periodic solution is unstable and has an unstable manifold of dimension three, which is equivalent to say that is a repelling periodic orbit. If $A_2b_3N<0$ and $A_3b_3(A_0A_3-A_1A_2)<0$, then the fixed point of the Poincaré map has a stable manifold of dimension two, and the associated periodic solution is stable and has a stable manifold of dimension three, which is equivalent to say that is a attracting periodic orbit. Finally, if $A_2A_3N(A_0A_3-A_1A_2)<0$, then the fixed point of the Poincaré application is a saddle point with a stable manifold of degree one and an unstable manifold of degree one, and the corresponding periodic solution is unstable and has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders.

On the other hand, the Jacobian matrix evaluated at (R_2,Z_2) has eigenvalues equal to $\pi(2A_0A_3-A_1A_2\pm\sqrt{K_1})/(2a_3^2b_3A_1\beta^5N)$, and so its stability is as follows. If $K_1>0$, $b_3A_1N(2A_0A_3-A_1A_2+\sqrt{K_1})>0$ and $b_3A_1N(2A_0A_3-A_1A_2-\sqrt{K_1})>0$ or if $K_1<0$ and $b_3A_1N(2A_0A_3-A_1A_2)>0$, then the fixed point of the Poincaré map has an unstable manifold of dimension two, and the periodic solution is unstable and has an unstable manifold of dimension three. If $K_1>0$, $b_3A_1N(2A_0A_3-A_1A_2+\sqrt{K_1})<0$ and $b_3A_1N(2A_0A_3-A_1A_2\sqrt{K_1})<0$ or if $K_1<0$ and $b_3A_1N(2A_0A_3-A_1A_2)<0$, then the fixed point of the Poincaré map has an unstable manifold of dimension two, and the periodic solution is stable and has a stable manifold of dimension three. If $K_1>0$ and $-\sqrt{K_1}<2A_0A_3-A_1A_2<\sqrt{K_1}$, then the fixed point of the Poincaré map is a saddle point with a stable manifold of degree one and an unstable manifold of degree one, and the associated periodic solution is unstable and has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders. If $K_1<0$ and $A_1A_2=2A_0A_3$, the fixed point of the Poincaré map asociated with the periodic orbit is linearly stable, and we cannot decide about the stability of the periodic orbit.

Combining the above information of the eigenvalues of the Jacobian matrix for both (R_1,Z_1) and (R_2,Z_2) we get statements (a)–(e) in the theorem.

Now we shall go back through the changes of variables and we obtain two periodic solutions, for $j=1,2$, $(x_j(t,\varepsilon),y_j(t,\varepsilon),z_j(t,\varepsilon))$ bifurcating from $(1,1,1)$ with a period tending to 2π when $\varepsilon\to 0$. Moreover, $(x_j(t,\varepsilon),y_j(t,\varepsilon),z_j(t,\varepsilon))=(1,1,1)+O(\varepsilon)$ for $j=1,2$. This completes the proof of the theorem. □

Proof of Theorem 1.4. We consider system (1.1) under conditions (iii) of Proposition 1.1. In order to study the zero-Hopf bifurcation we perturb the parameters a_1, a_2, a_3, c_2 and c_3 which define the zero-Hopf equilibrium under conditions (iii) as follows

$$a_1=\varepsilon a_{11},\quad a_2=\varepsilon a_{21},\quad a_3=\varepsilon a_{31},\quad c_2=-\frac{b_2^2+\beta^2}{b_3}+\varepsilon c_{21},\quad c_3=-b_2+\varepsilon c_{31},$$

where ε is a parameter to be taken sufficiently small.

We write the lineal part of system (2.1) at the origin in its real Jordan normal form, and the associated system becomes system (1) of file ss[[3]]. Then we write the system in cylindrical coordinates obtaining system (2) of file ss[[3]], and we do the scaling $(r,Z)\to(\varepsilon R,\varepsilon Z)$ obtaining system (3) in file ss[[3]].

As in the proof of Theorem 1.3 in order to apply Theorem A.1, we take the variable θ as the new independent variable obtaining a system

$$R'=\varepsilon F_{11}+O(\varepsilon^2),\quad Z'=\varepsilon F_{12}+O(\varepsilon^2) \tag{2.3}$$

which coefficients F_{11} and F_{12} are given in the file ss[[3]]. The averaged function of first order $f_1=(f_{11}(R,Z),f_{12}(R,Z))$ is

$$f_{11}=\frac{\pi R(B_0+B_1Z)}{b_3^2\beta^5},\qquad f_{12}=\frac{\pi(B_2Z+B_3Z^2+B_4R^2)}{b_3^2\beta^5},$$

with B_i, for $i = 0, ..., 4$, and K_2 given in Appendix B.Solving the equation $(f_{11}(R, Z), f_{12}(R, Z)) = (0, 0)$ we obtain two solutions $(R_1, Z_1) = (0, -B_2/B_3)$ and $(R_2, Z_2) = \left(\pm\sqrt{B_0(B_1B_2 - B_0B_3)}/(B_1\sqrt{B_4}), -B_0/B_1\right)$. Again we consider always the positive expression of R_2.

We compute the Jacobian matrix of f_1 and we get

$$\begin{pmatrix} \dfrac{\pi(B_0 + B_1Z)}{b_3^2\beta^5} & \dfrac{\pi RB_1}{b_3^2\beta^5} \\ \dfrac{2\pi RB_4}{b_3^2\beta^5} & \dfrac{\pi(B_2 + 2B_3Z)}{b_3^2\beta^5} \end{pmatrix},$$

whose determinant is $\pi^2(-2B_1B_4R^2 + (B_0 + B_1Z)(B_2 + 2B_3Z))/(b_3^4\beta^{10})$. The determinant at the solution (R_1, Z_1) is $\pi^2B_2(B_1B_2 - B_0B_3)/(b_3^4B_3\beta^{10})$, and at the solutions (R_2, Z_2) is $2\pi^2B_0(B_0B_3 - B_1B_2)/(b_3^4B_1\beta^{10})$.

From the hypothesis considered these determinants are nonzero, so it follows from Theorem A.1 that for ε sufficiently small, system (2.2) has two solutions $(R_1(\theta, \varepsilon), Z_1(\theta, \varepsilon))$ and $(R_2(\theta, \varepsilon), Z_2(\theta, \varepsilon))$ such that $(R_j(\theta, \varepsilon), Z_j(\theta, \varepsilon)) \to (R_j, Z_j)$ for $j = 1, 2$ when $\varepsilon \to 0$.

The Jacobian matrix evaluated at the solution (R_1, W_1) has eigenvalues equal to $-\pi B_2/(b_3^2\beta^5)$ and $\pi(B_0B_3 - B_1B_2)/(b_3^2B_3\beta^5)$. We study the stability of the associated periodic orbit which is provided by these eigenvalues. If $B_2 < 0$ and $B_3(B_0B_3 - B_1B_2) > 0$, the associated periodic solution is unstable and has an unstable manifold of dimension three. If $B_2 > 0$ and $B_3(B_0B_3 - B_1B_2) < 0$, then the associated periodic solution is stable and has a stable manifold of dimension three. Finally if $B_2B_3(B_0B_3 - B_1B_2) > 0$, the periodic solution is unstable and has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders.

On the other hand, the Jacobian matrix evaluated at (R_2, Z_2) has eigenvalues equal to $\pi(B_1B_2 - 2B_0B_3 \pm \sqrt{K_2})/(2b_3^2B_1\beta^5)$, and so if $K_2 > 0$, $B_1(B_1B_2 - 2B_0B_3 + \sqrt{K_2}) > 0$ and $B_1(B_1B_2 - 2B_0B_3 - \sqrt{K_2}) > 0$, or if $K_2 < 0$ and $B_1(B_1B_2 - 2B_0B_3) > 0$, then the associated periodic solution is unstable and has an unstable manifold of dimension three. If $K_2 > 0$, $B_1(B_1B_2 - 2B_0B_3 + \sqrt{K_2}) < 0$ and $B_1(B_1B_2 - 2B_0B_3 - \sqrt{K_2}) < 0$, or if $K_2 < 0$ and $B_1(B_1B_2 - 2B_0B_3) < 0$, then the associated periodic solution is stable and has a stable manifold of dimension three. If $K_2 > 0$ and $-\sqrt{K_2} < B_1B_2 - 2B_0B_3 < \sqrt{K_2}$, then the associated periodic solution is unstable and has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders. If $K_2 < 0$ and $B_1B_2 - 2B_0B_3 = 0$, the fixed point of the Poincaré map associated with the periodic orbit is linearly stable, and we cannot decide about the stability of the periodic orbit.

Combining the above information of the eigenvalues of the Jacobian matrix for both (R_1, Z_1) and (R_2, Z_2) we get statements (a)–(f) of the theorem.

Now we shall go back through the changes of variables and we obtain two periodic solutions,for $j = 1, 2$, $(x_j(t, \varepsilon), y_j(t, \varepsilon), z_j(t, \varepsilon))$ bifurcating from $(1, 1, 1)$ with a period tending to 2π when $\varepsilon \to 0$. Moreover, $(x_j(t, \varepsilon), y_j(t, \varepsilon), z_j(t, \varepsilon)) = (1, 1, 1) + O(\varepsilon)$ for $j = 1, 2$. This completes the proof of the theorem. □

Proof of Theorem 1.5. We consider system (1.1) under conditions (v) of Proposition 1.1. In order to study the zero-Hopf bifurcation we perturb the parameters a_2, a_3, b_2, b_3 and c_3 which define the zero-Hopf equilibrium point into the form

$$a_2 = -\frac{a_1^2 + \beta^2}{b_1} + \varepsilon a_{21}, \quad a_3 = \varepsilon a_{31}, \quad b_2 = -a_1 + \varepsilon b_{21}, \quad b_3 = \varepsilon b_{31} \quad c_3 = \varepsilon c_{31}$$

where ε is a sufficiently small parameter.

We write the system with the linear part at the origin in its real Jordan normal form, then we write it in cylindrical coordinates, and finally we do the scaling $(r, Z) \to (\varepsilon R, \varepsilon Z)$. Thus we obtain respectively systems (1), (2) and (3) of file ss[[5]]. Taking θ as the new independent variable we obtain a system in the form

$$R' = \varepsilon F_{11} + O(\varepsilon^2), \quad Z' = \varepsilon F_{12} + O(\varepsilon^2) \tag{2.4}$$

with coefficients F_{11} and F_{12} given in the file ss[[5]]. As in previous proofs we are in conditions to apply Theorem A.1. Now the averaged function of first order $f_1 = (f_{11}(R, Z), f_{12}(R, Z))$ is

$$f_{11} = \frac{\pi R(D_0 + D_1Z)}{b_1\beta^5}, \qquad f_{12} = \frac{\pi(D_2Z + D_3Z^2 + D_4R^2)}{b_1(a_1c_1 + b_1c_2)\beta^5},$$

with D_i, for $i = 0, ..., 4$, and K_4 given in Appendix B. We look for the solutions of $(f_{11}(R, Z), f_{12}(R, Z)) = (0, 0)$, and we obtain $(R_1, Z_1) = (0, -D_2/D_3)$ and $(R_2, Z_2) = \left(\pm\sqrt{D_0(D_1D_2 - D_0D_3)}/(D_1\sqrt{D_4}), -D_0/D_1\right)$, considering the positive expression of R_2.

We compute the Jacobian matrix of f_1 and we get

$$\begin{pmatrix} \dfrac{\pi(D_0 + D_1Z)}{b_1\beta^5} & \dfrac{\pi RD_1}{b_1\beta^5} \\ \dfrac{2\pi RD_4}{b_1\beta^5(a_1c_1 + b_1c_2)} & \dfrac{\pi(D_2 + 2D_3Z)}{b_1\beta^5(a_1c_1 + b_1c_2)} \end{pmatrix},$$

whose determinant is $\pi^2(-2D_1D_4R^2 + (D_0 + D_1Z)(D_2 + 2D_3Z))/(b_1^2\beta^{10}(a_1c_1 + b_1c_2))$. The determinant at the solution (R_1, Z_1) is $\pi^2D_2(D_1D_2 - D_0D_3)/(b_1^2D_3\beta^{10}\ (a_1c_1 + b_1c_2))$, and at the solution (R_2, Z_2) is $2\pi^2D_0(D_0D_3 - D_1D_2)/(b_1^2D_1\beta^{10}(a_1c_1 + b_1c_2))$, and both are nonzero by the hypotheses. By Theorem A.1, for ε sufficiently small system (2.4) has two solutions $(R_1(\theta, \varepsilon), Z_1(\theta, \varepsilon))$ and $(R_2(\theta, \varepsilon), Z_2(\theta, \varepsilon))$ such that $(R_j(\theta, \varepsilon), Z_j(\theta, \varepsilon)) \to (R_j, Z_j)$ for $j = 1, 2$ when $\varepsilon \to 0$.

The Jacobian matrix evaluated at the solution (R_1, W_1) has eigenvalues equal to $-\pi D_2/(b_1\beta^5(a_1c_1+b_1c_2))$ and $\pi(D_0D_3 - D_1D_2)/(b_1D_3\beta^5)$. We study the stability of the associated periodic orbit which is provided by these eigenvalues. If $b_1D_2(a_1c_1 + b_1c_2) < 0$ and $b_1D_3(D_0D_3 - D_1D_2) > 0$, the associated periodic solution is unstable and has an unstable manifold of dimension three. If $b_1D_2(a_1c_1+b_1c_2) > 0$ and $b_1D_3(D_0D_3-D_1D_2) < 0$, then the associated periodic solution is stable and has a stable manifold of dimension three. Finally if $D_2D_3(a_1c_1 + b_1c_2)(D_0D_3 - D_1D_2) > 0$, the associated periodic solution is unstable and has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders.

On the other hand, the Jacobian matrix evaluated at (R_2, Z_2) has eigenvalues equal to $\pi(D_1D_2-2D_0D_3 \pm \sqrt{K_4})/(2b_1D_1\beta^5(a_1c_1+b_1c_2))$. Then if $K_4 > 0$, $b_1D_1(a_1c_1+b_1c_2)(D_1D_2-2D_0D_3+\sqrt{K_4}) > 0$ and $b_1D_1(a_1c_1+b_1c_2)(D_1D_2-2D_0D_3-\sqrt{K_4}) > 0$, or if $K_4 < 0$ and $b_1D_1(a_1c_1+b_1c_2)(D_1D_2-2D_0D_3) > 0$, then the associated periodic solution is unstable and has an unstable manifold of dimension three. If $K_4 > 0$, $b_1D_1(a_1c_1 + b_1c_2)(D_1D_2 - 2D_0D_3 + \sqrt{K_4}) < 0$ and $b_1D_1(a_1c_1 + b_1c_2)(D_1D_2 - 2D_0D_3 - \sqrt{K_4}) < 0$, or if $K_4 < 0$ and $b_1D_1(a_1c_1 + b_1c_2)(D_1D_2 - 2D_0D_3) < 0$, then the associated periodic solution is stable and has a stable manifold of dimension three. If $K_4 > 0$ and $-\sqrt{K_4} < D_1D_2 - 2D_0D_3 < \sqrt{K_4}$, then the associated periodic solution is unstable and has a stable manifold formed by two cylinders and an unstable manifold formed by two cylinders. If $K_4 < 0$ and $D_1D_2 = 2D_0D_3$, the fixed point of the Poincaré map associated with the periodic orbit is linearly stable, and we cannot decide about the stability of the periodic orbit.

Combining the above information we get statements (a)–(e) in the theorem, and going back through the changes of variables we obtain two periodic solutions $(x_j(t, \varepsilon), y_j(t, \varepsilon), z_j(t, \varepsilon))$ for $j = 1, 2$ bifurcating from $(1, 1, 1)$ with a period tending to 2π when $\varepsilon \to 0$. Moreover, $(x_j(t, \varepsilon), y_j(t, \varepsilon), z_j(t, \varepsilon)) = (1, 1, 1) + O(\varepsilon)$ for $j = 1, 2$. This completes the proof of the theorem. □

3. Examples

3.1. Examples of Theorem 1.2

We give examples showing that the conditions provided by Theorem 1.2 are non-empty. The system

$$\begin{aligned} \dot{x} &= x\left(1 - 2(x-1) + \frac{11032749}{65536}(x-1)^2 - 6(y-1) + (x-1)(y-1) - z\right), \\ \dot{y} &= y\left(1 + 5(x-1) - \frac{193}{16}(y-1) - z\right), \\ \dot{z} &= z\left(-\frac{1048639}{128}(x-1)^2 - \frac{94765}{768}(y-1) + (z-1)\left(\frac{225}{16} + \varepsilon\right) + (x-1)\left(\frac{12865}{48} + \frac{279}{32}\varepsilon\right)\right), \end{aligned}$$

has two limit cycles with the stability given in (a) of Theorem 1.2. The following systems verify, respectively, the four sets of conditions of statement (b) of Theorem 1.2, so all of them have two limit cycles with the stability given in (b).

$$
\begin{aligned}
\dot{x} &= x\left(1+2(x-1)^2-x-2(y-1)+2(x-1)(y-1)+2(y-1)^2-2(z-1)+2(x-1)(z-1)\right.\\
&\quad \left.+(y-1)(z-1)\right),\\
\dot{y} &= y\left(-2+2(x-1)+y+z\right),\\
\dot{z} &= z\left(-\frac{11}{8}(x-1)+(y-1)^2-(z-1)^2-(y-1)\varepsilon+(z-1)\varepsilon\right),
\end{aligned}
$$

$$
\begin{aligned}
\dot{x} &= x\left(-1+2(x-1)^2+x-2(y-1)+2(y-1)^2+2(z-1)+2(x-1)(z-1)\right),\\
\dot{y} &= y\left(-2+x+z\right),\\
\dot{z} &= z\left(-1+(y-1)^2+y-(z-1)^2+(x-1)\left(-\frac{1}{2}-2\varepsilon\right)+(z-1)(-1-\varepsilon)\right),
\end{aligned}
$$

$$
\begin{aligned}
\dot{x} &= x\left(1+2(x-1)+\frac{1}{2}(y-1)+(y-1)^2-100(x-1)(z-1)-z\right),\\
\dot{y} &= y\left(-4(x-1)-\frac{10}{9}(y-1)-2(z-1)\right),\\
\dot{z} &= z\left(\frac{668}{495}(x-1)+(y-1)\left(\frac{2953}{8910}+\varepsilon\right)+(z-1)\left(-\frac{8}{9}+5\varepsilon\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
\dot{x} &= x\left(1+2(x-1)+\frac{1}{2}(y-1)+2(y-1)^2-100(x-1)(z-1)-z\right),\\
\dot{y} &= y\left(-4(x-1)-\frac{10}{9}(y-1)-2(z-1)\right),\\
\dot{z} &= z\left(\frac{2953}{8910}(y-1)+(x-1)\left(\frac{668}{495}-2\varepsilon\right)+(z-1)\left(-\frac{8}{9}+2\varepsilon\right)\right),
\end{aligned}
$$

The following systems verify, respectively, the three sets of conditions in statement (c) of Theorem 1.2, so all of them have two limit cycles with the stability given in (c).

$$
\begin{aligned}
\dot{x} &= x\left(-2+(x-1)^2+x+z\right),\\
\dot{y} &= y\left(-3+(x-1)^2+x+(x-1)(y-1)+y+z\right),\\
\dot{z} &= z\left((x-1)(y-1)+(x-1)(-2+\varepsilon)+(y-1)(-2+\varepsilon)+(z-1)(-2+\varepsilon)\right),
\end{aligned}
$$

$$
\begin{aligned}
\dot{x} &= x\left(2(x-1)^2+x+2(x-1)(y-1)+2(y-1)^2-y+2(z-1)+2(x-1)(z-1)+2(y-1)(z-1)\right),\\
\dot{y} &= y\left(-3+x+y+z\right),\\
\dot{z} &= z\left(\frac{1}{5}(y-1)+(y-1)^2-(z-1)^2+(x-1)\left(-\frac{7}{5}-2\varepsilon\right)+(z-1)(-2-\varepsilon)\right),
\end{aligned}
$$

$$
\begin{aligned}
\dot{x} &= x\left(-1+x+(y-1)^2-y+(z-1)^2+z\right),\\
\dot{y} &= y\left(-2+x+y\right),\\
\dot{z} &= z\left(-1+y+(z-1)(-2-\varepsilon)+(x-1)(-3+\varepsilon)\right).
\end{aligned}
$$

The following systems verify, respectively, the three sets of conditions in statement (d) of Theorem 1.2, so all of them have two limit cycles with the stability given in (d).

$$\begin{aligned}
\dot{x} &= x\,(2-y-z)\,,\\
\dot{y} &= y\left(-2+(x-1)^2+x+(x-1)(y-1)+y\right),\\
\dot{z} &= z\left(1+\frac{1}{4}(x-1)+(x-1)(y-1)-z+(y-1)\left(-\frac{3}{4}-\varepsilon\right)\right),
\end{aligned}$$

$$\begin{aligned}
\dot{x} &= x\left(1-x+\frac{1}{2}(y-1)\right),\\
\dot{y} &= y\,(-3+x+y+z)\,,\\
\dot{z} &= z\big(-(x-1)^2-\frac{5}{2}(y-1)-(x-1)(y-1)+(z-1)^2+(z-1)\varepsilon+(x-1)(5+\varepsilon)\big),
\end{aligned}$$

$$\begin{aligned}
\dot{x} &= x\left(-2+(x-1)^2+x+z\right),\\
\dot{y} &= y\left(-3+(x-1)^2+x+(x-1)(y-1)+y+z\right),\\
\dot{z} &= z\,(-2(y-1)-2(z-1)+(x-1)(-2+\varepsilon))\,.
\end{aligned}$$

Finally, the system

$$\begin{aligned}
\dot{x} &= x\,(2-x-z)\,,\\
\dot{y} &= y\left(2-\frac{69}{5}(x-1)^2-x-y-\frac{231}{10}(z-1)^2\right),\\
\dot{z} &= z\big(-\frac{8}{5}(x-1)^2+2(y-1)-\frac{58}{5}(x-1)(y-1)-\frac{109}{5}(y-1)^2+2(z-1)-(z-1)^2+(x-1)(4-\varepsilon)\big),
\end{aligned}$$

has two limit cycles with the stability given in (e) of Theorem 1.2.

3.2. Examples of Theorem 1.3

We give examples showing that the conditions provided by Theorem 1.3 are non-empty. The system

$$\begin{aligned}
\dot{x} &= x\left(2+\frac{1}{2}(x-1)^2-x-(x-1)(y-1)-z-(1+\varepsilon)(y-1)\right),\\
\dot{y} &= y\,(3-x-y-z)\,,\\
\dot{z} &= z\left(\frac{3}{2}(x-1)+\frac{7}{2}(y-1)+2(z-1)+4(x-1)(z-1)\right),
\end{aligned}$$

has two limit cycles with the stability given in (a) of Theorem 1.3. The following systems verify, respectively, the four sets of conditions in statement (b) of Theorem 1.3, so all of them have two limit cycles with the stability given in (b).

$$\begin{aligned}
\dot{x} &= x\left(2-\frac{7}{8}(x-1)^2-x+(x-1)(y-1)-z+(-1+\varepsilon)(y-1)\right),\\
\dot{y} &= y\,(3-x-y-z)\,,\\
\dot{z} &= z\left(\frac{7}{4}(x-1)+\frac{37}{16}(y-1)+6(x-1)(z-1)+(2+5\varepsilon)(z-1)\right),
\end{aligned}$$

$$\begin{aligned}
\dot{x} &= x\left(2-\frac{1}{2}(x-1)^2 - x + (x-1)(y-1) - z + (-1+\varepsilon)(y-1)\right),\\
\dot{y} &= y\,(3-x-y-z)\,,\\
\dot{z} &= z\left(\frac{3}{2}(x-1)+\frac{7}{2}(y-1)+5(x-1)(z-1)+(2+\varepsilon)(z-1)\right),
\end{aligned}$$

$$\begin{aligned}
\dot{x} &= x\left(2-\frac{7}{8}(x-1)^2 - x - (x-1)(y-1) - z + (1+\varepsilon)(y-1)\right),\\
\dot{y} &= y\,(-1+x-y+z)\,,\\
\dot{z} &= z\left(\frac{7}{4}(x-1)-\frac{37}{16}(y-1)+5(x-1)(z-1)+(2-5\varepsilon)(z-1)\right),
\end{aligned}$$

$$\begin{aligned}
\dot{x} &= x\left(2+\frac{3}{4}(x-1)^2 - x + (x-1)(y-1) - z + (1+\varepsilon)(y-1)\right),\\
\dot{y} &= y\,(-1+x-y+z)\,,\\
\dot{z} &= z\,(-5(y-1)+(2-3\varepsilon)(z-1))\,,
\end{aligned}$$

The following systems verify, respectively, the three sets of conditions in statement (c) of Theorem 1.3, so all of them have two limit cycles with the stability given in (c).

$$\begin{aligned}
\dot{x} &= x\left(2+\frac{1}{2}(x-1)^2 - x - (x-1)(y-1) - z + (\varepsilon-1)(y-1)\right),\\
\dot{y} &= y\,(3-x-y-z)\,,\\
\dot{z} &= z\left(\frac{3}{2}(x-1)+\frac{7}{2}(y-1)+4(x-1)(z-1)+(2+\varepsilon)(z-1)\right),
\end{aligned}$$

$$\begin{aligned}
\dot{x} &= x\left(-x-(x-1)(y-1)+z-\left(\frac{3}{4}+\varepsilon\right)(y-1)\right),\\
\dot{y} &= y\left(x+\frac{3}{4}(y-1)-z\right),\\
\dot{z} &= z\left(\frac{17}{16}(y-1)+2(x-1)(z-1)+\left(\frac{1}{4}+\varepsilon\right)(z-1)\right),
\end{aligned}$$

$$\begin{aligned}
\dot{x} &= x\left(1+\frac{1}{2}(x-1)^2 - x - (x-1)(y-1) - 3(z-1) - (3+\varepsilon)(y-1)\right),\\
\dot{y} &= y\left(2-\frac{1}{3}(x-1)-y-z\right),\\
\dot{z} &= z\,(1+2(x-1)-y+2(x-1)(z-1)+(2+2\varepsilon)(z-1))\,,
\end{aligned}$$

The following systems verify, respectively, the three sets of conditions in statement (d) of Theorem 1.3, so all of them have two limit cycles with the stability given in (d).

$$\begin{aligned}
\dot{x} &= x\left(2-\frac{1}{2}(x-1)^2 - x - (x-1)(y-1) - z + (1+\varepsilon)(y-1)\right),\\
\dot{y} &= y\,(-1+x-y+z)\,,\\
\dot{z} &= z\left(\frac{3}{2}(x-1)-\frac{7}{2}(y-1)+\frac{27}{8}(x-1)(z-1)+(2-\varepsilon)(z-1)\right),
\end{aligned}$$

$$\begin{aligned}
\dot{x} &= x\left(3(x-1)^2-(x-1)(y-1)-\frac{1}{2}(z-1)+\left(-\frac{1}{4}+\varepsilon\right)(y-1)\right),\\
\dot{y} &= y\left(1-\frac{1}{2}(y-1)-z\right),\\
\dot{z} &= z\left(-\frac{1}{4}(x-1)+\frac{393}{1024}(y-1)+33(x-1)(z-1)+\left(\frac{1}{2}+29\varepsilon\right)(z-1)\right),
\end{aligned}$$

$$\begin{aligned}
\dot{x} &= x\left(2-2(x-1)^2-x-(x-1)(y-1)-z+(1+\varepsilon)(y-1)\right),\\
\dot{y} &= y\left(-1+x-y+z\right),\\
\dot{z} &= z\left(\frac{3}{2}(x-1)-\frac{7}{2}(y-1)-(x-1)(z-1)+(2-\varepsilon)(z-1)\right),
\end{aligned}$$

Finally, the system

$$\begin{aligned}
\dot{x} &= x\left(2-x-z+(-1+\varepsilon)(y-1)\right),\\
\dot{y} &= y\left(3-x-y-(z-1)^2-z\right),\\
\dot{z} &= z\left(\frac{3}{2}(x-1)+\frac{7}{2}(y-1)+\frac{44}{3}(x-1)(z-1)+(2+\varepsilon)(z-1)\right),
\end{aligned}$$

has two limit cycles with the stability given in (e) of Theorem 1.3.

3.3. Examples of Theorem 1.4

We give examples showing that the conditions provided by Theorem 1.4 are non-empty.

The system

$$\begin{aligned}
\dot{x} &= x\left(2(x-1)^2-(y-1)^2-2\varepsilon(x-1)+\varepsilon(y-1)\right),\\
\dot{y} &= y\left(3-x-y-z\right),\\
\dot{z} &= z\left(-1+2(y-1)+z\right),
\end{aligned}$$

has two limit cycles with the stability given in (a) of Theorem 1.4.

The system

$$\begin{aligned}
\dot{x} &= x\left(-2(x-1)^2+(y-1)^2-\varepsilon(y-1)\right),\\
\dot{y} &= y\left(3-x-y-z\right),\\
\dot{z} &= z\left(-1+2(y-1)+z\right),
\end{aligned}$$

verfies the first set of conditions in statement (b) of Theorem 1.4, and the system

$$\begin{aligned}
\dot{x} &= x\left(4(x-1)^2-2(x-1)(y-1)-(y-1)^2-\varepsilon(y-1)\right),\\
\dot{y} &= y\left(3-x-y-z\right),\\
\dot{z} &= z\left(-1+2(y-1)+z\right),
\end{aligned}$$

verifies the second set of conditions. In both of them there exist two limit cycles with the stability given in (b). The system

$$\begin{aligned}
\dot{x} &= x\left(-\frac{7}{2}(x-1)^2+4(x-1)(y-1)-(y-1)^2-2\varepsilon(x-1)+\varepsilon(y-1)\right),\\
\dot{y} &= y\left(1-x-y+z\right),\\
\dot{z} &= z\left(-1-2(y-1)+z\right),
\end{aligned}$$

verfies the first set of conditions in statement (c) of Theorem 1.4, and the system

$$\begin{aligned}
\dot{x} &= x\left(-3(x-1)^2 + (y-1)^2 - 2\varepsilon(x-1) + \varepsilon(y-1)\right), \\
\dot{y} &= y\left(1 - x - y + z\right), \\
\dot{z} &= z\left(-1 - 2(y-1) + z\right),
\end{aligned}$$

verifies the second set of conditions. In both of them there exist two limit cycles with the stability given in (c). The following systems verify, respectively, the three sets of conditions in statement (d) of Theorem 1.4, so all of them have two limit cycles with the stability given in (d).

$$\begin{aligned}
\dot{x} &= x\left(-2(x-1)^2 + (y-1)^2 - (x-1)(z-1) - 4\varepsilon(x-1) + \varepsilon(z-1)\right), \\
\dot{y} &= y\left(2 - x - z\right), \\
\dot{z} &= z\left(y - 1\right),
\end{aligned}$$

$$\begin{aligned}
\dot{x} &= x\left(-2(x-1)^2 + (y-1)^2 - \varepsilon(y-1)\right), \\
\dot{y} &= y\left(3 - x - y - z\right), \\
\dot{z} &= z\left(-1 + 2(y-1) + z\right),
\end{aligned}$$

$$\begin{aligned}
\dot{x} &= x\left(-2(x-1)^2 + (y-1)^2 - (x-1)(z-1) + 4\varepsilon(x-1) - \varepsilon(z-1)\right), \\
\dot{y} &= y\left(2 - x - z\right), \\
\dot{z} &= z\left(y - 1\right).
\end{aligned}$$

The following systems verify, respectively, the three sets of conditions in statement (e) of Theorem 1.4, so all of them have two limit cycles with the stability given in (e).

$$\begin{aligned}
\dot{x} &= x\left(\frac{1}{2}(x-1)^2 - (y-1)^2 + 2\varepsilon(x-1) - \varepsilon(y-1)\right), \\
\dot{y} &= y\left(3 - x - y - z\right), \\
\dot{z} &= z\left(-1 + 2(y-1) + z\right),
\end{aligned}$$

$$\begin{aligned}
\dot{x} &= x\left(-\frac{1}{16}(x-1)^2 - \frac{1}{2}(x-1)(y-1) + (y-1)^2 + \varepsilon(y-1)\right), \\
\dot{y} &= y\left(1 - x - y - z\right), \\
\dot{z} &= z\left(-1 - 2(y-1) + z\right),
\end{aligned}$$

$$\begin{aligned}
\dot{x} &= x\left((y-1)^2 + (x-1)(z-1) - 2\varepsilon(x-1) - \varepsilon(z-1)\right), \\
\dot{y} &= y\left(-x + z\right), \\
\dot{z} &= z\left(y - 1\right).
\end{aligned}$$

Finally, the system

$$\begin{aligned}
\dot{x} &= x\left(-(x-1)^2 - (y-1)^2 - \varepsilon(z-1)\right), \\
\dot{y} &= y\left(2 - x - z\right), \\
\dot{z} &= z\left(-1 + y - (z-1)^2\right),
\end{aligned}$$

has two limit cycles with the stability given in (f) of Theorem 1.4.

3.4. Examples of Theorem 1.5

We give examples showing that the conditions provided by Theorem 1.5 are non-empty. The system

$$\begin{aligned}
\dot{x} &= x\left(1+2(x-1)^2-x+2(y-1)-(z-1)^2+\varepsilon(z-1)\right),\\
\dot{y} &= y\left(1-x+(1+2\varepsilon)(y-1)\right),\\
\dot{z} &= z\left(1-x\right),
\end{aligned}$$

has two limit cycles with the stability given in (a) of Theorem 1.5. The following systems verify, respectively, the four sets of conditions in statement (b) of Theorem 1.5, so all of them have two limit cycles with the stability given in (b).

$$\begin{aligned}
\dot{x} &= x\left(1-2(x-1)^2-x+2(y-1)+3(x-1)(z-1)+(z-1)^2-\varepsilon(z-1)\right),\\
\dot{y} &= y\left(1-x+\left(1-\frac{7\varepsilon}{8}\right)(y-1)\right),\\
\dot{z} &= z\left(1-x\right),
\end{aligned}$$

$$\begin{aligned}
\dot{x} &= x\left(1-2(x-1)^2-x+2(y-1)-5(x-1)(z-1)+(z-1)^2+\varepsilon(z-1)\right),\\
\dot{y} &= y\left(-x+y\right),\\
\dot{z} &= z\left(x-1\right),
\end{aligned}$$

$$\begin{aligned}
\dot{x} &= x\left(1+2(x-1)^2-x-2(y-1)+3(x-1)(z-1)-(z-1)^2-\varepsilon(z-1)\right),\\
\dot{y} &= y\left(-2+x+y\right),\\
\dot{z} &= z\left(x-1\right),
\end{aligned}$$

$$\begin{aligned}
\dot{x} &= x\left(1+2(x-1)^2-x-2(y-1)+5(x-1)(z-1)-(z-1)^2-\varepsilon(z-1)\right),\\
\dot{y} &= y\left(-2+x+y\right),\\
\dot{z} &= z\left(x-1\right),
\end{aligned}$$

The following systems verify, respectively, the three sets of conditions in statement (c) of Theorem 1.5, so all of them have two limit cycles with the stability given in (c).

$$\begin{aligned}
\dot{x} &= x\left(1+3(x-1)^2-x+2(y-1)-(x-1)(z-1)+\varepsilon(z-1)\right),\\
\dot{y} &= y\left(-x+y\right),\\
\dot{z} &= z\left(-1+x+2(z-1)^2\right),
\end{aligned}$$

$$\begin{aligned}
\dot{x} &= x\left(1-x-2(y-1)-(x-1)(z-1)+(z-1)^2-\varepsilon(z-1)\right),\\
\dot{y} &= y\left(-1+x+\left(1+\frac{7\varepsilon}{8}\right)(y-1)\right),\\
\dot{z} &= z\left(x-1\right),
\end{aligned}$$

$$\begin{aligned}
\dot{x} &= x\left(1+(x-1)^2-x-2(y-1)-(x-1)(z-1)-\varepsilon(z-1)\right),\\
\dot{y} &= y\left(-1+x+(1-4\varepsilon)(y-1)\right),\\
\dot{z} &= z\left(1-x-\frac{1}{2}(z-1)^2\right).
\end{aligned}$$

The following systems verify, respectively, the three sets of conditions in statement (d) of Theorem 1.5, so all of them have two limit cycles with the stability given in (d).

$$\begin{aligned}
\dot{x} &= x\left(1 - 3(x-1)^2 - x - 2(y-1) + (x-1)(z-1) - \varepsilon(z-1)\right),\\
\dot{y} &= y\left(-2 + x + y\right),\\
\dot{z} &= z\left(-1 + x - 2(z-1)^2\right),
\end{aligned}$$

$$\begin{aligned}
\dot{x} &= x\left(1 - 3(x-1)^2 - x + 2(y-1) + (x-1)(z-1) + \varepsilon(z-1)\right),\\
\dot{y} &= y\left(1 - x + (1+2\varepsilon)(y-1)\right),\\
\dot{z} &= z\left(1 - x + 2(z-1)^2\right),
\end{aligned}$$

$$\begin{aligned}
\dot{x} &= x\left(1 - (x-1)^2 - x + 2(y-1) + (x-1)(z-1) + \varepsilon(z-1)\right),\\
\dot{y} &= y\left(1 - x + (1+4\varepsilon)(y-1)\right),\\
\dot{z} &= z\left(1 - x + \frac{1}{2}(z-1)^2\right).
\end{aligned}$$

Finally, the system

$$\begin{aligned}
\dot{x} &= x\left(1 - x + 2(y-1) - (x-1)(z-1) + \frac{7}{8}\varepsilon(z-1)\right),\\
\dot{y} &= y\left(-x + y\right),\\
\dot{z} &= z\left(-1 + x + \frac{1}{2}(z-1)^2\right),
\end{aligned}$$

has two limit cycles with the stability given in (e) of Theorem 1.5.

A. Averaging theory

We summarize the averaging theory of first order, which provides sufficient conditions for the existence of periodic orbits for a periodic differential system depending on small parameters. For additional details and the proof of the result stated in this appendix, see [11, 12, 13] and [14, Theorems 11.5, 11.6].

Theorem A.1. *We consider the following differential system*

$$x'(t) = \varepsilon F_1(t, x) + \varepsilon^2 R(t, x, \varepsilon), \tag{A.1}$$

where $F_1 : \mathbb{R} \times D \to \mathbb{R}^n$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ *are continuous functions, T-periodic in the first variable and D is an open subset of $\mathbb{R}^n$. We define* $f_1 : D \to \mathbb{R}^n$ *as*

$$f_1(z) = \int_0^T F_1(s, z)ds, \tag{A.2}$$

and assume that:

1. *F_1 and R are locally Lipschitz with respect to x;*
2. *for $a \in D$ with $f_1(a) = 0$, there exists a neighboirhood V of a such that $f_1(z) \neq 0$ for all $z \in \overline{V} \backslash (a)$ and $d_B(f_1, V, 0) \neq 0$, where $d_B(f_1, V, 0)$ is the Brouwer degree.*

Then for $|\varepsilon| > 0$ sufficiently small, there exists a T-periodic solution $\varphi(\cdot, \varepsilon)$ of system (A.1) *such that $\varphi(\cdot, \varepsilon) \to a$ as $\varepsilon \to 0$. The kind of stability of the limit cycle is given by the eigenvalues of the Jacobian matrix at the point a.*

Note that a sufficient condition for showing that the Brouwer degree of a function f at a point a is nonzero, is that the Jacobian of the function f at a, when it is defined, is nonzero, see [15].

B. Notation

$$
\begin{aligned}
A_0 =& \, a_3\beta^2 \big((a_3^2 b_{11} - b_3^2 a_{21})(a_1(a_1+b_2) + a_3c_1) + a_3\beta^2(a_3b_{11} + b_3c_{31}) \big) \, , \\
A_1 =& \, a_1^6 \left(-2a_9b_3 + 2a_3b_9\right) + 2a_1^5 \big(b_3 \left(a_8b_3 - 4a_9b_2\right) - a_3^2b_6 + a_3 \left(a_6b_3 - b_3b_8 + 4b_2b_9\right)\big) \\
& + a_3^2 \big(2 \big(-b_3 \big(a_9b_2^2 + b_3(a_7b_3 - a_8b_2)\big) + a_3^3b_4 - a_3^2 \left(a_4b_3 - b_3b_5 + b_2b_6\right) \\
& + a_3b_3 \left(a_6b_2 - a_5b_3 + b_3b_7 - b_2b_8\right) + a_3b_2^2b_9\big) \, c_1^2 + c_1 \big(4a_3^2b_4 - 2a_3b_2b_6 + a_3b_3 \\
& (3b_5 - 2a_4 + c_6) + b_3 \big(a_6b_2 + b_2^2 + b_3(2b_7 - a_5 + c_8) + b_2(b_3 - b_8 - 2c_9)\big)\big) \, \beta^2 \\
& + (2a_3b_4 + b_3(b_5 + c_6))\beta^4\big) + a_1a_3 \big(-2a_3^3c_1(b_6c_1 - 2b_2b_4) + a_3^2 \, (2c_1 \, (-2b_2 \\
& (a_4b_3 - b_3b_5 + b_2b_6) + (a_6b_3 - b_3b_8 + 2b_2b_9)c_1) + (4b_2b_4 + (b_3 - 4b_6)c_1)\beta^2\big) \\
& a_3 \, \big(2c_1 \, \big(2b_2b_3(a_6b_2 - a_5b_3 + b_3b_7 - b_2b_8) + 2b_2^3b_9 + b_3(a_8b_3 - 2a_9b_2)c_1\big) \\
& + \, \big(-2a_4b_2b_3 - 2b_2^2b_6 + 4b_2b_9c_1 + b_2b_3(3b_5 + 2c_1 + c_6) + b_3c_1(3a_6 - 3b_8 - 2c_9)\big) \\
& \beta^2 + (b_3 - 2b_6)\beta^4\big) + b_3 \, \big(4b_2b_3(a_8b_2 - a_7b_3)c_1 + \big(a_6b_2^2 + b_2^3 + 2a_8b_3c_1 + b_2b_3 \\
& (2b_7 - a_5 + c_8) + b_2^2(b_3 - b_8 - 2c_9)\big) \, \beta^2 + (a_6 + b_2 - b_8 - 2c_9)\beta^4 - 4a_9b_2c_1 \, \big(b_2^2 \\
& + \, \beta^2)\big)\big) + a_1^4 \, \big(2a_3^3b_4 + 2a_3 \, \big(3a_6b_2b_3 - a_5b_3^2 + b_3^2b_7 - 3b_2b_3b_8 + 6b_2^2b_9 - 2a_9b_3c_1\big) \\
& + a_3^2 \, (2b_3b_5 - 2a_4b_3 - 6b_2b_6 + 4b_9c_1) + a_3(b_3 + 4b_9)\beta^2 - 2b_3 \, (b_3 \, (a_7b_3 - 3a_8b_2) \\
& + \, 2a_9(3b_2^2 + \beta^2))\big) + a_1^2 \, \big(4a_3^4b_4c_1 + 2a_3^3 \, \big(b_2^2b_4 - 4b_2b_6c_1 + c_1(2b_3b_5 - 2a_4b_3 + b_9c_1) \\
& + \, 2b_4\beta^2\big) + a_3 \, \big(2b_2^2b_3(a_6b_2 - a_5b_3 + b_3b_7 - b_2b_8) + 2b_2^4b_9 - 4b_3 \, \big(3a_9b_2^2 + b_3 \, (a_7b_3 \\
& - \, 2a_8b_2)\big) \, c_1 + \big(4a_6b_2b_3 + b_2^2(3b_3 + 4b_9) + b_3 \, (-4a_9c_1 + b_3(2b_7 - a_5 + c_8)) \\
& + \, b_2b_3(b_3 - 4(b_8 + c_9))\big) \, \beta^2 + (b_3 + 2b_9)\beta^4\big) + a_3^2 \, \big(-2b_2^3b_6 + 8b_2b_3(a_6 - b_8)c_1 \\
& - \, 2b_3c_1 \, (2a_5b_3 - 2b_3b_7 + a_9c_1) + 2b_2^2(b_3b_5 + 6b_9c_1) + (b_2(b_3 - 6b_6) + 4b_9c_1 \\
& + \, b_3(3b_5 + c_1 + c_6)) \, \beta^2 - 2a_4b_3(b_2^2 + \beta^2)\big) - 2b_3 \, \big(a_9(b_2^2 + \beta^2)^2 - b_2b_3 \, (-a_7b_2b_3 \\
& + \, a_8(b_2^2 + \beta^2))\big)\big) + a_1^3 \, \big(2a_3 \, \big(b_2b_3 \, (3a_6b_2 - 2a_5b_3 + 2b_3b_7 - 3b_2b_8) + 4b_2^3b_9 + 2b_3c_1 \\
& (a_8b_3 - 3a_9b_2)\big) + 4a_3^3(b_2b_4 - b_6c_1) + a_3 \, (3a_6b_3 + 3b_2b_3 - 3b_3b_8 + 8b_2b_9 - 2b_3c_9) \\
& \beta^2 + a_3^2 \, \big(-4a_4b_2b_3 - 6b_2^2b_6 + 4b_3(a_6 - b_8)c_1 + 4b_2(b_3b_5 + 3b_9c_1) + (b_3 - 4b_6)\beta^2\big) \\
& + 2b_3 \, \big(-4a_9b_2(b_2^2 + \beta^2) + b_3 \, \big(-2a_7b_2b_3 + a_8(3b_2^2 + \beta^2)\big)\big)\big) \, , \\
A_2 =& - 2a_3\beta^2 \, \big(\big(a_3^2b_{11} - b_3^2a_{21}\big) \, (a_1(a_1 + b_2) + a_3c_1) + a_3^2b_{11}\beta^2\big) \, \Big(\big(a_1(a_1 + b_2) + a_3c_1\big)^2 \\
& + \big(2a_1^2 + 2a_1b_2 + b_2^2 + 2a_3c_1\big) \, \beta^2 + \beta^4\Big) \, , \\
A_3 =& - 2 \, \Big(\big(a_1(a_1 + b_2) + a_3c_1\big)^2 + \big(2a_1^2 + 2a_1b_2 + b_2^2 + 2a_3c_1\big) \, \beta^2 + \beta^4\Big) \, \big(a_1^6 \, (a_3b_9 \\
& - \, a_9b_3) + a_1^5 \, \big(b_3 \, (a_8b_3 - 4a_9b_2) - a_3^2b_6 + a_3 \, (a_6b_3 - b_3b_8 + 4b_2b_9)\big) \\
& + a_3^2 \, \big(\big(-b_3 \, \big(a_9b_2^2 + b_3(a_7b_3 - a_8b_2)\big) + a_3^3b_4 - a_3^2 \, (a_4b_3 - b_3b_5 + b_2b_6) + a_3b_3 \\
& (a_6b_2 - a_5b_3 + b_3b_7 - b_2b_8) + a_3b_2^2b_9\big) \, c_1^2 + \big(b_3 \, (a_6b_2 - a_5b_3) + 2a_3^2b_4 \\
& + \, a_3 \, (b_3b_5 - 2a_4b_3 - b_2b_6)\big) \, c_1\beta^2 + (a_3b_4 - a_4b_3)\beta^4\big) + a_1a_3 \, \big(a_3^3c_1 \, (2b_2b_4 - b_6c_1) \\
& + \, a_3^2 \, \big(c_1 \, (-2b_2 \, (a_4b_3 - b_3b_5 + b_2b_6) + (a_6b_3 - b_3b_8 + 2b_2b_9) \, c_1) + 2\beta^2 \, (b_2b_4 \\
& - \, b_6c_1)\big) + a_3 \, \big(c_1 \, \big(2b_2b_3 \, (a_6b_2 - a_5b_3 + b_3b_7 - b_2b_8) + 2b_2^3b_9 + b_3c_1 \, (a_8b_3 \\
& - \, 2a_9b_2)\big) + \big(b_2b_3b_5 - 2a_4b_2b_3 - b_2^2b_6 + 2a_6b_3c_1 - b_3b_8c_1 + 2b_2b_9c_1\big) \, \beta^2 - b_6\beta^4\big) \\
& + \, b_3 \, \big(2b_2b_3c_1(a_8b_2 - a_7b_3) + \big(a_6b_2^2 - a_5b_2b_3 + a_8b_3c_1\big) \, \beta^2 + a_6\beta^4 - 2a_9b_2c_1 \, \big(b_2^2 \\
& + \, \beta^2)\big)\big) + a_1^4 \, \big(a_3^3b_4 + a_3^2 \, (b_3b_5 - a_4b_3 - 3b_2b_6 + 2b_9c_1) + a_3 \, \big(3a_6b_2b_3 - a_5b_3^2 \\
& + \, b_3^2b_7 - 3b_2b_3b_8 + 6b_2^2b_9 - 2a_9b_3c_1 + 2b_9\beta^2\big) - b_3 \, \big(b_3 \, (a_7b_3 - 3a_8b_2) + 2a_9 \, \big(3b_2^2
\end{aligned}
$$

$$
\begin{aligned}
&+\ \ \beta^2))\big) + a_1^3\left(2a_3^3\left(b_2b_4 - b_6c_1\right) + a_3^2\left(2b_2b_3(b_5 - a_4) - 3b_2^2b6 + 2a6b_3c_1\right.\right.\\
&-\ \ 2b_3b_8c_1 + 6b_2b_9c_1 - 2b_6\beta^2\big) + a_3\left(3b_2^2b_3(a_6 - b_8) - 2b_2b_3^2(b_7 - a_5) + 4b_2^3b_9\right.\\
&-\ \ 6a_9b_2b_3c_1 + 2a_8b_3^2c_1 + \left(2a_6b_3 - b_3b_8 + 4b_2b_9\right)\beta^2\big) + b_3\left(-4a_9b_2\left(b_2^2 + \beta^2\right)\right.\\
&+\ \ b_3\left(3a_8b_2^2 - 2a_7b_2b_3 + a_8\beta^2\right)\big)\big) + a_1^2\left(2a_3^4b_4c_1 + a_3^3\left(b_2^2b_4 - 4b_2b_6c_1 + c_1\left(2b_3b_5\right.\right.\right.\\
&-\ \ 2a_4b_3 + b_9c_1\right) + 2b_4\beta^2\big) - a_3^2\left(b_2^3b_6 + 4b_2b_3c_1(b_8 - a_6) + b_3c_1\left(2a_5b_3 - 2b_3b_7\right.\right.\\
&+\ \ a_9c_1\right) - b_2^2(b_3b_5 + 6b_9c_1) + (3b_2b_6 - b_3b_5 - 2b_9c_1)\beta^2 + a_4b_3(b_2^2 + 2\beta^2)\big)\\
&+\ \ a_3\left(b_2^2b_3^2b_7 - b_2^3b_3b_8 + b_2^4b_9 - 6a_9b_2^2b_3c_1 + 4a_8b_2b_3^2c_1 - 2a_7b_3^3c_1 + \beta^2\left(-b_2b_3b_8\right.\right.\\
&+\ \ 2b_2^2b_9 - 2a_9b_3c_1\right) + b_9\beta^4 - a_5b_3^2(b_2^2 + \beta^2) + a_6b_2b_3(b_2^2 + 3\beta^2)\big) + b_3\left(-a_9\left(b_2^2\right.\right.\\
&+\ \ \beta^2\right)^2 + b_2b_3\left(-a_7b_2b_3 + a_8(b_2^2 + \beta^2)\right)\big)\big)\Big)\,,\\
A_4 = &-\left(\left(a_1 + b_2\right)\left(a_1(a_1 + b_2) + a_3c_1\right)\beta + a_1\beta^2\right)^2\left(a_3^3b_4 - a_3^2\left(a_4b_3 - b_3b_5 + b_6(a_1 + b_2)\right)\right.\\
&-b_3\left(a_1^2a_9 + 2a_1a_9b_2 + a_9b_2^2 - a_1a_8b_3 - a_8b_2b_3 + a_7b_3^2 + a_9\beta^2\right) + a_3\left(a_6b_2b_3\right.\\
&-\ \ a_5b_3^2 + b_3^2b_7 - b_2b_3b_8 + a_1^2b_9 + b_2^2b_9 + a_1\left(a_6b_3 - b_3b_8 + 2b_2b_9\right) + b_9\beta^2\big)\big)\,,\\
K_1 = &\ A_1^2A_2^2 - 4A_0A_1A_2\left(A_3 + 2A_1\left(\left(a_1(a_1 + b_2) + a_3c_1\right)^2 + \beta^2\left(2a_1^2 + 2a_1b_2 + b_2^2 + 2a_3c_1\right)\right.\right.\\
&+\ \ \beta^4\big)\big) + 4A_0^2A_3\left(A_3 + 2A_1\left(\beta^2\left(2a_1^2 + 2a_1b_2 + 2a_3c_1 + b_2^2\right) + \left(a_1(a_1 + b_2)\right.\right.\right.\\
&+\ \ a_3c_1\right)^2 + \beta^4\big)\Big)\,,\\
N = &\ (a_1(a_1 + b_2) + a_3c_1)^2 + (2a_1^2 + 2a_1b_2 + b_2^2 + 2a_3c_1)\beta^2 + \beta^4 \neq 0.
\end{aligned}
$$

Under conditions (iii) of Proposition 1.1, the expressions of B_i with $i = 0, ..., 4$ and K_2 are the following.

$$
\begin{aligned}
B_0 &= \beta^2b_3\left((b_1b_2 + b_3c_1)(a_{31}b_2 - a_{21}b_3) + \beta^2(a_{31}b_1 + b_3c_{31})\right),\\
B_1 &= b_3(b_1b_2 + b_3c_1)(2(a_8b_2 - a_7b_3)(b_1b_2 + b_3c_1) + \beta^2(2a_8b_1 + a_6b_2 + b_2^2 + b_3(-a_5\\
&\quad + 2b_7 + c_8) + b_2(b_3 - b_8 - 2c_9))) + \beta^4b_3(b_3(b_5 + c_6) + b_1(a_6 + b_2 - b_8 - 2c_9))\\
&\quad - 2a_9(b_2b_3c_1 + b_1(b_2^2 + \beta^2))^2,\\
B_2 &= 2b_3\beta^2((b_1b_2 + b_3c_1)(a_{21}b_3 - a_{31}b_2) + \beta^2(a_{11}b_3 - a_{31}b_1)),\\
B_3 &= 2(b_3((a_7b_3 - a_8b_2)(b_1b_2 + b_3c_1)^2) - (a_8b_1 + a_6b_2 - a_5b_3)(b_1b_2 + b_3c_1)\beta^2\\
&\quad + (a_4b_3 - a_6b_1)\beta^4) + a_9(b_2b_3c_1 + b_1(b_2^2 + \beta^2))^2),\\
B_4 &= \beta^4(b_3(a_7b_3 - a_8b_2) + a_9(b_2^2 + \beta^2)),\\
K_2 &= B_1^2B_2(8B_0 + B_2) - 4B_0B_1(2B_0 + B_2)B_3 + 4B_0^2B_3^2.
\end{aligned}
$$

On the other hand, under conditions (iv) of Proposition 1.1, we redefine the expressions of B_i with $i = 0, ..., 4$ as follows:

$$
\begin{aligned}
B_0 &= \beta^2a_3(a_1a_2 + a_3c_2)(b_{31}a_1 - b_{11}a_3) + \beta^4a_3(b_{31}a_2 + a_3c_{31}),\\
B_1 &= a_3(a_1a_2 + a_3c_2)\left(2(b_6a_1 - b_4a_3)(a_1a_2 + a_3c_2) + \beta^2(2a_2b_6 + a_1b_8 + a_1^2 + a_3(-b_5\right.\\
&\quad + 2a_4 + c_6) + a_1(a_3 - a_6 - 2c_9))) + \beta^4a_3\left(a_3(a_5 + c_8) + a_2(b_8 + a_1 - a_6\right.\\
&\quad - 2c_9)) - 2b_9(a_1a_3c_2 + a_2(a_1^2 + \beta^2))^2,\\
B_2 &= 2a_3\beta^2((a_1a_2 + a_3c_2)(b_{11}a_3 - b_{31}a_1) + \beta^2(b_{21}a_3 - b_{31}a_2)),\\
B_3 &= 2(a_3((a_3b_4 - a_1b_6)(a_1a_2 + a_3c_2)^2) - (b_6a_2 + a_1b_8 - a_3b_5)(a_1a_2 + a_3c_2)\beta^2\\
&\quad + (a_3b_7 - a_2b_8)\beta^4) + b_9(a_1a_3c_2 + a_2(a_1^2 + \beta^2))^2),\\
B_4 &= \beta^4(a_3(a_3b_4 - a_1b_6) + b_9(a_1^2 + \beta^2)),
\end{aligned}
$$

The expression of K_2 is the same as under conditions (iii).

$$
\begin{aligned}
D_0 &= (a_1b_{31} - b_1a_{31})(a_1c_1 + b_1c_2)\beta^2 + (b_1b_{21} + c_1b_{31})\beta^4,\\
D_1 &= (a_1c_1 + b_1c_2)(2(a_1b_9 - a_9b_1)(a_1c_1 + b_1c_2) + (b_1(a_6 + b8) + 2b_9c_1)\beta^2),\\
D_2 &= 2(a_1c_1 + b_1c_2)\beta^2((a_{31}b_1 - a_1b_{31})(a_1c_1 + b_1c_2) + (b_1c_{31} - b_{31}c_1)\beta^2)\\
D_3 &= 2(a_1c_1 + b_1c_2)^2((a_9b_1 - a_1b_9)(a_1c_1 + b_1c_2) + (b_1c_9 - b_9c_1)\beta^2),\\
D_4 &= (a_9b_1 - a_1b_9)(a_1c_1 + b_1c_2)^3 - (a_1c_1 + b_1c_2)\left(a_1^3b_4 + a_1^2b_1(b_5 - a_4) + a_1\left(-a_5b_1^2\right.\right.\\
&\quad +b_1^2b_7 + b_1b_8c_1 + 2b_9c_1^2 - b_1b_6c_2 - b_1c_1c_9\left.\right) - b_1\left(a_7b_1^2 + a_8b_1c_1 + a_9c_1^2\right.\\
&\quad \left.\left.-a_6b_1c_2 - b_9c_1c_2 + b_1c_2c_9\right)\right)\beta^2 + \left(-b_9c_1^3 + a_1^2(b_1c_4 - 2b_4c_1) + a_1b_1\left(a_4c_1\right.\right.\\
&\quad \left.-b_5c_1 - b_4c_2 + b_1c_5\right) + b_1^3c_7 + b_1^2\left(-c_1b_7 + a_4c_2 - c_2c_6 + c_1c_8\right) + b_1c_1\left(-b_8c_1\right.\\
&\quad \left.\left.+b_6c_2 + c_1c_9\right)\right)\beta^4 + (b_1c_4 - b_4c_1)\beta^6,\\
K_4 &= (D_1D_2 - 2D_0D_3)^2 + 8a_1c_1D_0D_1(D_1D_2 - D_0D_3) + 8b_1c_2D_0D_1(D_1D_2 - D_0D_3).
\end{aligned}
$$

Acknowledgements

The first and third authors are partially supported by the Ministerio de Economía, Industria y Competitividad, Agencia Estatal de Investigación (Spain), grant MTM2016-79661-P (European FEDER support included, UE) and the Consellería de Educación, Universidade e Formación Profesional (Xunta de Galicia), grant ED431C 2019/10 with FEDER funds. The first author is also supported by the Ministerio de Educacion, Cultura y Deporte de España, contract FPU17/02125.

The second author is partially supported by the Ministerio de Economía, Industria y Competitividad, Agencia Estatal de Investigación grants MTM2016 - 77278 - P (FEDER) and MDM - 2014 - 0445, the Agència de Gestió d'Ajuts Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911.

The fourth author is partially supported by FCT/Portugal through UID/ MAT/ 04459/2013.

Bibliography

[1] A.J. Lotka, *Elements of physical biology*, Elements of physical biology, Science Progress in the Twentieth Century (1919- 1933) **82** (1926), 341–343.

[2] V. Volterra, *Variazioni e fluttuazioni del numero d'individui in specie animali conviventi*, Memoire della R. Accademia Nazionale dei Lincei **II** (1926), 558–560.

[3] A. Kolmogorov, *Sulla teoria di Volterra della lotta per l'esistenza*, Giornale dell' Istituto Italiano degli Attuari **7** (1936), 74–80.

[4] R. M. May, *Stability and Complexity in Model Ecosystems*, Princeton NJ, 1974.

[5] R.H. Hering, *Oscillations in Lotka-Volterra systems of chemical reactions*, J. Math. Chem. **5** (1990), 197–202.

[6] G. Laval, R. Pellat, *Plasma Physics*, Proceedings of Summer School of Theoretical Physics, Gordon and Breach, New York, 1975.

[7] F.H. Busse, *Transition to turbulence via the statistical limit cycle route*, Synergetics **39**, Springer-Verlag, Berlin, 1978.

[8] S. Solomon, P. Richmond, *Stable power laws in variable economies; Lotka-Volterra implies Pareto-Zipf*, The European Physical Journal B–Condensed Matter and Complex Systems **27** (2002), 257–261.

[9] B.R. de Freitas, J. Llibre, J.C. Medrado, *Limit cycles of continuous and discontinuous piecewise-linear differential systems in* $\mathbb{R}^3$, J. Comput. Appl. Math. **338** (2018), 311–323.

[10] J. Llibre, Y.P. Martínez and C. Valls, *Limit cycles bifurcating of Kolmogorov systems in* $\mathbb{R}^2$ *and in* $\mathbb{R}^3$, preprint, (2020).

[11] A. Buica, J. Llibre, *Averaging metods for finding periodic orbits via Brouwer degree*, Bull. Sci. Math. **28** (2004), 7–22.

[12] A. Buica, J. Llibre, O. Yu. Makarenkov, *On Yu.A. Mitropol'skii's Theorem on Periodic Solutions of System of Nonlinear Differential Equations with Nondifferentiable Right-Hand Sides*, Doklady Mathematics **78** (2008), 525–527.

[13] J. Llibre, D. D. Novaes, M. A. Teixeira, *Higher order averaging theory for finding periodic solutions via Brouwer degree*, Nonlinearity, **27** (2014), 563–583.

[14] F. Vehrulst, *Nonlinear differential equations and dynamical systems*, Universitext, Springer, 1996.

[15] N.G. Lloyd, *Degree theory*, Cambridge Trends in Mathematics, 73, Cambridge University Press, 1978.