PRICING INSURANCE IN ORDER TO MINIMIZING THE EXPECTED LOSS IN WEALTH VIA OPTIMAL CONTROL

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Abstract. In this paper we are interested in Pricing insurance in order to minimizing the expected loss in wealth via optimal control. The objective is to find the policy which maximizes the total wealth in company insurances. For this purpose, First, a dynamic model is introduced to describe the process of receiving premium and paying claims. Then, we introduce the premium variable as the problem control variable. Next, we define an appropriate objective function for the control variable and state variables in order to reduce expected losses and increase the wealth. In the end, one of the main variables is estimated by statistical methods and we solve the optimal control problem by PMP method and finally, numerical example are presented.

Keywords: Optimal control, Premium, Dynamical systems, expected loss, Optimization.

Introduction. In actuarial science, a premium principle equates the cost of a general insurance policy to the moments of the corresponding claim arrival and severity distributions[7]. In order to make a profit and cover their expenses, insurers add a loading to this cost price. Because many lines of insurance, especially automobile insurances, are highly competitive, the loading critically depends on the price of other insurers to ensure comparable costs of insurance policies. Insurance pricing is therefore an important factor in determining the type of insurance company customers select or change in the next insurance period. According to Taylor's model, that uses optimal control, the premium is set based on the average insurance market price[1]. The Taylor model is based on, a discrete, demand model for pricing in which the future average market price is exactly evaluated and new and existing premium holders are required to pay the same current premium rate[2].

It is difficult to determine premium price using a discrete deterministic model if the average market premium is a continuous stochastic process. Continuous time models can be sets up in several ways: one can either model the premium rate charged by the insurer for a unit of insurance cover, or charge a premium up front for a finite period of cover thereafter. In the former case, policyholders pay a premium p(t) continuously over the course of their policies[2]. It is important point to note is that pricing at a lower rate in the market can result in a negative premium[4]. This strategy of relatively low initial pricing in the market aims to generate sales; by controlling claims, a creditor's insurance can then increase after which he can increase his prices and profits.

Emms and Haberman disucss different modeling constraints[5]. All these models assume that there exists a single optimising insurer, whose price does not affect the premiums of other insurers. This however, is applicable to only small insurers in big markets. In mathematical finance, there is a similar supposition for the optimal wealth appropriation problem[6]. It is assumed that the stock rate does not affect the allocation of the investor's stock. Nevertheless, most insurance lines are under the influence of several large insurers who control each other's prices and constantly update their prices. In such markets, competitive pricing models are not responsive and insurers should therefore pay attention to the response of insurers to insurance prices. The competitive pricing model defines the demand function that determines the relationship between premiums and the average premium of the market[2]. If the insurer determines a premium price lower than the average market price, policies will be sold. If the whole insurance market determines a high premium, this would lead to notable sales for an optimising insurer. In reality however, customers do not pay more than the value of the insured, and they maybe at liberty to take or not to take out insurance. In some way, the demand for insurance policies depends on their premiums, and if the price of all policy sellers are far above the cost, our demand law dictates that very few policies will be sold.

In section 2, the motivation behind the modelling is presented and the notation for car insurance pricing is introduced. Section 3 We determinidtic the model. In section 4, we describes how to estimate claim size rate via statistical methods, Section 4, defines the demand function and solves the model and section 5 summarizes a conclusion.

1. Optimal control model preliminaries

In this section, based on the earlier work of Taylorand Emms [1,3], all prices (and claims) per unit of exposure varies according to the insured risk (where the exposure is the unit of risk for an insurer).

- Policyholders pay a premium p(t) continuously over the course of their policies
- We assume p(t) t≥0 per unit exposure, as a premium (p) at time t for a general insurance policy of fixed duration .
- We consider (t) per unit exposure as the ***'average market premium for a policy'*** of the same duration.
- q(t) is the '***exposure***' that expresses a measure of the insurance company's potential liabilities '.It reflects the number of currently in force insurance policies and the potential size of the claims on these policies.

- Suppose that G as the ***demand function***, which is associated with the premium.
- The reserve, w(t), represents the amount of ***current capital held by the insurance company***, which increases with the sale of the insurance policy and decreases with payment of claims.
- u(t)is the ***claim size rate per unit of exposure***

An insurer who initially tries to gain exposure faces the possibility that the market will follow the same course of action by setting a comparable premium, Consequently, we split up the drift in the market average premium based on the absence or presence of the market's reaction to an optimising insurer. If there is no reactive market, then we will we adopt the expected value principle (Paul Emms,2011), and assume that the average market premium is directly related to the losses, the average market premium p̄(t) is therefore equal to a constant γ that represents a fixed loss ratio per unit time multiplied by u(t)is the claim size rate per unit of exposure

$( ) = \ ^{-1} \ ( ),$

If we take $\lambda(p(t) - \bar{p}(t))dt$, to reepresent the change in the average premium market price that follows the market reaction and $\lambda \geq 0$ as the constant reaction of the market to the optimal price of the insurer, the ***market average premium constraint*** is therefore computed as follows [3]:

$$( ) = \ ^{-1} \ ( ) + \ (\ ( ) - \ ( )) \qquad (1)$$

Here, it is assumed that the premium u_t is charged at the beginning of a policy of length $l = \kappa^{-1}$ by the policyholder and each customer who has renewed his insurance policy is considered a new policyholder. Where q(t) is the 'exposure' and G is the demand function, which is associated with the premium, we assume that ***change in exposure over a time*** interval of length dt is given by[3]:

$$(t)=q(t)(G\text{-}k) \qquad (2)$$

Equation(2) measures the ability of an insurer under its current exposure. Suupposing that large insurers tend to gain greater exposure than small insurers with comparable premiums [3],

- the increase in reservew(t) (i.e. amount of current capital held by the insurance company) from selling insurance at time dt is the increase in exposure from selling policies q(t) multiplied by the current premium p(t),

  $q(t)\times G\times p(t)\times dt.$

the decrease in reserve by paying claims over time dt is the claim size rate u(t) per unit of exposure q(t)

$u(t)\times q(t)\times dt,$

Therefore, ***changes in reserve*** are as follows[3]where the constant α determines the loss of wealth due to returns to shareholders:

$$(t)=-\alpha w(t)+q(t)(Gp(t)-u(t)) \qquad (3)$$

The last equation of state in this model is u(t) that denotes the claim size rate of car insurance, so the last constraint of the model ***(changes in claim size)*** is as follows [3]:

$$(t)=u(t)(\mu+\sigma x_3 (t)) \qquad (4)$$

For simplicity we suppose the mean claim size rate u(t) is lognormally distributed with constant drift μ and volatility .

So, $= (\bar{p}( ), q( ), w( ), u( ))$ is the current status vector .

Since, insurance companies are always trying to reduce their expected loss, Therefore, in this paper, we control the premium so that it will reduce the expected loss.Therefore, the expected loss in wealth due to the:

$$[\int^{\infty} \qquad |\mathcal{F} \ ] \qquad (5)$$

Here $\mathcal{F}$ is the filtration derived from the Brownian motion w(t) and $= \quad ^{-(-)}$ where $>$ in order that the expectation is finite[4].

It is important to measure the amount of the expected loss in wealth for insurance companies especially in the car insurance, as insurance companies are always looking for solutions to reduce its amount, because a reduction in it, means that the insurance company has reached a low level of risk control. By considering the previous expression, we provide an optimal control in car insurance. For this purpose, we have the following model:

$$\min \ = \int (\int_0^{\infty} \qquad \mathcal{F} \ | \ ) \qquad (6)$$

s.t

$(t) = \gamma^{-1}u(t) + \lambda(p(t) - \bar{p}(t))$

$(t)=q(t)(G\text{-}k)$

$(t)=-\alpha w(t)+q(t)(Gp\text{-}u(t))$

$(t)=u(t)(\mu+\sigma w_3(t)).$

2. Deterministic the model

Although the relative premium is deterministic the insurer's premium is stochastic because the market average premium is a random process, So to determinidtic the relative premium folowing:

$_1( ) = \ (\ ( )|\mathcal{F}_0)$

$_2( ) = q(t)$

$_3( ) = \ (\ ( )|\mathcal{F}_0)$

$_4(\ ) = \ (\ (\ )|\mathcal{F}_0)$

The exposure q is a deterministic state variable and also, for deterministic the objective function[4]:

$$\int^{\infty} \qquad |\mathcal{F} \ = \frac{\quad}{\quad}$$

The optimisation problem is now deterministic and can be written in canonical form. Define the state vector by:

$$=(\ ),\quad = 1,2,3,4.$$

so that the state equation is:

$$= \ (\ ,\ ).$$

Therefore, the Deterministic model will be as follows:

$$\min \ = \int_0 \frac{_2(\ )\ _4(\ )}{\quad}$$

s.t

$_1(\ ) = \ ^{-1}\ _4(\ ) + \lambda(\ p(t) - \ _1(\ ))$

$x_2(t)=x_2(t)(G-k)$

$x_3(t)=-\alpha x_3(t)+x_2(t)(Gp(t)-x_4(t))$

$x_4(t)=x_4(t)(\mu+\sigma x_3(t))$.

3. Modifying the model to a simpler model based on statistical estimation

In this section, we use the statistical data of the insurance company to calculate the losses rate $x_4(t)$ to convert the model to a simpler model and then to solve it in the next section.

Table4.1 .the loss rate

year	Month	Losses car insurance	year	month	Losses car insuance
2012	April	6640	2014	August	3933
	May	6911		September	4493
	June	7106		October	4154
	July	7503		November	3860
2013	April	4148		December	4187
	May	4865		January	4367
	June	4442		February	4173
	July	4843		March	3866
	August	4452	2015	April	3113
	September	4439		May	3437
	October	4190		June	3323
	November	3560		July	3278
	December	3531		August	3655
	January	3444		September	3620
	February	3580		October	3414
	March	3284		November	3410
2014	April	3151		December	3882
	May	3393		January	3877
	June	3431		February	3963
	July	3737		March	3963

For this reason, to provide the losses incurred for three consecutive years from car policy in an insurance company and then calculation of the loss rate in 2012, 2013, 2014, 2015 compared to 4 months from 2012 the year (According to Table 4.1), we conclude that the above statistical data follows the exponential distribution with the distribution function $x_4(t)=1-e^{-0.1375t}$. for fit to the existing data from The kolmogorov-Smirnov statistic has been used. The numerical value of this statistic is equal 1.018 With a significant level of 0.463 Is obtained Which indicates that the existing datas at the significant level of 0.05 follow exponential distribution $x_4(t)=1-e^{-0.1375t}$. So, by this assumption and its substitution in model number (6), we get the following model

$$\min \ = \int_0 \frac{_2(\ )(1-e^{-0.1375t})}{\quad} \tag{7}$$

s.t

$x_1(t)=\gamma^{-1}(0.1375\ e^{-0.1375t})+\lambda(\ p(t)-x_1(t))$,

$x_2(t)=x_2(t)(G-k)$,

$x_3(t)=-\alpha x_3(t)+x_2(t)(Gp(t)-1+e^{-0.1375t})$

4. Solving the model

The model explained in Section 3, solving via PMP theorem and following algorithm and the results described in the next steps will be achieved. According to PMP, we consider the following equations:

$$p_i = -\frac{\partial H}{\partial x_i}, \tag{8}$$

$$\frac{\partial H}{\partial u^*} = 0 \tag{9}$$

- H is ***'Hamiltonian function'***.
- u(t) is ***'Control variable'***.
- are states of the system.
- We consider as the ***'Lagrange multipliers'*** and ***'co-states'*** of the system.

The algorithm used to solve the model is as follows:

1. Subdivide the interval $[\ _0, \ ]$ into N equal subintervals and assume a piecewise-constant control $u^{(0)}(t)=u^{(0)}(t_k)$, $t\in[t_k,t_{k+1}]k=0,1,\ldots,N-1$.
2. Applying the assumed control u^i to integrate the state equations from t_0 to t_f with initial conditions $x(t_0)=x_0$ and store the state trajectory $(\ )$.
3. Applying u^i and x^i to integrate costate equations backward, i.e., from $[t_0,t_f]$. The initial value $p^i(t_f)$ can be obtained by:

$$p^i(t_f) = \frac{\partial h}{\partial x}(x^{(i)}(t_f)). \tag{10}$$

Evaluate $\partial H^i(t)/\partial u$, $t\in[t_0,t_f]$ and store the vector.

4. If

$$\left\|\frac{\partial H^i}{\partial u}\right\| \leq \gamma \tag{11}$$

$$\left\|\frac{\partial H^i}{\partial u}\right\|^2 = \int_{t_0}^{t_f} \left[\left\|\frac{\partial H^i}{\partial u}\right\|\right]^T \left[\left\|\frac{\partial H^i}{\partial u}\right\|\right] dt \tag{12}$$

Then stop the iterative procedure. Here is a preselected small positive constant used as a tolerance. If (10) is not satisfied, adjust the piecewise-constant control function by:

$$u^{i+1}(t_k)=u^i(t_k) - \tau \frac{\partial H^i}{\partial u}(t_k), \quad k=0,1,\cdots,N-1 \tag{13}$$

Table 5.1. sample data set

Constant	value
Time horizon T	1 year
Depreciation of wealth α	0.05 p/a
Demand parameterisation a	1 p/a
Demand parameterisation b	1 p/a
Length of policy $l = k^{-1}$	1 year
Rate of market reaction λ	0.1 p/a
Loss ratio γ	0.9 p/a

Replace $u^{(i)}$ by $u^{(i+1)}$ and return to step2. Here, is the step size. In order to solve the problem and implement the above algorithm using sample data set in table 5.1, we verify in the following cases:
We set the initial conditions, designed state and demand function as $x(0)=(0,0.7,0.78)$, and $G=\frac{1}{ep(t)}$ respectively. Then the problem is solved and the value of the objective function is obtained $= 0.0486$. The state and control function are showed in Fig 5.1 and 5.2.

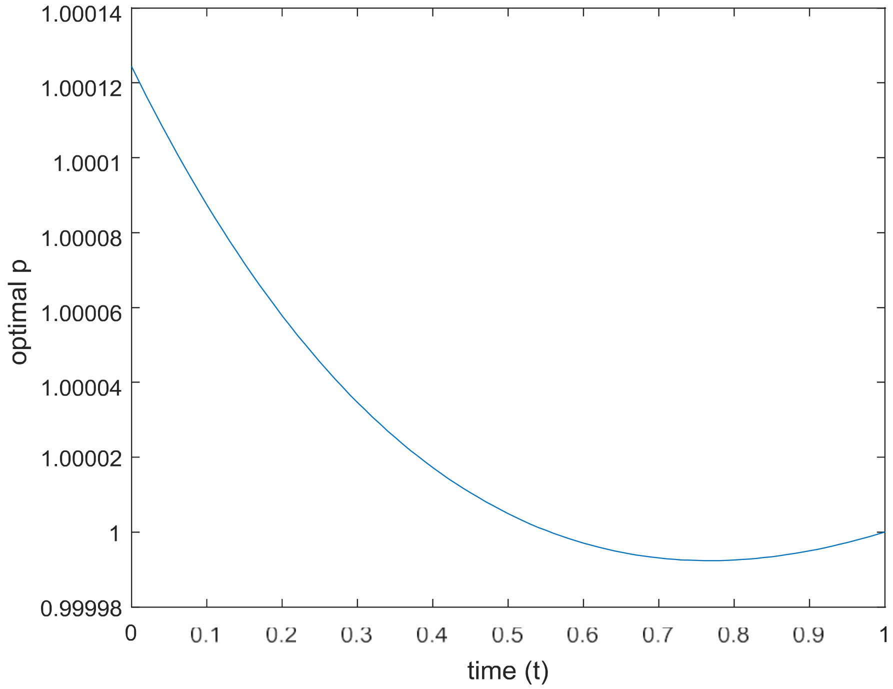

Fig5.1

You see that our target function has reached its minimizing and And the amount of premium at any moment is visible in Figure 5.1.

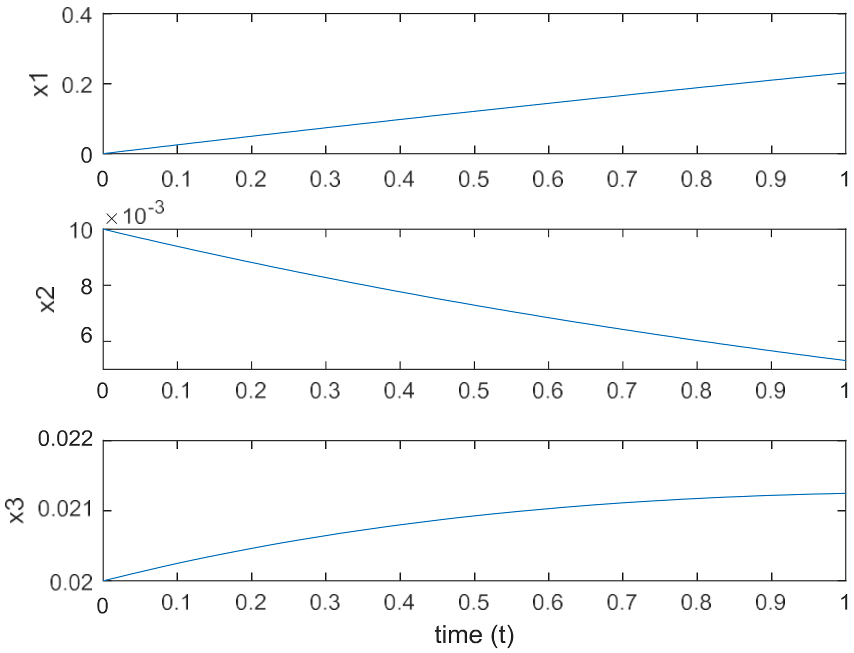

Fig.5.2

Figure 5.2 shows the equations of states at any moment of time that has made the research objective.

6. conclusion

In this paper, an optimal control problem was proposed based on an insurance model. Althoughthe deterministic control theory has been used to identify an optimal premium strategy for an insurer to the expected loss in wealth, in reality, the process of receiving premiums and paying losses have been described via a dynamic system. We therefore solve the model to determine insurance premiums using a sample dataset.

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