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A NEW CLASS OF SPACES WITH ALL FINITE POWERS LINDELÖF

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Abstract. We consider a new class of open covers and classes of spaces defined from them, called ι-spaces ("iota spaces"). We explore their relationship with ϵ-spaces (that is, spaces having all finite powers Lindelöf) and countable network weight. An example of a hereditarily ϵ-space whose square is not hereditarily Lindelöf is provided.

1. Introduction

A topological space in which each finite power is Lindelöf is called an *ϵ-space*. Equivalently, X is an ϵ-space if every open ω-cover of X has a countable ω-subcover, where a cover of a space X is an *ω-cover* if each finite subset of X is contained in an element of the cover. A natural generalization of an ω-cover can be defined by requiring that disjoint finite sets be separated by a member of the cover. We call a cover $\mathcal{U}$ of a space X an *ι-cover* if for every pair of disjoint finite sets $F, G \subseteq X$, there is a member $U \in \mathcal{U}$ such that $F \subseteq U$ and $G \cap U = \emptyset$. Notice that every space with a countable network is an ϵ-space. Furthermore, we will show that every T_2 space with a countable network has the property that every open ι-cover has a countable refinement that is also an ι-cover. Hence, we call such spaces with this property ι-spaces. The motivation for these definitions of ι-cover and ι-space arose when the third named author was trying to make the example in [8] zero dimensional to solve the D-space problem.

We will explore the relationship between ι-spaces and countable network weight, providing a ZFC example of a regular ι-space with no countable network. We also investigate the relationship between ϵ-spaces and ι-spaces, determining an additional property that makes them equivalent. We use the notion of an ι-cover to construct a hereditarily ϵ-space whose square is not hereditarily Lindelöf. Finally, we give an example of a non D-space that has a countable open ι-cover.

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2. Preliminaries

Definition 2.1. *A family of sets $\mathcal{U}$ is an ω-cover of X if for every $F \in [X]^{<\omega}$ there is $U \in \mathcal{U}$ such that $F \subset U$.*

Definition 2.2. *A family of sets $\mathcal{U}$ is an* ι-cover *(*n-ota cover*) of X if for every $F, G \in [X]^{<\omega}$ ($F, G \in [X]^n$) such that $F \cap G = \emptyset$ there is a member $U \in \mathcal{U}$ such that $F \subset U$ and $G \cap U = \emptyset$.*

In the following proposition we collect a few trivial facts about ι-covers and their relationship with ω-covers.

Proposition 2.3.

(1) *Every ι-cover is an ω-cover.*
(2) *Any open ω-cover of a T_1 topological space has an open refinement that is an ι-cover.*
(3) *Any fattening of an ι-cover is an ω-cover.*

Definition 2.4. *We call a space X an ϵ-space if every open ω-cover of X has a countable ω-subcover.*

Definition 2.5. *We call a space X an* ι-space *(*n-ota space*) if every open ι-cover (n-ota cover) of X has a countable refinement which is an ι-cover (n-ota cover).*

Remark 2.6. *In Definition* 2.5 *we used* refinement *rather than* subcover *because the class of spaces where every ι-cover has a countable ι-subcover coincides with the class of countable spaces. Indeed if X is uncountable and T_1 then $\{X \setminus F : F \in [X]^{<\omega}\}$ is an ι-cover without a countable ι-subcover.*

While every space X has a countable open ω-cover (simply consider $\{X\}$), not all spaces have countable ι-covers.

Example 2.7. *A compact T_2 space of size ω_1 without a countable ι-cover.*

Proof. Let $X = D \cup \{p\}$ be the one-point compactification of a discrete set of size $\aleph_1$, where p is the unique non-isolated point. Suppose by contradiction that X has a countable ι-cover $\mathcal{U}$ and let $\mathcal{U}_p = \{U \in \mathcal{U} : p \in U\}$. The set $\mathcal{U}_p$ is countable and every element of $\mathcal{U}_p$ is a cofinite set. Therefore, the set $\bigcap \mathcal{U}_p$ is uncountable and hence we can fix distinct points $x, y \in \bigcap \mathcal{U}_p$. But then $\mathcal{U}$ has no element containing $\{p, x\}$ and missing $\{y\}$. Therefore $\mathcal{U}$ is not an ι-cover. □

In view of Example 2.7 it makes sense to consider the following class of spaces.

Definition 2.8. *We call a space X an ι_w-space if it has a countable open ι-cover.*

Every ι-space is certainly an ι_w-space, but the converse is far from being true.

Proposition 2.9. *Let X be an ι_w-space. Then $|X| \leq \mathfrak{c}$.*

Proof. Let $\mathcal{U}$ be a countable open ι-cover for X. Define a map $f : X \to [\mathcal{U}]^\omega$ as follows: $f(x) = \{U \in \mathcal{U} : x \in U\}$. Since $\mathcal{U}$ is an ι-cover, f is a one-to-one map. Therefore $|X| \leq |[\mathcal{U}]^\omega| = \mathfrak{c}$. □

Corollary 2.10. *The discrete space of size κ is an ι_w-space if and only if $\kappa \leq \mathfrak{c}$.*

Proof. If $\kappa \leq \mathfrak{c}$ fix a separable metric topology τ on κ. Any ι-refinement of τ provides an open ι-cover of κ with the discrete topology. The converse follows from Proposition 2.9. □

There is, however, a natural relationship between ι-spaces and ι_w-spaces.

Theorem 2.11. *A space X is an ι-space if and only if it is an ϵ-space and an ι_w-space.*

Proof. The direct implication is trivial. To prove the converse implication, fix a countable ι-cover $\mathcal{C}$ for X and let $\mathcal{U}$ be any open ι-cover. Then $\mathcal{U}$ is also an ω-cover, and since X is an ϵ-space we can find a countable ω-subcover $\mathcal{V}$ of $\mathcal{U}$. The set $\{U \cap V : U \in \mathcal{C}, V \in \mathcal{V}\}$ is then a countable ι-refinement of $\mathcal{U}$. □

There are hereditarily Lindelöf spaces which are ι_w-spaces, but not ι-spaces. One such example is the Sorgenfrey line. Indeed, since its topology is a refinement of the topology of the real line, it has a countable ι-cover, but its square is not Lindelöf and hence it's not an ι-space. Note that by Theorem 2.11 if X is a subspace of the Sorgenfrey Line, then X is an ϵ-space if and only if X is an ι-space.

Theorem 2.12. *Let X be a Tychonoff space such that $C_p(X)$ is separable and has countable tightness. Then X is an ι-space.*

Proof. From [2], $C_p(X)$ has countable tightness if and only if X is an ϵ-space and $C_p(X)$ is separable if and only if X has a one-to-one continuous map onto a separable metrizable space. It's easy to see that this last condition is equivalent to X having a coarser second-countable topology. But this easily implies that X has a countable ι-cover, that is, X is an ι_w-space. □

Corollary 2.13. *Let X be a Tychonoff space. Suppose $C_p(X)$ is hereditarily separable. Then X is an ι-space.*

Proposition 2.14.

(1) *Let (X, τ) be an ι-space, then every closed subspace is an ι-space.*
(2) *Let (X, τ) be an ι_w-space, then every subspace is an ι_w-space.*

Proof. To prove (1) suppose Y is a closed subspace of the ι-space X. Fix an open cover $\mathcal{U}_Y$ of Y. Let $\mathcal{U} = \{U \in \tau : U \cap Y \in \mathcal{U}_Y\}$ and $\mathcal{U}^Y = \{U \in \tau : U \cap Y = \emptyset\}$.

Let

$$\mathcal{V} = \{(U \setminus F) \cup V : U \in \mathcal{U}, F \in [Y]^{<\omega}, V \in \mathcal{U}^Y\}$$

Then $\mathcal{V}$ is an ι-cover for the whole space X and the trace of any countable ι-refinement of $\mathcal{V}$ on Y is a countable ι-refinement of $\mathcal{U}_Y$.

The proof of (2) is similar and even easier. □

Corollary 2.15. *Let $\{X_i : i \in I\}$ be a family of spaces, where $|X_i| \geq 2$ and $|I| \geq \aleph_1$. Then $\prod_{i\in I} X_i$ is not an ι_w-space.*

Proof. Simply note that $\prod_{i\in I} X_i$ contains a copy of $2^{|I|}$, which in turn contains a copy of the one-point compactification of a discrete space of size $|I|$ and that this space is not an ι_w-space. □

Theorem 2.16. *Let $\{X_i : i < \omega\}$ be a countable family of ι_w-spaces. Then $X := \prod_{i<\omega} X_i$ is an ι_w-space.*

Proof. Let $\mathcal{U}_i$ be a countable open ι-cover for X_i. Consider two disjoint finite subsets F and G of X. For every $(x_n)_{n<\omega} \in F$ let $U_{x_i} \in \mathcal{U}_i$ be an open set containing x_i and missing $\{z \in \pi_{X_i}(F \cup G) : z \neq x_i\}$. Let now $U = \bigcup\{\prod_{i<\omega} U_{x_i} : (x_i)_{i<\omega} \in F\}$. Then U contains F and misses G. Indeed if $G \cap U$ were non-empty then there would be $(x_n)_{n<\omega} \in F$ such that $(\prod_{n<\omega} U_{x_n}) \cap G \neq \emptyset$, but this contradicts the definition of U_{x_n} for $n < \omega$.

It follows that $\{\bigcup\{\prod_{i<\omega} U_i : U_i \in \mathcal{F}_i\} : \mathcal{F}_i \in [\mathcal{U}_i]^{<\omega}\}$ is a countable open ι-cover for $\prod_{i<\omega} X_i$. □

Corollary 2.17. *Let $\{X_i : i \in I\}$ be a family of ι_w-spaces. Then $\prod_{i\in I} X_i$ is an ι_w-space if and only if $|I| \leq \omega$.*

3. Countable Network Weight

It is known that T_2 spaces with a countable network are ϵ-spaces, so using Theorem 2.11 we have the following.

Theorem 3.1. *Every T_2 space with a countable network is an ι-space.*

The converse is not true. We are going to present three counterexamples. The first one has the advantage of being simpler, the second one has the advantage of being regular, and the third one is only consistent, but we present it anyway, because the techniques used in verifying its properties might have independent interest.

Example 3.2. *There is a T_2 ι-space without a countable network.*

Proof. Let $\mathbb{R}_c$ be the real line with the topology generated by sets of the form $U \setminus C$, where U is a Euclidean open set and C is a countable set of reals.

Suppose by contradiction that $\{N_n : n < \omega\}$ is a countable network for $\mathbb{R}_c$. Without loss of generality we can assume that N_n is infinite for every n and use this to inductively pick $x_n \in N_n \setminus \{x_i : i < n\}$. then $\mathbb{R}_c \setminus \{x_i : i < \omega\}$ is an open set not containing any element of $\{N_n : n < \omega\}$. It follows that $\mathbb{R}_c$ does not have a countable network.

Now $\mathbb{R}_c$ is a refinement of the Euclidean topology on $\mathbb{R}$ and hence it is both an ϵ-space and an ι_w-space. Therefore, by Theorem 2.11, $\mathbb{R}_c$ is an ι-space.

□

Example 3.3. *There is a regular ι-space without a countable network within the usual axioms of ZFC.*

Proof. Let $X \subset \mathbb{R}$ be a subset of the reals. By Michael-type space $L(X)$ we mean the refinement of the usual topology on $\mathbb{R}$ obtained by isolating every point of $\mathbb{R} \setminus X$. By Theorem 2.11 every Michael-type space which is an ϵ-space is also an ι-space. It's easy to see, that if X is a Bernstein set (that is, a set which hits every uncountable closed set of the real line along with its complement), then $L(X)$ is Lindelöf, and Lawrence [6] proved that there is in ZFC a Bernstein set $X \subset \mathbb{R}$ such that $L(X)$ is an ϵ-space. The techniques used to construct the Bernstein set originated in [7] and Burke gives the details of the construction in [4].

□

The next construction preceded Theorem 2.11, but we include it because it may be of independent interest. It gives a recursive construction of an ι-space.

Example 3.4 (CH)**.** *There is a Michael space, M_X, that is an ι-space.*

Proof. For convenience, call $\mathcal{U}$ an open finite union (ofu)-ι-cover of $\mathbb{Q}$ if

(1) $\forall U \in \mathcal{U}$, $U = \bigcup_{i<n} I_i$ where $n \in \omega$, $I_i = (p_i, q_i)$, $p_i, q_i \in \mathbb{Q}$.
(2) $\mathbb{Q} \subseteq \bigcup \mathcal{U}$
(3) $\forall F, G \in [\mathbb{Q}]^{<\omega}$ such that $F \cap G = \emptyset$, $\exists U = \bigcup_{i<n} I_i \in \mathcal{U}$ such that $F \subseteq U$, $\bar{I}_i \cap G = \emptyset$, $\forall i < n$.

Let $\{\mathcal{U}_\alpha : \alpha < \omega_1\}$ enumerate all (ofu)-ι-covers of $\mathbb{Q}$. Define by recursion $X = \{x_\alpha : \alpha < \omega_1\}$ so that $\forall \alpha < \omega_1$, IH($\alpha$) holds, where

IH(α): $\forall \beta < \alpha, \mathcal{U}_\beta$ is an ι-cover of $\mathbb{Q} \cup \{x_\xi : \beta < \xi < \alpha\}$.

Let $x_0 \in \mathbb{R} \setminus \mathbb{Q}$.
Fix $\alpha < \omega_1$ and suppose $\{x_\xi : \xi < \alpha\}$ have been defined.
We must choose x_α so that IH($\alpha + 1$) is satisfied. That is, $\mathcal{U}_\beta$ is an ι-cover of $\mathbb{Q} \cup \{x_\xi : \beta < \xi \leq \alpha\}$, $\forall \beta \leq \alpha$.

Notation. For $\mathcal{U} \in \{\mathcal{U}_\beta : \beta \leq \alpha\}$, let $\mathbb{Q}_{\mathcal{U}} = \mathbb{Q} \cup \{x_\xi : \beta < \xi < \alpha\}$ where $\mathcal{U} = \mathcal{U}_\beta$.

Let $\mathcal{T} = \{(\mathcal{U}, F, G) \in \{\mathcal{U}_\beta : \beta \leq \alpha\} \times [\mathbb{Q} \cup \{x_\xi : \xi < \alpha\}]^{<\omega} \times [\mathbb{Q} \cup \{x_\xi : \xi < \alpha\}]^{<\omega} : F, G \in [\mathbb{Q}_\mathcal{U}]^{<\omega}, F \cap G = \emptyset\}$. Enumerate $\mathcal{T} = \{(\mathcal{U}_{\alpha n}, F_n, G_n) : n \in \omega\}$. For $n \in \omega$, let $\beta_n \leq \alpha$ such that $\mathcal{U}_{\alpha n} = \mathcal{U}_{\beta_n}$. Then, $F_n, G_n \in [\mathbb{Q} \cup \{x_\xi : \beta_n < \xi < \alpha\}]^{<\omega}$ and by IH(α), $\mathcal{U}_{\alpha n}$ is an ι-cover of $\mathbb{Q} \cup \{x_\xi : \beta_n < \xi < \alpha\}$.

Build sequences $\{q_n : n \in \omega\} \subseteq \mathbb{Q}$, $\{U_n^i : n \in \omega, i < 2\}$, $\{I_n : n \in \omega\}$ and $\{V_n : n \in \omega\}$ such that

(1) $U_n^i \in \mathcal{U}_{\alpha n} \forall i < 2$, $n \in \omega$.
(2) I_n, V_n open intervals.
(3) $q_0 \in \mathbb{Q} \setminus (F_0 \dot\cup G_0)$ and $q_n \in V_{n-1} \cap \mathbb{Q} \setminus (F_n \dot\cup G_n)$, $\forall n \geq 1$.
(4) $F_n \cup \{q_n\} \subseteq U_n^0$, $U_n^0 \cap G_n = \emptyset$ and $F_n \subseteq U_n^1$, $U_n^1 \cap (G_n \cup \{q_n\}) = \emptyset$.
(5) $q_0 \in I_0 \subseteq U_n^0$ and $q_n \in I_n \subseteq V_{n-1} \cap U_n^0$, $\forall n \geq 1$.
(6) $\overline{V}_n \subseteq I_n \setminus (U_n^1 \cup \{q_n\})$ such that $\mathrm{diam}(V_n) < \frac{1}{n}$ $(\forall n \geq 1)$

Let $q_0 \in \mathbb{Q} \setminus (F_0 \dot\cup G_0)$. Then $F_0 \cup \{q_0\}, G_0 \in [\mathbb{Q} \cup \{x_\xi : \beta_0 < \xi < \alpha\}]^{<\omega}$ so let $U_0^0 = \bigcup_{i<k_0} I_i \in \mathcal{U}_{\alpha 0}$ such that $F_0 \cup \{q_0\} \subseteq U_0^0$, $G_0 \cap U_0^0 = \emptyset$. Let $I_0 \in \{I_i : i < k_0\}$ such that $q_0 \in I_0$. Also, $F_0, G_0 \cup \{q_0\} \in [\mathbb{Q} \cup \{x_\xi : \beta_0 < \xi < \alpha\}]^{<\omega}$ so let $U_0^1 = \bigcup_{i<m_0} I_i$ such that $F_0 \subseteq U_0^1$, $(G_0 \cup \{q_0\}) \cap U_0^1 = \emptyset$. Let V_0 be an open interval such that $\overline{V}_0 \subseteq I_0 \setminus (U_0^1 \cup \{q_0\})$.
Fix $n \in \omega$ and suppose $\{q_m : m < n\}$, $\{U_m^i : i < 2, m < n\}$, $\{I_m : m < n\}$ and $\{V_m : m < n\}$ have been defined.
Let $q_n \in V_{n-1} \cap \mathbb{Q} \setminus (F_n \dot\cup G_n)$. Then $F_n \cup \{q_n\}, G_n \in [\mathbb{Q} \cup \{x_\xi : \beta_n < \xi < \alpha\}]^{<\omega}$ so let $U_n^0 = \bigcup_{i<k_n} I_i \in \mathcal{U}_{\alpha n}$ such that $F_n \cup \{q_n\} \subseteq U_n^0$, $G_n \cap U_n^0 = \emptyset$. Let $I_n \subseteq V_{n-1} \cap U_n^0$ be an open interval such that $q_n \in I_n$. Also, $F_n, G_n \cup \{q_n\} \in [\mathbb{Q} \cup \{x_\xi : \beta_n < \xi < \alpha\}]^{<\omega}$ so let $U_n^1 = \bigcup_{i<m_n} I_i$ such that $F_n \subseteq U_n^1$, $(G_n \cup \{q_n\}) \cap U_n^1 = \emptyset$. Let V_n be an open interval of diameter $< \frac{1}{n}$ such that $\overline{V}_n \subseteq I_n \setminus (U_n^1 \cup \{q_n\})$.
Let $x_\alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that $\bigcap_{n\in\omega} \overline{V}_n = \{x_\alpha\}$.
To see IH($\alpha + 1$) is satisfied, let $\beta \leq \alpha$ and notice $\mathcal{U}_\beta$ is an ι-cover of $\mathbb{Q} \cup \{x_\xi : \beta < \xi \leq \alpha\}$: Let $F, G \in [\mathbb{Q} \cup \{x_\xi : \beta < \xi \leq \alpha\}]^{<\omega}$ such that $F \cap G = \emptyset$. If $x_\alpha \in F$, let $F' = F \setminus \{x_\alpha\}$ and $m \in \omega$ such that $(\mathcal{U}_\beta, F', G) = (\mathcal{U}_{\alpha m}, F_m, G_m)$. Then, $U_m^0 \in \mathcal{U}_{\alpha m}$ such that $F_m \subseteq U_m^0, G_m \cap U_m^0 = \emptyset$ and $x_\alpha \in V_m \subseteq U_m^0$. Therefore, $U_m^0 \in \mathcal{U}_\beta$ such that $F \subseteq U_m^0$ and $G \cap U_m^0 = \emptyset$. If $x_\alpha \in G$, let $G' = G \setminus \{x_\alpha\}$ and $k \in \omega$ such that $(\mathcal{U}_\beta, F, G') = (\mathcal{U}_{\alpha k}, F_k, G_k)$. Then, $U_k^1 \in \mathcal{U}_{\alpha k}$ such that $F_k \subseteq U_k^1, G_k \cap U_k^1 = \emptyset$ and $x_\alpha \in V_k \subseteq I_k \setminus (U_k^1 \cup \{q_k\})$. Therefore, $U_k^1 \in \mathcal{U}_\beta$ such that $F \subseteq U_k^1$ and $G \cap U_k^1 = \emptyset$.

Therefore, by construction, $\mathcal{U}_\beta$ is an ι-cover of $\mathbb{Q} \cup \{x_\xi : \xi > \beta\}$, $\forall \beta < \omega_1$. So, $\mathcal{U}_\beta$ is an ι-cover of a tail of $\mathbb{Q} \cup X$.

Let $M_X = \mathbb{Q} \cup X$ with the Michael topology (usual basic open neighbourhoods for $\mathbb{Q}$ and isolate points of X). Let $\mathcal{U}$ be an open ι-cover of M_X.

Notation. For $F, G \in [X]^{<\omega}$ such that $F \cap G = \emptyset$, let $\mathcal{U}_{FG} = \{U \in \mathcal{U} : F \subseteq U, U \cap G = \emptyset\}$.

Claim. For any $F, G \in [X]^{<\omega}$ such that $F \cap G = \emptyset$, $\exists \alpha_{FG} < \omega_1$ such that $\mathcal{U}_{\alpha_{FG}} \prec \mathcal{U}_{FG}$.

Note that, $\mathcal{U}_{FG}$ is an ι-cover of $\mathbb{Q} \cup X \setminus (F \cup G)$. So, for each $F', G' \in [\mathbb{Q}]^{<\omega}$ such that $F' \cap G' = \emptyset$, let $U(F', G') \in \mathcal{U}_{FG}$ such that $F' \subseteq U(F', G')$ and $U(F', G') \cap G' = \emptyset$. For $x \in F'$, let $p_{x(F'G')}, q_{x(F'G')} \in \mathbb{Q}$ such that $x \in I_{x(F'G')} = (p_{x(F'G')}, q_{x(F'G')}) \subseteq U(F', G')$ but $\bar{I}_{x(F'G')} \cap G' = \emptyset$. Let $V_{F'G'} = \bigcup_{x \in F'} I_{x(F'G')}$. Then $\mathcal{V} = \{V_{F'G'} : F', G' \in [\mathbb{Q}]^{<\omega}, F' \cap G' = \emptyset\}$ is an (ofu)-ι-cover of $\mathbb{Q}$ that refines $\mathcal{U}_{FG}$. So, let $\alpha_{FG} < \omega_1$ such that $\mathcal{V} = \mathcal{U}_{\alpha_{FG}}$.

By a closing off argument, let $\bar{\alpha} < \omega_1$ such that $\mathcal{U}_{\bar{\alpha}} \prec \mathcal{U}_{FG}$, $\forall F, G \in [\{x_\xi : \xi \leq \bar{\alpha}\}]^{<\omega}$ such that $F \cap G = \emptyset$.

Claim. $\mathcal{U}' = \{U \cup F : U \in \mathcal{U}_{\bar{\alpha}}, F \in [\{x_\xi : \xi \leq \bar{\alpha}\}]^{<\omega}\}$ is a countable open refinement of $\mathcal{U}$ that is an ι-cover.

To see $\mathcal{U}' \prec \mathcal{U}$, let $U \cup F \in \mathcal{U}'$. Then $U \in \mathcal{U}_{\bar{\alpha}}$, $F \in [\{x_\xi : \xi \leq \bar{\alpha}\}]^{<\omega}$ so let $G \in [\{x_\xi : \xi \leq \bar{\alpha}\} \setminus F]^{<\omega}$ and since $\mathcal{U}_{\bar{\alpha}} \prec \mathcal{U}_{FG}$, let $U' \in \mathcal{U}_{FG}$ such that $U \subseteq U'$. Then, $U' \in \mathcal{U}$ such that $U \cup F \subseteq U'$.

To see $\mathcal{U}'$ is an ι-cover, let $F', G' \in [M_X]^{<\omega}$ such that $F' \cap G' = \emptyset$. Let $F'_1 = F' \cap \{x_\xi : \xi \leq \bar{\alpha}\}$, $F'_2 = F' \cap (\mathbb{Q} \cup \{x_\xi : \xi > \bar{\alpha}\})$, $G'_1 = G' \cap \{x_\xi : \xi \leq \bar{\alpha}\}$, $G'_2 = G' \cap (\mathbb{Q} \cup \{x_\xi : \xi > \bar{\alpha}\})$. Then, $F'_2, G'_2 \in [\mathbb{Q} \cup \{x_\xi : \xi > \bar{\alpha}\}]^{<\omega}$ such that $F'_2 \cap G'_2 = \emptyset$. So let $U \in \mathcal{U}_{\bar{\alpha}}$ such that $F'_2 \subseteq U$ and $U \cap G'_2 = \emptyset$. Then, $U \cup F'_1 \in \mathcal{U}'$ such that $F' \subseteq U \cup F'_1$ and $(U \cup F'_1) \cap G' = \emptyset$. □

Remark 3.5. *We constructed M_X so that any open ι-cover of M_X has a countable open refinement that ι-covers a tail of M_X, which is enough to show that M_X is an ι-space since the countable open refinement $\mathcal{U}'$ of $\mathcal{U}$ that ι-covers M_X is defined from an ι-cover of a tail or an almost ι-cover. This leads us to our next definition and some useful facts.*

Definition 3.6. *A space X is almost-ι if for every open ι-cover $\mathcal{U}$ of X, there is a countable open refinement $\mathcal{V}$ of $\mathcal{U}$ and $A \in [X]^\omega$ such that $\mathcal{V}$ is an ι-cover of $X \setminus A$.*

Note. Almost-ι is closed hereditary.

Lemma 3.7. *If X is almost-ι and has points regular G_δ then X is an ι-space.*

Before proving Lemma 3.7, we need the following:

Lemma 3.8. *If X is almost-ι and has points regular G_δ then $X \setminus F$ is almost-ι, $\forall F \in [X]^{<\omega}$.*

Proof. Fix $F \in [X]^{<\omega}$ and let $\mathcal{U}$ be any open ι-cover of $X \setminus F$. Since X has points regular G_δ, let $U_n \subseteq X$ be open such that $F \subseteq U_n$, $\forall n \in \omega$ and $F = \bigcap \overline{U_n}$. For $n \in \omega$, let $\mathcal{U}_n = \{U \cap X \setminus U_n : U \in \mathcal{U}\}$, which is an open ι-cover of $X \setminus U_n$. Thus, since $X \setminus U_n$ is almost-ι (being a closed subspace of an almost-ι space) let $\mathcal{U}'_n$ be a countable open refinement of $\mathcal{U}_n$ and $A_n \in [X \setminus U_n]^\omega$ such that $\mathcal{U}_n'$ is an ι-cover of $(X \setminus U_n) \setminus A_n$. Finally, $\forall n \in \omega$, let $\mathcal{V}_n = \{V \setminus \overline{U_n} : V \in \mathcal{U}_n'\}$. The following claim finishes the proof:

Claim. $\bigcup_{n\in\omega} \mathcal{V}_n$ is a countable open refinement of $\mathcal{U}$ that is an ι-cover of $(X \setminus F) \setminus (\bigcup_{n\in\omega} A_n)$.

Let $F', G' \in [(X \setminus F) \setminus (\bigcup_{n\in\omega} A_n)]^{<\omega}$ such that $F' \cap G' = \emptyset$. Then, $(F' \cup G') \cap F = \emptyset$ and $(F' \cup G') \cap \bigcup_{n\in\omega} A_n = \emptyset$. So, $\exists k \in \omega$ such that $(F' \cup G') \cap \overline{U_k} = \emptyset$ and $(F' \cup G') \cap A_k = \emptyset$. Thus, $F', G' \in [(X \setminus U_k) \setminus A_k]^{<\omega}$ such that $F' \cap G' = \emptyset$. So, let $U \in \mathcal{U}_k{}'$ such that $F' \subseteq U$ and $U \cap G' = \emptyset$. Then $U \setminus \overline{U_k} \in \mathcal{V}_k$ such that $F' \subseteq U \setminus \overline{U_k}$ and $(U \setminus \overline{U_k}) \cap G' = \emptyset$. □

Proof of Lemma 3.7. Let $\mathcal{U}$ be any open ι-cover of X. Let $\mathcal{M}$ be a countable elementary submodel of H_θ (for θ large enough) such that $\mathcal{U}, (X, \tau) \in \mathcal{M}$.

Claim. $\mathcal{V} = \{V \in \mathcal{M} \cap \tau : V \subseteq U \text{ for some } U \in \mathcal{U}\}$ is a countable open refinement of $\mathcal{U}$ that is an ι-cover of X.

Let $F', G' \in [X]^{<\omega}$ such that $F' \cap G' = \emptyset$. Let $F = F' \cap \mathcal{M}$ and $G = G' \cap \mathcal{M}$. By elementarity, $\mathcal{M} \models (X \setminus E \text{ is almost-}\iota,\ \forall E \in [X]^{<\omega})$. Thus, since $F, G \in \mathcal{M}$, $\mathcal{M} \models X \setminus (F \cup G)$ is almost-ι. Notice $\mathcal{U}_{FG} = \{U \in \mathcal{U} : F \subseteq U, U \cap G = \emptyset\} \in \mathcal{M}$ is an open ι-cover of $X \setminus (F \cup G)$. So, let $\mathcal{V}_{FG} \in \mathcal{M}$ be a countable open refinement of $\mathcal{U}_{FG}$ and $A_{FG} \in [X]^\omega \cap \mathcal{M}$ such that $\mathcal{V}_{FG}$ is an ι-cover of $(X \setminus (F \cup G)) \setminus A_{FG}$. Then, $\mathcal{V}_{FG}, A_{FG} \subseteq \mathcal{M}$. Thus $F' \setminus F, G' \setminus G \in [(X \setminus (F \cup G)) \setminus A_{FG}]^{<\omega}$ such that $F' \setminus F \cap G' \setminus G = \emptyset$. So let $V \in \mathcal{V}_{FG}$ such that $F' \setminus F \subseteq V$ and $V \cap G' \setminus G = \emptyset$. Since $\mathcal{V}_{FG}$ refines $\mathcal{U}_{FG}$, $\exists U \in \mathcal{U}_{FG}(V \subseteq U)$. So, by elementarity, let $U \in \mathcal{U}_{FG} \cap \mathcal{M}$ such that $V \subseteq U$. Also, $\mathcal{M} \models (x \text{ is regular } G_\delta,\ \forall x \in X)$, so in particular, since $F \in \mathcal{M}$ is finite, let $U_n \in \mathcal{M}$ such that $F \subseteq U_n$, $\forall n \in \omega$ and $F = \bigcap_{n\in\omega} \overline{U_n}$. Then, since $G' \cap F = \emptyset$, $\exists n \in \omega$ such that $G' \cap \overline{U_n} = \emptyset$ and $V \cup (U_n \cap U) \in \mathcal{V}$ such that $F' \subseteq V \cup (U_n \cap U)$ and $G' \cap (V \cup (U_n \cap U)) = \emptyset$. □

Returning again to the relationship with countable network weight, we have seen some (consistent) counterexamples, but restricting ourselves to the hereditary property raises the natural question.

Question 3.9. *Is every hereditarily ι-space a space with a countable network?*

4. ϵ-spaces

Theorem 2.11 provides us with an instance when ϵ-spaces and ι-spaces are equivalent. We investigate what additional characteristics can be placed on an ϵ-space to ensure it is an ι-space.

Definition 4.1. *Let $\mathcal{U}$ be a cover of a space X. We say that $\mathcal{U}$ is a* regular 1-ota cover *if for every $x \neq y \in X$ there is $U \in \mathcal{U}$ such that $x \in U$ and $y \notin \overline{U}$.*

Lemma 4.2. *Let $\mathcal{U}$ be an ι-cover of the regular space X. Then $\mathcal{U}$ has a regular 1-ota refinement.*

Proof. For every $x \neq y \in X$ choose $U(x,y) \in \mathcal{U}$ such that $x \in U(x,y)$ and $y \notin U(x,y)$. Now let $V(x,y)$ be an open set such that $x \in V(x,y) \subset \overline{V(x,y)} \subset U(x,y)$. Then $\mathcal{V} = \{V(x,y) : x \neq y \in X\}$ is a regular 1-ota refinement of $\mathcal{U}$. □

Lemma 4.3. *Let X be a regular space such that $X^2 \setminus \Delta$ is Lindelöf. Then X is 1-ota.*

Proof. Let $\mathcal{U}$ be a 1-ota cover for X without a countable 1-ota refinement. Let $\mathcal{V}$ be a regular 1-ota refinement of $\mathcal{U}$ having minimal size $\kappa \geq \omega_1$.

Fix an enumeration $\{V_\alpha : \alpha < \kappa\}$ of $\mathcal{V}$ and let $\mathcal{V}_\alpha := \{V_\beta : \beta \leq \alpha\}$ and:

$$A(\mathcal{V}_\alpha) := \{(x,y) \in X^2 \setminus \Delta : (\forall U \in \mathcal{V}_\alpha)((x \in U \wedge y \in \overline{U}) \vee (x \in \overline{U} \wedge y \in U) \vee (\{x,y\} \cap U = \emptyset)\}.$$

Claim. $A(\mathcal{V}_\alpha) \neq \emptyset$ and $A(\mathcal{V}_\alpha)$ is closed in $X^2 \setminus \Delta$ for every $\alpha < \omega_1$.

Proof of Claim. The fact that $A(\mathcal{V}_\alpha) \neq \emptyset$ follows from the assumptions about $\mathcal{U}$. To prove that $A(\mathcal{V}_\alpha)$ is closed, let $(x,y) \notin A(\mathcal{V}_\alpha) \cup \Delta$. Then we can find $U_x, U_y \in \mathcal{V}_\alpha$ such that $x \in U_x$, $y \notin \overline{U_x}$, $y \in U_y$ and $x \notin \overline{U_y}$. Now $(U_x \setminus \overline{U_y}) \times (U_y \setminus \overline{U_x})$ is an open neighbourhood of (x,y) which misses $A(\mathcal{V}_\alpha)$. □

So $\{A(\mathcal{V}_\alpha) : \alpha < \kappa\}$ is an uncountable decreasing sequence of non-empty closed subsets of the Lindelöf space $X^2 \setminus \Delta$ and thus $\bigcap\{A(\mathcal{V}_\alpha) : \alpha < \omega_1\} \neq \emptyset$ which contradicts regularity of $\mathcal{V}$. □

Note that the fact that $X^2 \setminus \Delta$ is Lindelöf implies that X has a G_δ diagonal, whenever X is regular. Indeed, for every $x \in X^2 \setminus \Delta$, let U_x be an open neighbourhood of x such that $\overline{U_x} \cap \Delta = \emptyset$. The family $\{U_x : x \in X^2 \setminus \Delta\}$ covers $X^2 \setminus \Delta$, and hence there is a countable set $C \subset X^2 \setminus \Delta$ such that $X^2 \setminus \Delta = \bigcup\{\overline{U_x} : x \in C\}$ and hence $\Delta = \bigcap_{x \in C} X^2 \setminus \overline{U_x}$, which proves that Δ is a G_δ subset of X.

The following lemma is not new. For example, the proof of a more general statement can be found in [3]. We nevertheless include a quick direct proof of it for the reader's convenience.

Lemma 4.4. *Every countably compact 1-ota space X is metrizable.*

Proof. Let $\mathcal{U}$ be an ι-cover of X. By Lemma 4.2 we can assume that $\mathcal{U}$ is a regular ι-cover. Let $\mathcal{V}$ be a countable ι-refinement of $\mathcal{U}$ and let $\mathcal{B} = \{X \setminus \overline{\bigcup \mathcal{F}} : \mathcal{F} \in [\mathcal{V}]^{<\omega}\}$. The set $\mathcal{B}$ is countable. We claim that $\mathcal{B}$ is a base of X, proving that X is metrizable.

To see that let U be an open set and $x \in U$. For every $y \in X \setminus U$ choose an open set $U_y \in \mathcal{V}$ such that $y \in U_y$ and $x \notin \overline{U_y}$. The countable set $\{U_y : y \in X \setminus U\}$ covers

$X \setminus U$ so we can choose a finite set $F \subset X \setminus U$ such that $X \setminus U \subset \bigcup_{y \in F} U_y$. Then $x \in \bigcap_{y \in F} X \setminus \overline{U_y} \subset U$ and hence $\mathcal{B}$ is a base. □

Corollary 4.5. *There are no countably compact strong L-spaces.*

Proof. If X^2 is hereditarily Lindelöf then X is 1-ota and every countably compact 1-ota space is metrizable and thus separable. □

Corollary 4.6. *Every compact space with a G_δ diagonal is metrizable.*

Proof. If X is a compact space with a G_δ diagonal then $X^2 \setminus \Delta$ is σ-compact. Thus X is a compact 1-ota space and hence it's metrizable. □

Generalizing Lemma 4.3 provides us with a characterization that we are looking for.

Let $\Delta_n = \{(x_1, \dots, x_n) \in X^n : |\{x_1, x_2, \dots, x_n\}| < n\}$. Clearly, Δ_n is a closed subset of X^n.

Theorem 4.7. *Assume X is a regular space. If $X^{2n} \setminus \Delta_{2n}$ is Lindelöf for every $n \in \omega$ then X is an ι-space.*

Proof. Suppose that $X^{2n} \setminus \Delta_{2n}$ is Lindelöf. We prove that X is n-ota. Indeed, let $\mathcal{U}$ be a n-ota cover without a countable n-ota refinement. Let $\mathcal{V}$ be a regular n-ota refinement of $\mathcal{U}$ having minimal size $\kappa \geq \omega_1$. Enumerate $\mathcal{V}$ as $\{V_\alpha : \alpha < \aleph_1\}$ and let $\mathcal{V}_\alpha = \{V_\beta : \beta \leq \alpha\}$ and

$$A(\mathcal{V}_\alpha) = \{(x_1, \dots x_{2n}) \in X^{2n} \setminus \Delta_{2n} : (\forall U \in \mathcal{V}_\alpha)(\{x_1, \dots x_n\} \subseteq U \wedge \{x_{n+1}, \dots, x_{2n}\} \cap \overline{U} \neq \emptyset)$$
$$\vee (\{x_1, \dots, x_n\} \cap \overline{U} \neq \emptyset \wedge \{x_{n+1}, \dots x_{2n}\} \subseteq U) \vee (\{x_1, \dots, x_n\} \nsubseteq U \wedge \{x_{n+1}, \dots x_{2n}\} \nsubseteq U)\}$$

Claim. $A(\mathcal{V}_\alpha)$ is closed.

Proof of Claim. Let $(x_1, \dots, x_{2n}) \notin A(\mathcal{V}_\alpha) \cup \Delta_{2n}$. Then we can find sets U_1 and U_2 which are open in x and such that $\{x_1, \dots, x_n\} \subset U_1$, $\{x_{n+1}, \dots x_{2n}\} \cap \overline{U_1} = \emptyset$, $\{x_{n+1}, \dots, x_{2n}\} \subset U_2$ and $\{x_1, \dots, x_n\} \cap \overline{U_2} = \emptyset$. Then $(U_1^n \setminus \overline{U_2^n}) \times (U_2^n \setminus \overline{U_1^n})$ is an open neighbourhood of $(x_1, \dots x_n, x_{n+1}, \dots, x_{2n})$ which misses $A(\mathcal{V}_\alpha)$. □

So $\{A(\mathcal{V}_\alpha) : \alpha < \kappa\}$ is an uncountable decreasing chain of closed sets in $X^{2n} \setminus \Delta_{2n}$, and hence it has non-empty intersection. This contradicts that $\mathcal{V}$ is a regular n-ota cover. So if $X^i \setminus \Delta_i$ is Lindelöf for every $i < \omega$ then X is i-ota for every $i < \omega$ and hence an ι-space.

□

Corollary 4.8. *Every ϵ-space with a G_δ diagonal is an ι-space.*

Proof. Suppose X has a G_δ diagonal. Then Δ_n is a finite union of G_δ sets, and thus G_δ. It follows that $X^n \setminus \Delta_n$ is a countable union of Lindelöf spaces, and thus Lindelöf. So X is an ι-space by Theorem 4.7. □

Corollary 4.9. *If $X^{2n} \setminus \Delta_{2n}$ is Lindelöf for some n, then X is Lindelöf.*

Proof. By the proof of Theorem 4.7, X is an n-ota space. But every n-ota space is Lindelöf. Indeed, let $\mathcal{U}$ be an open cover for X. Let $\mathcal{V}$ be the set of all n-sized unions from $\mathcal{U}$. Let $\mathcal{G}$ be an n-ota refinement of $\mathcal{V}$ and $\mathcal{F}$ be a countable refinement of $\mathcal{G}$. Then $\mathcal{F}$ naturally induces a countable refinement of the original cover $\mathcal{U}$. □

Recall that a space X is a *Lindelöf Σ-space* if it has a cover $\mathcal{C}$ by compact sets, and a countable family $\mathcal{N}$ of closed subsets of X which is a *network modulo $\mathcal{C}$*, that is, for every $C \in \mathcal{C}$ and every open set U such that $C \subset U$ there is $N \in \mathcal{N}$ such that $C \subset N \subset U$. We will use this notion to provide an instance of when being an ι-space and having a countable network are equivalent. The proof of the following theorem is similar to the proof that every Lindelöf Σ-space is stable (see, for example, [10]).

Theorem 4.10. *Let X be a regular Lindelöf Σ-space. If X is an ι-space then X has a countable network.*

Proof. Let $\mathcal{C}$ be a cover of X consisting of compact sets and $\mathcal{N}$ be a countable family which is a countable network for X modulo $\mathcal{C}$. Since X is regular, we can use Lemma 4.2 to fix a countable regular ι-cover $\mathcal{U}$ for X. Let $\mathcal{G} = \{\overline{U} : U \in \mathcal{U}\}$. We claim that the following family is a countable network for X:

$$\mathcal{B} = \left\{ \bigcap \mathcal{F} : \mathcal{F} \in [\mathcal{G} \cup \mathcal{U}]^{<\omega} \right\}.$$

To see that, let $x \in X$ and U be an open neighbourhood of x. Since $\mathcal{C}$ covers X, there is a $C \in \mathcal{C}$ such that $x \in C$. If $C \subset U$ then we can find $N \in \mathcal{N}$ such that $x \in C \subset N \subset U$ and we are done. Otherwise, the set $K = C \setminus U$ is compact non-empty. For every $y \in K$ choose a set $G_y \in \mathcal{G}$ such that $y \in G$ but $x \notin G$. Then $\{G_y \cap K : y \in K\}$ has empty intersection, and hence, by compactness of K there is a finite set $S \subset K$ such that $\bigcap_{y \in S} G_y \cap K = \emptyset$. Therefore, $\bigcap_{y \in S} G_y \cap C \setminus U = \emptyset$, and hence $C \subset X \setminus (\bigcap_{y \in S} G_y \setminus U)$. But then there is an $N \in \mathcal{N}$ such that $C \subset N \subset X \setminus (\bigcap_{y \in S} G_y \setminus U)$. Therefore $N \cap \bigcap_{y \in S} G_y \setminus U = \emptyset$ and hence $x \in N \cap \bigcap_{y \in S} G_y \subset U$, which is what we wanted, since $N \cap \bigcap_{y \in S} G_y \in \mathcal{B}$.

□

Question 4.11. *Is there a Lindelöf Hausdorff Σ-space without a countable network which is an ι-space?*

Note that a Lindelöf Σ-space which is an ι_w-space is also an ι-space, since countable products of Lindelöf Σ-spaces are Lindelöf Σ.

5. L-spaces

Since every ι-space is an ϵ-space and ϵ-spaces are characterized by having all finite powers Lindelöf, L-spaces are interesting spaces for us to consider. It is conjectured that even in ZFC there is an L-space that is not even an ϵ-space. That is, of course, it is conjectured that Justin Moore's L-space has a finite power which is not Lindelöf in ZFC. Certainly this is consistently known.

Proposition 5.1. *Consistently, every hereditarily Lindelöf ι-space is separable.*

Proof. Under MA_{ω_1}, every L-space has a finite power which is not Lindelöf. Now, every ι-space is an ϵ-space. □

To investigate the consistency of the negation of this statement, we focus on more general classes of spaces that yield L-spaces and provide many counterexamples in topology. These spaces are subspaces of products of the form 2^I, where I is a set of ordinals. So, the following notation is useful.

Notation.

- $Fn(I,2)$ is the set of finite partial functions from I into 2.
- For $\varepsilon \in Fn(I,2)$, $[\varepsilon] = \{f \in 2^I : \varepsilon \subseteq f\}$ denotes the basic clopen set determined by ε.
- If $b \in [I]^{<\omega}$ such that $b = \{\beta_i : i \in n = |b|\}$ and $\varepsilon \in 2^n$ then $\varepsilon * b$ denotes the element of $Fn(I,2)$ which has b as its domain and satisfies $\varepsilon * b(\beta_i) = \varepsilon(i)$, $\forall i \in n$.
- For any cardinal μ and $r \in \omega$ we denote by $\mathcal{D}^r_\mu(I)$ the collection of all sets $B \in [[I]^r]^\mu$ such that the members of B are pairwise disjoint. We write $\mathcal{D}_\mu(I) = \bigcup\{\mathcal{D}^r_\mu(I) : r \in \omega\}$ and if $B \in \mathcal{D}_\mu(I)$ then $n(B) = |b|$ for any $b \in B$.

Definition 5.2. *If $B \in \mathcal{D}_\mu(I)$ and $\varepsilon \in 2^{n(B)}$ then $[\varepsilon, B] = \bigcup\{[\varepsilon * b] : b \in B\}$ is called a $\mathcal{D}_\mu$-set in 2^I.*

Definition 5.3. *$X \subseteq 2^\lambda$ with $|X| > \omega$ is an HFC space if for every $B \in \mathcal{D}_\omega(\lambda)$ and $\varepsilon \in 2^{n(B)}$, $|X \setminus [\varepsilon, B]| \leq \omega$. That is, every $\mathcal{D}_\omega$-set in 2^λ finally covers X.*

Definition 5.4. *For any $k \in \omega$, a map $F : \kappa \times \lambda \to 2$ with $\kappa \geq \omega_1$ and $\lambda \geq \omega$ ($\lambda \geq \omega_1$) is called an HFC^k (HFC^k_w) matrix if for every $A \in \mathcal{D}^k_{\omega_1}(\kappa)$ and $B \in \mathcal{D}_\omega(\lambda)$ ($B \in \mathcal{D}_{\omega_1}(\lambda)$) and for any $\varepsilon_0, \ldots, \varepsilon_{k-1} \in 2^{n(B)}$ there exists $b \in B$ such that $|\{a \in A : \forall i \in k(f_{\alpha_i} \supseteq \varepsilon_i * b)\}| = \omega_1$, where $\{\alpha_i : i \in k\}$ is the increasing enumeration of the elements of a.*

F is a strong HFC (HFC$_w$) matrix if it is HFC^k (HFC^k_w) for all $k \in \omega$.

$X \subseteq 2^\lambda$ is a strong HFC (HFC$_w$) space if it is represented by a strong HFC (HFC$_w$) matrix, F. That is $X = \{f_\alpha : \alpha < \kappa\}$ where $f_\alpha(\gamma) = F(\alpha, \gamma)$, $\forall \gamma < \lambda$.

Theorem 5.5. [5] *If X is a strong HFC_w space (hence strong HFC) then X^k is hereditarily Lindelöf, $\forall k \in \omega$.*

Corollary 5.6. *Every strong HFC is an ι-space. In fact, if X^n is hereditarily Lindelöf, $\forall n \in \omega$ then X is an ι-space.*

Proof. Follows from Theorem 4.7. □

In contrast

Example 5.7 (CH)**.** *There is an HFC with no countable open ι-cover.*

Proof. By CH, enumerate the collection of $\mathcal{D}_\omega$-sets in 2^{ω_1} by $\{u_\alpha : \alpha < \omega_1\}$ so that for $n < \omega$, $\{\sigma_{ni} : i \in \omega\} \subseteq Fn(\omega_1, 2)$ such that $\{\mathrm{dom}(\sigma_{ni}) : i \in \omega\}$ is a pairwise disjoint collection of finite subsets of ω and $u_n = \bigcup_{i\in\omega}[\sigma_{ni}]$. Moreover, for $\alpha \geq \omega$, $\{\sigma_{\alpha i} : i \in \omega\} \subseteq Fn(\omega_1, 2)$ such that $\{\mathrm{dom}(\sigma_{\alpha i}) : i \in \omega\}$ is pairwise disjoint and $u_\alpha = \bigcup_{i\in\omega}[\sigma_{\alpha i}]$. For $\alpha \geq \omega$, let $\mathcal{U}_\alpha = \{u_\beta : \beta < \alpha, \mathrm{dom}(\sigma_{\beta i}) \subseteq \alpha, \forall i \in \omega\}$. Enumerate $\mathcal{U}_\alpha = \{v_{\alpha i} : i \in \omega\}$ where each $u \in \mathcal{U}_\alpha$ appears as infinitely many $v_{\alpha i}$'s. Construct HFCs $X = \{x_\alpha : \omega \leq \alpha < \omega_1\}$ and $Y = \{y_\alpha : \omega \leq \alpha < \omega_1\}$ by induction, defining $x \restriction \alpha$, $y \restriction \alpha$ at stage α and letting $x_\alpha(\gamma) = 0$, $\forall \gamma \geq \alpha$, $y_\alpha(\gamma) = 1$, $\forall \gamma \geq \alpha$.

For $\omega \leq \alpha < \omega_1$, define $\{\sigma_i^\alpha : i \in \omega\}$ such that

(i): if $v_{\alpha i} = u_\beta$ then $\sigma_i^\alpha = \sigma_{\beta j}$ for some $j \in \omega$.
(ii): $\{\mathrm{dom}(\sigma_i^\alpha) : i \in \omega\}$ is pairwise disjoint.

$v_{\alpha 0} \in \mathcal{U}_\alpha \Rightarrow v_{\alpha 0} = u_\beta$ for some $\beta < \alpha$ so let $\sigma_0^\alpha = \sigma_{\beta 0}$.

Fix $n > 0$ and suppose $\{\sigma_i^\alpha : i < n\}$ have been defined. Again, since $v_{\alpha n} \in \mathcal{U}_\alpha$, let $\gamma < \alpha$ such that $v_{\alpha n} = u_\gamma$, where $u_\gamma = \bigcup\{[\sigma_{\gamma i}] : i \in \omega\}$ with $\{\mathrm{dom}(\sigma_{\gamma i}) : i \in \omega\}$ pairwise disjoint. Thus, let $j_n \in \omega$ such that $\mathrm{dom}(\sigma_{\gamma j_n}) \cap \mathrm{dom}(\sigma_i^\alpha) = \emptyset$, $\forall i < n$ and $\sigma_n^\alpha = \sigma_{\gamma j_n}$.

For $\omega \leq \alpha < \omega_1$, define x_α, $y_\alpha \in 2^{\omega_1}$ as follows: $x_\alpha(\gamma) = y_\alpha(\gamma) = \sigma_i^\alpha(\gamma)$, $\forall \gamma \in \bigcup_{i\in\omega}\mathrm{dom}(\sigma_i^\alpha)$, $x_\alpha(\gamma) = y_\alpha(\gamma) = 0$, $\forall \gamma \in \alpha \setminus \bigcup_{i\in\omega}\mathrm{dom}(\sigma_i^\alpha)$ and as above, $x_\alpha(\gamma) = 0$, $\forall \gamma \geq \alpha$, $y_\alpha(\gamma) = 1$, $\forall \gamma \geq \alpha$.

Claim. $X \cup Y$ is an HFC with no countable open ι-cover.

To see $X \cup Y$ is an HFC, fix $\beta < \omega_1$ and show u_β is a final cover of $X \cup Y$. Note that $\forall \beta < \omega_1$, $\exists \delta < \omega_1$ such that $u_\beta \in \mathcal{U}_\delta$.

So, let $\delta_\beta = \min\{\delta < \omega_1 : u_\beta \in \mathcal{U}_\delta\}$. Then $X \cup Y \setminus (\{x_\gamma : \gamma < \delta_\beta\} \cup \{y_\gamma : \gamma < \delta_\beta\}) \subseteq u_\beta$.

Let $\mathcal{U} = \{U_n : n \in \omega\}$ be any countable open cover of $X \cup Y$. Since $X \cup Y$ is hereditarily Lindelöf (being HFC), let $\sigma_n(i) \in Fn(\omega_1, 2)$ such that $U_n = \bigcup_{i\in\omega}[\sigma_n(i)] \cap (X \cup Y)$, $\forall n \in \omega$. Let $\alpha_n = \sup(\bigcup_{i\in\omega}\mathrm{dom}(\sigma_n(i))) < \omega_1$ and $\alpha = \sup\{\alpha_n : n \in \omega\} < \omega_1$.

We claim that $\forall\beta > \alpha$, $x_\beta \in U_n \Leftrightarrow y_\beta \in U_n$, $\forall n \in \omega$ and hence $\mathcal{U}$ is not an ι-cover. Fix $\beta > \alpha$, $n \in \omega$.

$$\begin{aligned}
x_\beta \in U_n &\Leftrightarrow (\exists i \in \omega) x_\beta \in [\sigma_n(i)] \\
&\Leftrightarrow (\exists i \in \omega) \sigma_n(i) \subseteq x_\beta \\
&\Leftrightarrow (\exists i \in \omega) x_\beta(\gamma) = \sigma_n(i)(\gamma) \forall \gamma \in dom(\sigma_n(i)) \\
&\Leftrightarrow (\exists i \in \omega) y_\beta(\gamma) = \sigma_n(i)(\gamma) \forall \gamma \in dom(\sigma_n(i)) \\
&\Leftrightarrow (\exists i \in \omega) \sigma_n(i) \subseteq y_\beta \\
&\Leftrightarrow (\exists i \in \omega) y_\beta \in [\sigma_n(i)] \\
&\Leftrightarrow y_\beta \in U_n
\end{aligned}$$

□

This gives us another example of an L-space that is not an ι-space, in fact, not even an ι_w-space. Although we already know consistently (under MA_{ω_1}) that this space is not even an ϵ-space, the argument used to show the space has no countable open ι-cover will be used to show what we really want: there is a hereditarily ϵ-space that is not an ι_w-space. Naively we tried to extend this argument to a strong HFC space (a hereditarily ϵ-space), but along the way we discovered the missing ingredient. Thus Example 5.7 also provides an example of a certainly already known result.

Corollary 5.8. *There is a pair of strong HFCs whose union is not a strong HFC.*

Proof. Let $X = \{x_\alpha : \alpha < \omega_1\}$, $Y = \{y_\alpha : \alpha < \omega_1\}$ be the HFCs from Example 5.7. Let $f_X : X \to \omega_1$ such that $f_X(x_\alpha) = \alpha$ and $f_Y : Y \to \omega_1$ such that $f_Y(y_\alpha) = \alpha$.

Claim. $\exists A \in [\omega_1]^{\omega_1}$ such that $\{x_\alpha : \alpha \in A\}$, $\{y_\alpha : \alpha \in A\}$ are strong HFCs.

Proof of Claim. Let $\vec{N} = \langle N_\alpha : \alpha < \omega_1 \rangle$ be an ω_1-chain of countable elementary submodels of some H_θ such that $X, Y, f_X, f_Y \in N_0$ and $\beta < \alpha < \omega_1 \Rightarrow N_\beta \subsetneq N_\alpha$. Define by recursion $Z = \{z_\alpha : \alpha < \omega_1\} \subseteq X$, separated by $\vec{N}$:
Let $z_0 \in X \cap N_0$
Fix $\alpha > 0$ and suppose $\{z_\beta : \beta < \alpha\}$ have been defined such that $z_\beta \in X \cap N_\beta \setminus \bigcup_{\gamma<\beta} N_\gamma$. Since X is uncountable and $\bigcup_{\beta<\alpha} N_\beta$ is countable, $X \setminus \bigcup_{\beta<\alpha} N_\beta \neq \emptyset$. So, by elementarity, let $z_\alpha \in X \cap N_\alpha \setminus \bigcup_{\beta<\alpha} N_\beta \neq \emptyset$.
To see Z is separated by $\vec{N}$, let $\{z_\alpha, z_\beta\} \in [Z]^2$. Without loss of generality, suppose $\alpha < \beta$. Then, by construction, $N_\alpha \cap \{z_\alpha, z_\beta\} = \{z_\alpha\}$ and hence $\exists\alpha < \omega_1$ such that $|N_\alpha \cap \{z_\alpha, z_\beta\}| = 1$.
Then, by Theorem 2.1 of [9], Z is a strong HFC. Since $Z \in [X]^{\omega_1}$, let $A \in [\omega_1]^{\omega_1}$ such that $Z = \{x_\alpha : \alpha \in A\}$. We claim that $\{y_\alpha : \alpha \in A\}$ is separated by $\vec{N}$ and hence is a strong HFC (again by Theorem 2.1 of [9]).

Note. $\forall \alpha, \gamma < \omega_1$, $x_\alpha \in N_\gamma \Leftrightarrow y_\alpha \in N_\gamma$.

Proof of Note. Suppose $x_\alpha \in N_\gamma$. Since $f_X \in N_\gamma$, $f_X(x_\alpha) = \alpha \in N_\gamma$ and hence $x_\alpha \restriction \alpha \in N_\gamma$. Recall that y_α is definable from $x_\alpha \restriction \alpha$, $\alpha \in N_\gamma$ since $y_\alpha \restriction \alpha = x_\alpha \restriction \alpha$ and $y_\alpha(\gamma) = 1$, $\forall \gamma \geq \alpha$. Hence $y_\alpha \in N_\gamma$. Similarly, $y_\alpha \in N_\gamma \Rightarrow x_\alpha \in N_\gamma$. □

Then it is clear $\{y_\alpha : \alpha \in A\}$ is separated by $\vec{N}$. If $\{y_\alpha, y_\beta\} \in [\{y_\alpha : \alpha \in A\}]^2$ then $\{x_\alpha, x_\beta\} \in [Z]^2$ so $\exists \gamma < \omega_1$ such that $|N_\gamma \cap \{x_\alpha, x_\beta\}| = 1 \Leftrightarrow |N_\gamma \cap \{y_\alpha, y_\beta\}| = 1$ (by the note). □

Therefore, $\{x_\alpha : \alpha \in A\}, \{y_\alpha : \alpha \in A\}$ are strong HFCs and as in Example 5.7, $\{x_\alpha : \alpha \in A\} \cup \{y_\alpha : \alpha \in A\}$ has no countable ι-cover. Thus, $\{x_\alpha : \alpha \in A\} \cup \{y_\alpha : \alpha \in A\}$ is not an ι-space and hence is not a strong HFC by Corollary 5.6. □

Fortunately, considering strong HFC_w spaces and working a little harder provides us with the desired example. In [5], Juhász constructs a strong HFC_w space in a generic extension obtained by adding a Cohen or random real (in fact a generic extension with a slightly more general property). Using this same construction, we obtain two strong HFC_w spaces whose union is a hereditarily ϵ-space but has no countable open ι-cover, hence not ι_w. This gives an example of a space in which every subspace has any finite power Lindelöf, but there are two subspaces whose product is not Lindelöf. In particular, all squares of subspaces are Lindelöf, but there is a rectangle that is not Lindelöf; a hereditarily Lindelöf space whose square is not hereditarily Lindelöf.

In comparison to Definition 3.6,

Definition 5.9. *A space X is almost-ϵ if for every open ω-cover $\mathcal{U}$ of X, there is a countable $\mathcal{V} \subseteq \mathcal{U}$ and $A \in [X]^\omega$ such that $\mathcal{V}$ is an ω-cover of $X \setminus A$.*

Lemma 5.10. *If X is almost-ϵ then X is an ϵ-space.*

Proof. Let $\mathcal{U}$ be any open ω-cover of X and $\mathcal{M}$ be a countable elementary submodel of some H_θ (θ sufficiently large) such that $\mathcal{U}, (X, \tau) \in \mathcal{M}$.

Claim. $\mathcal{U} \cap \mathcal{M}$ is a countable ω-subcover of $\mathcal{U}$.

Let $F \in [X]^{<\omega}$ and consider $\mathcal{U}_{F\cap\mathcal{M}} = \{U \in \mathcal{U} : F \cap \mathcal{M} \subseteq U\} \in \mathcal{M}$ (since $F \cap \mathcal{M} \subseteq \mathcal{M}$ is finite). Notice that $\mathcal{U}_{F\cap\mathcal{M}}$ is an open ω-cover of X and since, by elementarity, $\mathcal{M} \models X$ is almost-ϵ, let $\mathcal{V} \in \mathcal{M}$ be countable and $A \in [X]^\omega \cap \mathcal{M}$ such that $\mathcal{V} \subseteq \mathcal{U}_{F\cap\mathcal{M}}$ is an ω-cover of $X \setminus A$. Since $A, \mathcal{V} \in \mathcal{M}$ are countable, $A, \mathcal{V} \subseteq \mathcal{M}$. In particular, $\mathcal{V} \subseteq \mathcal{U} \cap \mathcal{M}$. Also, since $A \subseteq \mathcal{M}$, $F \setminus \mathcal{M} \in [X \setminus A]^{<\omega}$ so let $V \in \mathcal{V} \subseteq \mathcal{U}_{F\cap\mathcal{M}}$ such that $F \setminus \mathcal{M} \subseteq V$. Then $V \in \mathcal{U} \cap \mathcal{M}$ such that $F \subseteq V$. □

The following alternate characterization of an HFC_w^k space is an adaptation of the characterization of an HFC_w space from [5].

Theorem 5.11. *For any $k \in \omega$, if $X \subseteq 2^\lambda$ with $|X| > \omega$ and $\lambda > \omega$ is HFC^k_w, then*

$$(*_k) \qquad \forall B \in \mathcal{D}_{\omega_1}(\lambda), \forall \varepsilon_0, \ldots, \varepsilon_{k-1} \in 2^{n(B)}, \exists C \in [B]^\omega, \exists \alpha \in \kappa$$
$$(\forall a = \{\alpha_i : i < k\} \in [\kappa \setminus \alpha]^k)(\exists b \in C) f_{\alpha_i} \supseteq \varepsilon_i * b, \forall i \in k.$$

Proof. Suppose, by way of contradiction, that there is $B \in \mathcal{D}_{\omega_1}(\lambda)$ and $\varepsilon_0, \ldots, \varepsilon_{k-1} \in 2^{n(B)}$ such that $\forall C \in [B]^\omega$ and $\forall \alpha < \kappa$, $\exists a = \{\alpha_i : i < k\} \in [\kappa \setminus \alpha]^k$ and $\exists j \in k$ such that $f_{\alpha_j} \not\supseteq \varepsilon_j * b$, $\forall b \in C$. Enumerate $B = \{b_\gamma : \gamma < \omega_1\}$ and let $C_\mu = \{b_\gamma : \gamma < \mu\}$, $\forall \mu < \omega_1$. Then $C_\mu \in [B]^\omega$, $\forall \mu < \omega_1$ so define by recursion $\{\alpha_\mu : \mu < \omega_1\} \subseteq \kappa$ so that $A = \{a_\mu : \mu < \omega_1\} \in \mathcal{D}^k_{\omega_1}(\kappa)$ where, by assumption, $a_\mu = \{\alpha^\mu_i : i < k\} \in [\kappa \setminus \alpha_\mu]^k$ such that $f_{\alpha^\mu_j} \not\supseteq \varepsilon_j * b$, $\forall b \in C_\mu$, for some $j < k$. Then, since X is HFC^k_w, $A \in \mathcal{D}^k_{\omega_1}(\kappa)$, $B \in \mathcal{D}_{\omega_1}(\lambda)$ and $\varepsilon_0, \ldots, \varepsilon_{k-1} \in 2^{n(B)}$, let $b \in B$ such that $|\{a \in A : \forall i \in k(f_{\alpha_i} \supseteq \varepsilon_i * b)\}| = \omega_1$. But then $b = b_\mu$ for some $\mu < \omega_1$ and $\{a \in A : \forall i \in k(f_{\alpha_i} \supseteq \varepsilon_i * b)\} \subseteq \{a_\gamma : \gamma \leq \mu\}$ (since $b = b_\mu \in C_\gamma$, $\forall \gamma > \mu$), which is countable and hence we have a contradiction. □

Theorem 5.12. *Con(ZFC) $\to$ Con(ZFC + $\exists$ hereditarily ϵ-space with no countable open ι-cover).*

Proof. Construct two HFC$_w$ spaces $X = \{x_\alpha : \alpha < \omega_1\}$ and $Y = \{y_\alpha : \alpha < \omega_1\}$, as in (4.2) of [5], so that $x_\alpha \restriction \alpha = y_\alpha \restriction \alpha = f_\alpha \restriction \alpha$, where $f_\alpha = F(\alpha, -) = r \circ h_\alpha$ and $\forall \gamma \geq \alpha$, $x_\alpha(\gamma) = 0, y_\alpha(\gamma) = 1$.

Claim. $X \cup Y$ is hereditarily-ϵ but has no countable open ι-cover.

Let $Z \subseteq X \cup Y$ and $\mathcal{U}$ be any open ω-cover of Z. Without loss of generality, $\mathcal{U}$ consists of finite unions of basic open sets in 2^{ω_1}. That is, $\forall U \in \mathcal{U}$, $U = \bigcup_{i<n_U}[\sigma^U_i]$ with $\sigma^U_i \in Fn(\omega_1, 2)$. Let $\mathcal{M}$ be a countable elementary submodel of H_θ (for some large enough θ) such that $\mathcal{U} \in \mathcal{M}$. We claim that $\mathcal{U} \cap \mathcal{M}$ is a countable ω-cover of $Z \setminus (Z \cap \mathcal{M})$ showing that Z is almost-ϵ and hence an ϵ-space by Lemma 5.10, as required. To see $\mathcal{U} \cap \mathcal{M}$ is an ω-cover, let $F \in [Z \setminus (Z \cap \mathcal{M})]^{<\omega}$. Enumerate $F = \{f_{\alpha_0}, \ldots, f_{\alpha_{n-1}}\}$ for some $n \in \omega$, where $f_{\alpha_i} = x_{\alpha_i} \in Z \cap X$ or $f_{\alpha_i} = y_{\alpha_i} \in Z \cap Y$ so that if $\exists \beta < \omega_1$ such that $x_\beta, y_\beta \in Z$, $f_\beta = x_\beta$. Notice that this enumeration is not a problem since if F is the original set and there is $U \in \mathcal{U} \cap \mathcal{M}$ such that $\{f_{\alpha_0}, \ldots, f_{\alpha_{n-1}}\} \subseteq U$, then, as above, since $\mathcal{U} \cap \mathcal{M}$ is countable, enumerate $\mathcal{U} \cap \mathcal{M} = \{U_k : k \in \omega\}$ where $U_k = \bigcup_{i<n_k}[\sigma^k_i]$ and $\gamma_k = \sup(\bigcup_{i<n_k} \mathrm{dom}(\sigma^k_i)) < \omega_1$. Let $\gamma = \sup\{\gamma_k : k \in \omega\}$. Then, as above, $\forall \beta > \gamma$, $x_\beta \in U_k \Leftrightarrow y_\beta \in U_k$, $\forall k \in \omega$. In particular, $x_\beta \in U \Leftrightarrow y_\beta \in U$ $\forall \beta > \gamma$. Note that γ is definable in $\mathcal{M}$ since $U_k \in \mathcal{M}$, $\forall k \in \omega$. Thus, since $F \notin \mathcal{M}$, $\alpha_i > \gamma$, $\forall i < n$ and hence $F \subseteq U$.

Since $F \in [Z]^{<\omega}$ and $\mathcal{U}$ is an ω-cover of Z, let $U \in \mathcal{U}$ such that $F \subseteq U$. If $U \in \mathcal{M}$ we are done, so suppose $U \notin \mathcal{M}$. Since $F \subseteq U = \bigcup_{i<n_U}[\sigma^U_i]$, we can refine U so that $F \subseteq \bigcup_{i<n}[\tau_i] \subseteq U$ with $f_{\alpha_i} \in [\tau_i]$ $\forall i < n$. We need to define $\tau_0, \ldots, \tau_{n-1} \in Fn(\omega_1, 2)$ such that $f_{\alpha_i} \in [\tau_i] \subseteq U$, $\forall i < n$:

Since $f_{\alpha_0} \in [\sigma^U_{i_0}]$ for some $i_0 < n_U$, let $\tau_0 = \sigma^U_{i_0}$.
Fix $m > 0$ and suppose $\{\tau_j : j < m\}$ have been defined. Since $f_{\alpha_m} \in [\sigma^U_{i_m}]$, if $\tau_j \neq \sigma^U_{i_m}$, $\forall j < m$, then $\tau_m = \sigma^U_{i_m}$. Otherwise, let $j < m$ such that $\tau_j = \sigma^U_{i_m}$. Let $N = \{i < m : f_{\alpha_m} \in [\tau_i]\}$ and $\gamma_m = \max\{\bigcup dom(\tau_i) : i < m\} < \omega_1$. Define $dom(\tau_m) = \bigcup_{i \in N} dom(\tau_i) \cup \{\gamma_m + 1\}$ and $\tau_m(\alpha) = \tau_i(\alpha) = f_{\alpha_m}(\alpha)$, $\forall \alpha \in dom(\tau_i), i \in N$, $\tau_m(\gamma_m + 1) = f_{\alpha_m}(\gamma_m + 1)$.
Then $F \subseteq \bigcup_{i<n}[\tau_i] \subseteq U$ and we will further refine $\bigcup_{i<n}[\tau_i]$ so that $F \subseteq \bigcup_{i<n}[\varepsilon_i * b] \subseteq \bigcup_{i<n}[\tau_i] \subseteq U$ for $b \in [\omega_1]^k$ ($k \in \omega$) and $\varepsilon_0, \dots \varepsilon_{n-1} \in 2^k$. Let $b = \bigcup_{i<n} dom(\tau_i) \in [\omega_1]^{<\omega}$, $k = |b|$ and enumerate $b = \{\beta_i : i < k\}$. For $i < n$, let $\varepsilon_i \in 2^k$ such that $\varepsilon_i(j) = f_{\alpha_i}(\beta_j)$, $\forall j < k$. Then $\varepsilon_i * b(\beta_j) = \varepsilon_i(j) = f_{\alpha_i}(\beta_j) \Rightarrow \varepsilon_i * b \subseteq f_{\alpha_i}$, $\forall i < n$. By absoluteness, $\varepsilon_0, \dots, \varepsilon_{n-1} \in \mathcal{M}$ and if $b \in \mathcal{M}$ then by elementarity, $\exists U \in \mathcal{U} \cap \mathcal{M}$ such that $F \subseteq U$, so suppose $b \notin \mathcal{M}$. Let $r = b \cap \mathcal{M}$. Then $b \setminus r \neq \emptyset$. Let $m = k - |r|$ and $\mathcal{D} = \{d \in [\omega_1]^m : \bigcup[\varepsilon_i * (r \cup d)] \subseteq V, \text{ for some } V \in \mathcal{U}\}$. Notice that $b \setminus r \in \mathcal{D}$ and since $b \setminus r \notin \mathcal{M}$, $\mathcal{D}$ is uncountable. Then $\mathcal{D}' = \{r \cup d : d \in \mathcal{D}\}$ is an uncountable family of finite subsets of ω_1 so let $\mathcal{B}' \subseteq \mathcal{D}'$ be an uncountable Δ-system with root $r = b \cap \mathcal{M}$ and let $\mathcal{B} = \{d \setminus r : d \in \mathcal{B}'\} \subseteq \mathcal{D}$. Then $\mathcal{B} \in \mathcal{D}_{\omega_1}(\omega_1)$ such that $\forall b' \in \mathcal{B}$, $|b'| = k - |r| = m$. Recall $b \setminus r = \{\beta_j : |r| \leq j < k\}$. So, reenumerate $b \setminus r = \{\gamma_j : j < m\}$ where $\gamma_j = \beta_{j+|r|}$, $\forall j < m$. For $i < n$, let $\varepsilon'_i \in 2^m$ such that $\varepsilon'_i(j) = \varepsilon'_i * b \setminus r(\gamma_j) = \varepsilon_i * b \setminus r(\beta_{j+|r|}) = \varepsilon_i(j + |r|)$, $\forall j < m$. Then, since $\mathcal{B} \in \mathcal{D}_{\omega_1}(\omega_1)$, $\varepsilon'_0, \dots, \varepsilon'_{n-1} \in 2^m$ and by elementarity $\mathcal{M} \models (*_k)$, let $\mathcal{C} \in [\mathcal{B}]^\omega \cap \mathcal{M}$ and $\alpha \in \omega_1 \cap \mathcal{M}$ such that $\forall a = \{\beta_i : i < n\} \in [\omega_1 \setminus \alpha]^n$, $\exists c \in \mathcal{C}$ such that $f_{\beta_i} \in [\varepsilon'_i * c]$, $\forall i < n$. In particular, since $F \notin \mathcal{M}$, $\alpha_0, \dots, \alpha_{n-1} \notin \mathcal{M}$ and since $\alpha \in \mathcal{M}$ we have that $\alpha_i > \alpha$, $\forall i < n$. Thus, $\{\alpha_i : i < n\} \in [\omega_1 \setminus \alpha]^n$ so let $c \in \mathcal{C}$ such that $f_{\alpha_i} \in [\varepsilon'_i * c]$, $\forall i < n$. But, since $\mathcal{C} \in \mathcal{M}$ is countable, $\mathcal{C} \subseteq \mathcal{M}$ and hence $c \in \mathcal{C} \cap \mathcal{M}$ such that $\varepsilon'_i * c \subseteq f_{\alpha_i}$, $\forall i < n$. Also, since $\varepsilon_i * b \subseteq f_{\alpha_i}$, $\forall i < n$, $f_{\alpha_i} \in [\varepsilon_i \restriction |r| * r] \cap [\varepsilon'_i * c]$, $\forall i < n$. We claim that $[\varepsilon_i \restriction |r| * r] \cap [\varepsilon'_i * c] = [\varepsilon_i * (r \cup c)]$ and hence $f_{\alpha_i} \in [\varepsilon_i * (r \cup c)]$, $\forall i < n \Rightarrow F \subseteq \bigcup_{i<n}[\varepsilon_i * (r \cup c)]$.

'$\subseteq$': Let $g \in [\varepsilon_i \restriction |r| * r] \cap [\varepsilon'_i * c]$. Then $\varepsilon_i \restriction |r| * r \subseteq g$ and $\varepsilon'_i * c \subseteq g$. Recall $r = \{\beta_j : j < |r|\}$ and enumerate $c = \{\gamma_j : j < m\}$. Then, $g(\beta_j) - \varepsilon_i \restriction |r| * r(\beta_j) = \varepsilon_i(j)$, $\forall j < |r|$ and $g(\gamma_j) = \varepsilon'_i * c(\gamma_j) = \varepsilon'_i(j) = \varepsilon_i(j + |r|)$, $\forall j < m$. Now, since $r \cup c = \{\beta_j : j < |r|\} \cup \{\gamma_j : j < m\}$, reenumerate $r \cup c = \{\alpha_j : j < k\}$ so that $\alpha_j = \beta_j$ for $j < |r|$ and $\alpha_j = \gamma_{j-|r|}$ for $|r| \leq j < k$. Then, for $j < |r|$, $g(\alpha_j) = g(\beta_j) = \varepsilon_i(j) = \varepsilon_i * (r \cup c)(\alpha_j)$ and for $j \geq |r|$, $g(\alpha_j) = g(\gamma_{j-|r|}) = \varepsilon_i((j - |r|) + |r|) = \varepsilon_i(j) = \varepsilon_i * (r \cup c)(\alpha_j)$.

'$\supseteq$': Let $g \in [\varepsilon_i * (r \cup c)]$. Then $g(\alpha_j) = \varepsilon_i * (r \cup c)(\alpha_j) = \varepsilon_i(j)$, $\forall j < k$, where α_j is defined as above, for $j < k$. Then, $g(\beta_j) = g(\alpha_j) = \varepsilon_i(j) = \varepsilon_i * (r \cup c)(\beta_j) = \varepsilon \restriction |r| * r(\beta_j)$, $\forall j < |r|$ and hence $g \in [\varepsilon_i \restriction |r| * r]$. Moreover, $g(\gamma_j) = g(\alpha_{j+|r|}) = \varepsilon_i(j + |r|) = \varepsilon'_i((j + |r|) - |r|) = \varepsilon'_i(j) = \varepsilon'_i * c(\gamma_j)$, $\forall j < m$ and hence $g \in [\varepsilon'_i * c]$.

Since $c \in \mathcal{C} \subseteq \mathcal{D}$, $\bigcup_{i<n}[\varepsilon_i*(r\cup c)] \subseteq V$ for some $V \in \mathcal{U}$. But $\bigcup_{i<n}[\varepsilon_i*(r\cup c)] \in \mathcal{M}$ (since $r, c, \varepsilon_0, \ldots, \varepsilon_{n-1} \in \mathcal{M}$) so by elementarity, let $V \in \mathcal{U}\cap\mathcal{M}$ such that $\bigcup_{i<n}[\varepsilon_i*(r\cup c)] \subseteq V$. Then $\exists V \in \mathcal{U} \cap \mathcal{M}$ such that $F \subseteq \bigcup_{i<n}[\varepsilon_i * (r \cup c)] \subseteq V$. □

6. D-spaces

The third named author first considered ι-covers when trying to make the T_2 hereditarily Lindelöf non D-space of [8] regular. Since this T_2 example is an ϵ-space, he asked in [8] whether every regular (hereditarily) ϵ-space is a D-space. We could ask the same about (hereditarily) ι-spaces. In fact, it remains unclear whether ι-covers could play a role in constructing such a regular, hereditarily Lindelöf non D-space.

Corollary 6.1. *There is an ι_w-space which is not a D-space.*

Proof. The example is taken from [1], but we nevertheless present the details of its construction for the reader's convenience. Erik van Douwen showed in [11] that one can put, on every subset of the real line, a locally compact locally countable topology with countable extent which is finer than the topology it inherits from the Euclidean one. Let $B \subset \mathbb{R}$ be a Bernstein set, that is, a set meeting every uncountable closed set along with its complement. Let $X = \mathbb{R}$, where points of B have neighbourhoods as in the van Douwen topology and points of $X \setminus B$ have their usual Euclidean neighbourhoods.

Claim 1. X is Lindelöf and a D-space.

Proof of Claim 1. To prove that X is Lindelöf, let $\mathcal{U}$ be an open cover of X. Let $V = \bigcup\{U \in \mathcal{U} : U \cap (X \setminus B) \neq \emptyset\}$. Then V covers all but countably many points of X. Indeed if $X \setminus V$ were uncountable, then $(X \setminus V) \cap (X \setminus B) \neq \emptyset$ which is a contradiction. But since the topology of $X\setminus B$ is Lindelöf, $\mathcal{V}$ has a countable subcover, and hence X is a Lindelöf space.

To prove that X is a D-space, we need the following lemma:

Lemma 6.2. *Every countable space Y is a D-space.*

Proof of Lemma. Let $N : Y \to \tau$ be a neighbourhood assignment and fix an enumeration $\{y_n : n < \omega\}$ of Y. Suppose you have picked points $\{y_{k_i} : i \leq n\}$. If $N(\{y_{k_i} : i \leq n\}) = Y$ then we are done, otherwise let $y_{k_{n+1}}$ be the least indexed point such that $y_{k_{n+1}} \notin N(\{y_{k_i} : i \leq n\})$. We claim that $N(\{y_{k_i} : i < \omega\}) = Y$ and $\{y_{k_i} : i < \omega\}$ is closed discrete. For the first claim, suppose by contradiction that there is a point $y \notin N(\{y_{k_i} : i < \omega\}$. Then there is a $j < \omega$ such that y is the least indexed point (in the original enumeration of Y) such that $y \notin \{y_{k_i} : i \leq j\}$. But then $y = y_{k_{j+1}}$, which is a contradiction. The fact that $\{y_{k_i} : i < \omega\}$ is closed discrete follows from the first claim and the fact that $N(y_{k_n}) \cap \{y_{k_i} : i < \omega\}$ is a finite set. □

Let $N : X \to \tau$ be an open neighbourhood assignment. Since $X \setminus B$ is a D-space we can find a closed discrete set $D_1 \subset X \setminus B$ such that $N(D_1) \supset X \setminus B$. Now $X \setminus N(D_1)$ is a countable closed set. So $X \setminus N(D_1)$ is a D-space and hence we can find $D_2 \subset X \setminus N(D_1)$ such that $N(D_2) \supset X \setminus N(D_1)$. Therefore $D = D_1 \cup D_2$ is a closed discrete set such that $N(D) = X$, so X is a D-space. □

Now let B_e be the Bernstein set B with its usual (Euclidean) topology.

Claim 2. $X \times B_e$ is an ι_w-space but not a D-space.

Proof of Claim 2. Since the topology on X refines the topology of the real line, and the real line has a countable open ι-cover, also X has a countable open ι-cover. So, it follows from Theorem 2.16 that $X \times B_e$ is an ι_w-space. To prove that $X \times B_e$ is not a D-space, note that it contains the closed copy $\{(x, x) : x \in B\}$ of the space B and that B is not a D-space, because it has countable extent, but it is uncountable and locally countable and thus not Lindelöf. □

Question 6.3. *Is every (hereditarily) ι-space a D-space?*

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