P-SPACES AND THE VOLTERRA PROPERTY

SANTI SPADARO

Abstract. We study the relationship between generalizations of P-spaces and Volterra (weakly Volterra) spaces, that is, spaces where every two dense G_δ have dense (non-empty) intersection. In particular, we prove that every dense and every open, but not every closed subspace of an almost P-space is Volterra and that there are Tychonoff non-weakly Volterra weak P-spaces. These results should be compared with the fact that every P-space is hereditarily Volterra. As a byproduct we obtain an example of a hereditarily Volterra space and a hereditarily Baire space whose product is not weakly Volterra. We also show an example of a Hausdorff space which contains a non-weakly Volterra subspace and is both a weak P-space and an almost P-space.

1. Introduction

A real-valued function f is called *pointwise discontinuous* if the set of all points where it is continuous is dense. In 1881, eighteen years before René-Louis Baire published the Baire category theorem [1], a twenty years old student of the *Scuola Normale Superiore di Pisa* named Vito Volterra proved that there are no two pointwise discontinous real-valued functions on $\mathbb{R}$ such that the set of all points of continuity of one is equal to the set of all discontinuity points of the other [17] (see also [5]). Volterra's theorem has inspired an interesting generalization of the Baire property.

Given $f : X \to \mathbb{R}$, let $C(f)$ be the set of all continuity points of f.

Definition 1.1. [7] *A topological space X is called* Volterra *(respectively,* weakly Volterra*) if for every pair of pointwise discontinuous functions $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ the set $C(f) \cap C(g)$ is dense in X (respectively, non-empty).*

Thus Volterra's theorem can be rephrased by stating that the real line is a Volterra space. Gauld and Piotrowski proved the following internal characterization of Volterra and weakly Volterra spaces. Recall that a set is called a G_δ *set* if it can be represented as a countable intersection of open sets.

Proposition 1.2. [7] *A space is Volterra (respectively, weakly Volterra) if and only if for every pair G and H of dense G_δ subsets of X, the set $G \cap H$ is dense (respectively, non-empty).*

Recall that a space is Baire if every countable intersection of dense open sets is dense. From the above characterization it's clear that every Baire space is Volterra. The problem of when a Volterra space is Baire has been studied extensively (see [3] and [8]).

2000 *Mathematics Subject Classification.* primary 54E52, 54G10; secondary 28A05.
Key words and phrases. Baire, Volterra, P-space, almost P-space, weak P-space, density topology.
Supported by an INdAM Cofund outgoing fellowship.

This note was inspired by the simple observation that every P-space (that is, a space where every G_δ set is open) is hereditarily Volterra. Weak P-spaces and almost P-spaces are the two most popular weakenings of P-spaces. We compare these properties with the notions of Volterra and weakly Volterra space. We find that every dense subset and every open subset of an almost P-space is Volterra, while weak P-spaces may fail to be weakly Volterra. Our example of a non-weakly Volterra weak P-space shows that the product of a hereditarily Baire space and a hereditarily Volterra space may fail to be weakly Volterra. Finally, we introduce the class of *pseudo P-spaces*, a natural new weakening of P-spaces and construct a Hausdorff Baire pseudo P-space with a non-weakly Volterra subspace. The existence of a Tychonoff space with the same properties is left as an open question.

2. P-spaces and generalizations

Definition 2.1.

(1) *A space X is called a* P-space *if every countable intersection of open subsets of X is open.*
(2) *A space X is called an* almost P-space *if every non-empty G_δ subset of X has non-empty interior.*
(3) *A space X is called a* weak P-space *if every countable subset of X is closed (and discrete).*

Every P-space is an almost P-space and a weak P-space. Examples of almost P-spaces which are not P-spaces abound. Walter Rudin [15] proved that the remainder of the Čech-Stone compactification of the natural numbers, ω^*, is an almost P-space. Since infinite weak P-spaces cannot be compact, ω^* provides an example of an almost P-space which is not a weak P-space. Steve Watson managed to construct in [18] even a compact almost P-space where every point is the limit of a non-trivial convergent sequence.

Definition 2.2. *Let $\mathcal{P}$ be a property of subsets of a topological space X. We say that X is* $\mathcal{P}$-hereditarily Volterra (Baire) *if every subspace of X satisfying $\mathcal{P}$ is Volterra (Baire). A space is* hereditarily Volterra (Baire) *if each one of its subspaces is Volterra (Baire).*

Contrast our Definition 2.2 with the common habit of calling a space *hereditarily Baire* if each of its closed subsets is Baire. For example, the real line is not hereditarily Baire according to our definition.

Since every subspace of a P-space is a P-space, the following proposition is clear.

Proposition 2.3. *Every P-space is hereditarily Volterra.*

Proposition 2.4. *Every almost P-space is dense-hereditarily Volterra and open-hereditarily Volterra.*

Proof. Let X be an almost P-space. We claim that X is Volterra. Indeed, let G and H be dense G_δ subspaces of X. We claim that $Int(G) \cap H$ is a dense set. Since H is dense and $Int(G)$ is open we have that $\overline{Int(G) \cap H} = \overline{Int(G)}$. So if $Int(G) \cap H$ were not dense then $X \setminus \overline{Int(G)}$ would be a non-empty open set, and thus it would have to meet G. Therefore, $G \cap (X \setminus \overline{Int(G)})$ would be a non-empty G_δ set with empty interior. But that contradicts the fact that X is an almost P-space.

To prove the statement of the proposition it now suffices to recall a result of R. Levy [12] stating that every open set and every dense set of an almost P-space is an almost P-space. □

Almost P-spaces need not be hereditarily Volterra.

Example 2.5. *There is a Baire regular almost P-space with a closed non-weakly Volterra subspace.*

Proof. R. Levy [11] constructed a Baire regular almost P-space containing a closed copy of the rational numbers, and the rational numbers are not weakly Volterra. □

On the other hand, weak P-spaces need not even be weakly Volterra. The construction of our counterexample will exploit the density topology on the real line. We recall its definition.

Definition 2.6. *A measurable set $A \subset \mathbb{R}$ has density d at x if the limit:*

$$\lim_{h \to 0} \frac{m(A \cap [x-h, x+h])}{2h}$$

exists and is equal to d. We denote by $d(x, A)$ the density of A at x and let $\phi(A) = \{x \in \mathbb{R} : d(x, A) = 1\}$.

Definition 2.7. *The family of all measurable sets $A \subset \mathbb{R}$ such that $\phi(A) \supset A$ defines a topology on $\mathbb{R}$ called the* density topology *and denoted by $\mathbb{R}_d$.*

Since the density topology is finer than the Euclidean topology on the real line, every point is a G_δ set in $\mathbb{R}_d$. Moreover, every measure zero set is easily seen to be closed in $\mathbb{R}_d$. In particular, the density topology is a weak P-space (see [16] for a comprehensive study of the density topology).

Recall that a space is *resolvable* if it contains two disjoint dense sets. Dontchev, Ganster and Rose [4] proved that the density topology is resolvable (this was later improved by Luukkainen [13] who proved that $\mathbb{R}_d$ even contains a pairwise disjoint family of dense sets of size continuum). In the following lemma we review all properties of the density topology that are relevant to us here.

Lemma 2.8. *$\mathbb{R}_d$ is a Tychonoff resolvable weak P-space with points G_δ.*

We also need the following lemma of Gruenhage and Lutzer.

Lemma 2.9. [8] *Suppose $\mathcal{U}$ is a point-finite collection of open subsets of a space X and that each $U \in \mathcal{U}$ contains a G_δ set $G(U)$. Then $\bigcup\{G(U) : U \in \mathcal{U}\}$ is a G_δ set.*

Example 2.10. *There is a non-weakly Volterra Tychonoff weak P-space.*

Proof. Let $X = \{f \in 2^{\omega_1} : |f^{-1}(1)| < \omega\}$ with the topology inherited from the countably supported product topology on 2^{ω_1}. Let $U_n = X \setminus \{f \in 2^{\omega_1} : |f^{-1}(1)| \leq n\}$, and note that U_n is an open dense set in X.

Use Lemma 2.8 to fix disjoint dense sets D_1 and D_2 inside $\mathbb{R}_d$.

Since $\mathbb{R}_d$ is a weak P-space and X is even a P-space, $X \times \mathbb{R}_d$ is a weak P-space. Note that the family $\{U_n \times \mathbb{R}_d : n < \omega\}$ is point-finite and $U_n \times \{x\}$ is a G_δ set contained in $U_n \times \mathbb{R}_d$ for every $x \in \mathbb{R}_d$. Thus, by Lemma 2.9, $\bigcup_{x \in D_1} U_n \times \{x\}$ and $\bigcup_{x \in D_2} U_n \times \{x\}$ are disjoint dense G_δ sets in $X \times \mathbb{R}_d$. □

Since every subspace of $\mathbb{R}_d$ is Baire (see [16]), Example 2.10 shows that the product of a hereditarily Volterra space and a hereditarily Baire space may fail to be weakly Volterra. This suggests the following question:

Question 2.11. *Are there hereditarily Baire spaces X and Y such that $X \times Y$ is not weakly Volterra?*

Note that there are metric Baire spaces whose square is not weakly Volterra (see [6], Example 3.9), but if an example answering 2.11 in the positive exists, none of its factors can be metric. Indeed, the product of a Baire space and a closed-hereditary Baire metric space is Baire (see [14]).

3. A new weakening of P-spaces

Definition 3.1. *We call a space X a* pseudo P-space *if it is both an almost P-space and a weak P-space.*

Example 3.2. *There are regular pseudo P-space which are not P-spaces.*

Proof. For one example, let X be the subspace of all weak P-points of ω^*. In [10], Kunen proved that X is a dense subset of ω^* and hence it is an almost P-space. Clearly X is a weak P-space. Since there is a weak P-point which is not a P-point in ω^*, X is not a P-space though.

Another example was essentially presented in [9]. Let X be a Lindelöf P-space without isolated points. Van Mill (see Lemma 3.1 of [9]) proved that there is a point $p \in \beta X \setminus X$ such that p is not in the closure of any countable subset of X. Then $X \cup \{p\}$ is a weak P-space. But, from the fact that X is a P-space it follows that $X \cup \{p\}$ is an almost P-space. Now, $X \cup \{p\}$ is not a P-space, or otherwise it would be a Lindelöf P-space, and thus each of its Lindelöf subspaces should be closed. But X is a non-closed Lindelöf subspace of $X \cup \{p\}$.

□

Pseudo P-spaces are in some sense very close to P-spaces, closer than almost P-spaces, so that suggests the following question.

Question 3.3. *Is there a regular pseudo P-space which is not hereditarily weakly Volterra?*

The following example provides a partial answer to this question.

Example 3.4. *There is a Hausdorff (non-regular) Baire pseudo P-space which is not hereditarily weakly Volterra.*

Proof. Let $X = \{f \in 2^{\omega_1} : |f^{-1}(1)| \leq \aleph_0\}$. Let $\mathcal{C}$ be the set of all countable partial functions from a countable subset of ω_1 to 2. For every $\sigma \in \mathcal{C}$ let $[\sigma] = \{f \in 2^{\omega_1} : \sigma \subset f\}$. Moreover, for every $n < \omega$ let $X_n = \{f \in 2^{\omega_1} : |f^{-1}(1)| = n\}$. Define a topology on X by declaring $\{[\sigma] \setminus X_n : \sigma \in \mathcal{C}, n < \omega\}$ to be a subbase.

Claim 1 X is a pseudo P-space.

Proof of Claim 1. The topology on X is a refinement of the countably supported box product topology on 2^{ω_1} and thus X is a weak P-space. To prove that X is an almost P-space, let $G = \bigcap \{U_n : n < \omega\}$ be a non-empty G_δ set and $x \in G$. For every $n < \omega$, choose α_n and a finite set $\mathcal{F}_n \subset \{X_k : k < \omega\}$ such that $V_n := [x \restriction \alpha_n] \setminus \bigcup \mathcal{F}_n \subset U_n$. Let $h \in \bigcap_{n<\omega} V_n$ be a function with infinite support and $\beta < \omega_1$ be an ordinal such that $\beta \geq \sup_{n<\omega} \alpha_n$. Then $[h \restriction \beta] \subset \bigcap_{n<\omega} V_n \subset \bigcap_{n<\omega} U_n$. △

Claim 2 The space X is Baire.

Proof of Claim 2. We prove that every meager set is nowhere dense. Let $\{N_n : n < \omega\}$ be a countable family of nowhere dense subsets of X. Let σ be a countable partial function with domain $\alpha < \omega_1$ and k be an integer: we are going to prove that the basic open set $[\sigma] \setminus \bigcup\{X_n : n \leq k\}$ is not contained in the closure of $\bigcup_{n<\omega} N_n$. Since N_0 is nowhere dense there must be a countable partial function σ_0 extending σ with domain $\alpha_0 > \alpha$ and an integer $k_0 < \omega$ such that $([\sigma_0] \setminus \bigcup\{X_k : k \leq k_0\}) \cap N_0 = \emptyset$.

Suppose we've found an increasing sequence of countable partial functions $\{\sigma_i : i < n\}$ and an increasing sequence of integers $\{k_i : i < n\}$. Since N_n is nowhere dense there must be a countable partial function σ_n extending σ_{n-1} and an integer $k_n > k_{n-1}$ such that $[\sigma_n] \cap N_n = \emptyset$. Let $\sigma_\omega = \bigcup_{i<\omega} \sigma_i$. Then $([\sigma_\omega] \setminus \bigcup\{X_k : k < \omega\}) \cap \bigcup_{n<\omega} N_n = \emptyset$ and $\emptyset \neq [\sigma_\omega] \subset ([\sigma] \setminus \bigcup\{X_n : n \leq k\})$. Thus $[\sigma] \setminus \bigcup\{X_n : n \leq k\}$ is not contained in $\overline{\bigcup_{n<\omega} N_n}$ and since the choice of σ and k was arbitrary, this shows that $\bigcup_{n<\omega} N_n$ is nowhere dense. △

Claim 3 Let $Y = \bigcup_{n<\omega} X_n \subset X$. Then Y is not weakly Volterra.

Proof of Claim 3. Let $G = \bigcap\{X \setminus X_k : k \text{ is even }\}$ and $H = \bigcap\{X \setminus X_k : k \text{ is odd }\}$. Then G and H are dense G_δ subsets of Y with empty intersection. △

□

As pointed out by Gary Gruenhage, Example 3.4 is not regular. For example, the closed set X_1 and the null function cannot be separated by disjoint open sets.

References

[1] R. Baire, *Sur les fonctions de variables réelles*, Ann. di Mat., **3** (1899), 1–123.
[2] J. Cao and D. Gauld, *Volterra spaces revisited*, J. Aust. Math. Soc. **79** (2005), 61–76.
[3] J. Cao and H. Junnila, *When is a Volterra space Baire?*, Topology Appl. **154** (2007), 527–532.
[4] J. Dontchev, M. Ganster and D. Rose, *Ideal Resolvability*, Topology Appl. **93** (1999), 1–16
[5] W. Dunham, *A historical gem from Vito Volterra*, Mathematics Magazine, **63** (1990), 234–237.
[6] D. Gauld, S. Greenwood and Z. Piotrowski, *On Volterra spaces III: Topological operations* Topology Proc. **23** (1998), 167–182.
[7] D. Gauld and Z. Piotrowski, *On Volterra spaces*, Far East J. Math. Sci. **1** (1993), 209–214.
[8] G. Gruenhage and D. Lutzer, *Baire and Volterra spaces*, Proc. Amer. Math. Soc. **128** (2000), 3115–3124.
[9] M. Henriksen and R.G. Woods, *Weak P-spaces and L-closed spaces*, Q & A in General Topology **6** (1988), 201–207.
[10] K. Kunen, *Weak P-points in* $\mathbb{N}^*$, Colloq. Math. Soc. János Bolyai, vol. 23, Topology (Budapest, 1978), 741–749.
[11] R. Levy, *Showering Spaces*, Pacific J. Math. **57** (1975), 223–232.
[12] R. Levy, *Almost P-spaces*, Canad. J. Math. **29** (1977), 284–288.
[13] J. Luukkainen, *The density topology is maximally resolvable*, Real Anal. Exchange **25** (1999), 419–420.
[14] W. Moors, *The product of a Baire space with a hereditarily Baire metric space is Baire*, Proc. Amer. Math. Soc. **134** (2006), 2161–2163
[15] W. Rudin, *Homogeneity problems in the theory of Cech compactifications*, Duke Math. J. **23** (1956), 409–419.
[16] F.D. Tall, *The Density Topology*, Pacific J. Math. **62** (1976), 275–284.
[17] V. Volterra, *Alcune osservasioni sulle funzioni punteggiate discontinue*, Giornale di Matematiche **19** (1881), 76–86.
[18] S. Watson, *A compact Hausdorff space without P-points in which G_δ sets have interior*, Proc. Amer. Math. Soc. **123** (1995), 2575–2577.

Department of Mathematics, Faculty of Science and Engineering, York University, Toronto, ON, M3J 1P3 Canada
E-mail address: santispadaro@yahoo.com