FREE SEQUENCES AND THE TIGHTNESS OF PSEUDORADIAL SPACES

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Abstract. Let $F(X)$ be the supremum of cardinalities of free sequences in X. We prove that the radial character of every Lindelöf Hausdorff almost radial space X and the set-tightness of every Lindelöf Hausdorff space are always bounded above by $F(X)$. We then improve a result of Dow, Juhász, Soukup, Szentmiklóssy and Weiss by proving that if X is a Lindelöf Hausdorff space, and X_δ denotes the G_δ topology on X then $t(X_\delta) \leq 2^{t(X)}$. Finally, we exploit this to prove that if X is a Lindelöf Hausdorff pseudoradial space then $F(X_\delta) \leq 2^{F(X)}$.

1. Introduction

Free sequences were one of the key tools in Arhangel'skii's celebrated solution of the Alexandroff-Urysohn problem on the cardinality of first-countable compacta. Later, they were discovered to have a definitive impact in various other aspects of the theory of cardinal invariants in topology. Recall that a sequence $\{x_\alpha : \alpha < \kappa\}$ in a topological space X is said to be *free* if $\overline{\{x_\alpha : \alpha < \beta\}} \cap \overline{\{x_\alpha : \beta \leq \alpha < \kappa\}} = \emptyset$, for every $\beta < \kappa$.

Let $F(X)$ be the supremum of the cardinalities of free sequences in X and let $t(X)$ be the tightness of X. Arhangel'skii proved that $F(X) = t(X)$ for every compact Hausdorff space X. For Lindelöf spaces this is not true anymore, as first noted by Okunev [13]. Indeed, let $\sigma(2^\kappa) = \{x \in 2^\kappa : |x^{-1}(1)| < \aleph_0\}$ and let $\mathbf{1} \in 2^\kappa$ be the function that is constantly equal to 1. Then $X = \sigma(2^\kappa) \cup \{\mathbf{1}\}$ with the topology induced from 2^κ is a space of tightness κ where free sequences are countable (the last assertion follows easily from the fact that $\sigma(2^\kappa)$ is a σ-compact space of countable tightness). However, the inequality $F(X) \leq t(X)$ is still true for every Lindelöf space X. More generally, if $L(X)$ denotes the *Lindelöf degree* of X, then $F(X) \leq L(X) \cdot t(X)$.

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There is at least another class of spaces for which we can prove the equality $F(X) = t(X)$. A topological space X is called *pseudoradial* if for every non-closed set $A \subset X$ there is a point $x \in \overline{A} \setminus A$ and a transfinite sequence $\{x_\alpha : \alpha < \kappa\} \subset A$ which converges to x. If for every non-closed set $A \subset X$ and every point $x \in \overline{A} \setminus A$ there is a transfinite sequence inside A which converges to x, then the space X is called *radial*. In [3], Bella proved that $t(X) \leq F(X)$ for every pseudoradial regular space X and hence $t(X) = F(X)$ for every Lindelöf pseudoradial regular space X.

Given a pseudoradial space X the *radial character of* X, $\sigma_C(X)$ is defined as the minimum cardinal κ such that every non-closed set contains a transfinite sequence of length at most κ converging outside. For radial spaces obviously $\sigma_C(X) = t(X)$, but this is true for an even larger class of spaces, that of almost radial spaces. A space X is said to be *almost radial* if, for every non-closed set $A \subset X$ there is a point $x \in \overline{A} \setminus A$ and a sequence $\{x_\alpha : \alpha < \kappa\}$ converging to x such that $x \notin \overline{\{x_\alpha : \alpha < \beta\}}$, for every $\beta < \kappa$ (we will call such a sequence a *thin sequence*). Bella's result implies that $t(X) = \sigma_C(X) \leq F(X)$ for every regular almost radial space X.

Since regularity was essential in the proof of his results, Bella [3] asked the following question:

Question 1. *Let X be a Hausdorff pseudoradial space. Is it true that $t(X) \leq F(X)$? Is it at least true that $\sigma_C(X) \leq F(X)$ for every Hausdorff almost radial space X?*

The above question is still open. We will prove that $t(X) \leq L(X) \cdot F(X)$ for every Hausdorff pseudoradial space and this will allow us to prove that $t(X) = F(X)$ for every Lindelöf pseudoradial Hausdorff space, thus removing the regularity assumption from one of Bella's results. Using the notion of *set-tightness* we will be able to extend our results outside of the pseudoradial realm.

In the second part of the paper we will prove a bound for the tightness of the G_δ-modification of a Lindelöf Hausdorff space. The G_δ-modification of X (or G_δ topology) is the topology on X generated by the G_δ-subsets of X. Studying which properties of X are preserved by this operation is a natural question that has been extensively studied in the literature. Of course the G_δ-topology is compact if and only if the space is finite, but there are already some interesting preservation results for the Lindelöf property. For example, Arhangel'skii proved that if X is a Lindelöf scattered space then X_δ is Lindelöf. More generally, one can ask whether the cardinal functions of X_δ can be bounded in terms of their value on X. Sometimes a neat ZFC bound is available.

For example, Juhász [10] proved that $c(X_\delta) \leq 2^{c(X)}$ for every compact Hausdorff space X, where $c(X)$ denotes the *cellularity* of X, Bella and the author [4] proved that $s(X_\delta) \leq 2^{s(X)}$ for every Hausdorff space X, where $s(X)$ is the *spread* of X and Carlson, Porter and Ridderbos [6] showed that $L(X_\delta) \leq 2^{F(X) \cdot L(X)}$ for every Hausdorff space X (the $F(X)$ in the exponent cannot be removed in the last result as there is even a compact space X such that $wL(X_\delta) > 2^{\aleph_0}$, see [15]). In other cases the existence of a bound may strongly depend on your set theory. For example, the authors of [12] proved that if $Nt(X)$ denotes the *Noetherian type* of X then $Nt(X_\delta) \leq 2^{Nt(X)}$ for every compact X if GCH is assumed and $Nt(X)$ has uncountable cofinality and showed, modulo very large cardinals, that both of their assumptions where essential.

The behavior of the tightness of the G_δ topology changes drastically depending on whether the space is Lindelöf or not. Dow, Juhász, Soukup, Szentmiklóssy and Weiss proved that, consistently, there can be countably tight regular spaces whose G_δ topology has arbitrarily large tightness, but for every Lindelöf regular space X we have $t(X_\delta) \leq 2^{t(X)}$.

Regularity was essential in their argument, but we will show how to remove it, and as a byproduct we will obtain that the natural bound $F(X_\delta) \leq 2^{F(X)}$ holds for a Lindelöf pseudoradial Hausdorff space X.

For undefined notions we refer to [9] and [11]. Background material on elementary submodels, which are used in the proof of Theorem 9, can be found in Dow's survey [7]. For general information about pseudoradial spaces we recommend Tironi's survey [16].

2. On the tightness of a pseudoradial space

Theorem 2. *Let X be a Hausdorff pseudoradial space. Then $t(X) \leq L(X) \cdot F(X)$.*

Proof. Let $\kappa = L(X) \cdot F(X)$ and suppose by contradiction that $t(X) > \kappa$. Then there is a κ-closed non-closed subset A of X. Since X is pseudoradial and A is not closed we can find a transfinite sequence $S \subset A$ and a point $p \in \overline{A} \setminus A$ such that S converges to p.

For every $x \in A$, let U_x and V_x be disjoint open sets such that $x \in U_x$ and $p \in V_x$. Let $\mathcal{U} = \{U_x : x \in A\}$.

Claim. There is a family $\mathcal{W} \in [\mathcal{U}]^{\leq \kappa}$ such that $S \subset \bigcup \mathcal{W}$.

Proof of Claim. Assume this is not the case. Then we will construct by recursion a free sequence $F \subset S$ such that $|F| = \kappa^+$.

Suppose that for some $\beta < \kappa^+$ we have picked points $\{x_\alpha : \alpha < \beta\} \subset S$ and open families $\{\mathcal{U}_\alpha : \alpha < \beta\}$ such that:

(1) $\overline{\{x_\alpha : \alpha < \tau\}} \subset \bigcup \mathcal{U}_\tau$, for every $\tau < \beta$.
(2) $|\mathcal{U}_\alpha| \leq \kappa$, for every $\alpha < \beta$.

The set $\overline{\{x_\alpha : \alpha < \beta\}}$ is contained in A and has Lindelöf degree at most κ, hence we can find a family $\mathcal{U}_\beta \subset \mathcal{U}$ such that $|\mathcal{U}| \leq \kappa$ and $\overline{\{x_\alpha : \alpha < \beta\}} \subset \bigcup \mathcal{U}_\beta$. Note that the family $\bigcup_{\alpha \leq \beta} \mathcal{U}_\alpha$ does not cover S and hence we can choose a point $x_\beta \in S \setminus \bigcup_{\alpha \leq \beta} \mathcal{U}_\alpha$.

Eventually $\{x_\alpha : \alpha < \kappa^+\}$ is a free sequence of cardinality κ^+. $\triangle$

Let λ be the length of the sequence S. We can assume that λ is a regular cardinal, and since A is κ-closed we must have $\lambda \geq \kappa^+$. Therefore there must be $U \in \mathcal{W}$ such that $|S \cap U| = \lambda$. Now $U = U_x$, for some $x \in A$ and V_x contains a final segment of S, because it is an open neighbourhood of p. But this contradicts the fact that U_x and V_x are disjoint. $\square$

Bella [3] obtained the following corollary with the additional assumption that X is a regular space.

Corollary 3. *Let X be a Lindelöf Hausdorff pseudoradial space. Then $t(X) = F(X)$.*

Recalling that $\sigma_C(X) = t(X)$ for every almost radial space, we have the following pair of corollaries.

Corollary 4. *Let X be a Lindelöf Hausdorff almost radial space. Then $\sigma_C(X) = F(X)$.*

Corollary 5. *Let X be a Lindelöf Hausdorff almost radial space. Then X is sequential if and only if $F(X) \leq \omega$.*

We can do without the pseudoradial assumption if we replace the tightness with a related cardinal invariant known as *set-tightness*. The set-tightness of a topological space X is defined as the minimum cardinal κ such that for every non-closed subset A of X and for every point $p \in \overline{A} \setminus A$, there is a κ-sized family $\{A_\alpha : \alpha < \kappa\}$ of subsets of A such that $p \in \overline{\bigcup\{A_\alpha : \alpha < \kappa\}}$, but $p \notin \bigcup\{\overline{A_\alpha} : \alpha < \kappa\}$.

The set-tightness was introduced in [2], where it was called *quasi-character*. Obviously $t_s(X) \leq t(X)$. Arhangel'skii and Bella [1] proved that $t(X) = t_s(X)$ for every compact Hausdorff space X.

Bella [3] proved that $t_s(X) \leq F(X)$ for every regular space X. We will prove that $t_s(X) \leq F(X) \cdot L(X)$ for every Hausdorff space X, so in particular, $t_s(X) \leq F(X)$ is true for every Lindelöf Hausdorff space X. Note that there are Hausdorff pseudoradial spaces where $t_s(X) < t(X)$ (see [14]), so the bound $t(X) \leq F(X) \cdot L(X)$ for every pseudoradial Hausdorff space X is not a consequence of $t_s(X) \leq F(X) \cdot L(X)$.

Theorem 6. *Let X be a Hausdorff space. Then $t_s(X) \leq F(X) \cdot L(X)$.*

Proof. Let $\kappa = L(X) \cdot F(X)$ and suppose by contradiction that $t_s(X) > \kappa$. Then there are a set $A \subset X$ and a point $p \in \overline{A} \setminus A$ such that if $\{A_\alpha : \alpha < \kappa\}$ is a κ-sized family of subsets of A such that $p \notin \bigcup \{\overline{A_\alpha} : \alpha < \kappa\}$ then $p \notin \overline{\bigcup \{A_\alpha : \alpha < \kappa\}}$.

For every $x \in X \setminus \{p\}$, let U_x and V_x be disjoint open sets such that $x \in U_x$ and $p \in V_x$. Let $\mathcal{U} = \{U_x : x \in X \setminus \{p\}\}$.

Claim. There is a family $\mathcal{W} \in [\mathcal{U}]^{\leq \kappa}$ such that $A \subset \bigcup \mathcal{W}$.

Proof of Claim. Assume this is not the case. Then we will construct a free sequence of cardinality κ^+ in X.

Suppose that for some $\beta < \kappa^+$ we have constructed points $\{x_\alpha : \alpha < \beta\} \subset A$ and open families $\{\mathcal{U}_\alpha : \alpha < \beta\}$ such that:

(1) $\overline{\{x_\alpha : \alpha < \tau\}} \subset \bigcup \mathcal{U}_\tau$, for every $\tau < \beta$.
(2) $|\mathcal{U}_\alpha| \leq \kappa$, for every $\alpha < \beta$.

Note that $p \notin \overline{\{x_\alpha : \alpha < \beta\}}$, so $\mathcal{U}$ is a cover of $\overline{\{x_\alpha : \alpha < \beta\}}$. Moreover $L(\overline{\{x_\alpha : \alpha < \beta\}}) \leq \kappa$ and therefore we can find a subcollection $\mathcal{U}_\beta$ of $\mathcal{U}$ having cardinality $\leq \kappa$ such that $\overline{\{x_\alpha : \alpha < \beta\}} \subset \bigcup \mathcal{U}_\beta$. By our assumption $\bigcup \{\bigcup \mathcal{U}_\alpha : \alpha \leq \beta\}$ does not cover A and hence we can pick a point $x_\beta \in A \setminus \bigcup \{\bigcup \mathcal{U}_\alpha : \alpha \leq \beta\}$

Eventually, $\{x_\alpha : \alpha < \kappa^+\}$ is a free sequence in X having cardinality κ^+, which is a contradiction. △

Let $\{U_\alpha : \alpha < \kappa\}$ be an enumeration of $\mathcal{W}$ and set $A_\alpha = U_\alpha \cap A$, for every $\alpha < \kappa$. Then $\{A_\alpha : \alpha < \kappa\}$ is a κ-sized family of subsets of A such that $p \notin \bigcup \{\overline{A_\alpha} : \alpha < \kappa\}$ but $p \in \overline{\bigcup \{A_\alpha : \alpha < \kappa\}}$, and that is a contradiction. □

Corollary 7. *Let X be a Lindelöf Hausdorff space. Then $t_s(X) = F(X)$.*

3. The G_δ topology

In [8], Dow, Juhász, Soukup, Szentmiklóssy and Weiss proved the following bound for the tightness of the G_δ topology of a Lindelöf regular space.

Theorem 8. [8] *Let X be a Lindelöf regular space. Then $t(X_\delta) \leq 2^{t(X)}$.*

In the next theorem we will show how to relax the regularity assumption in their bound.

Theorem 9. *Let (X, τ) be a Lindelöf Hausdorff space. Then $t(X_\delta) \leq 2^{t(X)}$.*

Proof. Let $A \subset X_\delta$ be a non-closed set and p be a point in the G_δ-closure of A. Let $\kappa = t(X)$, let θ be a large enough regular cardinal and let M be a κ-closed elementary submodel of $H(\theta)$ such that $X, \tau, A, p \in M$ and $|M| = 2^\kappa$. Note that if we call τ_δ the collection of all G_δ subsets of X we also have $\tau_\delta \in M$. We claim that $p \in Cl_\delta(A \cap M)$.

Indeed, let G be a G_δ set such that $p \in G$ and let $\{U_n : n < \omega\}$ be a sequence of open sets such that $G = \bigcap\{U_n : n < \omega\}$.

For every point $x \in \overline{A \cap M}$, such that $x \neq p$, let V_x and W_x be disjoint open sets such that $x \in V_x$ and $p \in W_x$. Since $t(X) \leq \kappa$, we can find, for every $x \in \overline{A \cap M}$ a $\leq \kappa$-sized subset $S_x \subset A \cap M$ such that $x \in \overline{S_x}$. Since M is κ-closed, we have $S_x \in M$ and hence $\overline{S_x} \in M$. Now $\overline{S_x}$ is Lindelöf and $\{V_x : x \in \overline{A \cap M} \setminus \{p\}\}$ is an open cover of $\overline{S_x}$. Therefore we can find a countable subset $C_x \subset \overline{A \cap M} \setminus \{p\}$ such that $\overline{S_x} \subset \bigcup\{V_x : x \in C_x\}$ and $p \in \bigcap\{W_x : x \in C_x\}$. Define $O'_x = \bigcup\{V_x : x \in C_x\}$ and $G'_x = \bigcap\{W_x : x \in C_x\}$. Note that O'_x is an open set, G'_x is a G_δ set and $O'_x \cap G'_x = \emptyset$. So we just showed that:

$$H(\theta) \models (\exists O)(\exists G)(O \in \tau \wedge G \in \tau_\delta \wedge \overline{S_x} \subset O \wedge p \in G \wedge O \cap G = \emptyset)$$

Note that all free variables in the above formula (namely $\overline{S_x}, \tau$ and τ_δ) are in M and hence by elementarity we also have:

$$M \models (\exists O)(\exists G)(O \in \tau \wedge G \in \tau_\delta \wedge \overline{S_x} \subset O \wedge p \in G \wedge O \cap G = \emptyset)$$

What this means is that we can find, for every $x \in \overline{A \cap M} \setminus \{p\}$ an open set $O_x \in M$ and a G_δ set $G_x \in M$ such that $\overline{S_x} \subset O_x$, $p \in G_x$ and $O_x \cap G_x = \emptyset$.

Note now that $\{O_x : x \in \overline{A \cap M} \setminus \{p\}\}$ is an open cover of the Lindelöf space $\overline{A \cap M} \setminus U_n$, for every $n < \omega$. Therefore we can find a countable subset C_n of $\overline{A \cap M} \setminus U_n$ such that $\overline{A \cap M} \setminus U_n \subset \bigcup\{O_x : x \in C_n\}$. Note that $H_n = \bigcap\{G_x : x \in C_n\}$ is a G_δ set and $H_n \cap \overline{A \cap M} \setminus U_n = \emptyset$. Let now $H = \bigcap\{H_n : n < \omega\} \in M$. Note that H is a G_δ set containing p and therefore $H \cap A \neq \emptyset$. Since $H, A \in M$, by elementarity we also have $H \cap A \cap M \neq \emptyset$. But $H \cap \overline{A \cap M} \setminus G = \emptyset$, that is $H \cap \overline{A \cap M} \subset G$. It follows that $G \cap A \cap M \neq \emptyset$, as we wanted.

□

Lemma 10. *(Carlson, Porter and Ridderbos* [6]*) Let X be a Lindelöf Hausdorff space. Then $L(X_\delta) \leq 2^{F(X)}$,*

Corollary 11. *Let X be a Lindelöf Hausdorff pseudoradial space. Then $F(X_\delta) \leq 2^{F(X)}$.*

Proof. Combining Theorem 2 with Theorem 9 we obtain $t(X_\delta) \leq 2^{t(X)} = 2^{F(X)}$. Recalling that $F(X_\delta) \leq L(X_\delta) \cdot t(X_\delta)$ and using Lemma 10 we obtain $F(X_\delta) \leq 2^{F(X)}$, as desired. □

The above corollary was proved in [5] with the additional assumption that X is a regular space.

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