CONVERGENCE OVER FRACTALS FOR THE SCHRÖDINGER EQUATION

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Abstract. We consider a fractal refinement of the Carleson problem for the Schrödinger equation, that is to identify the minimal regularity needed by the solutions to converge pointwise to their initial data almost everywhere with respect to the α-Hausdorff measure (α-a.e.). We extend to the fractal setting ($\alpha < n$) a recent counterexample of Bourgain [5], which is sharp in the Lebesque measure setting ($\alpha = n$). In doing so we recover the necessary condition from [23] for pointwise convergence α-a.e. and we extend it to the range $n/2 < \alpha \leq (3n+1)/4$.

1. Introduction

We consider the problem of pointwise convergence for the linear Schrödinger equation

$$\begin{cases} \partial_t u = i\frac{\hbar}{2}\Delta u \\ u(x,0) = f(x) \in H^s(\mathbb{R}^n); \end{cases}$$

here $\hbar = 1/(2\pi)$. A classical question is: what is the minimal regularity the initial datum must have for the solution u to converge almost everywhere (a.e.) to f? More precisely, which is the smallest $s \geq 0$ such that

$$\lim_{t\to 0} u(x,t) = f(x), \qquad \text{for a.e. } x \in \mathbb{R}^n \text{ and for all } f \in H^s(\mathbb{R}^n). \tag{1}$$

This problem was introduced by Carleson in [8], where he proved the validity of (1) for $s \geq 1/4$ in dimension $n = 1$; soon later Dahlberg and Kenig [10] proved this to be sharp. The considerably harder higher dimensional problem was subsequently studied by many authors [9, 6, 30, 34, 3, 26, 32, 33, 31, 22, 4, 25, 11, 24, 14].

Recently, the problem has been settled, up to the endpoint, thanks to the contributions of Bourgain [5] (see [27] for a nice detailed exposition), who proved the necessity of $s \geq \frac{n}{2(n+1)}$, and of Du–Guth–Li [13] and of Du–Zhang [15], who proved the sufficiency of $s > \frac{n}{2(n+1)}$ in dimensions $n = 2$ and $n \geq 3$, respectively. We mention that, besides

Bourgain's counterexample, the necessity of $s \geq \frac{n}{2(n+1)}$ can be proved also by different counterexamples [23].

In this paper we consider a fractal refinement of the Carleson problem. Given $\alpha \in (0, n]$, the goal is to identify the smallest $0 \leq s \leq n/2$ such that

$$\lim_{t \to 0} u(x,t) = f(x), \qquad \text{for } \alpha\text{-a.e. } x \in \mathbb{R}^n \text{ and for all } f \in H^s(\mathbb{R}^n), \tag{2}$$

where α-a.e. means almost everywhere with respect to the α-dimensional Hausdorff measure.

This fractal refinement of the Carleson problem was introduced in [29]. In [2], the authors gave a complete solutions for $\alpha \in [0, n/2]$, proving that $s > (n-\alpha)/2$ is necessary and sufficient for (2) to hold. The necessity of this condition depends on the Sobolev space framework, since for smaller s there exist initial data in $H^s(\mathbb{R}^n)$ that are not well defined on sets of dimension α; see [35]. On the other hand, for $s > (n-\alpha)/2$ one can make sense of the initial data and of the relative solution α-a.e.; we refer to the proof of Theorem 9 for details. When $\alpha \in (n/2, n]$, Du and Zhang [15] proved the best known sufficient condition for (2) to hold:

$$s > \frac{n}{2(n+1)}(n+1-\alpha). \tag{3}$$

As mentioned, this is optimal (up to the endpoint) when $\alpha = n$, but it is not clear yet whether this is optimal for α strictly smaller. It is worth mentioning that (3) is necessary for the α-a.e. pointwise convergence in the periodic setting [16], however in this setting it is still unknown if it is sufficient (not even for $\alpha = n$).

In [23] it was proved that for $(3n+1)/4 \leq \alpha \leq n$ the condition

$$s > \frac{n}{2(n+1)} + \frac{n-1}{2(n+1)}(n-\alpha)\,, \tag{4}$$

is necessary for (2) to hold. Here we extend this result to the full range $n/2 < \alpha \leq n$ (recall that for smaller α the problem has been solved in [2]); thus the result is new for $n/2 < \alpha \leq (3n+1)/4$. To prove this result, we use a modification of the Bourgain's counterexample rather than the counterexample in [23]. We consider this fact of independent interest. The possibility of adapting the Bourgain's counterexample to the fractal measure setting was also suggested by Lillian Pierce in [27].

Theorem 1. *Let $n \geq 2$ and $n/2 < \alpha \leq n$. Then for every*

$$s' < s := \frac{n}{2(n+1)} + \frac{n-1}{2(n+1)}(n-\alpha) \tag{5}$$

there exists a function $f \in H^{s'}(\mathbb{R}^n)$ such that

$$\limsup_{t\to 0^+} |e^{it\hbar\Delta/2} f(x)| = \infty \tag{6}$$

for x in a set of Hausdorff dimension $\geq \alpha$.

For $\alpha \in (n/2, n)$ we can in fact immediately improve the statement, saying that (6) occurs on a set with α-Hausdorff measure $= \infty$. This is because in (5) we have a strict inequality. Thus, given $\alpha' > \alpha$ and sufficiently close to α in such a way that

$$s' < s := \frac{n}{2(n+1)} + \frac{n-1}{2(n+1)}(n - \alpha'),$$

we would in fact prove that (6) occurs on a set of dimension $\geq \alpha'$. When $\alpha = n$ we can not self-improve the statement, however we know by [23] that (6) holds on a set of strictly positive Lebesgue measure.

A consequence of Theorem 1 is the necessity of the condition

$$s \geq \frac{n}{2(n+1)} + \frac{n-1}{2(n+1)}(n - \alpha)$$

for the validity of the maximal estimate

$$\int_{B_R} \sup_{t\in(0,1)} |e^{it\Delta} f(x)|^2 d\mu(x) \lesssim C_\mu R^{2s} \|f\|_2^2\,, \tag{7}$$

where $B_R \subset \mathbb{R}^n$ is a ball of radius $R > 1$, and μ is an α-dimensional measure on $B_R \subset \mathbb{R}^n$, *i.e.* a positive Borel measure that satisfies

$$\mu(B_r(x)) \lesssim C_\mu r^\alpha,$$

for all balls with center x and radius $r > 0$. One may see (7) as the weighted L^2 inequality

$$\int_{B_R} \sup_{t\in(0,1)} |\widehat{g d\sigma}(x)|^2 d\mu(x) \lesssim C_\mu R^{2s} \|g\|^2_{L^2(S)}, \tag{8}$$

where S is a bounded hypersurface in $\mathbb{R}^d := \mathbb{R}^{n+1}$ with non zero gaussian curvature (for instance, a portion of the paraboloid in the case of (7)) and $d\sigma$ is the measure induced on S by the Lebesgue measure. A closely related family of weighted L^2 estimates is

$$\int_{B_1} |\widehat{g d\sigma}(Rx)|^2 d\mu(x) \lesssim C_\mu R^{-\gamma} \|g\|^2_{L^2(S)}, \tag{9}$$

where B_1 is now a ball in $\mathbb{R}^d$ of radius 1, and μ is an α-dimensional measure on $B_1 \subset \mathbb{R}^d$. The problem here is to identify the largest γ such that (9) holds. Interestingly, these problems are very sensitive to the arithmetical structure of the hypersurface S. For instance, the known necessary conditions are different for the sphere and the paraboloid; see [21, 1, 25, 12, 28, 19].

In section 2 we introduce some preliminary results we need to prove the main lower bound for the solution of the Schödinger equation—Theorem 6—in section 3, where the initial data are basically the same described by Bourgain in [4]. Bourgain's counterexample is a function with frequencies at a fixed scale, so in section 4 functions at different scales are assembled into a single function, and we show that the interaction between different scales is negligible (Theorem 9). In section 5 we compute the dimension of the divergence sets, and we conclude the proof of Theorem 1 in section 6.

Notations.

- $e(z) = e^{iz}$.
- If $A \subset \mathbb{R}^n$, then $|A|$ is its Lebesgue measure, and if A is a discrete set, then $|A|$ is the cardinality. For example, if $I = [a, b] \subset \mathbb{Z}$ denotes the interval of integers $a \le k \le b$, then $|I|$ is the length of the interval.
- If $I = [a, b] \subset \mathbb{Z}$, for $a, b \in \mathbb{R}$, denotes an interval of integers, then we write $L(I) := \min_{k \in I} k$ and $R(I) := \max_{k \in I} k$.
- $B_r(x) \subset \mathbb{R}^n$ is a ball of radius r and center x—the center is usually omitted. $Q(x, l) \subset \mathbb{R}^n$ is a cube with side-length l and center x.
- If $x \lesssim y$, then $x \le Cy$ for some constant $C > 0$, and similarly for $x \gtrsim y$; if $x \simeq y$ then $x \lesssim y \lesssim x$. If $x \ll y$ then $x \le cy$, where c is a sufficiently small constant, and similarly for $x \gg y$.
- $\limsup_{k \to \infty} F_k := \bigcap_{N \ge 1} \bigcup_{k \ge N} F_k$.
- Hausdorff dimension of a set: for $0 < \alpha \le n$ and $\delta > 0$ we define the outer measure
$$\mathcal{H}^\alpha_\delta(F) := \inf\{ \sum_{B_r \in \mathcal{B}} r^\alpha \mid F \subset \bigcup_{B_r \in \mathcal{B}} B_r \text{ and } r < \delta \};$$
we do not exclude the case $\delta = \infty$. The α-dimensional Hausdorff measure of a set F is $\mathcal{H}^\alpha(F) := \lim_{\delta \to 0} \mathcal{H}^\alpha_\delta(F)$. The Hausdorff dimension of a set F is $\sup\{\alpha \mid \mathcal{H}^\alpha(F) > 0\}$.

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2. Preliminaries

We recall some classical estimates about exponential sums that we will use repeatedly in the rest of the paper.

We recall first a classical result about Gauss quadratic sums, whose proof can be consulted in Lemma 3.1 of [27].

Lemma 2 (Gauss quadratic sums)**.** *If $a, b, q \in \mathbb{Z}$ satisfy the conditions $(a, q) = 1$ and*

$$\begin{cases} b \in \mathbb{Z} & \text{when } q \text{ is an odd number,} \\ b \text{ is even} & \text{when } q \equiv 0\,(mod\,4), \\ b \text{ is odd} & \text{when } q \equiv 2\,(mod\,4), \end{cases} \tag{10}$$

then for the quadratic phase

$$f(r) := \frac{a}{q} r^2 + \frac{b}{q} r \tag{11}$$

it holds that

$$\left| \sum_{r=0}^{q-1} e(2\pi f(r)) \right| = c_q \sqrt{q}, \tag{12}$$

where $c_q = 1$ when q is odd, and $c_q = \sqrt{2}$ when q is even.

The following estimate due to Weyl will be useful to handle incomplete Gauss sums.

Lemma 3. *Let I be an integer interval. If $a, b, q \in \mathbb{Z}$ satisfy the conditions $(a, q) = 1$ and (10), then for the quadratic phase f in (11) it holds that*

$$\left| \sum_{k \in I} e(2\pi f(k)) \right| = C \frac{|I|}{\sqrt{q}} + \mathcal{O}(\sqrt{q \ln q}), \tag{13}$$

where $\frac{1}{2} \leq C \leq \sqrt{2}$.

Proof. We can assume that $L(I) := \min_{k \in I} k = 0$. In fact,

$$\sum_{k=L(I)}^{R(I)} e(2\pi(\frac{a}{q}k^2 + \frac{b}{q}k)) = e(2\pi(\frac{a}{q}L(I)^2 + \frac{b}{q}L(I))) \sum_{k=0}^{|I|-1} e(2\pi(\frac{a}{q}k^2 + \frac{b + 2aL(I)}{q}k)),$$

and the absolute value at both sides is the same; we observe that the parity of b and $b + 2aL(I)$ is preserved.

If $|I| < q$, then

$$\left| \sum_{k \in I} e(2\pi f(k)) \right| \leq C \sqrt{q \ln q}; \tag{14}$$

for the proof we refer to Lemma 3.2 of [27].

If $|I| \geq q$, then we can sum in blocks of length q. Let M be the largest integer that satisfies $Mq \leq |I|$, *i.e.* $Mq \leq |I| < (M+1)q$, then

$$\begin{aligned} I &= [0, Mq-1] \cup J \\ &= \Big(\bigcup_{m=0}^{M-1} [mq, mq+q-1] \Big) \cup J, \end{aligned}$$

where $|J| < q$. The sum over each block $[mq, mq+q-1]$ is a Guass quadratic sum, and we arrive to

$$\sum_{k \in I} e(2\pi f(k)) := M \sum_{r=0}^{q-1} e(2\pi f(r)) + \sum_{k \in J} e(2\pi f(k)).$$

By our election of M we have $M = C|I|/q$, for $\frac{1}{2} < C \leq 1$, and by (14) we have

$$\left| \sum_{k \in I} e(2\pi f(k)) \right| = C \frac{|I|}{q} \left| \sum_{r=0}^{q-1} e(2\pi f(r)) \right| + \mathcal{O}(\sqrt{q \ln q}).$$

Finally, we apply Lemma 2 to get (13). □

To deal with perturbations of quadratic sums, we will use the following Lemma, which is consequence of Abel's summation formula; see Lemma 2.3 of [16].

Lemma 4. *Let I be an integer interval. Let $a_k \geq 0$ be a sequence of real numbers and b_k be sequences of complex numbers such that*

(1) $a_{k+1} \leq a_k$,
(2) $\left|\sum_{k \in I'} b_k\right| \leq \mathcal{C}$, *for every interval* $I' \subseteq I$.

Then,

$$\left| \sum_{k \in I'} a_k b_k \right| \leq \mathcal{C} a_{L(I')}, \quad \text{for every interval } I' \subseteq I. \tag{15}$$

If (1) is replaced with $a_{k+1} \geq a_k$, then

$$\left| \sum_{k \in I'} a_k b_k \right| \leq \mathcal{C} a_{R(I')}, \quad \text{for every interval } I' \subseteq I.$$

3. The main lower bound

The initial data we consider are modifications of the Bourgain's counterexample in [5]. Let φ be a smooth positive function such that $\operatorname{supp}\hat{\varphi} \subset B_1(0)$ and $\varphi(0) = 1$. We define the function

$$f_D(x) := f_1(x_1)\tilde{f}(\tilde{x}) \tag{16}$$

where

$$f_1(x_1) = e(2\pi R x_1)\varphi(R^{\frac{1}{2}}x_1), \qquad \tilde{f}(\tilde{x}) := \prod_{j=2}^{n}\varphi(x_j)\Big(\sum_{\frac{R}{2D}<l_j<\frac{R}{D}} e(2\pi D l_j x_j)\Big)$$

where $l = (l_2,\dots,l_n) \in \mathbb{Z}^{n-1}$ and $x = (x_1,\tilde{x}) \in \mathbb{R}\times\mathbb{R}^{n-1}$. For now we set D as a free parameter, and we will choose its value later as a suitable power of R.

We need the following definition before investigating the divergence set of $e^{it\hbar\Delta/2}f_D$; compare with (10).

Definition 5 (Admissible fractions)**.** Let $p_1,\dots,p_n,q \in \mathbb{Z}$. A point $(p_1/q,\dots,p_n/q)$ is an admissible fraction if $(p_1,q) = 1$ and if

$$\begin{cases} (p_2,\dots,p_n) \in \mathbb{Z}^{n-1} & \text{when } q \text{ is an odd number,}\\ p_j \text{ are even} & \text{when } q \equiv 0 \,(\mathrm{mod}\, 4),\\ p_j \text{ are odd} & \text{when } q \equiv 2 \,(\mathrm{mod}\, 4). \end{cases} \tag{17}$$

Theorem 6. *Let $c \ll 1$ and let $q > 0$ be an integer such that $\frac{R}{Dq} \gg \sqrt{\ln q}$. If f is the initial datum (16), then*

$$\frac{|e^{it\hbar\Delta/2}f_D(x)|}{\|f_D\|_{L^2}} \gtrsim R^{\frac{1}{4}}\Big(\frac{R}{Dq}\Big)^{\frac{n-1}{2}} \tag{18}$$

for (x,t) such that $0 < t = 2p_1/(D^2q) \ll 1/R$ and

$$x \in E_{q,D} \cap [0,c]^n, \tag{19}$$

where $E_{q,D}$ is the set of points

$$x_1 \in 2\frac{p_1 R}{qD^2} + [-cR^{-\frac{1}{2}}, cR^{-\frac{1}{2}}] \quad \text{and} \quad x_j \in \frac{p_j}{Dq} + [-cR^{-1}, cR^{-1}],\ \ 2 \le j \le n; \tag{20}$$

here $(p_1/q,\dots,p_n/q)$ is an admissible fraction in the sense of Definition 5; see Fig. 1.

Proof. If $\hat{f}$ is an integrable functions, the solution of the Schrödinger equation with initial datum f can be represented as

$$e^{it\hbar\Delta/2}f(x) = \int \hat{f}(\xi)e(-\pi t|\xi|^2 + 2\pi x\cdot\xi)\,d\xi.$$

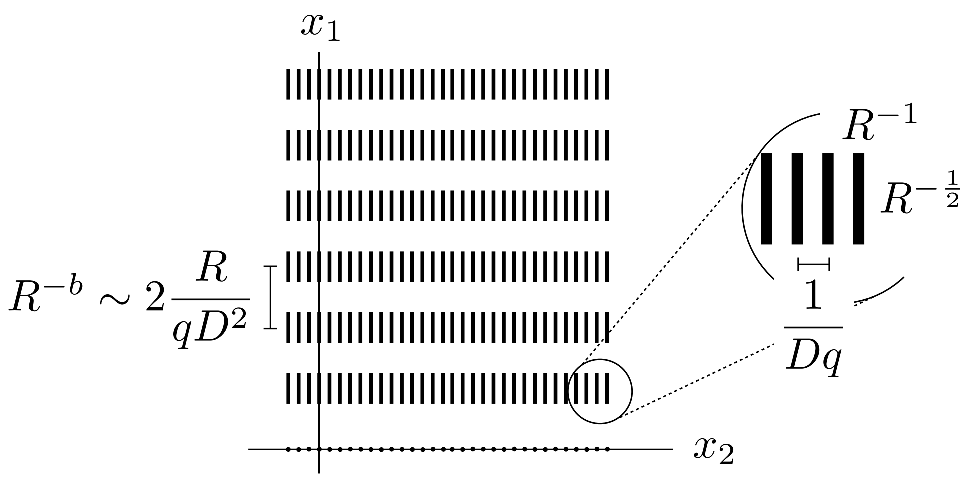


FIGURE 1. Set $E_{q,D}$ in Theorem 6. Some slabs may disappear to satisfy the conditions of admissibility.

We want to compute the modulus of $e^{it\hbar\Delta/2}f_D(x)$ in the region $|x| < c$ and $0 < t < c/R$. We note that

$$|e^{it\hbar\Delta/2}f_D(x)| = |e^{it\hbar\Delta/2}f_1(x_1)|\,|e^{it\hbar\Delta/2}\tilde{f}(\tilde{x})|\,.$$

A direct computation shows that for $|t| \leq c/R$ and

$$x_1 \in tR + [-cR^{-\frac{1}{2}}, cR^{-\frac{1}{2}}] \tag{21}$$

we have

$$|e^{it\hbar\Delta/2}f_1(x_1)| \simeq |\varphi(R^{\frac{1}{2}}(x_1 - tR))| \simeq 1 \tag{22}$$

Again, a direct computation gives ($\tilde{x} \in \mathbb{R}^{n-1}$)

(23)
$$e^{it\hbar\Delta/2}\tilde{f}(\tilde{x}) = \prod_{j=2}^{n}\int \hat{\varphi}(\xi_j)e(-\pi t\xi_j^2 + 2\pi x_j\xi_j) \sum_{\frac{R}{2D}<l_j<\frac{R}{D}} e(-\pi t|Dl_j|^2 + 2\pi Dl_j(x_j - t\xi_j))\,d\xi_j.$$

To estimate the absolute value of this product, we recall our hypotheses (20): $x_j = p_j/(Dq) + \varepsilon_j$, for $|\varepsilon_j| < c/R$. We split each factor in (23) into the main term

$$F_{\mathrm{main}}(t, p_j/q) := e^{it\hbar\Delta/2}\varphi(x_j) \sum_{\frac{R}{2D}<l_j<\frac{R}{D}} e(-\pi t|Dl_j|^2 + 2\pi l_j\frac{p_j}{q}) \tag{24}$$

and the perturbation

$$F_{\rm per}(t,x_j) := \int \hat{\varphi}(\xi_j) e(-\pi t\xi_j^2 + 2\pi x_j\xi_j) \sum_{\frac{R}{2D}<l_j<\frac{R}{D}} e(-\pi t|Dl_j|^2 + 2\pi l_j\frac{p_j}{q})(1-e(2\pi Dl_j(\varepsilon_j - t\xi_j))\, d\xi_j. \tag{25}$$

By hypothesis $t = 2p_1/(D^2q)$, so we can exploit Lemma 3 and the condition $R/(Dq) \gg \sqrt{\ln q}$ to estimate the main contribution (24) as

$$\begin{aligned} |F_{\rm main}| &\simeq |\sum_{\frac{R}{2D}<l_j<\frac{R}{D}} e\big(-2\pi(\frac{p_1}{q}l_j^2 - \frac{p_j}{q}l_j)\big)| \\ &\simeq \frac{R}{D\sqrt{q}}; \end{aligned} \tag{26}$$

we used $|e^{it\hbar\Delta/2}\varphi(x_j)| \simeq 1$.

We claim that the perturbation term (25) satisfies $|F_{\rm per}| \ll R/(D\sqrt{q})$, which, together with (23) and (26), leads to

$$|e^{it\hbar\Delta/2}\tilde{f}(\tilde{x})| \simeq \Big(\frac{R}{D\sqrt{q}}\Big)^{n-1}. \tag{27}$$

Then, we multiply by (22) to reach

$$|e^{it\hbar\Delta/2}f_D(x)| \simeq \Big(\frac{R}{D\sqrt{q}}\Big)^{n-1}. \tag{28}$$

Finally, we divide (28) by $\|f_D\|_2 \simeq R^{-\frac14}(R/D)^{\frac{n-1}{2}}$ to obtain (18), and so the statement of the Theorem follows up to the claim $|F_{\rm per}| \ll R/(D\sqrt{q})$.

To prove the upper bound $|F_{\rm per}| \ll R/(D\sqrt{q})$, where $F_{\rm per}$ was defined in (25), we begin with

$$\begin{aligned} |F_{\rm per}| &\lesssim \sup_{|\xi_j|\le 1} |\sum_{\frac{R}{2D}<l_j<\frac{R}{D}} e(-2\pi\frac{p_1}{q}l_j^2 + 2\pi l_j\frac{p_j}{q})(1-e(2\pi Dl_j(\varepsilon_j - t\xi_j))| \\ &= \sup_{|\xi_j|\le 1}\Big|\sum_{\frac{R}{2D}<l_j<\frac{R}{D}} e(-2\pi\frac{p_1}{q}l_j^2 + 2\pi\frac{p_j}{q}l_j)\phi_{\varepsilon,t,\xi_j}(l_j)\Big| \end{aligned}$$

where $(p_1,q)=1$, and $\phi_{\varepsilon_j,t,\xi_j}(l_j) = 1 - e(2\pi Dl_j(\varepsilon_j - t\xi_j))$.

By the triangle inequality, it suffices to prove

$$\Big|\sum_{\frac{R}{2D}<l_j<\frac{R}{D}} e(-2\pi\frac{p_1}{q}l_j^2 + 2\pi\frac{p_j}{q}l_j)\phi^i_{(\cdot)}(l_j)\Big| \lesssim c\frac{R}{D\sqrt{q}}, \qquad c\ll 1 \qquad i=1,2. \tag{29}$$

where

$\phi^1_{(\cdot)}(l_j) := 1-\cos(2\pi D l_j(\varepsilon_j - t\xi_j))$ and $\phi^2_{(\cdot)}(l_j) := |\sin(2\pi D l_j(\varepsilon_j - t\xi_j))|$.

Again by Lemma 3, and using $R/(Dq) \gg \sqrt{\ln q}$, we have that

(30)
$$\Big|\sum_{l_j\in I} e(-2\pi\frac{p_1}{q}l_j^2 + 2\pi\frac{p_j}{q}l_j)\Big| \lesssim \frac{|I|}{\sqrt{q}} + \sqrt{q\ln q}$$
$$\lesssim \frac{R}{D\sqrt{q}} + \sqrt{q\ln q} \lesssim \frac{R}{D\sqrt{q}}, \quad \forall\, |I| \le \frac{R}{D}.$$

On the other hand, the functions $\phi^i_{(\cdot)}(l_j)$ are real valued, positive, increasing in l_j, and satisfy
$$\phi^i_{(\cdot)}(l_j) \lesssim |D||l_j||\varepsilon_j - t\xi_j| \lesssim c, \qquad c \ll 1;$$
recall that $|l_j| \le \frac{R}{D}$, $|\varepsilon_j| \le \frac{c}{R}$, $|t| \le \frac{c}{R}$ and $|\xi_j| \le 1$ in the support of $\hat{\phi}_j$. Thus (29) follows by the second part of Lemma 4 taking
$$a_{l_j} = \phi^i_{(\cdot)}(l_j) \qquad \text{and} \qquad b_{l_j} = e(-2\pi\frac{p_1}{q}l_j^2 + 2\pi\frac{p_j}{q}l_j),$$
and the proof is concluded.

□

4. Construction of the Examples

According to Theorem 6, the function $\sup_{0<t<1}|e^{it\hbar\Delta/2}f_D|$ is large in the set
$$\bigcup_{1\le q\le Q}(E_{q,D}\cap[0,c]^n) \subset \mathbb{R}^n, \quad 0<c\ll 1,$$
as long as $\frac{R}{DQ} \gg \sqrt{\ln Q}$.

To cover the largest possible area, we should ensure that the collection of sets $E_{q,D}$, for $1\le q\le Q$, is essentially pairwise disjoint. In a unit cell $[0,1/D]^{n-1}$, the number of fractions $(p_2/(Dq),\dots,p_n/(Dq))$, for $1\le q\le Q$, is $\simeq Q^n$, and if we think of the fractions as if they were uniformly distributed, then the average distance between them is $\simeq 1/(Q^{\frac{n}{n-1}}D)$, so we impose the restriction

$$R^{-a} := \frac{1}{Q^{\frac{n}{n-1}}D} \ge R^{-1}. \tag{31}$$

We remark that $R/(DQ) = Q^{\frac{1}{n-1}}R^{1-a}$, so the condition $R/(Dq) \gg \sqrt{\ln q}$, for $1\le q\le Q$, is easily satisfied.

The slabs that form $E_{q,D}$ have dimensions $cR^{-\frac{1}{2}}\times cR^{-1}\times\cdots\times cR^{-1}$, for $c\ll 1$, and they do not overlap in the $\tilde{x}$-space because $R/(DQ)\gg 1$, however they may overlap in the x_1 direction. To exploit the whole area of the slabs, we impose the new restriction

$$R^{-b} := \frac{R}{QD^2} \geq R^{-\frac{1}{2}}; \tag{32}$$

see Figure 1.

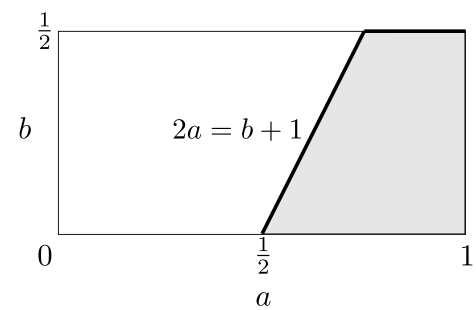
The conditions (31) and (32) allow us to solve for Q and D as

$$D = R^{(n-(n-1)a+nb)/(n+1)} \qquad \text{and} \qquad Q = R^{\frac{n-1}{n+1}(2a-b-1)}. \tag{33}$$

Since $Q \geq 1$, then we have to be sure that $2a \geq 1+b$, so we can write our conditions as

$$0 < a \leq 1, \quad 0 < b \leq \frac{1}{2} \quad \text{and} \quad 2a \geq 1+b; \tag{34}$$

in particular, $a \geq \frac{1}{2}$.



Definition 7 (Divergence Sets)**.** Let a and b satisfy the conditions (34), and let $\mathcal{A}_k$, for $k \geq k_0 \gg 1$, be the collection of slabs S such that:

(i) S has dimensions $cR_k^{-\frac{1}{2}} \times cR_k^{-1} \times \cdots \times cR_k^{-1}$, for $R_k = 2^k$ and $c \ll 1$.

(ii) S has center at

$$(2p_1 R_k/(qD_k^2), p_2/(D_k q), \ldots, p_n/(D_k q)),$$

where $(p_1/q, \ldots, p_n/q)$ is an admissible fraction (Definition 5) with $1 \leq q \leq Q_k$, and D_k and Q_k are given by (33).

A (a, b)-set of divergence F is defined as

$$F := \limsup_{k\to\infty} F_k, \qquad F_k := \bigcup_{S \in \mathcal{A}_k} S\,. \tag{35}$$

For fixed a and b, we define the initial datum

$$g_{a,b} = \sum_{k \geq k_0} R_k^{-s} \frac{k}{\|f_{D_k}\|_2} f_{D_k}, \tag{36}$$

where $R_k = 2^k$ and $k_0 \gg 1$. Inequality (18) dictates the value of s, and in terms of a and b we have

$$s := \frac{1}{4} + \frac{n-1}{2(n+1)}(n - (n-1)a - b). \tag{37}$$

Since

$$\|f\|^2_{H^{s'}(\mathbb{R}^n)} = \sum_{k \geq k_0} k R_k^{2(s'-s)} < \infty, \quad \text{for } s' < s, \tag{38}$$

we have that $f \in H^{s'}(\mathbb{R}^n)$ for every $s' < s$.

We have to prove that the different terms in the sum (36) do not interfere with each other. We need the following Lemma.

Lemma 8. *If the Fourier transform of $\varphi \in \mathcal{S}(\mathbb{R})$ is supported in $(-1, 1)$, then for every $N \geq 1$ it holds*

$$|e^{it\hbar\Delta/2}\varphi(x)| \leq C_N \frac{1}{|x|^N}, \qquad \textit{for } |x| > 2t. \tag{39}$$

Proof. We use the principle of non-stationary phase. We assume that $x > 2t$; the other case is similar. The solution is

$$e^{it\hbar\Delta/2}\varphi(x) = \int \hat{\varphi}(\xi) e(-\pi t|\xi|^2 + 2\pi x\xi)\, d\xi.$$

Since $\partial_\xi e(-\pi t|\xi|^2 + 2\pi x\xi) = -2\pi i(t\xi - x)e(-\pi t|\xi|^2 + 2\pi x\xi)$, then by repeated integration by parts we obtain

$$|e^{it\hbar\Delta/2}\varphi(x)| \leq C_N \frac{1}{|x-t|^N},$$

which is the statement of the Lemma. □

Before proving the main result of this section, we need to make an observation on the way we define solutions. For $f \in H^s$, we define solutions for Sobolev functions, in such a way that they are well defined on sets with large Hausdorff dimension. Recall that $Q(N)$ is the cube of side N centered at zero. We set

$$e^{it\hbar\Delta/2} f(x) = \lim_{N\to\infty} S_N(t)f(x), \tag{40}$$

where

$$S_N(t)f(x) = \int_{Q(N)} \hat{f}(\xi) e(-\pi t|\xi|^2 + 2\pi x\cdot\xi)\, d\xi. \tag{41}$$

The limit (40) is usually taken with respect to the L^2 norm, but here we take all the limits pointwise at each point x where they exist. When $f \in L^2(\mathbb{R})$, it is known that the limit exists pointwise for almost every $x \in \mathbb{R}$ and that it coincides with the L^2–limit. When $n = 1$, this result

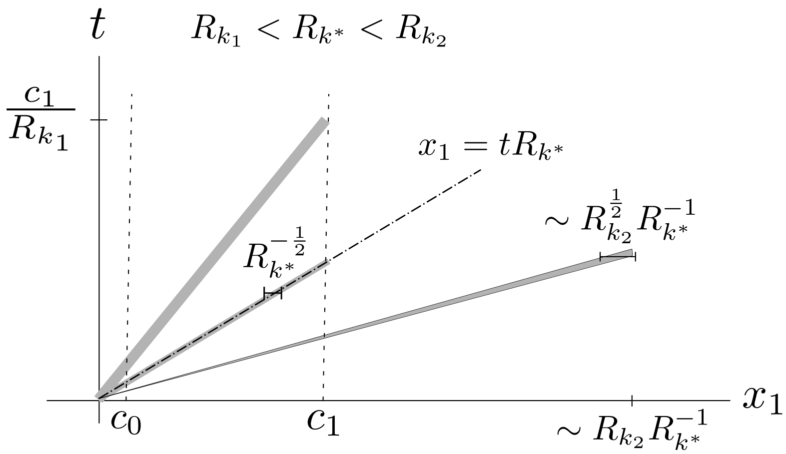


FIGURE 2. The gray lines represent the regions where the functions $e^{it\hbar\Delta/2}h_k$ concentrate.

is due to Carleson [7], whose proof extends to higher dimensions as proved, for instance, in [18]. Moreover, we can show that this limit exists γ-almost everywhere for every $f \in H^s$ with $s \in (0, n/2]$, as long as $\gamma > n - 2s$; see the appendix of [16]. This can be regarded as a refinement of Carleson's result, although it does not recover it.

Theorem 9. *If $g_{a,b}$ is the initial datum defined in* (36)*, then*

$$\limsup_{t \to 0^+} |e^{it\hbar\Delta/2} g_{a,b}(x)| = \infty \tag{42}$$

for every $x \in (F \cap ([c_0, c_1] \times [0, c_1]^{n-1})) \setminus \Omega$, where

- $c_0 := \frac{1}{10} c_1, \qquad c_1 \ll c \ll 1;$
- *F is a (a, b)-set of divergence;*
- *$\mathcal{H}^\gamma(\Omega) = 0$ for $\gamma > n - 2s$.*

Proof. We define $h_k := kR_k^{-s} f_{D_k}/\|f_{D_k}\|_2$, where $R_k := 2^k$. From the proof of Theorem 6 we know that for $t = 2p_1/(D_k^2 q) \ll 1/R_k$ the value of the solution at $x \in F_k \cap [0, c_1]^n = \bigcup_{1 \le q \le Q_k} E_{q,D_k} \cap [0, c_1]^n$ is

$$|e^{it\hbar\Delta/2} h_k(x)| \gtrsim k. \tag{43}$$

We fix $k^* \ge k_0 \gg 1$ and $x \in F_{k^*}$, and we know that

$$x_1 = tR_{k^*} + \mathcal{O}(R_{k^*}^{-\frac{1}{2}}), \qquad c_0 R_{k^*}^{-1} < t < c_1 R_{k^*}^{-1}. \tag{44}$$

It suffices to prove

$$|e^{it\hbar\Delta/2} h_k(x)| \lesssim R_k^{-1}, \quad \text{for} \quad k \neq k^*, \tag{45}$$

because then for $t = 2p_1/(D_{k^*}^2 q) \ll 1/R_{k^*}$ we would have, for all $k_1 \geq k^*$, the following (recall (41))

$$|S_{2^{k_1}}(t)g_{a,b}(x)| > |e^{it\hbar\Delta/2}h_{k^*}(x)| - \sum_{k_0 \leq k \neq k^* \leq k_1} |e^{it\hbar\Delta/2}h_k(x)| \gtrsim k^*, \tag{46}$$

as long as $k_0 \gg 1$; then in order to deduce (42) we note that for all $x \in F \cap ([c_0, c_1] \times [0, c_1]^{n-1})$ we can choose any $k^* \geq k_0 \gg 1$, and we we have a lower bound as (46) and the sequence of times $t = 2p_1/(D_{k^*}^2 q) \ll 1/R_{k^*}$ goes to zero as $k^* \to \infty$. More precisely, since we have

$$e^{it\hbar\Delta/2}f(x) = \lim_{N\to\infty} S_N(t)g_{a,b}(x) = \lim_{k_1\to\infty} S_{2^{k_1}}(t)g_{a,b}(x) \tag{47}$$

except possibly on sets Ω_t, $t = 2p_1/(D_{k^*}^2 q)$ with $\mathcal{H}^\gamma(\Omega_t) = 0$ and these sets are countably many, then (42) would follow by (46)-(47), taking

$$\Omega := \bigcup_{t=2p_1/(D_{k^*}^2 q)} \Omega_t.$$

It remains to prove (45). From (23), we see that we can bound $e^{it\hbar\Delta/2}\tilde{h}_k(\tilde{x})$ with the crude estimate

$$|e^{it\hbar\Delta/2}\tilde{f}_{D_k}(\tilde{x})| \lesssim \left(\frac{R_k}{D_k}\right)^{n-1},$$

so we can control each term $e^{it\hbar\Delta/2}h_k(x)$ as

$$\begin{aligned} |e^{it\hbar\Delta/2}h_k(x)| &\leq |[e^{itR_k\hbar\Delta/2}\varphi(R_k^{\frac{1}{2}}(x_1 - tR_k))]| kR_k^{\frac{1}{4}}\left(\frac{R_k}{D_k}\right)^{\frac{n-1}{2}} R_k^{-s} \\ &\lesssim |e^{itR_k\hbar\Delta/2}\varphi(R_k^{\frac{1}{2}}(x_1 - tR_k))| kQ_k^{\frac{n-1}{2}}, \end{aligned}$$

and we can apply Lemma 8 to φ.

We verify the hypotheses of Lemma 8 when $R_k < R_{k^*}$. By (44) we get

$$\begin{aligned} \frac{R_k^{\frac{1}{2}}(x_1 - tR_k)}{tR_k} &= \frac{R_k^{\frac{1}{2}}(t(R_{k^*} - R_k) + \mathcal{O}(R_{k^*}^{-\frac{1}{2}}))}{tR_k} \\ &\gtrsim R_k^{-\frac{1}{2}} R_{k^*} > 2, \end{aligned}$$

for $k, k^* \geq k_0 \gg 1$; hence, $|e^{it\hbar\Delta/2}h_k(x)| \lesssim_N kQ_k^{\frac{n-1}{2}} R_k^{-\frac{N}{2}} \lesssim R_k^{-1}$, for $N \gg 1$.

We verify now the hypotheses of Lemma 8 when $R_k > R_{k^*}$:

$$\begin{aligned} \frac{R_k^{\frac{1}{2}}(tR_k - x_1)}{tR_k} &= \frac{R_k^{\frac{1}{2}}(t(R_k - R_{k^*}) + \mathcal{O}(R_{k^*}^{-\frac{1}{2}}))}{tR_k} \\ &\gtrsim R_k^{\frac{1}{2}} > 2, \end{aligned}$$

for $k, k^* \geq k_0 \gg 1$; hence, $|e^{it\hbar\Delta/2}h_k(x)| \lesssim_N kQ_k^{\frac{n-1}{2}}(R_{k^*}R_k^{-\frac{3}{2}})^N \lesssim R_k^{-1}$, for $N \gg 1$. $\square$

5. Dimension of the Divergence Set

In the previous section we constructed initial data parameterized by a and b. To simplify matters, we choose those values of a and b for which computations are easier and exhaust all possible outcomes. Our choices are:

$$\text{(I)} \qquad \frac{1}{2} < a \leq \frac{3}{4} \qquad \text{and} \qquad b = 2a - 1 \tag{48}$$

$$\text{(II)} \qquad \frac{3}{4} < a \leq 1 \qquad \text{and} \qquad b = \frac{1}{2}. \tag{49}$$

We refer to these (a, b)-sets of divergence (Definition 7) as of type I and type II. We remark that for I we have $Q = 1$, and that $a = 1$ and $b = \frac{1}{2}$ is Bourgain's counterexample.

Theorem 10. *Let $0 < c_0 \leq 1$. If $F = \limsup_{k\to\infty} F_k$ is a (a, b)-set of divergence (Definition 7), then $\dim(F \cap [0, c_0]^n) \leq \alpha := \frac{1}{2} + (n-1)a + b$.*

Proof. Fix a scale $0 < \lambda \ll 1$ and choose k' such that $R_{k'}^{-1} < \lambda$. Since F_k is union of $\lesssim R_k^{(n-1)a+b}$ slabs with dimensions $R_k^{-\frac{1}{2}} \times R_k^{-1} \times \cdots \times R_k^{-1}$, and each slab can be covered by $R_k^{\frac{1}{2}}$ balls B_r, for $r = R_k^{-1}$, then we can find a collection $\mathcal{B}_k$ with $|\mathcal{B}_k| = R_k^\alpha$ of balls with radius R_k^{-1} covering F_k, so that

$$\mathcal{H}^\beta_\lambda(F) := \inf\{\sum_{B_\rho \in \mathcal{B}} \rho^\beta \mid F \subset \bigcup_{B_\rho \subset \mathcal{B}} B_\rho \text{ and } \rho < \lambda\} \leq \sum_{k \geq k'} \sum_{B_r \in \mathcal{B}_k} R_k^{-\beta},$$

and the last sum is smaller than $\sum_{k\geq k'} R_k^{\alpha-\beta}$, which tends to zero as $k' \to \infty$ whenever $\beta > \alpha$. $\square$

To prove the corresponding lower bound of $\dim F$, we employ the techniques in Section 4 of [24]. We recall a result of Falconer, which is consequence of Theorem 3.2 and Corollary 4.2 in [17].

Lemma 11. *Let $0 < c \leq 1$. Suppose that there exists a constant $C > 0$ such that, for all $\delta > 0$ and all cubes $Q(x, \delta) \subset [0, c]^n$, we have the density condition*

$$\liminf_{k\to\infty} \mathcal{H}^\beta_\infty(F_k \cap Q(x,\delta)) \geq C\delta^\beta,$$

where $\{F_k\}_{k\geq 0}$ is a sequence of open subsets of $B(0,1)$. Then, for all $\beta' < \beta$,

$$\mathcal{H}^{\beta'}(\limsup_{k\to\infty} F_k) > 0.$$

We prove now the lower bound of $\dim F$ in the easier case, in the case of sets of type I.

Theorem 12. *If $F = \limsup_{k\to\infty} F_k$ is a set of type I, that is, $\frac{1}{2} < a \leq \frac{3}{4}$ and $b = 2a - 1$, then $\dim F \cap [0, c_0]^n \geq \alpha$ where*

$$\alpha := \frac{1}{2} + (n-1)a + b. \tag{50}$$

Proof. From Lemma 11 it will be sufficient to show that

$$\mathcal{H}^\beta_\infty(F_k \cap Q(x,\delta)) \geq C\delta^\beta, \qquad \forall Q(x,\delta) \subseteq [0, c_0]^n, \tag{51}$$

holds for all k sufficiently large, where $\beta = \alpha - \varepsilon$ for $0 < \varepsilon \ll 1$. The size of k for which (51) holds will depend on δ. To prove (51) we define an auxiliary measure which is a uniform mass measure over $F_k \cap Q(x, \delta)$, namely

$$\mu_k(A) := \frac{|A \cap F_k \cap Q(x,\delta)|}{|F_k \cap Q(x,\delta)|}.$$

Note that μ_k depends on the set $F_k \cap Q(x, \delta)$, but we will only stress the dependence on k in the notation.

Assume we have proved

$$\mu_k(B_r) \leq Cr^\beta\delta^{-\beta} \tag{52}$$

for all sufficiently large k (the size of k will depend on δ). Using (52) we can prove (51) easily, noting that if $\mathcal{B}$ is a collection of balls B_r that covers F_k, then

$$1 = \mu_k(F_k \cap Q(x,\delta)) \leq \sum_{B_r\in\mathcal{B}} \mu_k(B_r) \leq C\delta^{-\beta} \sum_{B_r\in\mathcal{B}} r^\beta.$$

Thus we have reduced to prove (52). To do so we have to work at several scales. It will be useful to keep in mind that if $k \gg 1$ then

$$|F_k \cap Q(x,\delta)| \simeq R_k^{\alpha-n}\delta^n \tag{53}$$

and that $R_k \to \infty$ as $k \to \infty$. Many estimates below will be indeed justified taking k large enough, depending on δ.

(1) Scale $r < R_k^{-1}$ In the worst case a ball is entirely contained in a slab from F_k, so

$$\mu_k(B_r) \lesssim r^n R_k^{-\alpha+n}\delta^{-n} \le r^\alpha \delta^{-n} = r^\beta \delta^{-\beta}\, r^{\alpha-\beta}\delta^{\beta-n} < r^\beta\delta^{-\beta}\, R_k^{-(\alpha-\beta)}\delta^{\beta-n};$$

since $\alpha - \beta > 0$ and $r < R_k^{-1}$ we have $R_k^{-(\alpha-\beta)}\delta^{\beta-n} < 1$ for $k \gg_\delta 1$ thus (52) holds at this scale.

(2) Scale $R_k^{-1} < r < R_k^{-a}$. Recall that $R_k^{-a} < R_k^{-\frac{1}{2}}$, so a ball B_r cannot contain a slab. On the other hand, since $r < R_k^{-a}$ a ball B_r intersects at most one slab, so

$$\mu_k(B_r) \lesssim r R_k^{-(n-1)} R_k^{-\alpha+n}\delta^{-n} = r R_k^{-\alpha+1}\delta^{-n} = r^\alpha \frac{R_k^{-(\alpha-1)}}{r^{\alpha-1}}\delta^{-n} < r^\alpha\delta^{-n},$$

using $R_k^{-1} < r$ and $\alpha > 1$. Using also $r < R_k^{-a}$ we see that

$$\mu_k(B_r) \lesssim r^\beta R_k^{a(\beta-\alpha)}\delta^{-n} < r^\beta\delta^{-\beta}, \qquad k \gg_\delta 1.$$

(3) Scale $R_k^{-a} < r < R_k^{-\frac{1}{2}}$. A ball B_r intersects $\lesssim R_k^{(n-1)a} r^{n-1}$ slabs, so

$$\mu_k(B_r) \lesssim r^n R_k^{(n-1)a-n+1} R_k^{-\alpha+n}\delta^{-n} \le r^n R_k^{-b+\frac{1}{2}}\delta^{-n}.$$

where we used (50). Since $r < R_k^{-\frac{1}{2}}$ we have that

$$\mu_k(B_r) \lesssim r^\beta R_k^{\frac{1}{2}\beta - \frac{1}{2}n - b + \frac{1}{2}}\delta^{-n} < r^\beta R_k^{\frac{1}{2}(\beta-\alpha)}\delta^{-n}$$

where we used

$$\alpha := (n-1)a + b + \frac{1}{2} = \frac{n-3}{2}b + \frac{n}{2} + 1 + 2b - 1 < n + 2b - 1. \tag{54}$$

Thus

$$\mu_k(B_r) < r^\beta\delta^{-\beta}, \qquad k \gg_\delta 1.$$

(4) Scale $R_k^{-\frac{1}{2}} < r < R_k^{-b}$. A ball B_r contains $\lesssim R_k^{(n-1)a} r^{n-1}$ slabs, so recalling again (50) we get

$$\mu_k(B_r) \lesssim r^{n-1} R_k^{(n-1)a-n+\frac{1}{2}} R_k^{-\alpha+n}\delta^{-n} = r^{n-1}R_k^{-b}\delta^{-n} < r^{n-1+2b}\delta^{-n},$$

where we used $R_k^{-\frac{1}{2}} < r$. From $r < R_k^{-b}$ and (54) we have that

$$\mu_k(B_r) \lesssim r^\beta R_k^{-b(n+2b-1-\beta)}\delta^{-n} < r^\beta R_k^{-b(\alpha-\beta)}\delta^{-n} < r^\beta\delta^{-\beta}, \qquad k \gg_\delta 1.$$

(5) Scale $R_k^{-b} < r < \delta$. A ball intersects $\lesssim R_k^{(n-1)a+b} r^n$ slabs, so

$$\mu_k(B_r) \lesssim r^n R_k^{(n-1)a+b-n+\frac{1}{2}} R_k^{-\alpha+n}\delta^{-n} = r^n\delta^{-n} < r^\beta\delta^{-\beta}.$$

The inequality (52) thus holds, and so the statement of the Theorem. □

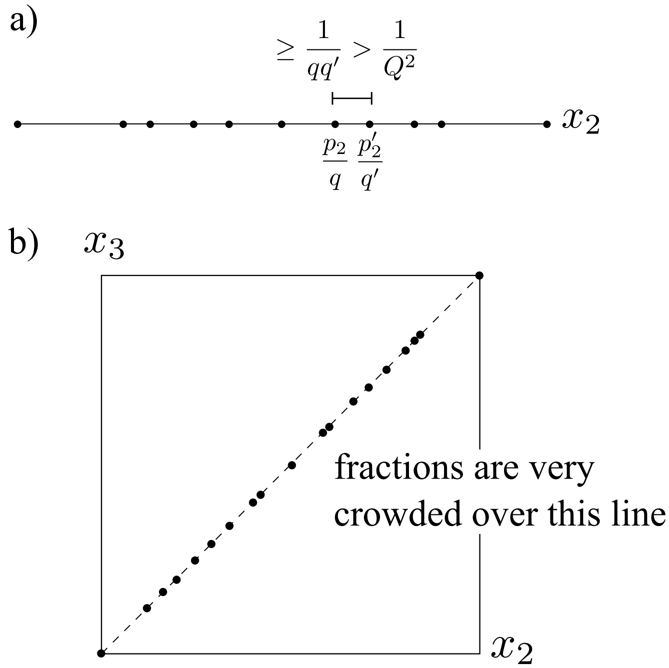


FIGURE 3. (a) When $n = 2$ the fractions are already well separated; Lemma 13 is unnecessary. (b) When $n \geq 3$ the fractions might concentrate around some regions, which prohibits the Frostman measure technique we used in Lemma 12.

.

The lower bound for type II sets is harder to prove, and we need a Lemma that assures us that for all F_k we can find a large sub-collection of slabs uniformly distributed. Similar arguments were used in Lemma 4.3 of [16] and in Sections 5.6–5.8 of [27].

Lemma 13. *Let $F = \limsup_{k\to\infty} F_k$ be a set of type II, that is, $\frac{3}{4} < a \leq 1$ and $b = \frac{1}{2}$. If $\mathcal{A}_k$ is the collection of slabs in $F_k \cap Q(x, \delta)$, for $\delta < 1$, then, for every $\varepsilon > 0$ and $k \gg_\varepsilon 1$, we can extract a sub-collection of slabs $\mathcal{A}_k' \subset \mathcal{A}_k$ such that*

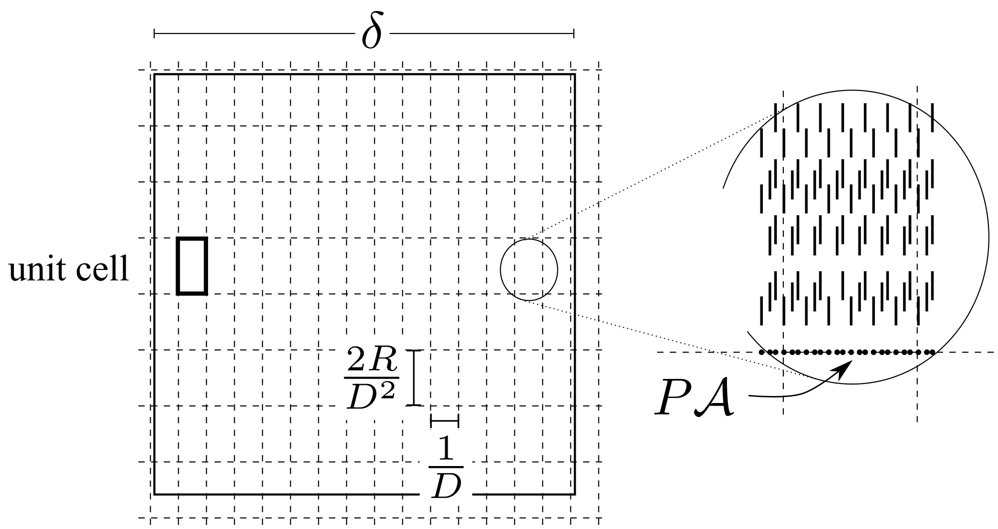
(i) $|\mathcal{A}_k'| \gtrsim R_k^{-\varepsilon} |\mathcal{A}_k|$.

(ii) If $x = (x_1, \tilde{x})$ and $y = (y_1, \tilde{y})$ are the centers of two slabs in $\mathcal{A}_k'$ and $\tilde{x} \neq \tilde{y}$, then $|\tilde{x} - \tilde{y}| \gtrsim 1/(Q_k^{\frac{n}{n-1}} D_k)$.

Proof. The sets $F_k := \bigcup_{s\in\mathcal{A}_k} s$ have a periodic structure. In fact, recall that the centers of the slabs are

$$(2p_1 R_k/(qD_k^2), p_2/(D_k q), \ldots, p_n/(D_k q)),$$

where $(p_1/q,\dots,p_n/q)$ is an admissible fraction (Definition 5); hence, F_k is made up of translation of the slabs in the unit cell $[0,2R_k/D_k^2]\times[0,1/D_k]^{n-1}$. We assume that k is so large that the number of unit cells not entirely contained in $Q(x,\delta)$ is negligible. Therefore, the number of slabs in $Q(x,\delta)$ is $|\mathcal{A}_k|\simeq D_k^{n+1}R_k^{-1}\delta^{-n}|\{\text{slabs per unit cell}\}|$, and the Lemma reduces to extract a large number of admissible fractions in $[0,1]^n$ with denominator $\le Q_k$.



We drop the subscript $k\gg 1$. Let $\mathcal{A}^0$ be the set of admissible fractions, and let $\mathcal{A}^1\subset\mathcal{A}^0$ be the collection of fractions $(p_1/q,\dots,p_n/q)$ with $q\equiv 0\,(\mathrm{mod}\ 4)$ and p_j even for $2\le j\le n$, so that $|\mathcal{A}^1|\simeq|\mathcal{A}^0|$.

We denote by $P\mathcal{A}^1$ the projection of $\mathcal{A}^1$ into the plane $(x_2,\dots,x_n)$, so $P\mathcal{A}^1$ is the set of fractions $(p_2/q,\dots,p_n/q)$ with $q\equiv 0\,(\mathrm{mod}\ 4)$ and even p_j. The Dirichlet's approximation Theorem asserts that for $2y\in\mathbb{R}^{n-1}$ there exists $(p_2',\dots,p_n')\in\mathbb{Z}^{n-1}$ such that

$$|2y-\frac{p_j'}{q'}|\le\frac{1}{q'(Q/4)^{\frac{1}{n-1}}},\qquad\text{for some } 1\le q'\le Q/4,\tag{55}$$

so if we write $q=4q'$ and $p_j=2p_j'$, then we can assert that for every $y\in\mathbb{R}^{n-1}$ there exists a fraction $(p_2/q,\dots,p_n/q)$, for $q\equiv 0\,(\mathrm{mod}\ 4)$ and p_j even, such that

$$|y-\frac{p_j}{q}|\le 2^{\frac{n+1}{n-1}}\frac{1}{qQ^{\frac{1}{n-1}}},\qquad\text{for some } 1\le q\le Q.$$

In general, a point $y\in[0,1]^{n-1}$ cannot be sufficiently well approximated by fractions if it satisfies (55) with a fraction $(p_2'/q',\dots,p_n'/q')$

with small q', so it is convenient to ignore those points. The volume in $[0,1]^{n-1}$ occupied by those undesirable points is less than

$$\sum_{1\leq q'\leq Q/2^{n+2}} \Big(\frac{1}{q'(Q/4)^{\frac{1}{n-1}}}\Big)^{n-1}(2q')^{n-1} = \frac{1}{2}. \tag{56}$$

Let $G := \{y \in [0,1]^{n-1} \mid y \text{ satisfies (55) for some } Q/2^{n+2} < q' \leq Q/4\}$, then by (56) the volume of G is $> \frac{1}{2}$. Cover G with cubes $Q(y,l)$, where $y \in G$ and $l := 2^{n+2+\frac{2}{n-1}}/Q^{\frac{n}{n-1}}$. By Vitali's covering Theorem we can find a disjoint collection of cubes $\{Q(y_j,l)\}_{1\leq j\leq N}$ such that

$$G \subset \bigcup_{j=1}^{N} Q(y_j,3l);$$

hence, $N \geq c_n Q^n$. We pick from within each $Q(y_j,l)$ a fraction and construct so a collection of fractions $\mathcal{C} \subset P\mathcal{A}^1$; we define $\mathcal{A}^2 \subset \mathcal{A}^1$ as the set of fractions such that $P\mathcal{A}^2 = \mathcal{C}$. By construction, $|P\mathcal{A}^2| \gtrsim Q^n$ and any two points in $P\mathcal{A}^2$ lie at distance $\gtrsim 1/Q^{\frac{n}{n-1}}$; the latter, after dilation by $1/D$, implies the condition *(ii)*.

The fractions in $\mathcal{A}^2$ that lie over $(p_2/q,\dots,p_n/q) \in P\mathcal{A}^2$ is in number at least $\varphi(q)$, where φ is the Euler's totient function. Since $\varphi(q) \geq q^{1-\varepsilon}$ for every $\varepsilon > 0$ and $q \gg_\varepsilon 1$—see Theorem 327 in [20]—then the number of fractions in $\mathcal{A}^2$ is $\geq Q^{1-\varepsilon}|P\mathcal{A}^2| \gtrsim Q^{n+1-\varepsilon} \simeq Q^{-\varepsilon}|\mathcal{A}^0|$, where $\mathcal{A}^0$ is the set of admissible fractions; this concludes the verification of condition *(i)*. ☐

Theorem 14. *Let* $0 < c_0 \leq 1$. *If* $F = \limsup_{k\to\infty} F_k$ *is a set of type II, that is,* $\frac{3}{4} < a \leq 1$ *and* $b = \frac{1}{2}$, *then* $\dim F \cap [0,c_0]^n \geq \alpha$ *where*

$$\alpha := 1 + (n-1)a. \tag{57}$$

Proof. We use the same method as in Theorem 12. For fixed $\varepsilon > 0$, let $\mathcal{A}'_k$ be the collection of slabs provided by Lemma 13, and let F'_k be the corresponding set. Given $Q(x,\delta) \subseteq [0,c_0]^n$, we define again a measure μ_k on $F_k \cap Q(x,\delta)$ that will be useful in the proof; the measure is

$$\mu_k(A) := \frac{|A \cap F'_k \cap Q(x,\delta)|}{|F'_k \cap Q(x,\delta)|}.$$

If $k \gg_\varepsilon 1$ then

$$|F'_k \cap Q(x,\delta)| \gtrsim R_k^{\alpha-n-\varepsilon}\delta^n.$$

We take $\beta := \alpha - 2n\varepsilon < \alpha - \varepsilon$. The goal is again to prove (52), from which we deduce Theorem 14 proceeding as we did in the proof of Theorem 12.

Since $b = \frac{1}{2}$, we can think of the slabs over $(p_2/(qD_k),\dots,p_n/(qD_k))$ as a single tube of length 1.

(1) Scale $r < R^{-1}$. In the worst case a ball is entirely contained in a slab from F_k, so
$$\mu_k(B_r) \lesssim r^n R_k^{-\alpha+n+\varepsilon}\delta^{-n} \le r^\alpha \delta^{-n} = r^\beta \delta^{-\beta}(r^{\alpha-\beta-\varepsilon}\delta^{\beta-n});$$
since $\alpha - \beta > \varepsilon$ and $r < R_k^{-1}$, then $\mu_k(B_r) < r^\beta\delta^{-\beta}$ whenever $k \gg_\delta 1$.

(2) Scale $R_k^{-1} < r < R_k^{-a}$. By the properties of separation of the slabs in $\mathcal{A}'_k$, a ball B_r intersects at most one slab—recall Lemma 13(ii) and (31)—so
$$\mu_k(B_r) \lesssim r R_k^{-(n-1)-\alpha+n+\varepsilon}\delta^{-n} = r R_k^{-\alpha+1+\varepsilon}\delta^{-n} < r^{\alpha-\varepsilon}\delta^{-n},$$
where we used $\alpha > 1$. Since $r < R_k^{-a}$ we see that
$$\mu_k(B_r) \lesssim r^\beta R_k^{a(\beta-\alpha+\varepsilon)}\delta^{-n}, \qquad k \gg_\delta 1.$$

(3) Scale $R_k^{-a} < r < R_k/D_k^2 = R_k^{\frac{n-1}{n+1}(2a-\frac{3}{2})-\frac{1}{2}}$. A ball intersects $\lesssim R_k^{(n-1)a} r^{n-1}$ "tubes" of length 1 and radius R_k^{-1}, so (recall (57))
$$\mu_k(B_r) \lesssim r^n R_k^{(n-1)a-(n-1)-\alpha+n+\varepsilon}\delta^{-n} = r^n R_k^\varepsilon \delta^{-n} = r^\beta\delta^{-\beta}(r^{n-\beta}R_k^\varepsilon \delta^{\beta-n});$$
since $r < R_k/D_k^2 \le R_k^{-\frac{1}{n+1}}$, we see that
$$\mu_k(B_r) \lesssim r^\beta \delta^{-\beta}, \qquad k \gg_\delta 1.$$

(4) Scale $R_k/D_k^2 < r < \delta$. A ball B_r contains $\simeq D_k^{n+1}R_k^{-1}r^n$ translations of the unit cell $[0, 2R_k/D_k^2] \times [0, 1/D_k]^{n-1}$. If V is the volume of F'_k per unit cell, then $|B_r \cap F'_k| \simeq V D_k^{n+1} R_k^{-1} r^n$ and
$$|Q(x,\delta) \cap F'_k| \simeq V D_k^{n+1}R_k^{-1}\delta^n;$$
hence
$$\mu_k(B_r) \lesssim r^n \delta^{-n} < r^\beta \delta^{-\beta}.$$

The inequality $\mu_k(B_r) \le C r^\beta \delta^{-\beta}$ holds for k sufficiently large (depending on δ), so the proof is complete. □

6. Conclusion of the proof

We are now ready to prove our statement combining the results from the previous section. First we take a, b as in (48)-(49) and recall that we have defined

$$\alpha := \frac{1}{2} + (n-1)a + b. \tag{58}$$

Note that we have a bijection between $a \in (1/2, 1]$ (which predicts also the value of b by (48)-(49)) and $\alpha \in (n/2, n]$, which is the range we are interested in (the case $\alpha = n$ was handled in [5]).

First we claim that given any

$$s' < s := \frac{n}{2(n+1)} + \frac{n-1}{2(n+1)}(n-\alpha) \tag{59}$$

we can find a solution $u(x,t)$ with initial datum $u_0 \in H^{s'}(\mathbb{R}^n)$ such that

$$\limsup_{t\to 0^+} |u(x,t)| = \infty$$

for $x \in (F \cap ([c_0, c_1] \times [0, c_1]^{n-1})) \setminus \Omega$, where F is an (a,b)-set of divergence, $0 < c_0 := \frac{1}{10}c_1 \ll 1$ and Ω has dimension $\leq n - 2s$. Indeed, it suffices to choose $u_0 := g_{a,b}$ defined in (36) so that $u_0 \in H^{s'}(\mathbb{R}^n)$ for

$$s' < s := \frac{1}{4} + \frac{n-1}{2(n+1)}(n-(n-1)a-b); \tag{60}$$

see (38)-(37). Since under (58) the inequality (60) becomes (59), then the claim follows invoking Theorem 9.

Thus, to conclude the proof, we need to show that

$$\dim\big((F \cap ([c_0, c_1] \times [0, c_1]^{n-1})) \setminus \Omega\big) \geq \alpha. \tag{61}$$

First, covering $(F \cap ([c_0, c_1] \times [0, c_1]^{n-1}))$ with $\simeq (c_1/c_0)^{n-1}$ cubes of side c_0, we see as consequence of Theorems 12 and 14 that

$$\dim(F \cap ([c_0, c_1] \times [0, c_1]^{n-1})) \geq \alpha.$$

On the other hand, we know that $\dim \Omega \leq n - 2s$; see Theorem 9. Thus, since for our choice (59) of s we have $\alpha > n - 2s$ when $\alpha > n/2$, then (61) follows and the proof is concluded.

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