Invariant measures for the DNLS equation

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Abstract We describe invariant measures associated to the integrals of motion of the periodic derivative nonlinear Schrödinger equation (DNLS) constructed in [2, 3]. The construction works for small L^2 data. The measures are absolutely continuous with respect to suitable weighted Gaussian measures supported on Sobolev spaces of increasing regularity. These results have been obtained in collaboration with Giuseppe Genovese (University of Zürich) and Daniele Valeri (University of Glasgow).

1 Introduction

We consider the periodic DNLS equation

$$\begin{cases} i\partial_t \psi + \psi'' = i\beta \left(\psi|\psi|^2\right)' \\ \quad \psi(x,0) = \psi_0(x)\,, \quad x \in \mathbb{T}\,, \end{cases} \tag{1}$$

where $\psi(x,t) : \mathbb{T} \times \mathbb{R} \to \mathbb{C}$, $\psi_0(x) : \mathbb{T} \to \mathbb{C}$, $\psi'(x,t)$ is the derivative of ψ with respect to x, and $\beta \in \mathbb{R}$. We write Φ_t for the associated flow-map. A (Gibbs) measure associated to the energy functional

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$$E_1[\psi] = \frac{1}{2}\|\psi\|^2_{\dot{H}^1} + \frac{3i}{4}\beta \int |\psi|^2\psi'\bar{\psi} + \frac{\beta^2}{4}\|\psi\|^6_{L^6}\,, \tag{2}$$

has been constructed in [8], while in [6] and [7] this measure has been proved to be invariant under Φ_t.

Let $k \geq 2$ be an integer and γ_k be the Gaussian measure induced by the random Fourier series

$$f^{\omega}(x) = \sum_{n\in\mathbb{Z}} \frac{e^{inx}}{\sqrt{1+n^{2k}}} g_n^{\omega}\,,$$

where g_n^{ω} are normalized independent complex Gaussian random variables. Notice that

$$f^{\omega} \in H^s \quad \omega\text{-almost surely for} \quad s < k - \frac{1}{2}$$

and

$$\|f^{\omega}\|_{H^{k-\frac{1}{2}}} = \infty \quad \omega\text{-almost surely}\,,$$

thus we may think that γ_k "concentrates" on $\bigcap_{s<k-\frac{1}{2}} H^s$. We fix $R_m > 0$, for $m = 0, \ldots, k-1$, and define the k-th Gibbs measure associated to the DNLS equation by

$$\rho_k(A) := \int_A \left(\prod_{m=0}^{k-1} \chi_{R_m}(E_m[\psi])\right) e^{-Q_k[\psi]}\gamma_k(d\psi)\,, \tag{3}$$

where χ_{R_m} are smooth cut–off of the interval $[-R_m, R_m]$,

$$Q_k[\psi] := E_k[\psi] - \frac{1}{2}\|\psi\|^2_{\dot{H}^k} \tag{4}$$

and $E_1, \ldots, E_k$ are integrals of motion of the DNLS equation; see (5), (6)

Recall from [5] (see also [1, 2]) that there exists an infinite sequence of integrals of motion $\{E_k[\psi]\}_{k\in\frac{1}{2}\mathbb{N}_0}$ for the DNLS equation (1). We will use the form of E_k found in [2]. In particular

$$
\begin{aligned}
E_0[\psi] &= \frac{1}{2}\|\psi\|^2_{L^2}\,,\\
E_{\frac{1}{2}}[\psi] &= \frac{i}{2}\int \psi'\bar{\psi} + \frac{\beta}{4}\|\psi\|^4_{L^4}\,,\\
E_1[\psi] &= \frac{1}{2}\|\psi\|^2_{\dot H^1} + \frac{3i}{4}\beta\int |\psi|^2\psi'\bar{\psi} + \frac{\beta^2}{4}\|\psi\|^6_{L^6}\,,\\
E_{\frac{3}{2}}[\psi] &= \frac{i}{2}\int \psi''\bar{\psi}' + \frac{\beta}{4}\int \left((\psi')^2\bar{\psi}^2 + 8|\psi|^2\psi'\bar{\psi}' + \psi^2(\bar{\psi}')^2\right)\\
&\quad + \frac{5i}{4}\beta^2\int |\psi|^4\psi'\bar{\psi} + \frac{5}{16}\beta^3\|\psi\|^8_{L^8}\,,\\
E_2[\psi] &= \frac{1}{2}\|\psi\|^2_{\dot H^2} + \frac{5i}{4}\beta\int |\psi|^2\left(\psi''\bar{\psi}' - \psi'\bar{\psi}''\right) + \frac{5}{4}\beta^2\int |\psi|^2\left((\psi')^2\bar{\psi}^2 + \psi^2(\bar{\psi}')^2\right)\\
&\quad + \frac{25}{4}\beta^2\int |\psi|^4\psi'\bar{\psi}' + \frac{35i}{16}\beta^3\int |\psi|^6\psi'\bar{\psi} + \frac{7}{16}\beta^4\|\psi\|^{10}_{L^{10}}\,.
\end{aligned}
\tag{5}
$$

Remark 1 These integrals of motion are slightly different from those appearing in the introduction of [2], where there is a typo in the coefficient of β of $E_1[\psi]$.

While we have in general (see agian [2]) that, for $k \geq 2$ and integer[1]

$$
E_k[\psi] = \frac{1}{2}\|\psi\|^2_{\dot H^k} + i\frac{2k+1}{4}\beta\int |\psi|^2\left(\psi^{(k)}\bar{\psi}^{(k-1)} - \psi^{(k-1)}\bar{\psi}^{(k)}\right) + \int \mathcal{R}_k\,, \tag{6}
$$

where $\mathcal{R}_k$ are polynomials in $\psi, \psi^{(1)}, \dots, \psi^{(k-1)}$ and their conjugate. We have denoted

$$
\psi^{(m)} := \partial_x^m \psi\,.
$$

2 Main results

The weighted gaussian measure (3) is well defined and absolutely continuous with respect to γ_k, as long as we restrict to functions with small L^2 norm. This is formalized in the next theorem.

Theorem 1 ([2])

We denote with

$$
F_k := \left(\prod_{m=0}^{k-1}\chi_{R_m}\left(E_m[\psi]\right)\right) e^{-Q_k[\psi]}
$$

the density of ρ_k. Let $k \geq 2$ and integer, $p \in (1,\infty)$. If $R_0 = R_0(p, |\beta|, k)$ is sufficiently small, then

$$
F_k \in L^p(\gamma_k)\,.
$$

[1] In this survey we never focus on half-integer conservation laws, on which almost no analytical results are available.

We can not infer $F_k \in L^\infty(\gamma_k)$. Indeed $R_0 = R_0(p, |\beta|) \to 0$ as $p \to \infty$. The small mass assumption $R_0 \ll 1$ is not surprising since the problem is L^2 critical. Regarding the well-posedness, the periodic DNLS equation has been shown to be locally well-posed for initial data in $H^{s \geq 1/2}$ in [4]. It is not clear if this result may extend to $s < 1/2$. Then, a standard procedure allows to globalise the local H^1 solutions with small mass $R_0 < \delta$ (notice (5) that the mass is conserved by Φ_t). taking advantage of the fact that in this regime E_1 dominates the H^1 norm.

Theorem 2 ([3])

Let $k \geq 2$ and integer, $p \in (1, \infty)$. Let $R_0 = R_0(p, |\beta|, k)$ be sufficiently small. There exists a probability measure $\tilde{\rho}_k$, absolutely continuous with respect to γ_k, which is preserved by Φ_t.

This result can be used to deduce informations on the long time behavior of solutions to DNLS. This follows by a direct application of the Poincaré recurrence theorem. Thus we have obtained

Corollary 1 *Let $k \geq 2$ and integer and let ψ a solution of the DNLS equation with initial datum $\psi(\cdot, 0) \in H^s$ with $s < k - \frac{1}{2}$. For $\tilde{\rho}_k$-a.e. (and so γ_k-a.e.) initial datum $\psi(\cdot, 0)$ there exists a divergent sequence $\{t_n\}_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty} \|\psi(\cdot, t_n) - \psi(\cdot, 0)\|_{H^s} = 0 .$$

An analog conclusion for $k = 1$ follows from combining the contributions in [8, 6, 7]. To the best of our knowledge, these are the sole known results on the long-time behavior of the DNLS equation.

2.1 Strategy of the Proof

An alternative formulation of our main Theorem 2 is that the DNLS equation has the structure of an infinite-dimensional (Hamiltonian) dynamical system. Since the earlier works of Bourgain and Zhidkov this concept has been made rigorous using a suitable finite dimensional approximation. In our case, this is done considering, for all $N \in \mathbb{N}$, the following equation

$$i\partial_t \psi_N + \psi_N'' = i\beta P_N(|\psi_N|^2 \psi_N)' , \tag{7}$$

with initial datum

$$\psi_N(\cdot, 0) := P_N \psi(\cdot, 0) , \tag{8}$$

where P_N is the projection on functions with less than N Fourier modes, namely

$$P_N\left(\sum_{n\in\mathbb{Z}} e^{inx} a_n\right) = \sum_{|n|\leq N} e^{inx} a_n$$

Notice that

$$\psi_N(\cdot,t) = P_N \psi_N(\cdot,t), \qquad \text{for all } t \geq 0\,.$$

Moreover, this system is actually Hamiltonian, but in does not preserve all the integrals of motion. This is often a major issue in this class of problems. However one expects the integrals of motion to be conserved in the limit $N \to \infty$. Following an approach developed by Tzvetkov and Visciglia for the Benjamin-Ono equation, we will show that the derivative of the integrals of motion along the flow of the truncated systems vanishes in the $L^2(\gamma_k)$ mean. Actually, as first observed in [11], one can reduce to consider only the derivative at the initial time, which is a crucial simplification.

It is helpful to recall that the integrals of motion of DNLS have the following form

$$E_k[\psi] = \frac{1}{2}\|\psi\|^2_{\dot{H}^k} - \frac{1}{2}\beta(2k+1)\Im \int \psi^{(k)}\bar{\psi}^{(k-1)}|\psi|^2 + \text{remainders}\,, \qquad k \geq 2\,, \tag{9}$$

where we consider as remainders all the terms that are in $L^\infty(\gamma_k)$. The difficulty to show the asymptotic (w.r.t. N) conservation of E_k comes form the second addendum in the r.h.s. of equation (9). Notably the integrals of motion of the Benjamin-Ono equation have an analog structure. However in that case a convenient cancellation coming from the symmetries of the problem simplifies substantially the computations [10, 11]. We cannot find a similar cancellation here. Nevertheless it is possible to eliminate the troubling term using the following gauge transformation.

$$(\Gamma_\alpha f)(x) := e^{i\alpha \mathcal{I}[f(x)]} f(x)\,. \tag{10}$$

where

$$\mathcal{I}[f(x)] := \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_\theta^x \left(|f(y)|^2 - \frac{\|f\|^2_{L^2}}{2\pi}\right) dy\,. \tag{11}$$

One can easily check that the (real) function $\mathcal{I}[f(x)]$ is the unique zero average (2π-periodic) primitive of $|f(x)|^2 - (2\pi)^{-1}\|f\|^2_{L^2}$. Note that $|f| = |\Gamma_\alpha f|$ and $\Gamma_\alpha f$ is 2π-periodic. Hence, Γ_α maps L^2 into L^2 preserving the norm (namely $\|\Gamma_\alpha f\|_{L^2} = \|f\|_{L^2}$). Using that $[f] = \mathcal{I}[\Gamma_\alpha(f)]$ one can easily show that the map $\alpha \to \Gamma_\alpha$ is a one parameter group of transformations on $(\mathbb{R},+)$, namely

$$\Gamma_0 = \mathbb{I} \qquad \text{and} \qquad \Gamma_{\alpha_1} \circ \Gamma_{\alpha_2} = \Gamma_{\alpha_1+\alpha_2}\,, \quad \text{for any} \alpha_1, \alpha_2 \in \mathbb{R}\,. \tag{12}$$

For any $s \geq 0$ the gauge transformation Γ_α is also an homeomorphism of H^s into itself.

A generic gauge choice (without specifying the value of α) yields the following expression for the integrals of motion of the gauged equation

$$\mathcal{E}_k[\phi] = \frac{1}{2}\|\phi\|_{\dot{H}^k}^2 + ik\alpha\mu\int \bar{\phi}^{(k)}\phi^{(k-1)}$$
$$-\frac{1}{2}\left((2k+2)\alpha + (2k+1)\beta\right)\Im\int \phi^{(k)}\bar{\phi}^{(k-1)}|\phi|^2 + \text{remainders},$$

where $\phi = \Gamma_\alpha\psi$ is the solution of the gauged equation and we shortened

$$\mu := \frac{1}{2\pi}\|\phi\|_{L^2}^2 = \frac{1}{2\pi}\|\psi\|_{L^2}^2 .$$

In general we will use the notation $\mu[f] := \frac{1}{2\pi}\|f\|_{L^2}^2$ and we will simply write μ if there will be no ambiguity. We recover (9) as $\alpha = 0$. Setting

$$\alpha = -\frac{2k+1}{2k+2}\beta, \tag{13}$$

we reduce to

$$\mathcal{E}_k[\phi] = \frac{1}{2}\|\phi\|_{\dot{H}^k}^2 - ik\frac{2k+1}{2k+2}\beta\mu\int \bar{\phi}^{(k)}\phi^{(k-1)} + \text{remainders}.$$

This form of the integrals of motion is much more suitable in order to prove the asymptotic conservation property and such a reduction is the main step in our proof. Of course also the flow of DNLS changes accordingly to the gauge transformation: indeed our gauge choice leads to a somewhat more involved form for the equation

$$i\partial_t\phi + \phi'' + 2i\alpha\mu\phi' = ic_1|\phi|^2\phi' + ic_2\phi^2\bar{\phi}' + c_3|\phi|^4\phi + c_4\mu|\phi|^2\phi + L[\phi]\phi, \tag{14}$$

where

$$c_1 = 2(\alpha+\beta), \qquad c_2 = 2\alpha+\beta, \qquad c_3 = -\alpha^2 - \frac{\alpha\beta}{2}, \qquad c_4 = -\alpha\beta \tag{15}$$

and

$$L[f] = \left(\frac{3\alpha\beta}{4\pi} + \frac{\alpha^2}{\pi}\right)\|f\|_{L^4}^4 - \alpha^2\mu[f]^2 + \frac{i\alpha}{\pi}\int_{\mathbb{T}} f'\bar{f}. \tag{16}$$

However, this does not introduce significant difficulties, as the form of the nonlinearity is essentially the same and we are working with rather regular solutions (let say in $H^{s>5/4}$). It is worthy to point out the difference with what is usually done in the low regularity theory for DNLS, where the choice of the gauge parameter $\alpha = -\beta$ aims to simplify the equation.

The next step is to define for any $k \geq 2$ a gauged Gibbs measure starting from the gauge-transformed (or *gauged*) integrals of motion and to prove its invariance w.r.t. the gauged flow. This requires some groundwork, namely a careful analysis of the DNLS-flow and gauge-flow maps, in order to adapt the strategy of [11].

The invariance of the gauged Gibbs measure under the gauged flow easily implies the invariance of its push-forward through Γ_α, under the DNLS flow. This will be our invariant measure $\tilde{\rho}_k$. We stress that in principle one expects $\rho_k = \tilde{\rho}_k$. The missing step to show the invariance of the Gibbs measures is the proof of absolute continuity of the pull-back $\gamma_k \circ \Gamma_\alpha$ w.r.t. γ_k with the explicit density. So far what we can prove is the following theorem:

Theorem 3 *Let $R_0 > 0$ small enough and*

$$\tilde{\gamma}_k(A) = \gamma_k(A \cap \{f \in L^2 : \mu[f] \leq R_0\}) .$$

Then for any $k \geq 2$ and integer and $\alpha \in \mathbb{R}$ the measure $\tilde{\gamma}_k \circ \Gamma_\alpha$ is absolutely continuous w.r.t. γ_k.

The absolute continuity of $\tilde{\rho}_k$ w.r.t. γ_k is a direct consequence of Theorem 3. The change of variable formula for $k = 1$ was established in [7]. This is however a very special case, as the typical trajectories for γ_1 are complex Brownian bridges, whose properties are crucially employed in the argument of [7]. For more regular processes one cannot expect to reproduce the same proof and some new idea is needed.

Recently Tzvetkov [9] proposed a strategy for proving quasi-invariance of the Gaussian measure under a one-parameter group of transformations via a *soft* argument, which does not provide the explicit density. We use this approach to prove Theorem 3, from which we deduce the absolutely continuity of $\tilde{\rho}_k$ w.r.t. γ_k. When $k \geq 2$, to prove that the Gibbs measures are invariant, one should know the exact form of the densities after the change of variables given by the gauge. As it happens in the case $k = 1$, these densities should complete exactly the part of the integrals of motion missing in the gauged Gibbs measure. We do not give here further details, leaving the discussion of this problem to future works.

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References

1. De Sole, A., Kac, V.G.: Non-local Poisson structures and applications to the theory of integrable systems. Jpn. J. Math. **8**(2), 233–347 (2013). DOI 10.1007/s11537-013-1306-z. URL https://doi.org/10.1007/s11537-013-1306-z
2. Genovese, G., Lucà, R., Valeri, D.: Gibbs measures associated to the integrals of motion of the periodic derivative nonlinear Schrödinger equation. Selecta Math. (N.S.) **22**(3), 1663–1702 (2016). DOI 10.1007/s00029-016-0225-2. URL https://doi.org/10.1007/s00029-016-0225-2

3. Genovese, G., Lucà, R., Valeri, D.: Invariant measures for the periodic derivative nonlinear Schrödinger equation. Mathematische Annalen (2018). DOI 10.1007/s00208-018-1754-0. URL https://doi.org/10.1007/s00208-018-1754-0
4. Herr, S.: On the Cauchy problem for the derivative nonlinear Schrödinger equation with periodic boundary condition. Int. Math. Res. Not. pp. Art. ID 96763, 33 (2006). DOI 10.1155/IMRN/2006/96763. URL https://doi.org/10.1155/IMRN/2006/96763
5. Kaup, D.J., Newell, A.C.: An exact solution for a derivative nonlinear Schrödinger equation. J. Mathematical Phys. **19**(4), 798–801 (1978). DOI 10.1063/1.523737. URL https://doi.org/10.1063/1.523737
6. Nahmod, A.R., Oh, T., Rey-Bellet, L., Staffilani, G.: Invariant weighted Wiener measures and almost sure global well-posedness for the periodic derivative NLS. J. Eur. Math. Soc. (JEMS) **14**(4), 1275–1330 (2012). DOI 10.4171/JEMS/333. URL https://doi.org/10.4171/JEMS/333
7. Nahmod, A.R., Rey-Bellet, L., Sheffield, S., Staffilani, G.: Absolute continuity of Brownian bridges under certain gauge transformations. Math. Res. Lett. **18**(5), 875–887 (2011). DOI 10.4310/MRL.2011.v18.n5.a6. URL https://doi.org/10.4310/MRL.2011.v18.n5.a6
8. Thomann, L., Tzvetkov, N.: Gibbs measure for the periodic derivative nonlinear Schrödinger equation. Nonlinearity **23**(11), 2771–2791 (2010). DOI 10.1088/0951-7715/23/11/003. URL https://doi.org/10.1088/0951-7715/23/11/003
9. Tzvetkov, N.: Quasiinvariant gaussian measures for one-dimensional hamiltonian partial differential equations. Forum of Mathematics, Sigma **3**, e28 (2015). DOI 10.1017/fms.2015.27
10. Tzvetkov, N., Visciglia, N.: Gaussian measures associated to the higher order conservation laws of the Benjamin-Ono equation. Ann. Sci. Éc. Norm. Supér. (4) **46**(2), 249–299 (2013) (2013). DOI 10.24033/asens.2189. URL https://doi.org/10.24033/asens.2189
11. Tzvetkov, N., Visciglia, N.: Invariant Measures and Long-Time Behavior for the Benjamin?Ono Equation. International Mathematics Research Notices **2014**(17), 4679–4714 (2013). DOI 10.1093/imrn/rnt094. URL https://doi.org/10.1093/imrn/rnt094