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REGULAR PAPER

Nonstandard finite difference schemes for general linear delay differential systems

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Summary

This paper deals with the construction of non-standard finite difference methods for coupled linear delay differential systems in the general case of non-commuting matrix coefficients. Based on an expression for the exact solution of the continuous initial value vector delay problem, a family of non-standard numerical methods of increasing orders is defined. Numerical examples show that the new proposed numerical methods preserve the stability properties of the exact solutions. This work extends previous results for delay systems with commuting matrix coefficients, allowing the use of the new numerical non-standard methods in more general problems.

KEYWORDS:
NSFD methods, coupled delay systems, non-commuting coefficients

1 | INTRODUCTION

Delay effects are necessarily present in most real phenomena, as instantaneous actions are unlikely to occur in the real world. Although instantaneous actions are implicitly assumed when modelling problems in the form of differential equations (DE) or systems, these type of models have proven their validity along the years in many practical applications, as there is a wide variety of situations where the time or space scales of the delays can be safely neglected in relation with the observation scales of interest. However, there are also plenty of phenomena where the presence of delays are to be necessarily considered, for otherwise their models would fail to properly reflect the states and dynamics of the systems. In these cases, the basic modelling tools are delay differential equations (DDE) or systems (see, e.g., [1,2,3,4] and references therein).

Analytic solutions for most real world DE models can not be obtained, and thus numerical approximate solutions are to be computed. In the case of DDE models, the presence of delays makes necessary the use of specifically designed or modified numerical methods [5]. Besides basic requirements of precision and computational efficiency, a desirable characteristic for a numerical method is the preservation of some dynamical property of interest present in the continuous model, that is, its dynamic consistency [6].

For DE models, the use of non-standard finite difference (NSFD) methods [7] has experienced an increase in the last years [8,9,10,11], in part for the possibility of designing methods with appropriate dynamic consistency properties [12]. A field of applications where these characteristics of dynamic consistency are particularly desirable is in epidemiology modelling, where NSFD methods have been applied both to DE and DDE models [13,14,15,16,17,18].

There are some heuristic rules to design NSFD methods with expected good properties, and an useful approach has also proven to be deriving NSFD methods from the expressions of exact numerical solutions for certain test equations [7,19]. For DDE models, the first example of a NSFD method derived from an exact solution of the linear scalar DDE initial value problem, in a restricted domain, was presented in Garba et al [20]. A full domain exact numerical solution for the same scalar problem was given in García et al[21], and a family of NSFD methods with good dynamic consistency properties were proposed from there. The same authors extended their work to the setting of vector DDE with commuting matrix coefficients[22].

In this work, we consider the coupled linear delay system

$$X'(t) = AX(t) + BX(t-\tau), \qquad t > 0, \tag{1}$$

$$X(t) = F(t), \qquad -\tau \le t \le 0, \tag{2}$$

where $X(t), F(t) \in \mathbb{R}^d$ and $A, B \in \mathbb{R}^{d\times d}$, with matrices A and B non-commuting in general.

Problem (1)-(2) has the general form obtained when linearising nonlinear DDE systems at equilibrium points, constituting the basic test problem to analyse stability of DDE systems and general dynamic properties of numerical methods developed for these systems[5,23]. To compute numerical solutions of (1)-(2), standard general purpose computational techniques adapted from similar methods for problems without delay, such as θ-methods or general Runge-Kutta methods, can be used[5,24,25], although stability properties are not guaranteed[5,26,27,28].

NSFD methods are specifically designed for particular classes of problems, and they are constructed following modelling rules, guided by the characteristics of exact numerical solutions, that have proven successful in preserving dynamical properties of the continuous solutions[7,19,29,30]. NSFD exact numerical schemes have been described for some low order linear systems without delay[31,32,33], and also for the DDE problem (1)-(2) in the particular case where the matrix coefficients A and B commute[22].

The aim of this work is to extend the results of[22] to general linear DDE systems, without requiring the coefficients to commute. As done in[22], and previously in[21] for scalar problems, we seek to obtain an exact scheme for the general linear system (1)-(2) that can guide the construction of a whole family of NSFD schemes of increasing orders, providing numerical solutions with as high precision as required.

The structure of this paper is as follows. In the next section different expressions for the exact solution of (1)-(2) will be presented. Then, in Section 3, a family of NSFD schemes of increasing orders will be defined. In Section 4 numerical examples will be presented showing the stability preserving properties of the new schemes. Finally, Section 5, the main conclusions of the work will be summarized.

2 | EXACT SOLUTION OF THE CONTINUOUS INITIAL VALUE VECTOR DELAY PROBLEM

In this section we will give an expression for the exact solution of the initial value problem (1)-(2) in the form of a perturbed difference system. Our expression will be independently proved, but it could also be derived from a constructive solution first proposed in[34], which is recalled in the next lemma with some minor modifications.

Lemma 1. Consider problem (1)-(2). Let $I \in \mathbb{R}^{d\times d}$ be the identity matrix, $C = A^{-1}B$, and assume A and $I + C$ invertible. Write

$$Q_1(t) = \left(e^{At} - I\right)(I + C), \qquad Q_m(t) = \int_0^t e^{A(t-s)} B Q_{m-1}(s)ds, \quad m > 1, \tag{3}$$

and

$$G(t) = I,\ t \in [-\tau, 0], \qquad G(t) = I + \sum_{k=1}^{m} Q_k(t - (k-1)\tau),\ t \in [(m-1)\tau, m\tau],\ m \ge 1. \tag{4}$$

Then, the solution of (1)-(2) for a continuous initial function $F(t)$ is given by $X(t) = F(t)$ for $t \in [-\tau, 0]$, and

$$X(t) = (G(t) + G(t-\tau)C)(I + C)^{-1}F(0) + \int_{-\tau}^{0} G'(t - \tau - s)(I + C)^{-1} C F(s)ds, \tag{5}$$

for $t \in [(m-1)\tau, m\tau]$ and $m \ge 1$.

Proof. The only differences in this lemma from Theorem 4.3 in[34] are some changes in notation, writing here $X(t)$, $F(t)$, and $Q_m(t)$, instead of $F(t)$, $f(t)$, and $Q(m,t)$, respectively, in[34], considering here the initial function defined in the interval $[-\tau, 0]$, accordingly to (2), instead of being defined in the interval $[0, \tau]$, as was in[34], and evaluating the integral term by parts, so that in this lemma it includes the derivative of $G(\cdot)$, instead of the derivative of $F(\cdot)$, as appears in Theorem 4.3 of[34], thus allowing to cancel some terms and resulting in a more compact expression. $\square$

To have the basis for a numerical scheme, for a given t in a certain interval of τ amplitude, $t \in [(m-1)\tau, m\tau]$, we seek an expression relating the value of the solution at a point $t+h$ in terms of the values in t and those in previous intervals, $t-k\tau$, $k = 1..m$. Our next theorem provides such an expression.

Theorem 1. Consider problem (1)-(2) with $F \in C^1[-\tau, 0]$. Assume, as in Lemma 1, that A and $(I+C)$ are nonsingular. Let $X(t) = F(t)$ for $t \in [-\tau, 0]$ and

$$X(t+h) = e^{Ah}X(t) + \sum_{p=1}^{m}\left(\sum_{r=p}^{\infty}\frac{h^r}{r!}K_{r,p}^m\right)X(t-p\tau) + \int_0^h Q_m(h-s)(I+C)^{-1}CF'(t-m\tau+s)ds, \tag{6}$$

for $m \geq 1$ and $(m-1)\tau \leq t < t+h \leq m\tau$, where the matrix functions $Q_m(\cdot)$ are as defined in (3), and the matrix constants $K_{r,p}^m$ are defined by

$$\forall m \geq 1: \qquad K_{r,s}^m = 0,\ r < s; \qquad K_{r,0}^m = A^r,\ r \geq 0; \tag{7}$$

$$K_{r+1,s}^m = AK_{r,s}^m + BK_{r,s-1}^m,\ 1 \leq s \leq m-1; \qquad K_{r+1,m}^m = K_{r,m-1}^m B, \tag{8}$$

Then, $X(t)$ is a well-defined continuous function satisfying (1) and (2).

To facilitate the proof of Theorem 1, in the next lemma some properties of the matrices $Q_m(.)$ and $K_{r,p}^m$ are collected. In what follows, $\|\cdot\|$ refers either to a vector norm or to the corresponding compatible matrix norm.

Lemma 2. The matrices defined in (3) and (7)-(8) satisfy the following properties:

1.
$$Q_1'(t) = AQ_1(t) + A + B, \qquad Q_k'(t) = AQ_k(t) + BQ_{k-1}(t),\ k \geq 2. \tag{9}$$

2.
$$\|K_{r,p}^m\| \leq \binom{r}{p}\|A\|^{r-p}\|B\|^p. \tag{10}$$

3.
$$K_{r,m-1}^m B = AK_{r,m}^m + BK_{r,m-1}^{m-1}. \tag{11}$$

Proof. Relations for the derivatives of $Q_k(t)$ given in (9) are immediate from their definition in (3).

Bounds given in (10) are trivial for $r < p$, where $K_{r,p}^m$ is the null $d \times d$ matrix, and for $p = 0$ and any $r \geq 0$, where $\|K_{r,0}^m\| = \|A^r\| \leq \|A\|^r$. Also, from the recursive relations in (8), it follows that $K_{1,1}^m = B$ for $m \geq 1$. Thus, assuming that for a given m (10) holds for all $r \leq n$ and $p \leq r$, one has, for $s \leq m-1$,

$$\begin{aligned}\|K_{n+1,s}^m\| &\leq \|A\|\|K_{n,s}^m\| + \|B\|\|K_{n,s-1}^m\| \leq \|A\|\binom{n}{s}\|A\|^{n-s}\|B\|^s + \|B\|\binom{n}{s-1}\|A\|^{n-(s-1)}\|B\|^{s-1} \\ &= \left(\binom{n}{s} + \binom{n}{s-1}\right)\|A\|^{n+1-s}\|B\|^s = \binom{n+1}{s}\|A\|^{n+1-s}\|B\|^s.\end{aligned}$$

Also, it holds that

$$\begin{aligned}\|K_{n+1,m}^m\| &\leq \|K_{n,m-1}^m\|\|B\| \leq \binom{n}{m-1}\|A\|^{n-(m-1)}\|B\|^{m-1}\|B\| \\ &= \binom{n}{m-1}|A\|^{n+1-m}\|B\|^m \leq \binom{n+1}{m}\|A\|^{n+1-m}\|B\|^m.\end{aligned}$$

Finally, regarding (11), one has from (8) that

$$K_{r,m-1}^m B = AK_{r-1,m-1}^m B + BK_{r-1,m-2}^m B,$$

$$AK_{r,m}^m = AK_{r-1,m-1}^m B,$$

and

$$BK_{r,m-1}^{m-1} = BK_{r-1,m-2}^{m-1}B.$$

Since $K_{r,s}^m = K_{r,s}^{m-1}$ for any $s < m-1$, in particular $K_{r-1,m-2}^m = K_{r-1,m-2}^{m-1}$, and thus (11) holds. $\square$

We can now proceed to the proof of Theorem 1.

Proof of Theorem 1. $X(t)$ satisfies (2) by definition. From the bounds given in (10) for the norm of $K^m_{r,p}$, it follows that the infinite sums in (6) converge absolutely, and they can be differentiated term-by-term with respect to h. Thus, $X(t)$ is a well-defined function and, for any fixed t such that $(m-1)\tau \le t < t+h \le m\tau$, it is immediate that $X(t+h)$, as a function of h, is continuous for $h \in [0, m\tau - t]$ and infinitely differentiable in the corresponding open interval.

When $m = 1$, i.e., for $0 \le t < t+h \le \tau$, a closed form for (6) can be obtained,

$$\begin{aligned} X(t+h) &= e^{Ah}X(t) + \left(\sum_{r=1}^{\infty} \frac{h^r}{r!} A^{r-1} B\right) X(t-\tau) + \int_0^h Q_1(h-s)(I+C)^{-1} C F'(t-\tau+s)ds, \\ &= e^{Ah}X(t) + \left(e^{Ah} - I\right) C X(t-\tau) + \int_0^h Q_1(h-s)(I+C)^{-1} C F'(t-\tau+s)ds. \end{aligned} \tag{12}$$

Thus, differentiating with respect to h, and taking into account (9) and that $C = A^{-1}B$, so that $A + B = A(I+C)$ and $AC = B$, one has

$$\begin{aligned} X'(t+h) &= AX(t+h) + BX(t-\tau) + \int_0^h (A+B)(I+C)^{-1} C F'(t-\tau+s)ds, \\ &= AX(t+h) + BF(t-\tau) + A(I+C)(I+C)^{-1} C \left(F(t+h-\tau) - F(t-\tau)\right) \\ &= AX(t+h) + BX(t+h-\tau). \end{aligned} \tag{13}$$

Since (13) is valid for any fixed $t \in [0, \tau]$, and in particular for $t = 0$, it follows that $X(\cdot)$ satisfies (1) in the interval $(0, \tau)$.

Consider now $(m-1)\tau \le t < t+h \le m\tau$, with $m > 1$. Differentiating both terms in (6) with respect to h, and taking into account (8) and (9), one gets

$$\begin{aligned} X'(t+h) &= Ae^{Ah}X(t) + \sum_{p=1}^{m} \left(\sum_{r=p}^{\infty} \frac{h^{r-1}}{(r-1)!} K^m_{r,p} \right) X(t-p\tau) + \int_0^h Q'_m(h-s)(I+C)^{-1} C F'(t-m\tau+s)ds \\ &= Ae^{Ah}X(t) + \sum_{p=1}^{m} \left(\sum_{r=p-1}^{\infty} \frac{h^{r}}{r!} K^m_{r+1,p} \right) X(t-p\tau) + \int_0^h \left(AQ_m(h-s) + BQ_{m-1}(h-s)\right)(I+C)^{-1} C F'(t-m\tau+s)ds \\ &= Ae^{Ah}X(t) + A\sum_{p=1}^{m-1} \left(\sum_{r=p-1}^{\infty} \frac{h^{r}}{r!} K^m_{r,p} \right) X(t-p\tau) + B\sum_{p=1}^{m-1} \left(\sum_{r=p-1}^{\infty} \frac{h^{r}}{r!} K^m_{r,p-1} \right) X(t-p\tau) + \sum_{r=m-1}^{\infty} \frac{h^r}{r!} K^m_{r,m-1} BX(t-m\tau) \\ &\quad + A\int_0^h Q_m(h-s)(I+C)^{-1} C F'(t-m\tau+s)ds + B\int_0^h Q_{m-1}(h-s)(I+C)^{-1} C F'(t-m\tau+s)ds. \end{aligned} \tag{14}$$

Now, taking into account that $K^m_{p-1,p} = 0$, and that $K^m_{r,s} = K^{m-1}_{r,s}$ for any $s < m-1$, one gets

$$\begin{aligned} X'(t+h) &= AX(t+h) - A\sum_{r=m-1}^{\infty} \frac{h^r}{r!} K^m_{r,m} X(t-m\tau) + B\sum_{r=0}^{\infty} \frac{h^r}{r!} A^r X(t-\tau) + B\sum_{p=2}^{m-1} \left(\sum_{r=p-1}^{\infty} \frac{h^{r}}{r!} K^{m-1}_{r,p-1} \right) X(t-\tau-(p-1)\tau) \\ &\quad + \sum_{r=m-1}^{\infty} \frac{h^r}{r!} K^m_{r,m-1} BX(t-m\tau) + B\int_0^h Q_{m-1}(h-s)(I+C)^{-1} C F'(t-m\tau+s)ds \\ &= AX(t+h) + Be^{Ah}X(t-\tau) + B\sum_{p=1}^{m-2} \left(\sum_{r=p}^{\infty} \frac{h^{r}}{r!} K^{m-1}_{r,p} \right) X(t-\tau-p\tau) + B\sum_{r=m-1}^{\infty} \frac{h^r}{r!} K^{m-1}_{r,m-1} X(t-\tau-(m-1)\tau) \\ &\quad + B\int_0^h Q_{m-1}(h-s)(I+C)^{-1} C F'(t-m\tau+s)ds + \sum_{r=m-1}^{\infty} \frac{h^r}{r!} \left(K^m_{r,m-1} B - AK^m_{r,m} - BK^{m-1}_{r,m-1} \right) X(t-m\tau) \\ &= AX(t+h) + BX(t-\tau), \end{aligned} \tag{15}$$

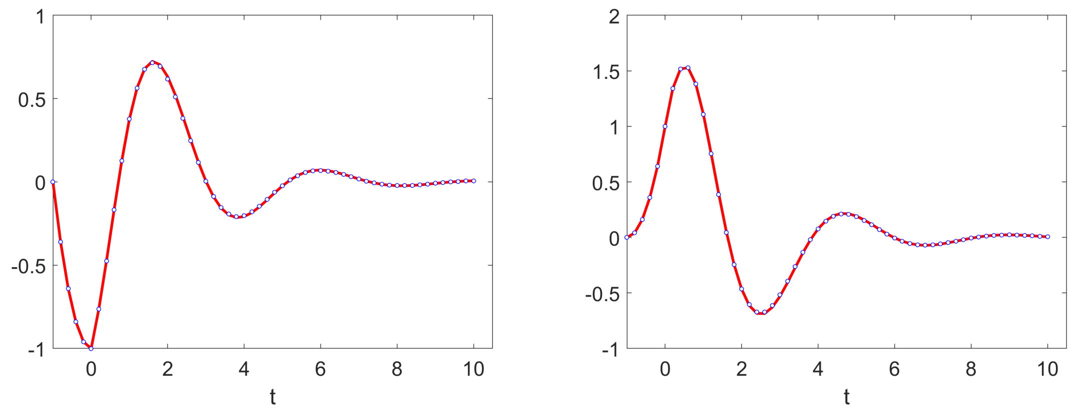

FIGURE 1 First (left) and second (right) component of the solution for Example 1. Exact continuous solution (lines) and high order approximation of the exact numerical solution (points).

since, from (11), the last term in (15) vanishes. Therefore, $X(\cdot)$ satisfies (1) in any open interval $((m-1)\tau, m\tau)$, and it is immediate that side derivatives coincide at the connecting points $m\tau$ for $m \geq 1$. □

From Theorem 1, an expression representing an exact numerical solution of problem (1)-(2) can immediately be obtained by considering a sequence of t values in a regular mesh, as presented in the next theorem.

Theorem 2. Consider problem (1)-(2), with conditions as in Theorem 1. Fix $N \geq 1$, take $h = \tau/N$, and consider the uniform mesh with points $t_n = nh$ and $X_n = X(t_n)$, for $n \geq -N$. Then, an exact numerical solution for problem (1)-(2) in the points of the mesh is given by $X_n = F(t_n)$ for $n = -N \ldots 0$, and

$$X_{n+1} = e^{Ah}X_n + \sum_{p=1}^{m}\left(\sum_{r=p}^{\infty}\frac{h^r}{r!}K_{r,p}^m\right)X_{n-pN} + \int_0^h Q_m(h-s)(I+C)^{-1}CF'(t_n - m\tau + s)ds, \tag{16}$$

for $(m-1)N \leq n < mN$ and $m \geq 1$.

Proof. For $k = -N \ldots 0$, i.e., in the initial interval, X_k provide the exact values of (1)-(2), by definition. Assume that, for a certain n with $(m-1)N \leq n < mN$ and $m \geq 1$, the sequence of values X_k, for $k = -N \ldots n$, coincides with the exact solution of (1)-(2). Then, the value of X_{n+1} computed using (16) is also exact, since from (6) in Theorem 1 one has

$$\begin{aligned} X_{n+1} &= X(t_n + h) = e^{Ah}X(t_n) + \sum_{p=1}^{m}\left(\sum_{r=p}^{\infty}\frac{h^r}{r!}K_{r,p}^m\right)X(t_n - p\tau) + \int_0^h Q_m(h-s)(I+C)^{-1}CF'(t_n - m\tau + s)ds \\ &= e^{Ah}X_n + \sum_{p=1}^{m}\left(\sum_{r=p}^{\infty}\frac{h^r}{r!}K_{r,p}^m\right)X_{n-pN} + \int_0^h Q_m(h-s)(I+C)^{-1}CF'(t_n - m\tau + s)ds, \end{aligned} \tag{17}$$

which is (16). □

Remark 1. We note that, writing $H_p^m = \sum_{r=p}^{\infty}\frac{h^r}{r!}K_{r,p}^m$ and $G(n) = \int_0^h Q_m(h-s)(I+C)^{-1}CF'(t_n - m\tau + s)ds$, expression (16) has the form of a perturbed difference system,

$$X_{n+1} = e^{Ah}X_n + \sum_{p=1}^{m} H_p^m X_{n-pN} + G(n). \tag{18}$$

Regarding practical computations, expression (16) has two main drawbacks. First, in a similar way to the exact numerical solutions previously proposed for scalar problems and for vector problems with commuting coefficients[21,22], there is an integral term that could be exactly computed only for particular initial functions F, but in general requires numerical approximations. More importantly, for the infinite sums in (16) no closed form is expected to be obtained except for particular problems, and in general they need to be truncated, albeit they can be approximated with as high order as required.

Example 1. An example of exact continuous and numerical solutions is presented in Figure 1 . Exact continuous solution (lines) is computed using expression (5) in Lemma 1, and a high order approximation to the exact numerical solution (points) is computed using expression (16) in Theorem 2, with $N = 5$ and truncating the infinite sums up to tenth order. This example correspond to problem (1)-(2) with $\tau = 1$ and

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 0.1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad F(t) = \begin{pmatrix} t^2 - 1 \\ (t+1)^2 \end{pmatrix}.$$

2.1 | Particular case of commuting matrix coefficients

In the special case of commuting matrix coefficients, an exact numerical scheme was presented in [22]. Next we will show that, in this particular case, the expression given in Theorem 2 reduces to that given in Theorem 1 in [22].

Our next lemma gives simplified expressions for the matrices $Q_k(t)$ and $K^m_{r,p}$ when A and B commute.

Lemma 3. When A and B commute, the matrices defined in (3) and (7)-(8) are given by the following expressions:

1.
$$Q_k(t) = \left(e^{At} \sum_{j=0}^{k-1} \frac{(-At)^j}{j!} - I \right) (-C)^{k-1}(I + C), \quad k \geq 1. \tag{19}$$

2.
$$K^m_{r,p} = \binom{r}{p} A^{r-p} B^p, \quad p \leq m-1,\ r \geq p. \tag{20}$$

3.
$$K^m_{r,m} = \binom{r-1}{m-1} A^{r-m} B^m, \quad r \geq m. \tag{21}$$

Proof. For $k = 1$, the expression for $Q_k(t)$ given in (19) coincides with its definition in (3). Assume the expression in (19) is valid for $1 \leq k \leq n$. Then, from (3), and taking into account that A and B commute, one has,

$$\begin{aligned} Q_{n+1}(t) &= \int_0^t \mathrm{e}^{A(t-s)} B \left(e^{As} \sum_{j=0}^{n-1} \frac{(-As)^j}{j!} - I \right) (-C)^{n-1}(I+C) ds = \mathrm{e}^{At} \int_0^t \left(\sum_{j=0}^{n-1} \frac{(-As)^j}{j!} - \mathrm{e}^{-As} \right) ds B (-C)^{n-1}(I+C) \\ &= \mathrm{e}^{At} \left(\sum_{j=0}^{n-1} \frac{(-A)^j t^{j+1}}{(j+1)!} - (-A^{-1}) \left(\mathrm{e}^{-At} - I \right) \right) B(-C)^{n-1}(I+C) \\ &= \mathrm{e}^{At} \left(\sum_{j=0}^{n-1} \frac{(-A)^{j+1} t^{j+1}}{(j+1)!} + I - \mathrm{e}^{-At} \right) (-A^{-1}) B (-C)^{n-1}(I+C) = \left(\mathrm{e}^{At} \sum_{j=0}^{n} \frac{(-At)^j}{j!} - I \right) (-C)^n (I+C). \end{aligned}$$

For $p = 0$, it is immediate that the expression for $K^m_{r,p}$ given in (20) reduces to its definition in (7). Assuming that for a given m (20) holds for all $r \leq n$ and $p \leq r$, one has from (8), for $s \leq m-1$,

$$\begin{aligned} K^m_{n+1,s} &= AK^m_{n,s} + BK^m_{n,s-1} = A \binom{n}{s} A^{n-s} B^s + B \binom{n}{s-1} A^{n-(s-1)} B^{s-1} \\ &= \left(\binom{n}{s} + \binom{n}{s-1} \right) A^{n+1-s} B^s = \binom{n+1}{s} A^{n+1-s} B^s. \end{aligned}$$

Finally, regarding (21), from (8) and (20) one gets, for any m and $r \geq m$,

$$K^m_{r,m} = K^m_{r-1,m-1} B = \binom{r-1}{m-1} A^{(r-1)-(m-1)} B^{m-1} B = \binom{r-1}{m-1} A^{r-m} B^m.$$

□

We can show now that the expression for an exact scheme given in Theorem 1 in [22], for problem (1)-(2) with commuting matrix coefficients, can be obtained as a corollary from expression (16) in Theorem 2.

Corollary 1. When A and B commute, expression (16) in Theorem 2 can be written in the form

$$X_{n+1} = \mathrm{e}^{Ah}\sum_{k=0}^{m-1}\frac{B^k h^k}{k!}X_{n-kN} + \frac{B^m}{(m-1)!}\int_{t_n-m\tau}^{t_n-m\tau+h}(t_n - m\tau + h - s)^{m-1}\mathrm{e}^{A(t_n-m\tau+h-s)}F(s)ds. \tag{22}$$

Proof. Using the expressions given in Lemma 3, integrating by parts after a change of variable in the integral, and taking into account that $X_{n-m\tau} = F(t_n - m\tau)$ for $(m-1)\tau \le t_n < m\tau$, one gets from (16)

$$\begin{aligned}
X_{n+1} &= e^{Ah}X_n + \sum_{p=1}^{m-1}\left(\sum_{r=p}^{\infty}\frac{h^r}{r!}\binom{r}{p}A^{r-p}B^p\right)X_{n-pN} + \sum_{r=m}^{\infty}\frac{h^r}{r!}\binom{r-1}{m-1}A^{r-m}B^m X_{n-mN} \\
&\quad + \int_{t_n-m\tau}^{t_n-m\tau+h} Q_m(t_n - m\tau + h - s)(I+C)^{-1}CF'(s)ds \\
&= e^{Ah}X_n + \sum_{p=1}^{m-1}\frac{B^p h^p}{p!}\left(\sum_{r=p}^{\infty}\frac{h^{r-p}}{(r-p)!}A^{r-p}\right)X_{n-pN} + \sum_{r=m}^{\infty}\frac{(Ah)^r}{r!}\binom{r-1}{m-1}A^{-m}B^m X_{n-mN} \\
&\quad - Q_m(h)(I+C)^{-1}CF(t_n - m\tau) + \int_{t_n-m\tau}^{t_n-m\tau+h} Q_m'(t_n - m\tau + h - s)(I+C)^{-1}CF(s)ds \\
&= \mathrm{e}^{Ah}\sum_{p=0}^{m-1}\frac{B^p h^p}{p!}X_{n-pN} + \left(\sum_{r=m}^{\infty}\frac{(Ah)^r}{r!}\binom{r-1}{m-1}C^m - \left(e^{Ah}\sum_{j=0}^{m-1}\frac{(-Ah)^j}{j!} - I\right)(-C)^{m-1}(I+C)(I+C)^{-1}C\right)X_{n-mN} \\
&\quad + \int_{t_n-m\tau}^{t_n-m\tau+h} Q_m'(t_n - m\tau + h - s)(I+C)^{-1}CF(s)ds.
\end{aligned}$$

We will show next that, in the last expression, the term multiplying X_{n-mN} vanishes, and the term in the integral coincides with that in (22), which will complete the proof. Regarding the integral term, from (21) one has

$$\begin{aligned}
Q_m'(t) &= e^{At}\left(A\sum_{j=0}^{m-1}\frac{(-At)^j}{j!} + \sum_{j=1}^{m-1}\frac{(-A)^j t^{j-1}}{(j-1)!}\right)(-C)^{m-1}(I+C) \\
&= e^{At}A\left(\sum_{j=0}^{m-1}\frac{(-At)^j}{j!} - \sum_{j=0}^{m-2}\frac{(-A)^j t^j}{(j)!}\right)(-1)^{m-1}(A^{-1}B)^{m-1}(I+C) = e^{At}t^{m-1}A\frac{B^{m-1}}{(m-1)!}(I+C),
\end{aligned}$$

and thus,

$$\int_{t_n-m\tau}^{t_n-m\tau+h} Q_m'(t_n - m\tau + h - s)(I+C)^{-1}CF(s)ds = \frac{B^m}{(m-1)!}\int_{t_n-m\tau}^{t_n-m\tau+h}(t_n - m\tau + h - s)^{m-1}\mathrm{e}^{A(t_n-m\tau+h-s)}F(s)ds.$$

Regarding the term multiplying X_{n-mN}, we note that

$$\sum_{j=0}^{m-1}\frac{(-Ah)^j}{j!} = \mathrm{e}^{-Ah} - \sum_{j=m}^{\infty}\frac{(-Ah)^j}{j!}.$$

so that

$$\begin{aligned}
&\left(e^{Ah}\sum_{j=0}^{m-1}\frac{(-Ah)^j}{j!} - I\right)(-C)^{m-1}C = -\mathrm{e}^{Ah}\sum_{j=m}^{\infty}\frac{(-Ah)^j}{j!}(-1)^{m-1}C^m = \sum_{i=0}^{\infty}\frac{(Ah)^i}{i!}\sum_{j=m}^{\infty}\frac{(-1)^j(Ah)^j}{j!}(-1)^m C^m \\
&= \sum_{r=m}^{\infty}\frac{(Ah)^r}{r!}\left(\sum_{j=m}^{r}\frac{r!}{j!(r-j)!}(-1)^j\right)(-1)^m C^m,
\end{aligned}$$

and using the binomial identity

$$\sum_{j=m}^{r}\binom{r}{j}(-1)^j = \binom{r-1}{m-1}(-1)^m,$$

one gets

$$\sum_{r=m}^{\infty} \frac{(Ah)^r}{r!} \binom{r-1}{m-1} C^m,$$

which cancels the first sum in the term multiplying X_{n-mN}. □

3 | NON-STANDARD FINITE DIFFERENCE SCHEMES

Based on the exact numerical solution given by Theorem 2, we propose in the next theorem a family of non-standard numerical schemes that avoid the drawbacks pointed out in the previous section, providing computationally efficient numerical methods with as high precision as required.

Theorem 3. Consider problem (1)-(2), with conditions as in Theorem 2 and with the same mesh and notation defined there. Make $X_n = F(t_n)$ for $n = -N \dots 0$, and for a fixed $M \geq 1$ compute the values of X_n, for $n = 1 \dots MN$, with the exact scheme of Theorem 2 or with any numerical method of at least local order $M+1$. Then, computing successive values X_n, for $n > MN$, with the expression

$$X_{n+1} = e^{Ah} X_n + \sum_{p=1}^{M} \left(\sum_{r=p}^{M} \frac{h^r}{r!} K_{r,p}^m \right) X_{n-pN}, \tag{23}$$

where $m = [n/N] + 1$, defines a numerical scheme of global order M.

Proof. Let X_n^e be the exact solution given in Theorem 2 and X_n the numerical solution computed according to Theorem 3. Assume that, for a certain n, $\|X_j - X_j^e\| = O(h^{M+1})$ for $j = -N \dots n$, which is certainly the case for $n \leq MN$. Then, for $n+1 > MN$, one has

$$\begin{aligned} \|X_{n+1} - X_{n+1}^e\| \leq{}& \|e^{Ah}\| \|X_n - X_n^e\| + \sum_{p=1}^{M} \left(\sum_{r=p}^{M} \frac{h^r}{r!} \|K_{r,p}^m\| \right) \|X_{n-pN} - X_{n-pN}^e\| \\ &+ \sum_{p=1}^{M} \left(\sum_{r=M+1}^{\infty} \frac{h^r}{r!} \|K_{r,p}^m\| \right) \|X_{n-pN}^e\| + \sum_{p=M+1}^{m} \left(\sum_{r=p}^{\infty} \frac{h^r}{r!} \|K_{r,p}^m\| \right) \|X_{n-pN}^e\| \\ &+ \int_0^h \|Q_m(h-s)\| \|(I+C)^{-1}C\| \|F'(t_n - m\tau + s)\| ds, \end{aligned} \tag{24}$$

and we will show that each term in (24) is $O(h^{M+1})$. By the induction hypothesis, this is the case for the differences $\|X_{n-pN} - X_{n-pN}^e\|$, for $p = 0 \dots M$. From the bounds for $\|K_{r,p}^m\|$ given in (10), one gets

$$\sum_{r=p}^{M} \frac{h^r}{r!} \|K_{r,p}^m\| \leq \sum_{r=p}^{M} \frac{h^r}{r!} \binom{r}{p} \|A\|^{r-p} \|B\|^p = \frac{h^p}{p!} \|B\|^p \sum_{r=p}^{M} \frac{h^{r-p}}{(r-p)!} \|A\|^{r-p} \leq \frac{h^p}{p!} \|B\|^p e^{\|A\|h},$$

$$\sum_{p=1}^{M} \left(\sum_{r=M+1}^{\infty} \frac{h^r}{r!} \|K_{r,p}^m\| \right) \leq \sum_{p=1}^{M} \frac{h^{M+1}}{p!} \|B\|^p \|A\|^{M+1-p} \left(\sum_{r=M+1}^{\infty} \frac{h^{r-(M+1)}}{(r-p)!} \|A\|^{r-(M+1)} \| \right) \leq h^{M+1} e^{\|A\|h} \sum_{p=1}^{M} \frac{\|B\|^p}{p!} \|A\|^{M+1-p},$$

and, similarly,

$$\sum_{p=M+1}^{m} \left(\sum_{r=p}^{\infty} \frac{h^r}{r!} \|K_{r,p}^m\| \right) \leq \sum_{p=M+1}^{m} \frac{h^p}{p!} \|B\|^p e^{\|A\|h} \leq h^{M+1} e^{\|A\|h} \|B\|^{M+1} \sum_{p=0}^{m-(M+1)} \frac{\|B\|^p}{p!}.$$

Finally, regarding the integral term, we will show by induction that $\|Q_m(h)\| = O(h^m)$. Since the other terms in the integral are either constant or bounded, as we assume that F' is continuous, it will follow that the integral is $O(h^{m+1})$, and hence $O(h^{M+1})$, since $m > M$. For $m = 1$ one gets

$$\|Q_1(t)\| \leq \| \left(\mathrm{e}^{At} - I \right) \| \|(I+C)\| \leq \|A\| h e^{\|A\|h} \|(I+C)\| = O(h).$$

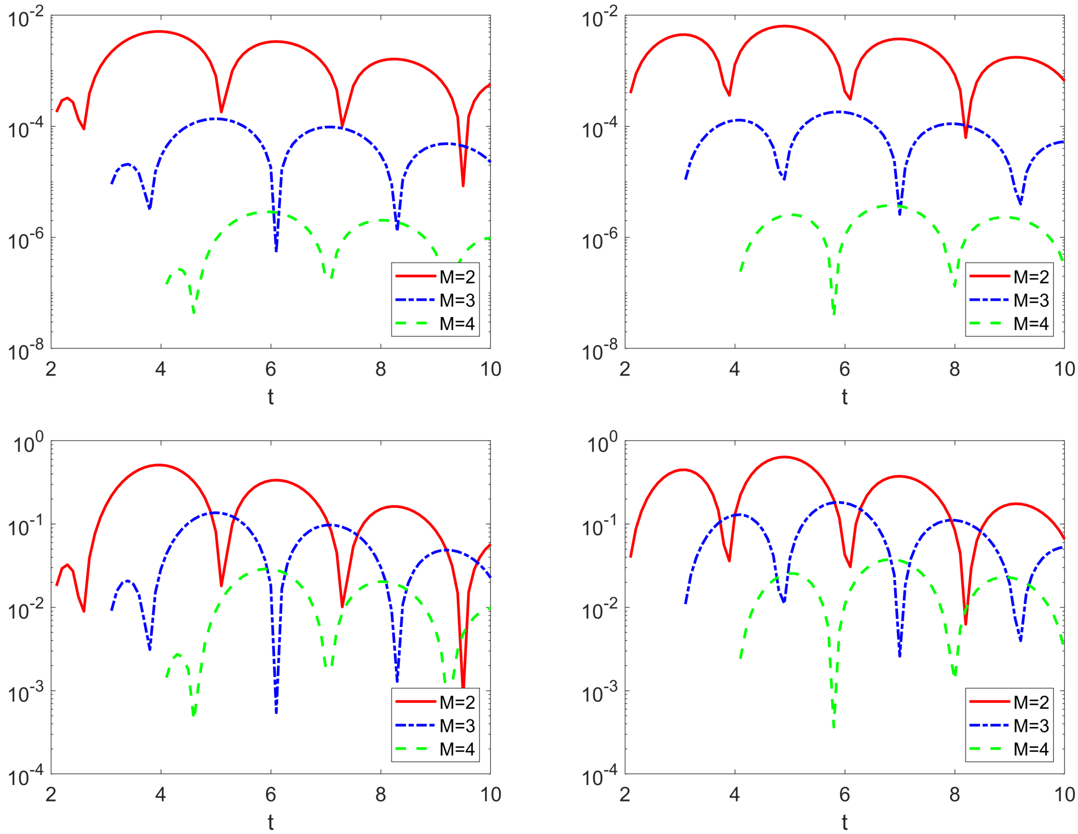

FIGURE 2 Absolute errors (top) for the two components (left and right) of the numerical solutions of Example 1 computed with schemes of three different orders ($M = 2$, $M = 3$, and $M = 4$), and corresponding errors divided by h^M (bottom).

Assuming $\|Q_j(h)\| = O(h^j)$ for $j = 1 \dots k$, from (3),

$$\|Q_{k+1}(h)\| \leq \int_0^h \|e^{A(h-s)}B\| \|Q_k(s)\| ds = O(h^{k+1}),$$

since $\|e^{A(h-s)}B\|$ is bounded in $[0, h]$, and by the induction hypothesis $\|Q_k(s)\| = O(h^k)$. □

In the next two figures the order estimations of the new methods given in Theorem 3 are illustrated for the problem in Example 1. Figure 2 (top) shows, for each component of the solution, the absolute errors of the numerical solutions computed with the methods defined in Theorem 3 for three different orders. When these errors are divided by h^M the curves overlap (Figure 2 , bottom), in agreement with the schemes being of order M, as established in Theorem 3.

It is to be noted that since the error curves in Figure 2 correspond to different methods, the corresponding bounding constant differ, and thus a perfect overlap in the relative error curves should not be expected. Figure 3 shows, for the method of third order ($M = 3$), the maximum norm for the errors of the numerical solutions (Figure 3 , left), and corresponding errors divided by h^3 (Figure 3 , right), for three different mesh sizes. In agreement with Theorem 3, the relative errors show a perfect overlap, as in this case the three curves correspond to the same method, with the same bounding constant.

Numerical values of maximum absolute errors and computational order estimates for the numerical solutions displayed in Figures 1 and 2 are presented in Table 1 . From the maximum absolute errors in the interval $[0, 10]$ for each mesh size $h = \tau/N$,

$$E_h = \max_{n=0\dots 10N} \|X_n - X_n^e\|,$$

computational orders are estimated as

$$\frac{\ln(E_h) - \ln(E_{h/2})}{\ln(2)}.$$

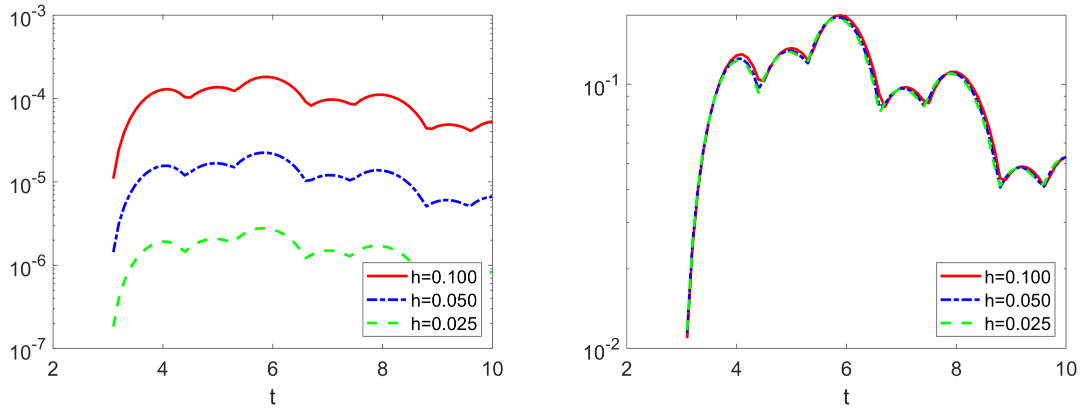

FIGURE 3 Maximum norms of the error (left) for the numerical solutions of Example 1 computed with the scheme of third order ($M = 3$) for three different mesh sizes ($h = 0.1$, $h = 0.05$, and $h = 0.025$), and corresponding errors divided by h^3 (right).

TABLE 1 Maximum absolute errors (upper values) and order estimates (lower values) for numerical solutions of Example 1 computed with NSFD schemes of three different orders ($M = 2$, $M = 3$, and $M = 4$), and with standard methods backward Euler, trapezoidal rule, and dde23, for different mesh sizes.

$\mathbf{h = \tau/N}$	**M = 2**	**M = 3**	**M = 4**	**backward Euler**	**trapezoidal rule**	**dde23***
$h = 0.025$	3.94×10^{-4}	2.78×10^{-6}	1.44×10^{-8}	7.45×10^{-2}	4.79×10^{-4}	8.13×10^{-6}
	-	-	-	-	-	-
$h = 0.05$	1.58×10^{-3}	2.24×10^{-5}	2.32×10^{-7}	1.42×10^{-1}	1.91×10^{-3}	6.39×10^{-5}
	2.01	3.01	4.01	0.93	2.00	2.91
$h = 0.1$	6.40×10^{-3}	1.82×10^{-4}	3.76×10^{-6}	2.59×10^{-1}	7.63×10^{-3}	4.82×10^{-4}
	2.01	3.02	4.02	0.87	2.00	2.97

*Stepsize restricted with option MaxStep=h.

As shown in Table 1 , computational order estimates for the NSFD defined in Theorem 3 agree with the theoretical results. For comparison purposes, Table 1 also shows corresponding values for two θ-methods, backward Euler and trapezoidal rule, and the BS(2,3) Runge-Kutta algorithm dde23 [35], incorporated in Matlab [25]. The second and third order standard methods both showed higher errors than the NSFD schemes of corresponding order.

4 | NUMERICAL EXAMPLES OF STABILITY PROPERTIES

For delay differential models, one of the main qualitative properties to preserve in a numerical method is delay-dependent stability, the so called $\tau(0)$-stability, so that the numerical solution is asymptotically stable for each delay value that makes the exact continuous solution to be asymptotically stable. Proving that a given numerical method is $\tau(0)$-stable is a difficult task, usually relying on specific properties of the method. Thus, for instance, the proof of $\tau(0)$-stability given in García et al [22] for a class of NSFD methods for problem (1)-(2) with commuting matrix coefficients was essentially based on this special property. Given the generality of the problems afforded in this work, it seems much difficult to devise a formal proof of $\tau(0)$-stability for the general new NSFD methods proposed in this work. Notwithstanding, as shown in the examples presented in the next figures, the new NSFD methods seem to possess rather good stability preserving properties, suggesting that a proof of $\tau(0)$-stability for these methods could be attained in future works, at least for some particular classes of problems.

Figure 4 shows the long-term behaviour of the numerical solution for Example 1 computed with the NSFD third order scheme for different delay values. Stability of the DDE system (1) with coefficients A and B of Example 1 is characterized in Gu et al [23] (Example 5.11; see also Li et al [36], Example 1.2). This system is unstable without delay, and become asymptotically

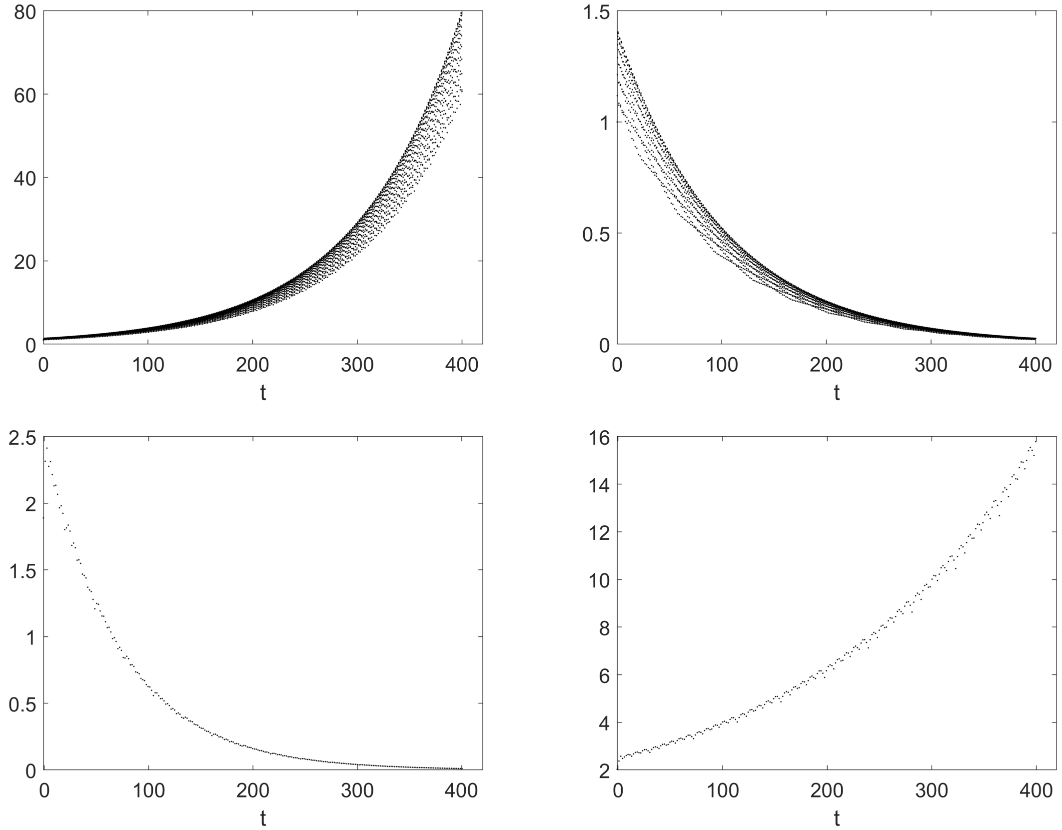

FIGURE 4 Maximum value in each τ-interval for the maximum norm of the numerical solutions of Example 1 computed with different delay values (from left to right and top to bottom, $\tau = 0.08$, $\tau = 0.12$, $\tau = 1.70$, $\tau = 1.74$).

stable when a positive delay τ is present, if and only if $\tau \in (0.1002, 1.7178)$. It is to be noted that the delay values used in the numerical solutions presented in Figure 4 are very close to the limits of stability, either the left limit (Figure 4 , top) or the right limit (Figure 4 , right), with the numerical solutions reproducing the expected asymptotic behaviour.

As shown in the next example, the new proposed NSFD methods also preserve asymptotic stability for problems with more complex behaviours.

Example 2. Figure 5 shows long-term behaviours of the numerical solutions, computed with the NSFD third order scheme for different delay values, for problem (1)-(2) with

$$A = \begin{pmatrix} -1 & 13.5 & -1 \\ -3 & -1 & -2 \\ -2 & -1 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} -5.9 & 7.1 & -70.3 \\ 2 & -1 & 5 \\ 2 & 0 & 6 \end{pmatrix}, \quad F(t) = \begin{pmatrix} t-0.1 \\ (t+0.1)^2 \\ t-2 \end{pmatrix}.$$

Stability properties for DDE system (1) with A and B in Example 2 were discussed by Olgac and Sipahi [37] (see also Li et al [36], Example 1.3). This system is stable when there is no delay, and then switches stability three times at increasing delay values, from stable to unstable (at $\tau = 0.1624$), to stable again (at $\tau > 0.1859$), and finally remains unstable for all $\tau \geq 0.2219$. As shown in Figure 5 , where panels from left to right and top to bottom correspond to increasing delay values across the different stability regions, with τ values close to their limits, the numerical solutions provided by the new NSFD schemes proposed in this work correctly reproduce the asymptotic behaviour of the exact solution.

To illustrate the computational efficiency of the NSFD schemes defined in this work, CPU times for computing numerical solutions for Example 1 with $\tau = 0.12$, for increasing lengths of computing intervals, are presented in Table 2 . Computations were performed in a personal computer (CPU i7-5500U 2.40GHz, 2 processors), so that computation times for the different methods should only be considered in relative terms. NSFD schemes were computed with $N = 5$, so that numerical solutions where obtained in a mesh with stepsize $h = 0.024$, which was also used for both θ-methods. Since dde23 computes the solution

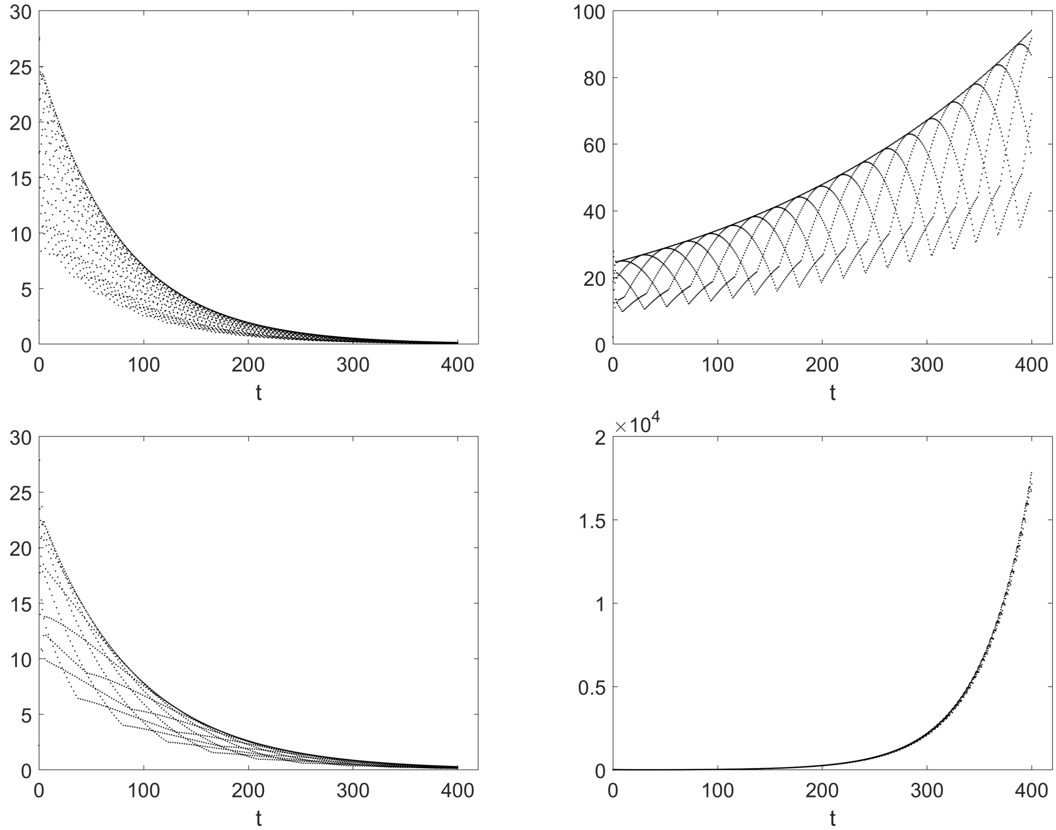

FIGURE 5 Maximum value in each τ-interval for the maximum norm of the numerical solutions of Example 2 computed with different delay values (from left to right and top to bottom, $\tau = 0.150$, $\tau = 0.175$, $\tau = 0.200$, $\tau = 0.223$).

with variable stepsizes, as large as possible to combine precision and efficiency in reaching the final computation time, times to evaluate the solutions in the same mesh used by the other methods are also presented. Times for dde23 restricted with a maximum stepsize of $h = 0.024$, so that the method in practice uses this mesh size except for a few points, are also presented in Table 2 .

As shown in Table 2 , the fastest method, as expected, was the first order backward Euler, which is included as a reference. Explicit forward Euler was faster, but it did not preserve asymptotic stability in this example and was excluded from comparisons. Basic computation times for second order NSFD scheme ($M = 2$) were only slightly higher than for backward Euler, and shorter than for the second order trapezoidal rule. It is to be noted that the NSFD schemes of order M proposed in this work require an initial computation in the first M intervals, displayed in Table 2 as T_0, which adds to total CPU times. However, T_0 values shown in Table 2 are upper bounds, since they correspond to evaluations of exact solutions and, as indicated in Teorem 3, the NSFD schemes can also be initialised with much faster evaluations at the mesh points in the first M intervals using any method of order at least $M + 1$. Although using increasingly coarse meshes, efficiency of the third order dde23 was largely degraded for longer integration times, even if evaluation times in the same fine mesh as other methods is not considered, with CPU times much longer than total times of the four order NSFD scheme. Much worse was the behaviour of dde23 when stepsize restriction was imposed.

5 | CONCLUSIONS

The results developed in this work are the first, to our knowledge, that propose non-standard numerical schemes for the general initial value coupled linear delay differential problem (1)-(2) based on an expression for the exact numerical solution, as given in Theorem 2. Although this expression does not provide in the general case a practical exact scheme, due to the presence of

TABLE 2 CPU times (seconds) for computing numerical solutions of Example 1, with $\tau = 0.12$, for NSFD schemes of three different orders ($M = 2$, $M = 3$, and $M = 4$; $N = 5$), and standard methods backward Euler, trapezoidal rule, and dde23.

Method		$\mathbf{t \in [0, 500]}$	$\mathbf{t \in [0, 5000]}$	$\mathbf{t \in [0, 10000]}$	$\mathbf{t \in [0, 50000]}$
$M = 2$	$T_0 = 0.95$†	0.08	0.70	1.37	6.78
$M = 3$	$T_0 = 2.45$†	0.17	0.91	1.85	8.64
$M = 4$	$T_0 = 4.41$†	0.22	1.31	2.59	1.28×10^1
bacward Euler		0.06	0.50	0.96	4.71
trapezoidal rule		0.12	0.75	1.54	7.26
dde23	T_1‡	0.79	6.07	1.34×10^1	1.87×10^2
	T_2‡	0.36	2.02×10^1	6.98×10^1	8.94×10^2
	mean stepsize	0.16	0.66	0.88	1.19
dde23 restricted*		9.76	1.37×10^3	6.06×10^3	2.69×10^5

†Computation time for the exact solution in the first M intervals.
‡T_1: Computation time in the coarse mesh defined by the method. T_2: Evaluation time in the finer mesh ($h = 0.024$).
*Stepsize restricted with option MaxStep=$\tau/N = 0.024$.

infinite sums, it paves the way to obtain closed form exact numerical solutions for different classes of problems, depending on the dimensions or particular structures of the matrix coefficients.

The family of NSFD schemes defined in Theorem 3 provide computationally efficient methods with as high order of precision as required. For a given order of approximation M, as needed in the problem at hand, the new NSFD methods work efficiently by concentrating the computational burden in the first M intervals of τ-amplitude, while maintaining the required precision for long computation times with very low computational effort.

As shown in the examples presented in Section 4, the numerical solutions provided by the new NSFD schemes proposed in this work seem to possess good properties in terms of preserving delay-dependent stability of the corresponding exact continuous solutions. Although a formal analysis of their delay-dependent stability properties in the general case seems hard to carry out, and it has not been attempted in this paper, the numerical examples suggest that, at least for some particular classes of problems, $\tau(0)$-stability of the new methods could be proved in future works.

The results of this work extend to the general problem (1)-(2) previous analogous methods proposed for restricted classes of problems, as those requiring scalar[21] or commuting coefficients[22], this way allowing their application to more general problems. In particular, the NSFD methods of this work can be applied to high order linear delay differential equations, as they can be converted in the usual way into a vector problem with non-commuting matrix coefficients. Additionally, it could be possible to extend the type of results for deterministic systems obtained in this work to the random setting, considering problems as those in[38,39] for delay scalar equations, with coefficients being random variables and initial functions being stochastic processes.

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Author contributions

M.A.C. and F.R. conceived the research. M.A.C. prepared the figures. F.R. wrote the initial draft. All authors contributed to methodology, analysis, and revision of the final text.

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Conflict of interest

The authors declare no potential conflict of interests.

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$\square$