THE $\mathfrak{sl}_3$-WEB ALGEBRA

M. MACKAAY, W. PAN AND D. TUBBENHAUER

ABSTRACT. In this paper we use Kuperberg's $\mathfrak{sl}_3$-webs and Khovanov's $\mathfrak{sl}_3$-foams to define a new algebra K^S, which we call the $\mathfrak{sl}_3$-web algebra. It is the $\mathfrak{sl}_3$ analogue of Khovanov's arc algebra.

We prove that K^S is a graded symmetric Frobenius algebra. Furthermore, we categorify an instance of q-skew Howe duality, which allows us to prove that K^S is Morita equivalent to a certain cyclotomic KLR-algebra of level 3. This allows us to determine the split Grothendieck group $K_0^{\oplus}(\mathcal{W}^S)_{\mathbb{Q}(q)}$, to show that its center is isomorphic to the cohomology ring of a certain Spaltenstein variety, and to prove that K^S is a graded cellular algebra.

CONTENTS

1. INTRODUCTION

In this paper, we define the $\mathfrak{sl}_3$ analogue of Khovanov's arc algebras H^n, introduced in [32]. We call them *web algebras* and denote them by K^S, where S is a *sign string* (string of $+$ and $-$ signs). Instead of arc diagrams, which give a diagrammatic presentation of the representation theory of $U_q(\mathfrak{sl}_2)$, we use $\mathfrak{sl}_3$-webs, introduced by Kuperberg [42]. These webs give a diagrammatic presentation of the representation theory of $U_q(\mathfrak{sl}_3)$. Instead of $\mathfrak{sl}_2$-cobordisms, which Bar-Natan

used [1] to give his formulation of Khovanov's link homology, we use Khovanov's [34] $\mathfrak{sl}_3$-foams.

We prove the following main results regarding K^S.

(1) K^S is a graded symmetric Frobenius algebra (Theorem 3.9).
(2) We give an explicit degree preserving algebra isomorphism between the cohomology ring of the Spaltenstein variety X_μ^λ and the center $Z(K^S)$ of K^S, where λ and μ are two weights determined by S (Theorem 4.8).
(3) Let $V^S = V^{s_1} \otimes \cdots \otimes V^{s_n}$, where V^+ is the basic $U_q(\mathfrak{sl}_3)$-representation and V^- its dual. Kuperberg [42] proved that W^S, the space of $\mathfrak{sl}_3$-webs whose boundary is determined by S, is isomorphic to $\mathrm{Inv}_{U_q(\mathfrak{sl}_3)}(V^S)$, the space of invariant tensors in V^S.

Choose an arbitrary $k \in \mathbb{N}$ and let $n = 3k$. Let us denote by $V_{(3^k)}$ the irreducible $U_q(\mathfrak{sl}_n)$-module with highest weight $3\omega_k$, where ω_k is the k-th fundamental $\mathfrak{sl}_n$-weight. As the reader will have noticed, we actually use the corresponding $\mathfrak{gl}_n$-weight (3^k). This is natural from the point of view of skew Howe duality, as we will explain in the paper.

The $\mathfrak{gl}_n$-weights of $V_{(3^k)}$ belong to $\Lambda(n,n)_3$, which is the set of n-part compositions of n whose parts are integers between 0 and 3. These weights, denoted μ_S, correspond bijectively to *enhanced sign sequences* of length n, denoted by S as before, and are in bijective correspondence to the semi-standard Young tableaux with k rows and 3 columns.

Define the *web module*

$$W_{(3^k)} = \bigoplus_{\mu_S \in \Lambda(n,n)_3} W^S,$$

where W^S is defined as before after deleting the entries of μ_S which are equal to 0 or 3.

By q-skew Howe duality, which we will explain at the beginning of Section 5, there is an $\dot{U}_q(\mathfrak{sl}_n)$-action on $W_{(3^k)}$ such that

$$V_{(3^k)} \cong \bigoplus_{\mu_S \in \Lambda(n,n)_3} W^S$$

as $\dot{U}_q(\mathfrak{sl}_n)$-modules.

In Section 5 we categorify this result. Let $R_{(3^k)}$ be the cyclotomic level-three Khovanov-Lauda Rouquier algebra (cyclotomic KLR algebra for short) with highest $\mathfrak{gl}_n$-weight (3^k), and let

$$\mathcal{V}_{(3^k)} = R_{(3^k)}\text{-}\mathbf{Mod}_{\mathrm{gr}} \quad \text{and} \quad {}_p\mathcal{V}_{(3^k)} = R_{(3^k)}\text{-}\mathrm{p}\mathbf{Mod}_{\mathrm{gr}}$$

be its categories of finite dimensional, graded modules and finite dimensional, graded, projective modules respectively. We define grading shifts by

$$M\{t\}_i = M_{i-t}$$

for any $M \in \mathcal{V}_{(3^k)}$ and $t \in \mathbb{Z}$. We denote the split Grothendieck group of ${}_p\mathcal{V}_{(3^k)}$ by

$$K_0^\oplus({}_p\mathcal{V}_{(3^k)}),$$

which becomes a $\mathbb{Z}[q, q^{-1}]$-module by defining

$$q^t[M] = [M\{t\}]$$

for any $M \in {}_p\mathcal{V}_{(3^k)}$ and $t \in \mathbb{Z}$. For the rest of this paper, we will always work with

$$K_0^\oplus({}_p\mathcal{V}_{(3^k)})_{\mathbb{Q}(q)} = K_0^\oplus({}_p\mathcal{V}_{(3^k)}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q).$$

Brundan and Kleshchev [7] (see also [31], [45], [68] and [69]) proved that there is a strong $\mathfrak{sl}_n$-2-representation on $\mathcal{V}_{(3^k)}$, which can be restricted to ${}_p\mathcal{V}_{(3^k)}$ such that

$$K_0^\oplus({}_p\mathcal{V}_{(3^k)})_{\mathbb{Q}(q)} \cong V_{(3^k)}$$

as $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$-modules.

We prove (Proposition 5.15) that there exists a strong $\mathfrak{sl}_n$-2-representation on

$$\mathcal{W}_{(3^k)} = \bigoplus_{\mu_S \in \Lambda(n,n)_3} \mathcal{W}^S,$$

where

$$\mathcal{W}^S = K^S\text{-}\mathbf{Mod}_{\mathrm{gr}}\ ^1.$$

This 2-representation can be restricted to

$$ {}_p\mathcal{W}_{(3^k)} = \bigoplus_{\mu_S \in \Lambda(n,n)_3} {}_p\mathcal{W}^S,$$

where

$$ {}_p\mathcal{W}^S = K^S\text{-}\mathrm{p}\mathbf{Mod}_{\mathrm{gr}}.$$

By a general result due to Rouquier [60], which we recall in Proposition 2.23, we get

(1.1) $$ {}_p\mathcal{V}_{(3^k)} \cong {}_p\mathcal{W}_{(3^k)}.$$

(4) In particular, this proves that the split Grothendieck groups of both categories are isomorphic (Corollary 5.17). It follows that we have

$$K_0^\oplus\left({}_p\mathcal{W}^S\right)_{\mathbb{Q}(q)} \cong W^S,$$

for any S such that $\mu_S \in \Lambda(n,n)_3$.

(5) As proved in Corollary 5.17, the equivalence in (1.1) implies that $R_{(3^k)}$ and

$$K_{(3^k)} = \bigoplus_{\mu_S \in \Lambda(n,n)_3} K^S$$

are Morita equivalent (Proposition 5.18), i.e. we have

(1.2) $$\mathcal{V}_{(3^k)} \cong \mathcal{W}_{(3^k)}$$

as strong $\mathfrak{sl}_n$-2-representations.

(6) In Corollary 5.21, we show that (1.2) and the cellularity of $R_{(3^k)}$, due to Hu and Mathas [29], imply that K^S is a graded cellular algebra for any S.

(7) We also show that (1.1) and Brundan and Kleshchev's results in [7] imply that the indecomposable objects in ${}_p\mathcal{W}^S$, with a suitable normalization of their gradings, correspond bijectively to the dual canonical basis elements in $\mathrm{Inv}(V^S)$ (Theorem 5.31).

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[1] The idea for this 2-representation was suggested by Mikhail Khovanov to M. M. in 2008 and its basic ideas were worked out modulo 2 in the unpublished preprint [47].

The first result is easy to prove and similar to the case for H^n. Some of the other results are much harder to prove for K^S than their analogues are for H^n (e.g. see Remark 5.24). In order to prove the second and the last result, we introduce a "new trick", i.e. we use a deformation of K^S, called G^S. This deformation is induced by Gornik's [26] deformation of Khovanov's original $\mathfrak{sl}_3$-foam relations. One big difference between G^S and K^S is that the former algebra is *filtered* whereas the latter is *graded*. As a matter of fact, K^S is the associated graded algebra of G^S. The usefulness of G^S relies on the fact that G^S is semisimple as an algebra, i.e. forgetting the filtration (see Proposition 3.13).

Let us explain the connection to some of the existing work in the literature.

We first comment on the relation of our results with known results in the $\mathfrak{sl}_2$ case. Khovanov [32] introduced the arc algebras H^n in his work on the generalization of his celebrated categorification of the Jones link polynomial to tangles. As he showed,

$$K_0^\oplus(H^n\text{-p}\mathbf{Mod}_{\mathrm{gr}}) \cong \mathrm{Inv}_{\dot{\mathbf{U}}_q(\mathfrak{sl}_2)}(V^{\otimes 2n}),$$

where V is the fundamental $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$-module. The Grothendieck classes of the indecomposable graded H^n-modules, with a suitable normalization of their grading, correspond bijectively to the dual canonical basis elements of the invariant tensor space. The proof of these facts is completely elementary and does not require any categorified skew Howe duality.

Huerfano and Khovanov categorified the irreducible level-two $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$-representations with highest weight $2\omega_k$ in [30], using the arc algebras and categorified skew Howe duality (without calling it that explicitly), but without explaining the relation with the level-two cyclotomic KLR algebras which had not yet been invented at that time.

That relation only appeared in the work by Brundan and Stroppel [11], who studied the representation theory of the arc algebras in great detail in [9], [10], [11], [12] and [13].

Khovanov showed that the center of the arc algebra H^n is isomorphic to the cohomology ring of the (n, n)-Springer variety X^n and Stroppel and Webster showed that H^n can be realized using the intersection cohomology of X^n.

The results in this paper are the $\mathfrak{sl}_3$ analogues of some of the results in the papers cited above.

There are several results for $\mathfrak{sl}_2$ for which we have not yet found the $\mathfrak{sl}_3$ analogues, e.g. we have not defined the quasi-hereditary cover of K^S in this paper. The quasi-hereditary cover of H^n is due to Chen and Khovanov [17] and Stroppel [64] and was studied by Brundan and Stroppel in [9],[10] and [11]. Furthermore, in [12] Brundan and Stroppel found a remarkable representation theoretic relation between general linear super groups and certain generalized arc algebras. This is another result for which we do not have an $\mathfrak{sl}_3$-analogue.

Let us now comment on the connection with other work on categorified $\mathfrak{sl}_n$-representations and link homologies, for $n \geq 3$. There are essentially three diagrammatic or combinatorial approaches which give $\mathfrak{sl}_n$-link homologies (there are other approaches using representation theory or algebraic geometry for example, but we will not consider those in this introduction).

(1) There is the approach using matrix factorizations due to Khovanov and Rozansky [34], which was proved to be equivalent to an approach using foams [32], [40], [49] and [52].
(2) There is Webster's approach using cyclotomic tensor algebras, which generalize the cyclotomic KLR-algebras, see [69] and [70].

(3) There is an approach using Chuang-Rouquier complexes over cyclotomic quotients [19] (its details have only been worked out and written up completely for $n = 2, 3$ in [44], but see our remarks below for the general case. Note that Chuang and Rouquier did not prove invariance under the third Reidemeister move nor did they discuss braid closures, i.e. knots and links in [19])

Only for $n = 2$ and $n = 3$ it is known that all three approaches give isomorphic link homologies. For $n \geq 4$ they are conjectured to be isomorphic, but only the first and the third approaches are known to give isomorphic link homologies. Let us explain this in a bit more detail.

The paper by Lauda, Queffelec and Rose [44] appeared online a little after our paper became available and is completely independent. They used a slightly different type of $\mathfrak{sl}_3$-foams in order to define their version of categorified level-three skew Howe duality and used it to relate the first and third aforementioned approach to $\mathfrak{sl}_3$-link homologies.

Although link homologies have attracted a lot of attention, it seems important to us to understand the bigger picture of the categorified representation theory behind the link homologies.

In this paper we do not work out the application to link homology, but concentrate on the indecomposable projective modules and the center of K^S. In Proposition 5.27 we actually prove a conjecture due to Morrison and Nieh [56] about the relation between the basis webs in B^S and the indecomposables in $_p\mathcal{W}^S$ (see Remark 5.28). For related results in this direction, see the work by Robert in [57] and [58].

It is not so hard to generalize our results in this paper to the case for $\mathfrak{sl}_n$, with $n \geq 2$, using matrix factorizations instead of foams. As a matter of fact, this has been done in the meanwhile by Mackaay and Yonezawa in [48] and [53]. General $\mathfrak{sl}_n$-foams have not been defined yet (for a partial case, see [49]), so they cannot be used. But the results in [53] will probably be helpful to define a finite set of relations on $\mathfrak{sl}_n$-foams and to prove that these relations are consistent and sufficient.

Having such a definition of $\mathfrak{sl}_n$-foams is important, just as it is important to have generators and relations for any interesting algebra. In our opinion, the categorified skew Howe duality using foams would be a proper categorification of Cautis, Kamnitzer and Morrison's results on quantum skew Howe duality in [15]. Furthermore, $\mathfrak{sl}_n$-foams might be very helpful in computing the Khovanov-Rozansky link homologies effectively (i.e. using computers).

Although Mackaay and Yonezawa did not work out the details, it is fairly straightforward to show that their definitions and results in [53] imply that the equivalence between level-n cyclotomic KLR algebras and $\mathfrak{sl}_n$-web algebras maps the colored Chuang-Rouquier complex of a braid over a cyclotomic KLR algebra to the corresponding colored Khovanov-Rozansky complex (due to Wu [71] and Yonezawa [72] in the colored case). This would imply the above claim that the third aforementioned construction of $\mathfrak{sl}_n$-link homologies works for all $n \geq 2$ and that the first and the third construction give isomorphic $\mathfrak{sl}_n$-link homologies.

In Proposition 4.4 in [70], Webster proved that his and Khovanov's $\mathfrak{sl}_3$-link homologies are isomorphic, but the proof is quite sophisticated and relies on Mazorchuk and Stroppel's approach to link homology using functors and natural transformations on certain blocks of category $\mathcal{O}$ [55]. Our results in this paper might help to give an elementary and direct isomorphism between Webster's and Khovanov's $\mathfrak{sl}_3$-link homologies. In order to do that, one would have to use yet to be defined bimodules over the cyclotomic tensor algebras which categorify the $\mathfrak{sl}_3$-webs.

For $n \geq 4$ Webster conjectured his $\mathfrak{sl}_n$-link homology to be isomorphic to Khovanov and Rozansky's, but did not prove it. Following the same reasoning as for $n = 3$, such a proof should now be within reach.

Finally, let us mention one interesting open question w.r.t. $\mathfrak{sl}_3$-web algebras. In [22], Fontaine, Kamnitzer and Kuperberg studied spiders using an algebro-geometric approach. For $\mathfrak{sl}_3$ these spiders are exactly the webs in our paper. Given a sign string S, the *Satake fiber* $F(S)$, denoted $F(\overrightarrow{\lambda})$ in [22], is isomorphic to the Spaltenstein variety X_μ^λ mentioned above. Let us point out the difference in these notations that otherwise might confuse the reader: the λ in [22] is equal to μ in our paper, which is also equivalent to S. Given a web w with boundary corresponding to S, Fontaine, Kamnitzer and Kuperberg also defined a variety $Q(D(w))$, called the *web variety*. One interesting question is the following (asked to us by Kamnitzer).

Question 1.1. For any two basis webs $u, v \in B^S$, does there exist a degree preserving algebra isomorphism
$$\bigoplus_{u,v \in B^S} H^*(Q(D(u))) \otimes_{F(S)} H^*(Q(D(v))) \cong \bigoplus_{u,v \in B^S} {}_u K_v ?$$
Here
$$K^S = \bigoplus_{u,v \in B^S} {}_u K_v$$
is the decomposition of K^S in Section 3. The product on
$$\bigoplus_{u,v \in B^S} H^*(Q(D(u))) \otimes_{F(S)} H^*(Q(D(v)))$$
is given by convolution.

If the answer to this question is affirmative, then that would be the $\mathfrak{sl}_3$ analogue of the aforementioned result for $\mathfrak{sl}_2$ due to Stroppel and Webster [65]. Our Theorem 4.8 could be a first step towards answering Kamnitzer's question.

This paper is organized as follows.
  (1) In Section 2, we recall the definitions and some fundamental properties of webs, foams and categorified quantum algebras and their categorical representations. The reader who already knows all this material well enough can just leaf through it, in order to understand our notations and conventions. Other readers might perhaps find it helpful as a brief introduction to the rapidly growing literature on categorification, although it is far from self-contained.
  (2) In Section 3, we define K^S and prove the first of our aforementioned main results.
  (3) In Section 4, we first study the relation between column strict tableaux and webs with flows. Using this relation, we prove our second main result.
  (4) In Section 5, we explain skew Howe duality in our context and categorify the case relevant to this paper. This leads to the other main results.
  (5) Sections 4 and 5 are largely independent of each other. However, the proof of Theorem 4.8 requires Proposition 5.18. The proof of Proposition 5.27, which is a key ingredient for the proof of Theorem 5.31, requires Lemma 4.7.

(6) In "Appendix 1", we collect some technical facts from the literature on filtered algebras, filtered modules and their associated graded counterparts. These are needed at various places in the paper.

2. BASIC DEFINITIONS AND BACKGROUND

2.1. **Webs.** In [42], Kuperberg describes the representation theory of $U_q(\mathfrak{sl}_3)$ using oriented trivalent graphs, possibly with boundary, called *webs*. Boundaries of webs consist of univalent vertices (the ends of oriented edges), which we will usually put on a horizontal line (or various horizontal lines), e.g. such a web is shown below.

(2.1)

We say that a web has n free strands if the number of non-trivalent vertices is exactly n. In this way, the boundary of a web can be identified with a *sign string* $S = (s_1, \ldots, s_n)$, with $s_i = \pm$, such that upward oriented boundary edges get a "+" and downward oriented boundary edges a "−" sign. Webs without boundary are called *closed* webs.

Any web can be obtained from the following elementary webs by glueing and disjoint union.

(2.2)

Fixing a boundary S, we can form the $\mathbb{Q}(q)$-vector space W^S, spanned by all webs with boundary S, modulo the following set of local relations (due to Kuperberg [42]).

(2.3) $$\bigcirc \;=\; [3]$$

(2.4) $=\; [2]$

(2.5)

Recall that

$$[a] = \frac{q^a - q^{-a}}{q - q^{-1}} = q^{a-1} + q^{a-3} + \cdots + q^{-(a-1)} \in \mathbb{N}[q, q^{-1}]$$

denotes the *quantum integer*.

Remark 2.1. Throughout this paper we will use q as the quantum parameter, which we take to be equal to the parameter v in [35]. Note that in that paper Khovanov and Kuperberg use the different convention $v = -q^{-1}$.

By abuse of notation, we will call all elements of W^S webs. From relations (2.3), (2.4) and (2.5) it follows that any element in W^S is a linear combination of webs with the same boundary and without circles, digons or squares. These are called *non-elliptic webs*. As a matter of fact, the non-elliptic webs form a basis of W^S, which we call B^S. Therefore, we will simply call them *basis webs*.

Following Brundan and Stroppel's [9] notation for arc diagrams, we will write w^* to denote the web obtained by reflecting a given web w horizontally and reversing all orientations.

(2.6)

By uv^*, we mean the planar diagram containing the disjoint union of u and v^*, where u lies vertically above v^*.

(2.7)

By v^*u, we shall mean the closed web obtained by glueing v^* on top of u, when such a construction is possible (i.e. the number of free strands and orientations on the strands match).

(2.8)

In the same vein, by $v_1^* u_1 v_2^* u_2$ we denote the following web.

(2.9)

To make the connection with the representation theory of $U_q(\mathfrak{sl}_3)$, we recall that a sign string $S = (s_1, \ldots, s_n)$ corresponds to

$$V^S = V^{s_1} \otimes \cdots \otimes V^{s_n},$$

where V^+ is the fundamental representation and V^- its dual. The latter is also isomorphic to $V^+ \wedge V^+$, a fact which we will need later on.

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Both V^+ and V^- have dimension three. In this interpretation, webs correspond to intertwiners and we have
$$W^S \cong \mathrm{Inv}_{U_q(\mathfrak{sl}_3)}(V^S) \cong \mathrm{Hom}_{U_q(\mathfrak{sl}_3)}(\mathbb{C}_q, V^S).$$
Here $\mathbb{C}_q$ is the trivial representation.

Thus, the elements of B^S give a basis of $\mathrm{Inv}_{U_q(\mathfrak{sl}_3)}(V^S)$. Using the embedding $\mathrm{Inv}_{U_q(\mathfrak{sl}_3)}(V^S) \subset V^S$, one can expand a non-elliptic web in terms of elementary tensors. In Theorem 2 of [35], Kuperberg and Khovanov prove an important result about this expansion, which we will reproduce in Theorem 2.6.

Kuperberg showed in [42] (see also [35]) that basis webs are indexed by closed weight lattice paths in the dominant Weyl chamber of $\mathfrak{sl}_3$. It is well-known that any path in the $\mathfrak{sl}_3$-weight lattice can be presented by a pair consisting of a sign string $S = (s_1, \ldots, s_n)$ and a *state string* $J = (j_1, \ldots, j_n)$, with $j_i \in \{-1, 0, 1\}$ for all $1 \leq i \leq n$. Given a pair (S, J) representing a closed dominant path, a unique basis web (up to isotopy) is determined by a set of inductive rules called the *growth algorithm*. We briefly recall the algorithm as described in [35]. In fact, the algorithm can be applied to any path, but we will only use it for closed dominant paths.

Definition 2.2. (The growth algorithm) Given (S, J), a web w_J^S is recursively generated by the following rules.

(1) Initially, the web consists of n parallel strands whose orientations are given by the sign string. If $s_i = +$, then the i-th strand is oriented upwards; if $s_i = -$, it is oriented downwards.

(2) The algorithm builds the web downwards. Suppose we have already applied the algorithm $k - 1$ times. For the k-th step, do the following. If the bottom boundary string contains a neighboring pair of edges matching the top of one of the following webs (called H, arc and Y respectively), then glue the corresponding H, arc or Y to the relevant bottom boundary edges.
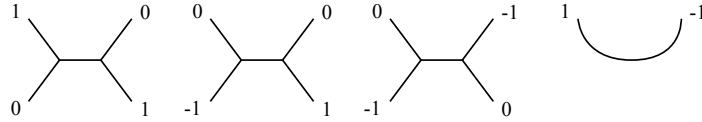

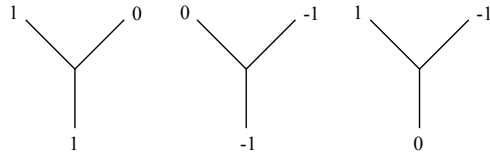FIGURE 1. Top strands have different signs.

FIGURE 2. Top strands have same sign.

These rules apply for any compatible orientation of the edges in the webs. Therefore, we have drawn them without any specific orientations. Below, whenever we write down an equation involving webs without orientations, we mean that the equation holds for all possible orientations. For future use, we will call the rules above the *H, arc and Y-rule*. The growth algorithm stops if no further rules can be applied.

If (S, J) represents a closed dominant path, then the growth algorithm produces a basis web.

For example, the growth algorithm converts $S = (+-+-+++)$ and $J = (1, 1, 0, 0, -1, 0, -1)$ into the following basis web.

(2.10)
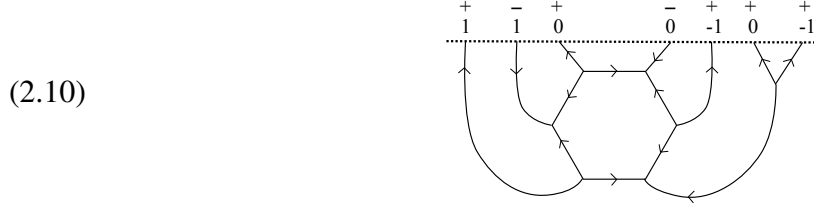
In addition, the growth algorithm has an inverse, called the *minimal cut path algorithm* [35], which we will not use in this paper.

Following Khovanov and Kuperberg in [35], we define a *flow f* on a web w to be an oriented subgraph that contains exactly two of the three edges incident to each trivalent vertex. The connected components of the flow are called the *flow lines*. The orientation of the flow lines need not agree with the orientation of w. Note that if w is closed, then each flow line is a closed cycle. At the boundary, the flow lines can be represented by a state string J. By convention, at the i-th boundary edge, we set $j_i = +1$ if the flow line is oriented upward, $j_i = -1$ if the flow line is oriented downward and $j_i = 0$ there is no flow line. The same convention determines a state for each edge of w.

Remark 2.3. Every flow determines a unique 3-coloring of w, with colours $-1, 0, 1$, satisfying the property that, for any trivalent vertex of w, the colors of the three incident edges are all distinct. These colorings are called *admissible* in [26].

Conversely, any such 3-coloring determines a unique flow on w. This correspondence determines a bijection between flows and admissible 3-colorings on w.

This remark will be important in Section 5.2 and in Section 5.3.

We will also say that any flow f that is compatible with a given state string J on the boundary of w *extends* J.

Given a web with a flow, denoted w_f, Khovanov and Kuperberg [35] attribute a *weight* to each trivalent vertex and each arc in w_f, as in Figures 2.11 and 2.12. The total weight of w_f is by definition the sum of the weights at all trivalent vertices and arcs.

(2.11)

(2.12)

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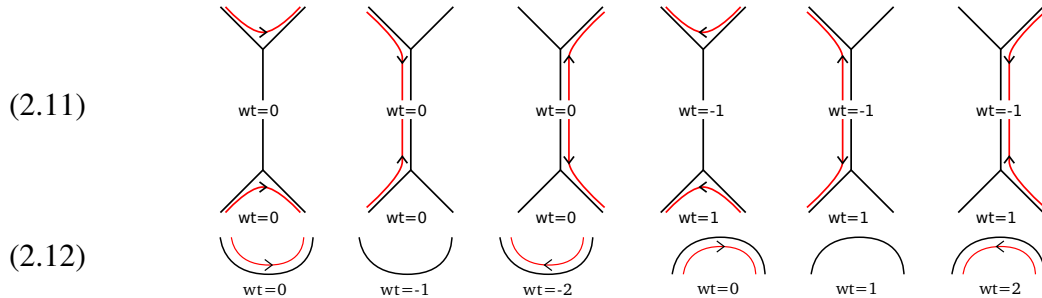
For example, the following web has weight -4.

(2.13)
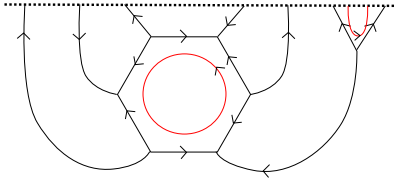
We can extend the table in (2.11) and (2.12) to calculate weights determined by flows on H's, so that it becomes easier to compute the weight of w_f when w is expressed using the growth algorithm (Definition 2.2).

Definition 2.4. [35] **(Canonical flows on basis webs)** Given a basis web w expressed using the growth algorithm. We define the *canonical flow* on w by the following rules.

(2.14)
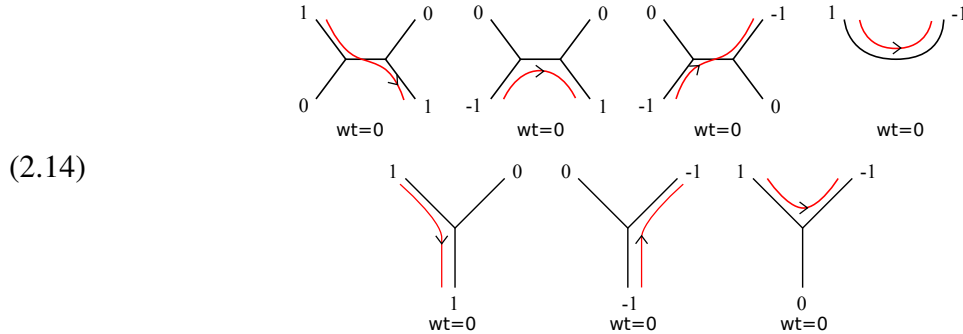
The canonical flow does not depend on the particular instance of the growth algorithm that we have chosen to obtain w.

Observe that the definition of the canonical flows implies the following lemma.

Lemma 2.5. *A basis web with the canonical flow has weight zero.*

Khovanov and Kuperberg [35] use a particular basis for V^+, denoted $\{e_1^+, e_2^+, e_3^+\}$, and also one for V^-, denoted $\{e_1^-, e_2^-, e_3^-\}$. Given (S, J), let

$$e_J^S = e_{j_1}^{s_1} \otimes \cdots \otimes e_{j_n}^{s_n}$$

be the *elementary tensor* corresponding to (S, J). Interpreting the webs in W^S as invariant tensors in V^S, we can write any web as a linear combination of elementary tensors. For each state string J', one can consider all flows on a given basis web $w = w_J^S$ which are compatible with J' on the boundary. The coefficient $c(S, J, J')$ of $e_{J'}^S$ in the linear combination corresponding to w satisfies

$$c(S, J, J') = \sum_f q^{\mathrm{wt}(w_f)},$$

where the sum is taken over all flows on w which are compatible with J' on the boundary.

Khovanov and Kuperberg prove the following theorem (Theorem 2 in [35]), which will be important for us in Section 5.

Theorem 2.6. *(**Khovanov-Kuperberg**) Given (S, J), we have*

$$w_J^S = e_J^S + \sum_{J' < J} c(S, J, J') e_{J'}^S$$

for some coefficients $c(S, J, J') \in \mathbb{N}[q, q^{-1}]$, where the state strings J and J' are ordered lexicographically.

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Remark 2.7. Khovanov and Kuperberg [35] show that B^S is not equal to the dual canonical basis of W^S. This follows from the fact that $c(S, J, J') \notin q^{-1}\mathbb{N}[q^{-1}]$, for general $J' < J$. In their Section 8, they give explicit counter-examples of elements $w \in B^S$ which admit non-canonical weight zero flows.

2.2. **Foams.** In this subsection we review the category $\mathbf{Foam}_3$ of $\mathfrak{sl}_3$-foams introduced by Khovanov in [34]. As a matter of fact, we will also need a deformation of Khovanov's original category, due to Gornik [26] in the context of matrix factorizations, and studied in [51] in the context of foams. Therefore, we introduce a parameter $c \in \mathbb{C}$ in $\mathbf{Foam}_3^c$, just as in [51], such that we get Khovanov's original category for $c = 0$ and, for any $c \neq 0$, the category $\mathbf{Foam}_3^c$ is isomorphic to Gornik's deformation (his original deformation was for $c = 1$). A big difference between these two specializations is that $\mathbf{Foam}_3^c$ is graded for $c = 0$ and filtered for any $c \neq 0$. In fact, for any $c \neq 0$, the associated graded category of $\mathbf{Foam}_3^c$ is isomorphic to $\mathbf{Foam}_3^0$.

We recall the following definitions as they appear in [51]. We note that the diagrams accompanying these definitions are taken, also, from [51].

A *pre-foam* is a cobordism with singular arcs between two webs. A singular arc in a pre-foam U is the set of points of U which have a neighborhood homeomorphic to the letter Y times an interval. Note that singular arcs are disjoint. Interpreted as morphisms, we read pre-foams from bottom to top by convention. Thus, pre-foam composition consists of placing one pre-foam on top of the other. The orientation of the singular arcs is, by convention, as in the diagrams below (called the *zip* and the *unzip* respectively).

We allow pre-foams to have dots that can move freely about the facet on which they belong, but we do not allow a dot to cross singular arcs.

By a *foam*, we mean a formal $\mathbb{C}$-linear combination of isotopy classes of pre-foams modulo the ideal generated by the set of relations $\ell = (3D, NC, S, \Theta)$ and the *closure relation*, as described below.

(3D)

(NC)

(S)

(Θ)

$$\beta\text{---}\bigcirc = \begin{cases} 1, & (\alpha, \beta, \delta) = (1, 2, 0) \text{ or a cyclic permutation,} \\ -1, & (\alpha, \beta, \delta) = (2, 1, 0) \text{ or a cyclic permutation,} \\ 0, & \text{else.} \end{cases}$$

The *closure relation*, i.e. any $\mathbb{C}$-linear combination of foams with the same boundary, is equal to zero if and only if any way of capping off these foams with a common foam yields a $\mathbb{C}$-linear combination of closed foams whose evaluation is zero.

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The relations in ℓ imply the following identities (for detailed proofs see [34]).

(Bamboo)

(RD)

(Bubble)

(DR)

(SqR)

(Dot Migration)
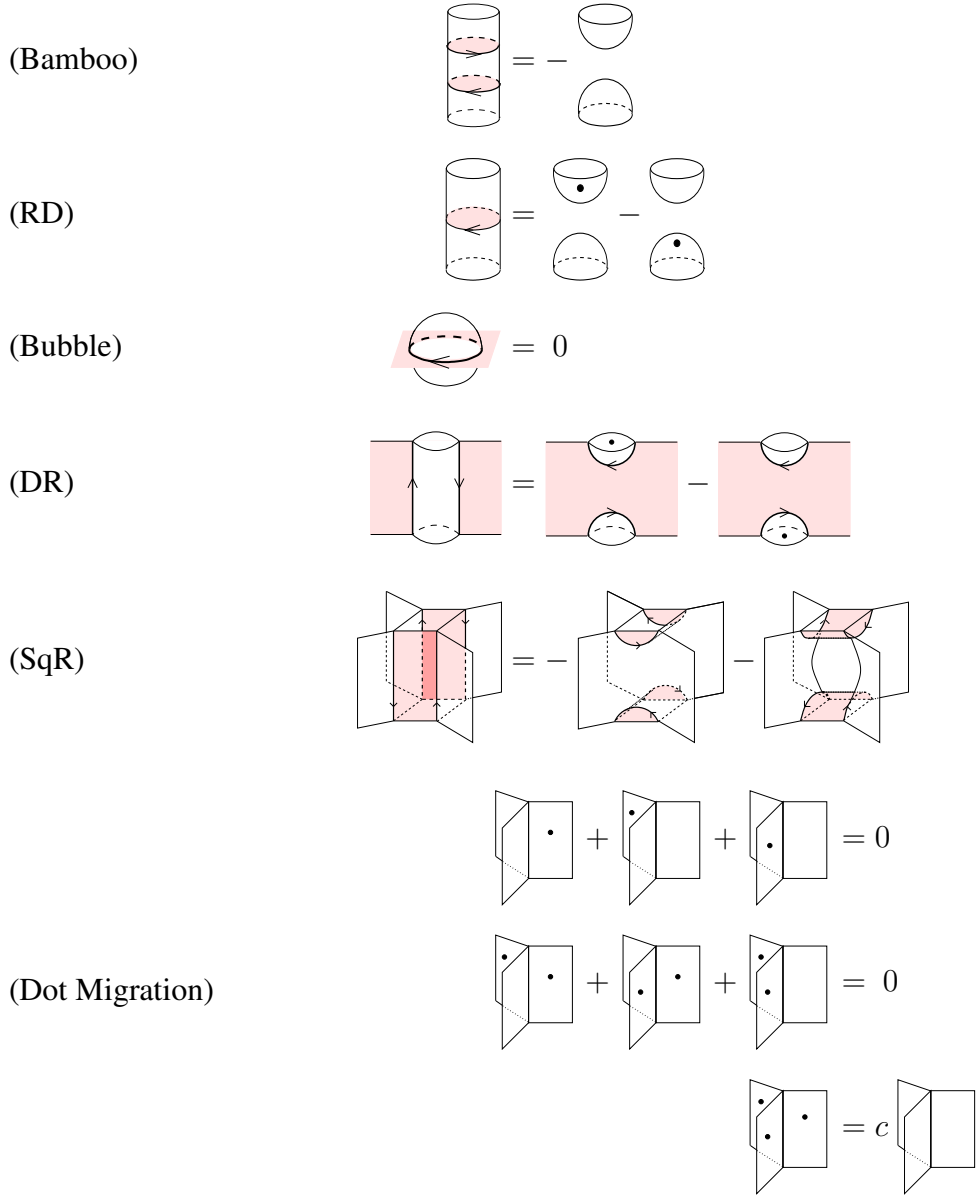
Definition 2.8. For any $c \in \mathbb{C}$, let $\mathbf{Foam}_3^c$ be the category whose objects are webs Γ lying inside a horizontal strip in $\mathbb{R}^2$, which is bounded by the lines $y = 0, 1$ containing the boundary points of Γ. The morphisms of $\mathbf{Foam}_3^c$ are $\mathbb{C}$-linear combinations of foams lying inside the horizontal strip bounded by $y = 0, 1$ times the unit interval. We require that the vertical boundary of each foam is a set (possibly empty) of vertical lines.

The q-*grading* of a foam U is defined as

$$q(U) = \chi(\partial U) - 2\chi(U) + 2d + b,$$

where χ denotes the Euler characteristic, d is the number of dots on U and b is the number of vertical boundary components. This makes $\mathbf{Foam}_3^0$ into a graded category. For any $c \neq 0$, this makes $\mathbf{Foam}_3^c$ into a filtered category, whose associated graded category is isomorphic to $\mathbf{Foam}_3^0$.

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Definition 2.9. [34] **(Foam Homology)** Given a web w the *foam homology* of w is the complex vector space, $\mathcal{F}^c(w)$, spanned by all foams

$$U : \emptyset \to w$$

in **Foam**$_3^c$.

The complex vector space $\mathcal{F}^c(w)$ is filtered/graded by the q-grading on foams and has rank $\langle w \rangle_{\mathrm{Kup}}$, where $\langle w \rangle_{\mathrm{Kup}}$ is the *Kuperberg bracket* computed recursively by the rules below.

(1) $\left\langle w \amalg \bigcirc \right\rangle_{\mathrm{Kup}} = [3]\langle w \rangle_{\mathrm{Kup}}.$

(2) $\langle \!\!-\!\!\!\prec\!\!\!\succ\!\!\!-\!\! \rangle_{\mathrm{Kup}} = [2]\langle \!\!-\!\!\!-\!\!\! \rangle_{\mathrm{Kup}}.$

(3) $\left\langle \boxed{\ } \right\rangle_{\mathrm{Kup}} = \left\langle \,\right)\left(\, \right\rangle_{\mathrm{Kup}} + \left\langle \asymp \right\rangle_{\mathrm{Kup}}.$

The relations above correspond to the decomposition of $\mathcal{F}^c(w)$ into direct summands. The idempotents corresponding to these direct summands are the terms on the r.h.s. of the relations (NC), (DR) and (SqR), respectively. For any $c \neq 0$, the complex vector space $\mathcal{F}^c(w)$ is filtered and its associated graded vector space is $\mathcal{F}^0(w)$. See [32] and [51] for details.

Remark 2.10. Given $u, v \in B^S$, the observations above and Theorem 2.6 show that there exists a homogeneous basis of $\mathcal{F}^0(u^*v)$ parametrized by the flows on u^*v. We have, in fact, constructed such a basis, but it is not unique. There is also no "preferred choice", unless one requires the basis to have other nice properties, e.g. in the $\mathfrak{sl}_2$ case, Brundan and Stroppel prove that there is a cellular basis of H^n. The construction of a "good" basis of the $\mathfrak{sl}_3$-web algebra K^S (and similarly for Gornik's deformation G^S) is still work in progress and will, hopefully, be the contents of a subsequent paper. Although we do not need such a basis in this paper, it is important that the reader keep this remark in mind while reading Section 5.

2.3. Quantum 2-algebras.

2.3.1. *The quantum general and special linear algebras.* First we recall the quantum general and special linear algebras. Most parts in this section are copied from section two and three in [50].

The $\mathfrak{gl}_n$-weight lattice is isomorphic to $\mathbb{Z}^n$. Let $\epsilon_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{Z}^n$, with 1 being on the i-th coordinate, and $\alpha_i = \epsilon_i - \epsilon_{i+1} = (0, \ldots, 1, -1, \ldots, 0) \in \mathbb{Z}^n$, for $i = 1, \ldots, n-1$. Recall that the Euclidean inner product on $\mathbb{Z}^n$ is defined by $(\epsilon_i, \epsilon_j) = \delta_{i,j}$.

Definition 2.11. For $n \in \mathbb{N}_{>1}$ the *quantum general linear algebra* $\mathbf{U}_q(\mathfrak{gl}_n)$ is the associative unital $\mathbb{Q}(q)$-algebra generated by K_i and K_i^{-1}, for $1, \ldots, n$, and $E_{\pm i}$ (beware that some authors use F_i instead of E_{-i}), for $i = 1, \ldots, n-1$, subject to the relations

$$K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1,$$

$$E_i E_{-j} - E_{-j} E_i = \delta_{i,j} \frac{K_i K_{i+1}^{-1} - K_i^{-1} K_{i+1}}{q - q^{-1}},$$

$$K_i E_{\pm j} = q^{\pm(\epsilon_i, \alpha_j)} E_{\pm j} K_i,$$

$$E_{\pm i}^2 E_{\pm j} - (q + q^{-1}) E_{\pm i} E_{\pm j} E_{\pm i} + E_{\pm j} E_{\pm i}^2 = 0, \qquad \text{if} \quad |i - j| = 1,$$

$$E_{\pm i} E_{\pm j} - E_{\pm j} E_{\pm i} = 0, \qquad \text{else.}$$

Definition 2.12. For $n \in \mathbb{N}_{>1}$ the *quantum special linear algebra* $\mathbf{U}_q(\mathfrak{sl}_n) \subseteq \mathbf{U}_q(\mathfrak{gl}_n)$ is the unital $\mathbb{Q}(q)$-subalgebra generated by $K_i K_{i+1}^{-1}$ and $E_{\pm i}$, for $i = 1, \ldots, n-1$.

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Recall that the $\mathbf{U}_q(\mathfrak{sl}_n)$-weight lattice is isomorphic to $\mathbb{Z}^{n-1}$. Suppose that V is a $\mathbf{U}_q(\mathfrak{gl}_n)$-weight representation with weights $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$, i.e.

$$V \cong \bigoplus_\lambda V_\lambda,$$

and K_i acts as multiplication by q^{λ_i} on V_λ. Then V is also a $\mathbf{U}_q(\mathfrak{sl}_n)$-weight representation with weights $\overline{\lambda} = (\overline{\lambda}_1, \ldots, \overline{\lambda}_{n-1}) \in \mathbb{Z}^{n-1}$ such that $\overline{\lambda}_j = \lambda_j - \lambda_{j+1}$ for $j = 1, \ldots, n-1$.

Conversely, given a $\mathbf{U}_q(\mathfrak{sl}_n)$-weight representation with weights $\mu = (\mu_1, \ldots, \mu_{n-1})$, there is not a unique choice of $\mathbf{U}_q(\mathfrak{gl}_n)$-action on V. We can fix this by choosing the action of $K_1, \cdots, K_n$. In terms of weights, this corresponds to the observation that, for any $d \in \mathbb{Z}$, the equations

(2.15) $$\lambda_i - \lambda_{i+1} = \mu_i,$$

(2.16) $$\sum_{i=1}^n \lambda_i = d,$$

determine $\lambda = (\lambda_1, \ldots, \lambda_n)$ uniquely, if there exists a solution to (2.15) and (2.16) at all. To fix notation, we define the map $\phi_{n,d} \colon \mathbb{Z}^{n-1} \to \mathbb{Z}^n \cup \{*\}$ by

$$\phi_{n,d}(\mu) = \lambda,$$

if (2.15) and (2.16) have a solution, and we put $\phi_{n,d}(\mu) = *$ otherwise.

Note that $\mathbf{U}_q(\mathfrak{gl}_n)$ and $\mathbf{U}_q(\mathfrak{sl}_n)$ are both Hopf algebras, which implies that the tensor product of two of their representations is a representation again. Moreover, the duals of representations are representations and there is a trivial representation.

Both $\mathbf{U}_q(\mathfrak{gl}_n)$ and $\mathbf{U}_q(\mathfrak{sl}_n)$ have plenty of non-weight representations, but we won't discuss them in the paper. Therefore we can restrict our attention to the Beilinson-Lusztig-MacPherson [2] idempotent version of these quantum groups, denoted $\dot{\mathbf{U}}_q(\mathfrak{gl}_n)$ and $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$ respectively. It is worth noting that such algebras can be seen as 1-categories.

To understand their definition, recall that K_i acts as q^{λ_i} on the λ-weight space of any weight representation. For each $\lambda \in \mathbb{Z}^n$ adjoin an idempotent 1_λ to $\mathbf{U}_q(\mathfrak{gl}_n)$ and add the relations

$$1_\lambda 1_\mu = \delta_{\lambda,\nu} 1_\lambda,$$
$$E_{\pm i} 1_\lambda = 1_{\lambda \pm \alpha_i} E_{\pm i},$$
$$K_i 1_\lambda = q^{\lambda_i} 1_\lambda.$$

Definition 2.13. The idempotented quantum general linear algebra is defined by

$$\dot{\mathbf{U}}_q(\mathfrak{gl}_n) = \bigoplus_{\lambda,\mu \in \mathbb{Z}^n} 1_\lambda \mathbf{U}_q(\mathfrak{gl}_n) 1_\mu.$$

Let $I = \{1, 2, \ldots, n-1\}$. In the sequel we use *signed sequences* $\underline{i} = (\alpha_1 i_1, \ldots, \alpha_m i_m)$, for any $m \in \mathbb{N}$, $\alpha_j \in \{\pm 1\}$ and $i_j \in I$. The set of signed sequences we denote SiSeq.

For such an $\underline{i} = (\alpha_1 i_1, \ldots, \alpha_{n-1} i_{n-1})$ we define

$$E_{\underline{i}} = E_{\alpha_1 i_1} \cdots E_{\alpha_{n-1} i_{n-1}}$$

and we define $\underline{i}_\Lambda \in \mathbb{Z}^n$ to be the n-tuple such that

$$E_{\underline{i}} 1_\mu = 1_{\mu + \underline{i}_\Lambda} E_{\underline{i}}.$$

Similarly, for $\mathbf{U}_q(\mathfrak{sl}_n)$, adjoin an idempotent 1_μ for each $\mu \in \mathbb{Z}^{n-1}$ and add the relations

$$1_\mu 1_\nu = \delta_{\mu,\nu} 1_\lambda,$$
$$E_{\pm i} 1_\mu = 1_{\mu \pm \overline{\alpha}_i} E_{\pm i}, \quad \text{with } \overline{\alpha}_i = \alpha_i - \alpha_{i+1},$$
$$K_i K_{i+1}^{-1} 1_\mu = q^{\mu_i} 1_\mu.$$

Definition 2.14. The idempotented quantum special linear algebra is defined by

$$\dot{\mathbf{U}}_q(\mathfrak{sl}_n) = \bigoplus_{\mu,\nu \in \mathbb{Z}^{n-1}} 1_\mu \mathbf{U}_q(\mathfrak{sl}_n) 1_\nu.$$

Note that $\dot{\mathbf{U}}_q(\mathfrak{gl}_n)$ and $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$ are both non-unital algebras, because their units would have to be equal to the infinite sum of all their idempotents.

Furthermore, the only $\mathbf{U}_q(\mathfrak{gl}_n)$ and $\mathbf{U}_q(\mathfrak{sl}_n)$-representations which factor through $\dot{\mathbf{U}}_q(\mathfrak{gl}_n)$ and $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$ respectively are the weight representations. Finally, note that there is no embedding of $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$ into $\dot{\mathbf{U}}_q(\mathfrak{gl}_n)$, because there is no embedding of the $\mathfrak{sl}_n$-weights into the $\mathfrak{gl}_n$-weights.

2.3.2. *The q-Schur algebra.* Let $d \in \mathbb{N}$ and let V be the natural n-dimensional representation of $\mathbf{U}_q(\mathfrak{gl}_n)$. Define

$$\Lambda(n,d) = \{\lambda \in \mathbb{N}^n \mid \sum_{i=1}^n \lambda_i = d\} \quad \text{and}$$

$$\Lambda^+(n,d) = \{\lambda \in \Lambda(n,d) \mid d \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}.$$

Recall that the weights in $V^{\otimes d}$ are precisely the elements of $\Lambda(n,d)$, and that the highest weights are the elements of $\Lambda^+(n,d)$. The highest weights correspond exactly to the irreducibles V_λ that show up in the decomposition of $V^{\otimes d}$.

We can define the q-Schur algebra as follows.

Definition 2.15. The *q-Schur algebra* $S_q(n,d)$ is the image of the representation $\psi_{n,d}$ defined by

$$\psi_{n,d} \colon \mathbf{U}_q(\mathfrak{gl}_n) \to \mathrm{End}_{\mathbb{C}}(V^{\otimes d}).$$

For $\lambda \in \Lambda^+(n,d)$, the $\mathbf{U}_q(\mathfrak{gl}_n)$-action on V_λ factors through the projection

$$\psi_{n,d} \colon \mathbf{U}_q(\mathfrak{gl}_n) \to S_q(n,d).$$

This way we obtain all irreducible representations of $S_q(n,d)$. Note that this also implies that all representations of $S_q(n,d)$ have a weight decomposition. As a matter of fact, it is well-known that

$$S_q(n,d) \cong \prod_{\lambda \in \Lambda^+(n,d)} \mathrm{End}_{\mathbb{C}}(V_\lambda).$$

Therefore $S_q(n,d)$ is a finite dimensional, semisimple, unital algebra and its dimension is equal to

$$\sum_{\lambda \in \Lambda^+(n,d)} \dim(V_\lambda)^2 = \binom{n^2 + d - 1}{d}.$$

Since $V^{\otimes d}$ is a weight representation, $\psi_{n,d}$ gives rise to a homomorphism $\dot{\mathbf{U}}_q(\mathfrak{gl}_n) \to S_q(n,d)$, for which we use the same notation. This map is still surjective and Doty and Giaquinto, in Theorem 2.4 of [20], showed that the kernel of $\psi_{n,d}$ is equal to the ideal generated by all idempotents 1_λ such that $\lambda \notin \Lambda(n,d)$. Clearly the image of $\psi_{n,d}$ is isomorphic to $S_q(n,d)$. By the above observations, we see that $S_q(n,d)$ has a Serre presentation. As a matter of fact, by Corollary 4.3.2 in [18], this

presentation is simpler than that of $\dot{\mathbf{U}}_q(\mathfrak{gl}_n)$, i.e. one does not need to impose the last two Serre relations, involving cubical terms, because they are implied by the other relations and the finite dimensionality.

Lemma 2.16. $S_q(n,d)$ *is isomorphic to the associative, unital $\mathbb{Q}(q)$-algebra generated by 1_λ, for $\lambda \in \Lambda(n,d)$, and $E_{\pm i}$, for $i = 1, \ldots, n-1$, subject to the relations*

$$(2.17) \qquad\qquad 1_\lambda 1_\mu = \delta_{\lambda,\mu} 1_\lambda,$$

$$(2.18) \qquad\qquad \sum_{\lambda \in \Lambda(n,d)} 1_\lambda = 1,$$

$$(2.19) \qquad\qquad E_{\pm i} 1_\lambda = 1_{\lambda \pm \alpha_i} E_{\pm i}, \quad \text{with } \alpha_i = \epsilon_i - \epsilon_{i+1} = (0, \ldots, 1, -1, \ldots, 0),$$

$$(2.20) \qquad E_i E_{-j} - E_{-j} E_i = \delta_{ij} \sum_{\lambda \in \Lambda(n,d)} [\overline{\lambda}_i] 1_\lambda.$$

We use the convention that $1_\mu X 1_\lambda = 0$, if μ or λ is not contained in $\Lambda(n,d)$. Again $[a]$ denotes the quantum integer from before.

Although there is no embedding of $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$ into $\dot{\mathbf{U}}_q(\mathfrak{gl}_n)$, the projection

$$\psi_{n,d} \colon \mathbf{U}_q(\mathfrak{gl}_n) \to S_q(n,d)$$

can be restricted to $\mathbf{U}_q(\mathfrak{sl}_n)$ and is still surjective. This gives rise to the surjection

$$\psi_{n,d} \colon \dot{\mathbf{U}}_q(\mathfrak{sl}_n) \to S_q(n,d),$$

defined by

$$(2.21) \qquad\qquad \psi_{n,d}(E_{\pm i} 1_\lambda) = E_{\pm i} 1_{\phi_{n,d}(\lambda)},$$

where $\phi_{n,d}$ was defined below equations (2.15) and (2.16). By convention we put $1_* = 0$.

2.3.3. *The general and special quantum 2-algebras.* We note that a lot of this section is copied from [50]. The reader can find even more details there.

Let $\mathcal{U}(\mathfrak{sl}_n)$ be Khovanov and Lauda's [36] diagrammatic categorification of $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$. In [50] it was shown that there is a quotient 2-category of $\mathcal{U}(\mathfrak{sl}_n)$, denoted by $\mathcal{S}(n,n)$, which categorifies $S_q(n,n)$.

We recall the definition of these categorified quantum algebras and some notions from above. As before, let $I = \{1, 2, \ldots, n-1\}$. Again, we use *signed sequences* $\underline{i} = (\alpha_1 i_1, \ldots, \alpha_m i_m)$, for any $m \in \mathbb{N}$, $\alpha_j \in \{\pm 1\}$ and $i_j \in I$, and the set of signed sequences is denoted SiSeq.

For $\underline{i} = (\alpha_1 i_1, \ldots, \alpha_m i_m) \in$ SiSeq we define $\underline{i}_\Lambda = \alpha_1(i_1)_\Lambda + \cdots + \alpha_m(i_m)_\Lambda$, where

$$(i_j)_\Lambda = (0, 0, \ldots, 1, -1, 0 \ldots, 0),$$

such that the vector starts with $i_j - 1$ and ends with $k - 1 - i_j$ zeros. We also define the symmetric $\mathbb{Z}$-valued bilinear form on $\mathbb{C}[I]$ by $i \cdot i = 2$, $i \cdot (i+1) = -1$ and $i \cdot j = 0$, for $|i - j| > 1$. Recall that $\overline{\lambda}_i = \lambda_i - \lambda_{i+1}$.

We first recall the definition, given in [50], of the 2-category which conjecturally categorifies $\dot{\mathbf{U}}_q(\mathfrak{gl}_n)$. It is a straightforward adaptation of Khovanov and Lauda's $\mathcal{U}(\mathfrak{sl}_n)$.

Definition 2.17. $\mathcal{U}(\mathfrak{gl}_n)$ is an additive $\mathbb{C}$-linear 2-category. The 2-category $\mathcal{U}(\mathfrak{gl}_n)$ consists of

- Objects are $\lambda \in \mathbb{Z}^n$.

The hom-category $\mathcal{U}(\mathfrak{gl}_n)(\lambda, \lambda')$ between two objects λ, λ' is an additive $\mathbb{C}$-linear category consisting of the following.

- Objects[2] of $\mathcal{U}(\mathfrak{gl}_n)(\lambda, \lambda')$, i.e. for 1-morphism in $\mathcal{U}(\mathfrak{gl}_n)$ from λ to λ' is a formal finite direct sum of 1-morphisms

$$\mathcal{E}_{\underline{i}}\mathbf{1}_\lambda\{t\} = \mathbf{1}_{\lambda'}\mathcal{E}_{\underline{i}}\mathbf{1}_\lambda\{t\} = \mathcal{E}_{\alpha_1 i_1} \cdots \mathcal{E}_{\alpha_m i_m}\mathbf{1}_\lambda\{t\}$$

for any $t \in \mathbb{Z}$ and signed sequence $\underline{i} \in \mathrm{SiSeq}$ such that $\lambda' = \lambda + \underline{i}_\Lambda$ and $\lambda, \lambda' \in \mathbb{Z}^n$.

- Morphisms of $\mathcal{U}(\mathfrak{gl}_n)(\lambda, \lambda')$, i.e. for 1-morphisms $\mathcal{E}_{\underline{i}}\mathbf{1}_\lambda\{t\}$ and $\mathcal{E}_{\underline{j}}\mathbf{1}_\lambda\{t'\}$ in $\mathcal{U}(\mathfrak{gl}_n)$, the hom sets $\mathcal{U}(\mathfrak{gl}_n)(\mathcal{E}_{\underline{i}}\mathbf{1}_\lambda\{t\}, \mathcal{E}_{\underline{j}}\mathbf{1}_\lambda\{t'\})$ of $\mathcal{U}(\mathfrak{gl}_n)(\lambda, \lambda')$ are graded $\mathbb{C}$-vector spaces given by linear combinations of degree $t - t'$ diagrams, modulo certain relations, built from composites of the following.

(i) Degree zero identity 2-morphisms 1_x for each 1-morphism x in $\mathcal{U}(\mathfrak{gl}_n)$; the identity 2-morphisms $1_{\mathcal{E}_{+i}\mathbf{1}_\lambda}\{t\}$ and $1_{\mathcal{E}_{-i}\mathbf{1}_\lambda}\{t\}$, for $i \in I$, are represented graphically by
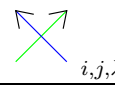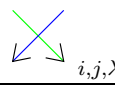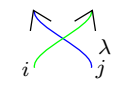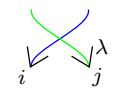

for any $\lambda + i_\Lambda \in \mathbb{Z}^n$ and any $\lambda - i_\Lambda \in \mathbb{Z}^n$, respectively.

More generally, for a signed sequence $\underline{i} = (\alpha_1 i_1, \alpha_2 i_2, \ldots \alpha_m i_m)$, the identity $1_{\mathcal{E}_{\underline{i}}\mathbf{1}_\lambda\{t\}}$ 2-morphism is represented as

where the strand labeled i_k is oriented up if $\alpha_k = +$ and oriented down if $\alpha_k = -$. We will often place labels with no sign on the side of a strand and omit the labels at the top and bottom. The signs can be recovered from the orientations on the strands.

(ii) Recall that $- \cdot -$ is the bilinear form from above. For each $\lambda \in \mathbb{Z}^n$ the 2-morphisms

Notation:				
2-morphism:				
Degree:	$i \cdot i$	$i \cdot i$	$-i \cdot j$	$-i \cdot j$

---

[2]We refer to objects of the category $\mathcal{U}(\mathfrak{gl}_n)(\lambda, \lambda')$ as 1-morphisms of $\mathcal{U}(\mathfrak{gl}_n)$. Likewise, the morphisms of $\mathcal{U}(\mathfrak{gl}_n)(\lambda, \lambda')$ are called 2-morphisms in $\mathcal{U}(\mathfrak{gl}_n)$.

Notation:	$_{i,\lambda}$	$_{i,\lambda}$	$_{i,\lambda}$	$_{i,\lambda}$
2-morphism:				
Degree:	$1 - \overline{\lambda}_i$	$1 + \overline{\lambda}_i$	$1 + \overline{\lambda}_i$	$1 - \overline{\lambda}_i$

- Biadjointness and cyclicity.
    (i) $\mathbf{1}_{\lambda+i_\Lambda}\mathcal{E}_{+i}\mathbf{1}_\lambda$ and $\mathbf{1}_\lambda\mathcal{E}_{-i}\mathbf{1}_{\lambda+i_\Lambda}$ are biadjoint, up to grading shifts, i.e.

(2.22)

(2.23)

(ii)

(2.24)

(iii) All 2-morphisms are cyclic with respect to the above biadjoint structure. This is ensured by the relations (2.24), and, for arbitrary i, j, the relations

(2.25)

Note that we can take either the first or the last diagram above as the definition of the up-side-down crossing. The cyclic condition on 2-morphisms, expressed by (2.24) and (2.25), ensures that diagrams related by isotopy represent the same 2-morphism in $\mathcal{U}(\mathfrak{gl}_n)$.

It will be convenient to introduce degree zero 2-morphisms.

(2.26)

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(2.27)

where the second equality in (2.26) and (2.27) follow from (2.25).

(iv) All dotted bubbles of negative degree are zero. That is,

(2.28) $\quad$ $\lambda = 0, \quad$ if $m < \overline{\lambda}_i - 1, \quad$ $\lambda = 0, \quad$ if $m < -\overline{\lambda}_i - 1$

for all $m \in \mathbb{Z}_+$, where a dot carrying a label m denotes the m-fold iterated vertical composite of $\uparrow_{i,\lambda}$ or $\downarrow_{i,\lambda}$ depending on the orientation. A dotted bubble of degree zero equals ± 1, that is

(2.29) $\quad$ $\lambda = (-1)^{\lambda_{i+1}}, \quad$ for $\overline{\lambda}_i \geq 1, \quad$ $\lambda = (-1)^{\lambda_{i+1}-1}, \quad$ for $\overline{\lambda}_i \leq -1$.

(v) For the following relations we employ the convention that all summations are increasing, so that a summation of the form $\sum_{f=0}^{m}$ is zero if $m < 0$.

(2.30)

(2.31)

for all $\lambda \in \mathbb{Z}^n$ (see (2.26) and (2.27) for the definition of sideways crossings). Notice that for some values of λ the dotted bubbles appearing above have negative labels. A composite of $\uparrow_{i,\lambda}$ or $\downarrow_{i,\lambda}$ with itself a negative number of times does not make sense. These dotted bubbles with negative labels, called *fake bubbles*, are formal symbols inductively defined by the equation

(2.32)

and the additional condition

 $= (-1)^{\lambda_{i+1}} \quad$ and $\quad$ $= (-1)^{\lambda_{i+1}-1}, \quad$ if $\overline{\lambda}_i = 0$.

Although the labels are negative for fake bubbles, one can check that the overall degree of each fake bubble is still positive, so that these fake bubbles do not violate the positivity of dotted bubble axiom. The above equation, called the infinite Grassmannian relation, remains valid even in high degree when most of the bubbles involved are not fake bubbles.

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(vi) The NilHecke relations are

(2.33)
$$\bigvee_{i\ \ i}\lambda = 0, \qquad \bigtimes\!\bigtimes_{i\ \ i\ \ i}\lambda = \bigtimes\!\bigtimes_{i\ \ i\ \ i}\lambda$$

(2.34)
$$\Big\uparrow_{i}\ \Big\uparrow_{i}\lambda \;=\; \bigtimes_{i\ \ i}\lambda \;-\; \bigtimes_{i\ \ i}\lambda \;=\; \bigtimes_{i\ \ i}\lambda \;-\; \bigtimes_{i\ \ i}\lambda.$$

- We have for $i \neq j$

(2.35)
$$\bigvee_{i\ \ j}\lambda = \Big\uparrow_{i}\ \bigvee_{j}\lambda \quad \text{and} \quad \bigvee_{i\ \ j}\lambda = \bigvee_{i}\ \Big\uparrow_{j}\lambda$$

- (i) We have for $i \neq j$

(2.36)
$$\bigvee_{i\ \ j}\lambda = \begin{cases} \Big\uparrow_{i}\ \Big\uparrow_{j}\lambda, & \text{if } i\cdot j = 0, \\[2ex] (i-j)\left(\Big\uparrow_{i}\ \Big\uparrow_{j}\lambda - \Big\uparrow_{i}\ \Big\uparrow_{j}\lambda \right), & \text{if } i\cdot j = -1. \end{cases}$$

Notice that $(i-j)$ is just a sign, which takes into account the standard orientation of the Dynkin diagram.

(2.37)
$$\bigtimes_{i\ \ j}\lambda = \bigtimes_{i\ \ j}\lambda \quad \text{and} \quad \bigtimes_{i\ \ j}\lambda = \bigtimes_{i\ \ j}\lambda.$$

(ii) Unless $i = k$ and $i\cdot j = -1$, we have

(2.38)
$$\bigtimes\!\bigtimes_{i\ \ j\ \ k}\lambda = \bigtimes\!\bigtimes_{i\ \ j\ \ k}\lambda$$

(iii) We have for $i\cdot j = -1$

(2.39)
$$\bigtimes\!\bigtimes_{i\ \ j\ \ i}\lambda - \bigtimes\!\bigtimes_{i\ \ j\ \ i}\lambda = (i-j)\Big\uparrow_{i}\ \Big\uparrow_{j}\ \Big\uparrow_{i}\lambda.$$

- The additive, linear composition functor $\mathcal{U}(\mathfrak{gl}_n)(\lambda, \lambda') \times \mathcal{U}(\mathfrak{gl}_n)(\lambda', \lambda'') \to \mathcal{U}(\mathfrak{gl}_n)(\lambda, \lambda'')$ is given on 1-morphisms of $\mathcal{U}(\mathfrak{gl}_n)$ by

(2.40)
$$\mathcal{E}_{\underline{j}}\mathbf{1}_{\lambda'}\{t'\} \times \mathcal{E}_{\underline{i}}\mathbf{1}_{\lambda}\{t\} \mapsto \mathcal{E}_{\underline{ji}}\mathbf{1}_{\lambda}\{t + t'\}$$

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for $\underline{i}_\Lambda = \lambda - \lambda'$, and on 2-morphisms of $\mathcal{U}(\mathfrak{gl}_n)$ by juxtaposition of diagrams, e.g.
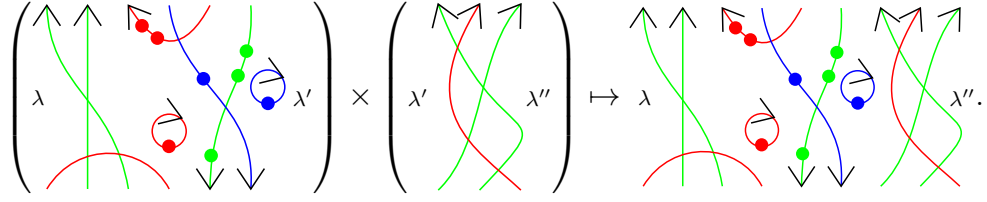
This concludes the definition of $\mathcal{U}(\mathfrak{gl}_n)$.

Note that for two 1-morphisms x and y in $\mathcal{U}(\mathfrak{gl}_n)$ the 2-hom-space $\mathrm{Hom}_{\mathcal{U}(\mathfrak{gl}_n)}(x, y)$ only contains 2-morphisms of degree zero and is therefore finite dimensional. Following Khovanov and Lauda we introduce the graded 2-hom-space

$$\mathrm{HOM}_{\mathcal{U}(\mathfrak{gl}_n)}(x, y) = \bigoplus_{t \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{U}(\mathfrak{gl}_n)}(x\{t\}, y),$$

which is infinite dimensional. We also define the 2-category $\mathcal{U}(\mathfrak{gl}_n)^*$ which has the same objects and 1-morphisms as $\mathcal{U}(\mathfrak{gl}_n)$, but for two 1-morphisms x and y the vector space of 2-morphisms is defined by

$$(2.41) \qquad\qquad \mathcal{U}(\mathfrak{gl}_n)^*(x, y) = \mathrm{HOM}_{\mathcal{U}(\mathfrak{gl}_n)}(x, y).$$

It should be noted that $\mathcal{U}(\mathfrak{sl}_n)$ is defined just as $\mathcal{U}(\mathfrak{gl}_n)$, but labeling all the regions of the diagrams with $\mathfrak{sl}_n$-weights, i.e. elements of $\mathbb{Z}^{n-1}$. Note that one also has to re-normalize the signs of the left cups and caps, so that the bubble relations all become dependent on the $\mathfrak{sl}_n$-weights. For much more details, see [50].

2.3.4. *The q-Schur 2-algebra.* The categorification of $S_q(n, n)$ is now obtained from $\mathcal{U}(\mathfrak{gl}_n)$ by taking a quotient.

Definition 2.18. The 2-category $\mathcal{S}(n, n)$ is the quotient of $\mathcal{U}(\mathfrak{gl}_n)$ by the ideal generated by all 2-morphisms containing a region with a label not in $\Lambda(n, n)$.

We remark that we only put real bubbles, whose interior has a label outside $\Lambda(n, n)$, equal to zero. To see what happens to a fake bubble, one first has to write it in terms of real bubbles with the opposite orientation using the infinite Grassmannian relation (2.32).

A main result of [50], given in Theorem 7.11 in that paper, is the following.

Theorem 2.19. *Let $\dot{\mathcal{S}}(n, n)$ denote the Karoubi envelope of $\mathcal{S}(n, n)$. The $\mathbb{Q}(q)$-linear map*

$$\gamma_S \colon S_q(n, n) \to K_0^\oplus(\dot{\mathcal{S}}(n, n))_{\mathbb{Q}(q)},$$

determined by

$$\gamma_S(E_{\underline{i}} 1_\lambda) = [\mathcal{E}_{\underline{i}} 1_\lambda]$$

is an isomorphism of algebras.

Recall also (see Definition 4.1 in [50]) that there is an essentially surjective and full additive 2-functor

$$\Psi_{n,n} \colon \mathcal{U}(\mathfrak{sl}_n) \to \mathcal{S}(n, n),$$

whose precise definition is not relevant here. Up to signs related to cups and caps, it is obtained by mapping any string diagram to itself and applying $\phi_{n,n}$ to the labels of the regions. By convention, any diagram with a region labeled $*$ is taken to be zero. It is important to note that

$$K_0^\oplus(\Psi_{n,n})_{\mathbb{Q}(q)} \colon K_0^\oplus(\mathcal{U}(\mathfrak{sl}_n))_{\mathbb{Q}(q)} \to K_0^\oplus(\dot{\mathcal{S}}(n,n))_{\mathbb{Q}(q)}$$

corresponds to the aforementioned surjective homomorphism

$$\psi_{n,n} \colon \dot{\mathbf{U}}_q(\mathfrak{sl}_n) \to S_q(n,n).$$

2.3.5. *The cyclotomic KLR-algebras.* In this subsection, we recall the definition of the cyclotomic KLR-algebras, due to Khovanov and Lauda [38] and, independently, to Rouquier [60]. We also recall two important results about them.

Fix $\nu \in \mathbb{Z}_{\leq 0}[I]$. Let $\mathrm{Seq}(\nu)$ be the set of all sequences $\underline{i} = (-i_1, -i_2, \cdots, -i_m)$, such that $i_k \in I$ for each k and $\nu_j = \#\{k \mid i_k = j\}$.

Definition 2.20. For any $\underline{i}, \underline{j} \in \mathrm{Seq}(\nu)$ and any $\mathfrak{gl}_n$-weight $\lambda \in \mathbb{Z}^n$, let

$$_{\underline{i}}R^\nu_{\underline{j}} \subset \mathrm{End}_{\mathcal{U}(\mathfrak{gl}_n)}(\mathcal{E}_{\underline{j}}1_\lambda, \mathcal{E}_{\underline{i}}1_\lambda)$$

be the subalgebra containing only diagrams which are oriented downwards. So, only strands oriented downwards with dots and crossings are allowed. No strands oriented upwards, no cups and no caps. The relations in $\mathcal{U}(\mathfrak{gl}_n)$ involving only downward strands do not depend on λ. Therefore, the definition above makes sense. In [38], the authors do not label the regions of the diagrams.

Then R^ν is defined as

$$R^\nu = \bigoplus_{\underline{i},\underline{j} \in \mathrm{Seq}(\nu)} {}_{\underline{i}}R^\nu_{\underline{j}}.$$

The ring R is defined as

$$R = \bigoplus_{\nu \in \mathbb{Z}_{\leq 0}[I]} R^\nu.$$

As remarked above, the definition of R^ν does not depend on λ. However, when we use a particular λ, we will write $R^\nu 1_\lambda$.

Note that R^ν is unital, whereas R has infinitely many idempotents.

Let R^ν-p$\mathbf{Mod}_{\mathrm{gr}}$ be the category of graded, finitely generated, projective R^ν-modules and define

$$R\text{-p}\mathbf{Mod}_{\mathrm{gr}} = \bigoplus_{\nu \in \mathbb{Z}_{\leq 0}[I]} R^\nu\text{-p}\mathbf{Mod}_{\mathrm{gr}}.$$

In Proposition 3.18 in [38], Khovanov and Lauda showed that R-p$\mathbf{Mod}_{\mathrm{gr}}$ categorifies the negative half of $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$ and R^ν-p$\mathbf{Mod}_{\mathrm{gr}}$ categorifies the ν-root space.

We can now recall the definition of the *cyclotomic KLR-algebras*. The reader can find more details in [38] or [60], for example.

Definition 2.21. Choose a dominant $\dot{\mathbf{U}}_q(\mathfrak{gl}_n)$-weight $\lambda \in \Lambda(n,n)^+$. Let R^ν_λ be the quotient algebra of $R^\nu 1_\lambda$ by the ideal generated by all diagrams of the form

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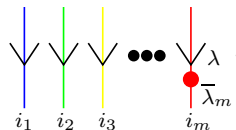
Recall that $\overline{\lambda}_m = \lambda_m - \lambda_{m+1}$, the m-th entry of the $\mathfrak{sl}_n$-weight corresponding to λ.

Define

$$R_\lambda = \bigoplus_{\nu \in \mathbb{Z}_{\leq 0}[I]} R_\lambda^\nu.$$

Note that we mod out by relations involving dots on the last strand, rather than the first strand as in [38]. This is to make the definition compatible with the other definitions in our paper.

It turns out that R_λ is a finite dimensional, unital algebra (Corollary 2.2 in [6]). Let R_λ-p$\mathbf{Mod}_{\mathrm{gr}}$ be its category of finite dimensional, graded, projective modules and R_λ-$\mathbf{Mod}_{\mathrm{gr}}$ its category of all finite dimensional modules.

There is a strong $\mathfrak{sl}_n$-2-representation on R_λ-$\mathbf{Mod}_{\mathrm{gr}}$, which can be restricted to R_λ-p$\mathbf{Mod}_{\mathrm{gr}}$ (Section 4.4 in [7]). This basically means that the 1-morphisms of $\mathcal{U}(\mathfrak{sl}_n)$ act as endofunctors and the 2-functors as natural transformations, such that the relations in $\mathcal{U}(\mathfrak{sl}_n)$ are preserved. For a precise definition of a strong 2-representation see [16], which is almost equal to Rouquier's definition of a 2-representation in [60]) but satisfies one extra condition.

Brundan and Kleshchev [7] proved the following result, which was first conjectured by Khovanov and Lauda (see also [31], [45], [68] and [69]). From now on, we will always use the notation

$$\mathcal{V}_\lambda = R_\lambda\text{-p}\mathbf{Mod}_{\mathrm{gr}}.$$

Since $\mathcal{V}_\lambda$ is a strong $\mathfrak{sl}_2$-2-representation, $K_0^\oplus(\mathcal{V}_\lambda)$ is a $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$-module.

Theorem 2.22. *There exists a degree preserving isomorphism of $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$-modules*

$$\gamma_V \colon V_\lambda \to K_0^\oplus(\mathcal{V}_\lambda)_{\mathbb{Q}(q)}.$$

Here V_λ is the irreducible $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$-module with highest weight $\overline{\lambda}$.

In Section 5.3.4, we will recall some extra properties of γ_V which Brundan and Kleshchev showed.

Rouquier showed that, in a certain sense, R_λ is the universal categorification of V_λ. For a proof of the following result, see Lemma 5.4, Proposition 5.6 and Corollary 5.7 in [60].

Proposition 2.23. *Let $\mathcal{V}$ be any additive, idempotent complete category, which allows an integrable, graded, categorical action by $\mathcal{U}(\mathfrak{sl}_n)$ (for the precise definition see [60]). Suppose V_h is a highest weight object in $\mathcal{V}$, i.e an object that is killed by $\mathcal{E}_{+i}$, for all $i \in I$, and $\mathrm{End}_{\mathcal{V}}(V_h) \cong \mathbb{C}$. Suppose also that any object in $\mathcal{V}$ is a direct summand of XV_h, for some object $X \in \mathcal{U}(\mathfrak{sl}_n)$. Then there exists an equivalence of categorical $\mathcal{U}(\mathfrak{sl}_n)$-representations*

$$\Phi \colon \mathcal{V}_\lambda \to \mathcal{V}.$$

We will not spell out the precise definition of Φ in this paper. Let us just remark that, up to natural isomorphism, Φ is uniquely determined by the fact that it sends the highest weight object to the highest weight object and intertwines the $\mathfrak{sl}_n$-2-representations.

There are some subtle differences between Rouquier's approach to categorification and Khovanov and Lauda's. However, Proposition 2.23 holds in both setups, as already remarked by Webster in Section 1.4 in [69]. The proof of Proposition 2.23 consists of Rouquier's remarks in Section 5.1.2 and of the contents of his proofs of Lemma 5.4 and Proposition 5.6 in [60], which only rely

on the assumptions in the statement of our Proposition 2.23 and the fact that $\mathcal{E}_{+i}$ and $\mathcal{E}_{-i}$ are biadjoint in $\mathcal{U}(\mathfrak{sl}_n)$, for any $i \in I$. The precise definition of the units and the counits, i.e. the cups and the caps, is not relevant for the validity of the proof. Note that we have included the hypothesis

$$\mathrm{End}_{\mathcal{V}}(V_h) \cong \mathbb{C}$$

in Proposition 2.23, which is not one of Rouquier's assumptions. There are categorifications of V_λ without that property. See Conjecture 7.16 in [50] for example. However, in order to get a categorification which is really equivalent to $\mathcal{V}_\lambda$, i.e. with hom-spaces of the same graded dimension, one needs to add that assumption because it holds in the latter category.

3. THE $\mathfrak{sl}_3$-WEB ALGEBRA $\mathcal{W}_c^S$

For the rest of this section, let S be a fixed sign string of length $\ell(S)$. We are going to define the *web algebra* $\mathcal{W}_c^S$.

Definition 3.1. (Web algebra) For $u, v \in B^S$, we define

$$_u\mathcal{W}_v^c = \mathcal{F}^c(u^*v)\{\ell(S)\},$$

where $\{\ell\}$ denotes a grading shift upwards by $\ell(S)$ degrees.

The *web algebra* $\mathcal{W}_c^S$ is defined by

$$\mathcal{W}_c^S = \bigoplus_{u,v \in B^S} {}_u\mathcal{W}_v^c.$$

The multiplication on $\mathcal{W}_c^S$ is defined by taking

$$_u\mathcal{W}_{v_1}^c \otimes {}_{v_2}\mathcal{W}_w^c \to {}_u\mathcal{W}_w^c$$

to be zero, if $v_1 \neq v_2$, and by the map to be defined in Definition 3.3, if $v_1 = v_2 = v$.

Remark 3.2. In Proposition 3.6 we prove that the multiplication foam always has degree n, so the degree shift in the definition above makes $\mathcal{W}_0^S$ into a graded algebra and, for any $c \neq 0$, it makes $\mathcal{W}_c^S$ into a filtered algebra.

Definition 3.3. (Multiplication of closed webs) The *multiplication*

$$_u\mathcal{W}_v^c \otimes {}_v\mathcal{W}_w^c \to {}_u\mathcal{W}_w^c$$

is induced by the *multiplication foam*

$$m_{u,v,w} \colon u^*vv^*w \xrightarrow{Id_{u^*}m_vId_w} u^*w,$$

where $m_v \colon vv^* \to \mathrm{Vert}_n$, with Vert_n being the web of n parallel oriented vertical line segments, is defined by the following inductive algorithm.

(1) Express v using the growth algorithm, label each level of the growth algorithm starting from zero. Then form vv^*.

(2) At the kth level in the growth algorithm, *resolve* the corresponding pair of arc, H or Y-rules in v and v^* by applying the foams.

(3.1)

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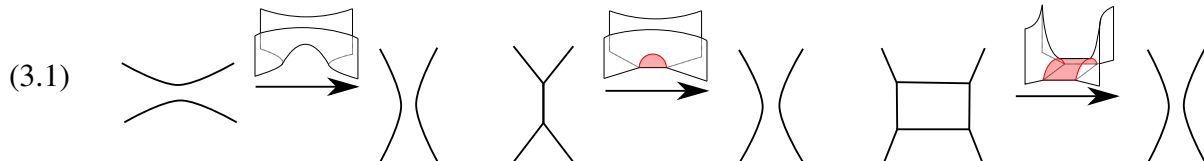
Note that at the last level in the growth algorithm of v, only pairs of arcs are present.

Example 3.4. Let w and v be the following webs.

(3.2)
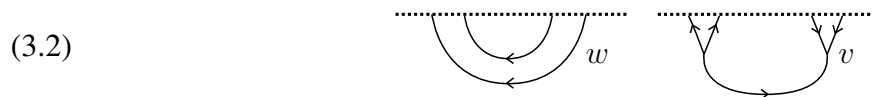
The multiplication foam $m_{w,v,v}$ is given by the following steps.

(3.3)
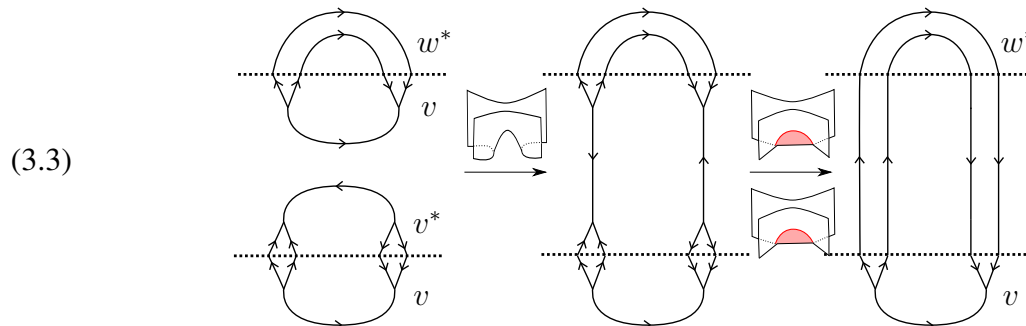
Proposition 3.5. *The foam m_v in Definition 3.3 only depends on the isotopy type of v.*

Proof. We have to show that m_v is independent of the way v is expressed using the growth algorithm (Definition 2.2). Let G_1 and G_2 be two different expressions of v using the growth algorithm. We have to compare G_1 and G_2 walking backwards in the growth algorithm. Note that we only have to worry about two consecutive steps in the same region of v. Reordering steps in "distant" regions of v corresponds to an isotopy which simply alters the height function on m_v. With these observations, the only possible remaining difference between the last two steps in G_1 and G_2 is the following.

(3.4)

If the last two steps in G_1 and G_2 are equal, we have to go further back in the growth algorithm. Besides two-step differences of the same sort as above, we can encounter another one of the following sort.

(3.5)
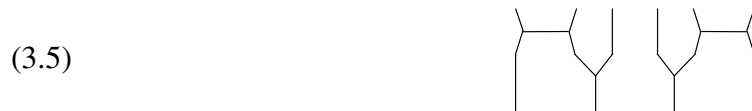
We have to check that the above two-step differences in G_1 and G_2 correspond to equivalent foams. In the first case, the foams in the multiplication algorithm are given by
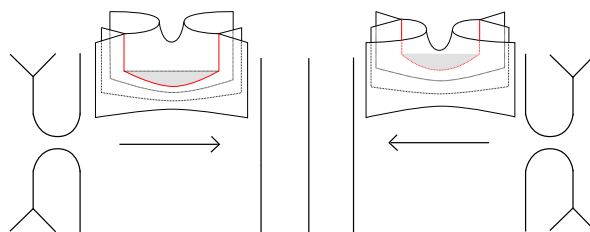
FIGURE 3. A possible local difference between m_{G_1} and m_{G_2}.

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In the second case, we get
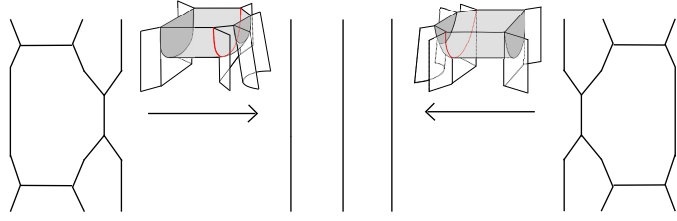
FIGURE 4. The other possible local difference between m_{G_1} and m_{G_2}.

The two foams in Figure 3 are isotopic - one foam can be produced from the other by sliding the red singular arc over the saddle as illustrated below.

(3.6)
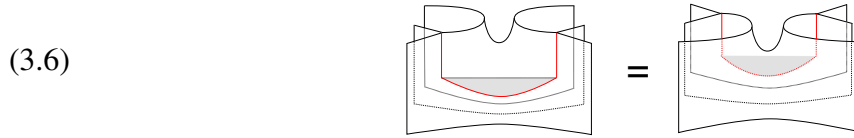
The two foams in Figure 4 are also isotopic - one foam can be produced from the other by moving the red singular arc to the right or to the left as illustrated below.

(3.7)
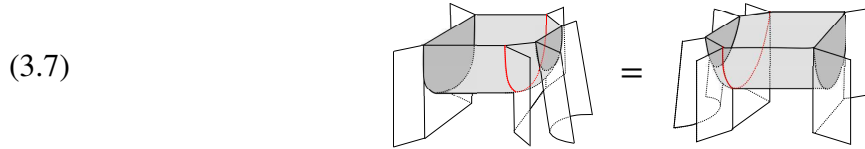
The cases above are the only possible ones, so their verification provides the proof. $\qquad\square$

Proposition 3.6. *The foam m_v has q-grading $\ell(S)$.*

Proof. We proceed by backward induction on the level of the growth algorithm expressing v. At the final level of the growth algorithm, the only possible rule is the arc rule. Resolving a corresponding pair of arcs in v and v^* results in two new vertical strands and is obtained by a saddle point cobordism, which has q-grading 2.

Let n_k be the number of vertical strands and m_v^k be the foam after resolving the last k rules in the growth algorithm of v. Suppose that n_k is equal to the q-degree of m_v^k. In the next step of the multiplication we can have three cases.

(1) The resolution of a pair of arc-rules. In this case we have $n_{k+1} = n_k + 2$ and m_v^{k+1} is obtained from m_v^k by adding a saddle, which adds 2 to the q-grading.
(2) The resolution of a pair of Y-rules. In this case we have $n_{k+1} = n_k + 1$ and m_v^{k+1} is obtained from m_v^k by adding an unzip, which adds 1 to the q-grading.
(3) The resolution of a pair of H-rules. In this case we have $n_{k+1} = n_k$ and m_v^{k+1} is obtained from m_v^k by adding a square foam, which adds 0 to the q-grading.

$\qquad\square$

There is a useful alternative definition of $\mathcal{W}_S^c$, which we give below. As a service to the reader, we state it as a lemma and prove that it really is equivalent to our definition above. Both definitions have their advantages and disadvantages, so it is worthwhile to catalogue both in this paper.

Lemma 3.7. *For any $c \in \mathbb{C}$ and any $u, v \in B^S$, we have a grading preserving isomorphism*

$$\mathbf{Foam}_3^c(u, v) \cong {}_u\mathcal{W}_v^c.$$

Using this isomorphism, the multiplication

$${}_u\mathcal{W}_v^c \otimes {}_{v'}\mathcal{W}_w^c \to {}_u\mathcal{W}_w^c$$

corresponds to the composition

$$\mathbf{Foam}_3^c(u, v) \otimes \mathbf{Foam}_3^c(v', w) \to \mathbf{Foam}_3^c(u, w),$$

if $v = v'$, and is zero otherwise.

Proof. The isomorphism of the first claim is sketched in the following figure.
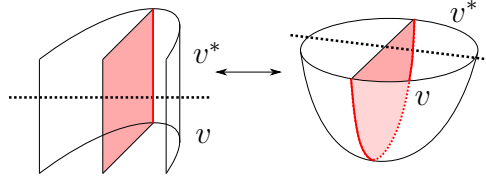
The proof of the second claim follows from analysing what the isomorphism does to the resolution of a pair of arc, Y or H-rules in the multiplication foam. This is done below.

$\square$
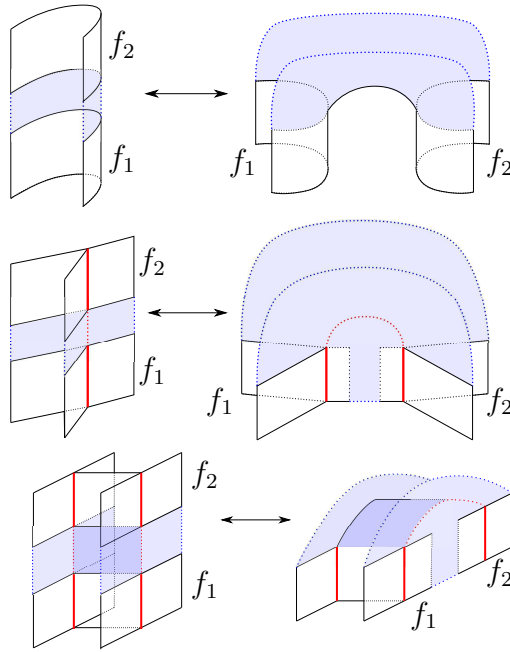
Note that Lemma 3.7 implies that $\mathcal{W}_c^S$ is associative and unital, something that is not immediately clear from Definition 3.1. For any $u \in B^S$, the identity $1_u \in \mathbf{Foam}_3^c(u, u)$ defines an idempotent. We have

$$1 = \sum_{u \in B^S} 1_u \in \mathcal{W}_c^S.$$

Alternatively, one can see $\mathcal{W}_c^S$ as a category whose objects are the elements in B^S such that the module of morphisms between $u \in B^S$ and $v \in B^S$ is given by $\mathbf{Foam}_3^c(u, v)$. In this paper we will mostly see $\mathcal{W}_c^S$ as an algebra, but will sometimes refer to the category point of view.

In this paper, we will study $\mathcal{W}_c^S$ for two special values of $c \in \mathbb{C}$.

Definition 3.8. Let K^S and G^S be the complex algebras obtained from $\mathcal{W}_c^S$ by setting $c = 0$ and $c = 1$, respectively. We call them *Khovanov's web algebra* and *Gornik's web algebra*, respectively, to distinguish them throughout the paper.

Note that G^S is a filtered algebra. Its associated graded algebra is K^S. By Lemma 3.7, both K^S and G^S are finite dimensional, unital, associative algebras. They also have similar decompositions as shown below.

$$K^S = \bigoplus_{u,v \in B^S} {}_uK_v\,, \qquad G^S = \bigoplus_{u,v \in B^S} {}_uG_v.$$

We now recall the definition of complex, graded and filtered Frobenius algebras. Let A be a finite dimensional, graded, complex algebra and let $\mathrm{Hom}_\mathbb{C}(A, \mathbb{C})$ be the complex vector space of grading preserving maps. The *dual* of A is defined by

$$A^\vee = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_\mathbb{C}(A, \mathbb{C}\{n\}),$$

where $\{n\}$ denotes an upward degree shift of size n. Note that $A^\vee$ is also a graded module, such that

(3.8) $$(A^\vee)_i = (A_{-i})^\vee,$$

for any $i \in \mathbb{Z}$. Then A is called a *graded, symmetric Frobenius algebra of Gorenstein parameter* ℓ, if there exists an isomorphism of graded (A, A)-bimodules

$$A^\vee \cong A\{-\ell\}.$$

If A is a complex, finite dimensional, filtered algebra, let $\mathrm{Hom}_\mathbb{C}(A, \mathbb{C})$ be the complex vector space of filtration preserving maps. The *dual* of A is defined by

$$A^\vee = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_\mathbb{C}(A, \mathbb{C}\{n\}),$$

where $\{n\}$ denotes an upward suspension of size n. Note that $A^\vee$ is also a filtered module, such that

(3.9) $$(A^\vee)_i = (A_{-i})^\vee,$$

for any $i \in \mathbb{Z}$. Then A is called a *filtered, symmetric Frobenius algebra of Gorenstein parameter* ℓ, if there exists an isomorphism of filtered (A, A)-bimodules

$$A^\vee \cong A\{-\ell\}.$$

For more information on graded Frobenius algebras, see [67] and the references therein, for example. We do not have a good reference for filtered Frobenius algebras, but it is a straightforward generalization of the graded case. We will explain some basic results on the character theory of filtered and graded symmetric Frobenius algebras in Section 5.

Theorem 3.9. *For any sign string S, the algebra K^S is a graded symmetric Frobenius algebra and G^S is a filtered symmetric Frobenius algebra, both of Gorenstein parameter $2\ell(S)$.*

Proof. First, let $c = 0$. We take, by definition, the trace form

$$\mathrm{tr}\colon K^S \to \mathbb{C}$$

to be zero on ${}_uK_v$, when $u \neq v \in B^S$. For any $v \in B^S$, we define

$$\mathrm{tr}\colon {}_vK_v \to \mathbb{C}$$

by closing any foam f_v with 1_v, e.g. as pictured below.
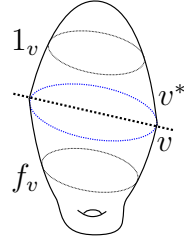
Equivalently, in $\mathbf{Foam}_3^0(v, v)$, closing f_v by 1_v,
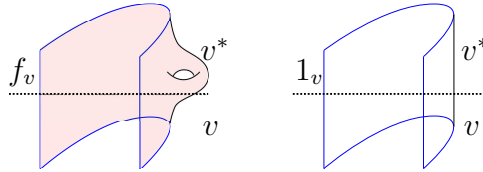
gives
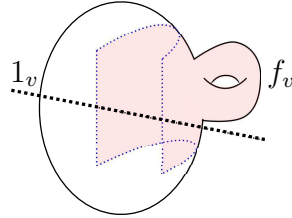
The fact that the trace form is non-degenerate follows immediately from the closure relation in Subsection 2.2.

The fact that $\mathrm{tr}(gf) = \mathrm{tr}(fg)$ holds follows from sliding f around the closure until it appears on the other side of g, e.g. as shown below.
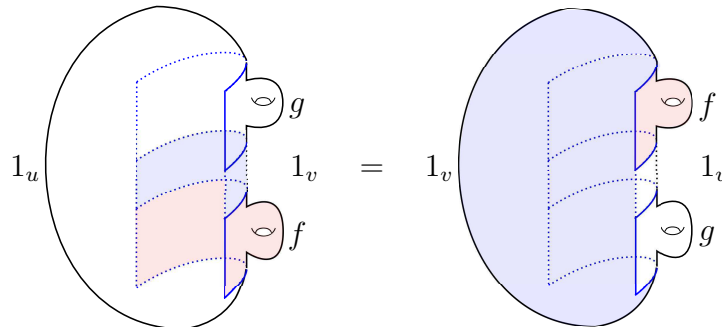
Note that a closed foam can only have non-zero evaluation if it has degree zero. Therefore, for any $u \in B^S$ and any two homogeneous elements $f \in \mathcal{F}^0(u^*v)$ and $g \in \mathcal{F}^0(v^*u)$, we have $\mathrm{tr}(fg) \neq 0$ unless $\deg(f) = -\deg(g)$. By the shift $\ell(S)$ in

$$ {}_uK_v = \mathcal{F}^0(u^*v)\{\ell(S)\} $$

and by (3.8), this implies that the non-degenerate trace form on K^S gives rise to a graded (K^S, K^S)-bimodule isomorphism

$$(3.10) \qquad (K^S)^\vee \cong K^S\{-2\ell(S)\}.$$

Now, let $c = 1$. Then the construction above also gives a non-degenerate bilinear form on G^S. Moreover, it induces a filtration preserving bijective $\mathbb{C}$-linear map of filtered (G^S, G^S)-bimodules

$$(3.11) \qquad G_s\{-2n\} \to G_s^\vee.$$

The associated graded map is precisely the isomorphism in (3.10). By Proposition 5.36, this implies that the map in (3.11) is a strict isomorphism of filtered (G^S, G^S)-bimodules. □

We now explain some of Gornik's results, which are relevant for G^S. Recall that $R^1_{u^*v}$ is the commutative ring associated to u^*v, generated by the edge variables of u^*v and mod out by the ideal, which, for each trivalent vertex in u^*v, is generated by the relations

$$(3.12) \qquad x_1 + x_2 + x_3 = 0, \qquad x_1 x_2 + x_1 x_3 + x_2 x_3 = 0, \qquad x_1 x_2 x_3 = 1,$$

where x_1, x_2 and x_3 are the edge variables around the vertex. The algebra $R^1_{u^*v}$ acts on $_uG_v$ in such a way that each edge variable corresponds to adding a dot on the incident facet. See [26], [34] and [51] for the precise definition and more details.

In what follows, 3-colorings will always be assumed to be admissible and we therefore omit the adjective. Theorem 3 in [26] proves the following.

Theorem 3.10. *(**Gornik**) There is a complete set of orthogonal idempotents $e_T \in R^1_{u^*v}$, indexed by the 3-colorings T of u^*v. The number of 3-colorings of u^*v is exactly equal to $\dim_q(_uG_v)$.*

These idempotents are not filtration preserving, but as an $R^1_{u^*v}$-module (i.e. forgetting the filtration on $_uG_v$ and its left $_uG_u$ and right $_vG_v$-module structures) we have

$$_uG_v \cong \bigoplus_T \mathbb{C}e_T.$$

Let us have a closer look at Gorniks idempotents. First of all, in the proof of Theorem 3 in [26] Gornik notes that for any edge i and any 3-coloring T of u^*v, we have

$$(3.13) \qquad x_i e_T = \zeta^{T_i} e_T \in R^1_{u^*v},$$

where ζ is a primitive third root of unity, x_i is the edge variable and T_i the color of the edge (see (4) in [51] for this result in the context of foams).

Furthermore, a 3-coloring of u^*v actually corresponds to a pair of 3-colorings of u and v^* that match at the boundary. Of course, there is a bijective correspondence between 3-colorings of u and v^*, so we see that a 3-coloring of u^*v corresponds to a matching pair of 3-colorings of u and v.

Recall that $_uG_v$ is a left $_uG_u$-module and a right $_vG_v$-module. Let T_1 and T_2 be a pair of matching 3-colorings of u and v, respectively, which together give a 3-coloring T of u^*v. Then the action of e_T on any $f: u \to v$ can be written as

$$e_{T_1} f e_{T_2}.$$

To show that this notation really makes sense, define *Gornik's symmetric idempotent* associated to T_1 as

$$e_{u,T_1} = e_{T_1} 1_u e_{T_1}.$$

So we let the Gornik idempotent associated to the symmetric 3-coloring of u^*u, given by T_1 both on u and u^*, act on 1_u. Then we have

$$e_{T_1} f e_{T_2} = e_{u,T_1} f e_{v,T_2},$$

where on the right-hand side we really mean composition.

We immediately see that

$$e_{T_1} 1_u e_{T_2} = 0 \Leftrightarrow T_1 \neq T_2$$

and

$$e_{T_1} 1_u e_{T_1} e_{T_2} 1_u e_{T_2} = \delta_{T_1,T_2} e_{T_1} 1_u e_{T_1} \quad \text{and} \quad \sum_T e_T 1_u e_T = 1_u,$$

where the sum is over all 3-colorings of u. This shows that the $e_{u,T}$, for all 3-colorings T of a given $u \in B^S$, are orthogonal idempotents in $_u G_u$. It also implies that

$$e_{T_1} 1_u e_{T_1} = e_{T_1} 1_u = 1_u e_{T_1},$$

so it is enough to label just the source or just the target of 1_u. For this purpose, we define R_u^1 to be "half" of $R_{u^*u}^1$, i.e. the subring which is only generated by the edge variables of u. To be precise, we have

$$R_{u^*u}^1 \cong R_u^1 \otimes_S R_u^1,$$

where $\otimes_S$ indicates that we impose the relation $x \otimes 1 = 1 \otimes x$, for any x corresponding to a boundary edge of u.

If u has no closed cycles, then all the 3-colorings of u^*u are symmetric, because they are completely determined by the colors on the boundary of u. In that case

$$e_T \mapsto e_{u,T}$$

defines an isomorphism of algebras $R_u^1 \cong {}_u G_u$. In particular, $_u G_u$ is commutative. This is not true in general, but we can prove the following.

Lemma 3.11. *For any $u \in B^S$, the map*

$$x \mapsto x 1_u$$

defines a strict embedding of filtered R_u^1-modules

$$\iota \colon R_u^1 \to {}_u G_u.$$

In particular, we see that $(R_u^1)_0 \cong \operatorname{im}(\iota)_0 \cong \mathbb{C} 1_u$.

Proof. The map is clearly a homomorphism of filtered algebras.

The relations (Dot Migration) correspond precisely to the relations in R_u^1, because the only singular edges in 1_u are the ones corresponding to the trivalent vertices of u. This shows that it is a strict embedding. $\square$

For any $u \in B^S$, we define the graded ring

$$R_u^0 = E(R_u^1).$$

This ring is the one which appears in Khovanov's original paper [34]. In R_u^0 we have the relations

(3.14) $\qquad x_1 + x_2 + x_3 = 0, \qquad x_1 x_2 + x_1 x_3 + x_2 x_3 = 0, \qquad x_1 x_2 x_3 = 0.$

The reader should compare them to (3.12).

There are no analogues of the Gornik idempotents in R_u^0, but we do have an analogue of Lemma 3.11.

Lemma 3.12. *For any $u \in B^S$, the map*

$$x \mapsto x1_u$$

defines an embedding of graded R_u^0-modules

$$E(\iota) \colon R_u^0 \to {}_u K_u.$$

In particular, we see that $(R_u^0)_0 \cong \operatorname{im}(E(\iota))_0 \cong \mathbb{C}1_u$.

Another interesting consequence of Theorem 3.10 is the following.

Proposition 3.13. *As a complex algebra, i.e. without taking the filtration into account, G^S is semisimple.*

Proof. For any $u \in B^S$ and any 3-coloring T of u, define the projective G^S-module

$$P_{u,T} = (G^S)e_{u,T},$$

where $e_{u,T}$ is Gornik's symmetric idempotent in G^S defined above. Theorem 3.10 and our subsequent analysis of Gornik's idempotents show that the $P_{u,T}$ form a complete set of indecomposable, projective G^S-modules. Furthermore, we have

$$\operatorname{Hom}_{G^S}(P_{u,T}, P_{v,T'}) \cong e_{u,T}(G^S)e_{v,T'} \cong \begin{cases} \mathbb{C}, & \text{if } T \text{ and } T' \text{ match at } S, \\ \{0\}, & \text{else.} \end{cases}$$

This shows that $P_{u,T} \cong P_{v,T'}$ if and only if T and T' match at the common boundary. It also shows that if $P_{u,T} \not\cong P_{v,T'}$, then

$$\operatorname{Hom}_{G^S}(P_{u,T}, P_{v,T'}) = \operatorname{Hom}_{G^S}(P_{v,T'}, P_{u,T}) = \{0\}.$$

Finally, it shows that each $P_{u,T}$ has only one composition factor, i.e. $P_{u,T}$ is irreducible.

It is well-known that this implies that G^S is semisimple, e.g. see Proposition 1.8.5 in [3]. $\square$

By Proposition 3.13, it is clear that for each $u \in B^S$ and each coloring T of u, the corresponding block in G^S is isomorphic to $\operatorname{End}(P_{u,T})$. In Section 4, we will determine the central idempotents of G^S.

4. THE CENTER OF THE WEB ALGEBRA AND THE COHOMOLOGY RING OF THE SPALTENSTEIN VARIETY

For the rest of this section, choose arbitrary but fixed non-negative integers $n \geq 2$ and $k \leq n$, such that $d = 3k \geq n$. Let

$$\Lambda(n, d) = \left\{ \mu \in \mathbb{N}^n \mid \sum_{i=1}^n \mu_i = d \right\}$$

be the set of *compositions* of d of length n. By $\Lambda^+(n, d) \subset \Lambda(n, d)$ we denote the subset of *partitions*, i.e. all $\mu \in \Lambda(n, d)$ such that

$$\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \geq 0.$$

Also for the rest of this section, choose an arbitrary but fixed sign string S of length n. We associate to S a unique element $\mu = \mu_S \in \Lambda(n, d)$, such that

$$\mu_i = \begin{cases} 1, & \text{if } s_i = +, \\ 2, & \text{if } s_i = -. \end{cases}$$

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Let $\Lambda(n,d)_{1,2} \subset \Lambda(n,d)$ be the subset of compositions whose entries are all 1 or 2. For any sign string S, we have $\mu_S \in \Lambda(n,d)_{1,2}$.

Let $\lambda = (3^k) \in \Lambda(n,d)$. Let Col_μ^λ be the set of column strict tableaux of shape λ and type μ, both of length n. It is well-known that there is a bijection between Col_μ^λ and the tensor basis of

$$V^\mu = V^{\mu_1} \otimes \cdots \otimes V^{\mu_n},$$

where $V^1 = V^+$ and $V^2 = V^1 \wedge V^1 \cong V^-$ (see Section 3 in [55], for example). However, we are interested in tensors as summands in the decomposition of elements in B^S. Therefore, we prove Proposition 4.2 in Subsection 4.1. The reader, who is not interested in the details of the proof of this proposition, can choose to skip this subsection at a first reading and just read the statement of the proposition.

4.1. **Tableaux and flows.** Let p_S be the number of positive entries and n_S the number of negative entries of S. By definition, we have that $d = p_S + 2n_S$. The key idea in this subsection is to reduce all proofs to the case where $n_S = 0$.

Definition 4.1. Fix any state string J of length n, we define a new state string $\hat{J}$ of length d by the following algorithm.

(1) Let $_0\hat{J}$ be the empty string.
(2) For $1 \le i \le n$, let $_i\hat{J}$ be the result of concatenating j_i to $_{i-1}\hat{J}$ if $\mu_i = 1$. If $\mu_i = 2$ then
   (a) concatenate $(1,0)$ to $_{i-1}\hat{J}$ if $j_i = 1$.
   (b) concatenate $(0,-1)$ to $_{i-1}\hat{J}$ if $j_i = -1$.
   (c) concatenate $(1,-1)$ to $_{i-1}\hat{J}$ if $j_i = 0$.

We set $\hat{J} = {}_n\hat{J}$. Lastly, for any $c \in \{-1,0,1\}$, we define $\hat{J}^c$ to be the number of entries in $\hat{J}$ that is equal to c.

Proposition 4.2. *There is a bijection between Col_μ^λ and the set of state strings J such that there exists a $w \in B^S$ and a flow f on w which extends J.*

The proof of Proposition 4.2 follows directly from Lemmas 4.3 and 4.4.

Lemma 4.3. *There is a bijection between Col_μ^λ and state strings J of length n such that*

(4.1) $$\hat{J}^{-1} = \hat{J}^0 = \hat{J}^1.$$

where the $\hat{J}^c$ are as defined in Definition 4.1.

Proof. Given a state string J satisfying (4.1), we first give an algorithm to build a 3-column tableau Y_J, filled with integers from 1 to n. Afterwards, we show that Y_J has shape λ.

Begin by labeling the three columns with $1, 0$ and -1, reading from left to right. We are going to build up Y_J from top to bottom. Start by taking Y_J to be the empty tableau. Then, from $i = 1$ to $i = n$, do the following:

(1) If $\mu_i = 1$, add one box labeled i to column j_i in Y_J.
(2) If $\mu_i = 2$, add two boxes labeled i to columns c_1 and c_2, such that $c_1 \ne c_2$ and $c_1 + c_2 = j_i$.

We have to show that Y_J belongs to Col_μ^λ. Since the algorithm builds up from top to bottom, Y_J is strictly column increasing. To see that Y_J has shape λ, we need to show that every row in Y_J has three entries. Observe that the number of filled boxes in column c of Y_J is exactly equal to $\hat{J}^c$.

Since we have assumed condition (4.1), all three columns have the same length, therefore every row in Y_J must have exactly three entries.

Conversely, let $T \in \mathrm{Col}_\mu^\lambda$. We define a state string J as follows.

$$j_i = \sum_{i \text{ appears in column } c} c.$$

Since μ corresponds to a sign string and T is column strict, we see that, for each $1 \leq i \leq n$, i can appear at most twice in T but never twice in the same column. Thus, $j_i \in \{-1, 0, 1\}$, i.e. J is a state string. It follows from the definition of $\hat{J}$ that $\hat{J}^c$ is equal to the length of column c of T. Since T is of shape λ, the number of boxes in each column is the same. Hence, condition (4.1) holds for J.

It is straightforward to check that the above two constructions are inverse to each other and therefore determine a bijection. □

Lemma 4.4. *A state string J corresponds to the boundary state of a flow on a web $w \in B^S$ if and only if condition (4.1) holds for J.*
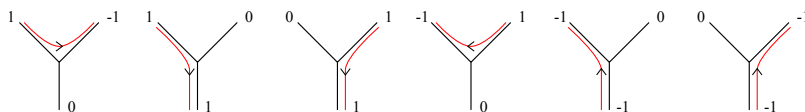
Proof. Let $w \in B^S$ be equipped with a flow with boundary state string J. We are going to show that J satisfies condition (4.1) by induction on n. For $n = 2$, w can only be an arc. In this case it is simple to check that all flows on w have corresponding boundary state strings satisfying condition (4.1).

For $n > 2$, we express w using the growth algorithm in an arbitrary, but fixed way, with the restriction that only one rule is applied per level. Let $_kJ$ denote the boundary state string at the beginning of the k-th level in the growth algorithm and $_k\hat{J}$ the associated string as in Definition 4.1. Similarly, let $_k\mu$ denote the composition corresponding to the sign string at the k-th level. Let us compare $_{k+1}J$ and $_kJ$. They can only differ in the following ways.

(1) In case an arc-rule is applied at the k-th level, $_kJ$ can be obtained from $_{k+1}J$ by inserting the substring $(1, -1)$, $(0, 0)$ or $(-1, 1)$ between the i-th and $i + 1$-th entries in $_{k+1}J$. $_k\mu$ can be obtained from $_{k+1}\mu$ by inserting the substring $(1, 2)$ or $(2, 1)$ between the i-th and $i + 1$-th entries in $_{k+1}\mu$.

(4.2)

(2) In case a Y-rule is applied, $_kJ$ can be obtained from $_{k+1}J$ by replacing the i-th entry in $_{k+1}J$ with a length two substring whose sum is equal to the i-th entry. $_k\mu$ can be obtained from $_{k+1}\mu$ by replacing the i-th entry in $_{k+1}\mu$ with the substring $(3 - _{k+1}\mu_i, 3 - _{k+1}\mu_i)$.

(4.3)

(3) In case an H-rule is applied, $_k\mu$ can be obtained from $_{k+1}\mu$ by replacing a substring $(1, 2)$ or $(2, 1)$, at the i-th and $(i+1)$-th position in $_{k+1}\mu$, with $(3 - _{k+1}\mu_i, 3 - _{k+1}\mu_{i+1})$. $_kJ$ can be obtained from $_{k+1}J$ by replacing a substring of length two in $_{k+1}J$ at the i-th and $(i+1)$-th

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position according to the schema.

(4.4)

(4.5)

It is straightforward to check that $_{k+1}J$ satisfies condition (4.1), with composition $_{k+1}\mu$, if and only if $_kJ$ does, with composition $_k\mu$. Take, for example, an instance where a Y-rule is applied; suppose also that the i-th entry in $_{k+1}\mu$ is 2 and the i-th entry in $_{k+1}J$ is 0. Thus, the i-th entry in $_{k+1}J$ contributes a pair $(1,-1)$ to $_{k+1}\hat{J}$. By (4.3), $_kJ$ is obtained from $_{k+1}J$ by replacing the i-th entry in $_{k+1}J$ with $(1,-1)$ and the i-th entry in $_{k+1}\mu$ with $(1,1)$. We see that $_k\hat{J}$ is in fact exactly equal to $_{k+1}\hat{J}$. Therefore $_{k+1}\hat{J}$ satisfies condition (4.1) if and only if $_k\hat{J}$ does. Similar analysis apply to all cases in (4.2), (4.3) and (4.4).

Let k be the first level in the growth algorithm of w where a Y or an arc-rule is applied. From the $(k+1)$-th level down we have a non-elliptic web w' with flow, whose boundary state string $_{k+1}J$ and composition $_{k+1}\mu$ both have length less than n. Thus, by our induction hypothesis, $_{k+1}J$, with composition $_{k+1}\mu$, satisfies condition (4.1).

By the above argument, then $_iJ$ also satisfy condition (4.1), for any $0 \le i \le k$. In particular, $J = {_0J}$ satisfies that condition, which is what we had to prove.

Conversely, let J satisfy condition (4.1), with composition μ. We show, by induction on n, that there is a $w \in B^S$ with flow whose boundary state string is exactly J. More specifically, we first construct a $w \in W^S$ and then show that w is non-elliptic, i.e. $w \in B^S$.

For $n = 2$, then w must be an arc. It is simple to check that if J satisfies condition (4.1), J is the boundary state of a flow on an arc.

For $n > 2$, suppose it is possible to apply an arc or Y-rule to the pair μ and J, depicted in (4.2) and (4.3). Then we obtain a new pair μ' and J' with length less than n. Thus, by induction, there exist a web $w' \in W_{S'}$ and flow extending J'. Gluing the arc or Y on top of w' results in a web $w \in W^S$ with a flow extending J.

Suppose, then, that it is not possible to apply an arc or Y-rule to μ and J. This means that one of the following must hold.

(1) μ does not contain a substring of type $(1,2)$ or $(2,1)$ and $J = (1,...,1)$, $J = (-1,...,-1)$ or $J = (0,...,0)$.

(2) μ contains at least one substring of the form $(1,2)$ or $(2,1)$. For every substring in μ of the form $(1,2)$ or $(2,1)$, the corresponding substring in J is $(\pm 1, \pm 1)$, $(0,1)$ or $(1,0)$. For every substring in μ of the form $(1,1)$ or $(2,2)$, the corresponding substring in J is $(1,1)$, $(-1,-1)$ or $(0,0)$.

Case 1 contradicts the assumption that J satisfies condition (4.1).

Case 2 contains several subcases, each of which contains details which are slightly different. However, the general idea is the same for all of them and is very simple: apply H-moves until you can apply an arc or a Y-rule and finish the proof by induction.

We first suppose, without loss of generality, that μ contains a substring $(\mu_i, \mu_{i+1}) = (1,2)$ and that the corresponding substring in J is $(j_i, j_{i+1}) = (1,1)$ (the subcase for $(j_i, j_{i+1}) = (-1,-1)$ is

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analogous). We see that $(1,1)$ in J contributes a substring $(1,0,1)$ to $\hat{J}$. Thus, our assumption that $\hat{J}$ satisfies condition (4.1) implies that $\hat{J}$ contains at least one more entry equal to -1. This means that for some $r \neq i, i+1$, $1 \leq r \leq n$, one of the following is true.

(a) $j_r = -1$, $\mu_r = 1$, denoted for brevity by

(4.6)

(b) $j_r = -1$, $\mu_r = 2$, denoted

(4.7)

(c) $j_r = 0$, $\mu_r = 2$, denoted

(4.8)

Without loss of generality, let us assume $i+1 < r$. Consider subcases (a) and (b). If $j_m \neq 0$ for all $i+1 < m < r$, then it is possible to apply an arc or Y-move to J and μ, contrary to our assumption in case 2. Thus, in all three scenarios above it suffices to analyze the following two configurations:

(4.9)

Let $i + 1 < r \leq n$ be smallest integer where $j_r = 0$. We must have that $\mu_{r-1} = 3 - \mu_r$ and $j_{r-1} = \pm 1$. For any other values of μ_{r-1} and j_{r-1} we would be able to apply an arc or a Y-move, contradicting our assumptions for case 2. In both situations, we can apply an H-rule to the substrings (j_{r-1}, j_r) and (μ_{r-1}, μ_r) as shown below.

(4.10)

This results in new sign and state strings, each with length n satisfying condition (4.1). The application of the H-rule in (4.10) moves the zero at the r-th position to the $r-1$ position. Either we can now apply an arc or Y-rule to the new strings or by repeatedly applying an H-rule in the manner of (4.10), we obtain one of the following pairs.

(4.11)

To either of the above diagrams we can apply a Y-rule, after which we can use induction.

To complete our analysis of case 2, now suppose, without loss of generality, that μ contains a substring $(\mu_i, \mu_{i+1}) = (1,2)$ and that the corresponding substring in J is $(j_i, j_{i+1}) = (1,0)$ (the subcases for $(0, \pm 1)$ or $(-1, 0)$ are analogous).

We see that $(1,0)$ in J contributes a substring $(1,1,-1)$ to $\hat{J}$. Thus, our assumption that $\hat{J}$ satisfies condition (4.1) implies that $\hat{J}$ contains at least one more entry equal to -1. This means that for some r, with $1 \leq r \leq n$, one of the following is true.

(a) $j_r = -1$, $\mu_r = 1$, denoted for brevity by

(4.12)

(b) $j_r = -1$, $\mu_r = 2$, denoted

(4.13)

(c) $j_r = 0$, $\mu_r = 2$, denoted

(4.14)

For subcases (a) and (b), if $\mu_{i+2} = 1$ and $j_{i+2} = -1$, we may apply an H-rule to (μ_{i+1}, μ_{i+2}), (j_{i+1}, j_{i+2}) to obtain a new pair μ' and J'. Subsequently we can apply a Y-rule to the i-th and $(i+1)$-th entries of μ' and J':

(4.15)

After applying the Y-rule, we can use induction.

Otherwise, we can show, just as before, that all three scenarios above reduce to an analysis of the following two configurations.

(4.16)

That is, we can assume that μ contains a substring $(\mu_i, \mu_{i+1}) = (1, 2)$ with the corresponding substring in J being $(j_i, j_{i+1}) = (1, 0)$, and for some $0 < r \neq i + 1 < n$ we have $j_r = 0$. In particular, this tells us that there exist a $0 < r \neq i + 1 < n$ such that $\mu_r = 1$ and $j_r = 0$.

(4.17)

This has to hold because otherwise $\hat{J}$ cannot satisfy condition (4.1). Let us assume r to be the smallest integer such that $i + 1 < r$, $\mu_r = 1$ and $j_r = 0$. By our assumption that we cannot apply an arc or Y-rule to J and μ, we see that $\mu_{r-1} = 2$ and $j_{r-1} = \pm 1$. Applying an H-rule to (j_{r-1}, j_r) and (μ_{r-1}, μ_r)

(4.18)

results in new sign and state strings, also with length n, satisfying condition (4.1). The application of the H-rule in the above case moves the zero at the r-th position to the $r-1$-th position. Either

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we can now apply an arc or a Y-rule to the new sign and state strings, or by repeatedly apply an H-rule in the manner of (4.10), we obtain a pair as below.

(4.19)

$$\cdots \Big\downarrow \Big\uparrow \cdots$$

to which we can apply an arc-rule. Finally, apply induction.

It remains to show that the web w (with flow) produced from the above algorithm is an element of B^S, that is, w does not contain digons or squares. We note that, just as in [35], in the expression of w using the arc, Y and H-rules, digons can only appear as the result of applying an arc-rule to the bottom of an H-rule, i.e. we have

(4.20)

A square can only result from the following sequence of arc, Y and H-rules.

(4.21)

Note that in the above, we do not consider the case in which we apply an H-rule to the bottom of another H-rule. This is because such a case cannot arise in our construction of w.

Recall that in our inductive construction of w, we only apply H-rules equipped with the following flows.

(4.22)

We can immediately see that is it not possible to apply an arc-rule with flow to the bottom of such an H-rule as shown below.

(4.23)

Since we only use the above two H-rules with flow, the induced flows on squares are as follows.

(4.24)

(4.25)

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(4.26)

In each case, one can check that it is possible to apply an arc-rule to the state and sign strings (the same analysis applies to the cases where the faces above are given the opposite edge orientations). However, recall that an H-rule is used in our construction only in the case for which it is not possible to apply any other rules to the boundary. This implies that none of the above faces can appear during the construction of w. □

Implicit in the proof of Lemma 4.4 is a procedure to construct, from a state string J satisfying condition (4.1), a non-elliptic web w with flow extending J, such that $\partial w = \mu$. Note that this procedure is not deterministic. That is, it is possible to produce different webs with flows extending J by making different choices in the construction.

Example 4.5. The procedure is exemplified below. If we choose to replace the substrings as indicated in the right figure, the tableau on the left gives rise to the web with flow next to it.

(4.27)
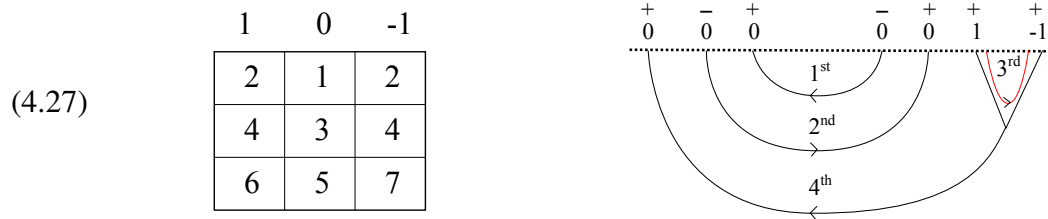
However, for other choices the same tableau generates the following web with flow.

(4.28)
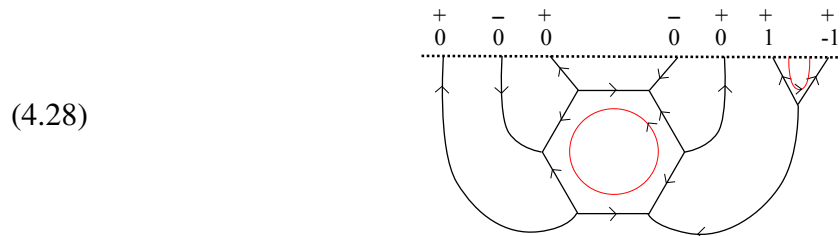
As a matter of fact, we could also invert the orientation of the flow in the internal cycle. The resulting web with flow would still correspond to the same tableau.

However, when we restrict to semi-standard tableaux, the procedure gives a unique web with flow, the canonical flow. One can check that the procedure implicit in Lemma 4.4, restricts to the same bijection between Std_μ^λ and non-elliptic webs as defined by Russell in [61].

4.2. $Z(G^S)$ **and** $E(Z(G^S))$. In this subsection, S continues to be a fixed sign string of length n. Moreover, we continue to use some of the other notations and conventions from the previous subsection as well, e.g. $d = 3k \geq n$ etc. Let μ be the composition associated to S and let S_μ be the corresponding parabolic subgroup of the symmetric group S_d.

Let $Z(K^S)$ be the center of K^S and let X_μ^λ be the Spaltenstein variety, with $\lambda = (3^k)$ and the notation as in [8]. If $n_s = 0$, then $X_\mu^\lambda = X^\lambda$, the latter being the Springer fiber associated to λ.[3]

In Theorem 4.8, we are going to prove that $H^*(X_\mu^\lambda)$ and $Z(K^S)$ are isomorphic as graded algebras.

Recall the following result by Tanisaki [66]. Let $P = \mathbb{C}[x_1, \ldots, x_d]$ and let I^λ be the ideal generated by

(4.29) $$\left\{ e_r(i_1, \ldots, i_m) \;\middle|\; \begin{array}{l} m \geq 1, 1 \leq i_1 < \cdots < i_m \leq d \\ r > m - \lambda_{d-m+1} - \cdots - \lambda_n \end{array} \right\},$$

where $e_r(i_1, \ldots, i_m) \in P$ is the r-th elementary symmetric polynomial. Write

$$R^\lambda = P/I^\lambda.$$

Tanisaki showed that

$$H^*(X^\lambda) \cong R^\lambda.$$

Note that S_μ acts on P by permuting the variables and that it maps I^λ to itself. Moreover, let $P^\mu = P^{S_\mu} \subset P$ be the subring of polynomials which are invariant under S_μ.

For $1 \leq i_1 \leq \cdots \leq i_m \leq n$ and $r \geq 1$, we let $e_r(\mu, i_1, \ldots, i_m)$ denote the r-th elementary symmetric polynomials in the variables $X_{i_1} \cup \cdots \cup X_{i_m}$, where

$$X_p = \{x_k \mid \mu_1 + \cdots + \mu_{p-1} + 1 \leq k \leq \mu_1 + \cdots + \mu_p\}.$$

So, we have

$$e_r(\mu, i_1, \ldots, i_m) = \sum_{r_1 + \cdots + r_m = r} e_{r_1}(\mu; i_1) \cdots e_{r_m}(\mu; i_m).$$

If $r = 0$, we set $e_r(\mu, i_1, \ldots, i_m) = 1$ and if $r < 0$, we set $e_r(\mu, i_1, \ldots, i_m) = 0$. Let I_μ^λ be the ideal generated by

(4.30) $$\left\{ e_r(\mu, i_1, \ldots, i_m) \;\middle|\; \begin{array}{l} m \geq 1, 1 \leq i_1 < \cdots < i_m \leq d \\ r > m - \mu_{i_1} + \cdots + \mu_{i_m} - \lambda_{l+1} - \cdots - \lambda_n \\ \text{where } l = \#\{i \mid \mu_i > 0, i \neq i_1, \ldots, i_m\} \end{array} \right\}.$$

Note that $I_\mu^\lambda \subseteq I^\lambda$ holds. Write

$$R_\mu^\lambda = P^\mu / I_\mu^\lambda.$$

Brundan and Ostrik [8] proved that

$$H^*(X_\mu^\lambda) \cong R_\mu^\lambda.$$

First we want to show that R_μ^λ acts on K^S. Clearly, P^μ acts on K^S, by converting polynomials into dots on the facets meeting S.

Lemma 4.6. *The ideal I_μ^λ annihilates any foam in K^S.*

Proof. The following argument demonstrates that it suffices to show this for the case when $n_s = 0$. Let $u, v \in B^S$. For each $1 \leq i \leq n$ with $s_i = -$, glue a Y onto the i-th boundary edge of u and v, respectively. Call these new webs $\hat{u}$ and $\hat{v}$, respectively. Note that $\partial \hat{u} = \partial \hat{v} = \hat{S}$, where $\hat{S} = (+^d)$. Let $f \in {}_uK_v$ be any foam. For each $1 \leq i \leq n$ with $s_i = -$, glue a digon foam on top of the i-th facet of f meeting S. The new foam $\hat{f}$, obtained in this way, belongs to ${}_{\hat{u}}K_{\hat{v}}$. Note that we

---

[3]When comparing to Khovanov's result for $\mathfrak{sl}_2$, the reader should be aware that he labels the Springer fiber by λ^T, the transpose of λ.

can re-obtain f by capping off $\hat{f}$ with dotted digon foams. Any polynomial $p \in I^\lambda_\mu \subseteq I^\lambda$ acting on f also acts on $\hat{f}$ (using the relations in (Dot Migration) on the digon foams). So, if we know that $p\hat{f} = 0$, then it follows that $pf = 0$.

Thus, without loss of generality, assume that $n_s = 0$. We are now going to show that I^λ annihilates K^S.

As follows from Definition in (4.29), I^λ is generated by the elementary symmetric polynomials $e_r(x_{i_1}, \ldots, x_{i_m})$, for the following values of m and r.

$$
\begin{aligned}
m &= 2n + 1 & ; & \quad r > 2n - 2, \\
m &= 2n + 2 & ; & \quad r > 2n - 4, \\
&\;\vdots & & \;\;\vdots \;\; \vdots \\
m &= 3n - 1 & ; & \quad r > 2, \\
m &= 3n & ; & \quad r > 0.
\end{aligned}
$$

Note that for $m = 3n$, we simply get all completely symmetric polynomials of positive degree in the variables $x_1, \ldots, x_d$. Any such polynomial p annihilates any foam $f \in {_u}K_v$, because by the complete symmetry of p, the dots can all be moved to the three facets around one singular edge. The relations (Dot Migration) then show that p kills f.

Now suppose $m = 3n - \ell$, for $\ell > 0$. So we must have $r > 2\ell$. The argument we are going give does not depend on the particular choice of $i_1, \ldots, i_m \subseteq \{1, 2, \ldots, d\}$, so, without loss of generality, let us assume that $(i_1, \ldots, i_m) = (1, \ldots, m)$.

Let f be any foam in ${_u}K_v$.

First assume that $\ell = 1$.

$$
\begin{aligned}
&e_r(x_1, \ldots, x_{d-1})f \\
={} &- e_{r-1}(x_1, \ldots, x_{d-1})x_d f \\
={} &e_{r-2}(x_1, \ldots, x_{d-1})x_d^2 f \\
&\;\;\vdots \\
={} &(-1)^r x_d^r f,
\end{aligned}
$$

All these equations follow from the fact that, for any $j > 0$, we have

$$
e_j(x_1, \ldots, x_d) = e_j(x_1, \ldots, x_{d-1}) + e_{j-1}(x_1, \ldots, x_{d-1})x_d,
$$

and the fact that $e_j(x_1, \ldots, x_d)f = 0$, as we proved above in the previous case for $m = 3n$. Since in this case we have $r > 2$, we see that

$$
(-1)^r x_d^r f = 0,
$$

by Relation (3D). This finishes the proof for this case.

In general, for $\ell \geq 1$, we get that $e_r(x_1, \ldots, x_{d-\ell})f$ is equal to a linear combination of terms of the form

$$
x_{d-\ell+1}^{r_1} x_{d-\ell+2}^{r_2} \cdots x_d^{r_\ell} f,
$$

with $r_1 + \cdots + r_\ell = r$. Since $r > 2\ell$, there exists a $1 \leq j \leq \ell$ such that $r_j > 2$, in each term. So each term kills f, by Relation (3D). This finishes the proof. $\qquad\square$

Note that Lemma 4.6 shows that there is a well-defined homomorphism of graded algebras $c_S \colon R_\mu^\lambda \to Z(K^S)$, defined by

$$c_S(p) = p1.$$

Similarly, there is a filtration preserving homomorphism

$$P^\mu \to Z(G^S)$$

defined by $p \mapsto p1$. This homomorphism does not descend to R_μ^λ, because the relations in G^S are deformations of those in K^S, but the associated graded homomorphism maps P^μ to $E(Z(G^S))$ and we have

$$E(P^\mu 1) = R_\mu^\lambda 1.$$

Before giving our following result, we recall that Brundan and Ostrik [8] showed that

$$\dim H^*(X_\mu^\lambda) = \#\mathrm{Col}_\mu^\lambda.$$

They actually gave a concrete basis, but we do not need it here.

Lemma 4.7. *We have*

$$\dim Z(G^S) = \#\mathrm{Col}_\mu^\lambda.$$

Proof. Let J be any state-string satisfying condition (4.1). We define

(4.31) $$z_J = \sum_{u \in B^S} \sum_T e_{u,T} \in G^S,$$

where the second sum is over all 3-colorings of u extending J.

First we show that $z_J \in Z(G^S)$. For any $u, v \in B^S$, let $f \in {}_uG_v$. Choose two arbitrary compatible colorings T_1 and T_2 of u and v, respectively. Assume that $e_{T_1}fe_{T_2} \neq 0$. Then we have

$$z_J e_{T_1} f e_{T_2} = \begin{cases} e_{T_1} f e_{T_2}, & \text{if } T_1 \text{ extends } J, \\ 0 & \text{else.} \end{cases}$$

We also have

$$e_{T_1} f e_{T_2} z_J = \begin{cases} e_{T_1} f e_{T_2}, & \text{if } T_2 \text{ extends } J, \\ 0 & \text{else.} \end{cases}$$

This shows that $z_J \in Z(G^S)$, because T_1 and T_2 are compatible, and so T_1 extends J if and only if T_2 extends J.

Note that

$$\sum_J z_J = 1 \quad \text{and} \quad z_J z_{J'} = \delta_{J,J'} z_J.$$

In particular, the z_J's are linearly independent.

For any state-string J satisfying condition (4.1), the central idempotent z_J belongs to $P^\mu 1$. In order to see this, first note that, for any $u \in B^S$, the element

$$z_J 1_u = \sum_{T \text{ extends } J} e_{u,T},$$

belongs to $P^\mu 1_u$. This holds, because only the colors of the boundary edges of u are fixed. We can sum over all possible 3-colorings of the other edges, which implies that these edges only contribute

a factor 1 to $z_J 1_u$. Furthermore, we see that $z_J 1_u = p_J 1_u$, for a fixed polynomial $p_J \in P^\mu$, i.e. p_J is independent of u. Therefore, we have

$$z_J = \sum_{u \in B^S} p_J 1_u = p_J 1 \in P^\mu 1.$$

It remains to show that $Z(G^S) z_J = \mathbb{C} z_J$. Let $z \in Z(G^S)$. By the orthogonality of Gornik's symmetric idempotents, we have

$$z = \sum_{u,T} e_{u,T} z e_{u,T}.$$

By Theorem 3.10, we know that

$$e_{u,T} z e_{u,T} = \lambda_{u,T}(z) e_{u,T},$$

for a certain $\lambda_{u,T}(z) \in \mathbb{C}$. Therefore, we have

$$z = \sum_{u,T} \lambda_{u,T}(z) e_{u,T} \in \bigoplus_{u,T} \mathbb{C} e_{u,T}.$$

By Lemma 4.4, we know that $z_J \neq 0$. This shows that

$$\{z_J \mid J \text{ satisfying condition (4.1)}\}$$

forms a basis of $Z(G^S)$. By Proposition 4.2, the claim of the lemma follows. $\qquad\square$

Theorem 4.8. *The degree preserving algebra homomorphism*

$$c_S \colon R^\lambda_{\mu_S} \to Z(K^S)$$

is an isomorphism.

Proof. In Corollary 5.19 it will be shown that

$$\dim H^*(X^\lambda_{\mu_S}) = \dim Z(K^S),$$

so it suffices to show that c_S is injective.

Lemma 4.6 shows that (as graded complex algebras)

$$R^\lambda_\mu 1 \subset Z(K^S).$$

As already mentioned above, Brundan and Ostrik [8] showed that

$$H^*(X^\lambda_\mu) \cong R^\lambda_\mu$$

as graded complex algebras.

The proof of Lemma 4.7 shows that the filtration preserving homomorphism

$$P^\mu \to Z(G^S),$$

defined by $p \mapsto p1$, is surjective. Note the $E(\cdot)$ is not a map. However, a filtered algebra A and its associated graded $E(A)$ are isomorphic as vector spaces. In particular, they satisfy

$$\dim A = \dim E(A).$$

Therefore, since $p \mapsto p1$ is a surjection of vector spaces, we have

$$\dim Z(G^S) = \dim E(Z(G^S)) = \dim E(P^\mu 1) = \dim P^\mu 1.$$

Recall that $E(P^\mu 1) = R^\lambda_\mu 1$ and $\dim Z(G^S) = \dim R^\lambda_\mu$. This shows

$$\dim R^\lambda_\mu 1 = \dim P^\mu 1 = \dim Z(G^S) = \dim R^\lambda_\mu,$$

which implies that the map c_S is injective. □

5. WEB ALGEBRAS AND THE CYCLOTOMIC KLR ALGEBRAS

5.1. **Howe duality.** We first recall classical Howe duality briefly. Our main references are [28] and [27], where the reader can find the proofs of the results which we recall below and other details.

Let us briefly explain Howe duality.[4] The two natural actions of $\mathrm{GL}_m = \mathrm{GL}(m, \mathbb{C})$ and of $\mathrm{GL}_n = \mathrm{GL}(n, \mathbb{C})$ on $\mathbb{C}^m \otimes \mathbb{C}^n$ commute and the two groups are each others commutant. We say that the actions of GL_m and GL_n are *Howe dual*.

More interestingly, their actions on the symmetric powers

$$S^p \left(\mathbb{C}^m \otimes \mathbb{C}^n \right)$$

and on the alternating powers

$$\Lambda^p \left(\mathbb{C}^m \otimes \mathbb{C}^n \right)$$

are also Howe dual, for any $p \in \mathbb{N}$. These are called the *symmetric* and the *skew* Howe duality of GL_m and GL_n, respectively. In this paper, we are only considering skew Howe duality.

Note that skew Howe duality implies that we have the following decomposition into irreducible $\mathrm{GL}_m \times \mathrm{GL}_n$-modules.

$$\tag{5.1} \Lambda^p \left(\mathbb{C}^m \otimes \mathbb{C}^n \right) \cong \bigoplus_\lambda V_\lambda \otimes W_{\lambda'},$$

where λ ranges over all partitions with p boxes and at most m rows and n columns and λ' is the transpose of λ.

Here V_λ is the unique irreducible GL_m-module of highest weight λ and $W_{\lambda'}$ is the unique irreducible GL_n-module of highest weight λ'.

Without giving a full proof of (5.1), which can be found in Section 4.1 of [28], we note that it is easy to write down the highest weight vectors in the decomposition of

$$\Lambda^p \left(\mathbb{C}^m \otimes \mathbb{C}^n \right).$$

Define

$$\epsilon_{ij} = \epsilon_i \otimes \epsilon_j,$$

for any $1 \leq i \leq m$ and $1 \leq j \leq n$. Here the ϵ_i and the ϵ_j are the canonical basis elements of $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively. Let λ be one of the highest GL_m weights in (5.1). Write $\lambda = (\lambda_1, \ldots, \lambda_m)$ with $n \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$. Then

$$\begin{aligned} v_{\lambda,\lambda'} &= \left(\epsilon_{11} \wedge \cdots \wedge \epsilon_{1\lambda_1} \right) \wedge \left(\epsilon_{21} \wedge \cdots \wedge \epsilon_{2\lambda_2} \right) \wedge \left(\epsilon_{m1} \wedge \cdots \wedge \epsilon_{m\lambda_m} \right) \\ &= \pm \left(\epsilon_{11} \wedge \cdots \wedge \epsilon_{\lambda'_1 1} \right) \wedge \left(\epsilon_{12} \wedge \cdots \wedge \epsilon_{\lambda'_2 2} \right) \wedge \left(\epsilon_{1n} \wedge \cdots \wedge \epsilon_{\lambda'_n n} \right) \end{aligned}$$

is a highest $\mathrm{GL}_m \times \mathrm{GL}_n$-weight. By convention, we exclude factors ϵ_{ij} for which $\lambda_i = 0$ or $\lambda'_j = 0$.

Now restrict to SL_m and assume that $p = mk$, for some $k \in \mathbb{N}$. By Schur's lemma, the decomposition in (5.1) implies that

$$\tag{5.2} \mathrm{Inv}_{\mathrm{GL}_m} \left(\Lambda^p \left(\mathbb{C}^m \otimes \mathbb{C}^n \right) \right) \cong \mathrm{Hom}_{\mathrm{SL}_m} \left(\mathbb{C}, \Lambda^p \left(\mathbb{C}^m \otimes \mathbb{C}^n \right) \right) \cong W_{(m^k)},$$

where $\mathbb{C}$ denotes the trivial representation.

Decompose

$$\mathbb{C}^n \cong \mathbb{C}\epsilon_1 \oplus \mathbb{C}\epsilon_2 \oplus \cdots \oplus \mathbb{C}\epsilon_n$$

---

[4]We follow Kamnitzer's exposition in "The ubiquity of Howe duality", which is online available at https://sbseminar.wordpress.com/2007/08/10/the-ubiquity-of-howe-duality/.

into its one-dimensional $\mathfrak{gl}_n$-weight spaces. Then we have

$$(5.3) \qquad \Lambda^p\left(\mathbb{C}^m \otimes \mathbb{C}^n\right) \cong \bigoplus_{(p_1,\ldots,p_n)\in\Lambda(n,p)} \Lambda^{p_1}\left(\mathbb{C}^m\right) \otimes \Lambda^{p_2}\left(\mathbb{C}^m\right) \otimes \cdots \otimes \Lambda^{p_n}\left(\mathbb{C}^m\right)$$

as $\mathrm{GL}_m \times T$-modules, where T is the diagonal torus in GL_n.

This decomposition implies that

$$(5.4) \qquad \mathrm{Inv}_{\mathrm{SL}_m}\left(\Lambda^{p_1}\left(\mathbb{C}^m\right) \otimes \Lambda^{p_2}\left(\mathbb{C}^m\right) \otimes \cdots \otimes \Lambda^{p_n}\left(\mathbb{C}^m\right)\right) \cong W(p_1,\ldots,p_n),$$

where $W(p_1,\ldots,p_n)$ denotes the $(p_1,\ldots,p_n)$-weight space of $W_{(m^k)}$.

In the next subsection, we use Kuperberg's $\mathfrak{sl}_3$-webs to give a q-version of the isomorphism in (5.4), for $U_q(\mathfrak{sl}_3)$ and $U_q(\mathfrak{gl}_n)$ with $n = 3k$ and $k \in \mathbb{N}$ arbitrary but fixed.

When we finished the first complete version of this paper, the only available results on the quantum version of skew Howe duality were those in Section 6.1 in [14]. Cautis's paper does not contain Definition 5.2 nor Lemma 5.4, which are our main results in the next subsection.

Independently and around the same time as our preprint appeared, Cautis, Kamnitzer and Morrison finished a paper on $\mathfrak{sl}_n$-webs in which they gave the $\mathfrak{sl}_n$-generalization of Definition 5.2 and Lemma 5.4. Their paper is now the best reference for quantum skew Howe duality in general. We therefore refer to their paper for the general case of quantum skew Howe duality (for some extra details the reader might also want to have a look at [48]) and restrict ourselves to the case of interest to us in this paper.

As already mentioned in the introduction, Lauda, Queffelec and Rose [44] wrote an independent paper in which they defined and used $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$-foams to categorify special cases of quantum skew Howe duality. Their $\mathfrak{sl}_3$ case is very similar to ours, but is used for the purpose of studying $\mathfrak{sl}_3$-knot homology. The decategorification of their results also contains the analogue of Definition 5.2 and Lemma 5.4.

5.2. **The uncategorified story.**

5.2.1. *Enhanced sign sequences.* In this section we slightly generalize the notion of a sign sequence/string. We call this generalization *enhanced sign sequence* or *enhanced sign string*. Note that, with a slight abuse of notation, we use $\hat{S}$ for sign strings and S for enhanced sign string throughout the whole section.

Definition 5.1. An *enhanced sign sequence/string* is a sequence $S = (s_1,\ldots,s_n)$ with entries $s_i \in \{\circ, -1, +1, \times\}$, for all $i = 1,\ldots n$. The corresponding weight $\mu = \mu_S \in \Lambda(n,d)$ is given by the rules

$$\mu_i = \begin{cases} 0, & \text{if } s_i = \circ, \\ 1, & \text{if } s_i = 1, \\ 2, & \text{if } s_i = -1, \\ 3, & \text{if } s_i = \times. \end{cases}$$

Let $\Lambda(n,d)_3 \subset \Lambda(n,d)$ be the subset of weights with entries between 0 and 3. Recall that $\Lambda(n,d)_{1,2}$ denotes the subset of weights with only 1 and 2 as entries.

Let $n = d = 3k$. For any enhanced sign string S such that $\mu_S \in \Lambda(n,n)_3$, we define $\hat{S}$ to be the sign sequence obtained from S by deleting all entries that are equal to $\circ$ or $\times$ and keeping the linear ordering of the remaining entries. Similarly, for any $\mu \in \Lambda(n,n)_3$, let $\hat{\mu}$ be the weight

obtained from μ by deleting all entries which are equal to $\circ$ or 3. Thus, if $\mu = \mu_S$, for a certain enhanced sign string S, then $\hat{\mu} = \mu_{\hat{S}}$. Note that $\hat{\mu} \in \Lambda(m, d)_{1,2}$, for a certain $0 \le m \le n$ and $d = 3(k - (n - m))$.

Note that for any semi-standard tableau $T \in \mathrm{Std}_\mu^{(3^k)}$, there is a unique semi-standard tableau $\hat{T} \in \mathrm{Std}_{\hat{\mu}}^{(3^{k-(n-m)})}$, obtained by deleting any cell in T whose label appears three times and keeping the linear ordering of the remaining cells within each column.

Conversely, let $\mu' \in \Lambda(m, d)_{1,2}$, with $m \le n$ and $d = 3(k - (n - m))$. In general, there is more than one $\mu \in \Lambda(n, n)_3$ such that $\hat{\mu} = \mu'$, but at least one. Choose one of them, say μ_0. Then, given any $T' \in \mathrm{Std}_{\mu'}^{(3^{k-(n-m)})}$, there is a unique $T \in \mathrm{Std}_{\mu_0}^{(3^k)}$ such that $\hat{T} = T'$.

The construction of T is as follows. Suppose that i is the smallest number such that $(\mu_0)_i = 3$.

(1) In each column c of T', there is a unique vertical position such that all cells above that position have label smaller than i and all cells below that position have label greater than i. Insert a new cell labeled i precisely in that position, for each column c.

(2) In this way, we obtain a new tableau of shape $(3^{k-(n-m)+1})$. It is easy to see that this new tableau is semi-standard. Now apply this procedure recursively for each $i = 1, \dots, n$, such that $(\mu_0)_i = 3$.

(3) In this way, we obtain a tableau T of shape (3^k). Since in each step the new tableau that we get is semi-standard, we see that T belongs to $\mathrm{Std}_{\mu_0}^{(3^k)}$.

Note also that $\hat{T} = T'$. This shows that for a fixed $\mu \in \Lambda(n, n)_3$, we have a bijection

$$\mathrm{Std}_\mu^{(3^k)} \ni T \longleftrightarrow \hat{T} \in \mathrm{Std}_{\hat{\mu}}^{(3^{k-(n-m)})}.$$

Given an enhanced sign sequence S, such that $\mu_S \in \Lambda(n, n)_3$, we define

$$W^S = W^{\hat{S}}.$$

In other words, as a vector space W^S does not depend on the $\circ$ and $\times$-entries of S. However, they do play an important role below. Similarly, we define

$$B^S = B^{\hat{S}} \quad \text{and} \quad K^S = K^{\hat{S}}.$$

5.2.2. *An instance of q-skew Howe duality.* Let $V_{(3^k)}$ be the irreducible $U_q(\mathfrak{gl}_n)$-module of highest weight (3^k). By restriction, $V_{(3^k)}$ is also a $U_q(\mathfrak{sl}_n)$-module and, since it is a weight representation, it is a $\dot{U}(\mathfrak{sl}_n)$-module, too. It is well-known (see [25] and [54]) for example) that

$$\dim V_{(3^k)} = \sum_{\mu \in \Lambda(n,n)_3} \#\mathrm{Std}_\mu^{(3^k)}.$$

Note that a tableau of shape (3^k) can only be semi-standard if its filling belongs to $\Lambda(n, n)_3$, so strictly speaking we could drop the 3-subscript. More precisely, if

$$V_{(3^k)} = \bigoplus_{\mu \in \Lambda(n,n)_3} V^\mu$$

is the $U_q(\mathfrak{gl}_n)$-weight decomposition of $V_{(3^k)}$, then

$$\dim V^\mu = \#\mathrm{Std}_\mu^{(3^k)}.$$

Note that the action of $U_q(\mathfrak{gl}_n)$ on $V_{(3^k)}$ descends to $S_q(n,n)$ and recall that there exists a surjective algebra homomorphism

$$\psi_{n,n}\colon \dot{U}_q(\mathfrak{sl}_n) \to S_q(n,n).$$

The action of $\dot{U}_q(\mathfrak{sl}_n)$ on $V_{(3^k)}$ is equal to the pull-back of the action of $S_q(n,n)$ via $\psi_{n,n}$.
  Define

$$W_{(3^k)} = \bigoplus_{S\in\Lambda(n,n)_3} W^S.$$

Below, we will show that $S_q(n,n)$ acts on $W_{(3^k)}$. Pulling back the action via $\psi_{n,n}$, we see that $W_{(3^k)}$ is a $\dot{U}_q(\mathfrak{sl}_n)$-module. We will also show that

$$W_{(3^k)} \cong V_{(3^k)}$$

as $S_q(n,n)$-modules, and therefore also as $\dot{U}_q(\mathfrak{sl}_n)$-modules, and that the space W^S corresponds to the μ_S-weight space of $V_{(3^k)}$.

Let us define the aforementioned left action of $S_q(n,n)$ on $W_{(3^k)}$. The reader should compare this action to the categorical action on the objects in Section 4.2 in [50]. Note that our conventions in this paper are different from those in [50].

Definition 5.2. Let

$$\phi\colon S_q(n,n) \to \mathrm{End}_{\mathbb{Q}(q)}\left(W_{(3^k)}\right)$$

be the homomorphism of $\mathbb{Q}(q)$-algebras defined by gluing the following webs on top of the elements in $W_{(3^k)}$.

We use the convention that vertical edges labeled 1 are oriented upwards, vertical edges labeled 2 are oriented downwards and edges labeled 0 or 3 are erased.

The orientation of the horizontal edges is uniquely determined by the orientation of the vertical edges. With these conventions, one can check that the horizontal edge is always oriented from right to left for E_{+i} and from left to right for E_{-i}.

Furthermore, let $\lambda \in \Lambda(n,n)$ and let S be any sign string such that $\mu_S \in \Lambda(n,n)_3$. For any $w \in W^S$, we define

$$\phi(1_\lambda)w = 0, \quad \text{if} \quad \mu_S \neq \lambda.$$

By $\phi(1_\lambda)w$ we mean the left action of $\phi(1_\lambda)$ on w.
  In particular, for any $\lambda > (3^k)$, we have $\phi(1_\lambda) = 0$ in $\mathrm{End}_{\mathbb{Q}(q)}\left(W_{(3^k)}\right)$.

Let us give two examples to show how these conventions work. We only write down the relevant entries of the weights and only draw the important edges. We have

$$E_{+1}1_{(22)} \mapsto$$

$$E_{-2}E_{+1}1_{(121)} \mapsto$$
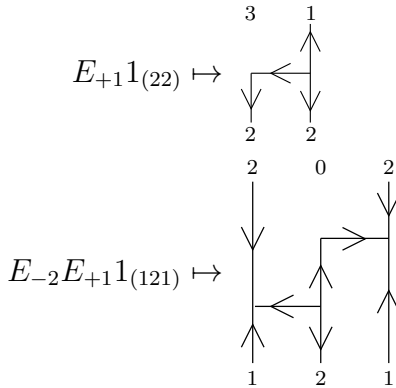
Remark 5.3. Note that the introduction of enhanced sign strings is necessary for the definition of ϕ to make sense. Although as a vector space W^S does not depend on the entries of S which are equal to $\circ$ or $\times$, the $S_q(n,n)$ action on W^S does depend on them.

Lemma 5.4. *The map ϕ in Definition 5.2 is well-defined.*

Proof. It follows immediately from its definition that ϕ preserves the three relations (2.17), (2.18) and (2.19).

Checking case by case, one can easily show that ϕ preserves (2.20) by using the relations (2.3), (2.4) and (2.5). We do just one example and leave the other cases to the reader. The figure below shows the image of the relation

$$E_1 E_{-1}1_{(21)} - E_{-1}E_1 1_{(21)} = 1_{(21)}$$

under ϕ.
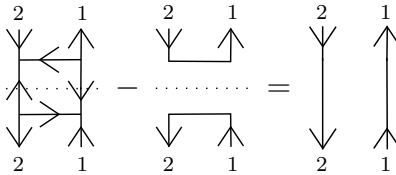
This relation is exactly the third Kuperberg relation in (2.5). □

Lemma 5.5. *The map ϕ gives rise to an isomorphism*

$$\phi \colon V_{(3^k)} \to W_{(3^k)}$$

of $S_q(n,n)$-modules.

Proof. Note that the empty web $w_h = w_{(3^k)}$, which generates $W_{(\times^k, 0^{2k})} \cong \mathbb{Q}(q)$, is a highest weight vector.

The map ϕ induces a homomorphism of $S_q(n,n)$-modules

$$\phi \colon S_q(n,n)1_{(3^k)} \to W_{(3^k)},$$

defined by

$$\phi(x1_{(3^k)}) = \phi(x)w_h.$$

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It is well-known that

$$V_{(3^k)} \cong S_q(n,n)1_{(3^k)}/(\mu > (3^k)),$$

where $(\mu > (3^k))$ is the ideal generated by all elements of the form $x1_\mu y1_{(3^k)}$ such that we have $x, y \in S_q(n,n)$ and μ is some weight greater than (3^k). Since we are using $\mathfrak{sl}_3$-webs, the kernel of ϕ contains $(\mu > (3^k))$, so ϕ descends to $V_{(3^k)}$. Since $V_{(3^k)}$ is irreducible and ϕ is clearly non-zero, the map $\phi \colon V_{(3^k)} \to W_{(3^k)}$ is injective.

As we already remarked above, we have

$$\dim V_{(3^k)} = \sum_{\mu_S \in \Lambda(n,n)_3} \#\mathrm{Std}_{\mu_S}^{(3^k)}$$

$$= \sum_{\mu_S \in \Lambda(n,n)_3} \dim W^S = \dim W_{(3^k)}.$$

Therefore, we have

$$V_{(3^k)} \cong \phi\left(S_q(n,n)\right)w_h \cong W_{(3^k)},$$

which finishes the proof. $\qquad\square$

We want to explain two more facts about the isomorphism in Lemma 5.5, which we will need later.

Recall that there is an inner product on $V_{(3^k)}$. First of all, there is a q-antilinear involution (*bar involution*) on $\mathbb{Q}(q)$ determined by

$$\overline{f(q)} = f(q^{-1}),$$

for any $f(q) \in \mathbb{Q}(q)$. Recall Lusztig's q-antilinear algebra anti-automorphism τ of $S_q(n,n)$ defined by

$$\tau(1_\lambda) = 1_\lambda, \quad \tau(1_{\lambda+\alpha_i}E_i 1_\lambda) = q^{-1-\overline{\lambda}_i}1_\lambda E_{-i}1_{\lambda+\alpha_i}, \quad \tau(1_\lambda E_{-i}1_{\lambda+\alpha_i}) = q^{1+\overline{\lambda}_i}1_{\lambda+\alpha_i}E_i 1_\lambda.$$

The *q-Shapovalov form* $\langle \cdot, \cdot \rangle_{\mathrm{Shap}}$ on $V_{(3^k)}$ is the unique q-sesquilinear form such that

- $\langle v_h, v_h \rangle_{\mathrm{Shap}} = 1$, for a fixed highest weight vector v_h.
- $\langle xv, v' \rangle_{\mathrm{Shap}} = \langle v, \tau(x)v' \rangle_{\mathrm{Shap}}$, for any $x \in S_q(n,n)$ and any $v, v' \in V_{(3^k)}$.
- $\langle f(q)v, g(q)v' \rangle_{\mathrm{Shap}} = \overline{f(q)}g(q)\langle v, v' \rangle_{\mathrm{Shap}}$, for any any $v, v' \in V_{(3^k)}$ and $f(q), g(q) \in \mathbb{Q}(q)$.

We can also define an inner product on $W_{(3^k)}$, using the Kuperberg bracket. Let S be any enhanced sign string S, such that $\mu_S \in \Lambda(n,n)_3$. Denote the length of the sign string $\hat{S}$ by $\ell(\hat{S})$.

Definition 5.6. Define the q-sesquilinear *normalized Kuperberg form* by

- $\langle w_h, w_h \rangle_{\mathrm{Kup}} = 1$, for a fixed highest weight vector w_h.
- $\langle u, v \rangle_{\mathrm{Kup}} = q^{\ell(\hat{S})}\langle u^* v \rangle_{\mathrm{Kup}}$, for any $u, v \in B^S$.
- $\langle f(q)u, g(q)v \rangle_{\mathrm{Kup}} = \overline{f(q)}g(q)\langle u, v \rangle_{\mathrm{Kup}}$, for any $u, v \in B^S$ and $f(q), g(q) \in \mathbb{Q}(q)$.

The following lemma motivates the normalization of the Kuperberg form.

Lemma 5.7. *The isomorphism of $S_q(n,n)$-modules*

$$\phi \colon V_{(3^k)} \to W_{(3^k)}$$

is an isometry.

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Proof. First note that

$$\langle (E_{\pm i}u)^*v\rangle_{\text{Kup}} = \langle u^*E_{\mp i}v\rangle_{\text{Kup}},$$

for any $u, v \in W^S$ and any $i = 1, \ldots, n$, which is exactly (2) from above. This shows that the result of the lemma holds up to normalization.

Our normalization of the Kuperberg form matches the normalization of the q-Shapovalov form. One can easily check this case by case. Let us just do two examples. Let $i = 1$. Then one has $E_1 1_{(a,b,\ldots)} = 1_{(a+1,b-1,\ldots)} E_1$. If $(a, b, \ldots) \in \Lambda(n, n)_3$ such that $a - b = -1$, then

$$\ell(\widehat{(a, b)}) = \ell(\widehat{(a + 1, b - 1)}),$$

where ℓ indicates the length of the sign sequence. This matches

$$\tau(E_1 1_{(a,b)}) = 1_{(a,b)} E_{-1}.$$

If $(a, b) = (2, 1)$, then $E_1 1_{(2,1,\ldots)} = 1_{(3,0,\ldots)} E_1$. Note that

$$\ell(\widehat{(2, 1, \ldots)}) = \ell(\widehat{(3, 0, \ldots)}) + 2.$$

This $+2$ cancels exactly with the -2, which appears as the exponent of q in

$$\tau(E_1 1_{(2,1,\ldots)}) = q^{-2} 1_{(2,1,\ldots)} E_{-1}.$$

$\square$

We will need one more fact about ϕ. For any $i = 1, \ldots, n$ and any $a \in \mathbb{N}$, let

$$E_{\pm i}^{(a)} = \frac{E_{\pm i}^a}{[a]!}$$

denote the *divided power* in $S_q(n, n)$. Recall the following relations for the divided powers.

$$(5.5) \qquad E_{\pm i}^{(a)} E_{\pm i}^{(b)} 1_\lambda = \begin{bmatrix} a + b \\ a \end{bmatrix} E_{\pm i}^{(a+b)} 1_\lambda,$$

$$(5.6) \qquad E_{+i}^{(a)} E_{-i}^{(b)} 1_\lambda = \sum_{j=0}^{\min(a,b)} \begin{bmatrix} a - b + \lambda_i - \lambda_{i+1} \\ j \end{bmatrix} E_{-i}^{(b-j)} E_{+i}^{(a-j)} 1_\lambda,$$

$$(5.7) \qquad E_{-i}^{(b)} E_{+i}^{(a)} 1_\lambda = \sum_{j=0}^{\min(a,b)} \begin{bmatrix} b - a - (\lambda_i - \lambda_{i+1}) \\ j \end{bmatrix} E_{+i}^{(a-j)} E_{-i}^{(b-j)} 1_\lambda.$$

Here $[a]!$ denotes the *quantum factorial* and $\begin{bmatrix} a \\ b \end{bmatrix}$ denotes the *quantum binomial*.

The images of the divided powers under

$$\phi \colon S_q(n, n) \to \text{End}(W_{(3^k)})$$

are easy to compute. For example, we have (for simplicity, we only draw two of the strands and write $E = E_{+i}$)

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Therefore, we get

$$\phi(E^{(2)}1_{(0,2)}) = \qquad .$$

Another interesting example is

$$\phi(E^2 1_{(0,3)}) = \quad {}_1\!\!\begin{array}{c} \overset{2}{} \quad \overset{1}{} \\ \boxed{} \\ \underset{0}{} \quad \underset{3}{} \end{array}\!\!{}_2 \; = \; [2]\;\searrow\!\!\smile \; ,$$

which shows that

$$\phi(E^{(2)}1_{(03)}) = \quad \searrow\!\!\smile \; .$$

The final example we will consider is $\phi(E^{(3)}1_{(0,3)})$. We see that

$$\phi(E^3 1_{(0,3)}) = \quad {}^2_1\!\!\begin{array}{c} \overset{3}{} \quad \overset{0}{} \\ \boxed{} \\ \underset{0}{} \quad \underset{3}{} \end{array}\!\!{}^1_2 \; = \; \bigcirc\!\!\Leftarrow \; = \; [3]! \qquad .$$

Thus, we have

$$\phi(E^{(3)}1_{(0,3)}) = \qquad ,$$

which is the unique empty web from $(\circ, \times)$ to $(\times, \circ)$.

Note that (5.6) and (5.7) imply that, for any $a \in \mathbb{N}$, we have

(5.8) $\qquad E^{(a)}_{-i} E^{(a)}_{+i} 1_{(\dots,0,a,\dots)} = 1_{(\dots,0,a,\dots)} \quad$ and $\quad E^{(a)}_{+i} E^{(a)}_{-i} 1_{(\dots,a,0,\dots)} = 1_{(\dots,a,0,\dots)}$

in $S_q(n,n)$. Similarly, let $S_q(n,n)/I$, where I denotes the two-sided ideal generated by all 1_μ such that $\mu > (3^k)$. Again by (5.6) and (5.7), we have

(5.9) $\qquad E^{(3-a)}_{-i} E^{(3-a)}_{+i} 1_{(\dots,a,3,\dots)} = 1_{(\dots,a,3,\dots)} \quad$ and $\quad E^{(3-a)}_{+i} E^{(3-a)}_{-i} 1_{(\dots,3,a,\dots)} = 1_{(\dots,3,a,\dots)}$

in $S_q(n,n)/I$. One can check that ϕ maps the two sides of the equations in (5.8) and (5.9) to isotopic diagrams. For example, ϕ maps

$$E^{(2)}_- E^{(2)}_+ 1_{(0,2)} = 1_{(0,2)}$$

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to
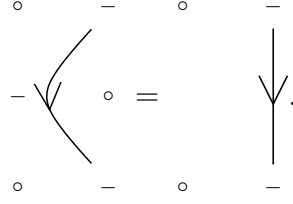
The proof of the following lemma is based on an algorithm, which we call the *enhanced inverse growth algorithm*. The result is needed later to show essential surjectivity in Theorem 5.16.

Lemma 5.8. *Let S be any enhanced sign string such that $\mu_S \in \Lambda(n,n)_3$. For any $w \in B^S$, there exists a product of divided powers x, such that*

$$\phi(x1_{(3^k)}) = w.$$

Proof. Choose any $w \in B^S$. We consider $w \in B^S_{(\times^k, \circ^{2k})}$, i.e. a non-elliptic web with (empty) lower boundary determined by $(\times^k, \circ^{2k})$ and upper boundary determined by S. Express w using the growth algorithm, in an arbitrary way. Suppose there are m steps in this instance of the growth algorithm. The element x is built up in $m+2$ steps, i.e. an initial step, one step for each step in the growth algorithm, and a last step. During the construction of x, we always keep track of the $\circ$s and $\times$s. At each step the strands of w are numbered according to their position in x.

If the H, Y or arc-move is applied to two non-consecutive strands, we first have to apply some divided powers, as in (5.8) and (5.9), to make them consecutive. Let $x_k \in S_q(n,n)$ be the element assigned to the k-th step and let μ^k be the weight after the k-step, i.e. $x_k = 1_{\mu^{k-1}} x_k 1_{\mu^k}$. The element x we are looking for is the product of all x_k.

(1) Take $x_0 = 1_{\mu_S}$.

(2) Suppose that the k-th step in the growth algorithm is applied to the strands i and $i + r$, for some $r \in \mathbb{N}_{>0}$. This means that the entries of μ^{k-1} satisfy $\mu_j \in \{0,3\}$, for all $j = i+1, \ldots, i+r-1$. Let x'_k be the product of divided powers which "swap" the $(\mu_{i+1}, \ldots, \mu_{i+r-1})$ and μ_{i+r}. So, we first swap μ_{i+r-1} and μ_{i+r}, then μ_{i+r-2} and μ_{i+r} etc. Now, the rule in the growth algorithm, still corresponding to the k-th step, can be applied to the strands i and $i + 1$.

(3) Suppose that it is an H-rule. If the bottom of the H is a pair (up-arrow down-arrow), then take $x_k = x'_k E_{+i}$. If the bottom of the H is a pair (down-arrow up-arrow), then take $x_k = x'_k E_{-i}$.

(4) Suppose that the rule, corresponding to the k-th step in the growth algorithm, is a Y-rule. If the bottom strand of Y is oriented downward, then take $x_k = x'_k E_{-i}$. If it is oriented upward, take $x_k = x'_k E_{+i}$. Note that these two choices are not unique. They depend on where you put 0 or 3 in μ^k. The choice we made corresponds to taking $(\mu^k_i, \mu^k_{i+1}) = (2,0)$ in the first case and $(\mu^k_i, \mu^k_{i+1}) = (1,3)$ in the second case. Other choices would be perfectly fine and would lead to equivalent elements in $S_q(n,n)1_{(3^k)}/(\mu > (3^k))$.

(5) Suppose that the rule, corresponding to the k-th step in the growth algorithm, is an arc-rule. If the arc is oriented clockwise, take $x_k = x'_k E^{(2)}_{-i}$. If the arc is oriented counter-clockwise, take $x_k = x'_k E_{-i}$. Again, these choices are not unique. They correspond to taking $(\mu^k_i, \mu^k_{i+1}) = (3,0)$ in both cases.

(6) After the m-th step in the growth algorithm, which is the last one, we obtain μ^m, which is a sequence of 3's and 0's. Let x_{m+1} be the product of divided powers which reorders the entries of μ^m, so that $\mu^{m+1} = (3^k)$.

(7) Take $x = 1_{\mu_S} x_1 x_2 \cdots x_{m+1} 1_{(3^k)} \in S_q(n, n)$. Note that x is of the form $E_{\underline{i}} 1_{(3^k)}$. From the analysis of the images of the divided powers under ϕ, it is clear that

$$\phi(x) = w.$$

□

We do a simple example to illustrate Lemma 5.8. Let

Then the algorithm in the proof of Lemma 5.8 gives

$$x = 1_{(111)} E_{-1} E_{-2} E_{-1} 1_{(300)},$$

or as a picture (read from bottom to top)

We are now ready to start explaining the categorified story.

5.3. **And its categorification.** Let $\mathcal{W}^S = K^S\text{-pMod}_{\mathrm{gr}}$ be the category of all finite dimensional projective graded K^S-modules. In what follows, it will sometimes be useful to consider homomorphisms of arbitrary degree, so we define

$$\mathrm{HOM}_B(M, N) = \bigoplus_{t \in \mathbb{Z}} \mathrm{Hom}_B(M\{t\}, N),$$

for any finite dimensional associative unital graded algebra B and any finite dimensional graded B-modules M and N. Note that for almost all $t \in \mathbb{Z}$ we have $\mathrm{Hom}_B(M\{t\}, N)$, so $\mathrm{HOM}_B(M, N)$ is still finite dimensional. We also define

$$\dim_q \mathrm{HOM}_B(M, N) = \sum_{t \in \mathbb{Z}} q^t \dim \mathrm{Hom}_B(M\{t\}, N).$$

Moreover, we need the following notions throughout the rest of the section.

Suppose that S is an enhanced sign string such that $\mu_S \in \Lambda(n, n)_3$. For any $u \in B^S$, let

$$P_u = \bigoplus_{w \in B^S} {}_u K_w.$$

Then we have

$$K^S = \bigoplus_{u \in B^S} P_u,$$

and so P_u is an object in $\mathcal{W}^S$, for any $u \in B^S$. Note that, for any $u, v \in B^S$, we have

$$\mathrm{HOM}(P_u, P_v) \cong {}_u K_v,$$

where an element in ${}_u K_v$ acts on P_u by composition on the right-hand side.

Similarly, we can define

$$_uP = \bigoplus_{w \in B^S} {}_uK_w,$$

which is a right graded projective K^S-module.

Remark 5.9. Just one warning: The reader should not confuse P_u with $P_{u,T}$ in Section 3.

5.3.1. *The definition of* $\mathcal{W}_{(3^k)}$. Recall that S denotes an enhanced sign string. Define

$$K_{(3^k)} = \bigoplus_{\mu_S \in \Lambda(n,n)_3} K^S$$

and

$$\mathcal{W}_{(3^k)} = K_{(3^k)}\text{-p}\mathbf{Mod}_{\mathrm{gr}} \cong \bigoplus_{\mu_S \in \Lambda(n,n)_3} \mathcal{W}^S.$$

The main goal of this section is to show that $\mathcal{W}_{(3^k)}$ is a strong $\mathfrak{sl}_n$-2-representation and that

$$\mathcal{W}_{(3^k)} \cong \mathcal{V}_{(3^k)}$$

as strong-$\mathfrak{sl}_n$ 2-representations.

This will imply that

$$K_0^{\oplus}(\mathcal{W}_{(3^k)})_{\mathbb{Q}(q)} \cong V_{(3^k)}.$$

Note that

$$K_0^{\oplus}(\mathcal{W}_{(3^k)})_{\mathbb{Q}(q)} \cong \bigoplus_{\mu_S \in \Lambda(n,n)_3} K_0^{\oplus}(\mathcal{W}^S)_{\mathbb{Q}(q)}.$$

We will show that this corresponds exactly to the $U_q(\mathfrak{gl}_n)$-weight space decomposition of $V_{(3^k)}$. In particular, this will show that

$$(5.10) \qquad\qquad K_0^{\oplus}(\mathcal{W}^S)_{\mathbb{Q}(q)} \cong W^S,$$

for any enhanced sign sequence S such that $\mu_S \in \Lambda(n,n)_3$.

First, we have to recall the definitions of *sweet* bimodules.

5.3.2. *Sweet bimodules.* Note that the following definitions and results are the $\mathfrak{sl}_3$ analogues of those in Section 2.7 in [32].

Definition 5.10. Given rings R_1 and R_2, a (R_1, R_2)-bimodule N is called *sweet* if it is finitely generated and projective as a left R_1-module and as a right R_2-module.

If N is a sweet (R_1, R_2)-bimodule, then the functor

$$N \otimes_{R_2} -: R_2\text{-}\mathbf{Mod} \to R_1\text{-}\mathbf{Mod}$$

is exact and sends projective modules to projective modules. Given a sweet (R_1, R_2)-bimodule M and a sweet (R_2, R_3)-bimodule N, then the tensor product $M \otimes_{R_2} N$ is a sweet (R_1, R_3)-bimodule.

Let S and S' be two enhanced sign strings. Then $\widehat{B}_S^{S'}$ denotes the set of all monomial webs whose boundary is divided into a lower part, determined by S, and an upper part, determined by S'. By a *monomial* web we mean a web given by one diagram. Let $B_S^{S'} \subset \widehat{B}_S^{S'}$ be the subset of non-elliptic webs.

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For any $w \in \widehat{B}_S^{S'}$, define a graded finite dimensional $(K^{S'}, K^S)$-bimodule $\Gamma(w)$ by

$$\Gamma(w) = \bigoplus_{u \in B^{S'}, v \in B^S} {}_u\Gamma(w)_v,$$

with

$${}_u\Gamma(w)_v = \mathcal{F}^c(u^*wv)\{\ell(\hat{S})\}.$$

The left and right actions of K^S on $\Gamma(w)$ are defined by applying the multiplication foam in 3.3 to

$${}_rK_u \otimes {}_u\Gamma(w)_v \to {}_r\Gamma(w)_v \quad \text{and} \quad {}_u\Gamma(w)_v \otimes {}_vK_r \to {}_u\Gamma(w)_r.$$

Let $w \in \widehat{B}_S^{S'}$. Then $w = c_1 w_1 + \cdots + c_t w_t$, for certain $w_i \in B_S^{S'}$ and $c_i \in \mathbb{N}[q, q^{-1}]$. Since all relations which are satisfied by the Kuperberg bracket have categorical analogues for foams, this shows that

$$\Gamma(w) \cong \Gamma(w_1)^{\oplus c_1} \oplus \cdots \oplus \Gamma(w_t)^{\oplus c_t}.$$

We have the following analogue of Proposition 3 in [32].

Proposition 5.11. *For any $w \in \widehat{B}_S^{S'}$, the graded $(K^{S'}, K^S)$-bimodule $\Gamma(w)$ is sweet.*

Proof. As a left K^S-module, we have

$$\Gamma(w) \cong \bigoplus_{v \in B^S} \Gamma(w)_v,$$

where

$$\Gamma(w)_v = \bigoplus_{u \in B^{S'}} {}_u\Gamma(w)_v.$$

So, as far as the left action is concerned, it is sufficient to show that $\Gamma(w)_v$ is a left projective $K^{S'}$-module. Note that, as a left $K^{S'}$-module, we have

$$\Gamma(w)_v \cong \bigoplus_{u \in B^{S'}} \mathcal{F}^0(u^*wv)\{\ell(\hat{S})\}.$$

Then $wv = c_1 u_1 + \cdots + c_t u_t$, for certain $u_i \in B^{S'}$ and $c_i \in \mathbb{N}[q, q^{-1}]$. By the remarks above, this means that

(5.11) $$\Gamma(w)_v \cong P_{u_1}^{\oplus c_1}\{\ell(\hat{S}) - \ell(\hat{S}')\} \oplus \cdots \oplus P_{u_t}^{\oplus c_t}\{\ell(\hat{S}) - \ell(\hat{S}')\}.$$

This proves that $\Gamma(w)$ is projective as a left $K^{S'}$-module.

The proof that $\Gamma(w)$ is projective as a right K^S-module is similar. $\qquad\square$

It is not hard to see that (see for example [32]), for any $w \in \widehat{B}_S^{S'}$ and $w' \in \widehat{B}_{S'}^{S''}$, we have

(5.12) $$\Gamma(w'w) \cong \Gamma(w') \otimes_{K^{S'}} \Gamma(w).$$

Lemma 5.12. *Let $w, w' \in \widehat{B}_S^{S'}$. An isotopy between w and w' induces an isomorphism between $\Gamma(w)$ and $\Gamma(w')$. Two isotopies between w and w' induce the same isomorphism if and only if they induce the same bijection between the connected components of w and w'.*

Lemma 5.13. *Let $w, w' \in \widehat{B}_S^{S'}$ and let $f \in \mathbf{Foam}_3^0(w, w')$ be a foam of degree t. Then f induces a bimodule map*

$$\Gamma(f) \colon \Gamma(w) \to \Gamma(w')$$

of degree t.

Proof. Note that, for any $u \in B^{S'}$ and $v \in B^S$, the foam f induces a linear map

$$\mathcal{F}^0(1_{u^*} f 1_v) \colon \mathcal{F}^0(u^* w v) \to \mathcal{F}^0(u^* w' v),$$

by glueing $1_{u^*} f 1_v$ on top of any element in $\mathcal{F}^0(u^* w v) = \mathbf{Foam}^0_3(\emptyset, u^* w v)$. This map has degree t, e.g. the identity has degree 0 because the multiplication in K^S is degree preserving. By taking the direct sum over all $u \in B^{S'}$ and $v \in B^S$, we get a linear map

$$\Gamma(f) \colon \Gamma(w) \to \Gamma(w').$$

With the shift $\ell(\hat{S})$ in the definition of $\Gamma(w)$ and $\Gamma(w')$, we get exactly $\deg \Gamma(f) = t$.

The fact that $\Gamma(f)$ is a left K^S-module map follows from the following observation. For any $u \in B^S$ and $v \in B^{S'}$, the linear map $\mathcal{F}^0(1_{u^*} f 1_v)$ corresponds to the linear map

$$\mathbf{Foam}^0_3(u, w v) \to \mathbf{Foam}^0_3(u, w' v)$$

determined by horizontally composing with $f 1_v$ on the right-hand side. This map clearly commutes with any composition on the left-hand side.

Analogously, the linear map $\mathcal{F}^0(1_{u^*} f 1_v)$ corresponds to the linear map

$$\mathbf{Foam}^0_3(w^* u, v) \to \mathbf{Foam}^0_3((w')^* u, v)$$

determined by horizontally composing with $f^* 1_u$ on the left-hand side. This map clearly commutes with any composition on the right-hand side.

These two observations show that $\Gamma(f)$ is a $(K^{S'}, K^S)$-bimodule map. $\qquad\square$

It is not hard to see that, for any $f \in \mathbf{Foam}^0_3(w, w')$ and $g \in \mathbf{Foam}^0_3(w', w'')$, we have

$$\Gamma(fg) = \Gamma(f)\Gamma(g).$$

And similarly, for any $u_1, u_2 \in \widehat{B}^{S'}_S$ and $u'_1, u'_2 \in \widehat{B}^{S''}_{S'}$ and for any $f \in \mathbf{Foam}^0_3(u_1, u_2)$ and $f' \in \mathbf{Foam}^0_3(u'_1, u'_2)$, we have a commuting square

$$
\begin{array}{ccc}
\Gamma(u_1 u'_1) & \xrightarrow{\Gamma(f \circ f')} & \Gamma(u_2 u'_2) \\
\cong \downarrow & & \cong \downarrow \\
\Gamma(u_1) \otimes_{K_{S'}} \Gamma(u'_1) & \xrightarrow{\Gamma(f) \otimes \Gamma(f')} & \Gamma(u_2) \otimes_K \Gamma(u'_2)
\end{array}
$$

where the vertical isomorphisms are as in (5.12).

5.3.3. *The strong $\mathfrak{sl}_n$-2-representation on $\mathcal{W}_{(3^k)}$.* We are now going to use sweet bimodules to define a strong $\mathfrak{sl}_n$-2-representation on $\mathcal{W}_{(3^k)}$. It suffices to define a functorial action of $\mathcal{S}(n, n)$ on $\mathcal{W}_{(3^k)}$, which then gives the desired strong $\mathfrak{sl}_n$-2-representation by pulling back along the 2-functor $\Psi_{n,n} \colon \mathcal{U}(\mathfrak{sl}_n) \to \mathcal{S}(n, n)$. For convenience, we consider $\mathcal{S}(n, n)$ to be a monoidal category rather than a 2-category in this section.

Definition 5.14. On objects: The functorial action of any object $\mathcal{E}_{\underline{i}} 1_\lambda$ in $\mathcal{S}(n, n)$ on $\mathcal{W}_{(3^k)}$ is defined by tensoring with the sweet bimodule (see Proposition 5.11)

$$\Gamma\left(\phi\left(E_{\underline{i}} 1_\lambda\right)\right).$$

Recall that $\phi \colon S_q(n, n) \to \mathrm{End}_{\mathbb{Q}(q)}(W_{(3^k)})$ was defined in Definition 5.2.

On morphisms: We give a list of the foams associated to the generating morphisms of $\mathcal{S}(n,n)$. Applying Γ to these foams determines the natural transformations associated to the morphisms of $\mathcal{S}(n,n)$.

As before, we only draw the most important part of the foams, omitting partial identity foams. Our conventions are the following.

(1) We read the regions of the morphisms in $\mathcal{S}(n,n)$ from right to left and the morphisms themselves from bottom to top.
(2) The corresponding foams we read from bottom to top and from front to back.
(3) Vertical front edges labeled 1 are assumed to be oriented upward and vertical front edges labeled 2 are assumed to be oriented downward.
(4) The convention for the orientation of the back edges is precisely the opposite.
(5) A facet is labeled 0 or 3 if and only if its boundary has edges labeled 0 or 3.

In the list below, we always assume that $i < j$. Finally, all facets labeled 0 or 3 in the images below have to be erased, in order to get real foams. For any $\lambda > (3^k)$, the image of the elementary morphisms below is taken to be zero, by convention.

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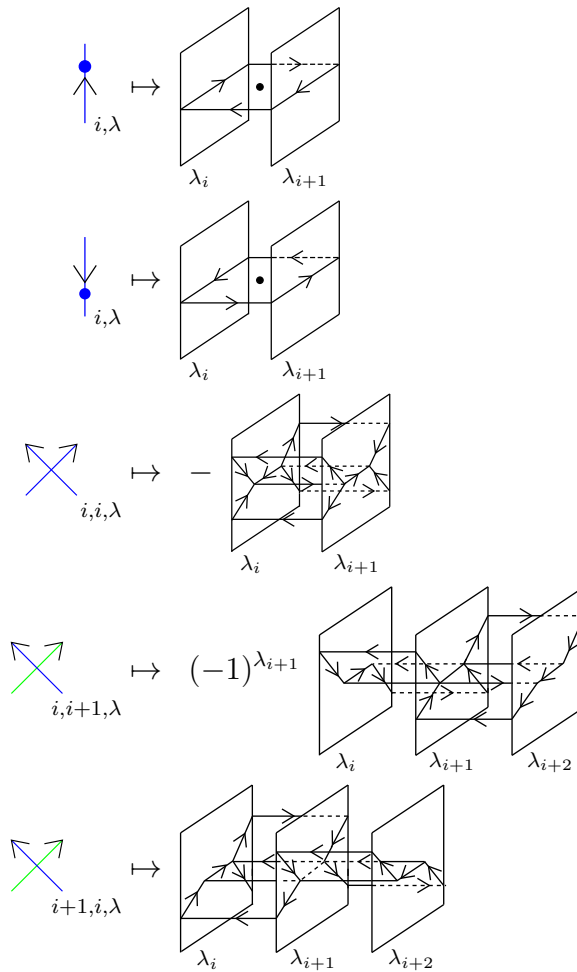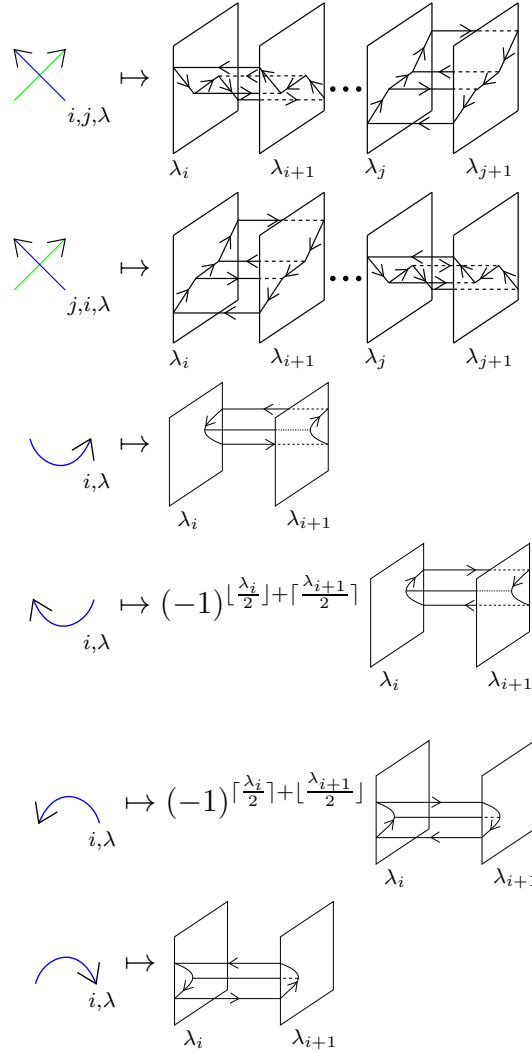
Proposition 5.15. *The formulas in Definition 5.14 determine a well-defined graded functorial action of $\mathcal{S}(n,n)$ on $\mathcal{W}_{(3^k)}$.*

Proof. A tedious but straightforward case by case check, for each generating morphism and each λ which give a non-zero foam, shows that each of the foams in Definition 5.14 has the same degree as the elementary morphism in $\mathcal{S}(n,n)$ to which it is associated. Note that it is important to erase the facets labeled 0 or 3, before computing the degree of the foams. We do just one example here. We have

We see that f has one facet labeled 0 and another labeled 3, so those two facets have to be erased. Therefore, f has 12 vertices, 14 edges and 3 faces, i.e.

$$\chi(f) = 12 - 14 + 3 = 1.$$

The boundary of f has 12 vertices and 12 edges, so

$$\chi(\partial f) = 12 - 12 = 0.$$

Note that the two circular edges do not belong to ∂f, because the circular facets have been removed. In this section we draw the foams horizontally, so b is the number of horizontal edges at the top and the bottom of f, which go from the front to the back. Thus, for f we have

$$b = 4.$$

Altogether, we get

$$q(f) = 0 - 2 + 4 = 2.$$

In order to show that the categorical action is well-defined, one has to check that it preserves all the relations in Definition 2.17. modulo 2 this was done in the proof of Theorem 4.2 in [47]. At the time there was a small issue about the signs in [36], which prevented the author to formulate and prove Theorem 4.2 in [47] over $\mathbb{C}$. That issue has now been solved (see [37] and [50] for more information) and in this paper we use the sign conventions from [50], which are compatible with those from [37]. We laboriously checked all these relations again, but now over $\mathbb{C}$ and with the signs above. The arguments are exactly the same, so let us not repeat them one by one here. Instead, we first explain how we computed the signs for the categorical action above and why they give the desired result over $\mathbb{C}$. After that, we will do an example. For a complete case by case check, we refer to the arguments used in the proof of Theorem 4.2 in [47]. The reader should check that our signs above remove the sign ambiguities in that proof.

One can compute the signs above as follows: first check the relations only involving strands of one color, i.e. the $\mathfrak{sl}_2$-relations. The first thing to notice is that the foams in the categorical action do not satisfy relation (2.34), i.e. for all λ, which give a non-zero foam, the sign is wrong. Therefore, one is forced to multiply the foam associated to
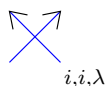
by -1, for all λ.

After that, compute the foams associated to the degree zero bubbles (real bubbles, not fake bubbles) and adjust the signs of the images of the left cups and caps accordingly. This way, most of the signs of the images of the left cups and caps get determined. The remaining ones can be determined by imposing the zig-zag relations in (2.22) and (2.23).

Of course, one could also choose to adjust the signs of the images of the right cups and caps. That would determine a categorical action that is naturally isomorphic to the one in this paper.

After these signs have been determined, one can check that all $\mathfrak{sl}_2$-relations are preserved by the categorical action.

The next and final step consists in determining the signs of
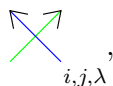
for $i \neq j$. First one can check that cyclicity is already preserved. The relations in (2.25) are preserved by the corresponding foams, which are all isotopic, with our sign choices for the foams associated to the left cups and caps. Therefore, cyclicity does not determine any more signs.

The relations in (2.35) are preserved on the nose, for $i = j$ and $|i - j| > 1$. For $|i - j| = 1$, they are only preserved up to a sign. Note that, since the corresponding foams are all isotopic, the signs actually come from the sign choice for the foams associated to the left cups and caps. Thus, whenever the total sign in the image of (2.35) becomes negative, one has to change the sign of one of the two crossings (not of both of course). Our choice has been to change the sign of the foam associated to

$$\text{\scriptsize (figure) } _{i,i+1,\lambda},$$
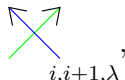
whenever necessary. Any other choice, consistent with all the previous sign choices, leads to a naturally isomorphic categorical action. It turns out that the sign has to be equal to $(-1)^{\lambda_{i+1}}$, after checking for all λ.

After this, one can check that all relations involving two or three colors are preserved by the categorical action. Note that we have not specified an image for the fake bubbles. As stressed repeatedly in [36], fake bubbles do not exist as separate entities. They are merely formal symbols, used as computational devices to keep the computations involving real bubbles tidy and short. As we are using $\mathfrak{sl}_3$-foams in this paper, most of the dotted bubbles are mapped to zero. Therefore, under the categorical action it is very easy to convert the fake bubbles in the relations in Definition 2.17 into linear combinations of real bubbles, using the infinite Grassmannian relation (2.32). Thus, there is no need to use fake bubbles in this paper.

Finally, let us do two examples, i.e. one involving only one color and another involving two colors.

The left side of the equation in (2.31), for $i = 1$ and $\lambda = (1, 2)$ (the other entries are omitted for simplicity), becomes
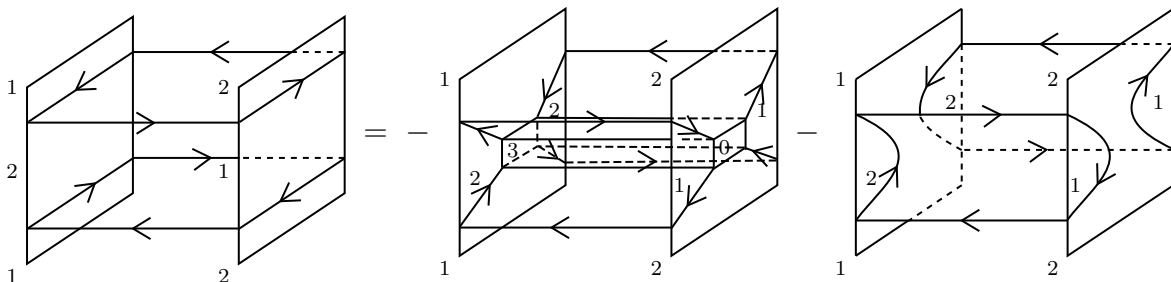
This foam equation is precisely the relation (SqR). Note that the signs match perfectly, because we have

$$\text{sign}\left(\text{\scriptsize (figure)}_{i,(12)}\right) = + \quad \text{and} \quad \text{sign}\left(\text{\scriptsize (figure)}_{i,(12)}\right) = -.$$

The equation in (2.36), for $(i, j) = (1, 2)$ and $\lambda = (121)$ (the other entries are omitted for simplicity), becomes

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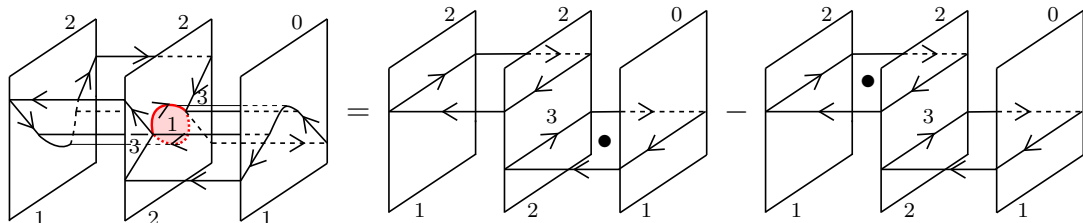
To see that this holds, apply the (RD) relation to the foam on the l.h.s., in order to remove the disc bounded by the red singular circle on the middle sheet. □

Let $W_h \cong \mathbb{C}$ be the unique indecomposable projective graded $K^{(\times^k, \circ^{2k})}$-module of degree zero. Recall that $K^{(\times^k, \circ^{2k})}$ is generated by the empty diagram, so W_h is indeed one-dimensional. It is the categorification of w_h, the highest weight vector in $W_{(3^k)}$.

As already remarked, we can pull back the functorial action of $\mathcal{S}(n, n)$ on $\mathcal{W}_{(3^k)}$ via

$$\Psi_{n,n} \colon \mathcal{U}(\mathfrak{sl}_n) \to \mathcal{S}(n, n).$$

We are now able to prove one of our main results. Recall that $\mathcal{V}$ is any additive, idempotent complete category, which allows an integrable, graded categorical action by $\mathcal{U}(\mathfrak{sl}_n)$.

Theorem 5.16. *There exists an equivalence of strong $\mathfrak{sl}_n$-2-representations*

$$\Phi \colon \mathcal{V}_{(3^k)} \to \mathcal{W}_{(3^k)}.$$

Proof. As we already mentioned above, we have

$$\mathrm{End}_{\mathcal{W}_{(3^k)}}(W_h) \cong \mathbb{C}.$$

Let Q be any indecomposable object in $\mathcal{W}_{(3^k)}$. There exists an enhanced sign string S such that Q belongs to $\mathcal{W}^S$. Therefore, there exists a basis web $w \in B^S$ and a $t \in \mathbb{Z}$, such that Q is a graded direct summand of $P_w\{t\}$. Without loss of generality, we may assume that $t = -\ell(\hat{S})$.

By Lemma 5.8 and Proposition 5.15, there exists an object of X in $\mathcal{S}(n, n)$ such that Q is a direct summand of XW_h. This holds, because in $\dot{\mathcal{S}}(n, n)$, the Karoubi envelope of $\mathcal{S}(n, n)$, the divided powers correspond to direct summands of ordinary powers. For more details on the categorification of the divided powers see [36] and [39].

Proposition 2.23 now proves the existence of Φ. □

An easy consequence of Theorem 5.16 is the following.

Corollary 5.17. *By Theorem 5.16, the $S_q(n, n)$-module map*

$$K_0^{\oplus}(\Phi)_{\mathbb{Q}(q)} \colon K_0^{\oplus}(\mathcal{V}_{(3^k)})_{\mathbb{Q}(q)} \to K_0^{\oplus}(\mathcal{W}_{(3^k)})_{\mathbb{Q}(q)}$$

is an isomorphism. □

The following consequence of Theorem 5.16 is very important and we thank Ben Webster for explaining its proof.

Proposition 5.18. *The graded algebras $K_{(3^k)}$ and $R_{(3^k)}$ are Morita equivalent.*

Proof. For each weight $\mu_S \in \Lambda(n, n)_3$, let $\nu_S = (\nu_1, \dots, \nu_n)$ be the unique element of $\mathbb{Z}_{\leq 0}[I]$ such that

$$\mu_S - (3^k) = \sum_{i=1}^{n} \nu_i \alpha_i.$$

We are going to show that for each such μ_S, the graded algebras K^S and $R_{(3^k)}^{\nu_S}$ are Morita equivalent. This proves the proposition after taking direct sums over all weights.

Let $\mu_S \in \Lambda(n, n)_3$. Define

$$\Theta_{\mu_S} = \bigoplus_{\underline{i} \in \mathrm{Seq}(\nu_S)} \mathcal{E}_{\underline{i}} W_h \in \mathcal{W}^S.$$

In the proof of Theorem 5.16, we already showed that every object in $\mathcal{W}^S$ is a direct summand of XW_h for some object $X \in \mathcal{S}(n, n)$. By the biadjointness of the $\mathcal{E}_i$ and $\mathcal{E}_{-i}$ in $\mathcal{S}(n, n)$ (i.e. up to a shift) and the fact that W_h is a highest weight object, it is not hard to see that XW_h itself is a direct summand of a finite direct sum of degree shifted copies of Θ_{μ_S}. This shows that every object in $\mathcal{W}^S$ is a direct summand of a finite direct sum of degree shifted copies of Θ_{μ_S}. Since K^S is a finite dimensional, complex algebra, every finite dimensional, graded K^S-module has a projective cover and is therefore a quotient of a finite direct sum of degree-shifted copies of Θ_{μ_S}. This shows that Θ_{μ_S} is a projective generator of K^S-$\mathbf{Mod}_{\mathrm{gr}}$.

Theorem 5.16 also shows that

$$\mathrm{End}_{K^S}\left(\Theta_{\mu_S}\right) \cong R^{\nu_S}_{(3^k)}$$

holds.

By a general result due to Morita, it follows that the above observations imply that K^S and $R^{\nu_S}_{(3^k)}$ are Morita equivalent. For a proof see Theorem 5.55 in [59], for example. $\qquad\square$

We can draw two interesting conclusions from Proposition 5.18.

In [6], Brundan and Kleshchev defined an explicit isomorphism between blocks of cyclotomic Hecke algebras and cyclotomic KLR-algebras. Theorem 3.2 in [4] implies that the center of the cyclotomic Hecke algebra, which under Brundan and Kleshchev's isomorphism corresponds to $R^{\nu_S}_{(3^k)}$, has the same dimension as $H^*(X^{(3^k)}_{\mu_S})$.

Corollary 5.19. *The center of K^S is isomorphic to the center of $R^{\nu_S}_{(3^k)}$. In particular, we have*

$$\dim Z(K^S) = \dim Z(R^{\nu_S}_{(3^k)}) = \dim H^*(X^{(3^k)}_{\mu_S}).$$

Proof. We only have to prove the first statement, which follows from the well known fact that Morita equivalent algebras have isomorphic centers. For a proof see for example Corollary 18.42 in [43]. $\qquad\square$

In Theorem 4.8 we used Corollary 5.19 to give an explicit isomorphism

$$H^*(X^{(3^k)}_{\mu_S}) \to Z(K^S).$$

Remark 5.20. Just for completeness, we remark that the aforementioned results in [4] and [6] together with the results in [5], which we have not explained, imply that

$$H^*(X^{(3^k)}_{\mu_S}) \cong Z(R^{\nu_S}_{(3^k)}),$$

so we have not proved anything new about $Z(R^{\nu_S}_{(3^k)})$.

Another interesting consequence of Proposition 5.18 is the following.

Corollary 5.21. K^S *is a graded cellular algebra.*

Proof. In Corollary 5.12 in [29], Hu and Mathas proved that $R^{\nu_S}_{(3^k)}$ is a graded cellular algebra.

In [41], König and Xi showed that "being a cellular algebra" is a Morita invariant property, provided that the algebra is defined over a field whose characteristic is not equal to two.

These two results together with Proposition 5.18 prove that the $\mathfrak{sl}_3$-web algebra K^S is indeed a graded cellular algebra. $\qquad\square$

The precise definition of a graded cellular algebra can be found in [29]. We will not recall it here. In a follow-up paper, we intend to discuss the cellular basis of K^S in detail and use it to derive further results on the representation theory of K^S.

Remark 5.22. Corollary 5.21 is the $\mathfrak{sl}_3$ analogue of Corollary 3.3 in [9], which proves that Khovanov's arc algebra H^m is a graded cellular algebra. It is easy to give a cellular basis of H^m. The proof of cellularity follows from checking a small number of cases by hand. For K^S, we tried to mimic that approach, but had to give up because the combinatorics got too complex.

5.3.4. *The Grothendieck group of* $W_{(3^k)}$. Recall that W^S has an inner product defined by the normalized Kuperberg form (see Definition 5.6). The *Euler form*

$$\langle [P], [Q] \rangle_{\mathrm{Eul}} = \dim_q \mathrm{HOM}(P, Q)$$

defines a q-sesquilinear form on $K_0^{\oplus}(\mathcal{W}^S)_{\mathbb{Q}(q)}$.

Lemma 5.23. *Let S be an enhanced sign sequence. Take*

$$\gamma_S \colon W^S \to K_0^{\oplus}(\mathcal{W}^S)_{\mathbb{Q}(q)}$$

to be the $\mathbb{Q}(q)$-linear map defined by

$$\gamma_S(u) = q^{-\ell(\hat{S})}[P_u],$$

for any $u \in B^S$. Then γ_S is an isometric embedding.
 This implies that the $\mathbb{Q}(q)$-linear map

$$\gamma_W = \bigoplus_{\mu(S) \colon \Lambda(n,n)_3} \gamma_S$$

defines an isometric embedding

$$\gamma_W \colon W_{(3^k)} \to K_0^{\oplus}(\mathcal{W}_{(3^k)})_{\mathbb{Q}(q)}.$$

Proof. Note that the normalized Kuperberg form, because of the relations 2.3, 2.4 and 2.5, and the Euler form are non-degenerate. For any pair $u, v \in B^S$, we have

$$\dim_q \mathrm{HOM}(P_u\{-\ell(\hat{S})\}, P_v\{-\ell(\hat{S})\}) = \dim_q \mathrm{HOM}(P_u, P_v) = \dim_q {}_uK_v = q^{\ell(\hat{S})}\langle u^*v\rangle_{\mathrm{Kup}}.$$

The factor $q^{\ell(\hat{S})}$ is a consequence of the grading shift in the definition of ${}_uK_v$.
 Thus, γ_S is an isometry. Since the normalized Kuperberg form is non-degenerate, this implies that γ_S is an embedding. $\square$

Remark 5.24. In Section 5.5 in [56], Morrison and Nieh showed that P_u is not necessarily indecomposable (see also [57]). This is closely related to the contents of Remark 2.7, as Morrison and Nieh showed. Therefore, the surjectivity of γ_W is not immediately clear and we need the results of the previous subsections to establish it below.
 The $\mathfrak{sl}_2$ case is much simpler. The projective modules of the arc algebras analogous to the P_u are all indecomposable. See Proposition 2 in [32] for the details.

Theorem 5.25. *The map*

$$\gamma_W \colon W_{(3^k)} \to K_0^{\oplus}(\mathcal{W}_{(3^k)})_{\mathbb{Q}(q)}$$

is an isomorphism of $S_q(n, n)$-modules.
 This also implies that, for each sign string S with $\mu_S \in \Lambda(n, n)_3$, the map

$$\gamma_S \colon W^S \to K_0^{\oplus}(\mathcal{W}^S)_{\mathbb{Q}(q)}$$

is an isomorphism.

Proof. The proof of the theorem is only a matter of assembling already known pieces.

By Proposition 5.15 and the fact that

$$\Gamma(u) = P_u\{-\ell(\hat{S})\},$$

for any $u \in B^S$, we see that γ_W intertwines the $S_q(n,n) \cong K_0^{\oplus}(\dot{\mathcal{S}}(n,n))_{\mathbb{Q}(q)}$-actions.

We already know that γ_W is an embedding, by Lemma 5.23.

Note that, by Theorem 2.22, Lemma 5.5 and Corollary 5.17, we have the following commuting square

$$
\begin{array}{ccc}
V_{(3^k)} & \xrightarrow{\gamma_V} & K_0^{\oplus}(\mathcal{V}_{(3^k)})_{\mathbb{Q}(q)} \\
\phi \downarrow & & K_0^{\oplus}(\Phi) \downarrow \\
W_{(3^k)} & \xrightarrow{\gamma_W} & K_0^{\oplus}(\mathcal{W}_{(3^k)})_{\mathbb{Q}(q)}.
\end{array}
$$

We already know that γ_V, ϕ and $K_0^{\oplus}(\Phi)_{\mathbb{Q}(q)}$ are isomorphisms. Therefore, γ_W has to be an isomorphism. This shows that K^S indeed categorifies the μ_S-weight space of $V_{(3^k)}$. $\qquad\square$

A good question is how to find the graded, indecomposable, projective modules of K^S. Before answering that question, we need a result on the 3-colorings of webs.

Let $w \in B^S$. Recall that there is a bijection between the flows on w and the 3-colorings of w, as already mentioned in Remark 2.3. Call the 3-coloring corresponding to the canonical flow of w, the *canonical 3-coloring*, denoted T_w.

Lemma 5.26. *Let $u, v \in B^S$. If there is a 3-coloring of v which matches T_u and a 3-coloring of u which matches T_v on the common boundary S, then $u = v$.*

Proof. This result is a direct consequence of Theorem 2.6. Recall that there is a partial order on flows, and therefore on 3-colorings by Remark 2.3. This ordering is induced by the lexicographical order on the state-strings on S, which are induced by the flows. Note that two matching colorings of u and v have the same order, by definition. On the other hand, Theorem 2.6 implies that any 3-coloring of u, respectively v, has order less than or equal to that of T_u and T_v respectively. Therefore, if there exists a 3-coloring of v matching T_u, the order of T_u must be less or equal than that of T_v.

Thus, if there exists a 3-coloring of v matching T_u and a 3-coloring of u mathing T_v, then T_u and T_v must have the same order. This implies that $u = v$, because canonical 3-colorings are uniquely determined by their order and the corresponding canonical flows determine the corresponding basis webs uniquely by the growth algorithm. $\qquad\square$

Proposition 5.27. *For each web $u \in B^S$, there exists a graded, indecomposable, projective K^S-module Q_u, which is unique up to degree-preserving isomorphism, such that*

$$P_u \cong Q_u \oplus \bigoplus_{J_v < J_u} Q_v^{\oplus d(S, J_u, J_v)}.$$

Here J_u is the state string associated to the canonical flow on u, the coefficients $d(S, J_u, J_v)$ belong to $\mathbb{N}[q, q^{-1}]$ and the state strings are ordered lexicographically.

Proof. Let $u \in B^S$. Then there is a complete decomposition of 1_u into orthogonal primitive idempotents

$$1_u = e_1 + \cdots + e_r.$$

By Theorem 5.37 and Corollary 5.38, we can lift this decomposition to G^S. We do not introduce any new notation for this lift, trusting that the reader will not get confused by this slight abuse of notation.

Let $z_u \in Z(G^S)$ be the central idempotent corresponding to J_u, as defined in the proof of Lemma 4.7. We claim that there is a unique $1 \le i \le r$, such that

$$(5.13) \qquad z_u e_j = \delta_{ij} z_u 1_u,$$

for any $1 \le j \le r$.

Let us prove this claim. Note that

$$(5.14) \qquad z_u 1_u = e_{u,T_u},$$

where T_u is the canonical coloring of u, i.e. J_u only allows one compatible coloring of u, which is T_u. Since $e_{u,T_u} \ne 0$, courtesy of Lemma 3.11, this implies that

$$(5.15) \qquad z_{uu} G_u = {}_u G_u z_u = z_{uu} G_u z_u = e_{u,T_u} G^S e_{u,T_u} \cong \mathbb{C},$$

by Theorem 3.10.

We also see that there has to exist at least one $1 \le i_0 \le r$ such that $z_u e_{i_0} \ne 0$. Then, by (5.15), there exists a non-zero $\lambda_{i_0} \in \mathbb{C}$, such that

$$z_u e_{i_0} = \lambda_{i_0} z_u 1_u = \lambda_{i_0} e_{u,T_u}.$$

For any $1 \le i, j \le r$, we have

$$z_u e_i z_u e_j = z_u^2 e_i e_j = z_u \delta_{ij} e_i.$$

This implies that i_0 is unique and $\lambda_{i_0} = 1$. In order to see that this is true, suppose there exist $1 \le i_0 \ne j_0 \le r$ such that $z_u e_{i_0} \ne 0$ and $z_u e_{j_0} \ne 0$. By (5.15), there exist non-zero $\lambda_{i_0}, \lambda_{j_0} \in \mathbb{C}$ such that

$$z_u e_{i_0} = \lambda_{i_0} z_u 1_u \quad \text{and} \quad z_u e_{j_0} = \lambda_{j_0} z_u 1_u.$$

However, this is impossible, because we get

$$z_u e_{i_0} z_u e_{j_0} = \lambda_{i_0} \lambda_{j_0} z_u 1_u \ne 0,$$

which contradicts the orthogonality of $z_u e_{i_0}$ and $z_u e_{j_0}$.

Thus, for each $u \in B^S$, there is a unique primitive idempotent $e_u \in \text{End}_{\mathbb{C}}(P_u)$ that is not killed by z_u, when lifted to G^S. We define Q_u to be the corresponding graded indecomposable projective K^S-module by

$$Q_u = K^S e_u,$$

which is clearly a direct summand of $P_u = K^S 1_u$.

Let us now show that, for any $u, v \in B^S$, we have

$$Q_u \cong Q_v \Leftrightarrow u = v.$$

If $u = v$, we obviously have $Q_u \cong Q_v$. Let us prove the other implication. Suppose $Q_u \cong Q_v$. From the above, recall that e_u and e_v can be lifted to G^S. By a slight abuse of notation, call these lifted idempotents e_u and e_v again. We have

$$z_u e_u = e_{u,T_u} \ne 0 \quad \text{and} \quad z_v e_v = e_{v,T_v} \ne 0.$$

Since $Q_u \cong Q_v$, we then also have

$$z_u e_v \ne 0 \quad \text{and} \quad z_v e_u \ne 0.$$

This can only hold if T_u gives a 3-coloring of v and T_v a 3-coloring of u. By Lemma 5.26, this implies that $u = v$.

Since
$$\dim K_0^{\oplus}(\mathcal{W}^S)_{\mathbb{Q}(q)} = \dim W^S = \#B^S,$$
by Theorem 5.25, the above shows that
$$\left\{ Q_u \mid u \in B^S \right\}$$
is a basis of $K_0^{\oplus}(\mathcal{W}^S)_{\mathbb{Q}(q)}$. For any $u, v \in B^S$, we have
$$z_u 1_u = z_u e_u \quad \text{and} \quad z_v 1_u = 0, \text{ if } J_v > J_u.$$
The second claim follows from the fact that there are no admissible 3-colorings of u greater than J_u. The proposition now follows. $\qquad\square$

Remark 5.28. Proposition 5.27 proves the conjecture about the decomposition of 1_u, which Morrison and Nieh formulate in the text between Conjectures 5.14 and 5.15 in [56].

Before giving the last result of this section, we briefly recall some facts about the *dual canonical basis* of W^S. For more details see [24] and [35]. There exists a q-antilinear involution $\tilde{\psi}$ on V^S (in [24] and [35] this involution is denoted ψ' and Φ, respectively). For any sign string S and any state string J, there exists a unique element $e_{\heartsuit J}^S \in V^S$ which is $\tilde{\psi}$-invariant and satisfies

(5.16)
$$e_{\heartsuit J}^S = e_J^S + \sum_{J' < J} c(S, J, J') e_{J'}^S,$$

with $c(S, J, J') \in q^{-1}\mathbb{Z}[q^{-1}]$. (Recall again that we use q instead of v, contrary to [22] and [35] where v indicates $-q^{-1}$.) The e_J^S are the elementary tensors, which were defined in 2.6. The basis $\left\{ e_{\heartsuit J}^S \right\}$ is called the *dual canonical basis* of V^S. Restriction to the dominant closed paths (S, J) gives the dual canonical basis of W^S (see Theorem 3 in [35] and the comments below it).

We have not given a definition of $\tilde{\psi}$, but we note that $\tilde{\psi}$ is completely determined by Proposition 2 in [35], which we recall now.

Proposition 5.29. *(Khovanov-Kuperberg) Each basis web $w \in B^S$ is invariant under $\tilde{\psi}$.*

Let us show that there exists a duality $\circledast$ on $\mathcal{W}_{(3^k)}$ such that $\gamma_W \colon W_{(3^k)} \to K_0^{\oplus}(\mathcal{W}_{(3^k)})_{\mathbb{Q}(q)}$ intertwines $\tilde{\psi}$ and $K_0^{\oplus}(\circledast)_{\mathbb{Q}(q)}$.

As Brundan and Kleshchev showed in Section 4.5 in [7], there is a duality $\circledast$ on $\mathcal{V}_{(3^k)}$. It is induced by Khovanov and Lauda's [36] algebra anti-automorphism
$$* \colon R_{(3^k)} \to R_{(3^k)},$$
given by reflecting the diagrams in the x-axis and inverting their orientations. Let M be a finite dimensional, graded $R_{(3^k)}$-module, then
$$M^{\circledast} = M^{\vee}$$
as a graded vector space and $R_{(3^k)}$ acts on $M^{\circledast}$ by
$$xf(y) = f(x^*y).$$
In Theorem 4.18 in [7] Brundan and Kleshchev showed that $\circledast$ commutes with the categorical actions of $\mathcal{E}_{+i}$ and $\mathcal{E}_{-i}$, for any $i = 1, \ldots, n-1$, and the map $\gamma_V \colon V_{(3^k)} \to K_0^{\oplus}(\mathcal{V}_{(3^k)})_{\mathbb{Q}(q)}$ intertwines the usual bar-involution on $V_{(3^k)}$, which we also denote by $\tilde{\psi}$, and $K_0(\circledast)_{\mathbb{Q}(q)}$.

Using our equivalence

$$\Phi \colon \mathcal{V}_{(3^k)} \to \mathcal{W}_{(3^k)}$$

from Theorem 5.16, we can define a duality $\circledast$ on $\mathcal{W}_{(3^k)}$ exactly as above. We use the anti-automorphism $* \colon K_{(3^k)} \to K_{(3^k)}$ defined by reflecting the foams in the vertical yz-plane, i.e. the plane parallel to the front and the back of the foams in Definition 5.14, and inverting the orientation of their edges. Note that $\circledast$ maps indeed projective modules to projective modules, because K^S is a symmetric Frobenius algebra for all S such that $\mu_S \in \Lambda(n,n)_3$. Note also that, by construction, the equivalence

$$\Phi \colon \mathcal{V}_{(3^k)} \to \mathcal{W}_{(3^k)}$$

intertwines the dualities $\circledast$ on both categories.

Lemma 5.30. *For any S such that $\mu_S \in \Lambda(n,n)_3$ and for any $v \in B^S$, we have*

$$P_v^{\circledast} \cong P_v\{-2\ell(\hat{S})\}.$$

Proof. Let $u, v \in B^S$. The proof of Theorem 3.9 shows that the map determined by

$$f \mapsto \mathrm{tr}(f^*-)$$

defines a degree-preserving isomorphism

$$_uK_v \to {}_uK_v^{\vee}\{2\ell(\hat{S})\}.$$

Taking the direct sum over all $v \in B^S$, we get a degree-preserving isomorphism of graded vector spaces

(5.17) $$P_v \cong P_v^{\vee}\{2\ell(\hat{S})\}.$$

Since $*$ is an anti-automorphism on K^S, we see that

$$gf \mapsto \mathrm{tr}((gf)^*-) = \mathrm{tr}(f^*g^*-) = g\,\mathrm{tr}(f^*-).$$

This shows that (5.17) is an isomorphism of graded K^S-modules. $\qquad\square$

By Proposition 5.29, we see that the isomorphism

$$\gamma_W \colon W_{(3^k)} \to K_0^{\oplus}(\mathcal{W}_{(3^k)})_{\mathbb{Q}(q)}$$

intertwines the bar involutions $\tilde{\psi}$ and $K_0^{\oplus}(\circledast)_{\mathbb{Q}(q)}$.

In Theorem 5.14 in [7], Brundan and Kleshchev also proved that γ_V maps the canonical basis elements of $V_{(3^k)}$ to the Grothendieck classes of certain indecomposables in $K_0^{\oplus}(\mathcal{V}_{(3^k)})_{\mathbb{Q}(q)}$. They used multipartitions λ to parametrize these indecomposables, which they denoted $Y(\lambda)\{-m\}$ for degree shifts m depending on λ.

These multipartitions corresponds bijectively to the semi-standard tableaux, which in turn correspond bijectively to the elements of B^S. We therefore can denote these indecomposables by $Y_u\{-\ell(\hat{S})\}$, for $u \in B^S$. By (4.26) in [7] (note that $\# \cong \{2\ell(\hat{S})\}\circledast$), we also have

(5.18) $$Y_u\{-\ell(\hat{S})\}^{\circledast} \cong Y_u\{-\ell(\hat{S})\}.$$

Finally, in Theorem 4.18 in [7] it was proved that γ_V intertwines the q-Shapovalov form and the Euler form. The q-Shapovalov form $\langle -, - \rangle_{\text{Shap}}$ on V^S is related to the symmetric $\mathbb{Q}(q)$-bilinear form $(-, -)_{\text{Lusz}}$ defined by

(5.19) $$(-, -)_{\text{Lusz}} = \overline{\langle -, \psi(-) \rangle}_{\text{Shap}}.$$

This bilinear form is used by Lusztig in [46]. By Proposition 19.3.3 in that book, this implies that the Euler form on $\mathcal{V}_{(3^k)}$ satisfies

$$(5.20) \qquad \langle [Y_u\{-\ell(\hat{S})\}], [Y_v\{-\ell(\hat{S})\}] \rangle_{\mathrm{Eul}} \in \delta_{u,v} + q\mathbb{Z}[q]$$

for any $u, v \in B^S$.

Theorem 5.31. *The basis*

$$\left\{ q^{-\ell(\hat{S})}[Q_u] \mid u \in B^S \right\}$$

corresponds to the dual canonical basis of $\mathrm{Inv}_{U_q(\mathfrak{sl}_3)}(V^S)$, *under the isomorphisms*

$$\mathrm{Inv}_{U_q(\mathfrak{sl}_3)}(V^S) \cong W^S \cong K_0^\oplus(\mathcal{W}^S)_{\mathbb{Q}(q)}.$$

Proof. Note that by Theorem 2.6 and Proposition 5.27, we have

$$(5.21) \qquad q^{-\ell(\hat{S})}[Q_u] = e_{J_u}^S + \sum_{J' < J_u} e(S, J_u, J') e_{J'}^S$$

for any $u \in B^S$. The dual canonical basis elements are uniquely determined by the conditions that they are bar-invariant and satisfy (5.16), so all we have to show is

$$(5.22) \qquad Q_u^\circledast\{-\ell(\hat{S})\} \quad \cong \quad Q_u\{-\ell(\hat{S})\}$$
$$(5.23) \qquad e(S, J_u, J') \quad \in \quad q^{-1}\mathbb{Z}[q^{-1}].$$

The equivalence $\Phi \colon \mathcal{V}_{(3^k)} \to \mathcal{W}_{(3^k)}$ maps indecomposables to indecomposables. By the observations above, this shows that for any $u \in B^S$ there exists a $t_u \in \mathbb{Z}$ such that

$$Q_u^\circledast \cong Q_u\{t_u\}.$$

By Proposition 5.27 and Lemma 5.30, we get $t_u = -2\ell(\hat{S})$ for all $u \in B^S$. Hence, this proves (5.22). In particular, this implies that Φ maps Y_u to Q_u for any $u \in B^S$.

In order to prove (5.23), we first recall Lusztig's symmetric $\mathbb{Q}(q)$-bilinear form $(-, -)_{\mathrm{Lusz}}$ form on V^S (see Section 19.1.1 and Section 27.3 in [46]). A short calculation shows that it is completely determined by

$$(5.24) \qquad (e_{J'}^S, e_{J''}^S)_{\mathrm{Lusz}} = \delta_{J', J''},$$

for any elementary tensors $e_{J'}^S$ and $e_{J''}^S$. Lusztig's symmetric bilinear form can be restricted to $W_{\hat{S}}^{\mathcal{Z}}$ and condition (5.23) is then equivalent to the condition

$$(5.25) \qquad (q^{-\ell(\hat{S})}[Q_u], q^{-\ell(\hat{S})}[Q_v])_{\mathrm{Lusz}} \in \delta_{u,v} + q^{-1}\mathbb{Z}[q^{-1}]$$

for any $u, v \in B^S$.

Similarly, we can define a symmetric $\mathbb{Q}(q)$-bilinear web form on W^S by

$$(u, v)_{\mathrm{Kup}} = q^{-\ell(\hat{S})}\langle u^* v \rangle_{\mathrm{Kup}}.$$

We claim that both bilinear forms are equal. Since $(-, -)_{\mathrm{Lusz}}$ and $(-, -)_{\mathrm{Kup}}$ are $\mathbb{Q}(q)$-bilinear and symmetric, it suffices to show that we have

$$(u, u)_{\mathrm{Lusz}} = (u, u)_{\mathrm{Kup}},$$

for any $u \in B^S$. Let $u \in B^S$ be arbitrary and write

$$u = e^S_{J_u} + \sum_{J' < J_u} c(S, J_u, J') e^S_{J'},$$

as in Theorem 2.6. Then, by (5.24), we get

(5.26)
$$(u, u)_{\text{Lusz}} = 1 + \sum_{J' < J} c(S, J, J')^2.$$

Now let us compute $(u, u)_{\text{Kup}}$. By definition, we have

$$(u, u)_{\text{Kup}} = q^{-\ell(S)} \langle u^* u \rangle_{\text{Kup}}.$$

Consider the way in which the coefficients $c(S, J_u, J')$ change under the symmetry $x \mapsto x^*$, for x any Y, cup or cap with flow. Comparing the corresponding weights in (2.11) and (2.12), we get

$$\text{wt}(x^*) = q^{(\ell(t(x)) - \ell(b(x)))} \text{wt}(x).$$

where $t(x)$ and $b(x)$ are the top and bottom boundary of x. Recall also that the canonical flow on u has weight 0 (see Lemma 2.5). It follows that

$$\begin{aligned}
(u, u)_{\text{Kup}} &= q^{-\ell(S)} \langle u^* u \rangle_{\text{Kup}} \\
&= q^{-\ell(S)} \left(q^{\ell(S)} + q^{\ell(S)} \sum_{J' < J_u} c(S, J_u, J')^2 \right) \\
&= 1 + \sum_{J' < J_u} c(S, J_u, J')^2.
\end{aligned}$$

This finishes the proof that $(-, -)_{\text{Lusz}} = (-, -)_{\text{Kup}}$.

Comparing the definition of the bilinear and the sesquilinear web form on W^S, we see that

$$(-, -)_{\text{Kup}} = \overline{\langle -, \tilde{\psi}(-) \rangle}_{\text{Kup}},$$

just as in (5.19) (recall that the elements in B^S are $\tilde{\psi}$-invariant). Since $(-, -)_{\text{Kup}} = (-, -)_{\text{Lusz}}$, we see that condition (5.25) is equivalent to the condition

(5.27)
$$\langle q^{-\ell(\hat{S})}[Q_u], q^{-\ell(\hat{S})}[Q_v] \rangle_{\text{Kup}} \in \delta_{u,v} + q\mathbb{Z}[q]$$

for any $u, v \in B^S$. As already shown above, Φ maps Y_u to Q_u for any $u \in B^S$. Since Φ is an equivalence, $K_0^{\oplus}(\Phi)_{\mathbb{Q}(q)}$ intertwiners the Euler forms. Therefore (5.20) implies (5.27), which we had already shown to be equivalent to (5.23). $\square$

Remark 5.32. Just for completeness, let us summarize the decategorification of the above results. What they show is that the quantum skew Howe duality isomorphism

$$\phi \colon V_{(3^k)} \to W_{(3^k)}$$

intertwines the respective q-sesquilinear forms and bar-involutions, and that it maps the canonical $\dot{U}_q(\mathfrak{sl}_n)$-basis to the dual canonical $\dot{U}_q(\mathfrak{sl}_3)$-basis.

Remark 5.33. It is also worth noting that Lemma 5.23, Proposition 5.27 and Theorem 5.31 imply that the change-of-basis matrix from Kuperberg's basis B^S to the dual canonical basis of W^S is unitriangular.

In this appendix, we have collected some basic facts about filtered algebras, the associated graded algebras and the idempotents in both. Our main sources are [62] and [63]. In this appendix, everything is defined over an arbitrary commutative associative unital ring R.

Let A be a finite dimensional, associative, unital R-algebra together with an increasing filtration of R-submodules

$$\{0\} \subset A_{-p} \subset A_{-p+1} \subset \cdots \subset A_0 \subset \cdots \subset A_{m-1} \subset A_m = A.$$

Actually, for any $t \in \mathbb{Z}$ we have a subspace A_t, where we extend the filtration above by

$$A_t = \begin{cases} \{0\}, & \text{if } t < -p, \\ A, & \text{if } p \geq m. \end{cases}$$

Note that in the language of [62], such a filtration is *discrete, separated, exhaustive and complete*. If $1 \in A_0$ and the multiplication satisfies $A_i A_j \subseteq A_{i+j}$, we say that A is an associative, unital, *filtered algebra*.

The *associated graded algebra* is defined by

$$E(A) = \bigoplus_{i \in \mathbb{Z}} A_i / A_{i-1},$$

and is also associative and unital. Although A and $E(A)$ are isomorphic R-modules, they are not isomorphic as algebras.

A finite dimensional, *filtered A-module* is a finite dimensional, unitary A-module M with an increasing filtration of R-submodules

$$\{0\} \subset M_{-q} \subset M_{-q+1} \subset \cdots \subset M_t = M,$$

such that $A_i M_j \subseteq M_{i+j}$, for all $i, j \in \mathbb{Z}$, after extending the finite filtration to a $\mathbb{Z}$-filtration as above.

We define the t-fold *suspension* $M\{t\}$ of M, which has the same underlying A-module structure, but a new filtration defined by

$$M\{t\}_r = M_{r+t}.$$

Given a filtered A-module M, the *associated graded module* is defined by

$$E(M) = \bigoplus_{i \in \mathbb{Z}} M_i / M_{i-1}.$$

An A-module map $f \colon M \to N$ is said to *preserve the filtrations* if $f(M_i) \subseteq N_i$, for all $i \in \mathbb{Z}$. Any such map $f \colon M \to N$ induces a grading preserving $E(A)$-module map $E(f) \colon E(M) \to E(N)$ in the obvious way.

This way, we get a functor

$$E \colon A\text{-}\mathbf{Mod}_{\mathrm{fl}} \to E(A)\text{-}\mathbf{Mod}_{\mathrm{gr}},$$

where A-**Mod**$_\text{fl}$ is the category of finite dimensional, filtered A-modules and filtration preserving A-module maps and $E(A)$-**Mod**$_\text{gr}$ is the category of finite dimensional, graded $E(A)$-modules and grading preserving $E(A)$-module maps.

Recall that A-**Mod**$_\text{fl}$ is not an abelian category, e.g. the identity map $M \to M\{1\}$ is a filtration preserving bijective A-module map, but does not have an inverse in A-**Mod**$_\text{fl}$.

In order to avoid such complications, one can consider a subcategory with fewer morphisms. An A-module map $f\colon M \to N$ is called *strict* if

$$f(M_i) = f(M) \cap N_i$$

holds, for all $i \in \mathbb{Z}$. Let A-**Mod**$_\text{st}$ be the subcategory of filtered A-modules and strict A-module homomorphisms.

Lemma 5.34. *The restriction of E to A-**Mod**$_\text{st}$ is exact.*

We also need to recall a simple result about bases. A basis $\{x_1, \ldots, x_n\}$ of a filtered algebra A is called *homogeneous* if, for each $1 \leq j \leq n$, there exists an $i \in \mathbb{Z}$ such that $x_j \in A_i \backslash A_{i-1}$. In that case, $\{\overline{x}_1, \ldots, \overline{x}_n\}$ defines a homogeneous basis of $E(A)$, where $\overline{x}_j \in A_i/A_{i-1}$. In order to avoid cluttering our notation, we always write $\overline{x}_j$ and then specify in which subquotient we take the equivalence class by saying that it belongs to A_i/A_{i-1}.

Given a homogeneous basis $\{y_1, \ldots, y_n\}$ of the associated graded $E(A)$, we say that a homogeneous basis $\{x_1, \ldots, x_n\}$ of A *lifts* $\{y_1, \ldots, y_n\}$ if $\overline{x}_j = y_j \in A_i/A_{i-1}$ holds, for each $1 \leq j \leq n$ and the corresponding $i \in \mathbb{Z}$. The result in the following lemma is well-known. However, we could not find a reference in the literature, so we provide a short proof here.

Lemma 5.35. *Let A be a finite dimensional, filtered algebra and $\{y_1, \ldots, y_n\}$ be a homogeneous basis of $E(A)$. Then there is a homogeneous basis $\{x_1, \ldots, x_n\}$ of A which lifts $\{y_1, \ldots, y_n\}$.*

Proof. We prove the lemma by induction with respect to the filtration degree q. Suppose $A_q = 0$, for all $q < -p$, and $A_q = A$, for all $q \geq m$. Then $E(A_{-p}) = A_{-p}$. Since $\{y_1, \ldots, y_n\}$ is a homogeneous basis of $E(A)$, a subset of this basis forms a basis of A_{-p}.

For each $-p + 1 \leq q \leq m$, choose elements in A_q which lift the homogeneous subbasis of $E(A_q)$. We claim that the union of the sets of these elements, for all $-p \leq q \leq m$, form a homogeneous basis of A which lifts $\{y_1, \ldots, y_n\}$. Call it $\{x_1, \ldots, x_n\}$. By definition, the x_j lift the y_j, for all $1 \leq j \leq n$. It remains to show that the x_j are all linearly independent. This is true for $q = -p$, as shown above.

Let $q > -p$ and suppose that the claim holds for $\{x_1, \ldots, x_{m_{q-1}}\}$, the subset of $\{x_1, \ldots, x_n\}$ which belongs to A_{q-1}. Let

$$\{x_1, \ldots, x_{m_q}\} = \{x_1, \ldots, x_{m_{q-1}}\} \cup \{x_{m_{q-1}+1}, \ldots, x_{m_q}\}$$

be the subset belonging to A_q. Suppose that

(5.28)
$$\sum_{j=1}^{m_q} \lambda_j x_j = 0,$$

with $\lambda_j \in R$. Then we have

$$\sum_{j=1}^{m_q} \lambda_j \overline{x}_j = \sum_{j=m_{q-1}+1}^{m_q} \lambda_j \overline{x}_j = \sum_{j=m_{q-1}+1}^{m_q} \lambda_j y_j = 0 \in A_q/A_{q-1}.$$

By the linear independence of the y_j, this implies that $\lambda_j = 0$, for all $m_{q-1} + 1 \leq j \leq m_q$. Thus, the linear combination in (5.28) becomes

$$\sum_{j=1}^{m_{q-1}} \lambda_j x_j = 0.$$

By induction, this implies that $\lambda_j = 0$, for all $1 \leq j \leq m_{q-1}$.

This shows that $\lambda_j = 0$, for all $1 \leq j \leq n$, so the x_j are linearly independent. $\qquad\square$

For a proof of the following proposition, see for example Proposition 1 in the appendix of [63].

Proposition 5.36. *Let M and N be filtered A-modules and $f\colon M \to N$ a filtration preserving A-linear map. If $E(f)\colon E(M) \to E(N)$ is an isomorphism, then f is an isomorphism (and therefore strict too).*

The most important fact about filtered, projective modules and their associated graded, projective modules, that we need in this paper, is Theorem 6 in [62]. Note that these projective modules are the projective objects in the category A-$\mathbf{Mod}_{\mathrm{st}}$.

Theorem 5.37. *(Sjödin) Let P be a finite dimensional, graded, projective $E(A)$-module. Then there exists a finite dimensional, filtered, projective A-module P', such that $E(P') = P$. Moreover, if M is a finite dimensional, filtered, A-module, then any degree preserving $E(A)$-module map $P \to E(M)\{t\}$, for some grading shift $t \in \mathbb{Z}$, lifts to a filtration preserving A-module map $P' \to M\{t\}$.*

We also recall the following corollary of Sjödin (Corollary in [62] after Lemma 20).

Corollary 5.38. *Let M be a finite dimensional, filtered, A-module, then any finite or countable set of orthogonal idempotents in*

$$\mathrm{im}(\phi) \subset \mathrm{Hom}_{E(A)}(E(M), E(M))$$

can be lifted to $\mathrm{Hom}_A(M, M)$, where ϕ is the natural transformation

$$\phi\colon E(\mathrm{Hom}_A(M, M)) \to \mathrm{Hom}_{E(A)}(E(M), E(M)).$$

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M.M.: *CAMGSD, Instituto Superior Técnico, Lisboa, Portugal; Universidade do Algarve, Faro, Portugal*
email: mmackaay@ualg.pt

W.P.: *Courant Research Center "Higher Order Structures", University of Göttingen, Göttingen, Germany; Saint Mary's College of California, Moraga, USA*
email: wp1@stmary-ca.edu

D.T.: *Courant Research Center "Higher Order Structures", University of Göttingen, Göttingen, Germany; Mathematisches Institut, Georg-August-Universität Göttingen, Göttingen, Germany*
email: dtubben@uni-math.gwdg.de