SIMPLE TRANSITIVE 2-REPRESENTATIONS OF SMALL QUOTIENTS OF SOERGEL BIMODULES

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Abstract. In all finite Coxeter types but $I_2(12)$, $I_2(18)$ and $I_2(30)$, we classify simple transitive 2-representations for the quotient of the 2-category of Soergel bimodules over the coinvariant algebra which is associated to the two-sided cell that is the closest one to the two-sided cell containing the identity element. It turns out that, in most of the cases, simple transitive 2-representations are exhausted by cell 2-representations. However, in Coxeter types $I_2(2k)$, where $k \geq 3$, there exist simple transitive 2-representations which are not equivalent to cell 2-representations.

1. Introduction and description of the results

Classical representation theory makes a significant emphasis on problems and techniques related to the classification of various classes of representations. The recent "upgrade" of classical representation theory, known as 2*-representation theory*, has its abstract origins in [BFK, CR, KL, Ro]. The first classification result in 2-representation theory was obtained in [CR, Subsection 5.4.2] where certain "minimal" 2-representations of the Chuang-Rouquier 2-analogue of $U(\mathfrak{sl}_2)$ were classified.

The series [MM1, MM2, MM3, MM4, MM5, MM6] of papers initiated a systematic study of the so-called *finitary* 2-categories which are natural 2-analogues of finite dimensional algebras. The penultimate paper [MM5] of this series proposes the notion of a *simple transitive* 2*-representation* which seems to be a natural 2-analogue for the classical notion of a simple module. Furthermore, [MM5, MM6] classifies simple transitive 2-representations for a certain class of finitary 2-categories with a weak involution which enjoy particularly nice combinatorial properties, the so-called *(weakly) fiat* 2-categories with *strongly regular two-sided cells*. Examples of the latter 2-categories include projective functors for finite dimensional self-injective algebras, finitary quotients of finite type 2-Kac-Moody algebras and Soergel bimodules (over the coinvariant algebra) in type A. The classification results of [MM5, MM6] assert that, for such 2-categories, every simple transitive 2-representation is, in fact, equivalent to a so-called *cell* 2*-representation*, that is a natural subquotient of the regular (principal) 2-representation defined combinatorially in [MM1, MM2]. In [KM1], this classification was used to describe projective functors on parabolic category $\mathcal{O}$ in type A.

The problem of classifying simple transitive 2-representations was recently studied for several classes of finitary 2-categories which are not covered by the results in [MM5, MM6]. In particular, in [Zi] it is shown that cell 2-representations exhaust the simple transitive 2-representations for the 2-category of Soergel bimodules in Weyl type B_2. In [MZ], a similar result was proved for the 2-categories of projective functors for the two smallest non-self-injective finite dimensional algebras. Some other related results can be found in [GM1, GM2, Zh1, Zh2].

An essential novel step in this theory was made in the recent paper [MaMa] where a classification of the simple transitive 2-representations was given for certain 2-subquotient categories of Soergel bimodules in two dihedral Coxeter types. In one of the cases, it turned out that cell 2-representations do not exhaust the simple transitive 2-representations. A major part of [MaMa] is devoted to an explicit construction of the remaining simple transitive 2-representation. This construction involves a subtle interplay of various category theoretic tricks.

The present paper explores to which extent the techniques developed in [MM5, Zi, MaMa] can be used to attack the problem of classification of the simple transitive 2-representations for 2-categories of Soergel bimodules over coinvariant algebras in the general case of finite Coxeter systems. We develop the approach and intuition described in [Zi, MaMa] further and single out a situation in which this approach seems to be applicable. The combinatorial structure of the 2-category of Soergel bimodules is roughly captured by the Kazhdan-Lusztig combinatorics of the so-called *two-sided Kazhdan-Lusztig cells*. The minimal, with respect to the two-sided order, two-sided cell corresponds to the identity element. If we take this minimal two-sided cell out, in what remains there is again a unique minimal two-sided cell. This is the two-sided cell which contains all simple reflections. The main object of study in the present paper is the unique "simple" quotient of the 2-category of Soergel bimodules in which only these two smallest two-sided cells survive. This is the 2-category which we call the *small quotient* of the 2-category of Soergel bimodules. Our main result is the following statement that combines the statements of Theorems 15, 20, 28, 31, 34 and 37.

Theorem 1. *Let $\underline{\mathscr{S}}$ be the small quotient of the 2-category of Soergel bimodules over the coinvariant algebra associated to a finite Coxeter system (W, S).*

(*i*) *If W has rank greater than two or is of Coxeter type $I_2(n)$, with $n = 4$ or $n > 1$ odd, then every simple transitive 2-representation of $\underline{\mathscr{S}}$ is equivalent to a cell 2-representation.*

(*ii*) *If W is of Coxeter type $I_2(n)$, with $n > 4$ even, then, apart from cell 2-representations, $\underline{\mathscr{S}}$ has two extra equivalence classes of simple transitive 2-representations, see the explicit construction in Subsection 7.2. If $n \neq 12, 18, 30$, these are all the simple transitive 2-representations.*

We note that Coxeter type $I_2(4)$ is dealt with in [Zi] and the result in this case is similar to Theorem 1(i). In fact, the construction in Subsection 7.2 also works in type $I_2(4)$, however, it results in 2-representations which turn out to be equivalent to cell 2-representations. The exceptional types $I_2(12)$, $I_2(18)$ and $I_2(30)$ exhibit some strange connection, which we do not really understand, to Dynkin diagrams of type E that, at the moment, does not allow us to complete the classification of simple transitive 2-representations in these types.

Theorem 1 is proved by a case-by-case analysis. Similarly to the general approach of [MM5, MM6, Zi, MaMa], in each case, the proof naturally splits into two major parts:

- the first part of the proof determines the non-negative integral matrices which represent the action of Soergel bimodules corresponding to simple reflections;

- to each particular case of matrices determined in the first part, the second part of the proof provides a classification of simple transitive 2-representations for which these particular matrices are realized.

Although being technically more involved, the second part of the proof is fairly similar to the corresponding arguments in [MM5, MM6, Zi, MaMa]. For Theorem 1(ii), an additional essential part is the construction of the two extra simple transitive 2-representations. This construction follows closely the approach of [MaMa, Section 5]. Apart from this, the main general difficulty in the proof of Theorem 1 lies in the first part of the proof. Both the difficulty of this part and the approach we choose to deal with this part depends heavily on each particular Coxeter type we study.

For Coxeter groups of rank higher than two, we use a reduction argument to the rank two case. For this reduction to work, we need the statement of Theorem 1 in Coxeter types $I_2(n)$, where $n = 3, 4, 5$. For $n = 3, 5$, we may directly refer to Theorem 1(i). As already mentioned above, the case $n = 4$ is treated in [Zi]. Behind this reduction is the observation that Soergel bimodules corresponding to the longest elements in rank two parabolic subgroups of W do not survive projection onto $\underline{\mathscr{L}}$.

The argument for the first part of the proof which we employ in the rank two case is based on an analysis of spectral properties of some integral matrices. We observe that Fibonacci polynomials are connected to minimal polynomials of certain integral matrices which encode the combinatorics of simple transitive 2-representations. Using the factorization of Fibonacci polynomials over $\mathbb{Q}$, see [Lev], and estimating the values of the maximal possible eigenvalues for the matrices which encode the action of Soergel bimodules corresponding to simple reflections in simple transitive 2-representations, allows us to deduce that the only possibility for these matrices are the ones appearing in cell 2-representations.

Apart from Theorem 1, we also prove the following general result. We refer to Subsection 2.4 and [CM, Subsection 3.2] for the definition of the apex and to Subsection 2.3 and [MM2, Subsection 4.2] for the definition of the abelianization $\overline{\mathbf{M}}$ of a 2-representation $\mathbf{M}$.

Theorem 2. *Let* $\mathbf{M}$ *be a simple transitive* 2*-representation of a fiat* 2*-category* $\mathscr{C}$ *with apex* $\mathcal{I}$*. Then, for every* 1*-morphism* $\mathrm{F} \in \mathcal{I}$ *and every object* X *in any* $\overline{\mathbf{M}}(\mathtt{i})$*, the object* $\mathrm{F}\,X$ *is projective. Moreover,* $\overline{\mathbf{M}}(\mathrm{F})$ *is a projective functor.*

Our proof of this statement is based crucially on the results from [KM2]. Theorem 2 will be very useful in further studies of simple transitive 2-representations. It applies directly to all cases we consider and substantially simplifies some of the arguments. For example, in Coxeter type $I_2(5)$, we originally had an independent argument for a similar result which involved a very technical statement that a certain collection of twenty seven linear inequalities is equivalent to the fact that all these inequalities are, in fact, equalities. This full argument was three pages long and just covered one Coxeter type. A similar argument in other Coxeter types of the form $I_2(n)$ seemed, for a long period of time, unrealistic.

Our results form a first step towards classification of simple transitive 2-representations of Soergel bimodules in all types. However, for the moment, the technical difficulty of solving the first part of the problem (as described above) in the general case seems too high. Already in type B_3 the classification of simple transitive 2-representations is not complete. It is also of course very natural to ask what happens in positive characteristics. However, the situation there is expected to be even more complicated.

The paper is organized as follows: In Section 2 we collect all necessary preliminaries on 2-categories and 2-representations. Section 3 collects preliminaries on 2-categories of Soergel bimodules. In Section 4 we collect several general results related to classification of simple transitive 2-representations of small quotients of Soergel bimodules. It also contains our proof of Theorem 2, see Theorem 11 in Subsection 4.5. Section 5 contains auxiliary results on some spectral properties of integral matrices. Coxeter type $I_2(n)$, for n odd, is studied in Section 6. Coxeter type $I_2(n)$, for n even, is studied in Section 7. Section 8 collects our study of all non-simply laced Coxeter types of rank higher than two (all simply laced types are covered by the results of [MM5, MM6]). Finally, in Section 9 we propose a new general construction of finitary 2-categories. The novel component of this construction is that indecomposable 2-morphisms are, in general, defined using decomposable functors. This allows us to give an alternative description of one interesting example of a finitary 2-category from [Xa, Example 8].

Warning. All Soergel bimodules considered in this paper are over the coinvariant algebra, not the polynomial algebra. Furthermore, our setup is ungraded.

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2. 2-categories and 2-representations

2.1. **Notation and conventions.** We work over the field $\mathbb{C}$ of complex numbers and denote $\otimes_{\mathbb{C}}$ by $\otimes$. A module always means a *left* module. All maps are composed from right to left.

2.2. **Finitary and fiat 2-categories.** For generalities on 2-categories, we refer the reader to [Le, Mac, Maz].

By a 2-category we mean a category enriched over the monoidal category **Cat** of small categories. In other words, a 2-category $\mathscr{C}$ consists of objects (which we denote by Roman lower case letters in a typewriter font), 1-morphisms (which we denote by capital Roman letters), and 2-morphisms (which we denote by Greek lower case letters), composition of 1-morphisms, horizontal and vertical compositions of 2-morphisms (denoted $\circ_0$ and $\circ_1$, respectively), identity 1-morphisms and identity 2-morphisms. These satisfy the obvious collection of axioms.

For a 1-morphism F, we denote by id_{F} the corresponding identity 2-morphism. We often write $\mathrm{F}(\alpha)$ for $\mathrm{id}_{\mathrm{F}} \circ_0 \alpha$ and α_{F} for $\alpha \circ_0 \mathrm{id}_{\mathrm{F}}$.

We say that a 2-category $\mathscr{C}$ is *finitary* if each category $\mathscr{C}(\mathtt{i},\mathtt{j})$ is an idempotent split, additive and Krull-Schmidt $\mathbb{C}$-linear category with finitely many isomorphism classes of indecomposable objects and finite dimensional morphism spaces, moreover, all compositions are assumed to be compatible with these additional structures, see [MM1, Subsection 2.2] for details.

If $\mathscr{C}$ is a finitary 2-category, we say that $\mathscr{C}$ is *fiat* provided that it has a weak involution $\star$ together with adjunction 2-morphisms satisfying the axioms of adjoint functors, for each pair $(\mathrm{F}, \mathrm{F}^\star)$ of 1-morphisms, see [MM1, Subsection 2.4] for details. Similarly, we say that $\mathscr{C}$ is *weakly fiat* provided that it has a weak anti-autoequivalence $\star$ (not necessarily involutive) together with adjunction 2-morphisms satisfying the axioms of adjoint functors, for each pair $(\mathrm{F}, \mathrm{F}^\star)$ of 1-morphisms, see [MM6, Subsection 2.5] for details.

2.3. **2-representations.** Let $\mathscr{C}$ be a finitary 2-category. The 2-category $\mathscr{C}$-afmod consists of all *finitary* 2*-representations* of $\mathscr{C}$ as defined in [MM3, Subsection 2.3]. Objects in $\mathscr{C}$-afmod are strict functorial actions of $\mathscr{C}$ on idempotent split, additive and Krull-Schmidt $\mathbb{C}$-linear categories which have finitely many isomorphism classes of indecomposable objects and finite dimensional spaces of morphisms. In $\mathscr{C}$-afmod, 1-morphisms are strong 2-natural transformations and 2-morphisms are modifications.

Similarly, we consider the 2-category $\mathscr{C}$-mod consisting of all *abelian* 2*-representations* of $\mathscr{C}$. These are functorial actions of $\mathscr{C}$ on categories equivalent to module categories over finite dimensional algebras, we again refer to [MM3, Subsection 2.3] for details. The 2-categories $\mathscr{C}$-afmod and $\mathscr{C}$-mod are connected by the diagrammatically defined *abelianization* 2*-functor*

$$\overline{\,\cdot\,} : \mathscr{C}\text{-afmod} \to \mathscr{C}\text{-mod},$$

see [MM2, Subsection 4.2] for details. For $\mathbf{M} \in \mathscr{C}$-afmod, objects in $\overline{\mathbf{M}}$ are diagrams of the form $X \longrightarrow Y$ over $\mathbf{M}(\mathtt{i})$'s and morphisms are quotients of the space of (solid) commutative squares of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \arrow[dl, dotted] \\ X \arrow[r] & Y' \end{tikzcd}$$

modulo the subspace for which the right horizontal map factorizes via some dotted map. The action of $\mathscr{C}$ is defined component-wise.

We say that two 2-representations are *equivalent* provided that there is a strong 2-natural transformation between them which restricts to an equivalence of categories, for each object in $\mathscr{C}$.

A finitary 2-representation $\mathbf{M}$ of $\mathscr{C}$ is said to be *transitive* provided that, for any indecomposable objects X and Y in $\coprod_{\mathtt{i}\in\mathscr{C}} \mathbf{M}(\mathtt{i})$, there is a 1-morphism F in $\mathscr{C}$ such that the object Y is isomorphic to a direct summand of the object $\mathbf{M}(\mathrm{F})\,X$. A transitive 2-representation $\mathbf{M}$ is said to be *simple transitive* provided that $\coprod_{\mathtt{i}\in\mathscr{C}} \mathbf{M}(\mathtt{i})$ does not have any non-zero proper ideals which are invariant under the functorial action of $\mathscr{C}$. Given a finitary 2-representation $\mathbf{M}$ of $\mathscr{C}$, the *rank* of $\mathbf{M}$ is the number of isomorphism classes of indecomposable objects in

$$\coprod_{\mathtt{i}\in\mathscr{C}} \mathbf{M}(\mathtt{i}).$$

For simplicity, we will often use the "module" notation $\mathrm{F}\,X$ instead of the corresponding "representation" notation $\mathbf{M}(\mathrm{F})\,X$.

2.4. **Combinatorics.** Let $\mathscr{C}$ be a finitary 2-category and $\mathcal{S}[\mathscr{C}]$ the corresponding *multisemigroup* in the sense of [MM2, Section 3]. The objects in $\mathcal{S}[\mathscr{C}]$ are isomorphism classes of indecomposable 1-morphisms in $\mathscr{C}$. For $\mathrm{F}, \mathrm{G} \in \mathcal{S}[\mathscr{C}]$, we set $\mathrm{F} \leq_L \mathrm{G}$ provided that G is isomorphic to a summand of $\mathrm{H} \circ \mathrm{F}$, for some 1-morphism H. The relation $\leq_L$ is called the *left* (pre)-order on $\mathcal{S}[\mathscr{C}]$. The *right* and *two-sided* (pre)-orders are defined similarly using multiplication on the right or on both sides. Equivalence classes with respect to these orders are called *cells* and the corresponding equivalence relations are denoted $\sim_L$, $\sim_R$ and $\sim_J$, respectively. As usual, for simplicity, we say "cells of $\mathscr{C}$" instead of "cells of $\mathcal{S}[\mathscr{C}]$".

Given a two-sided cell $\mathcal{J}$ in $\mathscr{C}$, we call $\mathscr{C}$ $\mathcal{J}$*-simple* provided that every non-zero two-sided 2-ideal in $\mathscr{C}$ contains the identity 2-morphisms for some (and hence for all) 1-morphisms in $\mathcal{J}$, see [MM2, Subsection 6.2] for details.

A two-sided cell is called *strongly regular*, see [MM1, Subsection 4.8], provided that

- different left (resp. right) cells inside $\mathcal{J}$ are not comparable with respect to $\leq_L$ (resp. $\leq_R$);
- the intersection of any left and any right cell inside $\mathcal{J}$ consists of exactly one element.

By [KM2, Corollary 19], the first condition is automatically satisfied for all fiat 2-categories.

By [CM, Subsection 3.2], each simple transitive 2-representation $\mathbf{M}$ of $\mathscr{C}$ has an *apex*, which is defined as a unique two-sided cell that is maximal in the set of all two-sided cells whose elements are not annihilated by $\mathbf{M}$.

2.5. **Cell 2-representations.** For every $\mathtt{i} \in \mathscr{C}$, we will denote by $\mathbf{P}_{\mathtt{i}} := \mathscr{C}_A(\mathtt{i}, {}_-)$ the corresponding *principal* 2-representation. For a left cell $\mathcal{L}$ in $\mathscr{C}$, there is $\mathtt{i} \in \mathscr{C}$ such that all 1-morphisms in $\mathcal{L}$ start at $\mathtt{i}$. Let $\mathbf{N}_{\mathcal{L}}$ denote the 2-subrepresentation of $\mathbf{P}_{\mathtt{i}}$ given by the additive closure of all 1-morphisms $\mathbf{F}$ satisfying $\mathbf{F} \geq_L \mathcal{L}$. Then $\mathbf{N}_{\mathcal{L}}$ has a unique maximal $\mathscr{C}$-invariant ideal and the corresponding quotient $\mathbf{C}_{\mathcal{L}}$ is called the *cell* 2-representation associated to $\mathcal{L}$. We refer the reader to [MM1, Section 4] and [MM2, Subsection 6.5] for more details.

2.6. **Matrices in the Grothendieck group.** Let $\mathscr{C}$ be a finitary 2-category and $\mathbf{M}$ a finitary 2-representation of $\mathscr{C}$. Fix a complete and irredundant list of representatives in all isomorphism classes of indecomposable objects in $\coprod_{\mathtt{i}} \mathbf{M}(\mathtt{i})$. Then, for every 1-morphism F in $\mathscr{C}$, we have the corresponding matrix $(\!\![\mathrm{F}]\!\!)$ which describes multiplicities in direct sum decompositions of the images of indecomposable objects under $\mathbf{M}(\mathrm{F})$.

If $\mathscr{C}$ is fiat, then each $\overline{\mathbf{M}}(\mathrm{F})$ is exact and we also have the matrix $[\![\mathrm{F}]\!]$ which describes composition multiplicities of the images of simple objects under $\overline{\mathbf{M}}(\mathrm{F})$. By adjunction, the matrix $[\![\mathrm{F}^\star]\!]$ is transposed to the matrix $(\!\![\mathrm{F}]\!\!)$.

2.7. **Based modules over positively based algebras.** In this subsection we recall some notation and results from [KM2]. Let A be a finite dimensional $\mathbb{C}$-algebra with a fixed basis $\{a_1, a_2, \dots, a_n\}$. Assume that A is *positively based* in the sense that all structure constants $\gamma_{i,j}^s$ with respect to this basis, defined via

$$a_i a_j = \sum_{s=1}^{n} \gamma_{i,j}^s a_s, \qquad \text{where } \gamma_{i,j}^s \in \mathbb{C},$$

are non-negative real numbers. For $s, j \in \{1, 2, \dots, n\}$, we set $a_s \geq_L a_j$ provided that $\gamma_{i,j}^s \neq 0$ for some i. We set $a_s \sim_L a_j$ provided that $a_s \geq_L a_j$ and $a_j \geq_L a_s$ at the same time. This defines the *left order* and the corresponding *left cells*. The right and two-sided orders and the right and two-sided cells are defined similarly (using multiplication from the right or from both sides) and denoted $\geq_R$, $\geq_J$, $\sim_R$ and $\sim_J$, respectively.

For each left cell $\mathcal{L}$, we have the corresponding *left cell* A-module $C_{\mathcal{L}}$. The module $C_{\mathcal{L}}$ contains a subquotient $L_{\mathcal{L}}$, called the *special subquotient*, which has composition multiplicity one in $C_{\mathcal{L}}$. This subquotient is defined as the unique subquotient of $C_{\mathcal{L}}$ which contains the Perron-Frobenius eigenvector of $C_{\mathcal{L}}$ for the linear operator $\sum_i a_i$. We have $L_{\mathcal{L}} \cong L_{\mathcal{L}'}$ if $\mathcal{L}$ and $\mathcal{L}'$ belong to the same two-sided cell. The latter justifies the notation $L_{\mathcal{J}} := L_{\mathcal{L}}$, where $\mathcal{J}$ is the two-sided cell containing $\mathcal{L}$.

An A-module V with a fixed basis $\{v_1, v_2, \dots, v_k\}$ is called *positively based* provided that each $a_i \cdot v_j$ is a linear combination of the v_m's with non-negative real coefficients. A positively based module V is called *transitive* provided that, for any v_i and v_j, there is an a_s such that v_j appears in $a_s \cdot v_i$ with a non-zero coefficient. For each transitive A-module V, there is a unique two-sided cell $\mathcal{J}(V)$ which is the maximum element (with respect to the two-sided order) in the set of all two-sided cells whose elements do not annihilate V. The two-sided cell $\mathcal{J}(V)$ is called the *apex* of V. The two-sided cell $\mathcal{J}(V)$ is *idempotent* in the sense that it contains some a_i, a_j and a_s (non necessarily different) such that $\gamma_{i,j}^s \neq 0$. The simple subquotient $L_{\mathcal{J}(V)}$ has composition multiplicity one in V and is also called the *special* subquotient of V.

2.8. **Decategorification.** For a finitary 2-category $\mathscr{C}$, we denote by $A_{\mathscr{C}}$ the complexification of the Grothendieck decategorification of $\mathscr{C}$. The algebra $A_{\mathscr{C}}$ is positively based with respect to the *natural basis* given by the classes of indecomposable 1-morphisms in $\mathscr{C}$, see [Maz, Subsection 1.2] or [KM2, Subsection 2.5] for details.

Given a 2-representation $\mathbf{M}$ of $\mathscr{C}$, the Grothendieck decategorification of $\mathbf{M}$ is, naturally, a positively based $A_{\mathscr{C}}$-module. Moreover, the latter is transitive if and only if $\mathbf{M}$ is transitive. This allows us to speak about the *apex* of $\mathbf{M}$ in the obvious way, see [CM, Subsection 3.2] for details.

We denote by $\mathrm{F}_{\mathrm{tot}}$ a multiplicity free direct sum of all indecomposable 1-morphisms in $\mathscr{C}$. Given a 2-representation $\mathbf{M}$ of $\mathscr{C}$, we set $\mathtt{M}_{\mathrm{tot}} := [\![\mathrm{F}_{\mathrm{tot}}]\!]$. Then $\mathbf{M}$ is transitive if and only if all coefficients of $\mathtt{M}_{\mathrm{tot}}$ are non-zero (and thus positive integers).

3. Soergel bimodules

3.1. **Soergel bimodules for finite Coxeter groups.** Let (W, S) be a finite irreducible Coxeter system and $\varphi : W \to \mathrm{GL}(V)$ be the geometric representation of W as in [Hu, Section 5.3]. Here V is a real vector space. We denote by $\leq$ the Bruhat order and by $\mathbf{l} : W \to \mathbb{Z}_{\geq 0}$ the length function. For $w \in W$, we denote by $\underline{w}$ the corresponding element in the Kazhdan-Lusztig basis of $\mathbb{Z}[W]$, see [KaLu, So2, EW]. Each $\underline{w}$ is a linear combination of elements in W with non-negative integer coefficients.

Let $\mathtt{C}$ be the (complexified) coinvariant algebra associated to (W, S, V). For $s \in S$, we denote by $\mathtt{C}^s$ the subalgebra of s-invariants in $\mathtt{C}$. A *Soergel* $\mathtt{C}$-$\mathtt{C}$*–bimodule* is a $\mathtt{C}$-$\mathtt{C}$–bimodule isomorphic to a bimodule from the additive closure of the monoidal category of $\mathtt{C}$-$\mathtt{C}$–bimodules generated by $\mathtt{C} \otimes_{\mathtt{C}^s} \mathtt{C}$, where s runs through S, see [So1, So2, Li]. We also note that by the *additive closure* of some X we mean the the full subcategory whose objects are isomorphic to finite direct sums of direct summands of X. Isomorphism classes of indecomposable Soergel bimodules are naturally indexed by $w \in W$, we denote by B_w a fixed representative from such a class. There is an isomorphism between $\mathbb{Z}[W]$ and the Grothendieck ring of the monoidal category of Soergel bimodules for (W, S, V). This isomorphism sends $\underline{w}$ to the class of B_w, see [So2, E, EW].

Let $\mathcal{C}$ be a small category equivalent to the category $\mathtt{C}$-mod. Define the 2-category $\mathscr{S} = \mathscr{S}(W, S, V, \mathcal{C})$ of *Soergel bimodules* associated to $(W, S, V, \mathcal{C})$ as follows:

- $\mathscr{S}$ has one object $\mathtt{i}$, which we can identify with $\mathcal{C}$;
- 1-morphisms in $\mathscr{S}$ are all endofunctors of $\mathcal{C}$ which are isomorphic to endofunctors given by tensoring with Soergel $\mathtt{C}$-$\mathtt{C}$–bimodules;
- 2-morphisms in $\mathscr{S}$ are natural transformations of functors (these correspond to homomorphisms of Soergel $\mathtt{C}$-$\mathtt{C}$–bimodules).

The 2-category $\mathscr{S}$ is fiat. The algebra $A_{\mathscr{S}}$ is isomorphic to $\mathbb{C}[W]$ and is positively based with respect to the Kazhdan-Lusztig basis in $\mathbb{C}[W]$. For $w \in W$, let θ_w denote a fixed representative in the isomorphism class of indecomposable 1-morphisms in $\mathscr{S}$ given by tensoring with B_w.

3.2. **Small quotients of Soergel bimodules.** Let (W, S), V, $\mathcal{C}$ and $\mathscr{S}$ be as above.

Lemma 3. *For any $s, t \in S$, we have $\theta_s \sim_J \theta_t$.*

Proof. As (W, S) is irreducible, we may assume that $st \neq ts$. As $\theta_s\theta_t \cong \theta_{st}$, we have $\theta_s \leq_R \theta_{st}$. Further, as $st \neq ts$, we have $\theta_{st}\theta_s \cong \theta_{sts} \oplus \theta_s$. This implies that $\theta_s \geq_R \theta_{st}$ and hence $\theta_s \sim_R \theta_{st}$. Similarly one shows that $\theta_t \sim_L \theta_{st}$. The claim follows. □

After Lemma 3, we may define the two-sided cell $\mathcal{J}$ of $\mathscr{S}$ as the two-sided cell containing θ_s, for all $s \in S$. As $\underline{s}$, where $s \in S$, generate $\mathbb{Z}[W]$, it follows that $\mathcal{J}$ is the unique minimal element, with respect to the two-sided order, in the set of all two-sided cells of $\mathscr{S}$ different from the two-sided cell corresponding to θ_e.

Let $\mathscr{I}$ be the unique 2-ideal of $\mathscr{S}$ which is maximal with respect to the property that it does not contain any id_{F}, where $\mathrm{F} \in \mathcal{J}$. The 2-category $\mathscr{S}/\mathscr{I}$ will be called the *small quotient* of $\mathscr{S}$ and denoted by $\underline{\mathscr{S}}$. By construction, the 2-category $\underline{\mathscr{S}}$ is fiat and $\mathcal{J}$-simple. It has two two-sided cells, namely, the two-sided cell corresponding to θ_e and the image of $\mathcal{J}$, which we identify with $\mathcal{J}$, abusing notation.

4. Generalities on simple transitive 2-representations of $\underline{\mathscr{S}}$

4.1. **Basic combinatorics of $\mathcal{J}$.** Here we describe the basics of the Kazhdan-Lusztig combinatorics related to the two-sided cell $\mathcal{J}$. Recall Lusztig's $\mathbf{a}$-function $\mathbf{a} : W \to \mathbb{Z}_{\geq 0}$, which is defined in [Lu1]. This function has, in particular, the following properties: it is constant on two-sided cells in W, and, moreover, we have the equality $\mathbf{a}(w) = \mathbf{l}(w)$ provided that w is the longest element in some parabolic subgroup of W.

Proposition 4.

(*i*) *The map $\mathcal{L} \mapsto \mathcal{L} \cap S$ is a bijection between the set of left cells in $\mathcal{J}$ and S.*

(*ii*) *For any left cell $\mathcal{L}$ in $\mathcal{J}$, the unique element in $\mathcal{L} \cap S$ is the Duflo involution in $\mathcal{L}$.*

(*iii*) *The map $\mathcal{R} \mapsto \mathcal{R} \cap S$ is a bijection between the set of right cells in $\mathcal{J}$ and S.*

(*iv*) *For any right cell $\mathcal{R}$ in $\mathcal{J}$, the unique element in $\mathcal{R} \cap S$ is the Duflo involution in $\mathcal{R}$.*

(*v*) *An element $w \in W$ such that $w \neq e$ belongs to $\mathcal{J}$ if and only if w has a unique reduced expression.*

(*vi*) *An element $w \in W$ belongs to $\mathcal{J}$ if and only if $\mathbf{a}(w) = 1$.*

If W is a Weyl group, all these results are mentioned in [Do] with references to [Lu1, Lu2, Lu3]. We note the difference in both the normalizations of the Hecke algebra and the choice of the Kazhdan-Lusztig basis in this paper and in [Do]. Below, we outline a general argument.

Proof. We start by observing that different simple reflections have different left (right) descent sets and hence are not right (left) equivalent.

Further, if $w \in W$ has more than one reduced expression, then any such reduced expression can be obtained from any other by means of the braid relations, see [Bo, §IV.1.5]. This means that, with respect to the Bruhat order, w is bigger than or equal to the longest element in some parabolic subgroup of W of rank two. As longest element in parabolic subgroup of W of rank two are not in $\mathcal{J}$ (since the value of $\mathbf{a}$ on such elements is strictly bigger than $1 = \mathbf{a}(s)$, where $s \in S$), we get $w \notin \mathcal{J}$.

By induction with respect to the rank of W, one checks that any two elements of the form xs and ys (resp. sx and sy) which, moreover, have unique reduced expressions, belong to the same left (resp. right) cell. Indeed, the basis of the induction is the rank two case which is standard. The induction step follows from the relation $\theta_s\theta_t\theta_s = \theta_{sts} \oplus \theta_s$ which is true for any pair s, t of non-commuting simple reflections. This, combined with the observation in the previous paragraph, yields (i), (iii), (v) and (vi).

That simple reflections are Duflo involutions in their cells follows directly from the definitions since the value of $\mathbf{a}$ on $\mathcal{J}$ is 1, implying (ii), (iv). This completes the proof. □

An important consequence of Proposition 4(v) is the following: if s and t are different elements in S, then the longest element in the parabolic subgroup of W generated by s and t has two different reduced expressions and thus does not belong to $\mathcal{J}$. We also refer the reader to [De, BW] for related results.

4.2. **Examples and special cases.** Using Proposition 4, one can describe elements in $\mathcal{J}$ explicitly as products of simple reflections. Here we list several examples and special cases under the following conventions:

- we provide a picture of the Coxeter diagram followed by the list of elements in $\mathcal{J}$ organized in a square table, each element is given by its unique reduced expression;
- vertices of the Coxeter diagram are numbered by positive integers and we write i for the corresponding simple reflection s_i;
- the left cell $\mathcal{L}_i$ containing i is the i-th column of the diagram;
- the right cell $\mathcal{R}_i$ containing i is the i-th row of the diagram;

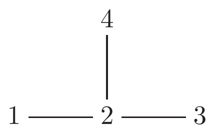
To verify these examples one could use the following general approach:

- check that all elements given in the example have a unique reduced expression;
- check that, making any step outside the list in the example, gives an element with more than one reduced expression;
- note that all elements in a left cell start with the same simple reflection on the right, and all elements in a right cell start with the same simple reflection on the left;
- use Proposition 4.

We start with types A_3 and D_4.

$$1 \text{ --- } 2 \text{ --- } 3$$

	$\mathcal{L}_1$	$\mathcal{L}_2$	$\mathcal{L}_3$
$\mathcal{R}_1$	1	12	123
$\mathcal{R}_2$	21	2	23
$\mathcal{R}_3$	321	32	3



	$\mathcal{L}_1$	$\mathcal{L}_2$	$\mathcal{L}_3$	$\mathcal{L}_4$
$\mathcal{R}_1$	1	12	123	124
$\mathcal{R}_2$	21	2	23	24
$\mathcal{R}_3$	321	32	3	324
$\mathcal{R}_4$	421	42	423	4

Similarly to the above, we have the following observation which follows directly from Proposition 4.

Corollary 5. *If the Coxeter diagram of W is simply laced, then the cell $\mathcal{J}$ is strongly regular. Namely, the intersection $\mathcal{L}_i \cap \mathcal{R}_j$ consists of $s_j s_{i_k} \dots s_{i_2} s_{i_1} s_i$, where $j - i_k - \dots - i_1 - i$ is the unique path in the diagram connecting i and j.*

Because of Corollary 5, in the simply laced case, any simple transitive 2-representation of $\underline{\mathscr{S}}$ is equivalent to a cell 2-representation by [MM5, Theorem 18]. We note that the original formulation of [MM5, Theorem 18] has an additional assumption, the so-called *numerical condition* which appeared in [MM1, Formula (10)]. This additional assumption is rendered superfluous by [MM6, Proposition 1].

Consequently,

in what follows, we assume that W is not simply laced.

In type G_2, we have the following.

$$1 \overset{6}{\text{ --- }} 2$$

	$\mathcal{L}_1$	$\mathcal{L}_2$
$\mathcal{R}_1$	$1, 121, 12121$	$12, 1212$
$\mathcal{R}_2$	$21, 2121$	$2, 212, 21212$

The general type B_n, for $n \geq 2$, corresponds to the Coxeter diagram

$$1 \overset{4}{\text{——}} 2 \text{——} 3 \text{——} \dots \text{——} n$$

and is given below:

	$\mathcal{L}_1$	$\mathcal{L}_2$	$\mathcal{L}_3$	$\dots$	$\mathcal{L}_n$
$\mathcal{R}_1$	$1, 121$	12	123	$\dots$	$12 \cdots n$
$\mathcal{R}_2$	21	$2, 212$	$23, 2123$	$\dots$	$23 \cdots n, 212 \cdots n$
$\mathcal{R}_3$	321	$32, 3212$	$3, 32123$	$\dots$	$34 \cdots n, 3212 \cdots n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\mathcal{R}_n$	$n \cdots 21$	$n \cdots 32, n \cdots 212$	$n \cdots 43, n \cdots 2123$	$\dots$	$n, n \cdots 212 \cdots n$

The remaining Weyl type F_4 looks as follows:

$$1 \text{——} 2 \overset{4}{\text{——}} 3 \text{——} 4$$

	$\mathcal{L}_1$	$\mathcal{L}_2$	$\mathcal{L}_3$	$\mathcal{L}_4$
$\mathcal{R}_1$	$1, 12321$	$12, 1232$	123	1234
$\mathcal{R}_2$	$21, 2321$	$2, 232$	23	234
$\mathcal{R}_3$	321	32	$3, 323$	$34, 3234$
$\mathcal{R}_4$	4321	432	$43, 4323$	$4, 43234$

The exceptional Coxeter type H_3 is the following:

$$1 \overset{5}{\text{——}} 2 \text{——} 3$$

	$\mathcal{L}_1$	$\mathcal{L}_2$	$\mathcal{L}_3$
$\mathcal{R}_1$	$1, 121$	$12, 1212$	$123, 12123$
$\mathcal{R}_2$	$21, 2121$	$2, 212$	$23, 2123$
$\mathcal{R}_3$	$321, 32121$	$32, 3212$	$3, 32123$

The exceptional Coxeter type H_4 has the Coxeter diagram

$$1 \overset{5}{\text{——}} 2 \text{——} 3 \text{——} 4$$

and the corresponding table looks as follows:

	$\mathcal{L}_1$	$\mathcal{L}_2$	$\mathcal{L}_3$	$\mathcal{L}_4$
$\mathcal{R}_1$	$1, 121$	$12, 1212$	$123, 12123$	$1234, 121234$
$\mathcal{R}_2$	$21, 2121$	$2, 212$	$23, 2123$	$234, 21234$
$\mathcal{R}_3$	$321, 32121$	$32, 3212$	$3, 32123$	$34, 321234$
$\mathcal{R}_4$	$4321, 432121$	$432, 43212$	$43, 432123$	$4, 4321234$

The general dihedral type with Coxeter diagram

$$1 \overset{k}{\text{——}} 2 \ ,$$

where $k \geq 3$, looks as follows:

	$\mathcal{L}_1$	$\mathcal{L}_2$
$\mathcal{R}_1$	$1, 121, \dots, 12 \cdots 21$	$12, 1212, 12 \cdots 12$
$\mathcal{R}_2$	$21, 2121, \dots, 21 \cdots 21$	$2, 212, \dots, 21 \cdots 12$

Note that the length of all elements in the latter table is strictly less than k. In particular, the diagonal boxes always contain $\lfloor \frac{k}{2} \rfloor$ elements while the off-diagonal boxes contain $\lfloor \frac{k}{2} \rfloor$ elements if k is odd and $\lfloor \frac{k}{2} \rfloor - 1$ elements if k is even.

4.3. **The principal element in $\mathbb{Z}[W]$.** Following [Zi], we consider the element

$$\mathbf{s} := \sum_{s\in S} \underline{s} \in \mathbb{Z}[W],$$

which we call the *principal* element. This element is the decategorification of

$$\mathrm{F}_{\mathrm{pr}} := \bigoplus_{s\in S} \theta_s.$$

The main aim of this subsection is to determine the form of the matrix

$$\mathtt{M} = \mathtt{M}_{\mathrm{pr}} := [\![\mathrm{F}_{\mathrm{pr}}]\!].$$

We start by recalling the following.

Lemma 6. *Let $\mathbf{M}$ be a transitive 2-representation of $\underline{\mathscr{S}}$ and $s \in S$. Then there is an ordering of indecomposable objects in $\mathbf{M}(\mathtt{i})$ such that*

$$[\![\theta_s]\!] = \left(\begin{array}{c|c} 2E & * \\ \hline 0 & 0 \end{array}\right),$$

where E is the identity matrix.

Proof. Mutatis mutandis the proof of [Zi, Lemma 6.4]. □

Lemma 7. *Let $\mathbf{M}$ be a transitive 2-representation of $\underline{\mathscr{S}}$ with apex $\mathcal{J}$ and P an indecomposable object in $\mathbf{M}(\mathtt{i})$. Then there exists a unique $s \in S$ having the property $\theta_s\, P \cong P \oplus P$.*

Proof. Assume that $\theta_s\, P \not\cong P \oplus P$ for every $s \in S$. From Lemma 6 it then follows that $\mathtt{M}$ has a zero row. Therefore the same row will be zero in any power of $\mathtt{M}$. Note that any indecomposable 1-morphism of $\underline{\mathscr{S}}$ which is not isomorphic to θ_e appears as a direct summand of some power of F_{pr}. If $\mathbf{M}$ had rank one, it would follow that all indecomposable 1-morphisms of $\underline{\mathscr{S}}$ which are not isomorphic to θ_e annihilate $\mathbf{M}$. This contradicts our assumption that $\mathcal{J}$ is the apex of $\mathbf{M}$. Therefore $\mathbf{M}$ has rank at least two. However, in this case, the same row which is zero in all powers of $\mathtt{M}$ must have a zero entry in $\mathtt{M}_{\mathrm{tot}}$. This contradicts transitivity of $\mathbf{M}$ and hence establishes existence of $s \in S$ such that $\theta_s\, P \cong P \oplus P$.

Assume that s and t are two different elements in S having the property that $\theta_s\, P \cong P \oplus P$ and $\theta_t\, P \cong P \oplus P$. Consider the parabolic subgroup W' of W generated by s and t. The additive closure of P is invariant under the action of the 2-subcategory of $\underline{\mathscr{S}}$ generated by θ_s and θ_t. The decategorification of the latter 2-representation gives a 1-dimensional $\mathbb{C}[W']$-module on which both $\underline{s}$ and $\underline{t}$ act as the scalar 2. Thus this is the trivial module and it is not annihilated by the element $\underline{w_0'}$, where w_0' is the longest element in W'. This contradicts $w_0' \notin \mathcal{J}$, see Proposition 4(v) and the remark after Proposition 4. The assertion of the lemma follows. □

Write $S = \{s_1, s_2, \dots, s_n\}$. Let $\mathbf{M}$ be a transitive 2-representation of $\underline{\mathscr{S}}$ with apex $\mathcal{J}$. Choose an ordering

$$P_1, P_2, \dots, P_m \tag{1}$$

on representatives of isomorphism classes of indecomposable projectives in $\mathbf{M}(\mathtt{i})$ such that, for all $i, j \in \{1, 2, \dots, n\}$ such that $i < j$ and for all $a, b \in \{1, 2, \dots, m\}$, the isomorphisms $\theta_{s_i}\, P_a \cong P_a \oplus P_a$ and $\theta_{s_j}\, P_b \cong P_b \oplus P_b$ imply $a < b$.

Proposition 8. *Let* $\mathbf{M}$ *be a transitive* 2*-representation of* $\underline{\mathscr{S}}$ *with apex* $\mathcal{J}$*. Then, with respect to the above ordering, we have*

$$\mathtt{M} = \left(\begin{array}{c|c|c|c|c} 2E_1 & B_{12} & B_{13} & \dots & B_{1n} \\ \hline B_{21} & 2E_2 & B_{23} & \dots & B_{2n} \\ \hline B_{31} & B_{32} & 2E_3 & \dots & B_{3n} \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline B_{n1} & B_{n2} & B_{n3} & \dots & 2E_n \end{array}\right), \tag{2}$$

where E_i*, for* $1 \leq i \leq n$*, are non-trivial identity matrices and, for* $1 \leq i \neq j \leq n$*, the matrix* B_{ij} *is non-zero if and only if* $s_i s_j \neq s_j s_i$*.*

Proof. After Lemmata 6 and 7, it remains to show that B_{ij} is non-zero if and only if $s_i s_j \neq s_j s_i$. If $s_i s_j = s_j s_i$, then B_{ij} is zero as $s_i s_j \notin \mathcal{J}$ and $\mathcal{J}$ is the apex of $\mathbf{M}$.

Assume that $s_i s_j \neq s_j s_i$ and $B_{ij} = 0$. Let as assume that $i > j$, the other case is similar. Taking into account the previous paragraph and the fact that the underlying graph of the Coxeter diagram of W is a tree, we may rearrange the order of the elements in S such that the matrix $\mathtt{M}$ has the following block form:

$$\left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right).$$

As any power of such a matrix will contain a non-empty zero south-west corner, we get a contradiction with the assumption that $\mathbf{M}$ is transitive. The claim follows. □

Let $\{s_1, s_2, \dots, s_n\}$ be an ordering of simple reflections in S such that the block $2E_i$ in (2) corresponds to s_i, for all $i = 1, 2, \dots, n$, then we have

$$[\![\theta_{s_1}]\!] = \left(\begin{array}{c|c|c|c|c} 2E_1 & B_{12} & B_{13} & \dots & B_{1n} \\ \hline 0 & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & 0 & \dots & 0 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & 0 & \dots & 0 \end{array}\right), \quad [\![\theta_{s_2}]\!] = \left(\begin{array}{c|c|c|c|c} 0 & 0 & 0 & \dots & 0 \\ \hline B_{21} & 2E_2 & B_{23} & \dots & B_{2n} \\ \hline 0 & 0 & 0 & \dots & 0 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & 0 & \dots & 0 \end{array}\right),$$

and so on.

4.4. **The special $\mathbb{C}[W]$-module for $\mathcal{J}$.** Here we describe the special $\mathbb{C}[W]$-module, in the sense of [KM2], see also Subsection 2.7, associated to $\mathcal{J}$.

Proposition 9. *The special* $\mathbb{C}[W]$*-module associated to* $\mathcal{J}$ *is* $V \otimes \mathbb{C}_{\mathrm{sign}}$*, where* $\mathbb{C}_{\mathrm{sign}}$ *is the sign* $\mathbb{C}[W]$*-module.*

If W is a Weyl group, this can be derived from [Do], if one takes into account the difference in the normalizations of the Hecke algebra and the choice of the Kazhdan-Lusztig basis.

Proof. Let s and t be two different simple reflections in W and W' the corresponding parabolic subgroup of rank two, with the longest element w_0'. The restriction of $V \otimes \mathbb{C}_{\mathrm{sign}}$ to W' decomposes, by construction, into a direct sum of $V' \otimes \mathbb{C}'_{\mathrm{sign}}$, where V' and $\mathbb{C}'_{\mathrm{sign}}$ are the geometric and the sign representations of W', respectively, and a number of copies of $\mathbb{C}'_{\mathrm{sign}}$. As both $V' \otimes \mathbb{C}'_{\mathrm{sign}}$ and $\mathbb{C}'_{\mathrm{sign}}$ are annihilated by $\underline{w_0'}$, it follows that $\underline{w_0'}$ annihilates $V \otimes \mathbb{C}_{\mathrm{sign}}$.

Let I be the ideal in $\mathbb{C}[W]$ with the $\mathbb{C}$-basis $\underline{w}$, where $w \notin \mathcal{J} \cup \{e\}$. From Proposition 4(v), it follows that I is generated by all $\underline{w_0'}$ as in the previous paragraph.

Therefore the previous paragraph implies that the vector space $V \otimes \mathbb{C}_{\mathrm{sign}}$ is, in fact, a non-zero $\mathbb{C}[W]/I$-module.

If W is simply laced, then the cell $\mathbb{C}[W]$-module corresponding to any cell in $\mathcal{J}$ is simple. Hence in this case the claim of the proposition follows directly from the previous paragraph.

Now note that the dimension of $V \otimes \mathbb{C}_{\mathrm{sign}}$ equals the number of left cells in $\mathcal{J}$. From detailed lists of elements in $\mathcal{J}$ provided in Subsection 4.2 we therefore have that $V \otimes \mathbb{C}_{\mathrm{sign}}$ is the only simple $\mathbb{C}[W]/I$-module of this dimension in types B_n and F_4. This implies the claim of the proposition for these types.

For the general dihedral type, one may note that $V \otimes \mathbb{C}_{\mathrm{sign}} \cong V$ and hence the claim of the proposition follows from [KM2].

It remains to consider the two exceptional types H_3 and H_4. We do explicit computations in both types. Note that, from the lists in Subsection 4.2 we see that there are two non-isomorphic simple $\mathbb{C}[W]/I$-modules of the same dimension.

Let us start with type H_3. The action of $\mathbf{s}$ (cf. Subsection 4.3) on $C_{\mathcal{L}_1}$ in the basis

$$1, 121, 21, 2121, 321, 32121,$$

taken from the corresponding table in Subsection 4.2, is given by the matrix

$$\begin{pmatrix} 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \end{pmatrix}.$$

The eigenvalue of maximal absolute value for this matrix is $2 + \frac{\sqrt{2\sqrt{5}+10}}{2}$. At the same time, the action of $\mathbf{s}$ on $V \otimes \mathbb{C}_{\mathrm{sign}}$ in the basis corresponding to 1, 2 and 3, is given by the matrix

$$\begin{pmatrix} 2 & -\frac{1+\sqrt{5}}{2} & 0 \\ -\frac{1+\sqrt{5}}{2} & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

As $2+\frac{\sqrt{2\sqrt{5}+10}}{2}$ is also an eigenvalue for this latter matrix, we obtain our statement in type H_3.

Type H_4 is treated similarly by noticing that the matrices

$$\begin{pmatrix} 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & -\frac{1+\sqrt{5}}{2} & 0 & 0 \\ -\frac{1+\sqrt{5}}{2} & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

which represent the action of $\mathbf{s}$ on $C_{\mathcal{L}_1}$ and $V \otimes \mathbb{C}_{\mathrm{sign}}$, respectively, have a common eigenvalue which is of maximal absolute value for both of them (this eigenvalue is approximately 3.98904). This completes the proof. $\square$

Remark 10. In type H_3, we used the matrix calculator that can be found at www.mathportal.org. As the outcome is given as a real number, it is easily checked by hand. In type H_4 we used the matrix calculator available at www.arndt-bruenner.de and cross-checked the result on the matrix calculator available at www.bluebit.gr.

4.5. **1-morphisms act as projective functors.** Our aim in this subsection is to prove the following very general result which extends and unifies [MM5, Lemma 12], [Zi, Lemma 6.14] and [MaMa, Lemma 10].

Theorem 11. *Let* $\mathbf{M}$ *be a simple transitive* 2*-representation of a fiat* 2*-category* $\mathscr{C}$*. Let, further,* $\mathcal{I}$ *be the apex of* $\mathbf{M}$ *and* $\mathrm{F} \in \mathcal{I}$*.*

(*i*) *For every object* X *in any* $\overline{\mathbf{M}}(\mathtt{i})$*, the object* $\mathrm{F}\,X$ *is projective.*

(*ii*) *The functor* $\overline{\mathbf{M}}(\mathrm{F})$ *is a projective functor.*

Proof. Without loss of generality we may assume that $\mathcal{I}$ is the maximum two-sided cell of $\mathscr{C}$.

Denote by Q the complexification of the split Grothendieck group of

$$\coprod_{\mathtt{i}\in\mathscr{C}} \mathbf{M}(\mathtt{i}).$$

Let $\mathbf{B}$ denote the distinguished basis in Q given by classes of indecomposable modules. Let $\mathrm{F}_1, \mathrm{F}_2, \dots, \mathrm{F}_k$ be a complete and irredundant list of indecomposable 1-morphisms in $\mathcal{I}$. For $i = 1, 2, \dots, k$, let $\mathcal{X}_\bullet^{(i)}$ be a minimal projective resolution of $\mathrm{F}_i\,X$. Further, for $j \geq 0$, denote by $v_j^{(i)}$ the image of $\mathcal{X}_j^{(i)}$ in Q. Note that $v_j^{(i)}$ is a linear combination of elements in $\mathbf{B}$ with non-negative integer coefficients.

As $\mathscr{C}$ is assumed to be fiat, there exist $i, j \in \{1, 2, \dots, k\}$ such that $\mathrm{F}_i \circ \mathrm{F}_j \neq 0$. Indeed, one can, for example, take any i and let F_j be the Duflo involution in the left cell of F_i, see [MM1, Proposition 17]. Therefore we may apply [KM2, Proposition 18], which asserts that the algebra $A_{\mathscr{C}}$ has a unique idempotent e of the form

$$\sum_{i=1}^{k} c_i[\mathrm{F}_i],$$

where all $c_i \in \mathbb{R}_{>0}$ and $[\mathrm{F}_i]$ denotes the image of F_i in $A_{\mathscr{C}}$. The vector space Q is, naturally, an $A_{\mathscr{C}}$-module.

Let $j \geq 0$ and set

$$v(j) := \sum_{i=1}^{k} c_i v_j^{(i)}.$$

Applying any 1-morphism G to a minimal projective resolution of some object Y and taking into account that the action of G is exact as $\mathscr{C}$ is fiat, gives a projective resolution of $\mathrm{G}\,Y$ which, a priori, does not have to be minimal. By construction and minimality of $\mathcal{X}_\bullet^{(i)}$, this implies that $ev(j) - v(j)$ is a linear combination of elements in $\mathbf{B}$ with non-negative real coefficients. Note that transitivity of $\mathbf{M}$ implies that, for any linear combination z of elements in $\mathbf{B}$ with non-negative real coefficients, the element ez is also a linear combination of elements in $\mathbf{B}$ with non-negative real coefficients, moreover, if $z \neq 0$, then $ez \neq 0$. Therefore the equality $e^2 = e$ yields $ev(j) = v(j)$. Consequently, the projective resolution $\mathrm{F}_i\mathcal{X}_\bullet^{(l)}$ of $\mathrm{F}_i(\mathrm{F}_l\,X)$ is minimal, for all i and l.

Now the proof is completed by standard arguments as in [MM5, Lemma 12]. From the previous paragraph it follows that homomorphisms $\mathcal{X}_1^{(l)} \to \mathcal{X}_0^{(l)}$, for

$l = 1, 2, \dots, k$, generate a $\mathscr{C}$-invariant ideal of $\mathbf{M}$ different from $\mathbf{M}$. Because of simple transitivity of $\mathbf{M}$, this ideal must be zero. Therefore all homomorphisms $\mathcal{X}_1^{(l)} \to \mathcal{X}_0^{(l)}$ are zero which implies $\mathcal{X}_0^{(l)} \cong \mathrm{F}_l\, X$. Claim (i) follows. Claim (ii) follows from claim (i) and [MM5, Lemma 13]. □

4.6. **The matrix $\mathtt{M}$ is symmetric.** Let $\mathbf{M}$ be a simple transitive 2-representation of $\underline{\mathscr{S}}$ with apex $\mathcal{J}$. Consider $\overline{\mathbf{M}}$ and let P_1, P_2,..., P_k be a full list of pairwise non-isomorphic indecomposable projectives in $\overline{\mathbf{M}}(\mathtt{i})$. Let L_1, L_2,..., L_k be their respective simple tops.

Proposition 12. *For every $i \in \{1, 2, \dots, k\}$, we have $\mathrm{F}_{\mathrm{pr}}\, L_i \cong P_i$.*

Proof. We know from Theorem 11(i) that $\mathrm{F}_{\mathrm{pr}}\, L_i$ is projective. From the matrix $\mathtt{M}$ described in Subsection 4.3, we see that there is a unique $s \in S$ such that $\theta_s\, L_i \neq 0$. Therefore $\mathrm{F}_{\mathrm{pr}}\, L_i \cong \theta_s\, L_i$. From the matrix $[\![\theta_s]\!]$, see Subsections 2.6 and 4.3 in combination with the remark that each θ_s is self-adjoint, we see that $\theta_s\, L_i$ has two subquotients isomorphic to L_i and all other subquotients are killed by θ_s. Note that $\theta_s\, L_i$ has length at least three, as each row of $\mathtt{M}$ must contain at least one non-zero block B_{ij} by Proposition 8. This means that $\theta_s\, L_i$ does contain some subquotients which are not isomorphic to L_i. If L_j is such a subquotient, then, by adjunction,

$$\mathrm{Hom}_{\overline{\mathbf{M}}(\mathtt{i})}(\theta_s\, L_i, L_j) = \mathrm{Hom}_{\overline{\mathbf{M}}(\mathtt{i})}(L_i, \theta_s\, L_j) = \mathrm{Hom}_{\overline{\mathbf{M}}(\mathtt{i})}(L_i, 0) = 0$$

and L_j does not appear in the top of $\theta_s\, L_i$. Similarly, L_j does not appear in the socle of $\theta_s\, L_i$ either. It follows that $\theta_s\, L_i$ can only have L_i in top and socle. This means that one of the L_i's is in the top of $\theta_s\, L_i$ and the other one is in the socle. In particular, $\theta_s\, L_i$ is indecomposable. Therefore $\theta_s\, L_i \cong P_i$ by the projectivity of $\theta_s\, L_i$. This completes the proof. □

Corollary 13. *Let $\mathbf{M}$ be a simple transitive 2-representation of $\underline{\mathscr{S}}$ with apex $\mathcal{J}$. Then the matrix $\mathtt{M}$ is symmetric.*

Proof. By Proposition 12, the row i of $\mathtt{M}$ describes composition multiplicities of simple modules in P_i. By definition, the column i of $\mathtt{M}$ describes the direct sum decomposition of $\mathrm{F}_{\mathrm{pr}}\, P_i$. Note that F_{pr} is exact as $\mathscr{S}$ is fiat. Therefore, from Proposition 12 we have that the multiplicity of P_j as a direct summand in $\mathrm{F}_{\mathrm{pr}}\, P_i$ coincides with the composition multiplicity of L_j in P_i. The claim follows. □

5. Some spectral properties of integral matrices

5.1. **Setup.** Denote by $\mathbb{M}$ the set of all matrices with non-negative integer coefficients, that is

$$\mathbb{M} := \bigcup_{k,m \in \mathbb{Z}_{>0}} \mathrm{Mat}_{k\times m}(\mathbb{Z}_{\geq 0}).$$

For each $X \in \mathbb{M}$, we can consider the symmetric matrices XX^{tr} and $X^{\mathrm{tr}}X$. Each of these two matrices is diagonalizable over $\mathbb{R}$, moreover, all eigenvalues are non-negative. Furthermore, these two matrices have the same spectrum (considered as a multiset) with only possible exception of the multiplicity of the eigenvalue zero. We denote by $\mathbf{m}_X$ the eigenvalue of XX^{tr} or, equivalently, $X^{\mathrm{tr}}X$ with the maximal absolute value. Note that $\mathbf{m}_X = 0$ if and only if X is the zero matrix. The aim of this section is to describe all $X \in \mathbb{M}$ such that $\mathbf{m}_X < 4$.

5.2. **Staircase matrices.** A *staircase matrix* is a matrix of the form

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & \dots & 0 & 0\\
0 & 1 & 1 & 0 & 0 & \dots & 0 & 0\\
0 & 0 & 1 & 1 & 0 & \dots & 0 & 0\\
0 & 0 & 0 & 1 & 1 & \dots & 0 & 0\\
0 & 0 & 0 & 0 & 1 & \dots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & 0 & 0 & \dots & 1 & 1\\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 1
\end{pmatrix}
\quad
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0\\
0 & 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0\\
0 & 0 & 1 & 1 & 0 & \dots & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 1 & \dots & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1
\end{pmatrix}
\tag{3}
$$

or its transpose. Note that each staircase matrix is of the size $k \times k$ or $k \times (k+1)$ or $(k+1) \times k$, for some positive integer k.

An *extended staircase matrix* is a matrix obtained from a staircase matrix by adding one additional row (or column, but not both) as shown below. In the pictures, the line separates the original staircase matrix from the added row (respectively, column). The picture below shows extended staircase matrices for the left matrix in (3).

$$
\left(\begin{array}{c|cccccccc}
1 & 1 & 1 & 0 & 0 & 0 & \dots & 0 & 0\\
0 & 0 & 1 & 1 & 0 & 0 & \dots & 0 & 0\\
0 & 0 & 0 & 1 & 1 & 0 & \dots & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 1 & \dots & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1
\end{array}\right)
\quad
\left(\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 0 & \dots & 0 & 0\\
0 & 1 & 1 & 0 & 0 & \dots & 0 & 0\\
0 & 0 & 1 & 1 & 0 & \dots & 0 & 0\\
0 & 0 & 0 & 1 & 1 & \dots & 0 & 0\\
0 & 0 & 0 & 0 & 1 & \dots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & 0 & 0 & \dots & 1 & 1\\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 1\\
\hline
0 & 0 & 0 & 0 & 0 & \dots & 0 & 1
\end{array}\right)
$$

The next picture shows extended staircase matrices for the right matrix in (3).

$$
\left(\begin{array}{c|ccccccccc}
1 & 1 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0\\
0 & 0 & 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 1 & 0 & \dots & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 1 & \dots & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1
\end{array}\right)
\quad
\left(\begin{array}{ccccccccc|c}
1 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0\\
0 & 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 1 & 0 & \dots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 1 & \dots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 & 1
\end{array}\right)
$$

For the transposes of (3), the above pictures should also be transposed. Note that each extended staircase matrix is of the size $k \times (k+1)$ or $k \times (k+2)$ or $(k+1) \times k$ or $(k+2) \times k$, for some positive integer k.

5.3. The main result.

Proposition 14. *Let $X \in \mathbb{M}$ be such that both XX^{tr} and $X^{\mathrm{tr}}X$ are irreducible and $\mathbf{m}_X < 4$. Then, using independent permutations of rows and columns, X can be reduced to a staircase matrix or an extended staircase matrix or one of the following matrices (or their transposes):*

$$
X_1 := \begin{pmatrix} 1 & 0 & 0\\ 1 & 1 & 1\\ 0 & 0 & 1 \end{pmatrix}, \quad
X_2 := \begin{pmatrix} 1 & 1 & 0 & 0\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad
X_3 := \begin{pmatrix} 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

The statement of Proposition 14 can be interpreted in the way that there is a natural correspondence between matrices X as in the proposition and simply laced Dynkin diagrams. Staircase matrices correspond to type A, extended staircase matrices correspond to type D and the three exceptional matrices correspond to type E. Starting from a matrix X appearing in the classification provided by Proposition 14, we can consider the matrix

$$\begin{pmatrix} 2E & X \\ X^{\mathrm{tr}} & 2E \end{pmatrix}$$

and pretend that it appears as $\mathtt{M}$ in some 2-representation. If we compute the underlying algebra of that 2-representation as in many examples later on, see e.g. Subsection 7.5, we will get a doubling of a simply laced Dynkin quiver.

5.4. **Proof of Proposition 14.** Let $X \in \mathbb{M}$ be such that both XX^{tr} and $X^{\mathrm{tr}}X$ are irreducible and $\mathbf{m}_X < 4$.

Assume that X has an entry that is greater than or equal to 2. Then XX^{tr} has a diagonal entry which is greater than or equal to 4. Let λ be the Perron-Frobenius eigenvalue of XX^{tr}. Then, by the Perron-Frobenius Theorem, the limit

$$\lim_{i\to\infty} \frac{(XX^{\mathrm{tr}})^i}{\lambda^i}$$

exists, which means that $\lambda \geq 4$, a contradiction. Therefore X is a 0-1-matrix.

For a matrix M, a submatrix N of M is the matrix obtained by taking entries in the intersection of a non-empty set of rows of M and a non-empty set of columns of M. If N is a submatrix of M, then, clearly, $\mathbf{m}_N \leq \mathbf{m}_M$. An argument similar to the one in the previous paragraph shows that X cannot have any submatrix which is equal to either of the following matrices, nor their transposes,

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}. \tag{4}$$

Taking the first matrix in (4) into account and using the fact that both XX^{tr} and $X^{\mathrm{tr}}X$ are irreducible, we see that, by independent permutation of row and columns, X can be reduced to the form

$$\begin{pmatrix} 1 & \dots & 1 & 1 & 0 & \dots & 0 & 0 & 0 & \dots \\ 0 & \dots & 0 & 1 & 1 & \dots & 1 & 1 & 0 & \dots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

So, from now on we may assume that X is of the latter form. Taking the second matrix in (4) into account, we see that each row and each column of X contains at most three non-zero entries.

Next we claim that the total number of rows and columns in X which contain three non-zero entries is at most one. For this we have to exclude, up to transposition, two types of possible submatrices in X. The first one is the matrix

$$\begin{pmatrix}
1 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0\\
0 & 1 & 1 & 0 & \dots & \dots & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots\\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \dots & \dots & 1 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & \dots & \dots & 0 & 1 & 1 & 1
\end{pmatrix}.$$

For this matrix it is easy to check that the vector $(2, 2, \dots, 2, 1, 1)^{\mathrm{tr}}$ is an eigenvector of $X^{\mathrm{tr}}X$ with eigenvalue 4, leading to a contradiction. The second one is the matrix

$$\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 1 & 0 & \dots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 1 & \dots & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots\\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 & 1
\end{pmatrix}.$$

For this matrix it is easy to check that the vector $(1, 1, \dots, 1)^{\mathrm{tr}}$ is an eigenvector of XX^{tr} with eigenvalue 4, leading again to a contradiction.

If X has neither rows nor columns with three non-zero entries, then X is a staircase matrix. By the above, if X has a row or a column with three non-zero entries, it is unique. Now, the claim of the proposition follows from the observation that, for the following matrices X:

$$\begin{pmatrix}
1 & 1 & 0 & 0\\
0 & 1 & 0 & 0\\
0 & 1 & 1 & 0\\
0 & 0 & 1 & 1\\
0 & 0 & 0 & 1
\end{pmatrix}, \qquad
\begin{pmatrix}
1 & 0 & 0\\
1 & 1 & 0\\
0 & 1 & 0\\
0 & 1 & 1\\
0 & 0 & 1
\end{pmatrix},$$

the matrix $X^{\mathrm{tr}}X$ has 4 as an eigenvalue.

6. Simple transitive 2-representations of $\mathscr{S}$ in Coxeter type $I_2(n)$ with n odd

6.1. **Setup and the main result.** In this section we assume that W is of Coxeter type $I_2(n)$, for some $n \geq 3$, and $S = \{s, t\}$ with the Coxeter diagram

$$s \overset{n}{\text{———}} t.$$

In this case there are three two-sided cells, namely

- the two-sided cell of the identity element;
- the two-sided cell of w_0;
- the two-sided cell $\mathcal{J}$.

Our aim in this section is to prove the following.

Theorem 15. *For $n \geq 3$ odd, every simple transitive 2-representation of $\mathscr{S}$ is a cell 2-representation.*

If $\mathbf{M}$ is a simple transitive 2-representation of $\mathscr{S}$ whose apex is not $\mathcal{J}$, then $\mathbf{M}$ is a cell 2-representation by the same argument as in [MM5, Theorem 18], see also [MaMa, Zi] for similar arguments. Taking [MM2, Theorem 19] into account, from now on, we may assume that $\mathbf{M}$ is a simple transitive 2-representation of $\underline{\mathscr{S}}$ with apex $\mathcal{J}$. Our goal is to prove that $\mathbf{M}$ is a cell 2-representation.

6.2. **Fibonacci polynomials in disguise.** For $i = 0, 1, 2, \dots$, we define, recursively, polynomials $f_i(x) \in \mathbb{Z}[x]$ as follows: $f_0(x) = 0$, $f_1(x) = 1$, and, for $i > 1$,

$$f_i(x) = \begin{cases} f_{i-1}(x) - f_{i-2}(x), & i \text{ is odd;} \\ x f_{i-1}(x) - f_{i-2}(x), & i \text{ is even.} \end{cases}$$

Coefficients of f_i are given by Sequence A115139 in [OEIS]. The values of f_i for small i are given here:

i	$f_i(x)$	factorization over $\mathbb{Q}$
0	0	
1	1	1
2	x	x
3	$x-1$	$x-1$
4	x^2-2x	$x(x-2)$
5	x^2-3x+1	x^2-3x+1
6	x^3-4x^2+3x	$x(x-1)(x-3)$
7	x^3-5x^2+6x-1	x^3-5x^2+6x-1
8	$x^4-6x^3+10x^2-4x$	$x(x-2)(x^2-4x+2)$
9	$x^4-7x^3+15x^2-10x+1$	$(x-1)(x^3-6x^2+9x-1)$
10	$x^5-8x^4+21x^3-20x^2+5x$	$x(x^2-3x+1)(x^2-5x+5)$
11	$x^5-9x^4+28x^3-35x^2+15x-1$	$x^5-9x^4+28x^3-35x^2+15x-1$
12	$x^6-10x^5+36x^4-56x^3+35x^2-6x$	$x(x-1)(x-2)(x-3)(x^2-4x+1)$.

For $i = 0, 1, 2, \dots$, we define, recursively, polynomials $g_i(x) \in \mathbb{Z}[x]$ in the following way: $g_0(x) = 0$, $g_1(x) = 1$, and $g_i(x) = x g_{i-1}(x) + g_{i-2}(x)$, for $i > 1$. These polynomials are called *Fibonacci polynomials*, see Sequence A011973 in [OEIS]. By comparing the two definitions, we have,

$$\begin{aligned} (-1)^{\lfloor \frac{i}{2} \rfloor} f_i(-x^2) &= g_i(x), \quad i \text{ is odd;} \\ \frac{(-1)^{\frac{i}{2}}}{x} f_i(-x^2) &= g_i(x), \quad i \text{ is even.} \end{aligned}$$

Now, from [Lev, Lemma 5], it follows that, for each $i \in \{1, 2, 3, \dots\}$, the polynomial $f_i(x)$ has a unique irreducible (over $\mathbb{Q}$) factor, denoted $\underline{f}_i(x)$, which is not an irreducible factor of any $f_j(x)$, for $j < i$. Furthermore,

$$f_i(x) = \prod_{d|i} \underline{f}_d(x).$$

From [Lev, Definition 1], it follows that, for $i > 2$, the polynomial $\underline{f}_i(x)$ has degree $\frac{\phi(i)}{2}$, where ϕ is Euler's totient function, and all roots of $\underline{f}_i(x)$ are positive real numbers less than 4. Furthermore, the sequence of maximal roots of $\underline{f}_i(x)$ is strictly increasing and converges to 4, when $i \to \infty$. For small i, the polynomials $\underline{f}_i(x)$ are given in the following table:

i	$\underline{f}_i(x)$
0	0
1	1
2	x
3	$x-1$
4	$x-2$
5	x^2-3x+1
6	$x-3$
7	x^3-5x^2+6x-1
8	x^2-4x+2
9	x^3-6x^2+9x-1
10	x^2-5x+5
11	$x^5-9x^4+28x^3-35x^2+15x-1$
12	x^2-4x+1
13	$x^6-11x^5+45x^4-84x^3+70x^2-21x+1$
14	$x^3-7x^2+14x-7$
15	$x^4-9x^3+26x^2-24x+1$.

We refer the reader to [Lev, WP] for further properties of the above polynomials.

6.3. **Disguised Fibonacci polynomials and 2-representations of $\underline{\mathscr{S}}$.** Recall that we are in Coxeter type $I_2(n)$, for some $n \geq 3$. Let $\mathbf{M}$ be a simple transitive 2-representation of $\underline{\mathscr{S}}$ with apex $\mathcal{J}$. By Subsection 4.3 and Corollary 13, we have

$$\mathtt{M} = \left(\begin{array}{c|c} 2E_1 & B \\ \hline B^{\mathrm{tr}} & 2E_2 \end{array}\right), \quad [\![\theta_s]\!] = \left(\begin{array}{c|c} 2E_1 & B \\ \hline 0 & 0 \end{array}\right), \quad [\![\theta_t]\!] = \left(\begin{array}{c|c} 0 & 0 \\ \hline B^{\mathrm{tr}} & 2E_2 \end{array}\right).$$

Lemma 16. *Both BB^{tr} and $B^{\mathrm{tr}}B$ are annihilated by $f_n(x)$.*

Proof. Set $C = B^{\mathrm{tr}}$. For simplicity, for $i \geq 1$, let $s_i = stst\dots$ be the element of length i and $t_i = tsts\dots$ be the element of length i. Using induction on i and the multiplication rule

$$(5) \qquad \theta_s\theta_{t_i} \cong \begin{cases} \theta_{s_2}, & i=1; \\ \theta_{s_{i+1}} \oplus \theta_{s_{i-1}}, & i>1; \end{cases} \qquad \theta_t\theta_{s_i} \cong \begin{cases} \theta_{t_2}, & i=1; \\ \theta_{t_{i+1}} \oplus \theta_{t_{i-1}}, & i>1; \end{cases}$$

one proves, by induction, that

$$[\![\theta_{s_i}]\!] = \left(\begin{array}{c|c} 2f_i(BC) & f_i(BC)B \\ \hline 0 & 0 \end{array}\right), \quad [\![\theta_{t_i}]\!] = \left(\begin{array}{c|c} 0 & 0 \\ \hline f_i(CB)C & 2f_i(CB) \end{array}\right),$$

if i is odd, and

$$[\![\theta_{s_i}]\!] = \left(\begin{array}{c|c} f_i(BC) & 2f_i(BC)C^{-1} \\ \hline 0 & 0 \end{array}\right), \quad [\![\theta_{t_i}]\!] = \left(\begin{array}{c|c} 0 & 0 \\ \hline 2f_i(CB)B^{-1} & f_i(CB) \end{array}\right),$$

if i is even (here C^{-1} just means that the rightmost appearances of C in $f_i(BC)$ should be deleted, and similarly for B^{-1} with respect to $f_i(CB)$). Note that $C(BC)^l = (CB)^l C$, for any $l \in \{0,1,2,\dots\}$, and therefore $Bf_i(CB)C = BCf_i(BC)$, for any i.

As $\mathbf{l}(w_0) = n$, the claim now follows from our assumption that $\mathbf{M}$ has apex $\mathcal{J}$ and therefore $[\![\theta_{w_0}]\!] = 0$. $\square$

Corollary 17. *If n is odd, then B is invertible.*

Proof. If n is odd, then $f_n(x)$ has a non-zero constant term. Therefore, from Lemma 16 we have that both BB^{tr} and $B^{\mathrm{tr}}B$ are invertible and thus B is invertible as well. □

Corollary 18. *Let $n \in \{3, 4, 5, \dots\}$ and $\mathbf{M}$ be a simple transitive 2-representation of $\underline{\mathscr{S}}$ in Coxeter type $I_2(n)$ with apex $\mathcal{J}$ and with the corresponding matrix $\mathtt{M}$. Let $n' \in \{3, 4, 5, \dots\}$ and $\mathbf{M}'$ be a simple transitive 2-representation of $\underline{\mathscr{S}}$ in Coxeter type $I_2(n')$ with apex $\mathcal{J}$ and with the corresponding matrix $\mathtt{M}'$. If $\mathtt{M} = \mathtt{M}'$, then $n = n'$.*

Proof. Let w denote the word $stst\dots$ of length n. Then $w = w_0$ in Coxeter type $I_2(n)$. From Lemma 16, we have that $\mathbf{M}'(\theta_w) = 0$. Therefore $w \notin \mathcal{J}$ in Coxeter type $I_2(n')$ and we obtain $n' \leq n$. By symmetry, we also have $n \leq n'$ and hence $n = n'$. □

6.4. **The explicit form of B.**

Proposition 19. *If $n = 2k + 1$, then, by independent permutations of rows and columns, the matrix B can be reduced to a $k \times k$ staircase matrix.*

Proof. If we can prove that, by independent permutations of rows and columns, B can be reduced to a square staircase matrix, then the fact that the size of such matrix is $k \times k$ follows from Corollary 18. From Corollary 17, we have that B is an invertible matrix, say of size $m \times m$. Set $Q = B^{\mathrm{tr}}B$. Then, from Lemma 16 and Subsection 6.2, we have that all eigenvalues of Q are positive real numbers that are strictly less than 4.

Therefore, by Proposition 14, the matrix B is either a staircase matrix or coincides with one of the matrices X_1 or X_3. If $B = X_3$, then

$$Q = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

The characteristic polynomial of Q is $x^4 - 7x^3 + 14x^2 - 8x + 1$ and is irreducible over $\mathbb{Q}$. The only odd i, for which $\phi(i) = 8$, is $i = 15$. However, we already know that $\underline{f}_{15}(x) = x^4 - 9x^3 + 26x^2 - 24x + 1$. Therefore $\underline{f}_i(Q) \neq 0$ for any odd i. This means that such B is not possible.

If $B = X_1$, we have

$$Q = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

The characteristic polynomial of Q has factorization $(x-1)(x^2-4x+1)$. This means that $\underline{f}_{12}(x)$ is a factor of any annihilating polynomial of Q and thus $\underline{f}_i(Q) \neq 0$ for any odd i. The claim of the proposition follows. □

6.5. **Proof of Theorem 15.** We are now ready to prove Theorem 15. Let $\mathbf{M}$ be a simple transitive 2-representation of $\underline{\mathscr{S}}$ with apex $\mathcal{J}$. Proposition 19 gives explicitly the matrix $\mathtt{M}$, in particular, it shows that this matrix is uniquely determined. Hence this matrix coincides with the corresponding matrix for the cell 2-representation $\mathbf{C}_{\mathcal{L}_s}$.

Consider the abelianization $\overline{\mathbf{M}}$. Now, starting from the simple module L_k, we can use Proposition 12 to get $\theta_s\, L_k \cong P_k$. Using the explicit formulae for the actions of

θ_s and θ_t together with (5), we see that, applying θ_w, for $w \in \mathcal{L}_s$, to L_k, we obtain all indecomposable projective objects in $\overline{\mathbf{M}}(\mathtt{i})$.

The rest of the proof is now similar to the corresponding parts of the proofs in the literature, see [MM5, Proposition 9], [MZ, Sections 6 and 9] or [MaMa, Subsection 4.9]. Let $\mathbf{N}$ be the additive 2-representation of $\mathscr{S}$ obtained by restricting the action of $\mathscr{S}$ to the category of projective objects in $\overline{\mathbf{M}}(\mathtt{i})$. Then $\mathbf{N}$ is equivalent to $\mathbf{M}$, see [MM2, Theorem 11]. There is a unique strong 2-natural transformation from $\mathbf{P}_{\mathtt{i}}$ to $\overline{\mathbf{M}}$ sending $\mathbb{1}_{\mathtt{i}}$ to L_k. It induces a strong 2-natural transformation from $\mathbf{C}_{\mathcal{L}_s}$ to $\mathbf{N}$. Since $\mathbf{C}_{\mathcal{L}_s}(\mathtt{i})$ is simple transitive with apex $\mathcal{J}$, everything that we established earlier for $\mathbf{M}$ (in particular, about the structure of projective modules etc.) also holds for $\mathbf{C}_{\mathcal{L}_s}$. In particular, the Cartan matrices of the underlying algebras of $\mathbf{C}_{\mathcal{L}_s}(\mathtt{i})$ and $\mathbf{N}(\mathtt{i})$ agree. Therefore the above 2-natural transformation is an equivalence between these two categories. This completes the proof.

7. Simple transitive 2-representations of $\mathscr{S}$ in Coxeter type $I_2(n)$ with n even

7.1. **Setup and preliminaries.** In this section we assume that W is of type $I_2(n)$ with $n = 2k > 4$ and $S = \{s, t\}$ with the Coxeter diagram

$$s \overset{n}{\longrightarrow} t.$$

Let $\mathbf{M}$ be a simple transitive 2-representation of $\underline{\mathscr{S}}$ with apex $\mathcal{J}$. By Subsection 4.3 and Corollary 13, we have

$$\mathtt{M} = \left(\begin{array}{c|c} 2E_1 & B \\ \hline B^{\mathrm{tr}} & 2E_2 \end{array}\right), \quad [\![\theta_s]\!] = \left(\begin{array}{c|c} 2E_1 & B \\ \hline 0 & 0 \end{array}\right), \quad [\![\theta_s]\!] = \left(\begin{array}{c|c} 0 & 0 \\ \hline B^{\mathrm{tr}} & 2E_2 \end{array}\right).$$

From Lemma 16, we have that both BB^{tr} and $B^{\mathrm{tr}}B$ are annihilated by $f_n(x)$.

The main result of the section is the following statement.

Theorem 20. *Assume that W is of type $I_2(n)$ with $n = 2k > 4$ and $n \neq 12, 18, 30$. Then every simple transitive 2-representation of $\mathscr{S}$ is equivalent to either a cell 2-representation or one of the 2-representations $\mathbf{N}_s^{(n)}$, $\mathbf{N}_t^{(n)}$ constructed in Subsection 7.2. All these 2-representations are pairwise non-equivalent.*

7.2. **Additional simple transitive 2-representations.** In this subsection we just assume that $n = 2k > 4$. The left cell $\mathcal{L}_s$ of the element $\theta_s \in \mathscr{S}$ consists of the elements θ_s, θ_{ts}, θ_{sts}, θ_{tsts} and so on, with $2k-1$ elements in total. Consider the corresponding cell 2-representation $\mathbf{C}_{\mathcal{L}_s}$. In this subsection we follow closely the approach of [MaMa, Subsection 5.8] to construct, starting from $\mathbf{C}_{\mathcal{L}_s}$, a new simple transitive 2-representation of $\mathscr{S}$, which we later on will denote by $\mathbf{N}_s^{(n)}$.

Set $\mathbf{Q} := \mathbf{C}_{\mathcal{L}_s}$ and consider $\overline{\mathbf{Q}}$. Let P_w, $w \in \mathtt{L} := \{s, ts, sts, tsts, \dots\}$, be representatives of the isomorphism classes of the indecomposable projective objects in $\overline{\mathbf{Q}}(\mathtt{i})$ and L_w, $w \in \mathtt{L}$, be the respective simple tops. Note that $|\mathtt{L}|$ is odd. With respect to this choice of a basis, we have

$$[\![\theta_s]\!] = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 2 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad [\![\theta_t]\!] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 2 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In particular, in this case the matrix B is a $k \times (k-1)$ staircase matrix. Consequently, we have the following Loewy filtrations of indecomposable projective modules:

$$\begin{tikzcd}[column sep=small]
L_s \arrow[d, dash] & & L_{ts} \arrow[dl, dash] \arrow[dr, dash] & & & & L_{sts} \arrow[dl, dash] \arrow[dr, dash] & \\
L_{ts} \arrow[d, dash] & L_s \arrow[dr, dash] & & L_{sts} \arrow[dl, dash] & & L_{ts} \arrow[dr, dash] & & L_{tsts} \arrow[dl, dash] \\
L_s & & L_{ts} & & & & L_{sts} &
\end{tikzcd}$$

and so on (the last module in the series is uniserial of length three, just like the first one). Let H denote the basic underlying algebra of $\overline{\mathbf{Q}}(\mathtt{i})$. Then the above implies that H is the quotient of the path algebra of

$$\begin{tikzcd}
\mathtt{s} \arrow[r, bend left] & \mathtt{ts} \arrow[l, bend left] \arrow[r, bend left] & \mathtt{sts} \arrow[l, bend left] \arrow[r, bend left] & \mathtt{tsts} \arrow[l, bend left] \arrow[r, bend left] & \dots \arrow[l, bend left]
\end{tikzcd}$$

modulo the relations that any path of the form $\mathtt{v} \to \mathtt{u} \to \mathtt{w}$ is zero if $v \neq w$ and all paths of the form $\mathtt{v} \to \mathtt{u} \to \mathtt{v}$ coincide. Let f_w, $w \in \mathtt{L}$, denote pairwise orthogonal primitive idempotents of H corresponding to P_w.

As $\mathbf{Q}$ is simple transitive, all θ_w, $w \in \mathcal{J}$, send simple objects in $\overline{\mathbf{Q}}(\mathtt{i})$ to projective objects in $\overline{\mathbf{Q}}(\mathtt{i})$ and act as projective endofunctors of $\overline{\mathbf{Q}}(\mathtt{i})$, by Theorem 11. In terms of H, the action of θ_s is given by tensoring with the H-H–bimodule

$$(Hf_s \otimes f_s H) \oplus (Hf_{sts} \otimes f_{sts} H) \oplus \dots$$

while the action of θ_t is, similarly, given by tensoring with the H-H–bimodule

$$(Hf_{ts} \otimes f_{ts} H) \oplus (Hf_{tsts} \otimes f_{tsts} H) \oplus \dots .$$

Set $\mathbf{Q}^{(0)} := \mathbf{Q}$ and let $\mathbf{Q}^{(1)}$ denote the 2-representation of $\mathscr{S}$ given by the action of $\mathscr{S}$ on the category of projective objects in $\overline{\mathbf{Q}}^{(0)}(\mathtt{i})$. Recursively, for $k \geq 1$, define $\mathbf{Q}^{(k)}$ as the 2-representation of $\mathscr{S}$ given by the action of $\mathscr{S}$ on the category of projective objects in $\overline{\mathbf{Q}}^{(k-1)}(\mathtt{i})$. For every $k \geq 0$, we have a strict 2-natural transformation $\Lambda_k : \mathbf{Q}^{(k-1)} \to \mathbf{Q}^{(k)}$ which sends an object X to the diagram $0 \to X$ and a morphism $\alpha : X \to X'$ to the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & X \arrow[d, "\alpha"] \\
0 \arrow[r] & X'.
\end{tikzcd}$$

Clearly, each such Λ_k is an equivalence. Denote by $\mathbf{K}$ the inductive limit of the directed system

$$\mathbf{Q}^{(0)} \xrightarrow{\Lambda_0} \mathbf{Q}^{(1)} \xrightarrow{\Lambda_1} \mathbf{Q}^{(2)} \xrightarrow{\Lambda_2} \dots . \tag{6}$$

Then $\mathbf{K}$ is a 2-representation of $\mathscr{S}$ which is equivalent to $\mathbf{Q}$.

Let x' be the element which you get by swapping s and t in the reduced expression for x.

Lemma 21. *There is a strict 2-natural transformation $\Psi : \mathbf{K} \to \mathbf{K}$ which is an equivalence and which acts on the isomorphism classes of the indecomposable projective objects by swapping each P_x with $P_{x'w_0}$, for $x \in \mathtt{L}$ such that $x'w_0 \neq x$, and fixing the unique P_x for which $x'w_0 = x$.*

Proof. Consider the unique strict 2-natural transformation $\Phi : \mathbf{P}_{\mathtt{i}} \to \overline{\mathbf{Q}}$ which sends $\mathbb{1}_{\mathtt{i}}$ to L_{tw_0}. Using Proposition 12 and the explicit formulae for the matrices representing the action of all θ_w, we have

$$\theta_s\, L_{tw_0} \cong P_{tw_0}, \quad \theta_{ts}\, L_{tw_0} \cong P_{stw_0}, \qquad \text{and so on.}$$

As the Cartan matrix of H is invariant under swapping P_x with $P_{x'w_0}$, for $x \in \mathtt{L}$ such that $x'w_0 \neq x$, and fixing the unique P_x for which $x'w_0 = x$, it follows by the usual arguments, see for example [MaMa, Subsection 4.9], that the 2-natural transformation Φ factors through $\mathbf{C}_{\mathcal{L}_s}$ and, therefore, gives a strict equivalence $\Phi^{(0)} : \mathbf{Q}^{(0)} \to \mathbf{Q}^{(1)}$. For $k \geq 0$, via abelianization, we get a strict equivalence $\Phi^{(k)} : \mathbf{Q}^{(k)} \to \mathbf{Q}^{(k+1)}$, which is compatible with (6). Now we can take Ψ as the inductive limit of $\Phi^{(k)}$. □

Consider a new finitary 2-representation $\mathbf{K}'$ of $\mathscr{S}$ defined as follows:

- Objects of $\mathbf{K}'(\mathtt{i})$ are sequences $(X_n, \alpha_n)_{n\in\mathbb{Z}}$, where X_n is an object in $\mathbf{K}(\mathtt{i})$ and $\alpha_n \colon \Psi(X_n) \to X_{n+1}$ an isomorphism in $\mathbf{K}(\mathtt{i})$, for all $n \in \mathbb{Z}$.
- Morphisms in $\mathbf{K}'(\mathtt{i})$ from $(X_n, \alpha_n)_{n\in\mathbb{Z}}$ to $(Y_n, \beta_n)_{n\in\mathbb{Z}}$ are sequences of morphisms $f_n \colon X_n \to Y_n$ in $\mathbf{K}(\mathtt{i})$ such that

$$\begin{tikzcd} \Psi(X_n) \arrow[r, "\alpha_n"] \arrow[d, "\Psi(f_n)"'] & X_{n+1} \arrow[d, "f_{n+1}"] \\ \Psi(Y_n) \arrow[r, "\beta_n"] & Y_{n+1} \end{tikzcd}$$

  commutes for all $n \in \mathbb{Z}$.
- The action of $\mathscr{S}$ on $\mathbf{K}'(\mathtt{i})$ is inherited from the action of $\mathscr{S}$ on $\mathbf{K}(\mathtt{i})$ component-wise.

The construction of $\mathbf{K}'(\mathtt{i})$ from $\mathbf{K}(\mathtt{i})$ is the standard construction which turns a category with an autoequivalence (in our case Ψ) into an equivalent category with an automorphism, cf. [Ke, BL].

We have the strict 2-natural transformation $\Pi : \mathbf{K}' \to \mathbf{K}$ given by projection onto the zero component of a sequence. This Π is an equivalence, by construction. We also have a strict 2-natural transformation $\Psi' : \mathbf{K}' \to \mathbf{K}'$ given by shifting the entries of the sequences by one, that is, sending $(X_n, \alpha_n)_{n\in\mathbb{Z}}$ to $(X_{n+1}, \alpha_{n+1})_{n\in\mathbb{Z}}$, with the similar obvious action on morphisms. Note that $\Psi' : \mathbf{K}'(\mathtt{i}) \to \mathbf{K}'(\mathtt{i})$ is an automorphism. The functor Ψ' acts on the isomorphism classes of indecomposable objects in $\mathbf{K}'(\mathtt{i})$ in the same way as Ψ does. As Ψ^2 is isomorphic to the identity functor on $\mathbf{K}(\mathtt{i})$, it follows, by construction, that $(\Psi')^2$ is isomorphic to the identity functor on $\mathbf{K}'(\mathtt{i})$.

Let $\mathbf{K}''$ be the 2-representation of $\mathscr{S}$ given by the action of $\mathscr{S}$ on the category of projective objects in $\overline{\mathbf{K}'}$. We denote by $\Psi'' : \mathbf{K}'' \to \mathbf{K}''$ the diagrammatic extension of Ψ' to $\mathbf{K}''$. Again, $\Psi'' : \mathbf{K}''(\mathtt{i}) \to \mathbf{K}''(\mathtt{i})$ is an automorphism and $(\Psi'')^2$ is isomorphic to the identity functor on $\mathbf{K}''(\mathtt{i})$. We need the following stronger statement.

Lemma 22. *Let* $\mathrm{Id} : \mathbf{K}'' \to \mathbf{K}''$ *denote the identity* 2*-natural transformation.*

(*i*) *There is an invertible modification* $\eta : \mathrm{Id} \to (\Psi'')^2$.

(*ii*) *For any* η *as in* (i), *we have* $\mathrm{id}_{(\Psi'')^2} \circ_0 \eta = \eta \circ_0 \mathrm{id}_{(\Psi'')^2}$.

(*iii*) *For any* $\eta' \in \mathrm{Hom}_{\mathscr{S}\text{-mod}}(\mathrm{Id}, (\Psi'')^2)$, *we have* $\mathrm{id}_{(\Psi'')^2} \circ_0 \eta' = \eta' \circ_0 \mathrm{id}_{(\Psi'')^2}$.

Proof. Let L be a simple object in $\overline{\mathbf{K}'}$ corresponding to P_1 (thus L is a simple corresponding to the Duflo involution in $\mathcal{L}_s$). Fix an isomorphism $\alpha : L \to (\Psi')^2(L)$. For $w \in \mathtt{L}$, set $\eta_{\theta_w L} := \mathrm{id}_{\theta_w} \circ_0 \alpha$. As $\{\theta_w L : w \in \mathtt{L}\}$ is a complete list of pairwise non-isomorphic indecomposable objects in $\mathbf{K}''(\mathtt{i})$, this uniquely defines a natural transformation $\eta : \mathrm{Id} \to (\Psi'')^2$.

If F is a 1-morphism in $\mathscr{S}$, then, for any $w \in \mathtt{L}$, we have $\mathrm{F} \circ \theta_w \cong \mathrm{F}_1 \oplus \mathrm{F}_2$, where $\mathrm{F}_1 L \in \mathrm{add}(\{\theta_w L : w \in \mathtt{L}\})$ and $\mathrm{F}_2 L = 0$. From the definition in the previous paragraph we thus get $\mathrm{F}(\eta_{\theta_w L}) = \eta_{\mathrm{F}_1 L}$ which implies that η is, in fact, a modification. Claim (i) follows. Claims (ii) and (iii) are now proved similarly to [MaMa, Lemma 17(2) and (3)]. □

Proposition 23. *There is an invertible modification $\eta : \mathrm{Id} \to (\Psi'')^2$ for which we have the equality $\mathrm{id}_{\Psi''} \circ_0 \eta = \eta \circ_0 \mathrm{id}_{\Psi''}$.*

Proof. Mutatis mutandis the proof of [MaMa, Proposition 18]. □

From now on we fix some invertible modification $\eta : \mathrm{Id} \to (\Psi'')^2$ as given by Proposition 23. Define a small category $\mathbf{L}(\mathtt{i})$ as follows:

- objects in $\mathbf{L}(\mathtt{i})$ are all 4-tuples (X, Y, α, β), where we have $X, Y \in \mathbf{K}''(\mathtt{i})$, while $\alpha : X \to \Psi''(Y)$ and $\beta : Y \to \Psi''(X)$ are isomorphism such that the following conditions are satisfied

$$\begin{cases} \eta_Y^{-1} \circ_1 \Psi''(\alpha) \circ_1 \beta &= \mathrm{id}_Y, \\ \beta \circ_1 \eta_Y^{-1} \circ_1 \Psi''(\alpha) &= \mathrm{id}_{\Psi''(X)}, \\ \eta_X^{-1} \circ_1 \Psi''(\beta) \circ_1 \alpha &= \mathrm{id}_X, \\ \alpha \circ_1 \eta_X^{-1} \circ_1 \Psi''(\beta) &= \mathrm{id}_{\Psi''(Y)}. \end{cases} \tag{7}$$

- morphisms in $\mathbf{L}(\mathtt{i})$ from (X, Y, α, β) to $(X', Y', \alpha', \beta')$ are pairs (ζ, ξ), where $\zeta : X \to X'$ and $\xi : Y \to Y'$ are morphisms in $\mathbf{K}''(\mathtt{i})$ such that the diagrams

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\zeta"'] & \Psi''(Y) \arrow[d, "\Psi''(\xi)"] \\ X' \arrow[r, "\alpha'"] & \Psi''(Y') \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y \arrow[r, "\beta"] \arrow[d, "\xi"'] & \Psi''(X) \arrow[d, "\Psi''(\zeta)"] \\ Y' \arrow[r, "\beta'"] & \Psi''(X') \end{tikzcd}$$

  commute;
- the composition and identity morphisms are the obvious ones.

The category $\mathbf{L}(\mathtt{i})$ comes equipped with an action of $\mathscr{S}$, defined component-wise, using the action of $\mathscr{S}$ on $\mathbf{K}(\mathtt{i})$. This is well-defined as the 2-natural transformation Ψ is strict by Lemma 21 and, moreover, η is a modification. We denote the corresponding 2-representation of $\mathscr{S}$ by $\mathbf{L}$.

Lemma 24. *Restriction to the first component of a quadruple defines a strict 2-natural transformation $\Upsilon : \mathbf{L} \to \mathbf{K}''$. This Υ is an equivalence,*

Proof. Mutatis mutandis the proof of [MaMa, Lemma 19]. □

Define an endofunctor Θ on $\mathbf{L}(\mathtt{i})$ by sending (X, Y, α, β) to (Y, X, β, α) with the obvious action on morphisms. From all symmetries in the definition of $\mathbf{L}(\mathtt{i})$, it follows that Θ is a strict involution and it also strictly commutes with the action of $\mathscr{S}$.

Next, consider the category $\mathbf{L}'(\mathtt{i})$ defined as follows:

- $\mathbf{L}'(\mathtt{i})$ has the same objects as $\mathbf{L}(\mathtt{i})$,
- morphisms in $\mathbf{L}'(\mathtt{i})$ are defined, for objects $X, Y \in \mathbf{L}(\mathtt{i})$, via

$$\mathrm{Hom}_{\mathbf{L}'(\mathtt{i})}(X, Y) := \mathrm{Hom}_{\mathbf{L}(\mathtt{i})}(X, Y) \oplus \mathrm{Hom}_{\mathbf{L}(\mathtt{i})}(X, \Theta(Y)),$$

- composition and identity morphisms in $\mathbf{L}'(\mathtt{i})$ are induced from those in $\mathbf{L}(\mathtt{i})$ in the obvious way, see [CiMa, Definition 2.3] for details.

The fact that Ψ preserves the isomorphism class of the indecomposable object P_{sts} implies that the endomorphism algebra of the corresponding object in $\mathbf{L}'(\mathtt{i})$ contains a copy of the group algebra of the group $\{\mathrm{Id}, \Theta\}$ and hence is not local. This means that $\mathbf{L}'(\mathtt{i})$ is not idempotent split. Denote by $\mathbf{N}_s^{(n)}(\mathtt{i})$ the idempotent completion of $\mathbf{L}'(\mathtt{i})$.

Proposition 25.

(*i*) *The category $\mathbf{N}_s^{(n)}(\mathtt{i})$ is a finitary $\mathbb{C}$-linear category.*

(*ii*) *The category $\mathbf{N}_s^{(n)}(\mathtt{i})$ is equipped with an action of $\mathscr{S}$ induced from that on $\mathbf{L}'(\mathtt{i})$.*

(*iii*) *The obvious functor $\Xi : \mathbf{L}(\mathtt{i}) \to \mathbf{N}_s^{(n)}(\mathtt{i})$ is a strict 2-natural transformation.*

Proof. The only thing which is different from [MaMa, Proposition 20] is the fact that the category $\mathbf{L}'(\mathtt{i})$ is not idempotent split. However, since we define $\mathbf{N}_s^{(n)}(\mathtt{i})$ as the idempotent completion of $\mathbf{L}'(\mathtt{i})$, it follows that $\mathbf{N}_s^{(n)}(\mathtt{i})$ is idempotent split and hence $\mathbf{N}_s^{(n)}(\mathtt{i})$ is a finitary $\mathbb{C}$-linear category. The rest is similar to [MaMa, Proposition 20]. $\square$

The above gives us the 2-representation $\mathbf{N}_s^{(n)}$ of $\mathscr{S}$. From the fact that $\mathbf{C}_{\mathcal{L}_s}$ is simple transitive, it follows that $\mathbf{N}_s^{(n)}$ is simple transitive. The underlying algebra $\mathbf{N}_s^{(n)}$ is the quotient of the path algebra of the following quiver:

$$\begin{tikzcd}
 & & & & \circ \arrow[d, shift left] & \\
\bullet \arrow[r, shift left] & \bullet \arrow[l, shift left] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & \bullet \arrow[l, shift left] \arrow[r, shift left] & \bullet \arrow[l, shift left] \arrow[r, shift left] \arrow[u, shift left] & \circ \arrow[l, shift left]
\end{tikzcd}$$

where we mod out by the relations that any path of the form $\mathtt{i} \to \mathtt{j} \to \mathtt{k}$ is zero if $i \neq k$ and all paths of the form $\mathtt{i} \to \mathtt{j} \to \mathtt{i}$ coincide. Here the vertices $\bullet$ correspond to pairs $\{P_x, P_{x'w_0}\}$, where $x \in \mathtt{L}$ is such that $x \neq x'w_0$, while the two vertices $\circ$ correspond to the unique $x \in \mathtt{L}$ for which $x'w_0 = x$.

If k is odd, then the unique $x \in \mathtt{L}$ for which $x'w_0 = x$ has the form $stst\dots s$. This implies that the matrix B is, up to permutation of rows and columns, an extended staircase matrix of size $\frac{k+3}{2} \times \frac{k-1}{2}$. If k is even, then the unique $x \in \mathtt{L}$ for which $x'w_0 = x$ has the form $tst\dots s$. This implies that the matrix B is, up to permutation of rows and columns, an extended staircase matrix of size $\frac{k}{2} \times \frac{k+2}{2}$. The explanation why B has only one row or column with three non-zero elements is the fact that the quiver above has only one vertex which is connected to three other vertices.

We denote by $\mathbf{N}_t^{(n)}$ the 2-representation of $\mathscr{S}$ constructed similarly starting from $\mathcal{L}_t$. Note that $\mathbf{C}_{\mathcal{L}_s}$ and $\mathbf{C}_{\mathcal{L}_t}$ are not equivalent as their decategorifications contain different one-dimensional simple W-modules and hence are not isomorphic. It is easy to check that the decategorifications of $\mathbf{N}_s^{(n)}$ and $\mathbf{N}_t^{(n)}$ also contain different one-dimensional simple W-modules. This implies that $\mathbf{N}_s^{(n)}$ and $\mathbf{N}_t^{(n)}$ are not equivalent. Comparing decategorifications, we, in fact, see that the 2-representation $\mathbf{C}_{\mathcal{L}_s}$, $\mathbf{C}_{\mathcal{L}_t}$, $\mathbf{N}_s^{(n)}$ and $\mathbf{N}_t^{(n)}$ are pairwise not equivalent.

7.3. **The matrix** B**.** Our aim in this section is to describe the matrix B from Subsection 7.1.

Proposition 26. *Let* $\mathbf{M}$ *be a simple transitive* 2*-representation of* $\underline{\mathscr{S}}$ *with apex* $\mathcal{J}$*. Assume that* $n \neq 12, 18, 30$*. Then, up to permutation of rows and columns, the matrix* B *is a staircase matrix of size* $k \times (k-1)$ *or an extended staircase matrix of size* $\frac{k}{2} \times \frac{k+2}{2}$ *(if* k *is even) or of size* $\frac{k+3}{2} \times \frac{k-1}{2}$ *(if* k *is odd).*

For example, if $n = 6$, then B is one of the following matrices:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

If $n = 8$, then B is one of the following matrices:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Proof. As $\mathbf{M}$ is simple transitive, the non-negative matrix $\mathtt{M}$ is primitive. From the computation in the proof of Lemma 16 we thus get that both BB^{tr} and $B^{\mathrm{tr}}B$ must be irreducible non-negative matrices. The combination of Lemma 16 with Subsection 6.2 implies that both BB^{tr} and $B^{\mathrm{tr}}B$ are diagonalizable with real eigenvalues, moreover, all eigenvalues are contained in the half-open interval $[0, 4)$.

From Proposition 14, we thus obtain that B is either a staircase matrix or an extended staircase matrix or coincides with X_1 or X_2 or X_3. Note that all staircase and extended staircase matrices appear as B for some type $I_2(l)$, where l can be arbitrary. Therefore, thanks to Corollary 18, to complete the proof of our proposition, it is enough to show that B cannot coincide with any of X_1, X_2 or X_3.

The minimal polynomial of the matrix $X_1X_1^{\mathrm{tr}}$ is $(x-1)(x^2-4x+1) = \underline{f}_3(x)\underline{f}_{12}(x)$. The arguments in the proof of Corollary 18 imply that B can be equal to X_1 only in the case $n = 12$.

The minimal polynomial of the matrix $X_2X_2^{\mathrm{tr}}$ is $x^3 - 6x^2 + 9x - 3 = \underline{f}_{18}(x)$. The arguments in the proof of Corollary 18 imply that B can be equal to X_1 only in the case $n = 18$.

The minimal polynomial of the matrix $X_3X_3^{\mathrm{tr}}$ is $x^4 - 7x^3 + 14x^2 - 8x + 1 = \underline{f}_{30}(x)$. The arguments in the proof of Corollary 18 imply that B can be equal to X_1 only in the case $n = 30$.

As the cases $n = 12, 18, 30$ are excluded, the claim of the proposition follows. $\square$

7.4. **Proof of Theorem 20.** Let $\mathbf{M}$ be a simple transitive 2-representation of $\mathscr{S}$. Since $n \neq 12, 18, 30$, we can apply Proposition 26 to get four possibilities for B which are in a natural bijection with the 2-representations $\mathbf{C}_{\mathcal{L}_s}$, $\mathbf{C}_{\mathcal{L}_t}$, $\mathbf{N}_s^{(n)}$ and $\mathbf{N}_t^{(n)}$. That in the first two cases we have that $\mathbf{M}$ is equivalent to $\mathbf{C}_{\mathcal{L}_s}$ or, respectively, $\mathbf{C}_{\mathcal{L}_t}$, is proved similarly to Subsection 6.5. That in the last two cases we have that $\mathbf{M}$ is equivalent to $\mathbf{N}_s^{(n)}$ or, respectively, $\mathbf{N}_t^{(n)}$, is proved similarly to [MaMa, Subsection 5.10].

7.5. Exceptional types $I_2(12)$, $I_2(18)$ and $I_2(30)$. In Coxeter type $I_2(12)$, the proof of Proposition 26 leaves the possibility of $B = X_1$. In this case

$$\mathtt{M} = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

If we assume that a simple transitive 2-representation with such $\mathtt{M}$ exists, then the underlying algebra of this 2-representation must be the quotient of the path algebra of the following quiver (here the number of each vertex corresponds to the numbering of columns in $\mathtt{M}$):

$$\begin{tikzcd} & & \mathtt{5} \arrow[d, bend left] & & \\ \mathtt{1} \arrow[r, bend left] & \mathtt{4} \arrow[l, bend left] \arrow[r, bend left] & \mathtt{2} \arrow[l, bend left] \arrow[u, bend left] \arrow[r, bend left] & \mathtt{6} \arrow[l, bend left] \arrow[r, bend left] & \mathtt{3} \arrow[l, bend left] \end{tikzcd}$$

modulo the relations that any path of the form $\mathtt{i} \to \mathtt{j} \to \mathtt{k}$ is zero if $i \neq k$ and all paths of the form $\mathtt{i} \to \mathtt{j} \to \mathtt{i}$ coincide. This algebra is the quadratic dual of the preprojective algebra of the underlying Dynkin quiver of type E_6. This suggests a relation between type $I_2(12)$ and type E_6 which we do not understand. We do not know whether this hypothetical 2-representation exists and we see no reasons why it should not exist. On the decategorified level, the corresponding representation of the group algebra, on which the elements of the Kazhdan-Lusztig basis act via the corresponding non-negative matrices certainly does exist. Because of Subsection 8.3, there is also a possibility of connection between type $I_2(12)$ and type F_4.

Similarly, Coxeter types $I_2(18)$ and $I_2(30)$ are connected to Dynkin types E_7 and E_8, respectively. Because of Subsection 8.2, Coxeter type $I_2(30)$ could also be connected to Coxeter type H_4.

8. Simple transitive 2-representations of $\underline{\mathscr{S}}$ in other Coxeter types of rank higher than two

8.1. Coxeter type H_3. In this subsection we assume that W is of Coxeter type H_3 and $S = \{r, s, t\}$ with the Coxeter diagram

$$r \overset{5}{\text{———}} s \text{———} t.$$

Proposition 27. *Assume that W is of Coxeter type H_3. Let $\mathbf{M}$ be a simple transitive 2-representation of $\underline{\mathscr{S}}$ with apex $\mathcal{J}$. Then the isomorphism classes of indecomposable objects in $\mathbf{M}(\mathtt{i})$ can be ordered such that, with respect to the ordering r, s, t, we have*

$$\mathtt{M} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \end{pmatrix}.$$

Proof. To prove this statement we use reduction to rank two Coxeter subgroups. Let $\mathscr{C}_1$ denote the 2-full 2-subcategory of $\underline{\mathscr{S}}$ which is monoidally generated by θ_r and θ_s. By construction, the 2-category $\mathscr{C}_1$ has a quotient which is biequivalent to the 2-category $\underline{\mathscr{S}}$ in Coxeter type $I_2(5)$. Let $\mathscr{C}_2$ denote the 2-full 2-subcategory of $\underline{\mathscr{S}}$ which is monoidally generated by θ_s and θ_t. By construction, the 2-category $\mathscr{C}_2$ has a quotient which is biequivalent to the 2-category $\underline{\mathscr{S}}$ in Coxeter type A_2.

We can restrict the action of $\mathscr{C}_1$ to the additive closure $\mathcal{X}$, in $\mathbf{M}(\mathtt{i})$, of all indecomposable objects which are not annihilated by either θ_r or θ_s. From Theorem 15, $\mathscr{C}_1$ has a unique (up to equivalence) simple transitive 2-representation which does not annihilate any non-zero 1-morphisms. In this 2-representation, the actions of θ_r and θ_s are given by the following matrices:

$$\begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}. \tag{8}$$

By the combination of [KM2, Corollary 20] and [CM, Theorem 25], the actions of θ_r and θ_s on $\mathcal{X}$ are then given by a direct sum of blocks of the form (8).

We can restrict the action of $\mathscr{C}_2$ to the additive closure $\mathcal{Y}$, in $\mathbf{M}(\mathtt{i})$, of all indecomposable objects which are not annihilated by either θ_s or θ_t. From [MM5, Theorem 18], $\mathscr{C}_2$ has a unique (up to equivalence) simple transitive 2-representation which does not annihilate any non-zero 1-morphisms. In this 2-representation, the actions of θ_s and θ_t are given by the following matrices:

$$\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}. \tag{9}$$

By the combination of [KM2, Corollary 20] and [CM, Theorem 25], the actions of θ_s and θ_t on $\mathcal{Y}$ are then given by a direct sum of blocks of the form (9).

We need to combine blocks of the form (8) with blocks of the form (9) to get an irreducible non-negative matrix for the principal element $\mathbf{s}$, of size $3m$, for some m. The latter restriction is due to the fact that the decategorification of $\mathbf{M}$ is a sum of 3-dimensional simple W-modules. Note that (9) allows us to "connect" (in the sense of having a non-zero element on the intersection of the corresponding row and column) one indecomposable which is not annihilated by θ_s with one indecomposable which is not annihilated by θ_t. Therefore the only way is to connect each of the two indecomposables in some block of the form (8) which are not annihilated by θ_s with an external pair of two indecomposables which are not annihilated by θ_t. This gives exactly the matrices given in the formulation of the proposition (up to permutation of basis elements). The claim of the proposition follows. □

Theorem 28. *Assume that W is of Coxeter type H_3. Then every simple transitive 2-representation of $\underline{\mathscr{S}}$ is equivalent to a cell 2-representation.*

Proof. Let $\mathbf{M}$ be a simple transitive 2-representation of $\underline{\mathscr{S}}$ with apex $\mathcal{J}$. We assume that the equivalence classes of indecomposable objects in $\mathbf{M}(\mathtt{i})$ are ordered such that the decategorification matrices of θ_r, θ_s and θ_t are gives as in Proposition 27. Consider $\overline{\mathbf{M}}$ and the 2-representation $\mathbf{N}$ of $\underline{\mathscr{S}}$ given by restriction of the action to the category of projective objects in $\overline{\mathbf{M}}(\mathtt{i})$. Then $\mathbf{M}$ and $\mathbf{N}$ are equivalent by [MM2, Theorem 11]. We call indecomposable projectives in $\overline{\mathbf{M}}(\mathtt{i})$, in order, $P_1, P_2, \dots, P_6$ and their corresponding simple tops $L_1, L_2, \dots, L_6$. Using the explicit matrices given by Proposition 27 and arguments similar to the one used in Subsection 6.5, one shows that

$$\theta_r\, L_1 \cong P_1, \quad \theta_{sr}\, L_1 \cong P_3, \quad \theta_{rsr}\, L_1 \cong P_2, \quad \theta_{tsr}\, L_1 \cong P_5,$$
$$\theta_{srsr}\, L_1 \cong P_4, \quad \theta_{tsrsr}\, L_1 \cong P_6.$$

Now the proof is completed as in Subsection 6.5. There is a unique strong 2-natural transformation from $\mathbf{P}_{\mathtt{i}}$ to $\overline{\mathbf{M}}$ sending $\mathbb{1}_{\mathtt{i}}$ to L_1. It induces a strong 2-natural transformation from $\mathbf{C}_{\mathcal{L}_s}$ to $\mathbf{N}$. Comparing the Cartan matrices, we see that the latter 2-natural transformation is, in fact, an equivalence. Therefore $\mathbf{C}_{\mathcal{L}_s}$ and $\mathbf{M}$ are equivalent. □

Remark 29. The underlying algebra of a cell 2-representation of $\underline{\mathscr{S}}$ with apex $\mathcal{J}$ is the quotient of the path algebra of the following quiver (here the number $\mathtt{i}$ corresponds to P_i):

$$\begin{tikzcd}
 & \mathtt{1} \arrow[d, bend left] & & & \\
\mathtt{5} \arrow[r, bend left] & \mathtt{3} \arrow[l, bend left] \arrow[u, bend left] \arrow[r, bend left] & \mathtt{2} \arrow[l, bend left] \arrow[r, bend left] & \mathtt{4} \arrow[l, bend left] \arrow[r, bend left] & \mathtt{6} \arrow[l, bend left]
\end{tikzcd}$$

modulo the relations that any path of the form $\mathtt{i} \to \mathtt{j} \to \mathtt{k}$ is zero if $i \neq k$ and all paths of the form $\mathtt{i} \to \mathtt{j} \to \mathtt{i}$ coincide. The Loewy filtrations of the indecomposable projective modules for this algebra are:

$$\begin{tikzcd}[column sep=small]
1 \arrow[d, dash] \\ 3 \arrow[d, dash] \\ 1
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
 & 2 \arrow[dl, dash] \arrow[dr, dash] & \\
3 \arrow[dr, dash] & & 4 \arrow[dl, dash] \\
 & 2 &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
 & 3 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
1 \arrow[dr, dash] & 2 \arrow[d, dash] & 5 \arrow[dl, dash] \\
 & 3 &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
 & 4 \arrow[dl, dash] \arrow[dr, dash] & \\
2 \arrow[dr, dash] & & 6 \arrow[dl, dash] \\
 & 4 &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
5 \arrow[d, dash] \\ 3 \arrow[d, dash] \\ 5
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
6 \arrow[d, dash] \\ 4 \arrow[d, dash] \\ 6
\end{tikzcd}$$

This algebra is the quadratic dual of the preprojective algebra of the underlying Dynkin quiver of type D_6, cf. [Du]. For further connections between these two Coxeter groups, see [De] and references therein.

8.2. **Coxeter type H_4.** In this subsection we assume that W is of Coxeter type H_4 and $S = \{r, s, t, u\}$ with the Coxeter diagram

$$r \overset{5}{\text{——}} s \text{——} t \text{——} u.$$

Proposition 30. *Assume that W is of Coxeter type H_4. Let $\mathbf{M}$ be a simple transitive 2-representation of $\underline{\mathscr{S}}$ with apex $\mathcal{J}$. Then the isomorphism classes of indecomposable objects in $\mathbf{M}(\mathtt{i})$ can be ordered such that, with respect to the ordering r, s, t, u, we have*

$$\mathtt{M} = \begin{pmatrix}
2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 2 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 2
\end{pmatrix}.$$

Proof. Mutatis mutandis the proof of Proposition 27. □

Theorem 31. *Assume that W is of Coxeter type H_4. Then every simple transitive 2-representation of $\underline{\mathscr{S}}$ is equivalent to a cell 2-representation.*

Proof. Mutatis mutandis the proof of Theorem 28. □

Remark 32. The underlying algebra of a cell 2-representation of $\underline{\mathscr{S}}$ with apex $\mathcal{J}$ is the quotient of the path algebra of the following quiver (here the number $\mathtt{i}$ corresponds to P_i):

$$\begin{tikzcd}
 & & \mathtt{1} \arrow[d, shift left] & & & & \\
\mathtt{7} \arrow[r, shift left] & \mathtt{5} \arrow[l, shift left] \arrow[r, shift left] & \mathtt{3} \arrow[l, shift left] \arrow[u, shift left] \arrow[r, shift left] & \mathtt{2} \arrow[l, shift left] \arrow[r, shift left] & \mathtt{4} \arrow[l, shift left] \arrow[r, shift left] & \mathtt{6} \arrow[l, shift left] \arrow[r, shift left] & \mathtt{8} \arrow[l, shift left]
\end{tikzcd}$$

modulo the relations that any path of the form $\mathtt{i} \to \mathtt{j} \to \mathtt{k}$ is zero if $i \neq k$ and all paths of the form $\mathtt{i} \to \mathtt{j} \to \mathtt{i}$ coincide. This algebra is the quadratic dual of the preprojective algebra of the underlying Dynkin quiver of type E_8. For further connections between these two Coxeter groups, see [De] and references therein.

8.3. **Weyl type F_4.** In this subsection we assume that W is of Weyl type F_4 and $S = \{r, s, t, u\}$ with the Coxeter diagram

$$r \;\text{——}\; s \overset{4}{\text{——}} t \;\text{——}\; u.$$

Proposition 33. *Assume that W is of Weyl type F_4. Let $\mathbf{M}$ be a simple transitive 2-representation of $\underline{\mathscr{S}}$ with apex $\mathcal{J}$. Then the isomorphism classes of indecomposable objects in $\mathbf{M}(\mathtt{i})$ can be ordered such that, with respect to the ordering r, s, t, u, we have either*

$$\mathtt{M} = \begin{pmatrix} 2&0&1&0&0&0\\ 0&2&0&1&0&0\\ 1&0&2&0&1&0\\ 0&1&0&2&1&0\\ 0&0&1&1&2&1\\ 0&0&0&0&1&2 \end{pmatrix} \quad \textit{or} \quad \mathtt{M} = \begin{pmatrix} 2&1&0&0&0&0\\ 1&2&1&1&0&0\\ 0&1&2&0&1&0\\ 0&1&0&2&0&1\\ 0&0&1&0&2&0\\ 0&0&0&1&0&2 \end{pmatrix}.$$

Proof. Mutatis mutandis the proof of Proposition 27. □

Theorem 34. *Assume that W is of Weyl type F_4. Then every simple transitive 2-representation of $\underline{\mathscr{S}}$ is equivalent to a cell 2-representation.*

Proof. Mutatis mutandis the proof of Theorem 28. □

Remark 35. We have two different cell 2-representations of $\underline{\mathscr{S}}$ with apex $\mathcal{J}$ in Weyl type F_4. However, the underlying algebras of these cell 2-representation are isomorphic. This common algebra is the quotient of the path algebra of the following quiver (here the number $\mathtt{i}$ corresponds to P_i in the left matrix in Proposition 33):

$$\begin{tikzcd}
 & & \mathtt{6} \arrow[d, shift left] & & \\
\mathtt{1} \arrow[r, shift left] & \mathtt{3} \arrow[l, shift left] \arrow[r, shift left] & \mathtt{5} \arrow[l, shift left] \arrow[u, shift left] \arrow[r, shift left] & \mathtt{4} \arrow[l, shift left] \arrow[r, shift left] & \mathtt{2} \arrow[l, shift left]
\end{tikzcd}$$

modulo the relations that any path of the form $\mathtt{i} \to \mathtt{j} \to \mathtt{k}$ is zero if $i \neq k$ and all paths of the form $\mathtt{i} \to \mathtt{j} \to \mathtt{i}$ coincide. This algebra is the quadratic dual of the preprojective algebra of the underlying Dynkin quiver of type E_6.

8.4. **Coxeter type B_n.** In this subsection we assume that W is of type B_n and $S = \{r, s, t, u, \dots, v\}$ with Coxeter diagram

$$r \overset{4}{\text{———}} s \text{———} t \text{———} u \text{———} \dots \text{———} v.$$

Proposition 36. *Assume that W is of type B_n. Let $\mathbf{M}$ be a simple transitive 2-representation of $\underline{\mathscr{S}}$ with apex $\mathcal{J}$. Then the isomorphism classes of indecomposable objects in $\mathbf{M}(\mathtt{i})$ can be ordered such that, with respect to the ordering $r, s, t, u, \dots, v$, we have either*

$$\mathtt{M} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 2 \end{pmatrix}$$

or

$$\mathtt{M} = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 2 \end{pmatrix}.$$

Proof. Mutatis mutandis the proof of Proposition 27. ☐

Theorem 37. *Assume that W is of type B_n. Then every simple transitive 2-representation of $\underline{\mathscr{S}}$ is equivalent to a cell 2-representation.*

Proof. Mutatis mutandis the proof of Theorem 28. ☐

Remark 38. We have two different cell 2-representations of $\underline{\mathscr{S}}$ with apex $\mathcal{J}$ in type B_n. For the first one (which corresponds to the first choice in Proposition 36) the underlying algebra is the quotient of the path algebra of the following quiver:

$$\begin{tikzcd} & \mathtt{2} \arrow[d, bend left] & & & \\ \mathtt{1} \arrow[r, bend left] & \mathtt{3} \arrow[l, bend left] \arrow[u, bend left] \arrow[r, bend left] & \mathtt{4} \arrow[l, bend left] \arrow[r, bend left] & \dots \arrow[l, bend left] \arrow[r, bend left] & \mathtt{n+1}. \arrow[l, bend left] \end{tikzcd}$$

For the second one (which corresponds to the second choice in Proposition 36) the underlying algebra is the quotient of the path algebra of the following quiver:

$$\begin{tikzcd} \mathtt{2n-1} \arrow[r, bend left] & \dots \arrow[l, bend left] \arrow[r, bend left] & \mathtt{5} \arrow[l, bend left] \arrow[r, bend left] & \mathtt{3} \arrow[l, bend left] \arrow[r, bend left] & \mathtt{1} \arrow[l, bend left] \arrow[r, bend left] & \mathtt{2} \arrow[l, bend left] \arrow[r, bend left] & \mathtt{4} \arrow[l, bend left] \arrow[r, bend left] & \dots \arrow[l, bend left] \arrow[r, bend left] & \mathtt{2n-2}. \arrow[l, bend left] \end{tikzcd}$$

In both cases, we mod out by the relations that any path of the form $\mathtt{i} \to \mathtt{j} \to \mathtt{k}$ is zero if $i \neq k$ and all paths of the form $\mathtt{i} \to \mathtt{j} \to \mathtt{i}$ coincide. The first algebra is the quadratic dual of the preprojective algebra of the underlying Dynkin quiver of type D_{n+1}. The second algebra is the quadratic dual of the preprojective algebra of the underlying Dynkin quiver of type A_{2n-1}.

9. New examples of finitary 2-categories

9.1. **Symmetric modules.** Let $\Bbbk$ be an algebraically closed field. In this subsection we assume that $\mathrm{char}(\Bbbk) \neq 2$. Let A be a connected finite dimensional $\Bbbk$-algebra with a fixed automorphism $\iota : A \to A$ such that $\iota^2 = \mathrm{id}_A$. For $M \in A$-mod, denote by ${}^{\iota}M$ the A-module with the same underlying space as M but with the new action $\bullet$ of A twisted by ι:

$$a \bullet m := \iota(a) \cdot m, \quad \text{for all} \quad a \in A \;\text{ and }\; m \in M.$$

For a fixed A-module Q, consider the category $\mathcal{Q} := \mathrm{add}(Q \oplus {}^{\iota}Q)$. Define the category $\hat{\mathcal{Q}} = \hat{\mathcal{Q}}(A, \iota, Q)$ in the following way:

- the objects in $\hat{\mathcal{Q}}$ are all diagrams of the form

$$M \xrightarrow{\alpha} {}^{\iota}M \tag{10}$$

  where $M \in \mathcal{Q}$ and $\alpha : M \to {}^{\iota}M$ is an isomorphism in A-mod such that $\alpha^2 \cong \mathrm{id}_M$;
- morphisms in $\hat{\mathcal{Q}}$ are all commutative diagrams of the form

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & {}^{\iota}M \arrow[d, "\varphi"] \\ N \arrow[r, "\beta"] & {}^{\iota}N, \end{tikzcd}$$

  where $\varphi : M \to N$ is a homomorphism in A-mod (note that $\varphi : {}^{\iota}M \to {}^{\iota}N$ is a homomorphism in A-mod as well);
- identity morphisms in $\hat{\mathcal{Q}}$ are given by the identity maps;
- composition in $\hat{\mathcal{Q}}$ is induced from composition in $\mathcal{Q}$ in the obvious way.

We will call $\hat{\mathcal{Q}}$ the category of *ι-symmetric A-modules* over $\mathcal{Q}$. Directly from the definitions it follows that $\hat{\mathcal{Q}}$ is additive, $\Bbbk$-linear and idempotent split.

We note that, if M is an A-module such that $M \cong {}^{\iota}M$, then an isomorphism $\alpha : M \to {}^{\iota}M$ can always be chosen such that $\alpha^2 = \mathrm{id}_M$. Indeed, if $\alpha : M \to {}^{\iota}M$ is any isomorphism, then $\iota^2 = \mathrm{id}_A$ implies that α^2 is an automorphism of M. As α^{-2} is invertible, there exists an automorphism β of M which is a polynomial in α^{-2} such that $\beta^2 = \alpha^{-2}$. In particular, β commutes with α. Then $(\alpha\beta) : M \to {}^{\iota}M$ is an isomorphism and $(\alpha\beta)^2 = \mathrm{id}_M$.

Proposition 39. *The category $\hat{\mathcal{Q}}$ is Krull-Schmidt and has finitely many isomorphism classes of indecomposable objects.*

Proof. Let Q_1, Q_2,..., Q_n be a complete list of pairwise non-isomorphic indecomposable objects in $\mathcal{Q}$. For every Q_i, we either have $Q_i \cong {}^{\iota}Q_i$ or we have $Q_i \cong {}^{\iota}Q_j$, for some $j \neq i$. For each i, set $Q_i^{(\iota)} := Q_i \oplus {}^{\iota}Q_i$ and let $\alpha_i : Q_i^{(\iota)} \to {}^{\iota}Q_i^{(\iota)}$ be the homomorphism which swaps the components of the direct sum. Then

$$Q_i^{(\iota)} \xrightarrow{\alpha_i} {}^{\iota}Q_i^{(\iota)} \tag{11}$$

is an object in $\hat{\mathcal{Q}}$ which we denote by $(Q_i^{(\iota)}, \alpha_i)$.

If $Q_i \cong {}^\iota Q_j$, for some $j \neq i$, then $(Q_i^{(\iota)}, \alpha_i)$ is, clearly, indecomposable. Moreover, it is easy to see that $(Q_i^{(\iota)}, \alpha_i)$ and $(Q_j^{(\iota)}, \alpha_j)$ are isomorphic.

Assume now that $\varphi : Q_i \cong {}^\iota Q_i$ is an isomorphism such that $\varphi^2 = \mathrm{id}_{Q_i}$. Then $\iota^2 = \mathrm{id}_A$ implies that $\varphi : {}^\iota Q_i \cong Q_i$ is an isomorphism as well and hence the matrix

$$\Phi := \begin{pmatrix} 0 & \varphi \\ \varphi & 0 \end{pmatrix}$$

gives rise to an endomorphism of the object $\alpha_i : Q_i^{(\iota)} \to {}^\iota Q_i^{(\iota)}$. Note that Φ^2 is the identity on this object. Write $Q_i^{(\iota)} \cong X_i \oplus Y_i$, where X_i denotes the eigenspace of Φ for the eigenvalue 1 and Y_i denotes the eigenspace of Φ for the eigenvalue -1. Clearly, $X_i \cong Y_i \cong Q_i$, as A-modules. At the same time, the objects

$$X_i \xrightarrow{(\alpha_i)|_{X_i}} {}^\iota X_i \qquad \text{and} \qquad Y_i \xrightarrow{(\alpha_i)|_{Y_i}} {}^\iota Y_i$$

are both in $\hat{\mathcal{Q}}$ and are, clearly, indecomposable.

For each i such that $Q_i \cong {}^\iota Q_i$, we fix an isomorphism $\beta_i : Q_i \to {}^\iota Q_i$ such that $(\beta_i)^2 = \mathrm{id}_{Q_i}$. We claim that each indecomposable object in $\hat{\mathcal{Q}}$ is isomorphic to $(Q_i^{(\iota)}, \alpha_i)$, for some i such that $Q_i \not\cong {}^\iota Q_i$, or is isomorphic to one of the objects

$$Q_i \xrightarrow{\beta_i} {}^\iota Q_i \qquad \text{or} \qquad Q_i \xrightarrow{-\beta_i} {}^\iota Q_i\ , \tag{12}$$

for some i such that $Q_i \cong {}^\iota Q_i$. Indeed, consider an indecomposable object of the form (10). If M contains, as a direct summand, some $N \cong Q_i$ such that $Q_i \not\cong {}^\iota Q_i$, then $N \oplus \alpha(N)$ is a direct summand of M isomorphic to $(Q_i^{(\iota)}, \alpha_i)$ and hence M is isomorphic to the latter module. If M contains, as a direct summand, some $N \cong Q_i$ such that $Q_i \cong {}^\iota Q_i$, then from the previous paragraph it follows that M is isomorphic to $Q_i \xrightarrow{\alpha} {}^\iota Q_i$, for some isomorphism α such that $\alpha^2 = \mathrm{id}_{Q_i}$. We claim that each such object is isomorphic to one from the list (12). From the commutative diagram

$$\begin{tikzcd}[column sep=huge, row sep=huge]
Q_i \arrow[rr, "\alpha"] \arrow[d, "\begin{pmatrix} \mathrm{id}_{Q_i} \\ \alpha \end{pmatrix}"'] & & {}^\iota Q_i \arrow[d, "\begin{pmatrix} \mathrm{id}_{{}^\iota Q_i} \\ \alpha \end{pmatrix}"] \\
Q_i \oplus {}^\iota Q_i \arrow[rr, "\begin{pmatrix} 0 & \mathrm{id}_{{}^\iota Q_i} \\ \mathrm{id}_{Q_i} & 0 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \mathrm{id}_{Q_i} & \alpha \end{pmatrix}"'] & & {}^\iota Q_i \oplus Q_i \arrow[d, "\begin{pmatrix} \mathrm{id}_{{}^\iota Q_i} & \alpha \end{pmatrix}"] \\
Q_i \arrow[rr, "\alpha"] & & {}^\iota Q_i
\end{tikzcd}$$

and the fact that $\mathrm{char}(\Bbbk) \neq 2$, it follows that $Q_i \xrightarrow{\alpha} {}^\iota Q_i$ is a summand of (11). As we already established in the previous paragraph, (11) has two direct summands. So, it is enough to argue that the two objects in the list (12) are not isomorphic. The latter follows easily from the fact that β, being an automorphism of an indecomposable module, has only one eigenvalue and this eigenvalue is nonzero and hence β cannot be conjugate to $-\beta$ whose unique eigenvalue is different.

The claim of the proposition follows. $\square$

9.2. **Symmetric projective bimodules.** Let $\Bbbk$ be an algebraically closed field and A a finite dimensional $\Bbbk$-algebra with a fixed automorphism $\iota : A \to A$ such that $\iota^2 = \mathrm{id}_A$. We extend ι to an automorphism $\underline{\iota}$ of $A \otimes_{\Bbbk} A^{\mathrm{op}}$ component-wise and have $\underline{\iota}^2 = \mathrm{id}_{A\otimes_{\Bbbk} A^{\mathrm{op}}}$. As usual, we identify $A \otimes_{\Bbbk} A^{\mathrm{op}}$-modules and A-A–bimodules.

Consider the A-A–bimodule $Q := A \oplus (A \otimes_{\Bbbk} A)$ and the corresponding categories $\mathcal{Q}$ and $\hat{\mathcal{Q}}$. Note that Q is isomorphic to its twist by $\underline{\iota}$, as the latter amounts to simultaneously twisting both the left and the right actions of A on Q by ι. The category $\hat{\mathcal{Q}}$ is additive, $\Bbbk$-linear, idempotent split, Krull-Schmidt and has finitely many isomorphism classes of indecomposable objects by Proposition 39. The category of A-A–bimodules has the natural structure of a tensor category given by tensor product over A. This structure, applied component-wise, turns $\hat{\mathcal{Q}}$ into a tensor category. We denote by $\mathscr{C}_{(A,\iota)}$ the strictification of $\hat{\mathcal{Q}}$ as described, for example, in [Le, Subsection 2.3]. Then $\mathscr{C}_{(A,\iota)}$ is a strict tensor category or, equivalently, a 2-category with one object which we call $\mathtt{i}$.

Proposition 40. *The* 2*-category* $\mathscr{C}_{(A,\iota)}$ *is finitary, moreover, it is weakly fiat provided that* A *is self-injective.*

Proof. The fact that $\mathscr{C}_{(A,\iota)}$ is finitary follows directly from the construction and Proposition 39. If A is self-injective, then the 2-category $\mathscr{C}_A$ is weakly fiat, cf. [MM1, Subsection 7.3]. The weak anti-automorphism lifts from $\mathscr{C}_A$ to $\mathscr{C}_{(A,\iota)}$ in the obvious way. We claim that even adjunction morphisms can be lifted from $\mathscr{C}_A$ to $\mathscr{C}_{(A,\iota)}$. Indeed, let $X \in \mathrm{add}(A \oplus \big(A \otimes_{\Bbbk} A\big))$ and $\varphi : X \to A$ be a homomorphism of bimodules. Then the following diagram commutes:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
X \oplus {}^{\iota}X^{\iota} \arrow[r, "\begin{pmatrix} 0 & \mathrm{Id}_X \\ \mathrm{Id}_X & 0 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \varphi \\ \iota\circ\varphi \end{pmatrix}"'] & X \oplus {}^{\iota}X^{\iota} \arrow[d, "\begin{pmatrix} \varphi \\ \iota\circ\varphi \end{pmatrix}"] \\
A \arrow[r, "\iota"] & A
\end{tikzcd}$$

and hence the vertical arrows give 2-morphisms in $\mathscr{C}_{(A,\iota)}$. Taking φ to be adjunction morphisms in $\mathscr{C}_A$ gives rise, in this way, to adjunction morphisms in $\mathscr{C}_{(A,\iota)}$. This implies that $\mathscr{C}_{(A,\iota)}$ is weakly fiat. □

The novel component of this example compared to various examples which can be found in [MM1]–[MM6], especially to the 2-category $\mathscr{C}_A$ from [MM1, Subsection 7.3], is the fact that *indecomposable* 1-morphisms in $\mathscr{C}_{(A,\iota)}$ are given, in general, by *decomposable* endofunctors of A-mod. This, in particular, allows us to give an alternative construction for [Xa, Example 8], see the next subsection.

9.3. **An example.** Here we use Subsection 9.2 to construct an example of a fiat 2-category which has a left cell for which the Duflo involution (see [MM1, Proposition 17]) is not self-adjoint. The example is essentially the same as [Xa, Example 8], however, it is constructed using completely different methods.

Let A be the quotient of the path algebra of the quiver

$$\begin{tikzcd}
\mathtt{1} \arrow[r, bend left, "\alpha"] & \mathtt{2} \arrow[l, bend left, "\beta"]
\end{tikzcd}$$

modulo the ideal generated by the relations $\alpha\beta = \beta\alpha = 0$. Let ε_1 be the trivial path at $\mathtt{1}$ and ε_2 be the trivial path at $\mathtt{2}$. Let ι be the automorphism of A given by swapping ε_1 with ε_2 and α with β. Consider the corresponding 2-category $\mathscr{C}_{(A,\iota)}$.

From Subsection 9.1 it follows that indecomposable 1-morphisms in $\mathscr{C}_{(A,\iota)}$ are given by $\mathrm{F}_1 := (A,\iota)$ and $\mathrm{F}_2 := (A,-\iota)$ together with

$$\mathrm{G}_1 := \left(\big(A\varepsilon_1 \otimes_{\Bbbk} \varepsilon_1 A\big) \oplus \big(A\varepsilon_2 \otimes_{\Bbbk} \varepsilon_2 A\big), \begin{pmatrix} 0 & \iota \\ \iota & 0 \end{pmatrix} \right),$$

$$\mathrm{G}_2 := \left(\big(A\varepsilon_1 \otimes_{\Bbbk} \varepsilon_2 A\big) \oplus \big(A\varepsilon_2 \otimes_{\Bbbk} \varepsilon_1 A\big), \begin{pmatrix} 0 & \iota \\ \iota & 0 \end{pmatrix} \right).$$

It is easy to see that we have two two-sided cells, namely, $\{\mathrm{F}_1, \mathrm{F}_2\}$ and $\{\mathrm{G}_1, \mathrm{G}_2\}$. They both are, at the same time, both left and right cells. From [MM1, Subsection 7.3] it follows that the objects G_1 and G_2 are, in fact, biadjoint to each other. Therefore, $\mathscr{C}_{(A,\iota)}$ is a fiat 2-category and the Duflo involution of the left cell $\{\mathrm{G}_1, \mathrm{G}_2\}$ cannot be self-adjoint as neither G_1 nor G_2 is.

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