THE $\mathfrak{sl}_N$-WEB ALGEBRAS AND DUAL CANONICAL BASES

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ABSTRACT. In this paper, which is a follow-up to [38], I define and study $\mathfrak{sl}_N$-web algebras, for any $N \geq 2$. For $N = 2$ these algebras are isomorphic to Khovanov's [22] arc algebras and for $N = 3$ they are Morita equivalent to the $\mathfrak{sl}_3$-web algebras which I defined and studied together with Pan and Tubbenhauer [34].

The main result of this paper is that the $\mathfrak{sl}_N$-web algebras are Morita equivalent to blocks of certain level-N cyclotomic KLR algebras, for which I use the categorified quantum skew Howe duality defined in [38].

Using this Morita equivalence and Brundan and Kleshchev's [4] work on cyclotomic KLR-algebras, I show that there exists an isomorphism between a certain space of $\mathfrak{sl}_N$-webs and the split Grothendieck group of the corresponding $\mathfrak{sl}_N$-web algebra, which maps the dual canonical basis elements to the Grothendieck classes of the indecomposable projective modules (with a certain normalization of their grading).

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1. Introduction

In [14] Cautis, Kamnitzer and Morrison defined $\mathfrak{sl}_N$-webs and the relations they satisfy, for arbitrary $N \in \mathbb{N}_{\geq 2}$. In [38] Yonezawa and I defined certain $\mathfrak{sl}_N$-web spaces W_Λ for arbitrary $N \in \mathbb{N}_{\geq 2}$ and $\Lambda := N\omega_\ell$, where ω_ℓ is the ℓ-th fundamental $\mathfrak{sl}_m$-weight with $m = N\ell$ for arbitrary $\ell \in \mathbb{N}$. By quantum skew Howe duality, also due to Cautis, Kamnitzer and Morrison [14], we obtained a $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-action on W_Λ and showed that there exists an isomorphism of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-modules

$$V_\Lambda \to W_\Lambda. \tag{1}$$

Here V_Λ is the irreducible $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-module of highest weight Λ, obtained as a quotient of the Verma module with the same highest weight.

In the same paper, we also defined $\mathbb{C}$-linear additive $\mathfrak{sl}_N$-web categories $\mathcal{W}_\Lambda^\circ$, using colored $\mathfrak{sl}_N$-matrix factorizations. We showed that $\mathcal{W}_\Lambda^\circ$ is a strong $\mathfrak{sl}_m$ 2-representation and that there exists an equivalence of strong $\mathfrak{sl}_m$ 2-representations

$$\mathcal{V}_\Lambda^p \to \dot{\mathcal{W}}_\Lambda^\circ. \tag{2}$$

Here $\dot{\mathcal{W}}_\Lambda^\circ$ denotes the Karoubi envelope of $\mathcal{W}_\Lambda^\circ$ and $\mathcal{V}_\Lambda^p := R_\Lambda - \mathrm{pmod}_{\mathrm{gr}}$ is the category of finite-dimensional graded projective modules of the cyclotomic Khovanov-Lauda-Rouquier (KLR) algebra R_Λ. As we argued in [38], this result can be seen as a categorification of an instance of the quantum skew Howe duality defined in [14].

Brundan and Kleshchev [4] showed that there exists a bijective $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-intertwiner

$$\delta\colon V_\Lambda \to K_0^q(\mathcal{V}_\Lambda^p) \tag{3}$$

where K_0^q denotes the split Grothendieck group tensored with $\mathbb{C}(q)$ over $\mathbb{Z}[q,q^{-1}]$.

Using (2) and (3), Yonezawa and I defined a bijective $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-intertwiner

$$\delta^\circ\colon W_\Lambda \to K_0^q(\dot{\mathcal{W}}_\Lambda^\circ), \tag{4}$$

such that the following square commutes:

$$\begin{array}{ccc} V_\Lambda & \xrightarrow{\delta} & K_0^q(\mathcal{V}_\Lambda^p) \\ \downarrow & & \downarrow \\ W_\Lambda & \xrightarrow{\delta^\circ} & K_0^q(\dot{\mathcal{W}}_\Lambda^\circ) \end{array}. \tag{5}$$

The category $\mathcal{W}_\Lambda^\circ$ has infinitely many objects, but the space W_Λ is finite-dimensional. Therefore, in this paper I choose a finite basis of W_Λ and look at the full subcategory of $\mathcal{W}_\Lambda^\circ$ generated by the objects corresponding to these basis webs, denoted $\mathcal{W}_\Lambda^p$. I use some "general arguments" to show that

$$\mathcal{W}_\Lambda^p \cong \dot{\mathcal{W}}_\Lambda^\circ \tag{6}$$

and that there exists a $\mathbb{Z}$-graded finite-dimensional algebra H_Λ such that

$$\mathcal{W}^p_\Lambda \cong H_\Lambda - \mathrm{pmod}_{\mathrm{gr}}. \tag{7}$$

Although conceptually the path I sketched above is probably clearer, in the paper I will follow the inverse path. I will first define H_Λ (Definitions 5.3 and 7.1), then use (7) as the definition of $\mathcal{W}^p_\Lambda$ (Definitions 5.5 and 7.1) and finally prove (6) (Lemma 7.5).

By "general arguments" I mean that they would prove the analogous results for other bases of W_Λ. In this paper I have chosen the basis of W_Λ which corresponds to the Leclerc-Toffin [32] basis in V_Λ by quantum skew Howe duality. However, I could have used *Fontaine's basis* [18] of W_Λ, for example. This would give rise to a Morita equivalent web algebra and all results in this paper for H_Λ would have analogues for this web algebra.

Leclerc and Toffin [32] showed that their basis can be used to compute the canonical basis of V_Λ. Therefore, quantum skew Howe duality implies that the corresponding basis can be used to compute the dual canonical basis in W_Λ, as I will show in Section 4.[1] In particular, Leclerc and Toffin's results indicate how one should normalize the generating $\mathfrak{sl}_N$-intertwiners corresponding to the generating $\mathfrak{sl}_N$-webs, such that the isomorphism in (1) maps the canonical basis in V_Λ precisely to the dual canonical basis of W_Λ. As I will explain in Remark 4.3, in order to do this properly one has to switch to a new quantum parameter $v = -q^{-1}$, which first appeared in [19] and [25]. Note that this normalization of the intertwiners differs from the normalization in the Cautis-Kamnitzer-Morrison paper [14].

Just as R_Λ, the algebra H_Λ is a direct sum of blocks

$$H_\Lambda = \bigoplus_{\vec{k}\in\Lambda(m,m)_N} H(\vec{k}, N),$$

where

$$\Lambda(m,m)_N := \{\vec{k} = (k_1,\ldots,k_m) \in \{0,\ldots,N\}^m \mid k_1 + \cdots + k_m = m\} \tag{8}$$

is the set of $\mathfrak{gl}_m$-weights of V_Λ. I show that $H(\vec{k}, N)$ is a finite-dimensional graded symmetric Frobenius algebra, for each $\vec{k} \in \Lambda(m,m)_N$. In fact, $H(\vec{k}, N)$ is only isomorphic to its dual $H(\vec{k}, N)^\vee$ after a certain degree shift depending on $\vec{k}$. Therefore H_Λ is not a graded Frobenius algebra, strictly speaking.

By (2) and (6), we see that

$$\mathcal{V}^p_\Lambda \cong \mathcal{W}^p_\Lambda$$

holds. The main result in this paper (Theorem 7.7) is the extension of this equivalence to the categories of *all* finite-dimensional graded modules of R_Λ and H_Λ:

$$\mathcal{V}_\Lambda \cong \mathcal{W}_\Lambda. \tag{9}$$

In other words, R_Λ and H_Λ are Morita equivalent as graded algebras.

The equivalence in (9) allows me to define an anti-involution

$$*\colon H_\Lambda \to H_\Lambda$$

which corresponds to Khovanov and Lauda's [26] anti-involution on R_Λ, and use it to define a duality

$$\circledast\colon \mathcal{W}_\Lambda \to \mathcal{W}_\Lambda$$

[1]This idea first arose in a discussion with Daniel Tubbenhauer on web bases.

which corresponds to Brundan and Kleshchev's [4] duality on $\mathcal{V}_\Lambda$. Just as in their case, the duality on $\mathcal{W}_\Lambda$ can be restricted to $\mathcal{W}_\Lambda^p$, which induces a bar-involution on its Grothendieck group.

Brundan and Kleshchev [4] showed that δ in (3) intertwines the bar-involutions and that it sends the canonical basis elements in V_Λ to the Grothendieck classes of the indecomposable projective modules in $\mathcal{V}_\Lambda^p$ with a suitable normalization of their grading. I will show that the arguments above imply that there exists an isomorphism of $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$-modules

$$\delta' \colon W_\Lambda \to K_0^v(\mathcal{W}_\Lambda^p)$$

such that the square

$$\begin{array}{ccc} V_\Lambda & \xrightarrow{\delta} & K_0^v(\mathcal{V}_\Lambda^p) \\ \uparrow & & \uparrow \\ W_\Lambda & \xrightarrow{\delta'} & K_0^v(\mathcal{W}_\Lambda^p) \end{array} \tag{10}$$

commutes. Note that the vertical maps are inverted, compared to (5). This is just for convenience: the proof of Theorem 7.7 becomes slightly shorter this way.

By the above, it follows (Corollary 7.9) that δ' maps the dual canonical basis elements in W_Λ to the Grothendieck classes of the indecomposable projective modules in $\mathcal{W}_\Lambda^p$ which correspond to Brundan and Kleshchev's indecomposables in $\mathcal{V}_\Lambda^p$.

Finally, the equivalence in (9) implies (Corollary 7.10) that the center of $H(\vec{k}, N)$ is isomorphic to the cohomology of a certain N-block Spaltenstein variety, for each $\vec{k} \in \Lambda(m,m)_N$. This follows from the analogous result for R_Λ, due to Brundan, Kleshchev and Ostrik [3, 5, 6].

For $N = 2$ the web algebras were introduced by Khovanov [22], who called them *arc algebras*. For $N = 3$ Pan and Tubbenhauer and I defined web algebras in [34] using Kuperberg's web basis. This basis is not equal to Leclerc and Toffin's, so the $\mathfrak{sl}_3$-web algebras in that paper are not isomorphic to the ones in this paper, but they are Morita equivalent.

For $N = 2$ Huerfano and Khovanov [21] used the arc algebras to category V_Λ, using a (partial) categorification of quantum skew Howe duality. The representation theory of the arc algebras and its connection with geometry have been studied in depth by a variety of people [7, 8, 9, 10, 11, 17, 22, 24, 42, 43]. For $N = 3$ the analogues of some of the results for arc algebras have been proved in [34] but much less is known. The results in this paper generalize some of the known results for $N = 2, 3$.

For $N = 2$ Stroppel and Webster [43] proved that the arc algebras can be obtained from the intersection cohomology of the corresponding 2-block Springer varieties. One can ask (as Kamnitzer asked M. M. for $N = 3$) if $H(\vec{k}, N)$ can be obtained from the intersection cohomology of the $\mathfrak{sl}_N$ web varieties in [19].

Another open question is how to generalize the web algebras to *clasped webs* [13, 30]. For such "clasped web algebras" one can also ask about the relation with the intersection cohomology of the web varieties in [19], since the framework in that paper is quite general.

2. Notation and conventions

In this section I fix some notations and explain some conventions.

Let $\mathcal{C}^*$ be a $\mathbb{Z}$-graded $\mathbb{C}$-linear additive or Abelian category which admits translation (for a precise definition and some introductory remarks on this sort of categories, see [31] and the references therein for example). Then $\{t\}$ denotes a positive translation/shift of t units. For any Laurent polynomial $f(q) = \sum a_i q^i \in \mathbb{N}[q, q^{-1}]$, define

$$X^{\oplus f(q)} := \bigoplus_i (X\{i\})^{\oplus a_i}.$$

Let $\mathcal{C}$ be the subcategory of $\mathcal{C}^*$ with the same objects but only degree-zero morphisms. In this paper $\mathcal{C}$ will always have finite-dimensional hom-spaces.

Typical examples of such categories in this paper will be of the following sort. Let $\mathcal{C}$ be given by

$$A - \mathrm{mod}_{\mathrm{gr}} \quad \text{or} \quad A - \mathrm{pmod}_{\mathrm{gr}},$$

which are the categories of finite-dimensional graded modules and finite-dimensional graded projective modules of a finite-dimensional $\mathbb{Z}$-graded complex algebra A. Both these categories admit translation, where the grading shifts are defined by

$$X\{t\}_i := X_{i-t}$$

for any object $X \in \mathcal{C}$ and any $i, t \in \mathbb{Z}$.

For any pair of objects $X, Y \in \mathcal{C}$, let $\mathrm{Hom}(X, Y)$ be the hom-space in $\mathcal{C}$. Then the graded hom-space of $\mathcal{C}^*$ is given by

$$\mathrm{HOM}(X, Y) := \bigoplus_{t \in \mathbb{Z}} \mathrm{Hom}(X\{t\}, Y).$$

For simplicity, assume that $\mathcal{C}^*$ has finite-dimensional hom-spaces too. Define the *quantum dimension* of $\mathrm{HOM}(X, Y)$ by

$$\dim_q \mathrm{HOM}(X, Y) := \sum_{t \in \mathbb{Z}} q^t \dim \mathrm{Hom}(X\{t\}, Y) \in \mathbb{N}[q, q^{-1}].$$

Assume additionally that $\mathcal{C}$ is Krull-Schmidt. The split Grothendieck group $K_0(\mathcal{C})$ is by definition the Abelian group generated by the isomorphism classes of the objects in $\mathcal{C}$ modulo the relation

$$[X \oplus Y] = [X] + [Y],$$

for any objects $X, Y \in \mathcal{C}$. It becomes a $\mathbb{Z}[q, q^{-1}]$-module, by defining

$$q[X] = [X\{1\}],$$

for any object $X \in \mathcal{C}$. For any Laurent polynomial $f(q) = \sum a_i q^i \in \mathbb{N}[q, q^{-1}]$, we get

$$f(q)[X] = [X^{\oplus f(q)}].$$

Assume that $S = \{X_1, \ldots, X_s\}$ is a set of indecomposable objects in $\mathcal{C}$ such that

- any indecomposable object in $\mathcal{C}$ is isomorphic to $X_i\{t\}$ for a certain $i \in \{1, \ldots, s\}$ and $t \in \mathbb{Z}$;
- for all $i \neq j \in \{1, \ldots, s\}$ and all $t \in \mathbb{Z}$ we have

$$X_i \ncong X_j\{t\}.$$

Then it is well-known that $K_0(\mathcal{C})$ is freely generated by S.

In this paper I will mostly tensor $K_0(\mathcal{C})$ with $\mathbb{C}(q)$, so let

$$K_0^q(\mathcal{C}) := K_0(\mathcal{C}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}(q).$$

A q-sesquilinear form on a finite-dimensional complex vector space V is by definition a form

$$\langle \cdot,\cdot \rangle \colon V \times V \to \mathbb{C}(q)$$

satisfying

$$\begin{aligned} \langle f(q)v, v' \rangle &= f(q^{-1})\langle v, v' \rangle \\ \langle v, f(q)v' \rangle &= f(q)\langle v, v' \rangle, \end{aligned}$$

for any $f(q) \in \mathbb{C}(q)$ and $v, v' \in V$.

There exists a well-known q-sesquilinear form on $K_0^q(\mathcal{C})$, which is called the *Euler form*. It is defined by

$$\langle [X], [Y] \rangle := \dim_q \mathrm{HOM}(X, Y),$$

for any objects $X, Y \in \mathcal{C}$. Note that the Euler form takes values in $\mathbb{N}[q, q^{-1}]$.

3. The special linear quantum algebra and its fundamental representations

I briefly recall the special linear quantum algebra and the pivotal category of its fundamental representations. In the beginning I will use the "neutral" parameter $n \in \mathbb{N}_{\geq 2}$ for the quantum algebras and their representations. Later on I will always carefully choose between $n = N$ or $n = m$ in the different parts, which will be convenient for distinguishing the two sides of quantum skew Howe duality.

3.1. **The special linear quantum algebra.** Let $n \geq 2$ be an arbitrary integer and let

$$\alpha_i := (0, \ldots, 1, -1, \ldots, 0) \in \mathbb{Z}^n$$

with 1 on the i-th position, for $i = 1, \ldots, n-1$. Denote the Euclidean inner product on $\mathbb{Z}^n$ by $(\cdot, \cdot)$.

Definition 3.1. *For $n \in \mathbb{N}_{\geq 2}$ the* quantum special linear algebra $\mathbf{U}_q(\mathfrak{sl}_n)$ *is the associative unital $\mathbb{C}(q)$-algebra generated by $K_i^{\pm 1}, E_{\pm i}$, for $i = 1, \ldots, n-1$, subject to the relations*

$$K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1,$$

$$E_{+i}E_{-j} - E_{-j}E_{+i} = \delta_{i,j}\frac{K_i - K_i^{-1}}{q - q^{-1}},$$

$$K_i E_{\pm j} = q^{\pm(\alpha_i, \alpha_j)} E_{\pm j} K_i,$$

$$E_{\pm i}^2 E_{\pm j} - (q + q^{-1}) E_{\pm i} E_{\pm j} E_{\pm i} + E_{\pm j} E_{\pm i}^2 = 0, \qquad \text{if} \quad |i - j| = 1,$$

$$E_{\pm i} E_{\pm j} - E_{\pm j} E_{\pm i} = 0, \qquad \text{else}$$

Recall that $\mathbf{U}_q(\mathfrak{sl}_n)$ is a Hopf algebra with coproduct given by

$$\Delta(E_{+i}) = E_{+i} \otimes K_i + 1 \otimes E_{+i}, \quad \Delta(E_{-i}) = E_{-i} \otimes 1 + K_i^{-1} \otimes E_{-i}, \quad \Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1}$$

and antipode by

$$S(E_{+i}) = -E_{+i}K_i^{-1}, \quad S(E_{-i}) = -K_i E_{-i}, \quad S(K_i) = K_i^{-1}.$$

The counit is given by

$$\epsilon(E_{\pm i}) = 0, \quad \epsilon(K_i) = 1.$$

The Hopf algebra structure is used to define $\mathbf{U}_q(\mathfrak{sl}_n)$ actions on tensor products and duals of $\mathbf{U}_q(\mathfrak{sl}_n)$-modules.

Recall that the $\mathbf{U}_q(\mathfrak{sl}_n)$-weight lattice is isomorphic to $\mathbb{Z}^{n-1}$. For any $i = 1, \ldots, n-1$, the element K_i acts as q^{λ_i} on the λ-weight space of any weight representation.

Although I have not recalled the definition of $\mathbf{U}_q(\mathfrak{gl}_n)$, it is sometimes convenient to use $\mathbf{U}_q(\mathfrak{gl}_n)$-weights. Recall that the $\mathbf{U}_q(\mathfrak{gl}_n)$-weight lattice is isomorphic to $\mathbb{Z}^n$ and that any $\mathbf{U}_q(\mathfrak{gl}_n)$-weight $\vec{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ determines a unique $\mathbf{U}_q(\mathfrak{sl}_n)$-weight

$$\lambda = (k_1 - k_2, \ldots, k_{n-1} - k_n) \in \mathbb{Z}^{n-1}.$$

In this way, we get an isomorphism

$$\mathbb{Z}^n / \langle (1^n) \rangle \cong \mathbb{Z}^{n-1}. \tag{11}$$

Remark 3.2. *Since the $\mathbf{U}_q(\mathfrak{sl}_n)$ and $\mathbf{U}_q(\mathfrak{gl}_n)$-weights and weight lattices are equal to those of the corresponding classical algebras, I will often refer to them as the $\mathfrak{sl}_n$ and $\mathfrak{gl}_n$-weights and weight lattices.*

For weight representations, one can also use the idempotented version of $\mathbf{U}_q(\mathfrak{sl}_n)$, denoted $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$, due to Beilinson, Lusztig and MacPherson [2]. For $n = 2$, define $i' = (2)$. For $n > 2$, define

$$i' := \begin{cases} (2, -1, 0 \ldots, 0), & \text{for } i = 1; \\ (0, \ldots, -1, 2, -1, \ldots, 0), & \text{for } 2 \leq i \leq n-2; \\ (0, \ldots, 0, -1, 2), & \text{for } i = n-1. \end{cases}$$

Adjoin an idempotent 1_λ for each $\lambda \in \mathbb{Z}^{n-1}$ and add the relations

$$1_\lambda 1_\mu = \delta_{\lambda,\mu} 1_\lambda,$$
$$E_{\pm i} 1_\lambda = 1_{\lambda \pm i'} E_i,$$
$$K_i 1_\lambda = q^{\lambda_i} 1_\lambda.$$

Definition 3.3. *The* idempotented quantum special linear algebra *is defined by*

$$\dot{\mathbf{U}}_q(\mathfrak{sl}_n) = \bigoplus_{\lambda,\mu \in \mathbb{Z}^{n-1}} 1_\lambda \mathbf{U}_q(\mathfrak{sl}_n) 1_\mu.$$

Remark 3.4. *It is sometimes convenient to consider $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$ as a category, whose objects are the weights $\lambda \in \mathbb{Z}^{n-1}$. The hom-space between $\lambda, \mu \in \mathbb{Z}^{n-1}$ is equal to*

$$1_\lambda \mathbf{U}_q(\mathfrak{sl}_n) 1_\mu$$

and composition is given by multiplication.

3.2. **Fundamental representations.** In this section I recall the fundamental $\mathbf{U}_q(\mathfrak{sl}_n)$-representation theory, following [14, 39]. The basic $\mathbf{U}_q(\mathfrak{sl}_n)$-representation is denoted $\mathbb{C}_q^n$. It has a standard basis $\{x_1, \ldots, x_n\}$ on which the action is given by

$$E_{+i}(x_j) = \begin{cases} x_i, & \text{if } j = i+1; \\ 0, & \text{else.} \end{cases} \quad E_{-i}(x_j) = \begin{cases} x_{i+1}, & \text{if } j = i; \\ 0, & \text{else.} \end{cases}$$

$$K_i(x_j) = \begin{cases} qx_i, & \text{if } j = i; \\ q^{-1}x_{i+1}, & \text{if } j = i+1; \\ x_j, & \text{else.} \end{cases}$$

Using the basic representation, one can define all fundamental $\mathbf{U}_q(\mathfrak{sl}_n)$-representations. Define the *quantum exterior algebra*

$$\Lambda_q^\bullet(\mathbb{C}_q^n) := T\mathbb{C}_q^n / \langle\{x_i \otimes x_i,\ x_i \otimes x_j + qx_j \otimes x_i \mid 1 \le i < j \le n\}\rangle.$$

We denote the equivalence class of $x \otimes y$ by $x \wedge_q y$. Note that

$$\Lambda_q^\bullet(\mathbb{C}_q^n) = \bigoplus_{k=0}^{n} \Lambda^k(\mathbb{C}_q^n).$$

For each $0 \le k \le n$, the homogeneous direct summand $\Lambda_q^k(\mathbb{C}_q^n)$ is an irreducible $\mathbf{U}_q(\mathfrak{sl}_n)$-representation. For $k = 0, n$ it is the trivial representation and for $1 \le k \le n$ it is called the *k-th fundamental $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$-representation*. Recall that the dual of the k-th fundamental representation is isomorphic to the $(n-k)$-th fundamental representation.

For each k element subset $S \subset \{1, \dots n\}$, order the elements in decreasing order and define

$$x_S := x_{s_1} \wedge_q x_{s_2} \wedge_q \cdots \wedge_q x_{s_k}.$$

We call the x_S *elementary tensors*. The *standard basis* of $\Lambda^k(\mathbb{C}_q^n)$ is by definition

$$\{x_S \mid S \subset \{1, \dots n\},\ |S| = k\}.$$

Note that x_S has $\mathfrak{gl}_n$-weight $\nu_S = (\nu_1, \dots, \nu_n)$ with

$$\nu_j = \begin{cases} 1, & \text{if } j \in S; \\ 0, & \text{else.} \end{cases}$$

We call such a weight 1-*bounded* of type k, following [14]. Below I will also use the notation

$$x_\nu := x_S \quad \text{for} \quad \nu = \nu_S.$$

In general I want to consider tensor products of elementary tensors. Let $\vec{k} = (k_1, \dots, k_m)$ be an *n-bounded* $\mathfrak{gl}_m$-weight, i.e. satisfying $0 \le k_i \le n$ for all $i = 1, \dots, m$. Define

$$\Lambda_q^{\vec{k}}(\mathbb{C}_q^n) := \Lambda_q^{k_m}(\mathbb{C}_q^n) \otimes \cdots \otimes \Lambda_q^{k_1}(\mathbb{C}_q^n).$$

Remark 3.5. *We write the tensor factors backwards in $\Lambda_q^{\vec{k}}(\mathbb{C}_q)$, which is more convenient for skew Howe duality and its categorification in this paper.*

Let $\mathcal{S}(\vec{k}, n)$ be the set of m-tuples

$$\vec{\nu} = (\nu^1, \dots, \nu^m)$$

such that ν^i is a 1-bounded $\mathfrak{gl}_n$-weight of type k_i for each $i = 1, \dots, m$. For short I say that $\vec{\nu}$ is 1-bounded of type $\vec{k}$. The elements

$$x_{\vec{\nu}} := x_{\nu^m} \otimes \cdots \otimes x_{\nu^1}$$

form a basis of $\Lambda_q^{\vec{k}}(\mathbb{C}_q^n)$, which I also call the *standard basis*.

As already announced, I now will start using the parameter $N \in \mathbb{N}_{\geq 2}$ because intertwiners will only occur on one side of quantum skew Howe duality. I will also use introduce a second quantum parameter $v := -q^{-1}$, as explained in the introduction.

Let $\mathcal{R}ep(\mathrm{SL}_N)$ be the pivotal category whose objects are tensor products of fundamental $\mathbf{U}_q(\mathfrak{sl}_N)$-representations and their duals and whose morphisms are intertwiners. Cautis, Kamnitzer and Morrison defined a set of generating intertwiners in $\mathcal{R}ep(\mathrm{SL}_N)$. If S, T are disjoint subsets of $\{1, \ldots, N\}$, let

$$\ell(S,T) := |\{(i,j) \mid i \in S,\ j \in T \text{ and } i < j\}|.$$

following [25]. For any $0 \leq a, b \leq N$ such that $a + b \leq N$, define

- the usual linear copairing $c_a\colon \mathtt{1} \to \Lambda_q^a(\mathbb{C}_q^N) \otimes (\Lambda_q^a(\mathbb{C}_q^N))^*$, which is an intertwiner;
- the usual linear pairing $p_a\colon (\Lambda_q^a(\mathbb{C}_q^N))^* \otimes \Lambda_q^a(\mathbb{C}_q^N) \to \mathtt{1}$, which is an intertwiner;
- the intertwiner $M_{a,b}\colon \Lambda_q^a(\mathbb{C}_q^N) \otimes \Lambda_q^b(\mathbb{C}_q^N) \to \Lambda_q^{a+b}(\mathbb{C}_q^N)$ by

$$M_{a,b}(x_S \otimes x_T) := x_S \wedge_q x_T = \begin{cases} v^{\ell(T,S)} x_{S \cup T}, & \text{if } S \cap T = \emptyset; \\ 0, & \text{else;} \end{cases} \tag{12}$$

- the intertwiner $M'_{a,b}\colon \Lambda_q^{a+b}(\mathbb{C}_q^N) \to \Lambda_q^a(\mathbb{C}_q^N) \otimes \Lambda_q^b(\mathbb{C}_q^N)$ by

$$M'_{a,b}(x_S) := \sum_{T \subset S} v^{-\ell(T, S\setminus T)} x_T \otimes x_{S \setminus T}; \tag{13}$$

- the bijective intertwiner $D_a\colon \Lambda_q^a(\mathbb{C}_q^N) \to (\Lambda_q^{N-a}(\mathbb{C}_q^N))^*$ by

$$D_a(x_S)(x_T) := \begin{cases} v^{\ell(T,S)}, & \text{if } S \cap T = \emptyset; \\ 0, & \text{else.} \end{cases} \tag{14}$$

Note that the usual linear copairing and pairing above define intertwiners, whereas the usual linear copairing and pairing

$$\mathtt{1} \to (\Lambda_q^a(\mathbb{C}_q^N))^* \otimes \Lambda_q^a(\mathbb{C}_q^N) \quad \text{and} \quad \Lambda_q^a(\mathbb{C}_q^N) \otimes (\Lambda_q^a(\mathbb{C}_q^N))^* \to \mathtt{1}$$

are not intertwiners. To get the intertwiners between those representations one has use the copairing and pairing above and D_a, D_{N-a} and their inverses. For a good discussion about this point, see Section 3 in [39].

As remarked in the introduction, I have normalized the maps differently from the ones in [14]. This is on purpose and I will explain the reason in Section 4.3. Note that the maps above still define intertwiners, because

- this $M_{a,b}$ is equal to theirs multiplied by $(-q)^{-ab}$;
- this M'_{ab} is equal to theirs multiplied by q^{ab};
- this D_a is equal to theirs multiplied by $(-q)^{-a(N-a)}$.

The following result is due to Morrison (Theorem 3.5.8 in [39]).

Theorem 3.6 (Morrison)**.** *Any morphism in $\mathcal{R}ep(\mathrm{SL}_N)$ can be obtained by tensoring, composing and taking linear combinations of morphisms of the form c_a, p_a, $M_{a,b}$, $M'_{a,b}$ and D_a, for $a, b = 0, \ldots, N$.*

For later use, I explain the *state-sum model* for the intertwiners. The coefficients in the formula for the intertwiners $M_{a,b}$ and $M'_{a,b}$ are powers of v associated to triples of 1-bounded

$\mathfrak{gl}_N$-weights of type a,b and $a+b$. Similarly, the coefficient in the formula for D_a is a power of v associated to a pair of 1-bounded $\mathfrak{gl}_N$-weights of type a and $N-a$. By composing and tensoring these elementary intertwiners, we see that any intertwiner maps basis tensors to linear combinations of basis tensors and that the coefficient of each summand is a sum of powers of v. Each power of v is associated to a particular assignment of 1-bounded $\mathfrak{gl}_N$-weights to the edges of the web which represents the intertwiner. These assignments have to satisfy compatibility conditions, because intertwiners preserve the total weight.

Definition 3.7. *Let w be any monomial web and let $E(w)$ be the set of its edges. A* state *of w is a map $\sigma\colon E(w) \to \mathbb{Z}^N/\langle (1^N)\rangle$ such that*

(1) $\sigma(e) \in W(1^{k_e}) \bmod (1^N)$*, for any $e \in E(w)$ of color k_e;*
(2) $\sigma(e_1)+\sigma(e_2) \equiv \sigma(e_3) \bmod (1^N)$*, whenever e_1, e_2, e_3 meet at a triple vertex and e_1 and e_2 are both oriented in the same direction w.r.t. that vertex;*
(3) $\sigma(e_1)+\sigma(e_2) \equiv 0 \bmod (1^N)$*, whenever e_1 and e_2 meet at a tag.*

The set of states of w is denoted $\mathcal{S}(w)$.

In this way, we see that any intertwiner can be written as a *state-sum*.

3.3. **Tensors and tableaux.** For later use, I recall here two relations between certain basis tensors and column-strict tableaux.

For simplicity, I assume that $m = N\ell$, which is the case of interest in this paper. Let $\mathrm{Col}^{(N^\ell)}$ be the set of all *column-strict tableaux* of shape (N^ℓ) whose fillings are natural numbers between 1 and m. Recall that a tableau is called column-strict if its entries are strictly increasing along every column from top to bottom.

Let $\mathrm{Std}^{(N^\ell)} \subset \mathrm{Col}^{(N^\ell)}$ be the subset of *semi-standard* tableaux. These satisfy the additional condition that the fillings are weakly increasing along each row from left to right. The semi-standard tableaux parametrize the canonical and dual canonical bases and also Leclerc and Toffin's intermediate basis and its Howe dual, as I will show later.

For any N-bounded $\mathfrak{gl}_m$-weight $\vec{k} \in \Lambda(m,m)_N$ (defined in (8)), let $\mathrm{Col}^{(N^\ell)}_{\vec{k}} \subset \mathrm{Col}^{(N^\ell)}$ be the subset of tableaux of type $\vec{k}$. Recall that a tableau is of type $\vec{k}$ if its filling contains k_i times the entry i, for $i = 1, \ldots, m$.

For the first relation between tableaux and tensors, let $\mathcal{S}(\vec{k},N)_0 \subset \mathcal{S}(\vec{k},N)$ be the subset of elements ν such that

$$\sum_{i=1}^{m} \nu^i = (\ell^N).$$

There is a well-known bijection between $\mathcal{S}(\vec{k},N)_0$ and $\mathrm{Col}^{(N^\ell)}_{\vec{k}}$. If T is a tableau, let $T(j)$ denote the j-th column of T. The bijection $\vec{\nu}_T \leftrightarrow T$ is determined by the rule

$$\nu^i_j = 1 \Leftrightarrow i \in T(j)$$

for any $1 \leq i \leq m$ and $1 \leq j \leq N$. Write

$$x^T := x_{\vec{\nu}_T} \in \Lambda^{\vec{k}}_q(\mathbb{C}^N_q)$$

and note that x^T has $\mathfrak{sl}_N$-weight zero.

Example 3.8. *Let $N = 3$, $\ell = 4$, $m = 12$, and $\vec{k} = (2, 2, 1, 2, 1, 3, 1, 0, 0, 0, 0, 0)$. The tableau*

1	1	2
2	3	4
4	5	6
6	6	7

corresponds to

$$\vec{\nu} = ((110), (101), (010), (101), (010), (111), (001)).$$

In the second relation between tableaux and tensors, the basis tensors of $\Lambda_q^\ell(\mathbb{C}_q^m)^{\otimes N}$ of gl_m-weight $\vec{k}$ are parametrized by the tableaux in $\mathrm{Col}_{\vec{k}}^{(N^\ell)}$. In this case, the bijection is given by

$$\mu_j^i = 1 \leftrightarrow j \in T(i)$$

for any $1 \leq i \leq N$ and $1 \leq j \leq m$. For any $T \in \mathrm{Col}_{\vec{k}}^{(N^\ell)}$, write

$$x_T := x_{\vec{\mu}_T} \in \Lambda_q^\ell(\mathbb{C}_q^m)^{\otimes N}.$$

Example 3.9. *For the tableau in Example 3.8 we get*

$$\vec{\mu} = ((110101000000), (101011000000), (010101100000)).$$

Remark 3.10. *Thus to the same tableaux $T \in \mathrm{Col}_{\vec{k}}^{(N^\ell)}$ one can associate two different basis tensors in two different tensor spaces and the notation in this paper distinguishes the two:*

$$x^T \in \Lambda_q^{\vec{k}}(\mathbb{C}_q^N) \quad \textit{and} \quad x_T \in \Lambda_q^\ell(\mathbb{C}_q^m)^{\otimes N}.$$

Both types of basis tensor will be of interest in this paper, because quantum skew Howe duality relates them.

For later use, I recall that there is a total ordering on $\mathrm{Col}^{(N^\ell)}$. Consider columns as increasing sequences. Given two columns c and d, define

$$c = (c_1 < \cdots < c_\ell) \succ d = (d_1 < \cdots < d_\ell)$$

if the first c_i different from d_i is less than d_i. Order the columns from left to right. The lexicographical ordering w.r.t. to these two orderings gives a total ordering on $\mathrm{Col}^{(N^\ell)}$.

Example 3.11. *The greatest tableau in $\mathrm{Col}^{(N^\ell)}$ w.r.t. this ordering is the one whose columns are all equal to $(1, 2, \ldots, \ell)$. Let us denote it by T_Λ, because it corresponds to the highest weight vector of both V_Λ and W_Λ, as we will see.*

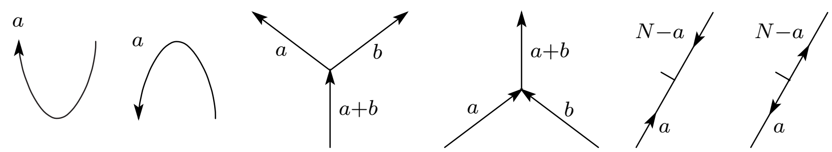
4. SL_N webs

The morphisms in $\mathcal{R}ep(\mathrm{SL}_N)$ can be represented graphically by $\mathfrak{sl}_N$-*webs*. These are certain oriented trivalent graphs, whose edges are colored by integers belonging to $\{0, \ldots, N\}$. Webs can be seen as morphisms in a pivotal category, which in the literature is called a *spider* or *spider category*, denoted $\mathcal{S}p(\mathrm{SL}_N)$.

4.1. **The SL_N spider.** Recently, Cautis, Kamnitzer and Morrison [14] gave a presentation of $\mathcal{S}p(\mathrm{SL}_N)$ in terms of generating webs and relations.

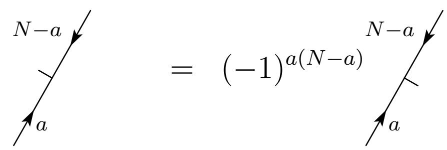
Definition 4.1 (Cautis-Kamnitzer-Morrison)**.** *The objects of $\mathcal{S}p(\mathrm{SL}_N)$ are finite sequences $\vec{k}$ of elements in $\{0^\pm, \ldots, (N)^\pm\}$.*

The hom-space $\mathrm{Hom}(\vec{k}, \vec{l})$ is the $\mathbb{C}(q)$-vector space freely generated by all diagrams, with lower and top boundary labeled from right to left by the entries of $\vec{k}$ and $\vec{l}$ respectively, which can be obtained by glueing and juxtaposing labeled cups and caps and the following elementary webs, together with the ones obtained by mirror reflections and arrow reversals:

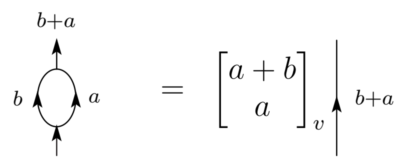


with all labels between 0 and N, modded out by planar isotopies (e.g. the zig-zag relations for cups and caps) and the following relations:

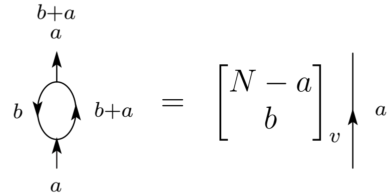
(15)



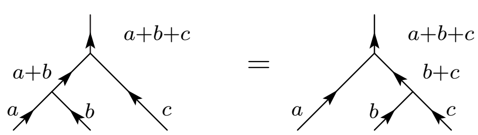
(16)



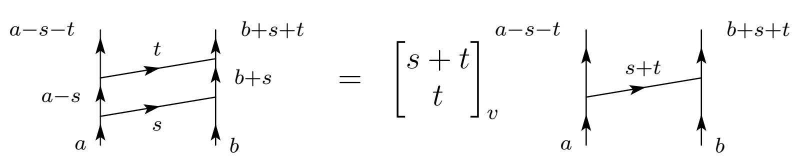
(17)



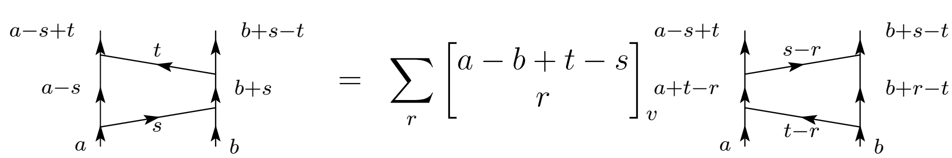
(18)



(19)



(20)

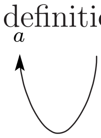


together with the analogous relations obtained by mirror reflections and arrow reversals.

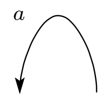
Let $\Gamma_N\colon \mathcal{S}p(\mathrm{SL}_N) \to \mathcal{R}ep(\mathrm{SL}_N)$ be the pivotal functor defined on objects by

$$\vec{k} = (k_1^{\epsilon_1}, \dots, k_m^{\epsilon_m}) \quad\mapsto\quad \Lambda_q^{\vec{k}}(\mathbb{C}_q^N) = (\Lambda_q^{k_m}(\mathbb{C}_q^N))^{\epsilon_m} \otimes \cdots \otimes (\Lambda_q^{k_1}(\mathbb{C}_q^N))^{\epsilon_m},$$

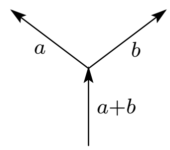
where $V^1 := V$ and $V^{-1} := V^*$ by definition. On morphisms, define Γ_N by



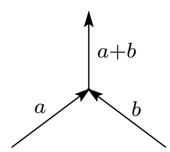
$\mapsto \; c_a$



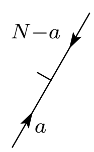
$\mapsto \; p_a$



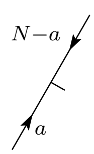
$\mapsto \; M'_{a,b}$



$\mapsto \; M_{a,b}$



$\mapsto \; D_a$



$\mapsto \; (-1)^{a(N-a)} D_a$

The following result can be found in [14] (Theorems 3.2.1 and 3.3.1):

Theorem 4.2 (Cautis-Kamnitzer-Morrison)**.** *The functor Γ_N is well defined and gives an equivalence of pivotal categories.*

Since I have changed the normalization of the intertwiners, the web relations in Definition 4.1 gain certain minus signs if the quantum parameter is taken to be q, as the reader can easily check. When we pass from q to v, these minus signs get absorbed by the v-binomial coefficients because

$$\begin{bmatrix} a+b \\ a \end{bmatrix}_v = (-1)^{ab} \begin{bmatrix} a+b \\ a \end{bmatrix}_q .$$

4.2. **Quantum skew Howe duality.** Let us briefly recall the instance of quantum skew Howe duality which was categorified in [38]. For more details see [14] and the references therein.

As explained in [14], there are $\mathbb{C}(q)$-linear isomorphisms

$$\Lambda_q^\bullet(\mathbb{C}_q^m)^{\otimes N} \cong \Lambda_q^\bullet(\mathbb{C}_q^m \otimes \mathbb{C}_q^N) \cong \Lambda_q^\bullet(\mathbb{C}_q^N)^{\otimes m}.$$

The commuting $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ and $\dot{\mathbf{U}}_q(\mathfrak{sl}_N)$-actions on $\Lambda_q^\bullet(\mathbb{C}_q^m \otimes \mathbb{C}_q^N)$ induce a $\dot{\mathbf{U}}_q(\mathfrak{sl}_N)$-action on $\Lambda_q^\bullet(\mathbb{C}_q^m)^{\otimes N}$, which commutes with the standard $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-action, and a $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-action on $\Lambda_q^\bullet(\mathbb{C}_q^N)^{\otimes m}$, which commutes with the standard $\dot{\mathbf{U}}_q(\mathfrak{sl}_N)$-action. The explicit definition of the "non-standard" actions is not immediately obvious and was worked out in [14].

Remark 4.3. *Unfortunately, the normalization of the intertwiners associated to the generating webs in* [14] *is not convenient for the purposes in this paper. For categorification we want all the coefficients in the web relations (for the webs without tags) and in the expressions for the generating intertwiners to be quantum natural numbers. Additionally, the top coefficient should always be equal to one for certain intertwiners if want we want a precise match between the dual canonical basis elements in W_Λ and the indecomposable objects in $\mathcal{W}_\Lambda^p$. This is impossible if one insists on having the same quantum parameter q for both $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ and $\dot{\mathbf{U}}_q(\mathfrak{sl}_N)$. Therefore I define commuting $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$ and $\dot{\mathbf{U}}_q(\mathfrak{sl}_N)$-actions on $\Lambda_q^\bullet(\mathbb{C}_q^N)^{\otimes m}$ in this paper. As already remarked in the introduction, this is consistent with the use of $v = -q^{-1}$ in* [19, 25].

The confusing part is that $\Lambda_q^\bullet(\mathbb{C}_q^N)^{\otimes m}$ is seen as a $\mathbb{C}(v)$-linear space in this approach. Perhaps the right way to think about it, is that both $\Lambda_v^\bullet(\mathbb{C}_v^m)^{\otimes N}$ and $\Lambda_q^\bullet(\mathbb{C}_q^N)^{\otimes m}$ as $\mathbb{C}(v)$-linear vector spaces are isomorphic to a direct summand of the $\mathbb{C}(v)$-linear Fock space generated by column-strict tableaux in [44]*. For each such tableau T, both the tensors x_T and x^T then become identified with the same "abstract" basis vector of the Fock space and they are no longer representatives of equivalence classes of tensors in exterior powers with different quantum parameters. On this Fock space there are two commuting actions of $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$ and $\dot{\mathbf{U}}_{-v^{-1}}(\mathfrak{sl}_N)$.*[2] *In fact, this Fock space is also used in* [4] *for the $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$-side of the story.*

Let m, d, N be arbitrary non-negative integers. Define

$$\Lambda(m,d) := \{\vec{k} \in \mathbb{N}^m \mid k_1 + \cdots + k_m = d\}$$

and

$$\Lambda(m,d)_N := \{\vec{k} \in \Lambda(m,d) \mid 0 \leq k_i \leq N \quad \text{for all } i = 1, \ldots, m\}.$$

Given a $\mathbf{U}_v(\mathfrak{sl}_m)$-weight $\lambda = (\lambda_1, \ldots, \lambda_{m-1})$, the isomorphism in (11) shows that in general there is not a unique way to lift λ to a $\mathbf{U}_q(\mathfrak{gl}_m)$-weight. However, for a fixed value of $d \in \mathbb{N}$ there exists a partially defined map

$$\phi_{m,d,N} \colon \mathbb{Z}^{m-1} \to \Lambda(m,d)_N$$

determined by

$$\phi_{m,d,N}(\lambda) = \vec{k}$$

[2] I thank Ben Webster for explaining this to me.

such that

$$\begin{aligned} k_i &\in \{0,\ldots,N\} \quad \text{for all } i=1,\ldots,m && (21)\\ k_i - k_{i+1} &= \lambda_i \quad \text{for all } i=1,\ldots,m, && (22)\\ \sum_{i=1}^{m} k_i &= d. && (23) \end{aligned}$$

Note that $\vec{k}$ might not exist. But if it does, it is necessarily unique. If it does not exist, put $\phi_{m,d,N}(\lambda) = *$ for convenience. For more information on this map in relation with quantum Schur algebras see [36].

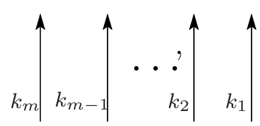
The following proposition follows from the results in Sections 4 and 5 in [14].

Proposition 4.4 (Cautis-Kamnitzer-Morrison)**.** *The $\mathbb{C}(v)$-linear functor*

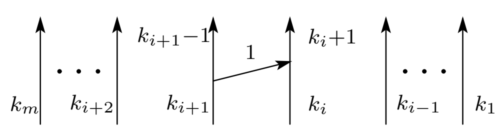
$$\gamma_{m,d,N} \colon \dot{\mathbf{U}}_v(\mathfrak{sl}_m) \to \mathcal{S}p(\mathrm{SL}_N)$$

determined by

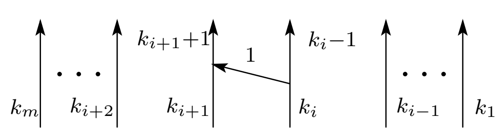
$1_\lambda \mapsto \begin{cases} \text{(diagram below),} & \text{if } \phi_{m,d,N}(\lambda) = \vec{k}; \\ 0, & \text{if } \phi_{m,d,N}(\lambda) = *. \end{cases}$



$E_{+i}1_\lambda \mapsto$



$E_{-i}1_\lambda \mapsto$

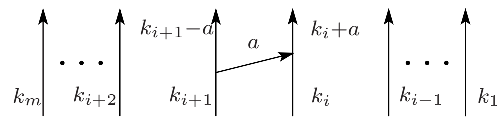


is well-defined and full.

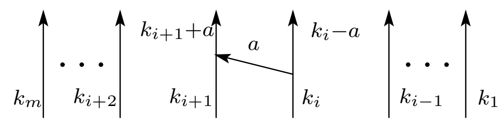
By (16) and (18), it is easy to determine the images of the divided powers

$$E_{+i}^{(a)} := E_{+i}^a/[a]! \quad \text{and} \quad E_{-i}^{(a)} = E_{-i}^a/[a]!$$

$E_{+i}^{(a)} 1_\lambda \quad \mapsto$



$E_{-i}^{(a)} 1_\lambda \quad \mapsto$



The images of products of divided powers in $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$ are special web diagrams, called *ladders* (see Section 5 in [14]).

Definition 4.5 (Cautis-Kamnitzer-Morrison)**.** *An* N-ladder with m uprights *is a rectangular* $\mathfrak{sl}_N$*-web diagram without tags, such that*

- *its vertical edges are all oriented upwards and lie on m parallel vertical lines running from bottom to top;*
- *it contains a certain number of horizontal oriented* rungs *connecting adjacent uprights.*

Since ladders do not have tags, at each trivalent vertex the sum of the labels of the incoming edges has to be equal to the sum of the labels of the outgoing edges.

By composing the functor $\dot{\mathbf{U}}_v(\mathfrak{sl}_m) \to \mathcal{S}p(\mathrm{SL}_N)$ in Proposition 4.4 with $\Gamma_N \colon \mathcal{S}p(\mathrm{SL}_N) \to \mathcal{R}ep(\mathrm{SL}_N)$, we get a well-defined action of $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$ on $\Lambda_q^\bullet(\mathbb{C}_q^N)^{\otimes m}$. For any $X \in \dot{\mathbf{U}}_v(\mathfrak{sl}_m)$ and $x_{\vec{v}} \in \Lambda_q^\bullet(\mathbb{C}_q^N)^{\otimes m}$, first map X to a ladder in $\mathcal{S}p(\mathrm{SL}_N)$ and then apply the corresponding intertwiner in $\mathcal{R}ep(\mathrm{SL}_N)$ to $x_{\vec{v}}$ if possible. If the domain of the intertwiner does not match the tensor type of $x_{\vec{v}}$, we declare the action to be zero. Since the $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$-action is defined by intertwiners, it commutes with the standard $\dot{\mathbf{U}}_q(\mathfrak{sl}_N)$-action on $\Lambda_q^\bullet(\mathbb{C}_q^N)^{\otimes m}$.

For the rest of this paper, let $N \geq 2$ and $m, \ell \geq 0$ be arbitrary but fixed integers such that $m = N\ell$. The extra parameter d in Proposition 4.4 is always taken to be equal to m.

The standard action of $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$ on $\Lambda_v^\ell(\mathbb{C}_v^m)^{\otimes N}$ is easy to describe in terms of tableaux. Choose $T \in \mathrm{Col}^{(N^\ell)}$ and $i \in \{1, \ldots, m-1\}$. Then

$$E_{-i} x_T = \sum_{T'} v^{-a_{T,T'}} x_{T'}$$

where the sum is over all tableaux $T' \in \mathrm{Col}^{(N^\ell)}$ obtained from T by changing one i into $i+1$. Suppose that T' is such a tableau and that it was obtained by changing the entry (k, l) of T, then

$$a_{T,T'} = |\{(k', l') \mid l' > l,\ T_{(k',l')} = i\}| - |\{(k', l') \mid l' > l,\ T_{(k',l')} = i+1\}|.$$

The action of E_i on x_T can be described similarly:

$$E_i x_T = \sum_{T''} v^{b_{T,T''}} x_{T''}$$

where the sum is taken over all tableaux in $T'' \in \mathrm{Col}^{(N^\ell)}$ obtained from T by changing one $i+1$ into i. Suppose that T'' is such a tableau and that it was obtained by changing the entry (k,l) of T, then

$$b_{T,T''} = |\{(k',l') \mid l' < l,\ T_{(k',l')} = i\}| - |\{(k',l') \mid l' < l,\ T_{(k',l')} = i+1\}|.$$

Proposition 4.6. *The action of $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$ on $\bigoplus_{\vec{k}\in\Lambda(m,m)_N} \Lambda_q^{\vec{k}}(\mathbb{C}_q^N)^{\otimes m}$ is given by*

$$E_{-i}x^T = \sum_{T'} v^{-a_{T,T'}} x^{T'} \quad E_i x^T = \sum_{T''} v^{b_{T,T''}} x^{T''},$$

for any $T \in \mathrm{Col}^{(N^\ell)}$. Here $T', T'', a_{T,T'}, b_{T,T''}$ are as above.

Proof. We compute the action of $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$ on $\bigoplus_{\vec{k}\in\Lambda(m,m)_N} \Lambda_q^{\vec{k}}(\mathbb{C}_q^N)^{\otimes m}$ using Proposition 4.4. Recall that

$$x^T = x_{S_m} \otimes \cdots \otimes x_{S_{i+1}} \otimes x_{S_i} \otimes \cdots \otimes x_{S_1},$$

where S_k is the set of numbers of the columns in T containing a filling equal to k. A small calculation shows that

$$E_{-i}x^T = \sum_{j\in S_i\setminus S_{i+1}} v^{-(\ell(j,S_i\setminus\{j\})-\ell(j,S_{i+1}))} x_{S_m} \otimes \cdots \otimes x_{S_{i+1}\cup\{j\}} \otimes x_{S_i\setminus\{j\}} \otimes \cdots x_{S_1}.$$

It follows immediately from the definitions that

$$a_{T,T'} = \ell(j, S_i\setminus\{j\}) - \ell(j, S_{i+1})$$

if $T' \in \mathrm{Col}^{(N^\ell)}$ corresponds to $x_{S_m} \otimes \cdots \otimes x_{S_{i+1}\cup\{j\}} \otimes x_{S_i\setminus\{j\}} \otimes \cdots x_{S_1}$.

Similarly, we have

$$E_i x^T = \sum_{j\in S_{i+1}\setminus S_i} v^{\ell(S_i,j)-\ell(S_{i+1}\setminus\{j\},j)} x_{S_m} \otimes \cdots \otimes x_{S_{i+1}\setminus\{j\}} \otimes x_{S_i\cup\{j\}} \otimes \cdots x_{S_1}$$

and

$$b_{T,T''} = \ell(S_i, j) - \ell(S_{i+1}\setminus\{j\}, j)$$

if $T'' \in \mathrm{Col}^{(N^\ell)}$ corresponds to $x_{S_m} \otimes \cdots \otimes x_{S_{i+1}\setminus\{j\}} \otimes x_{S_i\cup\{j\}} \otimes \cdots x_{S_1}$. □

As before, let $\Lambda := N\omega_\ell$, where ω_ℓ is the ℓ-th fundamental $\mathfrak{sl}_m$-weight. Let V_Λ be the irreducible $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$-module of highest weight Λ. It is well-known that V_Λ is isomorphic to a direct summand of $\Lambda_v^\ell(\mathbb{C}_v^m)^{\otimes N}$. Therefore, V_Λ is also isomorphic to a direct summand of the $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$-module $\Lambda_q^\bullet(\mathbb{C}_q^N)^{\otimes m}$. In order to see this explicitly, Let P_Λ be the set of $\mathfrak{sl}_m$-weights of V_Λ. Note that the restriction of $\phi_{m,m,N}$ to $P_\Lambda \subset \mathbb{Z}^{m-1}$ gives a bijection

$$\phi_{m,m,N}\colon P_\Lambda \to \Lambda(m,m)_N.$$

In particular, we have $\phi_{m,m,N}(\Lambda) = (N^\ell)$.

Definition 4.7. *For any $\vec{k}=(k_1,\ldots,k_m)\in\Lambda(m,m)_N$, define the* web space $W(\vec{k},N)$ *by*

$$W(\vec{k},N):=\mathrm{Hom}((N^\ell),\vec{k})$$

*in $\mathcal{S}p(\mathrm{SL}_N)$. Define also the $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$-*web module *with highest weight Λ by*

$$W_\Lambda:=\bigoplus_{\vec{k}\in\Lambda(m,m)_N} W(\vec{k},N).$$

Remark 4.8. *Note that $W(\vec{k},N)$ is isomorphic to the space of invariant $\dot{\mathbf{U}}_q(\mathfrak{sl}_N)$-tensors in $\Lambda_q^{\vec{k}}(\mathbb{C}_q^N)$.*

The action of $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$ on W_Λ is defined by applying the functor in Proposition 4.4 and glueing the ladders on top of the webs in W_Λ when the $\mathfrak{gl}_m$-weights match. If the $\mathfrak{gl}_m$-weights do not match, simply define the action to be zero.

For any $u\in W(\vec{k},N)$, let

$$u^*\in\mathrm{Hom}(\vec{k},(N^\ell))$$

be the web obtained via reflexion in the x-axis and reorientation. Note that u and u^* can be glued together such that $u^*u\in\mathrm{End}((N^\ell))$.

Let w_Λ be the web with ℓ vertical N-strands. Note that

$$\mathrm{End}((N^\ell))\cong\mathbb{C}(v)w_\Lambda,$$

so we can define a map

$$\mathrm{ev}\colon\mathrm{End}((N^\ell))\to\mathbb{C}(v).$$

In [38], the *v-sesquilinear web form* was defined by

$$\langle u,v\rangle:=v^{d(\vec{k})}\mathrm{ev}(u^*v)\in\mathbb{C}(v). \tag{24}$$

for any monomial webs $u,v\in W(\vec{k},N)$. The normalization factor is defined by

$$d(\vec{k})=1/2\left(N(N-1)\ell-\sum_{i=1}^m k_i(k_i-1)\right).$$

We assume that the web form is v-antilinear in the first entry and v-linear in the second. Note that all web relations have coefficients which are symmetric in v and v^{-1}, so this definition makes sense.

Recall also the v-antilinear algebra anti-automorphism τ on $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$ defined by

$$\tau(1_\lambda)=1_\lambda,\;\;\tau(1_{\lambda+i'}E_{+i}1_\lambda)=v^{-1-\lambda_i}1_\lambda E_{-i}1_{\lambda+i'},$$
$$\tau(1_\lambda E_{-i}1_{\lambda+i'})=v^{1+\lambda_i}1_{\lambda+i'}E_{+i}1_\lambda.$$

The *v-Shapovalov form* $\langle\cdot,\cdot\rangle$ on V_Λ is the unique v-sesquilinear form such that

(1) $\langle v_\Lambda,v_\Lambda\rangle=1$, for a fixed highest weight vector v_Λ;
(2) $\langle Xv,v'\rangle=\langle v,\tau(X)v'\rangle$, for any $X\in\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$ and any $v,v'\in V_\Lambda$.

In [38], the following corollary was proved.

Corollary 4.9. *With the action above, W_Λ is an irreducible $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$-representation with highest weight Λ. The v-sesquilinear web form is equal to the v-Schapovalov form.*

Remark 4.10. *To circumvent the v versus q problem, in the rest of the paper I will consider W_Λ as a $\mathbb{C}(v)$-vector space and write V_Λ for the quotient of $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$ by the kernel of its action on W_Λ. Let v_Λ for the image of 1_λ in V_Λ. We then have a well-defined $\mathbb{C}(v)$-linear $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$-isomorphism $V_\Lambda \to W_\Lambda$ such that*

$$Xv_\Lambda \mapsto Xw_\Lambda$$

for all $X \in \dot{\mathbf{U}}_v(\mathfrak{sl}_m)$.

4.3. **The dual canonical basis.** There is a well-known basis of $W(\vec{k}, N)$, the *dual canonical basis*. In this section I recall its definition, following the exposition in [20].

In Section 3.3 I recalled that, for any $\vec{k} = (k_1, \ldots, k_m) \in \Lambda(m,m)_N$, the zero weight basis tensors of $\Lambda_q^{\vec{k}}(\mathbb{C}_q^N)$ are in one-to-one correspondence with the elements of $\mathrm{Col}_{\vec{k}}^{(N^\ell)}$. I also recalled the total order on $\mathrm{Col}^{(N^\ell)}$, which of course restricts to $\mathrm{Col}_{\vec{k}}^{(N^\ell)}$ and induces a total order on the subspace of zero-weight basis tensors in $\Lambda_q^{\vec{k}}(\mathbb{C}_q^N)$.

Using the isomorphism $D_k\colon \Lambda^k(\mathbb{C}_q^N) \to (\Lambda^{N-k}(\mathbb{C}_q^N))^*$, we can extend this order to the zero weight basis tensors in

$$(\Lambda_q^{k_m}(\mathbb{C}_q^N))^{\epsilon_m} \otimes \cdots \otimes (\Lambda_q^{k_1}(\mathbb{C}_q^N))^{\epsilon_1},$$

for any object $(k_1^{\epsilon_1}, \ldots, k_m^{\epsilon_m})$ in $\mathcal{S}p(\mathrm{SL}_N)$.

The coproduct Δ on $\mathbf{U}_q(\mathfrak{sl}_N)$ in this paper is the same as that in [14] and [16]. The corresponding *quasi R-matrix*

$$\Theta \in 1\widehat{\otimes}1 + \mathbf{U}_q(\mathfrak{sl}_N)^+\widehat{\otimes}\mathbf{U}_q(\mathfrak{sl}_N)^- \tag{25}$$

is given in Section 10.1 D in [16] (where it is denoted $\tilde{R}_h$). Here $\widehat{\otimes}$ is a suitably completed tensor product. The precise formula for Θ is complicated and not needed in this paper, but let us give two simple examples of the action of Θ on tensor products of fundamental representations.

Example 4.11. *The action of Θ on $\mathbb{C}_q^2 \otimes \mathbb{C}_q^2$ is given by*

$$1 \otimes 1 + (q - q^{-1})E_1 \otimes E_{-1}$$

and its action on $\mathbb{C}_q^3 \otimes \Lambda_q^2\mathbb{C}_q^3$ is given by

$$1\otimes 1 + q\left(1-q^{-2}\right)\left(E_1 \otimes E_{-1} + \left(-E_1E_2 + q^{-1}E_2E_1\right) \otimes (-E_{-2}E_{-1} + qE_{-1}E_{-2}) + E_2 \otimes E_{-2}\right)$$
$$+q^2\left(1-q^{-2}\right)^2 E_1E_2 \otimes E_{-1}E_{-2}.$$

Lusztig showed in Theorem 4.1.2 in [33] that Θ is uniquely determined by its form in (25) and the property

$$\Theta\Delta(u) = \overline{\Delta}(u)\Theta, \tag{26}$$

for all $u \in \mathbf{U}_q(\mathfrak{sl}_N)$. Recall that the *bar-involution* on $\mathbf{U}_q(\mathfrak{sl}_N)$ is the q-antilinear algebra involution defined by

$$\overline{E_{\pm i}} = E_{\pm i},\ \overline{K_i} = K_i^{-1}$$

and $\overline{\Delta}$ is the coproduct defined by

$$\overline{\Delta}(u) := \overline{\Delta(\overline{u})},$$

where the bar-involution acts on $\mathbf{U}_q(\mathfrak{sl}_N)\widehat{\otimes}\mathbf{U}_q(\mathfrak{sl}_N)$ factorwise. By q-antilinear is meant that the map is $\mathbb{C}$-linear but sends q to q^{-1}.

The relation with Lusztig's coproduct Δ_L and quasi R-matrix Θ_L is easy to give:

$$\Delta_L = (\rho' \otimes \rho')\Delta\rho' \quad \Theta_L = (\rho'\widehat{\otimes}\rho')\Theta,$$

where ρ' is the q-antilinear algebra anti-involution defined by

$$\rho'(E_{+i}) = E_{-i},\ \rho'(E_{-i}) = E_{+i},\ \rho'(K_i) = K_i^{-1}.$$

Note that

$$\overline{\rho'(x)} = \rho'(\overline{x}).$$

Corollary 4.1.3 in [33] implies that

$$\Theta\overline{\Theta} = \overline{\Theta}\Theta = 1\widehat{\otimes}1. \tag{27}$$

For each $k \in \mathbb{N}$, the bar-involution on $\Lambda_q^k(\mathbb{C}_q^N)$ is given by

$$\psi(\sum_\nu a_\nu(q)x_\nu) = \sum_\nu a_\nu(q^{-1})x_\nu.$$

This bar-involution is compatible with the one on $\mathbf{U}_q(\mathfrak{sl}_N)$ in the sense that

$$\psi(Xz) = \overline{X}\ \psi(z)$$

for any $X \in \mathbf{U}_q(\mathfrak{sl}_N)$ and $z \in \Lambda_q^k(\mathbb{C}_q^N)$.

In order to extend the bar-involution to tensor products of fundamental representations, write any tensor z as $x \otimes y$, with x and y each having fewer tensor factors than z. Define inductively

$$\psi(x \otimes y) := \overline{\Theta}(\psi(x) \otimes \psi(y)).$$

One can show that this definition does not depend on how one chooses the factorization $z = x \otimes y$. Note that this really is an involution by (27), and that

$$\psi(X(\bullet \otimes \bullet)) = \overline{X}\psi(\bullet \otimes \bullet)$$

for any $X \in \mathbf{U}_q(\mathfrak{sl}_N)$.

Since

$$\Theta \in 1 \otimes 1 + \mathbf{U}_q(\mathfrak{sl}_N)^+\widehat{\otimes}\mathbf{U}_q(\mathfrak{sl}_N)^-,$$

it follows that, for any $T \in \mathrm{Col}_{\vec{k}}^{(N^\ell)}$, we have

$$\psi(x^T) = x^T + \sum_{T' \prec T} c^{T,T'}(q)x^{T'} \tag{28}$$

with $T' \in \mathrm{Col}_{\vec{k}}^{(N^\ell)}$ and $c^{T,T'}(q) \in \mathbb{C}(q)$.

As already remarked, the dual canonical basis elements of $W(\vec{k}, N)$ are parametrized by the subset

$$\mathrm{Std}_{\vec{k}}^{(N^\ell)} \subset \mathrm{Col}_{\vec{k}}^{(N^\ell)}$$

of semi-standard tableaux. From now on, I adopt Leclerc and Toffin's [32] convention to use Greek lower case letters for column-strict tableaux and Roman upper case letters for semi-standard tableaux. The following theorem is proved in Chapter 27 in [33], for example. Note that we switch to $v = -q^{-1}$ again.

Theorem 4.12 (Kashiwara, Lusztig)**.** *For any $T \in \mathrm{Std}_{\vec{k}}^{(N^\ell)}$, there exists a unique element $b^T \in W(\vec{k}, N)$ such that*

$$\psi(b^T) = b^T \tag{29}$$

$$b^T = x^T + \sum_{\tau \prec T} d^{\tau,T}(v) x^\tau \tag{30}$$

with $\tau \in \mathrm{Col}_{\vec{k}}^{(N^\ell)}$ and $d^{\tau,T}(v) \in v^{-1}\mathbb{Z}[v^{-1}]$. The property in (30) *is called the* negative exponent property*.*

The b^T form a basis of $W(\vec{k}, N)$ which is called the **dual canonical basis***.*

In the following examples I use the notation x_S where S is a subset of $\{1, \ldots, N\}$ as explained in Section 3.2.

Example 4.13. *Using the expressions in Example 4.11 one can easily check that*

$$x_2 \otimes x_1 + v^{-1} x_1 \otimes x_2 \in W((1,1),2)$$

and

$$x_3 \otimes x_{\{2,1\}} + v^{-1} x_2 \otimes x_{\{3,1\}} + v^{-2} x_1 \otimes x_{\{3,2\}} \in W((1,2),3)$$

are invariant $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$ and $\dot{\mathbf{U}}_q(\mathfrak{sl}_3)$ tensors which are ψ-invariant. Since they have the negative exponent property too, they are dual canonical basis elements.

Lusztig also defined a symmetric bilinear inner product on $\Lambda_q^{\vec{k}}(\mathbb{C}_q^N)$. On zero-weight vectors it is defined by

$$(x^\tau, x^{\tau'}) := \delta_{\tau,\tau'}$$

for any $\tau, \tau' \in \mathrm{Col}_{\vec{k}}^{(N^\ell)}$. The negative exponent property for the dual canonical basis elements is equivalent to the *almost orthogonality* property:

$$(b^T, b^{T'}) \in \delta_{T,T'} + v^{-1}\mathbb{Z}[v^{-1}],$$

for all $T, T' \in \mathrm{Std}_{\vec{k}}^{(N^\ell)}$.

Alternatively, one can use the v-sesquilinear form defined by

$$\langle x^\tau, x^{\tau'} \rangle := \overline{(x^\tau, \psi(x^{\tau'}))}.$$

Since the dual canonical basis elements are ψ-invariant, we have

$$\langle b^T, b^{T'} \rangle = \overline{(b^T, b^{T'})}$$

for any $T, T' \in \mathrm{Std}_{\vec{k}}^{(N^\ell)}$. In particular we have

$$\langle b^T, b^{T'} \rangle \in \delta_{T,T'} + v\mathbb{Z}[v],$$

for all $T, T' \in \mathrm{Std}_{\vec{k}}^{(N^\ell)}$. Note also that $\langle \cdot, \cdot \rangle$ is not symmetric, but satisfies

$$\langle x, x' \rangle = \langle \psi(x'), \psi(x) \rangle.$$

Just as we defined the v-sesquilinear web form $\langle \cdot, \cdot \rangle$ on $W(\vec{k}, N)$, we can also define a symmetric *v-bilinear web form* $(\cdot, \cdot)$ by

$$\langle \cdot, \cdot \rangle = \overline{(\cdot, \psi(\cdot))}.$$

Let $w \in W(\vec{k}, N)$ be any monomial web and write

$$w = \sum_{\tau \in \mathrm{Col}_{\vec{k}}^{(N^\ell)}} c^\tau(v) x^\tau$$

for certain coefficients $c^\tau(v) \in \mathsf{C}(v)$.

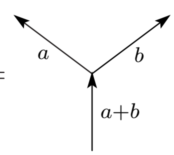
Lemma 4.14. *We have*

$$(w, w) = \sum_{\tau \in \mathrm{Col}_{\vec{k}}^{(N^\ell)}} c^\tau(v)^2.$$

This means that the symmetric v-bilinear web form is the restriction to $W(\vec{k}, n)$ of Lusztig's bilinear form on $\Lambda_q^{\vec{k}}(\mathsf{C}_q^N)$.

Proof. One way to compute $\mathrm{ev}(w^* w)$ is to use the state-sum model for intertwiners. Any state in $\mathcal{S}(w^* w)$ is given by a pair of states in $\mathcal{S}(w)$ and $\mathcal{S}(w^*)$ which match on the boundary. Therefore, it suffices to compare the powers of v associated to u and u^* for any given state when u is a cup or Y or Y^*-shaped. I am using here the fact that there exists a bijection between $\mathcal{S}(u)$ and $\mathcal{S}(u^*)$ by assigning exactly the same weight to two corresponding edges in u and u^*.

I only show the case for a Y-shaped web, the other cases being similar. Suppose that

$u =$ 

and that a certain state assigns to the bottom $(a+b)$-edge an $(a+b)$-element subset $S \subseteq \{1, \ldots, N\}$, to the upper a-edge an a-element subset $T \subset S$ and to the upper b-edge the b-element subset $S \backslash T$. For this state, the corresponding power of v is equal to

$$v^{-\ell(T, S\backslash T)}.$$

For u^*, with the corresponding state, we get

$$v^{\ell(S\backslash T, T)}.$$

Recall that

$$\ell(T, S\backslash T) + \ell(S\backslash T, T) = ab.$$

Now compute $d(\vec{k})$ at the top and the bottom of u. We have

$$a(a-1) + b(b-1) = (a+b)(a+b-1) - 2ab,$$

so

$$d(\vec{k}_{\mathrm{top}}) = v^{ab} d(\vec{k}_{\mathrm{bottom}}).$$

Doing the same analysis for cups and Y^*-shaped webs, we arrive at

$$\mathrm{ev}(w^* w) = v^{d(\vec{k})} \sum_{T \in \mathrm{Col}_{\vec{k}}^{(N^\ell)}} c^T(v)^2,$$

which implies the lemma. $\square$

Here are some more examples of dual canonical basis elements. In these examples I also use the notation of subsets.

Lemma 4.15. *The following elements are all dual canonical:*

(1)
$$b^{++}_{a(N-a)} := M'_{a(N-a)}(x_{\{N,\ldots,1\}}) = \sum_{|T|=a} v^{-\ell(T,T^c)} x_T \otimes x_{T^c},$$
for any $1 \leq a \leq N$*;*

(2)
$$b^{-+}_a := (D_{N-a} \otimes 1)(b^{++}_{(N-a)a}) \quad \textit{and} \quad b^{+-}_a := (1 \otimes D_{N-a})(b^{++}_{a(N-a)}),$$
for any $1 \leq a \leq N$*;*

(3)
$$t^{+++}_{abc} := (M'_{ab} \otimes 1)(b^{++}_{(N-c)c}),$$
for any $1 \leq a, b, c \leq N$ *such that* $a + b + c = N$*;*

(4)
$$t^{---}_{abc} := (1 \otimes D_{a+c} M_{ac} \otimes 1)(b^{-+}_a \otimes b^{+-}_c)$$
for any $1 \leq a, b, c \leq N$ *such that* $a + b + c = N$*.*

Proof. (1) Note that $b^{++}_{a(N-a)}$ has the negative exponent property with top term
$$x_{\{N,\ldots,N-a+1\}} \otimes x_{\{N-a,\ldots,1\}}.$$
Since $\mathrm{Hom}(\Lambda^N(\mathbb{C}^N_q), \Lambda^a(\mathbb{C}^N_q) \otimes \Lambda^{N-a}(\mathbb{C}^N_q))$ has dimension one, this implies that $b^{++}_{a(N-a)}$ has to be dual canonical.

(2) Follows directly from (1).

(3) Note that t^{+++}_{abc} has the negative exponent property with top term
$$x_{\{N,\ldots,b+c+1\}} \otimes x_{\{b+c,\ldots,c+1\}} \otimes x_{\{c,\ldots,1\}}$$
Again we have
$$\dim(\mathrm{Hom}(\Lambda^N(\mathbb{C}^N_q), \Lambda^a(\mathbb{C}^N_q) \otimes \Lambda^b(\mathbb{C}^N_q) \otimes \Lambda^c(\mathbb{C}^N_q))) = 1,$$
so t^{+++}_{abc} is indeed dual canonical.

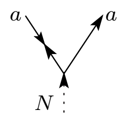
(4) Note that t^{---}_{abc} has the negative exponent property with top term
$$\hat{x}_{\{N,\ldots,a+1\}} \otimes \hat{x}_{\{N,\ldots,a+b+1,a\ldots,1\}} \otimes \hat{x}_{\{a+b,\ldots,1\}}$$
Also in this case the relevant hom-space is one-dimensional, so the result follows.

□

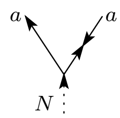
Not all dual canonical basis elements can be represented by monomial webs. However, one can prove the following analogue of Proposition 2 in [25]. I identify webs with intertwiners, so I call a certain web ψ-equivariant if the corresponding intertwiner is ψ-equivariant.

Proposition 4.16. *Any monomial web in* $\mathcal{S}p(\mathrm{SL}_N)$ *is* ψ*-equivariant. In particular, any monomial web in* $W(\vec{k}, N)$ *is* ψ*-invariant, for any* $m = N\ell$ *and* $\vec{k} \in \Lambda(m,m)_N$*.*

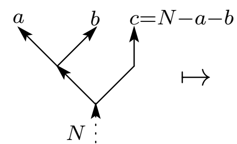
Proof. By Lemma 4.15, the proposition holds true for the following webs (the $\pm$-signs appear because the pictures do not specify whether left or right tags are used at the bi-valent vertices):



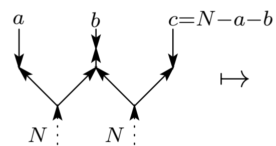
$\mapsto \quad \pm b_a^{-+}$



$\mapsto \quad \pm b_a^{+-}$



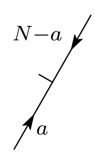
$\mapsto \quad t_{abc}^{+++}$



$\mapsto \quad \pm t_{abc}^{---}$

By the zig-zag relations for cups and caps, the ψ-invariance of the cups implies the ψ-equivariance of the caps.

By definition, the web



$\mapsto D_a$

is also ψ-equivariant.

Up to a sign, any monomial web can be obtained by gluing instances of the above ones, so we see that any monomial web is ψ-equivariant. $\square$

Remark 4.17. *For Lemma 4.15 and Proposition 4.16 to be true, the generating intertwiners in the image of* Γ_N *have to be normalized as in Section 3.2.*

4.4. **The canonical and the LT-bases and their Howe duals.** In this section I recall the canonical basis and Leclerc and Toffin's intermediate basis of V_Λ. The latter basis gives rise to a nice basis of $W(\vec{k}, N)$ by quantum skew Howe duality. Finally I will show that one can identify the canonical $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$-basis and the dual canonical $\dot{\mathbf{U}}_q(\mathfrak{sl}_N)$-basis of W_Λ.

Let us first briefly recall the *canonical basis* of the irreducible $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$-representation V_Λ. For more details the reader can consult [5, 9, 33] for example.

As was already remarked above, V_Λ is isomorphic to a direct summand of $\Lambda_v^\ell(\mathbb{C}_v^m)^{\otimes N}$. The highest weight vector corresponds to

$$v_\Lambda := x_{T_\Lambda} = x_{\{12\ldots\ell\}} \otimes \cdots \otimes x_{\{12\ldots\ell\}} \in \Lambda_v^\ell(\mathbb{C}_v^m)^{\otimes N}.$$

There is a unique v-antilinear bar-involution on V_Λ determined by the conditions

$$\widetilde{\psi}(v_\Lambda) = v_\Lambda$$
$$\widetilde{\psi}(Xv) = \overline{X}\tilde{\psi}(v),$$

for any $X \in \dot{\mathbf{U}}_v(\mathfrak{sl}_m)$ and $v \in V_\Lambda$.

Just as the dual canonical basis of W_Λ, the canonical basis of V_Λ is parametrized by $\mathrm{Std}^{(N^\ell)}$. For a proof of the following theorem see [32] and the references therein. Note that our coproduct differs from that in [32], so we get the negative exponent property rather than the positive one.

Theorem 4.18 (Kashiwara, Lusztig). *For each $T \in \mathrm{Std}^{(N^\ell)}$, there exists a unique element b_T such that*

$$\begin{aligned}\tilde{\psi}(b_T) &= b_T\\ b_T &= x_T + \sum_{T' \prec T} d_{\tau,T}(v) x_\tau\end{aligned}$$

with $\tau \in \mathrm{Col}^{(N^\ell)}$ and $d_{\tau,T}(v) \in v^{-1}\mathbb{Z}[v^{-1}]$. The elements b_T form the **canonical basis** *of V_Λ.*

As before, we have a symmetric v-bilinear form $(\cdot,\cdot)$ on $\Lambda_v^\ell(\mathbb{C}_v^m)^{\otimes N}$ determined by

$$(x_\tau, x_{\tau'}) = \delta_{\tau,\tau'}$$

for all $\tau, \tau' \in \mathrm{Col}^{(N^\ell)}$. Lemmas 17.1.3 and 26.2.2 in [33] show that the restriction of this form to V_Λ is the unique symmetric v-bilinear form on V_Λ satisfying

(1) $(v_\Lambda, v_\Lambda) = 1$;
(2) $\langle Xv, v' \rangle = \langle v, \rho(X)v' \rangle$, for any $X \in \dot{\mathbf{U}}_v(\mathfrak{sl}_m)$ and any $v, v' \in V_\Lambda$.

Here ρ is the v-linear anti-involution on $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$ defined in Section 19.1 in [33].

One can easily check that

$$\tau(X) = \overline{\rho(X)}$$

for any $X \in \dot{\mathbf{U}}_v(\mathfrak{sl}_m)$, so the relation between $(\cdot,\cdot)$ and the v-Shapovalov form is given by

$$(\cdot,\cdot) := \overline{\langle \cdot, \widetilde{\psi}(\cdot) \rangle}.$$

Note that the negative exponent property in the above theorem is equivalent to the condition

$$(b_T, b_{T'}) \in \delta_{T,T'} + v^{-1}\mathbb{Z}[v^{-1}].$$

Since the canonical basis elements are $\widetilde{\psi}$-invariant, this is also equivalent to

$$\langle b_T, b_{T'} \rangle = \overline{(b_T, b_{T'})} \in \delta_{T,T'} + v\mathbb{Z}[v]. \tag{31}$$

Leclerc and Toffin [32] (Section 4.1) defined a different basis of V_Λ, which I denote by B_Λ. The elements of B_Λ are also parametrized by the elements in $\mathrm{Std}^{(N^\ell)}$.

Suppose $T \in \mathrm{Std}^{(N^\ell)}$ is arbitrary. Let us recall how to construct the Leclerc-Toffin (LT) basis element $A_T \in B_\Lambda$. Let $1 \leq i_1 \leq \ell$ be the smallest integer such that the rows of $T = T_1$ with row number $\leq i_1$ contain entries equal to $i_1 + 1$. Denote the total number of such entries by $r_1 > 0$. Change all these entries to i_1 and denote the new semi-standard tableau by $T_2 \in \mathrm{Std}^{(N^\ell)}$. Let $1 \leq i_2 \leq \ell$ be the smallest integer such that the rows of T_2 with row number $\leq i_2$ contain $r_2 > 0$ entries equal to $i_2 + 1$. Then change these entries to i_2 and denote the new tableau by $T_3 \in \mathrm{Std}^{(N^\ell)}$. Continue this way until $T_s = T_\Lambda \in \mathrm{Std}^{(N^\ell)}$. Leclerc and Toffin define the basis element A_T as

$$A_T := E_{-i_1}^{(r_1)} \cdots E_{-i_s}^{(r_s)} v_\Lambda. \tag{32}$$

By definition, we have

$$\tilde{\psi}(A_T) = A_T, \tag{33}$$

for any $T \in \mathrm{Std}^{(N^\ell)}$.

Using the coproduct of $\mathbf{U}_v(\mathfrak{sl}_m)$, it is easy to work out the expansion of A_T on the standard basis of $\Lambda_v^\ell(\mathbb{C}_v^m)^{\otimes N}$. In Lemma 9 in [32], Leclerc and Toffin showed that

$$A_T = x_T + \sum_{\tau \prec T} \alpha_{\tau T}(v) x_\tau, \tag{34}$$

with $\tau \in \mathrm{Col}^{(N^\ell)}$ and certain coefficients $\alpha_{\tau T}(v) \in \mathbb{N}[v, v^{-1}]$. For any $\tau \in \mathrm{Col}^{(N^\ell)}$ and $T \in \mathrm{Std}^{(N^\ell)}$, the coefficient $\alpha_{\tau T}(v) = 0$ if τ and T do not have the same $\mathfrak{sl}_m$-weight.

Leclerc and Toffin gave an algorithm to compute the canonical basis of V_Λ which uses B_Λ as an intermediate basis. In Section 4.2 they showed that

$$b_T = A_T + \sum_{S \prec T} \beta_{ST}(v) A_S, \tag{35}$$

with $S \in \mathrm{Std}^{(N^\ell)}$ for certain bar-invariant coefficients $\beta_{ST}(v) \in \mathbb{Z}[v, v^{-1}]$. For any $S, T \in \mathrm{Std}^{(N^\ell)}$, the coefficient $\beta_{ST}(v) = 0$ if S and T do not have the same $\mathfrak{sl}_m$-weight.

By Corollary 4.9 we see that the quantum skew Howe duality functor in Proposition 4.4 maps the basis B_Λ of V_Λ to a web basis of W_Λ. The elements of this web basis are defined in exactly the same way as those of B_Λ by letting the divided powers of $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$ act on the highest weight web w_Λ. We call this web basis the *LT-web basis* and denote it by BW_Λ. We denote the element in BW_Λ associated to a tableau $T \in \mathrm{Std}^{(N^\ell)}$ by A^T.

For any $T \in \mathrm{Std}^{(N^\ell)}$, we have

$$\psi(A^T) = A^T \tag{36}$$

by Proposition 4.16.

Using the functor $\Gamma_N \colon \mathcal{S}p(\mathrm{SL}_N) \to \mathcal{R}ep(\mathrm{SL}_N)$ in Theorem 4.2, we can write every basis web A^T as a linear combination of standard basis elements in $\Lambda_q^\bullet(\mathbb{C}_q^N)^{\otimes m}$. The following proposition is the analogue of Theorem 2 in [25] and follows immediately from (34) and Proposition 4.6 in this paper.

Proposition 4.19. *For any $T \in \mathrm{Std}^{(N^\ell)}$, we have*

$$A^T = x^T + \sum_{\tau \prec T} \alpha_{\tau,T}(v) x^\tau,$$

with $\tau \in \mathrm{Col}^{(N^\ell)}$.

Note that, for any $\tau \in \mathrm{Col}^{(N^\ell)}$ and $T \in \mathrm{Std}^{(N^\ell)}$, we have

$$\sum_{\tau \prec S \prec T} \alpha_{\tau S}(v)\beta_{ST}(v) = d_{\tau T}(v),$$

with $S \in \mathrm{Std}^{(N^\ell)}$. By Theorems 4.12 and 4.18, equation (36) and Proposition 4.19 we obtain

$$d^{\tau T}(v) = d_{\tau T}(v),$$

for any $\tau \in \mathrm{Col}^{(N^\ell)}$ and $T \in \mathrm{Std}^{(N^\ell)}$, and the following result.

Proposition 4.20. *For any* $T \in \mathrm{Std}^{(N^\ell)}$, *we have*

$$b^T = A^T + \sum_{S \prec T} \beta_{ST}(v) A^S,$$

with $S \in \mathrm{Std}^{(N^\ell)}$.

Corollary 4.21. *Under the isomorphism in Remark 4.10 we have*

$$A_T \mapsto A^T \quad \text{and} \quad b_T \mapsto b^T,$$

for all $T \in \mathrm{Std}^{(N^\ell)}$.

5. Web categories and algebras

Wu [45] and independently Yonezawa [46, 47] defined $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z})$-graded matrix factorizations associated to colored $\mathfrak{sl}_N$-webs, generalizing the ground breaking work of Khovanov and Rozanksy [29]. The $\mathbb{Z}/2\mathbb{Z}$-grading is generally referred to as the *homological grading* and the $\mathbb{Z}$-grading as the *quantum grading.* A shift of 1 in the homological grading is indicated by $\langle 1\rangle$ and in the quantum grading by $\{1\}$.

In [38] Yonezawa and I recalled these matrix factorizations in great detail, so I refer to that paper and its references for the relevant background on graded matrix factorizations in general and the definitions of and the results on $\mathfrak{sl}_N$-web matrix factorizations in particular.

All the reader of this paper has to know, is that to any monomial $\mathfrak{sl}_N$-web $u \in W(\vec{k}, N)$ without tags (but with oriented N-colored edges) one can associate a matrix factorization $\hat{u}$ such that

$$\widehat{u^*} = \hat{u}_\bullet\{-d(\vec{k})\}\langle 1\rangle,$$

where $\hat{u}_\bullet$ is the dual matrix factorization. As explained in Section 5 in [38], this implies that

$$\mathrm{EXT}(\hat{u}, \hat{v}) \cong H(\hat{u}_\bullet \boxtimes_{R^k} \hat{v}) \cong H(\widehat{u^* v})\{d(\vec{k})\}\langle 1\rangle \tag{37}$$

for any monomial webs $u, v \in W(\vec{k}, N)$.

For the proof of the following theorem, which was recalled as Theorem 5.14 in [38], I refer to Sections 6 through 11 in [45] and Section 3 in [46].

Theorem 5.1 (Wu, Yonezawa)**.** *The matrix factorizations associated to webs without tags satisfy all relations in Definition 4.1, except the first one, up to homotopy equivalence. These equivalences are quantum degree preserving, but might involve homological degree shifts.*

Because of the last remark in Theorem 5.1, people working on $\mathfrak{sl}_N$-web matrix factorizations usually forget about the homological degree in their notation. I will follow that tradition and will never write homological shifts explicitly in the equations below.

5.1. **The graded web category.** Let $\vec{k} = (k_1, \ldots, k_m) \in \Lambda(m, m)_N$. In this section I recall the $\mathbb{C}$-linear category $\mathcal{W}^\circ(\vec{k}, N)$ from [38].

Definition 5.2. *The objects of $\mathcal{W}^\circ(\vec{k}, N)$ are by definition all matrix factorizations which are homotopy equivalent to direct sums of matrix factorizations of the form $\hat{u}$, where u is an N-ladder with m uprights in $W(\vec{k}, N)$.*

For any pair of objects $X, Y \in \mathcal{W}^\circ(\vec{k}, N)$, we define

$$\mathcal{W}^\circ(X, Y) := \mathrm{Ext}(X, Y).$$

Composition in $\mathcal{W}^\circ(\vec{k}, N)$ is induced by the composition of homomorphisms between matrix factorizations.

Note that $\mathcal{W}^\circ(\vec{k}, N)^*$ is a $\mathbb{Z}$-graded $\mathbb{C}$-linear additive category which admits translation and has finite-dimensional hom-spaces.

Identifying ladders with the corresponding matrix factorizations, we see that $\mathcal{W}^\circ(\vec{k}, N)$ is a full subcategory of the homotopy category of matrix factorization with fixed potential determined by $\vec{k}$ and N. The latter category is Krull-Schmidt by Propositions 24 and 25 in [29], so we can take the Karoubi envelope of $\mathcal{W}^\circ(\vec{k}, N)$, denoted $\dot{\mathcal{W}}^\circ(\vec{k}, N)$, which is also Krull-Schmidt.

5.2. **Graded web algebras.** For any $T \in \mathrm{Std}_{\vec{k}}^{(N^\ell)}$, let $\hat{A}^T$ denote the $\mathfrak{sl}_N$-matrix factorization associated to A^T in [38].

Definition 5.3. *For any pair $S, T \in \mathrm{Std}_{\vec{k}}^{(N^\ell)}$, define*

$${}_S H(\vec{k}, N)_T := \mathrm{EXT}(\hat{A}^S, \hat{A}^T) \ (\cong H(\widehat{(A^S)^* A^T})\{d(\vec{k})\}).$$

The web algebra $H(\vec{k}, N)$ *is defined by*

$$H(\vec{k}, N) := \bigoplus_{S,T \in \mathrm{Std}_{\vec{k}}^{(N^\ell)}} {}_S H(\vec{k}, N)_T,$$

with multiplication induced by the composition of maps between matrix factorizations.

Note that $H(\vec{k}, N)$ is a finite-dimensional graded unital associative algebra.

The following proposition is due to Buchweitz, see Proposition 10.1.5 and Example 10.1.6 in [12]. The ring R in Buchweitz's Example 10.1.6 is equal to the center of $H(\vec{k}, N)$ in our case. As I will show in Corollary 7.10 below, this center is isomorphic to the complex cohomology ring of the Spaltenstein variety $X_{\vec{k}}^{(N^\ell)}$ of partial flags in $\mathbb{C}^m$ of type $\vec{k}$ and nilpotent linear operator of Jordan type (ℓ^N). The Gorenstein parameter in the proposition below is therefore equal to twice the top dimension of $H^*(X_{\vec{k}}^{(N^\ell)})$ ("twice" because $\deg(x_i) = 2$), which is equal to

$$2d(\vec{k}) := N(N-1)\ell - \sum_{i=1}^{m} k_i(k_i - 1)$$

by Theorem 1.2 in [6] and the remarks following that theorem.

Proposition 5.4. *The web algebra $H(\vec{k},N)$ is a graded symmetric Frobenius algebra of Gorenstein parameter $2d(\vec{k})$. This means that*

$$H(\vec{k},N)^{\vee} \cong H(\vec{k},N)\{-2d(\vec{k})\}$$

as graded $H(\vec{k},N)-H(\vec{k},N)$ bimodules, where $H(\vec{k},N)^{\vee}$ is the graded dual.

To fix notation, let us already introduce the following two definitions here.

Definition 5.5.

$$\begin{aligned}\mathcal{W}(\vec{k},N) &:= H(\vec{k},N)-\mathrm{mod}_{\mathrm{gr}};\\ \mathcal{W}^{p}(\vec{k},N) &:= H(\vec{k},N)-\mathrm{pmod}_{\mathrm{gr}}.\end{aligned}$$

We also give some important examples of projective $H(\vec{k},N)$-modules.

Example 5.6. *For $T \in \mathrm{Std}_{\vec{k}}^{(N^\ell)}$, we define*

$$P^T := H(\vec{k},N)1_{\hat{A}^T}\{-d(\vec{k})\} \quad \textit{and} \quad {}^{T}P := 1_{\hat{A}^T}H(\vec{k},N)\{-d(\vec{k})\}.$$

Since $\sum_{T\in\mathrm{Std}_{\vec{k}}^{(N^\ell)}} 1_{\hat{A}^T} = 1$ in $H(\vec{k},N)$, we see that P^T and ${}^{T}P$ are graded left and right projective $H(\vec{k},N)$-modules, respectively. Both are finite-dimensional, of course.

We have normalized the grading of P^T and ${}^{T}P$ so that

$$P^T \cong (P^T)^{\vee} \quad \textit{and} \quad {}^{T}P \cong ({}^{T}P)^{\vee}$$

as graded vector spaces (see Proposition 5.4).

We note that P^T and ${}^{T}P$ can be decomposable as $H(\vec{k},N)$-modules in general.

6. Categorified quantum $\mathfrak{sl}_m$ and 2-representations

6.1. **Categorified $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$.** Khovanov and Lauda introduced diagrammatic 2-categories $\mathcal{U}(\mathfrak{g})$ which categorify the integral version of the corresponding idempotented quantum groups [27]. Independently, Rouquier also introduced similar 2-categories [41]. Subsequently, Cautis and Lauda [15] defined diagrammatic 2-categories $\mathcal{U}_Q(\mathfrak{g})$ with implicit scalers Q consisting of t_{ij}, r_i and s_{ij}^{pq} which determine certain signs in the definition of the categorified quantum groups.

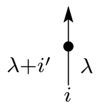
In this section, I recall $\mathcal{U}(\mathfrak{sl}_m) = \mathcal{U}_Q(\mathfrak{sl}_m)$ briefly. The implicit scalars Q are given by $t_{ij} = -1$ if $j = i+1$, $t_{ij} = 1$ otherwise, $r_i = 1$ and $s_{ij}^{pq} = 0$. This corresponds precisely to the signed version in [27, 28]. The other conventions here are the same as those in Section 3.

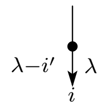
Definition 6.1 (Khovanov-Lauda)**.** *The 2-category $\mathcal{U}(\mathfrak{sl}_m)$ is defined as follows:*

- *The objects in $\mathcal{U}(\mathfrak{sl}_m)$ are the weights $\lambda \in \mathbb{Z}^{m-1}$.*

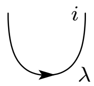
For any pair of objects λ and λ' in $\mathcal{U}(\mathfrak{sl}_m)$, the hom category $\mathcal{U}(\mathfrak{sl}_m)(\lambda,\lambda')$ is the graded additive $\mathbb{C}$-linear category consisting of:

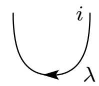
- *objects (1-morphisms in $\mathcal{U}(\mathfrak{sl}_m)$), which are finite formal sums of the form $\mathcal{E}_{\underline{i}}\mathbf{1}_\lambda\{t\}$ where $t \in \mathbb{Z}$ is the grading shift and $\underline{i}$ is a signed sequence such that $\lambda' = \lambda + \sum_{a=1}^{l} \epsilon_a i'_a$.*
- *morphisms from $\mathcal{E}_{\underline{i}}\mathbf{1}_\lambda\{t\}$ to $\mathcal{E}_{\underline{l}}\mathbf{1}_\lambda\{t'\}$ in $\mathcal{U}(\mathfrak{sl}_m)(\lambda,\lambda')$ (2-morphisms in $\mathcal{U}(\mathfrak{sl}_m)$) are $\mathbb{C}$-linear combinations of diagrams with degree $t'-t$ spanned by composites of the following diagrams:*



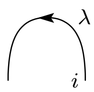
$: \mathcal{E}_{+i}\boldsymbol{1}_\lambda \to \mathcal{E}_{+i}\boldsymbol{1}_\lambda\{a_{ii}\}$


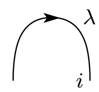
$: \mathcal{E}_{+i}\boldsymbol{1}_\lambda \to \mathcal{E}_{-i}\boldsymbol{1}_\lambda\{a_{ii}\}$



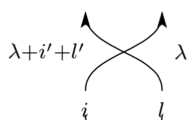
$: \boldsymbol{1}_\lambda \to \mathcal{E}_{(-i,+i)}\boldsymbol{1}_\lambda\{\lambda_i + 1\}$


$: \boldsymbol{1}_\lambda \to \mathcal{E}_{(+i,-i)}\boldsymbol{1}_\lambda\{-\lambda_i + 1\}$

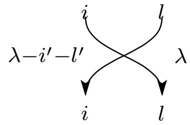


$: \mathcal{E}_{(-i,+i)}\boldsymbol{1}_\lambda \to \boldsymbol{1}_\lambda\{\lambda_i + 1\}$


$: \mathcal{E}_{(+i,-i)}\boldsymbol{1}_\lambda \to \boldsymbol{1}_\lambda\{-\lambda_i + 1\}$



$: \mathcal{E}_{(+i,+l)}\boldsymbol{1}_\lambda \to \mathcal{E}_{(+l,+i)}\boldsymbol{1}_\lambda\{-a_{il}\}$



$: \mathcal{E}_{(-i,-l)}\boldsymbol{1}_\lambda \to \mathcal{E}_{(-l,-i)}\boldsymbol{1}_\lambda\{-a_{il}\}$

As already remarked, the relations on the 2-morphisms are those of the signed version in [27, 28]*, which I do not recall here because I do not need them explicitly in this paper.*

Khovanov and Lauda's main result for type A in [27] was their Proposition 1.4.

Theorem 6.2 (Khovanov-Lauda)**.** *The linear map*

$$\gamma\colon \dot{\mathbf{U}}_v(\mathfrak{sl}_m) \to K_0^v(\mathcal{U}(\mathfrak{sl}_m))$$

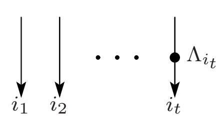
defined by

$$v^t E_{\underline{i}} 1_\lambda \to \mathcal{E}_{\underline{i}}\boldsymbol{1}_\lambda\{t\}$$

is an isomorphism of $\mathbb{C}(v)$*-algebras.*

6.2. **Cyclotomic KLR algebras and 2-representations.** Let Λ be a dominant $\mathfrak{sl}_m$-weight, V_Λ the irreducible $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-module of highest weight Λ and P_Λ the set of weights in V_Λ.

Definition 6.3 (Khovanov-Lauda, Rouquier)**.** *The* cyclotomic KLR algebra R_Λ *is the subquotient of* $\mathcal{U}(\mathfrak{sl}_m)$ *defined by the subalgebra of all diagrams with only downward oriented strands and right-most region labeled* Λ *modded out by the ideal generated by diagrams of the form*



Note that

$$R_\Lambda = \bigoplus_{\mu \in P_\Lambda} R_\Lambda(\mu),$$

where $R_\Lambda(\mu)$ is the subalgebra generated by all diagrams whose left-most region is labeled μ. Brundan and Kleshchev proved that R_Λ is finite-dimensional in Corollary 2.2 in [5]. We also define

$$\mathcal{V}_\Lambda := R_\Lambda - \mathrm{mod}_{\mathrm{gr}};$$

$$\mathcal{V}_\Lambda^p := R_\Lambda - \mathrm{pmod}_{\mathrm{gr}}.$$

Below I will use Cautis and Lauda's language of strong $\mathfrak{sl}_m$ 2-representations (see Definition 1.2 in [15]). For a comparison with Rouquier's [41] definition of a Kac-Moody 2-representation, see Cautis and Lauda's remark (1) below their Definition 1.2.

In Section 4.4 in [4] Brundan and Kleshchev defined a strong $\mathfrak{sl}_m$ 2-representation on $\mathcal{V}_\Lambda$, which can be restricted to $\mathcal{V}_\Lambda^p$.

They also defined a duality $\circledast\colon \mathcal{V}_\Lambda \to \mathcal{V}_\Lambda$. To define it, they used Khovanov and Lauda's graded anti-automorphism

$$*\colon R_\Lambda \to R_\Lambda,$$

which is defined by reflecting diagrams in the x-axis followed by inversion of the orientation. Let $M \in \mathcal{V}_\Lambda$. On the underlying vector space, we have

$$M^\circledast := M^\vee.$$

The action on $M^\circledast$ is defined by

$$xf(p) := f(x^*p).$$

(Note that I use a left action throughout the paper, contrary to Brundan and Kleshchev.) This duality can be restricted to $\mathcal{V}_\Lambda^p$ and induces a q-antilinear involution on $K_0^q(\mathcal{V}_\Lambda^p)$.

Brundan and Kleshchev [4] proved the following result (Theorems 4.18 and Theorem 5.14).

Theorem 6.4 (Brundan-Kleshchev)**.** *There exists a unique $\mathbb{C}(v)$-linear isomorphism*

$$\delta\colon V_\Lambda \to K_0^v(\mathcal{V}_\Lambda^p)$$

of $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$-modules, such that

- *δ intertwines the bar-involutions $\tilde{\psi}$ and $\circledast$;*
- *δ intertwines the v-Shapovalov form and the Euler form;*
- *for each $T \in \mathrm{Std}^{(N^\ell)}$, there exists an indecomposable projective module $Q_T \in \mathcal{V}_\Lambda^p$ such that $Q_T = \delta(b_T)$ (note that this implies that $Q_T^\circledast \cong Q_T$).*

Remark 6.5. *Note that Brundan and Kleshchev* [4] *use multipartitions instead of column-strict tableaux. There is a well-known bijection between these two combinatorial data, which I will not recall here. Using this bijection, their module $Y(T)$ in Theorem 5.14 satisfies*

$$Y(T) = Q_T\{d(\vec{k})\}$$

as follows from their Lemma 3.12. The Grothendieck classes of the $Y(T)$ correspond to what Brundan and Kleshchev call the quasi-canonical *basis of V_Λ.*

Let $T \in \mathrm{Std}^{(N^\ell)}$ and suppose that

$$A_T = E_{-i_1}^{(r_1)} \cdots E_{-i_l}^{(r_l)} v_\Lambda,$$

as in (32). Then we can define a projective module $P_T \in \mathcal{V}_\Lambda^p$ by

$$P_T := \mathcal{E}_{-i_1}^{(r_1)} \cdots \mathcal{E}_{-i_l}^{(r_l)} V(\Lambda), \tag{38}$$

where $V(\Lambda) = P_{T_\Lambda}$ is the highest weight object in $\mathcal{V}_\Lambda^p$. By (33) and Theorem 6.4, we have

$$P_T = \delta(A_T) \quad \text{and} \quad P_T^\circledast \cong P_T. \tag{39}$$

By (35) and Theorem 6.4, we have

$$[Q_T] = [P_T] + \sum_{S \prec T} \beta_{ST}(v)[P_S] \tag{40}$$

in $K_0^v(\mathcal{V}_\Lambda^p)$.

7. Categorified skew Howe duality

7.1. **Equivalences.** Recall the algebra $H(\vec{k},N)$ and the categories $\mathcal{W}(\vec{k},N)$, $\mathcal{W}^p(\vec{k},N)$ and $\mathcal{W}^\circ(\vec{k},N)$ which were defined in Definitions 5.2, 5.3 and 5.5.

Definition 7.1. *Define*

$$\begin{aligned} H_\Lambda &:= \bigoplus_{\vec{k}\in\Lambda(m,m)_N} H(\vec{k},N); \\ \mathcal{W}_\Lambda &:= \bigoplus_{\vec{k}\in\Lambda(m,m)_N} \mathcal{W}(\vec{k},N); \\ \mathcal{W}^p_\Lambda &:= \bigoplus_{\vec{k}\in\Lambda(m,m)_N} \mathcal{W}^p(\vec{k},N); \\ \mathcal{W}^\circ_\Lambda &:= \bigoplus_{\vec{k}\in\Lambda(m,m)_N} \mathcal{W}^\circ(\vec{k},N). \end{aligned}$$

We denote the Karoubi envelope of $\mathcal{W}^\circ_\Lambda$ by $\dot{\mathcal{W}}^\circ_\Lambda$.

In Definition 9.1 in [38] Yonezawa and I defined a strong $\mathfrak{sl}_m$ 2-representation on $\dot{\mathcal{W}}^\circ_\Lambda$. This was defined by sending $\mathbf{1}_{\lambda'}\mathcal{E}_{\underline{i}}\mathbf{1}_\lambda$ to the functor defined by tensoring with the matrix factorization

$$\widehat{E}'_{\underline{i},[\vec{k}]} := \widehat{E}_{\underline{i},[\vec{k}]}\{d(\vec{k})-d(\vec{k}')\}$$

which was associated to the ladder obtained by quantum skew Howe duality, for $\underline{i}=(\epsilon_1 i_1,\ldots,\epsilon_l i_l)$, $\lambda,\lambda'\in P_\Lambda$ and $\phi_{m,m,N}(\lambda)=\vec{k}$ and $\phi_{m,m,N}(\lambda')=\vec{k}'$.

The results in Section 9.2 of [38] can be summarized as follows:

Theorem 7.2. *$\dot{\mathcal{W}}^\circ_\Lambda$ is a strong additive $\mathfrak{sl}_m$ 2-representation equivalent to $\mathcal{V}^p_\Lambda$, such that the following square commutes*

$$\begin{array}{ccc} V_\Lambda & \xrightarrow{\delta} & K_0^v(\mathcal{V}^p_\Lambda) \\ \downarrow & & \downarrow \\ W_\Lambda & \xrightarrow{\delta^\circ} & K_0^v(\dot{\mathcal{W}}^\circ_\Lambda) \end{array} . \tag{41}$$

The map δ is the one from Theorem 6.4. The map δ° is determined by

$$A^T \mapsto q^{-d(\vec{k})}[\hat{A}^T] = [\mathcal{E}^{(r_1)}_{-i_1}\cdots\mathcal{E}^{(r_l)}_{-i_l}\hat{w}_\Lambda],$$

for any $T\in\mathrm{Std}^{(N^\ell)}$. The map $V_\Lambda\to W_\Lambda$ was defined in Remark 4.10. All maps in the square are isometric isomorphisms of $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$-modules.

In this section I am first going to show that $\dot{\mathcal{W}}^\circ_\Lambda$ is equivalent to $\mathcal{W}^p_\Lambda$. Recall the definition of $P^T\in\mathcal{W}^p_\Lambda$ in Example 5.6.

Definition 7.3. *We define a linear map*

$$\delta'_{\vec{k},N}\colon W(\vec{k},N)\to K_0^v(\mathcal{W}^p(\vec{k},N))$$

by

$$A^T\mapsto [P^T]\in K_0^v(\mathcal{W}^p(\vec{k},N))$$

for all $T\in\mathrm{Std}^{(N^\ell)}_{\vec{k}}$.

In Corollary 7.6 I will show that $\delta'_{\vec{k},N}$ is an isomorphism. For now, all I can show is injectivity.

Lemma 7.4. *The map $\delta'_{\vec{k},N}$ is injective.*

Proof. Recall that the Euler form

$$\langle [P],[Q]\rangle = \dim_v \mathrm{HOM}(P,Q)$$

is a non-degenerate v-sesquilinear form on $K_0^v(H(\vec{k},N))$. Furthermore, the v-sesquilinear web form gives a non-degenerate v-sesquilinear form on $W(\vec{k},N)$.

The map $\delta'_{\vec{k},N}$ is an isometry w.r.t. these two forms because we have

$$\dim_v \mathrm{HOM}(P^S,P^T) = \dim_v {}_S H(\vec{k},N)_T =$$
$$\dim_v \mathrm{EXT}(\hat{A}^S,\hat{A}^T) = v^{d(\vec{k})}\dim_v H^*(\widehat{(A^S)^*A^T}) =$$
$$v^{d(\vec{k})}\mathrm{ev}((A^S)^*A^T) = \langle A^S, A^T\rangle.$$

for any $S,T\in \mathrm{Std}_{\vec{k}}^{(N^\ell)}$, by Theorem 5.1.

Since isometries for non-degenerate forms are always injective, this proves the lemma. □

Lemma 7.5. *There exists an equivalence of additive categories*

$$\mathcal{W}^p(\vec{k},N)\cong \dot{\mathcal{W}}^\circ(\vec{k},N).$$

Proof. First define the image of projective modules associated to basis webs. For any $T\in \mathrm{Std}_{\vec{k}}^{(N^\ell)}$ and any $t\in\mathbb{Z}$, we define

$$\mathcal{W}^p(\vec{k},N)\ni P^T\{t\}\mapsto \hat{A}^T\{t-d(\vec{k})\}\in \dot{\mathcal{W}}^\circ(\vec{k},N). \tag{42}$$

By definition, for any pair $S,T\in\mathrm{Std}_{\vec{k}}^{(N^\ell)}$ we have

$$\mathrm{HOM}(P^S,P^T)=\mathrm{EXT}(\hat{A}^S\{-d(\vec{k})\},\hat{A}^T\{-d(\vec{k})\})=\mathcal{W}^\circ(\hat{A}^S\{-d(\vec{k})\},\hat{A}^T\{-d(\vec{k})\}),$$

where the latter is the hom-space in $\dot{\mathcal{W}}^\circ(\vec{k},N)^*$. So any intertwiner in $\mathrm{HOM}(P^S,P^T)$ can be mapped to the corresponding morphism in $\mathcal{W}^\circ(\hat{A}^S\{-d(\vec{k})\},\hat{A}^T\{-d(\vec{k})\})$.

Any indecomposable object in $\mathcal{W}^p(\vec{k},N)$ has to be isomorphic to a direct summand of $P^T\{t\}$, for some $T\in\mathrm{Std}_{\vec{k}}^{(N^\ell)}$ and $t\in\mathbb{Z}$, because by definition we have

$$H(\vec{k},N)=\bigoplus_{T\in\mathrm{Std}_{\vec{k}}^{(N^\ell)}} P^T\{d(\vec{k})\}.$$

Therefore (42) determines the desired embedding up to natural isomorphism.

To see this, let Q be an indecomposable object in $\mathcal{W}^p(\vec{k},N)$. There exists a $T\in\mathrm{Std}_{\vec{k}}^{(N^\ell)}$ and $t\in\mathbb{Z}$ such that Q is isomorphic to a direct summand of $P^T\{t\}$ given by a primitive idempotent $e_Q\in\mathrm{End}(P^T\{t\})$. We define

$$Q\mapsto (A^T\{t\},e_Q)\in\dot{\mathcal{W}}^\circ(\vec{k},N).$$

The choice of T and t need not be unique, but we simply choose one. A different choice will lead to a naturally isomorphic functor.

This gives a well-defined functor

$$\mathcal{W}^p(\vec{k},N)\hookrightarrow\dot{\mathcal{W}}^\circ(\vec{k},N), \tag{43}$$

which is fully faithful.

We now have to show that this functor is essentially surjective. It suffices to show that any indecomposable object in $\dot{\mathcal{W}}^\circ(\vec{k},N)$ is isomorphic to the image of an indecomposable object in $\mathcal{W}^p(\vec{k},N)$. Since the functor in (43) is fully faithful, it induces an embedding

$$K_0^v(\mathcal{W}^p(\vec{k},N)) \hookrightarrow K_0^v(\dot{\mathcal{W}}^\circ(\vec{k},N))$$

which sends Grothendieck classes of indecomposables to Grothendieck classes of indecomposables. Therefore, all we have to do is show that this map between the Grothendieck groups is surjective. We do this by proving that both vector spaces have the same dimension.

In Lemma 7.4, I showed that the linear map

$$\delta'_{\vec{k},N} \colon W(\vec{k},N) \to K_0^v(\mathcal{W}^p(\vec{k},N))$$

from Definition 7.3 is injective.

So we have a sequence of two embeddings

$$W(\vec{k},N) \hookrightarrow K_0^v(\mathcal{W}^p(\vec{k},N)) \hookrightarrow K_0^v(\dot{\mathcal{W}}^\circ(\vec{k},N)) \tag{44}$$

and their composite is an isomorphism by Theorem 7.2. This shows that both embeddings in (44) are isomorphisms, which finishes the proof of this lemma. □

In the proof of Lemma 7.5 we have also obtained the following result.

Corollary 7.6. *The linear map* $\delta'_{\vec{k},N} \colon W(\vec{k},N) \to K_0^v(\mathcal{W}^p(\vec{k},N))$ *is an isomorphism.*

Let $\delta' \colon W_\Lambda \to K_0^v(\mathcal{W}_\Lambda^p)$ be the bijective $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$-intertwiner defined by

$$\delta' := \bigoplus_{\vec{k}\in\Lambda(m,m)_N} \delta'_{\vec{k},N}.$$

We can now prove our main result.

Theorem 7.7. *There exists a strong* $\mathfrak{sl}_m$ *2-representation on the Abelian category* $\mathcal{W}_\Lambda$*, which can be restricted to* $\mathcal{W}_\Lambda^p$*.*

There exists a duality on $\mathcal{W}_\Lambda$*, also denoted* $\circledast$*, which commutes with the action of* $\mathcal{E}_{\pm i}\boldsymbol{1}_\lambda$*, for any* $i = 1,\ldots,m-1$ *and* $\lambda \in \mathbb{Z}^{m-1}$*. This duality can be restricted to* $\mathcal{W}_\Lambda^p$*.*

Moreover, up to natural isomorphism, there exists a unique equivalence

$$\mathcal{W}_\Lambda \to \mathcal{V}_\Lambda$$

of Abelian strong $\mathfrak{sl}_m$ *2-representations intertwining the dualities, which can be restricted to an equivalence*

$$\mathcal{V}_\Lambda^p \to \mathcal{W}_\Lambda^p$$

such that the following square commutes

$$\begin{array}{ccc} W_\Lambda & \xrightarrow{\delta'} & K_0^v(\mathcal{W}_\Lambda^p) \\ \downarrow & & \downarrow \\ V_\Lambda & \xrightarrow{\delta} & K_0^v(\mathcal{V}_\Lambda^p) \end{array} \; . \tag{45}$$

All maps in (45) *are* $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$*-intertwiners, isometries and intertwine the relevant involutions.*

Proof. In (38) I defined a projective module $P_T \in \mathcal{V}^p_\Lambda$, for any $T \in \mathrm{Std}^{(N^\ell)}$. By Theorem 7.2, we have

$$\mathrm{HOM}(P_S, P_T) \cong \mathrm{EXT}(\hat{A}^S, \hat{A}^T) \tag{46}$$

for any $S, T \in T \in \mathrm{Std}^{(N^\ell)}_{\vec{k}}$. Let

$$P_\Lambda := \bigoplus_{\vec{k}\in\Lambda(m,m)_N} \bigoplus_{T\in\mathrm{Std}^{(N^\ell)}_{\vec{k}}} P_T\{d(\vec{k})\}.$$

Then (46) implies

$$\mathrm{END}(P_\Lambda) \cong H^{\mathrm{opp}}_\Lambda,$$

so P_Λ is a $R_\Lambda - H_\Lambda$ bimodule.

By (35), Theorem 6.4 and (40), we see that P_Λ is a projective generator of $\mathcal{V}_\Lambda$.

By one of Morita's main results on Morita equivalence (see Theorem 5.55 in [40], for example), the above observations show that the exact functor $\mathcal{W}_\Lambda \to \mathcal{V}_\Lambda$ defined by

$$M \mapsto P_\Lambda \otimes_{H_\Lambda} M \tag{47}$$

is an equivalence. It also maps projective modules to projective modules, so it restricts to an equivalence $\mathcal{W}^p_\Lambda \to \mathcal{V}^p_\Lambda$. Note that under the equivalence (47), we have

$$P^T \mapsto P_T$$

for all $T \in \mathrm{Std}^{(N^\ell)}$.

Up to natural isomorphism, there is now a unique strong $\mathfrak{sl}_m$ 2-representation on $\mathcal{W}_\Lambda$ such that the equivalence in (47) becomes an intertwiner and the square in (45) commutes. We can describe this action concretely as follows.

For any signed sequence $\underline{i} = (\epsilon_1 i_1, \ldots, \epsilon_l i_l)$ of simple $\mathfrak{sl}_m$ roots and any $\vec{k} \in P_\Lambda$, we define the $H(\vec{k}', N) - H(\vec{k}, N)$ bimodule

$$H_{\underline{i}}(\vec{k}, N) := \bigoplus_{S\in\mathrm{Std}^{(N^\ell)}_{\vec{k}'}, T\in\mathrm{Std}^{(N^\ell)}_{\vec{k}}} \mathrm{EXT}(\hat{A}^S, \widehat{E}_{\underline{i},[\vec{k}]} \boxtimes_{R^{\vec{k}}} \hat{A}^T)\{d(\vec{k}) - d(\vec{k}')\},$$

where $\vec{k}' = \vec{k} + \sum_{j=1}^{l} \epsilon_j \alpha_{i_j}$. The grading shift $d(\vec{k}) - d(\vec{k}')$ matches precisely Brundan and Kleshchev's grading shift in Lemma 4.4 in [4]. To see this, take for example $\underline{i} = (-i)$. Then $\vec{k}' = \vec{k} - \alpha_i$, so

$$d(\vec{k}) - d(\vec{k}') = 1 - (k_i - k_{i+1}).$$

This is exactly the shift corresponding to the functor $K_i^{-1}\langle 1\rangle$ on $\mathcal{V}_\Lambda$ in Lemma 4.4 in [4].

From the equivalence in (47) it follows that $H_{\underline{i}}(\vec{k}, N)$ is *sweet*, meaning that it is projective as a left module and as a right module (but not as a bimodule!). Therefore, tensoring with $H_{\underline{i}}(\vec{k}, N)$ defines an exact endofunctor on $\mathcal{W}_\Lambda$ which can be restricted to $\mathcal{W}^p_\Lambda$.

Furthermore, we have

$$H_{\underline{i}}(\vec{k}, N) \cong$$

$$H_{\epsilon_1 i_1}(\vec{k} + \sum_{j}^{l-1} \epsilon_j \alpha_j, N) \otimes_{H(\vec{k}+\sum_j^{l-1}\epsilon_j\alpha_j, N)} \cdots \otimes_{H(\vec{k}+\epsilon_l\alpha_l, N)} H_{\epsilon_l i_l}(\vec{k}, N).$$

Note that by Theorem 9.2 in [38] any 2-morphism in $\mathcal{U}(\mathfrak{sl}_m)$ gives rise to a homomorphism of matrix factorizations

$$\widehat{E}_{\underline{i},[\vec{k}]} \to \widehat{E}_{\underline{i}',[\vec{k}]}$$

which induces a bimodule map

$$H_{\underline{i}}(\vec{k},N) \to H_{\underline{i}'}(\vec{k},N).$$

This relation between 2-morphisms and bimodule maps is compatible with compositions, tensor products and units.

It now follows that the square of isomorphisms in (45) commutes. Moreover, all arrows are isometric $\dot{\mathbf{U}}_v(\mathfrak{sl}_m)$-intertwiners.

We can also define a duality on $\mathcal{W}_\Lambda$ such that the maps in the square (45) intertwine the various bar-involutions, as I now explain.

By the duality $\circledast$ on $\mathcal{V}_\Lambda$ and (46) we get a graded anti-automorphism

$$*\colon H_\Lambda \to H_\Lambda.$$

Just as Brundan and Kleshchev did for $\mathcal{V}_\Lambda$ (see the text above Theorem 6.4), we can therefore define a duality $\circledast\colon \mathcal{W}_\Lambda \to \mathcal{W}_\Lambda$.

By definition the equivalence in (47) intertwines the dualities on $\mathcal{W}_\Lambda$ and $\mathcal{V}_\Lambda$. In particular, we see that $(P^T)^{\circledast} \cong P^T$. By Proposition 4.16 it follows that $\delta'\colon W_\Lambda \to K_0^v(\mathcal{W}_\Lambda^p)$ intertwines ψ and $\circledast$.

$\square$

Remark 7.8. *Let $S,T \in \mathrm{Std}^{(N^\ell)}$. If we had a definition of $\mathfrak{sl}_N$-foams like we have for $N = 2$ [1], $N = 3$ [23] and in a limited case for $N \geq 4$ [35], and if we had an isomorphism between the space of $\mathfrak{sl}_N$-foams from A^S to A^T and the elements of $\mathrm{EXT}(\hat{A}^S, \hat{A}^T)$ as in [29, 35, 37], we could describe $*\colon H_\Lambda \to H_\Lambda$ more directly. As in [34], it would be given by the symmetry on foams given by reflection in a plane paralel to the source and target webs together with orientation reversal.*

7.2. **Two consequences.**

For any $T \in \mathrm{Std}^{(N^\ell)}$, let $Q^T \in \mathcal{W}_\Lambda^p$ correspond to the indecomposable $\circledast$-invariant projective module $Q_T \in \mathcal{V}_\Lambda^p$ (Theorem 6.4) under the equivalence in Theorem 7.7. Note that Q^T is also indecomposable and $\circledast$-invariant (i.e. up to isomorphism, of course), because the equivalence intertwines the duality.

Under the same equivalence P^T corresponds to P_T. By (40) and the commutativity of the square in (45), we get

$$[Q^T] = [P^T] + \sum_{S \prec T} \beta_{ST}(v)[P^S]$$

and the following corollary.

Corollary 7.9. *Under the isomorphism $\delta'\colon W_\Lambda \to K_0^v(\mathcal{W}_\Lambda^p)$ we have*

$$b^T \mapsto [Q^T],$$

for any $T \in \mathrm{Std}^{(N^\ell)}$.

Analogously to the case for $N = 2,3$, Theorem 7.7 also implies that we can determine the center of $H(\vec{k},N)$. By the work of Brundan, Kleshchev and Ostrik [3, 5, 6] it is known that the center of $R_\Lambda(k_1 - k_2, \ldots, k_{m-1} - k_m)$ is isomorphic to

$$H^*(X^\Lambda_{\vec{k}}).$$

As before, $X^\Lambda_{\vec{k}}$ is the Spaltenstein variety of partial flags in $\mathbb{C}^m$ of type $\vec{k}$ and nilpotent linear operator of Jordan type (ℓ^N). Since Morita equivalent algebras have isomorphic centers, we get:

Corollary 7.10. *For any $\vec{k} \in \Lambda(m,m)_N$, there exists an isomorphism of graded complex algebras*

$$Z(H(\vec{k}, N)) \cong H^*(X^\Lambda_{\vec{k}}).$$

I could give an explicit degree preserving isomorphism, as was done in [24] for $N = 2$ and in [34] for $N = 3$, but I do not use it here and will therefore omit it.

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