AUGMENTED INVARIANT-EKF DESIGNS FOR SIMULTANEOUS STATE AND DISTURBANCE ESTIMATION

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AUGMENTED INVARIANT-EKF DESIGNS FOR SIMULTANEOUS STATE AND DISTURBANCE ESTIMATION

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Abstract: In this thesis, we study Invariant-EKF designs for invariant systems with disturbances. We identify two sets of sufficient conditions that preserve the invariance of systems when additive dynamic disturbances are applied. A first order approximation of the filtering covariance matrices is proposed that more accurately represents the uncertainties for the Invariant-EKF. Applying the developed theory, three different IEKF designs are presented for a unicycle robot under linear disturbances. Monte Carlo simulations demonstrate the contribution of the first order approximation and also illustrate the performance improvement of all three designs over the standard Extended Kalman Filter.

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CHAPTER I

INTRODUCTION

1.1 Motivation

Estimation and filtering of nonlinear dynamic systems is an important and ongoing topic in academia and in industry. Estimation is the idea of combining available measurements with knowledge of the system dynamics to produce an estimate of the full system state, usually for the purpose of feedback control. When uncertainties are present, either in the dynamic model or from noisy sensor measurements, these estimators can also work as filters, providing the state estimate as a mean and uncertainty from an assumed distribution. If the dynamic and measurement models are linear then the Kalman filter is a minimum mean-squared estimator. For the case of nonlinear models, extensions of the Kalman filter have been developed. The Extended Kalman Filter (EKF) is the widely accepted standard approach, even though it loses the theorectical stability characteristics of the linear filter due to its use of linearization. Another popular technique is the Unscented Kalman Filter (UKF) which propagates the mean and uncertainty through the nonlinear dynamics by utilizing a set of deterministically chosen 'sigma points' that represent the distribution. Both of these approaches have their benefits and will usually provide decent performance. However, neither of these estimators utilize symmetries that can exist in nonlinear dynamic systems.

Symmetries in dynamic systems represent quantities that remain unchanged by certain transformations. For this work, we are interested in continuous transformations, which can be represented by Lie groups. For dynamic models which represent the motion of physical systems, it is common that there exists symmetries corresponding to relative position and orientation. In other words, the dynamics are invariant with respect to translations and rotations of the coordinate system. The goal of invariant observers is to leverage this invariance information to improve estimation performance.

We are primarily interested in applications related to mobile robotic systems. Unmanned systems are quickly becoming an important part of everyday life. There is a wide range of civilian, commercial and military applications that already use unmanned systems. Aerial video and photography, facility inspection and monitoring, search and rescue and aerial delivery are just of few of many applications. As these mobile systems become more ingrained in our society, taking on more responsibilities, they will need to be capable of navigating in dynamic environments. Unmanned aerial vehicles (UAVs) flying in the presence of wind and autonomous underwater vehicles (AUVs) navigating in ocean currents are examples of such systems. For systems in these situations, the ability to simultaneously estimate the system state and disturbance information will be valuable. Invariant observers are well suited for this application due to the fact that most mobile robotic systems possess some inherent symmetry, as stated above. Our interest is to determine whether we can utilize the concept of system invariance to design novel observers capable of simultaneously estimating the full state and disturbances online based on impartial noisy measurements.

1.2 Literature Review

Considering system symmetries for observer design can be traced back to the early 2000s. The idea of using an invariant error was first proposed in [1]. The idea was motivated from a system describing a chemical process where the output was a ratio of concentrations, and therefore invariant to scaling of the individual components. Around this time there was also considerable interest designing observers on the special orthogonal group for attitude estimation of spacecraft [24] and UAVs [23], [14], [20], [21]. A general framework for designing symmetry preserving observers was given in [8], [9]. This approach, the details of which will be given in the next chapter, provides a geometric framework for designing nonlinear observers with interesting convergence properties, due to the use of an invariant error. This theory led directly to the development of a symmetry preserving EKF, known as the Invariant EKF or IEKF [11]. The main benefit of this design is that the observer gain matrix, converges to constant values, around so-called 'permanent trajectories', because by design the A and C matrices become constant around such trajectories.

Since then, the IEKF has gained attention as a tool well suited for applications in localization of mobile robots and sensor fusion for navigation of unmanned aerial vehicles. In [11] the authors apply the IEKF to the problem of estimating the attitude and velocity of an aircraft using GPS velocity and measurements from on board gyroscopes and accelerometers. The authors in [25] design a symmetry preserving observer for fusing measurements from several sensors in different coordinate frames for attitude heading systems for aircraft. Reference [3] develops an IEKF for use with a low cost Kinect depth camera to perform Scan-Matching aided localization of a mobile ground robot. They compare the performance of the IEKF to the Multiplicative EKF (MEKF) and show that the IEKF has better performance. In [12] the authors apply the IEKF to the problem of relative localization for multiple mobile robots. Reference [30] uses the IEKF in a visual inertial navigation system. In [31] the authors show that an IEKF based SLAM (simultaneous localization and mapping) algorithm has better consistency and convergence properties over other EKF based SLAM techniques.

Furthermore, the success of the IEKF has resulted in continuing research into symmetric properties of nonlinear systems and how to exploit them for observer design. The authors in [29] provide checkable sufficient conditions on kinematic systems with symmetries to determine whether considered systems can be lifted to invariant systems on symmetry groups. In [10] a separation principle for invariant systems on Lie groups is established which holds for a larger set of time-varying trajectories than for the typical nonlinear case. The authors in [4] propose a matrix Lie group framework for the IEKF and show that for a class of systems referred to as 'group affine', the Lie logarithm of the invariant error obeys a true linear system. The authors use this fact to prove local stability around any trajectory. In [5] the authors generalize the idea of linear systems to include systems defined on a general group. They show that in this context invariant systems could be viewed as pure integrators and therefore, constitute a more restrictive class of systems than those that can possess linear qualities. In [22] the authors provide a unifying theory that connects invariant, group affine and equivariant systems on Lie groups. They prove that any kinematic system defined on a Lie group can be embedded into another system by extending the input space through a process they call the equivariant input extension. They also provide a filter design, known as the Eq Filter, which for group affine or invariant systems specializes directly to the IEKF.

1.3 Contribution

The contribution of this thesis is as follows. First, we expand upon the theory of invariant systems by developing two sets of sufficient conditions that preserve the invariance of systems under dynamic disturbances. Next, we propose a first order approximation of the standard filtering covariance matrices to more accurately represent the uncertainties needed for the IEKF. Then, using the developed theory, we provide two IEKF designs for a unicycle robot with disturbances that correspond to the two sets of sufficient conditions identified. We provide simulation results that compare the performance of both designs to the EKF and results that demonstrate the benefit of the proposed covariance approximation. In addition to the two designs, we also show that the same dynamic model can be embedded into a matrix Lie group framework that results in a third IEKF design.

1.4 Outline

This thesis is organized as follows. In Chapter II we review some topics related to this research. This includes the Extended Kalman Filter, Lie Groups, Invariant Systems and approaches to nonlinear observer design that incorporate symmetries. In Chapter III we present several ideas for designing augmented IEKFs. In 3.1 are three propositions for preserving the invariant properties of systems when disturbances are applied. In 3.2 we address the issue of rotating the IEKF filter covariances. In Chapter IV we apply the pre-existing theory from Chapter II and the developed theory from Chapter III, to the problem of estimating the state and disturbances for a unicycle robot. We provide two IEKF designs based on the approaches in 3.1 and a third IEKF design based on the matrix IEKF formulation reviewed in 2.4. Then, we provide simulation results and discussion that compare the performance of the three IEKF designs with that of the traditional EKF.

CHAPTER II

CONCEPT OVERVIEW

In this section we provide an overview of concepts that are relevant to this research.

2.1 Extended Kalman Filter (EKF)

As previously mentioned, the Extended Kalman Filter or EKF is the widely accepted standard for filtering of nonlinear systems. It takes the formulation of the original linear Kalman Filter and performs a linearization at each step about the current state estimate. Consider the following general nonlinear system:

$$x_k = f(x_{k-1}, u_k) + w_k \tag{1}$$

$$y_k = h(x_k, u_k) + v_k \tag{2}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^q$ is the input and $y \in \mathbb{R}^p$ is the output. w and v are additive zero mean Gaussian noises with covariances Q and R, respectively. The system is assumed to be observable. The discrete time formulation is broken into two steps: the prediction and correction. Let $\hat{x}$ be the estimate of x and P be the corresponding covariance. The following equations are implemented at every time step.

Prediction:

$$\hat{x}_k^- = f(\hat{x}_{k-1}^+, u_k) \tag{3}$$

$$P_k^- = \Phi_{k-1} P_{k-1}^+ \Phi_{k-1}^\top + Q \tag{4}$$

where

$$\Phi_{k-1} = \left.\frac{\partial f}{\partial x}\right|_{\hat{x}^+_{k-1}} \tag{5}$$

Correction

$$K_k = P^-_k C^\top_k \left(C_k P^-_k C^\top_k + R\right)^{-1} \tag{6}$$

$$\hat{x}^+_k = \hat{x}_{k-1} + K_k \left(y - h(\hat{x}_k, u_k)\right) \tag{7}$$

$$P^+_k = \left(I - K_k C_k\right) P^-_k \tag{8}$$

where

$$C_k = \left.\frac{\partial h}{\partial x}\right|_{\hat{x}^-_k} \tag{9}$$

The EKF works by combining information of the system dynamics with available measurements. It does this through the Kalman gain, K, which is calculated as the optimal gain that minimizes a quadratic cost function of the Q and R matrices. Therefore, it encodes information about the relative uncertainties of the model and the measurements. The EKF is a recursive algorithm that only requires information from the previous time step to work. This is desirable as it does not require storing large amounts of data, even for systems that need to run for large periods of time. These equations are relevant, because the IEKF uses the same general structure, with some changes.

Although the EKF is the most widely used nonlinear filter, it is not the only approach available. Another technique is the Unscented Kalman Filter or UKF. The UKF has a prediction and correction step and the gain minimizes the same quadratic cost, however it does not utilize a linearization process. Instead, it uses a deterministic set of 'sigma points' that captures the mean, covariance and potentially higher moments of the distribution. These sigma points are propagated through the nonlinear dynamics and measurement equations and the posterior distributions are calculated using the Unscented Transform.

2.2 Lie Groups and Invariant Systems

In the mathematical field of group theory, a group is simply a collection of things that have common properties. More formally, a group is a set of mathematical objects along with an operation that must satisfy four axioms: closure, associativity, identity and inverse. Groups can be continuous or discrete. Lie groups are a special type of group. They are continuous differentiable manifolds where the group operations of multiplication and inversion are smooth maps.

Every Lie group has a corresponding Lie algebra, which is a vector space that is the tangent space to the group at the identity element. Lie algebras have the same dimensions as their Lie group. Let G be an n dimensional Lie group and $\mathcal{G}$ be its Lie algebra. The Lie algebra maps to the group by the matrix exponential such that $\forall a \in \mathcal{G}, \exists b \in G,$ s.t. $b = \text{expm}(a)$. There is also a linear mapping $\mathcal{L}_g : \mathbb{R}^n \mapsto \mathcal{G}$ which we will use later on. This allows for a mapping from $\mathbb{R}^n$ to the group G. For example, let $\xi \in \mathbb{R}^n$. Then $\exists\ g \in G,$ s.t. $g = \text{expm}\left(\mathcal{L}_g(\xi)\right)$.

Consider the general nonlinear system

$$\dot{x} = f(x, u) \tag{10}$$

$$y = h(x, u) \tag{11}$$

where the state x belongs to a general manifold Σ. Let G be a Lie group and denote an element by $g \in G$. Define group actions on the states, inputs and outputs respectively as

$$X = \varphi_g(x) \qquad U = \psi_g(u) \qquad Y = \varrho_g(y). \tag{12}$$

From [8], the dynamics are said to be *invariant* if

$$f(\varphi_g(x), \psi_g(u)) = \frac{\partial}{\partial x}\varphi_g(x) f(x, u). \tag{13}$$

The above equation can be understood as a statement that the original nonlinear dynamics remain satisfied, when the states and inputs have undergone a transformation.

The outputs are said to be *equivariant* if

$$\varrho_g(y) = h(\varphi_g(x), \psi_g(u)). \tag{14}$$

Equivalently, the conditions (13),(14) represented by the transformed variables simply reads,

$$\dot{X} = f(X, U) \tag{15}$$

$$Y = h(X, U). \tag{16}$$

Conceptually, this means that the original system dynamics are unchanged by a transformation of variables defined by a Lie group. This can help provide some intuition into whether certain systems possess symmetries. As previously mentioned mobile robotic systems usually possess symmetries with regard to position and orientation. This is due to the fact that the dynamics of these systems do not explicitly depend on a globally defined position or attitude.

2.3 Symmetry Preserving Observers

If a system satisfies conditions (13),(14) with respect to a certain group G, then a symmetry preserving observer can be constructed using the method given in [8]. Below is a summary of the method presented in that paper.

Assume that (10)–(11) with transformations defined in (12) satisfies (13),(14). First, split the transformation on the states $\varphi_g(x)$ into $\varphi_g^a(x)$ and $\varphi_g^b(x)$ such that $\varphi_g^a(x)$ is invertible with respect to g. Solving

$$\varphi_g^a(x) = c \tag{17}$$

for g, where c is a constant, results in $g = \gamma(x)$, a mapping $\gamma : \Sigma \to G$, known as the moving frame. Next, a complete set of invariants can be found by

$$I(\hat{x}, u) = \left(\varphi^b_{\gamma(\hat{x})}(\hat{x}), \psi_{\gamma(\hat{x})}(u)\right). \tag{18}$$

The invariant output error is calculated from

$$E(\hat{x}, u, y) = \varrho_{\gamma(\hat{x})}(h(\hat{x}, u)) - \varrho_{\gamma(\hat{x})}(h(x, u)). \tag{19}$$

Then the invariant frame is given by

$$W(\hat{x}) = [w_1, ..., w_n] \tag{20}$$

where

$$w_i = \left(\frac{\partial}{\partial x}\varphi_{\gamma(x)}(x)\right)^{-1} \cdot \frac{\partial}{\partial x_i}, \quad i = 1, ..., n \tag{21}$$

and $\frac{\partial}{\partial x_i}$ is a basis for Σ. The invariant frame $W(\hat{x})$ is a set of n independent G-invariant vector fields. Then the general form of an invariant observer equation is

$$\dot{\hat{x}} = f(\hat{x}, u) + W(\hat{x})LE(\hat{x}, u, y). \tag{22}$$

Define the invariant state error as

$$\sigma = \varphi_{\gamma(\hat{x})}(x) - \varphi_{\gamma(\hat{x})}(\hat{x}). \tag{23}$$

A remarkable result from this theory is that for an observer with equation (22) and error coordinate defined by (23), the invariant error dynamics can be represented by the general function

$$\dot{\sigma} = \Gamma(\sigma, I(\hat{x}, u)) \tag{24}$$

which is only a function of the invariant error σ and the previously defined invariants (18). For nonlinear systems which possess symmetries, this result provides a different approach to analyzing the convergence of the errors. For the purposes of our research, at this point, we take the dynamics of (24) and linearize around $\sigma = 0$ to get a matrix for use in calculating the gain matrix L, from the standard Riccati equation of the EKF.

2.4 Invariant Extended Kalman Filter (IEKF)

In the previous section, the IEKF was formulated by considering the group action of G on a system defined on a general manifold. If the original system dynamics can instead be defined directly on a matrix Lie group then a matrix formulation of the IEKF may be used. This requires that the state be written as an element of the group, $\mathcal{X} \in G$. The dynamics now become

$$\dot{\mathcal{X}} = f(\mathcal{X}) \tag{25}$$

where $f()$ is now a function that encodes the original system dynamics, but maps from the group to the tangent space of the group. The theory from this section is provided by [4].

In the paper, the authors extend the notion of linearity to include dynamic systems defined on Lie groups and determine a sufficient condition referred to as the group affine condition. In this paper the left invariant and right invariant errors are defined similarly as

$$\eta^L = \mathcal{X}^{-1}\widehat{\mathcal{X}} \tag{26}$$

$$\eta^R = \widehat{\mathcal{X}}\mathcal{X}^{-1}. \tag{27}$$

There is a theorem that states, for $a, b \in G$ if the following group affine condition is satisfied

$$f(ab) = f(a)b + af(b) - af(I)b \tag{28}$$

then the dynamics of the left or right invariant errors are independent of the state trajectory.

In addition to the group affine condition, the measurements must also be able to be written as left invariant

$$Y = \mathcal{X}d_1, ..., \mathcal{X}d_p \tag{29}$$

or right invariant

$$Y = \mathcal{X}^{-1}d_1, ..., \mathcal{X}^{-1}d_p \tag{30}$$

where d_i are known vectors.

Moving forward, we will just include the specifics for left invariant systems, since that is what was applied for this research, however there is a parallel formulation for right invariant systems.

For a system defined as in (25) that satisfies (28) and has left invariant measurements, the following matrix formulation of the IEKF can be applied.

Prediction:

$$\frac{d}{dt}\widehat{\mathcal{X}} = f(\widehat{\mathcal{X}}) \tag{31}$$

Correction:

$$\widehat{\mathcal{X}}^{+} = \widehat{\mathcal{X}}\text{expm}\left(\mathcal{L}_g\left(L\begin{pmatrix}\widehat{\mathcal{X}}^{-1}Y_1 - d_1 \\ \vdots \\ \widehat{\mathcal{X}}^{-1}Y_p - d_p\end{pmatrix}\right)\right). \tag{32}$$

Alternatively, if we let

$$\xi_c = L\begin{pmatrix}\widehat{\mathcal{X}}^{-1}Y_1 - d_1 \\ \vdots \\ \widehat{\mathcal{X}}^{-1}Y_p - d_p\end{pmatrix} \tag{33}$$

be a correction vector, then the correction step could be seen as

$$\widehat{\mathcal{X}}^{+} = \widehat{\mathcal{X}}\text{expm}\left(\mathcal{L}_g(\xi_c)\right) \tag{34}$$

or simply as

$$\widehat{\mathcal{X}}^{+} = \widehat{\mathcal{X}}\mathcal{X}_c \tag{35}$$

an intuitive correction in the context of Lie groups.

The dynamics of the invariant error are given by

$$\dot{\eta} = f(\eta) - f(I)\eta. \tag{36}$$

Define a vector $\xi \in \mathbb{R}^n$ by $\eta = \text{expm}(\mathcal{L}_g(\xi))$, which is the Lie logarithm of the invariant error. One of the remarkable results from [4] is that if the group affine condition is satisfied, the measurements are invariant and an update equation of the form of (22) is used, then the Lie logarithm of the invariant error follows truly linear dynamics. Thus,

$$\dot{\xi} = A\xi \tag{37}$$

$$\xi^+ = \xi + L \begin{pmatrix} -\mathcal{L}_g(\xi)d_1 \\ \vdots \\ -\mathcal{L}_g(\xi)d_p \end{pmatrix} \tag{38}$$

is an underlying linear error system for which the Kalman gain L can be calculated. It is this knowledge that allowed the authors to prove local stability of the IEKF for any trajectory.

In addition to the two previously mentioned formulations, the authors in [22] provided their own formulation called the equivariant or Eq Filter. This design is applicable to equivariant systems on Lie groups. However, in the paper they also provide a method of embedding any kinematic system on a Lie group into an equivariant system by means they call an *Equivariant Input Extension*. Therefore, they propose that their design can be applied to any kinematic system defined on a Lie group. To the author's knowledge, the definition of *equivariance* in [22] is equivalent to the definition of *invariance* in [8]. The proposed Eq Filter, may differ in the sense that no explicit condition on the measurements is stated. Instead, the resulting filter derivation could rely on a linearization of the measurement equations evaluated at the current estimate of the state. Nevertheless, it is a new approach to the design of symmetry preserving observers that deserves attention.

CHAPTER III

AUGMENTED IEKF DESIGNS

In this chapter, we provide some theoretical contributions related to invariant systems and the IEKF. In the first section, we identify two sets of sufficient conditions that preserve the invariant properties of systems under additive dynamic disturbances. In the second section, we propose a correction to the IEKF covariances to better represent uncertainties in the invariant frame.

3.1 Augmenting Invariant Systems

Consider the following nonlinear system

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned} \tag{39}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^q$ and $y \in \mathbb{R}^p$. Let G be an n dimensional Lie group, such that $g \in G$, and define local transformations on the state and input by $\varphi_g(x)$ and $\psi_g(u)$, respectively. By definition, the system (39) is *invariant* with respect to G if $f(\varphi_g(x), \psi_g(u)) = \frac{\partial}{\partial x}\varphi_g(x) f(x,u)$ for all g, x and u. Similarly, the output is said to be *equivariant* with respect to G if there exists a transformation ϱ_g such that $h(\varphi_g(x), \psi_g(u)) = \varrho_g(h(x,u))$ [8].

Assumption 1 *The system* (39) *is invariant with respect to the transformations* $\varphi_g(x)$ *and* $\psi_g(u)$.

Consider (39) cascaded with nonlinear dynamic disturbances d

$$\begin{aligned} \dot{x} &= f(x,u) + Cd \\ \dot{d} &= J(d) \\ y &= h(x,u) \end{aligned} \tag{40}$$

where $d \in \mathbb{R}^m$, $C \in \mathbb{R}^{n\times m}$ and $J(\cdot)$ is a smooth nonlinear function. Note that we choose to write the disturbances affecting the states as Cd instead of an arbitrary nonlinear function $g(d)$. The nonlinear disturbance model ($\dot{d} = J(d),\ z = Cd$) is general since a nonlinear dynamical system with nonlinear outputs of full row-rank can be converted to a system with linear outputs through a nonlinear coordinate transformation, e.g., based on its normal forms [17, Section 13.2] [28].

Define two transformations $\beta_g(C) :\ G\times\mathbb{R}^{n\times m} \mapsto \mathbb{R}^{n\times m}$ and $\xi_g(d) :\ G\times\mathbb{R}^m \mapsto \mathbb{R}^m$. We next derive sufficient conditions on $\beta_g(C)$ and $\xi_g(d)$ such that the cascaded system (40) remains invariant under the group actions $(\varphi_g(x),\ \psi_g(u),\ \beta_g(C),\ \xi_g(d))$. We do this by proposing two different approaches outlined in Proposition 1 and Proposition 2. Proposition 1 takes $\xi_g(\cdot)$ to be the identity operator and examines invariance conditions on $\beta_g(\cdot)$. Proposition 2 takes $\beta_g(\cdot)$ to be the identity operator and examines invariance conditions on $\xi_g(\cdot)$. Our results rely on the following assumption.

Assumption 2 *$\varphi_g(x)$ and $\xi_g(d)$ are linear in x and d, respectively.*

We define

$$\alpha(g) = \frac{\partial}{\partial x}\varphi_g(x) \in \mathbb{R}^{n\times n} \tag{41}$$

$$\kappa(g) = \frac{\partial}{\partial d}\xi_g(d) \in \mathbb{R}^{m\times m}. \tag{42}$$

Proposition 1 *Suppose that Assumption 1 and 2 hold. Then* (40) *is invariant with respect to G if $\beta_g(C)$ and $\xi_g(d)$ are selected as $\beta_g(C) = \alpha(g)C$ and $\xi_g(d) = d$, respectively.*

Proof. It follows from the definition of invariance that (40) is invariant if the following two equations hold:

$$f(\varphi_g(x), \psi_g(u)) + \beta_g(C)\xi_g(d) = \frac{\partial}{\partial x}\varphi_g(x)\left(f(x,u) + Cd\right) \tag{43}$$

$$J(\xi_g(d)) = \frac{\partial}{\partial d}\xi_g(d)J(d). \tag{44}$$

Let $\xi_g(d) = d$. Then (44) is trivially satisfied. Since $f(x,u)$ is invariant with respect to G, (43) reduces to

$$\beta_g(C)d = \frac{\partial}{\partial x}\varphi_g(x)Cd = \alpha(g)Cd. \tag{45}$$

Since $\beta_g(C)$ can only be a function of g, it follows from (45) that $\frac{\partial}{\partial x}\varphi_g(x)$ cannot be a function of x, which means that $\varphi_g(x)$ is linear in x. Therefore, invariance with respect to G is preserved by leaving the disturbances (d) unchanged and transforming C with a transformation defined by $\beta_g(C) = \alpha(g)C$. ■

From the proof, we see that invariance can be preserved in the augmented system (40) by performing a transformation on the system parameter C, instead of on the disturbances d. In Proposition 2 below, we preserve the invariance property by performing a transformation directly on d instead of on C.

Proposition 2 *Suppose that Assumption 1 and 2 hold. Then* (40) *is invariant with respect to* G *if* $\beta_g(C)$ *and* $\xi_g(d)$ *satisfy* $\beta_g(C) = C$, $C\kappa(g) = \alpha(g)C$ *and* $J(\kappa(g)d) = \kappa(g)J(d)$.

Proof. Let $\beta_g(C) = C$. From Assumption 1 it follows that $\varphi_g(x) = \alpha(g)x$ and $\xi_g(d) = \kappa(g)d$. Then (43) reduces to $C\kappa(g)d = \alpha(g)Cd$ which implies that $C\kappa(g) = \alpha(g)C$ must be satisfied. The second equation (44) becomes $J(\kappa(g)d) = \kappa(g)J(d)$. ■

Proposition 1 and 2 provide two approaches to defining transformations that preserve invariance of a nonlinear system when state dynamic disturbances are included.

Both approaches assume that the original group action is linear with respect to the states. Motivated by internal model control and disturbance rejection literature (see e.g., [15], [16]), we next focus on dynamic disturbances resulting from a linear dynamic model, i.e., $J(d) = Ad$, $A \in \mathbb{R}^{m\times m}$. In this case, the conditions in Proposition 2 become $C\kappa(g) = \alpha(g)C$ and $A\kappa(g) = \kappa(g)A$, the second signifying that the Lie bracket of the vector fields Ad and $\xi_g(d)$ must be zero. Assuming that the disturbances affecting the individual dimensions of x share the same generating model, i.e., $J(d) = Ad = (I_n \otimes \mathcal{A})d$, Proposition 3 below establishes a sufficient condition on $(C, \alpha(g))$ such that the augmented system (40) remains invariant.

Proposition 3 *Suppose that Assumption 1 and 2 hold. Suppose that* $J(d) = Ad = (I_n \otimes \mathcal{A})d$ *and* $C = blkdiag\{I_s \otimes \mathcal{C}, 0_{(n-s)\times \frac{m(n-s)}{n}}\}$, $0 < s \leq n$, *where* $\mathcal{A} \in \mathbb{R}^{\frac{m}{n}\times\frac{m}{n}}$ *and* $\mathcal{C} \in \mathbb{R}^{1\times\frac{m}{n}}$. *Assume that* $\alpha(g)$ *has the following form*

$$\alpha(g) = \begin{pmatrix} \alpha_1(g) & \alpha_2(g) \\ 0 & \alpha_3(g) \end{pmatrix}, \tag{46}$$

where $\alpha_1(g) \in \mathbb{R}^{s\times s}$, $\alpha_2(g) \in \mathbb{R}^{s\times(n-s)}$ *and* $\alpha_2(g) \in \mathbb{R}^{(n-s)\times(n-s)}$. *Then* (40) *is invariant with respect to* G *by choosing* $\beta_g(C) = C$ *and*

$$\kappa(g) = \begin{pmatrix} \alpha_1(g) \otimes I_{\frac{m}{n}} & 0 \\ 0 & 0 \end{pmatrix}. \tag{47}$$

Proof. Using Kronecker product properties and the forms of C and $\alpha(g)$, we have

$$\alpha(g)C = \begin{pmatrix} \alpha_1(g)(I_s \otimes \mathcal{C}) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \alpha_1(g) \otimes \mathcal{C} & 0 \\ 0 & 0 \end{pmatrix}, \tag{48}$$

which can be further rewritten as

$$\begin{pmatrix} \alpha_1 \otimes \mathcal{C} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (I_s \otimes \mathcal{C})(\alpha_1 \otimes I_{\frac{m}{n}}) & 0 \\ 0 & 0 \end{pmatrix} = C\kappa, \tag{49}$$

where we have dropped the dependency of $\alpha_1(g)$ and $\kappa(g)$ on g. Similarly, we verify that

$$(I_n \otimes \mathcal{A})\kappa = \text{blkdiag}\{(I_s \otimes \mathcal{A})(\alpha_1 \otimes I_{\frac{m}{n}}), 0\} \tag{50}$$

$$= \text{blkdiag}\{\alpha_1 \otimes \mathcal{A}, 0\} \tag{51}$$

$$= \text{blkdiag}\{(\alpha_1 \otimes I_{\frac{m}{n}})(I_s \otimes \mathcal{A}), 0\} \tag{52}$$

$$= \kappa(I_n \otimes A). \tag{53}$$

Thus, the invariant conditions in Proposition 2 are satisfied with the choice of $\kappa(g)$ in (47). ∎

In Proposition 3, the first s elements $(0 < s \leq n)$ of x are affected by the disturbances. The disturbance affecting each element of x is generated from the same dynamic system specified by $(\mathcal{A}, \mathcal{C})$ with possibly different initial conditions. When $s = n$, Proposition 3 holds for any $\alpha(g)$. When $s < n$, $\alpha(g)$ needs to satisfy (46) to ensure invariance. The condition (46) means that after the transformation $\varphi_g(x)$, the last $n - s$ elements of x remain unaffected by the disturbances.

In Chapter IV, we employ Proposition 1 and 2 to design invariant EKFs for a unicycle robot under linear dynamic disturbances and compare their performance using simulations. We show that 1) IEKF 1 (the design based on Proposition 1) is applicable and provides improvement over a standard EKF; 2) If the invariant conditions in Proposition 2 are satisfied, IEKF 2 (based on Proposition 2) should be considered since it can provide further improvement in transient performance. We also show that when applied to the unicycle robot problem, Proposition 2 allows a broader class of dynamic disturbances than Proposition 3.

3.2 Rotated Covariances

Instead of using the typical linear output error used by the EKF, the IEKF uses an invariant output error, which is defined as a group action on the output space. Because of this, the measurement covariance, R, no longer accurately represents the uncertainty of the transformed output. Therefore, we propose that the original measurement covariance should be transformed to better represent the correct measurement covariance.

Let $X = [x_1, ..., x_n]^\top$ be the full system state. The general equation for the invariant output error is given in (19). Here we choose to define it more specifically as

$$E = T(\hat{X})\left(\hat{Y} - Y\right). \tag{54}$$

where we make the assumption that it can be written as the product of a matrix and the linear error. This is not the most general case, however it is still applicable for certain systems, including the unicycle model used in the next chapter.

We now derive the transformation rule for the measurement noise matrix R. We use the notation $\mathcal{N}(\mu, \Sigma)$ to denote the Gaussian distribution with mean μ and covariance Σ.

Proposition 4 *Let* $\epsilon = \hat{Y} - Y \sim \mathcal{N}(0, R)$. *Let* $\hat{X}$ *be the estimate of* X *such that* $\delta X = \hat{X} - X \sim \mathcal{N}(0, P)$. *We assume that* ϵ *and* δX *are uncorrelated. Then, the covariance of the invariant output error up to first order accuracy is given by*

$$cov(T(\hat{X})\epsilon) = T(X)RT(X)^\top + \sum_{i=1}^{n}\sum_{j=1}^{n} P_{ij}\frac{\partial T}{\partial x_i}R\frac{\partial T}{\partial x_j}^\top \tag{55}$$

for a sufficiently small P.

Proof. We have

$$\mathrm{cov}(T(\hat{X})\epsilon) = \mathrm{cov}(T(X + \delta X)\epsilon),$$

where $\epsilon \sim \mathcal{N}(0, R)$. Since $\delta X \sim \mathcal{N}(0, P)$ with P sufficiently small and since δX is uncorrelated with ϵ, we use the first order approximation and obtain

$$\mathbb{E}(T(X + \delta X)\epsilon) = \mathbb{E}\left(\left(T(X) + \frac{\partial T}{\partial x_1}\delta x_1 + ... + \frac{\partial T}{\partial x_n}\delta x_n\right)\epsilon\right) = 0$$

The covariance of $T(\hat{X})\epsilon$ is computed as

$$\operatorname{cov}(T(\hat{X})\epsilon) = \mathbb{E}(T(\hat{X})\epsilon\epsilon^\top T(\hat{X})^\top) \tag{56}$$

$$= \mathbb{E}\left(\left(T(X) + \frac{\partial T}{\partial x_1}\delta x_1 + ... + \frac{\partial T}{\partial x_n}\delta x_n\right)\epsilon\epsilon^\top \left(T(X) + \frac{\partial T}{\partial x_1}\delta x_1 + ... + \frac{\partial T}{\partial x_n}\delta x_n\right)^\top\right) \tag{57}$$

$$= T(X)RT(X)^\top + \mathbb{E}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial T}{\partial x_i}\delta x_i\epsilon\epsilon^\top\delta x_j^\top\frac{\partial T}{\partial x_j}^\top\right) \tag{58}$$

$$= T(X)RT(X)^\top + \sum_{i=1}^{n}\sum_{j=1}^{n}P_{ij}\frac{\partial T}{\partial x_i}R\frac{\partial T}{\partial x_j}^\top \tag{59}$$

Because δX is zero mean with a covariance P, it follows from [13] that $\mathbb{E}(\delta x_i\epsilon\epsilon^\top\delta x_j^\top) = P_{ij}R$ and thus (55) is recovered. ■

This same process can be applied to find the transformation rule for the initial state covariance P_0, where the initial state error at time $t = 0$ is given by $\eta_0 = \hat{X}_0 - X_0 \sim \mathcal{N}(0, P_0)$. For finding the transformation law for the process noise matrix, Q, the same steps are taken where instead of $T(\hat{X})$, we use $W(\hat{X})^\top$. Suppose that the process noise is given by $\nu \sim \mathcal{N}(0, Q)$. Extending the process in the proof of Proposition 4 results in the transformations for the state covariance and process noise matrices given by

$$\operatorname{cov}(W(\hat{X}_0)^\top\eta_0) \approx W(X_0)^\top P_0 W(X_0) + \sum_{i=1}^{n}\sum_{j=1}^{n}P_{ij}\frac{\partial W}{\partial x_i}^\top P_0\frac{\partial W}{\partial x_j} \tag{60}$$

$$\operatorname{cov}(W(\hat{X})^\top\nu) = W(X)^\top QW(X) + \sum_{i=1}^{n}\sum_{j=1}^{n}P_{ij}\frac{\partial W}{\partial x_i}^\top Q\frac{\partial W}{\partial x_j}. \tag{61}$$

The transformation on the initial state covariance, (60), is an approximation, due to the correlation between $\hat{X}_0$ and η_0. In implementation, for all transformations we replace X with its estimate $\hat{X}$, assuming that they are close. Note that [4] uses only the first term in (55) in their examples (Section IV-B-3), which corresponds to the zeroth-order approximation of the covariance. Through simulations in Section

4.5, we demonstrate the significant improvement due to the second term when the measurement noise is non-isotropic.

Having found the rotated covariances, we present the IEKF algorithm in Algorithm 1 below. Algorithm 1 follows the standard steps of an EKF except line 2, 7, and 9 where the covariance matrices are modified, line 5 where the linearized A_k is computed based on the invariant error dynamics (see A_k in (80) and (96) for the two IEKF designs), and line 11 where the update equation is modified with transformations of the innovation.

Algorithm 1 The IEKF

1: Initialize $\hat{X}_0$, P_0 in the original coordinates.

2: $P = W(\hat{X}_0)^\top P_0 W(\hat{X}_0) + \sum_{i=1}^n \sum_{j=1}^n P_{ij} \frac{\partial W}{\partial x_i}^\top P_0 \frac{\partial W}{\partial x_j}$

3: **for** $k = 1$ **to** n **do**

4: $\quad \hat{X}_k^- = f(\hat{X}_{k-1}^+, U)$

5: $\quad$ Compute A_k

6: $\quad$ Compute H_k

7: $\quad Q_{rot} = W(\hat{X})^\top Q W(\hat{X}) + \sum_{i=1}^n \sum_{j=1}^n P_{ij} \frac{\partial W}{\partial x_i}^\top Q \frac{\partial W}{\partial x_j}$

8: $\quad P_k^- = A_k P_{k-1}^+ A_k^\top + Q_{rot}$

9: $\quad R_{rot} = T(\hat{X}) R T(\hat{X})^\top + \sum_{i=1}^n \sum_{j=1}^n P_{ij} \frac{\partial T}{\partial x_i} R \frac{\partial T}{\partial x_j}^\top$

10: $\quad L_k = P_k^- H_k^\top \left(H_k P_k^- H_k^\top + R_{rot}\right)^{-1}$

11: $\quad \hat{X}_k^+ = \hat{X}_k^- + W(\hat{\theta}) L_k T(\hat{\theta})(Y - h(\hat{X}_k^-, U))$

12: $\quad P_k^+ = (I - L_k H_k) P_k^-$

13: **end for**

CHAPTER IV

UNICYCLE ROBOT UNDER LINEAR DISTURBANCES

In this chapter, we explore three IEKF designs for a unicycle model subject to disturbances modeled as the output of a linear time-invariant system. A unicycle robotic model is widely used to model the kinematics of a differential drive mobile vehicle, underwater vehicle motion [27], and the simplified kinematics of a fixed wing aerial vehicle in planar flight [6]. Furthermore, some applications include estimating the states of these types of vehicles for the purpose of localization [7], trajectory tracking [18], or flow field reconstruction [2], [26]. The linear disturbance models can represent uniform flow and sinusoidal wave disturbances with known frequencies.

We design three augmented IEKFs for the unicycle to estimate both its heading and disturbance based on position information. The first two designs are based on the two identified scenarios where the augmented dynamics are invariant. We show that the first design is applicable to general linear dynamic disturbances while the second design is restricted to a class of systems that satisfy 'rotational invariance' conditions on the dynamics and the output matrices. The third design is based on a matrix implementation of the IEKF, where the system dynamics are defined on a matrix Lie group and the observer state is an element of the defined group.

4.1 Problem Formulation

Consider a unicycle robot subject to velocity disturbances. The kinematic model of the robot is given by

$$\begin{aligned}\dot{x} &= v\cos\theta + C_x d\\ \dot{y} &= v\sin\theta + C_y d \\ \dot{\theta} &= \omega,\end{aligned} \tag{62}$$

where (x, y) is the position of the robot, θ is the heading, v is the linear velocity and ω is the turning rate. We assume that $(C_x d, C_y d)$ are outputs from a linear system given by

$$\dot{d} = Ad \tag{63}$$

where $d \in \mathbb{R}^{m\times 1}$, $A \in \mathbb{R}^{m\times m}$ and $C_x, C_y \in \mathbb{R}^{1\times m}$. The matrices A, C_x, and C_y are assumed known and constant. For example, $C_x d$ and $C_y d$ can represent constant disturbances and sinusoidal disturbances with known frequencies.

The robot is equipped with a positioning device, such as a GPS or a suite of range and bearing sensors, measuring its position (x, y). The position measurement can be in a global frame or with respect to a known landmark. In the latter case, without loss of generality, we assume that the landmark is at the origin. Then (x, y) represents the relative position between the robot and the landmark. The measurement equation of the system is

$$Y = \begin{bmatrix} x \\ y \end{bmatrix}. \tag{64}$$

We augment (62) with the disturbance dynamics (63) and obtain

$$\begin{aligned}
\dot{x} &= v\cos\theta + C_x d \\
\dot{y} &= v\sin\theta + C_y d \\
\dot{\theta} &= \omega \\
\dot{d} &= Ad.
\end{aligned} \tag{65}$$

In [8], it was shown that the undisturbed form of (62) is invariant with respect to actions of the special Euclidean group $SE(2)$, the group of translations and rotations in 2 dimensions. With the additive disturbances, our objective is to design an IEKF to estimate both the states and the disturbances. In the following two sections, we design two invariant IEKFs that correspond with Propositions 1 and 3 introduced in Chapter III.

4.2 IEKF Design 1

Let G be the group $SE(2)$. Any element g of G can be represented by (x_g, y_g, θ_g). Let $X = [x, y, \theta]^\top$ and define two transformations as

$$\varphi_g(X) = \begin{pmatrix} x\cos\theta_g - y\sin\theta_g + x_g \\ x\sin\theta_g + y\cos\theta_g + y_g \\ \theta + \theta_g \end{pmatrix} \tag{66}$$

$$\xi_g(d) = d. \tag{67}$$

Notice that the disturbances d remain unchanged by the transformation. We now use the result of Proposition 1 to find the transformations on C_x and C_y. Since the disturbances do not affect θ, we concatenate C_x and C_y with a row of zeros and define

$\beta_g(\cdot)$ as

$$\beta_g\left(\begin{bmatrix} C_x \\ C_y \\ 0 \end{bmatrix}\right) = \frac{\partial}{\partial X}\varphi_g(X)\begin{bmatrix} C_x \\ C_y \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta_g & -\sin\theta_g & 0 \\ \sin\theta_g & \cos\theta_g & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} C_x \\ C_y \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} C_x\cos\theta_g - C_y\sin\theta_g \\ C_x\sin\theta_g + C_y\cos\theta_g \\ 0 \end{bmatrix}. \tag{68}$$

Let $U = (v, \omega, A)$ and define a transformation of $\mathcal{U}$ as

$$\psi_g(U) = \begin{pmatrix} v \\ \omega \\ A \end{pmatrix}. \tag{69}$$

Corollary 1 *The system* (65) *is invariant with respect to* $SE(2)$*.*

Proof. Note that the undisturbed system (62) without (d_x, d_y) is invariant with respect to $SE(2)$. With the transformations defined in (67) and (68), it follows from Proposition 1 that the augmented system (65) is invariant with respect to $SE(2)$. ■

Following the methods outlined in [8], $\varphi_g(X)$ can be split into $\varphi_g^a(X)$ and $\varphi_g^b(X)$ such that $\varphi_g^a(X)$ is invertible with respect to g. Setting $\varphi_g^a(X) = 0$ and solving for g result in

$$\begin{pmatrix} x_g \\ y_g \\ \theta_g \end{pmatrix} = \gamma\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} = \begin{pmatrix} -x\cos\theta - y\sin\theta \\ x\sin\theta - y\cos\theta \\ -\theta \end{pmatrix}. \tag{70}$$

The invariants are

$$I(\hat{X}, U) = \left(\varphi^b_{\gamma(\hat{X})}\left(\hat{X}\right), \psi_{\gamma(\hat{X})}(U)\right) =$$
$$\left(v, \omega, C_x \cos\hat{\theta} + C_y \sin\hat{\theta}, -C_x \sin\hat{\theta} + C_y \cos\hat{\theta}, A\right), \tag{71}$$

where $\hat{X}$ is the estimate of X. The invariant output error is given by

$$\begin{aligned} E &= \varrho_g(\hat{x}, \hat{y}) - \varrho_g(x, y) \\ &= \begin{pmatrix} \hat{x}\cos\theta_g - \hat{y}\sin\theta_g + x_g - x\cos\theta_g + y\sin\theta_g - x_g \\ \hat{x}\sin\theta_g + \hat{y}\cos\theta_g + y_g - x\sin\theta_g - y\cos\theta_g - y_g \end{pmatrix} \\ &= T(\hat{\theta}) \begin{bmatrix} \hat{x} - x \\ \hat{y} - y \end{bmatrix}, \end{aligned} \tag{72}$$

where

$$T(\hat{\theta}) = \begin{bmatrix} \cos\hat{\theta} & \sin\hat{\theta} \\ -\sin\hat{\theta} & \cos\hat{\theta} \end{bmatrix}. \tag{73}$$

The invariant frame is given by

$$W(\hat{\theta}) = \begin{bmatrix} T(\hat{\theta})^\top & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_m \end{bmatrix}. \tag{74}$$

Thus, the observer equation has the following form

$$\dot{\hat{X}} = f(\hat{X}) + W(\hat{\theta}) \cdot L \cdot T(\hat{\theta}) \left(Y - \hat{Y}\right), \tag{75}$$

where L is a gain matrix to be designed. For notation convenience, we let

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \\ L_{31} & L_{32} \\ L_{d1} & L_{d2} \end{bmatrix}, \tag{76}$$

where L_{ij} are scalars for $i = 1, 2, 3,\ j = 1, 2$, and $L_{d1}, L_{d2} \in \mathbb{R}^{m\times 1}$. The invariant state error is given by

$$
\begin{aligned}
\sigma(\hat{X}, X) &= \varphi_{\gamma(\hat{X})}(X) - \varphi_{\gamma(\hat{X})}(\hat{X}) \\
&= \begin{pmatrix} x\cos\hat{\theta} + y\sin\hat{\theta} - \hat{x}\cos\hat{\theta} - \hat{y}\sin\hat{\theta} \\ -x\sin\hat{\theta} + y\cos\hat{\theta} + \hat{x}\sin\hat{\theta} - \hat{y}\cos\hat{\theta} \\ \theta - \hat{\theta} \\ d - \hat{d} \end{pmatrix} \\
&= W(\hat{\theta})^\top \begin{bmatrix} x - \hat{x} \\ y - \hat{y} \\ \theta - \hat{\theta} \\ d - \hat{d} \end{bmatrix}.
\end{aligned} \tag{77}
$$

To find the invariant error dynamics, we differentiate (77) and obtain

$$
\begin{aligned}
\dot{\sigma} = W(\hat{\theta})^\top \Bigg[&\begin{pmatrix} v\cos\theta + C_x d - v\cos\hat{\theta} - C_x\hat{d} \\ v\sin\theta + C_y d - v\sin\hat{\theta} - C_y\hat{d} \\ 0 \\ Ad - A\hat{d} \end{pmatrix} \\
&- W(\hat{\theta}) L \begin{bmatrix} \cos\hat{\theta} & \sin\hat{\theta} \\ -\sin\hat{\theta} & \cos\hat{\theta} \end{bmatrix} \begin{bmatrix} x - \hat{x} \\ y - \hat{y} \end{bmatrix} \Bigg] + \begin{bmatrix} \dot{\hat{\theta}}\sigma_y \\ -\dot{\hat{\theta}}\sigma_x \\ 0 \\ 0 \end{bmatrix},
\end{aligned} \tag{78}
$$

which yields

$$
\begin{aligned}
\dot{\sigma}_x &= v\left(\cos\sigma_\theta - 1\right) + \omega\sigma_y + \left(C_x\cos\hat{\theta} + C_y\sin\hat{\theta}\right)\sigma_d \\
&\quad + L_{11}\sigma_x + L_{12}\sigma_y + L_{31}\sigma_x\sigma_y + L_{32}\sigma_y^2 \\
\dot{\sigma}_y &= v\sin\sigma_\theta - \omega\sigma_x + \left(-C_x\sin\hat{\theta} + C_y\cos\hat{\theta}\right)\sigma_d \\
&\quad + L_{21}\sigma_x + L_{22}\sigma_y - L_{31}\sigma_x^2 - L_{32}\sigma_x\sigma_y \\
\dot{\sigma}_\theta &= L_{31}\sigma_x + L_{32}\sigma_y \\
\dot{\sigma}_d &= A\sigma_d + L_{d1}\sigma_x + L_{d2}\sigma_y.
\end{aligned} \tag{79}
$$

Note that the invariant error dynamics (79) depend only on σ and the invariants $I(\hat{X}, U)$ in (71).

Linearizing (79) around $\sigma = 0$ yields the state matrix needed for implementing the IEKF at time step k:

$$
A_k = \begin{bmatrix} 0 & \omega_k & 0 & C_x\cos\hat{\theta}_k + C_y\sin\hat{\theta}_k \\ -\omega_k & 0 & v_k & -C_x\sin\hat{\theta}_k + C_y\cos\hat{\theta}_k \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A \end{bmatrix}. \tag{80}
$$

The A_k matrix is used in the IEKF algorithm to propagate the state covariance matrix. Next, we illustrate a second IEKF design for (65) based on Proposition 3.

4.3 IEKF Design 2

Compared with the design in Section 4.2, this design assumes the same state transformation $\varphi_g(X)$ in (66) and introduces transformations on the disturbances. We define

$$\beta_g\left(\begin{bmatrix} C_x \\ C_y \\ 0 \end{bmatrix}\right) = \begin{bmatrix} C_x \\ C_y \\ 0 \end{bmatrix} \tag{81}$$

$$\xi_g(d) = \left(R(\theta_g) \otimes I_{\frac{m}{2}}\right) d \tag{82}$$

where

$$R(\theta_g) = \begin{bmatrix} \cos\theta_g & -\sin\theta_g \\ \sin\theta_g & \cos\theta_g \end{bmatrix} \tag{83}$$

and $I_{\frac{m}{2}}$ is the identity matrix of dimension $\frac{m}{2}$. Notice that $\xi_g(d)$ is linear in d. Applying Proposition 2, we note that in order to preserve the invariance property, we need

$$\begin{bmatrix} C_x \\ C_y \\ 0 \end{bmatrix} \left(R(\theta_g) \otimes I_{\frac{m}{2}}\right) = \begin{bmatrix} \cos\theta_g & -\sin\theta_g & 0 \\ \sin\theta_g & \cos\theta_g & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_x \\ C_y \\ 0 \end{bmatrix} \tag{84}$$

and

$$A\left(R(\theta_g) \otimes I_{\frac{m}{2}}\right) = \left(R(\theta_g) \otimes I_{\frac{m}{2}}\right) A. \tag{85}$$

Proposition 5 *Equations* (84) *and* (85) *are satisfied if* A *and* $[C_x^\top \; C_y^\top]^\top$ *satisfy*

$$A = \begin{bmatrix} \mathcal{M} & \mathcal{N} \\ -\mathcal{N} & \mathcal{M} \end{bmatrix}, \tag{86}$$

$$\begin{bmatrix} C_x \\ C_y \end{bmatrix} = \begin{bmatrix} \mathcal{D} & \mathcal{E} \\ -\mathcal{E} & \mathcal{D} \end{bmatrix}, \tag{87}$$

where $\mathcal{M}, \mathcal{N} \in \mathcal{R}^{\frac{m}{2}\times\frac{m}{2}}$ *and* $\mathcal{D}, \mathcal{E} \in \mathcal{R}^{1\times\frac{m}{2}}$ *are arbitrary matrices.*

Proof. Let

$$\begin{bmatrix} C_x \\ C_y \end{bmatrix} = \begin{bmatrix} \mathcal{C}_1 & \mathcal{C}_2 \\ \mathcal{C}_3 & \mathcal{C}_4 \end{bmatrix}, \tag{88}$$

where $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4 \in \mathcal{R}^{1\times\frac{m}{2}}$. Then (84) becomes

$$\begin{bmatrix} C_x \\ C_y \end{bmatrix} = \begin{bmatrix} \cos\theta_a & \sin\theta_a \\ -\sin\theta_a & \cos\theta_a \end{bmatrix} \begin{bmatrix} \mathcal{C}_1 & \mathcal{C}_2 \\ \mathcal{C}_3 & \mathcal{C}_4 \end{bmatrix} \begin{bmatrix} I_{\frac{m}{2}}\cos\theta_a & -I_{\frac{m}{2}}\sin\theta_a \\ I_{\frac{m}{2}}\sin\theta_a & I_{\frac{m}{2}}\cos\theta_a \end{bmatrix}.$$

Multiplying the matrices together and simplifying the 2 independent equations lead to

$$\begin{bmatrix} -\sin^2\theta_a & -\cos\theta_a\sin\theta_a \\ \cos\theta_a\sin\theta_a & -\sin^2\theta_a \\ \cos\theta_a\sin\theta_a & -\sin^2\theta_a \\ \sin^2\theta_a & \cos\theta_a\sin\theta_a \end{bmatrix}^\top \begin{bmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \\ \mathcal{C}_3 \\ \mathcal{C}_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{89}$$

Thus, $[\mathcal{C}_1^T, \mathcal{C}_2^\top, \mathcal{C}_3^\top, \mathcal{C}_4^\top]^\top$ must lie in the non-trivial null spaces spanned by

$$\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \tag{90}$$

which means that $\mathcal{C}_1 = \mathcal{C}_4$ and $\mathcal{C}_2 = -\mathcal{C}_3$. Therefore, we have

$$\begin{bmatrix} C_x \\ C_y \end{bmatrix} = \begin{bmatrix} \mathcal{C}_1 & \mathcal{C}_2 \\ -\mathcal{C}_2 & \mathcal{C}_1 \end{bmatrix}. \tag{91}$$

A similar analysis of (85) shows that A must also have the specific form in (86). ■

Thus, the cascaded system (65) remains invariant under the transformations given in (81)–(82) if A and $\begin{bmatrix} C_x^\top & C_y^\top \end{bmatrix}^\top$ satisfy (86) and (87), respectively. Characterizing what linear systems can be transformed to satisfy (86) and (87) is beyond the scope of this thesis. However, we note that an important case where (86) and (87) are satisfied is when $\mathcal{N}$ and $\mathcal{E}$ are zero matrices, which means that the disturbances along the x and y directions are decoupled and share the same dynamic model. The case where $\mathcal{N}$ and $\mathcal{E}$ are zero can also be proved using Proposition 3. Note that Proposition 3 is applicable to any group operations satisfying (46). However, because Proposition 5 is specific to $\varphi_g(X)$ in (66) and the rotation operation (82), it allows $\mathcal{N}$ and $\mathcal{E}$ to be nonzero, thereby encompassing a wider class of disturbance systems than Proposition 3.

For the remainder of the section, we assume that the disturbance subsystem is in the form of (86)–(87). Applying the same process as in Section 4.2, we obtain the observer equation as

$$\dot{\hat{X}} = f(\hat{X}) + W(\hat{\theta}) \cdot L \cdot T(\hat{\theta}) \left(Y - \hat{Y} \right), \tag{92}$$

where $T(\hat{\theta})$ is the same as (73). The invariant frame is now given by

$$W\left(\hat{\theta}\right) = \begin{bmatrix} T(\hat{\theta})^\top & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & T(\hat{\theta})^\top \otimes I_{\frac{m}{2}} \end{bmatrix}. \tag{93}$$

The invariant error is given in (77) with the the invariant frame now defined by (93), where the error of the disturbances is also rotated. This results in the following invariant error dynamics

$$\begin{aligned} \dot{\sigma}_x &= v\left(\cos\sigma_\theta - 1\right) + \omega\sigma_y + C_x\sigma_d + L_{11}\sigma_x + L_{12}\sigma_y + L_{31}\sigma_x\sigma_y + L_{32}\sigma_y^2 \\ \dot{\sigma}_y &= v\sin\sigma_\theta - \omega\sigma_x + C_y\sigma_d + L_{21}\sigma_x + L_{22}\sigma_y - L_{31}\sigma_x^2 - L_{32}\sigma_x\sigma_y \\ \dot{\sigma}_\theta &= L_{31}\sigma_x + L_{32}\sigma_y \\ \dot{\sigma}_d &= A_\omega\sigma_d + L_{d1}\sigma_x + L_{d2}\sigma_y \end{aligned} \tag{94}$$

where

$$A_\omega = A + \left(\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \otimes I_{\frac{m}{2}} \right). \tag{95}$$

Linearizing (94) around $\sigma = 0$ results in the state matrix needed for implementing this IEKF design:

$$A_k = \begin{bmatrix} 0 & \omega_k & 0 & C_x \\ -\omega_k & 0 & v_k & C_y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_\omega \end{bmatrix}. \tag{96}$$

Note that unlike (80), the state matrix given in (96) is not a function of the estimated state $\hat{\theta}$.

4.4 Matrix IEKF

In this section, we derive an IEKF to estimate the states of (62) and the disturbance in (63) and explain the necessary conditions that must be imposed to achieve true linear error dynamics.

Let G be the matrix Lie Group of double direct spatial isometries. We define our system state, $\mathcal{X} \in G$, as

$$\mathcal{X} = \begin{bmatrix} \cos\theta & -\sin\theta & x & d_1^\top \\ \sin\theta & \cos\theta & y & d_2^\top \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I_{\frac{m}{2}} \end{bmatrix} \tag{97}$$

where $I_{\frac{m}{2}}$ is the identity matrix of dimension $\frac{m}{2}$ and $d_1, d_2 \in \mathcal{R}^{\frac{m}{2}\times 1}$ such that

$$d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}. \tag{98}$$

With the state defined as $\mathcal{X}$, the augmented system (65) is represented by defining a matrix function f such that

$$\frac{d}{dt}\mathcal{X} = f\left(\mathcal{X}\right) \tag{99}$$

where

$$f(\mathcal{X}) = \begin{bmatrix} -\omega\sin\theta & -\omega\cos\theta & v\cos\theta + C_x d & d^\top A_1^\top \\ \omega\cos\theta & -\omega\sin\theta & v\sin\theta + C_y d & d^\top A_2^\top \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{100}$$

in which A_1 and A_2 are defined as the top and bottom $\frac{m}{2}$ rows of A, respectively, i.e.,

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}. \tag{101}$$

An IEKF that acts on this matrix state exists only if m is an even number. Thus, the decomposition in (101) is valid. The measurements given in (64) are

$$Y = \mathcal{X} q \tag{102}$$

where $q = [0, 0, 1, 0, \cdots, 0]^\top$. Note that the measurements are left invariant. For any true state, $\mathcal{X}$, and state estimate, $\hat{\mathcal{X}}$, we define the left invariant error as

$$\eta = \mathcal{X}^{-1}\hat{\mathcal{X}} = \begin{bmatrix} \cos\eta_\theta & -\sin\eta_\theta & \eta_x & \eta_{d_1}^\top \\ \sin\eta_\theta & \cos\eta_\theta & \eta_y & \eta_{d_2}^\top \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \tag{103}$$

where

$$\begin{aligned} \eta_x &= (\hat{x} - x)\cos\theta + (\hat{y} - y)\sin\theta \\ \eta_y &= -(\hat{x} - x)\sin\theta + (\hat{y} - y)\cos\theta \\ \eta_\theta &= \hat{\theta} - \theta \\ \eta_{d_1} &= \left(\hat{d}_1 - d_1\right)\cos\theta + \left(\hat{d}_2 - d_2\right)\sin\theta \\ \eta_{d_2} &= -\left(\hat{d}_1 - d_1\right)\sin\theta + \left(\hat{d}_2 - d_2\right)\cos\theta. \end{aligned} \tag{104}$$

Recall the group affine condition for invariant systems on Lie groups

$$f(ab) = f(a)b + af(b) - af(I)b. \tag{105}$$

Next, we'll show that imposing (105) on the dynamics of $\mathcal{X}$ leads to the same restrictions on A and $\begin{bmatrix} C_x^\top & C_y^\top \end{bmatrix}^\top$ that were required for IEKF design 2.

Define $a, b \in G$ as

$$a = \begin{bmatrix} \cos\theta_a & -\sin\theta_a & x_a & d_{1a}^\top \\ \sin\theta_a & \cos\theta_a & y_a & d_{2a}^\top \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \tag{106}$$

$$b = \begin{bmatrix} \cos\theta_b & -\sin\theta_b & x_b & d_{1b}^\top \\ \sin\theta_b & \cos\theta_b & y_b & d_{2b}^\top \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}. \tag{107}$$

Then

$$ab = \begin{bmatrix} \cos\theta_{ab} & -\sin\theta_{ab} & x_{ab} & d_{1ab}^\top \\ \sin\theta_{ab} & \cos\theta_{ab} & y_{ab} & d_{2ab}^\top \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \tag{108}$$

where

$$\begin{aligned} x_{ab} &= x_a + \cos\theta_a x_b - \sin\theta_a y_b \\ y_{ab} &= y_a + \sin\theta_a x_b + \cos\theta_a y_b \\ \theta_{ab} &= \theta_a + \theta_b \\ d_{1ab}^\top &= d_{1a}^\top + d_{1b}^\top \cos\theta_a - d_{2b}^\top \sin\theta_a \\ d_{2ab}^\top &= d_{2a}^\top + d_{1b}^\top \sin\theta_a + d_{2b}^\top \cos\theta_a. \end{aligned}$$

Substituting (106), (107) and (108) into (105) and examining the $(1,3)^{\text{th}}$ and $(2,3)^{\text{th}}$

elements lead to

$$v \begin{bmatrix} \cos\theta_{ab} \\ \sin\theta_{ab} \end{bmatrix} + \begin{bmatrix} C_x \\ C_y \end{bmatrix} d_{ab} =$$

$$\begin{bmatrix} -\omega x_b \sin\theta_a - \omega y_b \cos\theta_a \\ \omega x_b \cos\theta_a - \omega y_b \sin\theta_a \end{bmatrix} + v \begin{bmatrix} \cos\theta_a \\ \sin\theta_a \end{bmatrix} + \begin{bmatrix} C_x \\ C_y \end{bmatrix} d_a$$

$$+ \begin{bmatrix} \cos\theta_a & -\sin\theta_a \\ \sin\theta_a & \cos\theta_a \end{bmatrix} \begin{bmatrix} v\cos\theta_b + C_x d_b \\ v\sin\theta_b + C_y d_b \end{bmatrix} - v \begin{bmatrix} \cos\theta_a \\ \sin\theta_a \end{bmatrix}$$

$$- \begin{bmatrix} -\omega x_b \sin\theta_a - \omega y_b \cos\theta_a \\ \omega x_b \cos\theta_a - \omega y_b \sin\theta_a \end{bmatrix},$$

which is simplified to

$$\begin{bmatrix} C_x \\ C_y \end{bmatrix} \begin{bmatrix} d_{1ab} \\ d_{2ab} \end{bmatrix} = \begin{bmatrix} C_x \\ C_y \end{bmatrix} \begin{bmatrix} d_{1a} \\ d_{2a} \end{bmatrix} + \begin{bmatrix} \cos\theta_a & -\sin\theta_a \\ \sin\theta_a & \cos\theta_a \end{bmatrix} \begin{bmatrix} C_x \\ C_y \end{bmatrix} \begin{bmatrix} d_{1b} \\ d_{2b} \end{bmatrix}.$$

Substituting the expressions for d_{1ab} and d_{2ab} yields

$$\begin{bmatrix} C_x \\ C_y \end{bmatrix} \left(\begin{bmatrix} d_{1a} \\ d_{2a} \end{bmatrix} + \begin{bmatrix} I_{\frac{m}{2}} \cos\theta_a & -I_{\frac{m}{2}} \sin\theta_a \\ I_{\frac{m}{2}} \sin\theta_a & I_{\frac{m}{2}} \cos\theta_a \end{bmatrix} \begin{bmatrix} d_{1b} \\ d_{2b} \end{bmatrix} \right) = \tag{109}$$

$$= \begin{bmatrix} C_x \\ C_y \end{bmatrix} \begin{bmatrix} d_{1a} \\ d_{2a} \end{bmatrix} + \begin{bmatrix} \cos\theta_a & -\sin\theta_a \\ \sin\theta_a & \cos\theta_a \end{bmatrix} \begin{bmatrix} C_x \\ C_y \end{bmatrix} \begin{bmatrix} d_{1b} \\ d_{2b} \end{bmatrix},$$

which results in

$$\begin{bmatrix} C_x \\ C_y \end{bmatrix} \begin{bmatrix} I_{\frac{m}{2}} \cos\theta_a & -I_{\frac{m}{2}} \sin\theta_a \\ I_{\frac{m}{2}} \sin\theta_a & I_{\frac{m}{2}} \cos\theta_a \end{bmatrix} = \begin{bmatrix} \cos\theta_a & -\sin\theta_a \\ \sin\theta_a & \cos\theta_a \end{bmatrix} \begin{bmatrix} C_x \\ C_y \end{bmatrix}. \tag{110}$$

A similar analysis of the elements corresponding with d_1 and d_2 in (105) reveals the necessary conditions on the matrix A:

$$A \begin{bmatrix} I_{\frac{m}{2}} \cos\theta_a & -I_{\frac{m}{2}} \sin\theta_a \\ I_{\frac{m}{2}} \sin\theta_a & I_{\frac{m}{2}} \cos\theta_a \end{bmatrix} = \begin{bmatrix} I_{\frac{m}{2}} \cos\theta_a & -I_{\frac{m}{2}} \sin\theta_a \\ I_{\frac{m}{2}} \sin\theta_a & I_{\frac{m}{2}} \cos\theta_a \end{bmatrix} A. \tag{111}$$

Notice how (110) and (111) are identical conditions as were required for IEKF design 2, Proposition 5.

We continue this section by showing the LIEKF formulation, under the assumptions of A and $\begin{bmatrix} C_x^\top & C_y^\top \end{bmatrix}^\top$ in Proposition 5.

The prediction and update equations of the LIEKF are given by [4]

$$\frac{d}{dt}\hat{\mathcal{X}} = f\left(\hat{\mathcal{X}}\right) \tag{112}$$

$$\hat{\mathcal{X}}^+ = \hat{\mathcal{X}}\exp\left(K\left[\hat{\mathcal{X}}^{-1}\mathbf{y} - q\right]\right) \tag{113}$$

where K is the Kalman gain matrix, computed from the Ricatti equation with matrices that will be given at the end of the section. With respect to the left invariant error defined in (103), the above equations can be written as

$$\frac{d}{dt}\eta = f(\eta) - f(I)\,\eta \tag{114}$$

$$\eta^+ = \eta\exp\left(L\left[\eta^{-1}q - q\right]\right). \tag{115}$$

Expanding the expression in (114) results in the following matrix where the individual elements are the expressions for the error of the corresponding original states with respect to the invariant error.

$$\dot{\eta} = \begin{bmatrix} 0 & -\dot{\eta}_\theta & \dot{\eta}_x & \dot{\eta}_{d_1} \\ \dot{\eta}_\theta & 0 & \dot{\eta}_y & \dot{\eta}_{d_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{116}$$

where

$$\dot{\eta}_x = v\left(\cos\eta_\theta - 1\right) + \omega\eta_y + C_x \begin{bmatrix}\eta_{d_1}^\top & \eta_{d_2}^\top\end{bmatrix}^\top$$
$$\dot{\eta}_y = v\sin\eta_\theta - \omega\eta_x + C_y \begin{bmatrix}\eta_{d_1}^\top & \eta_{d_2}^\top\end{bmatrix}^\top$$
$$\dot{\eta}_\theta = 0$$
$$\dot{\eta}_{d_1} = \begin{bmatrix}\eta_{d_1}^\top & \eta_{d_2}^\top\end{bmatrix} A_1^\top - \omega\eta_{d_2}^\top$$
$$\dot{\eta}_{d_1} = \begin{bmatrix}\eta_{d_1}^\top & \eta_{d_2}^\top\end{bmatrix} A_2^\top + \omega\eta_{d_1}^\top.$$

The expressions in (116) are still nonlinear functions of η. We next show that the dynamics of the Lie algebra of η is indeed linear. Towards this end, we derive the Lie algebra of G.

Proposition 6 *Let $\eta \in G$. The corresponding Lie algebra of G satisfying $\exp\left(\mathcal{L}_g(\xi)\right) = \eta$ is given by*

$$\mathcal{L}_g(\xi) = \begin{bmatrix} 0 & -\xi_3 & \xi_1 & \xi_{d_1}^\top \\ \xi_3 & 0 & \xi_2 & \xi_{d_2}^\top \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{117}$$

where

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_{d_1} \\ \xi_{d_2} \end{bmatrix} = \begin{bmatrix} \frac{\eta_y\eta_\theta}{2} - \frac{\eta_x\eta_\theta\sin\eta_\theta}{2(\cos\eta_\theta - 1)} \\ \frac{-\eta_x\eta_\theta}{2} - \frac{\eta_y\eta_\theta\sin\eta_\theta}{2(\cos\eta_\theta - 1)} \\ \eta_\theta \\ \frac{\eta_{d_2}\eta_\theta}{2} - \frac{\eta_{d_1}\eta_\theta\sin\eta_\theta}{2(\cos\eta_\theta - 1)} \\ \frac{-\eta_{d_1}\eta_\theta}{2} - \frac{\eta_{d_2}\eta_\theta\sin\eta_\theta}{2(\cos\eta_\theta - 1)} \end{bmatrix}, \tag{118}$$

in which η has the form of (103).

Proof. We show $\exp\left(\mathcal{L}_g(\xi)\right) = \eta$ with ξ and η defined in (118) and (103). We organize (117) into a block matrix

$$\mathcal{L}_g(\xi) = \begin{bmatrix} G_1 & G_2 \\ 0 & 0 \end{bmatrix} \tag{119}$$

where

$$G_1 = \begin{bmatrix} 0 & -\xi_3 \\ \xi_3 & 0 \end{bmatrix} \tag{120}$$

and

$$G_2 = \begin{bmatrix} \xi_1 & \xi_{d_1}^\top \\ \xi_2 & \xi_{d_2}^\top \end{bmatrix}. \tag{121}$$

Recall that for the Lie group of planar rotations, $SO(2)$, the Lie algebra is

$$\mathcal{L}_{so(2)}(\theta) = \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix} \tag{122}$$

which is then mapped to an element of the group by the matrix exponential map

$$\exp\left(\mathcal{L}_{so(2)}(\theta)\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{123}$$

Thus, G_1 is the Lie algebra of $SO(2)$.

Using the series expansion of the matrix exponential we get

$$\exp\left(\mathcal{L}_g(\xi)\right) = \sum_{i=0}^{\infty} \frac{1}{i!} \left(\mathcal{L}_g(\xi)\right)^i. \tag{124}$$

Substituting (119) into (124) yields

$$\exp\left(\mathcal{L}_g(\xi)\right) = \tag{125}$$

$$= I + \begin{bmatrix} G_1 & G_2 \\ 0 & 0 \end{bmatrix} + \frac{1}{2!} \begin{bmatrix} G_1^2 & G_1 G_2 \\ 0 & 0 \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} \sum_{i=0}^{\infty} \frac{1}{i!} G_1^i & G_2 + \frac{1}{2!} G_1 G_2 + \frac{1}{3!} G_1^2 G_2 + \cdots \\ 0 & I \end{bmatrix}. \tag{126}$$

Since G_1 is the Lie algebra of $SO(2)$, the top left block element of (126) is simply

$$\begin{bmatrix} \cos\xi_3 & -\sin\xi_3 \\ \sin\xi_3 & \cos\xi_3 \end{bmatrix}. \tag{127}$$

Setting $\exp\left(\mathcal{L}_g(\xi)\right) = \eta$ and noting η in (103), we obtain $\xi_3 = \eta_\theta$.

Letting

$$S = \begin{bmatrix} \cos\eta_\theta & -\sin\eta_\theta \\ \sin\eta_\theta & \cos\eta_\theta \end{bmatrix}. \tag{128}$$

and using algebraic manipulations to simplify the top right element of (126), we further obtain

$$\begin{aligned} & G_2 + \frac{1}{2!}G_1G_2 + \frac{1}{3!}G_1^2G_2 + \dots \\ &= \left(I + \frac{1}{2!}G_1 + \frac{1}{3!}G_1^2 + \dots\right)G_2 \\ &= \left(G_1 + \frac{1}{2!}G_1^2 + \frac{1}{3!}G_1^3 + \dots\right)G_1^{-1}G_2 \\ &= \left[\left(I + G_1 + \frac{1}{2!}G_1^2 + \frac{1}{3!}G_1^3 + \dots\right) - I\right]G_1^{-1}G_2 \\ &= (S - I)\,G_1^{-1}G_2. \end{aligned} \tag{129}$$

Setting the simplified equation (129) equal to the corresponding block of η in (103) yields

$$G_1^{-1}G_2 = (S - I)^{-1}\begin{bmatrix} \eta_x & \eta_{d_1}^\top \\ \eta_y & \eta_{d_2}^\top \end{bmatrix}. \tag{130}$$

In (130) the left hand side is a $2 \times \frac{m}{2}$ matrix whose elements are functions of the elements of ξ, and the right hand side is a $2 \times \frac{m}{2}$ matrix whose elements are the elements of η. Simplifying this equation gives

$$\begin{bmatrix} \xi_2 & \xi_{d_2}^\top \\ -\xi_1 & -\xi_{d_1}^\top \end{bmatrix} = \begin{bmatrix} \frac{-\eta_\theta}{2} & \frac{-\eta_\theta\sin\eta_\theta}{2(\cos\eta_\theta - 1)} \\ \frac{\eta_\theta\sin\eta_\theta}{2(\cos\eta_\theta - 1)} & \frac{-\eta_\theta}{2} \end{bmatrix}\begin{bmatrix} \eta_x & \eta_{d_1}^\top \\ \eta_y & \eta_{d_2}^\top \end{bmatrix}. \tag{131}$$

Solving this equation gives an expression for each element of the vector, ξ, in terms of the elements of the matrix, η, which are given in (118). ■

To find the dynamics of ξ, we differentiate both sides of (118) with respect to time. Due to the form of the equations and the fact that $\dot{\eta}_\theta = 0$, $\dot{\xi}$ has the following convenient form

$$\begin{aligned}
\dot{\xi}_1 &= \frac{\dot{\eta}_y \eta_\theta}{2} - \frac{\dot{\eta}_x \eta_\theta \sin \eta_\theta}{2\left(\cos \eta_\theta - 1\right)} \\
\dot{\xi}_2 &= \frac{-\dot{\eta}_x \eta_\theta}{2} - \frac{\dot{\eta}_y \eta_\theta \sin \eta_\theta}{2\left(\cos \eta_\theta - 1\right)} \\
\dot{\xi}_3 &= \dot{\eta}_\theta \\
\dot{\xi}_{d_1} &= \frac{\dot{\eta}_{d_2} \eta_\theta}{2} - \frac{\dot{\eta}_{d_1} \eta_\theta \sin \eta_\theta}{2\left(\cos \eta_\theta - 1\right)} \\
\dot{\xi}_{d_2} &= \frac{-\dot{\eta}_{d_1} \eta_\theta}{2} - \frac{\dot{\eta}_{d_2} \eta_\theta \sin \eta_\theta}{2\left(\cos \eta_\theta - 1\right)}.
\end{aligned} \tag{132}$$

Substituting the $\dot{\eta}$ terms in (116) and rearranging yields

$$\begin{aligned}
\dot{\xi}_1 =& \, \omega \left(\frac{-\eta_x \eta_\theta}{2} - \frac{\eta_y \eta_\theta \sin \eta_\theta}{2\left(\cos \eta_\theta - 1\right)}\right) + \mathcal{D} \left(\frac{\eta_{d_2} \eta_\theta}{2} - \frac{\eta_{d_1} \eta_\theta \sin \eta_\theta}{2\left(\cos \eta_\theta - 1\right)}\right) \\
&+ \mathcal{E} \left(\frac{-\eta_{d_1} \eta_\theta}{2} - \frac{\eta_{d_2} \eta_\theta \sin \eta_\theta}{2\left(\cos \eta_\theta - 1\right)}\right) \\
\dot{\xi}_2 =& - \omega \left(\frac{\eta_y \eta_\theta}{2} - \frac{\eta_x \eta_\theta \sin \eta_\theta}{2\left(\cos \eta_\theta - 1\right)}\right) + v\eta_\theta - \mathcal{E} \left(\frac{\eta_{d_2} \eta_\theta}{2} - \frac{\eta_{d_1} \eta_\theta \sin \eta_\theta}{2\left(\cos \eta_\theta - 1\right)}\right) \\
&+ \mathcal{D} \left(\frac{-\eta_{d_1} \eta_\theta}{2} - \frac{\eta_{d_2} \eta_\theta \sin \eta_\theta}{2\left(\cos \eta_\theta - 1\right)}\right) \\
\dot{\xi}_3 =& \, 0 \\
\dot{\xi}_{d_1} =& \, \mathcal{M} \left(\frac{\eta_{d_2} \eta_\theta}{2} - \frac{\eta_{d_1} \eta_\theta \sin \eta_\theta}{2\left(\cos \eta_\theta - 1\right)}\right) + \left(\mathcal{N} + I\omega\right) \left(\frac{-\eta_{d_1} \eta_\theta}{2} - \frac{\eta_{d_2} \eta_\theta \sin \eta_\theta}{2\left(\cos \eta_\theta - 1\right)}\right) \\
\dot{\xi}_{d_2} =& \, - \left(\mathcal{N} + I\omega\right) \left(\frac{\eta_{d_2} \eta_\theta}{2} - \frac{\eta_{d_1} \eta_\theta \sin \eta_\theta}{2\left(\cos \eta_\theta - 1\right)}\right) + \mathcal{M} \left(\frac{-\eta_{d_1} \eta_\theta}{2} - \frac{\eta_{d_2} \eta_\theta \sin \eta_\theta}{2\left(\cos \eta_\theta - 1\right)}\right)
\end{aligned}$$

which is indeed linear in ξ, i.e.,

$$\frac{d}{dt}\xi = \begin{bmatrix} 0 & \omega & 0 & C_x \\ -\omega & 0 & v & C_y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_\omega \end{bmatrix} \xi, \tag{133}$$

where

$$A_\omega = A + \begin{bmatrix} 0 & I_{\frac{m}{2}}\omega \\ -I_{\frac{m}{2}}\omega & 0 \end{bmatrix}. \tag{134}$$

The Kalman Filter update equation (113) becomes

$$\xi^+ = \xi + L\left(-\mathcal{L}_g(\xi)q\right) \tag{135}$$

where

$$\mathcal{L}_g(\xi)q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\xi. \tag{136}$$

Thus, we have shown that the error coordinate ξ follows linear dynamics. The Kalman gain K can then be computed based on linear Kalman filter equations. Algorithm 2 below is the IEKF algorithm for the matrix implementation which includes the rotated covariance terms in Section 3.2. Applying the algorithm to the unicycle problem, A_k and H_k in lines 6 and 7 are computed from (133) and (136), respectively.

In summary, we have shown that the system in (65) is group affine with respect to G under the conditions (84)–(85) of A and $\begin{bmatrix} C_x^\top & C_y^\top \end{bmatrix}^\top$. When these conditions are satisfied and with the measurements given in (64), a LIEKF design is proposed. Through a connection to the Lie algebra, an underlying error coordinate system is shown to possess truly linear dynamics.

Algorithm 2 The Matrix IEKF

Initialize $\hat{X}_0$, P_0 in the original coordinates.

Create $\hat{\mathcal{X}}_0$ from $\hat{X}_0$

$P = W(\hat{X}_0)^\top P_0 W(\hat{X}_0) + \sum_{i=1}^n \sum_{j=1}^n P_{ij} \frac{\partial W}{\partial x_i}^\top P_0 \frac{\partial W}{\partial x_j}$

for $k = 1$ **to** n **do**

  $\hat{\mathcal{X}}_k^- = f(\hat{\mathcal{X}}_{k-1}^+)$

  Compute A_k

  Compute H_k

  $Q_{rot} = W(\hat{X})^\top Q W(\hat{X}) + \sum_{i=1}^n \sum_{j=1}^n P_{ij} \frac{\partial W}{\partial x_i}^\top Q \frac{\partial W}{\partial x_j}$

  $P_k^- = A_k P_{k-1}^+ A_k^\top + Q_{rot}$

  $R_{rot} = T(\hat{X}) R T(\hat{X})^\top + \sum_{i=1}^n \sum_{j=1}^n P_{ij} \frac{\partial T}{\partial x_i} R \frac{\partial T}{\partial x_j}^\top$

  $K = P^- H_k^\top \left(H_k P_k^- H_k^\top + R_{rot}\right)^{-1}$

  $\hat{\mathcal{X}}_k^+ = \hat{\mathcal{X}}_k^- \exp\left(K \left[\hat{\mathcal{X}}^{-1}\mathbf{y} - q\right]\right)$

  $P_k^+ = (I - LH_k)P_k^-$

  Extract actual state estimate $(\hat{X}_k^+)$ from elements of $\hat{\mathcal{X}}_k^+$

end for

4.5 Simulations

In this section, we compare the performances of the proposed IEKF designs against the EKF in a simulation environment. Each graph represents a Monte Carlo simulation with 100 trials. The simulations were run with the robot maintaining a constant linear velocity and constant turning rate, collecting measurements at a rate of 10 Hz. A sample trajectory is shown below with the oscillating being caused by a disturbance.

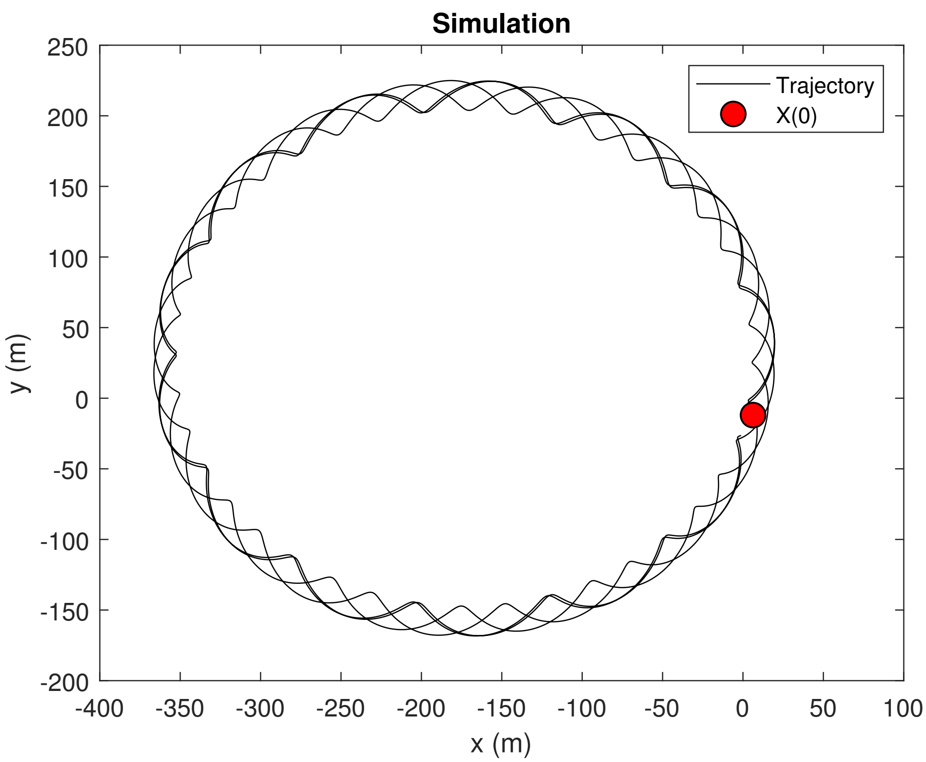

Figure 1: Sample simulation trajectory.

The following parameters were used in all the simulations:

$$v = 13\,\mathrm{m/s}$$
$$\omega = 4\,\mathrm{deg/s}$$
$$\mu_0 = \mathbf{0}_{n\times 1}$$
$$P_0 = \mathrm{diag}(10^2, 10^2, (\pi/2)^2, 2^2, 2^2, 2^2, 2^2)$$
$$\mathcal{X}_0 \sim \mathcal{N}(\mu_0, P_0)$$
$$\hat{\mathcal{X}}_0 = \mu_0.$$

Our metric of performance is the root mean square error (RMSE) of each filter's estimate with simulated 'truth' data, calculated at every time step. Let $x_i(t)$ be the ith element of $\mathcal{X}$ at time t. Then the RMSE of $x_i(t)$ is given by

$$RMSE_i(t) = \sqrt{\frac{\sum_{j=1}^{n} \left(x_i(t) - \hat{x}_i(t)\right)^2}{n}} \tag{137}$$

where n is the number of trials.

The simulated measurement noise in the truth data was generated from a zero mean Gaussian distribution with a non-isotropic covariance given by

$$R = \begin{bmatrix} 9 & 8 \\ 8 & 9 \end{bmatrix}. \tag{138}$$

The disturbances were generated from a linear time-invariant model with

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{139}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}. \tag{140}$$

The outputs from the linear system are two signals containing both sinusoidal oscillations plus a constant offset. Note that (139) and (140) satisfy the specific form given in Proposition 5. Thus, all IEKF designs can be applied.

4.5.1 Effect of Transformed Noise

First, using the defined invariant output error for the unicycle (72) and Monte Carlo simulations we compare the covariance transformation proposed in Proposition 4 with a numerically generated one. 2,000,000 simulated measurement errors were drawn from the original distribution, $\epsilon \sim \mathcal{N}(0, R)$ and rotated by a nominal angle θ with an uncertainty $\delta\theta \sim \mathcal{N}(0, q_\theta)$. After the rotation, the 'true' covariance was computed numerically. This was compared with three different approaches to handling the covariance. The first approach, denoted as '0-term' does not transform the original R matrix. The second approach, referred to as '1-term', includes only the first term on the right side of (55), excluding the first derivative terms. This '1-term' approach corresponds to noise covariance used in [4, Section IV-B-3]. Lastly, 'Both terms' refers to the case when the transformation in (55) is used entirely.

To determine 'closeness' of the various approaches to the 'true' transformed covariance, we employ two metrics: the Frobenius norm and a geodesic distance explored in [19]. Let $\tilde{R}$ and $\hat{R}$ denote the numerically computed covariance and the covariance computed from an analytical transformation, respectively. The Frobenius norm is then

$$\|\tilde{R} - \hat{R}\|_F = \sqrt{\sum_{i=1}^{p} \pi_i^2(\tilde{R} - \hat{R})} \tag{141}$$

where p is the number of measurements and $\pi_i(\tilde{R} - \hat{R})$ are the singular values of the matrix $\tilde{R} - \hat{R}$. The geodesic distance is a geometric distance between positive definite matrices given by

$$\delta_2(\tilde{R}, \hat{R}) = \sqrt{\sum_{i=1}^{p} \log^2 \lambda_i(\tilde{R}^{-1}\hat{R})} \tag{142}$$

where $\lambda_i(\tilde{R}^{-1}\hat{R})$ are the generalized eigenvalues of $(\tilde{R}, \hat{R})$.

	$q_\theta = 1e-3$		$q_\theta = 1e-2$		$q_\theta = 1e-1$	
	Frob	Geo	Frob	Geo	Frob	Geo
0-term	19.5582	3.6379	19.3992	3.5461	17.8555	3.0485
1-term	1.2567	0.3034	0.3780	0.1626	5.6348	1.3447
Both terms	1.2560	0.3016	0.3476	0.0693	5.2216	0.7720

Table 1.: Table of norms comparing different noise transformations for unicycle invariant error.

In the table, we consider cases with different uncertainty of the rotation angle, q_θ. Both norms show that the addition of the first order term in the transformation results in the best outcome. The smaller q_θ is, the contribution of the additional term is less significant. As q_θ increases however, the significance of the contribution is more noticeable.

We demonstrate the effect of the different rotated noise terms on the performance of IEKF1 and IEKF2. In the simulation the filters are run using the 3 different approaches previously mentioned, but applied to the measurement covariance R, the model uncertainty Q and the initial state covariance P_0.

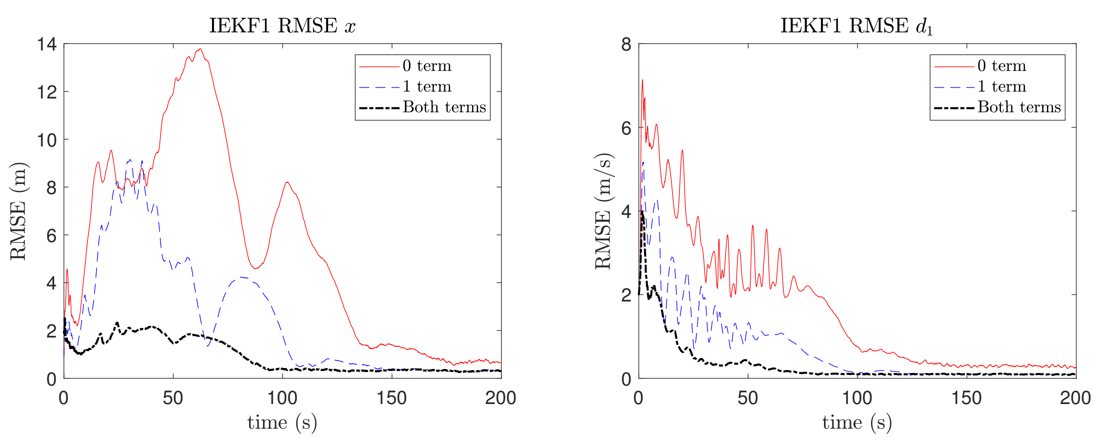

Figure 2: Effect of rotated noise terms for IEKF1.

From Figure. 2 we see this comparison for IEKF1. Using both terms from equations (55), (60) and (61) results in the best transient performance and fastest convergence rate for estimating x and d_1. The rest of the states all have similar trends.

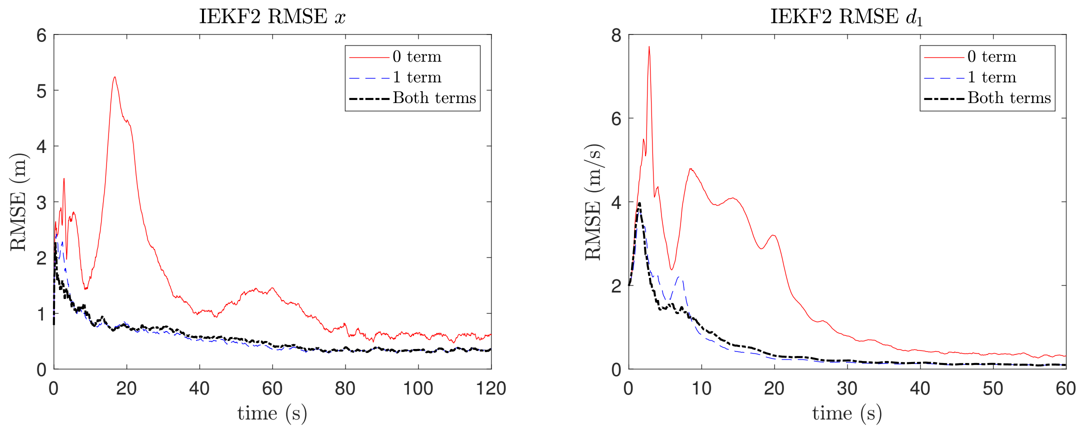

Figure 3: Effect of rotated noise terms for IEKF2.

Figure. 3 shows the same comparison for IEKF2. From Figure. 3, we see that both changes to the covariances improve the transient performance over the nominal case. For IEKF2, the addition of the first order correction term has a less significant impact than it does for IEKF1. However, it still improves the performance at the beginning of the simulation, as seen in the left graph of Figure. 3. Again, only graphs of the states x and d_1 are provided, however these trends extend to all the remaining states. Since it has been shown through simulations that adding the full noise correction given by equations (55), (60) and (61) improves the performance of both IEKF1 and IEKF2, this implementation is included in the performance comparisons in the following section.

4.5.2 EKF/IEKF Comparison

We now show the comparisons between the EKF and all three IEKF designs.

As can be seen from Figure 4, all three IEKF designs show significant improvement over the EKF for all states shown. The additional two disturbances, d_3 and d_4, not

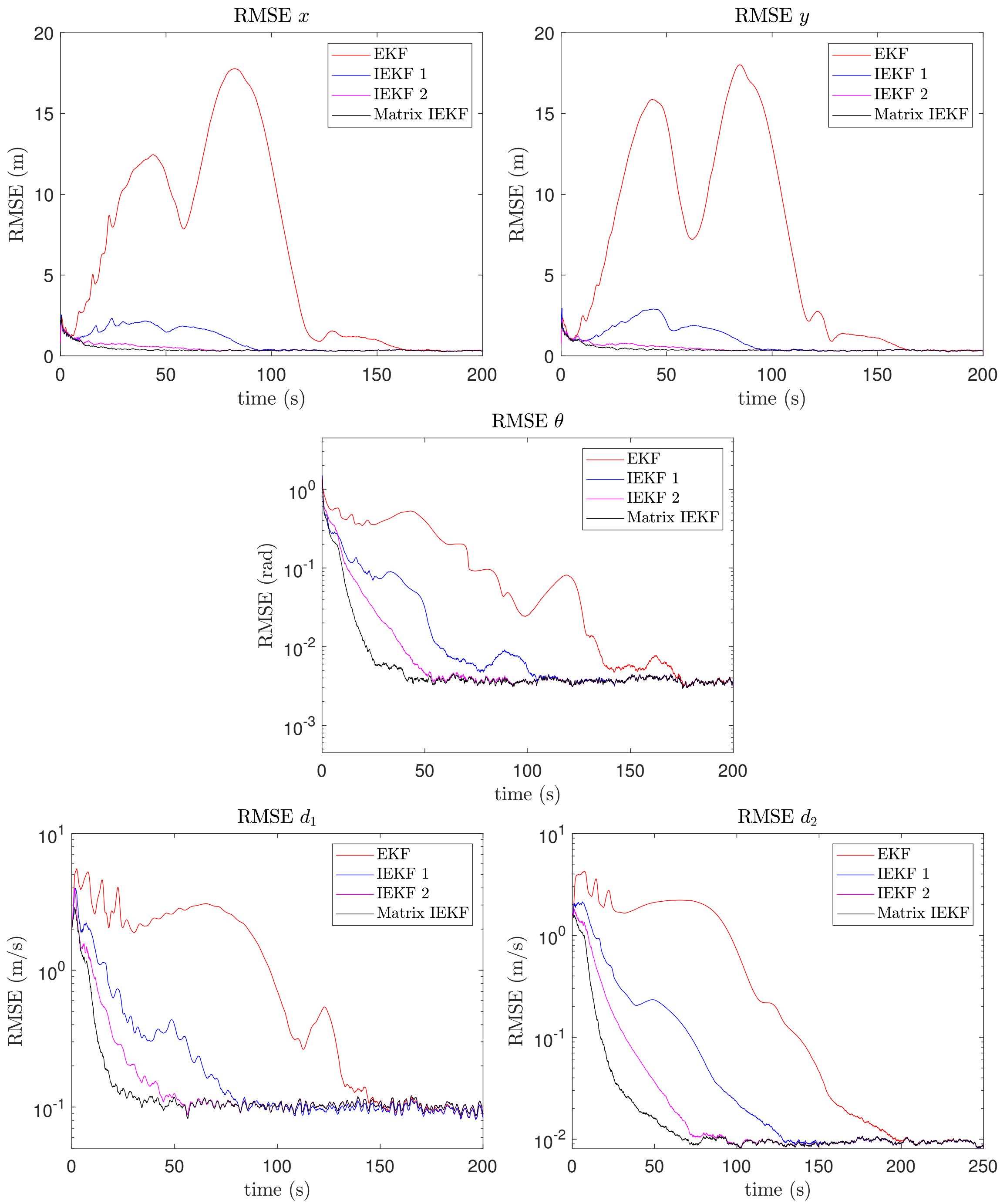

Figure 4: RMSE comparison of EKF, IEKF1, IEKF2 and Matrix IEKF when the disturbance conditions are satisfied.

shown have the exact same trends. From the graphs, the relative performance from best to worst goes: Matrix IEKF, IEKF2, IEKF1 and EKF. All four filters, converge to the same steady state RMSE values. Although the matrix IEKF clearly has the best estimation performance here, its performance is fairly similar to that of IEKF2. This is not surprising since both filters have the same invariant error dynamics. As mentioned previously, if the disturbance model satisfies the conditions in Proposition 5, then IEKF2 or the matrix IEKF should be used over IEKF1 since their error dynamics do not depend at all on the trajectory. However, even with IEKF1 we still see a significant improvement over the standard EKF for this case.

The results in Figure 5 show similar trends when the disturbance model varies from trial to trial. This demonstrates the effectiveness of all three designs when the disturbance conditions are satisfied. The disturbance models were generated by random sampling the $\mathcal{M}, \mathcal{N}, \mathcal{D}$ and $\mathcal{E}$ blocks from (86)–(87) and checking that the resulting system was both stable and observable. Then, the real parts of the eigenvalues of A were scaled to be within $(-.1,\ -.01)$ so that the generated disturbances would not decay quickly. Determining the systems this way results in matrices that satisfy the form of (86)–(87), but are more general than (139)–(140), in the sense that all of the blocks $\mathcal{M}, \mathcal{N}, \mathcal{D}, \mathcal{E}$ are nonzero.

4.5.3 Disturbance Condition Not Satisfied

The next set of graphs show the results for the case when the conditions in Proposition 5 are not satisfied. In this case, the group action used in IEKF2, along with the dynamics does not satisfy the invariance condition. Similarly, the group affine condition for the matrix IEKF is also not satisfied. Therefore, those designs would not be recommended for this case, and the graphs in Figures 6 and 7 demonstrate why. The linear disturbance model used in this simulation, was calculated by performing a similarity transformation on a system of the form, (86)–(87). The transformation

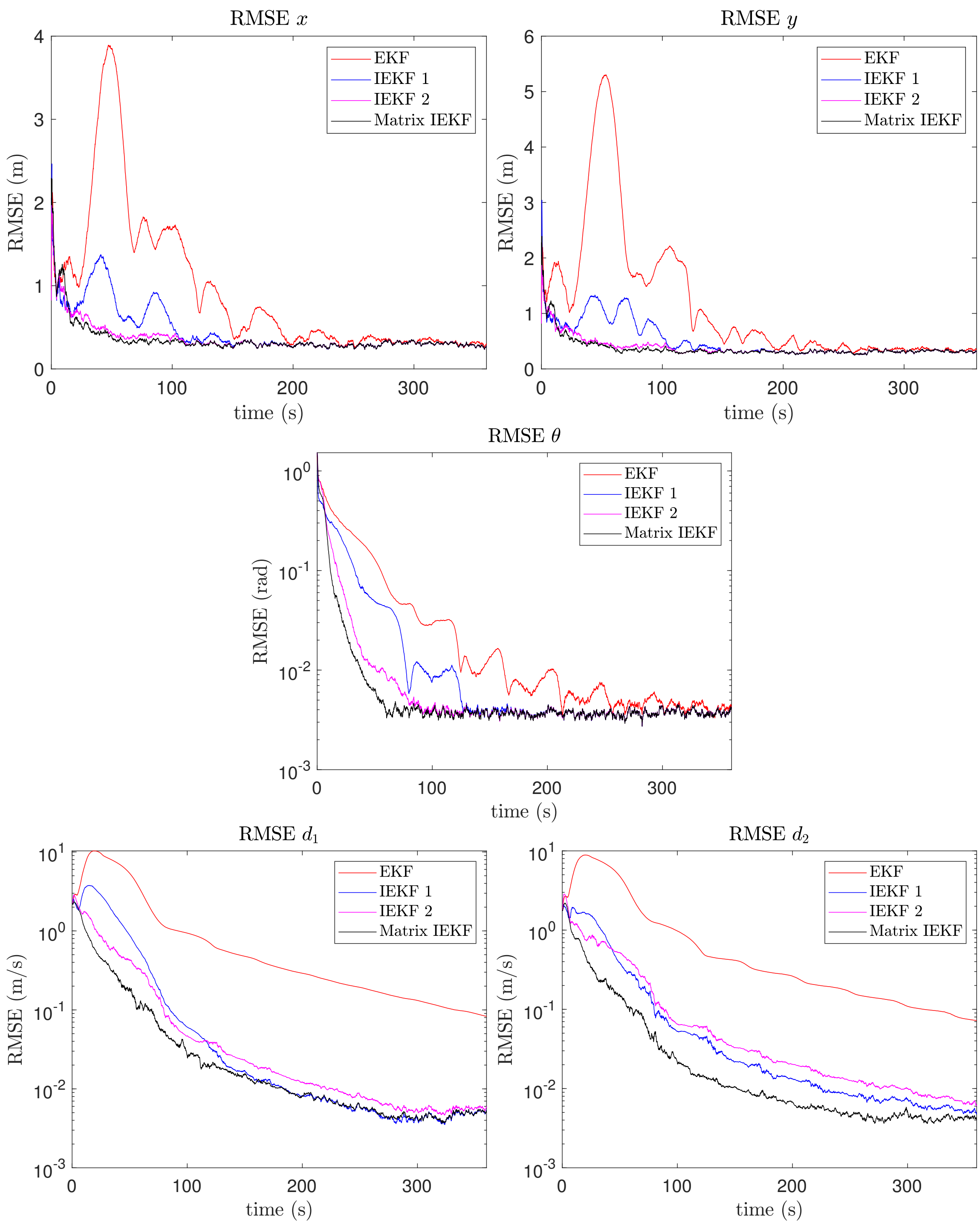

Figure 5: RMSE comparison of EKF, IEKF1, IEKF2 and Matrix IEKF when the disturbance conditions are satisfied with varying disturbance models.

matrix was calculated using the following equation:

$$P = (I + \epsilon D) \tag{143}$$

where ϵ is a relatively small value, 0.3 here, and D is a skew symmetric matrix, whose elements are randomly generated. This equation was used with the idea, that for small ϵ, such a transformation P, would preserve the 2-norm of the original disturbance d.

The original disturbance has the general form

$$\dot{d} = Ad \tag{144}$$

$$\begin{bmatrix} d_x \\ d_y \end{bmatrix} = Cd \tag{145}$$

where A and C are of the form of (86)–(87). After a similarity transformation where

$$\tilde{d} = P^{-1}d \tag{146}$$

the new system becomes

$$\dot{\tilde{d}} = \tilde{A}\tilde{d} \tag{147}$$

$$\begin{bmatrix} \tilde{d}_x \\ \tilde{d}_y \end{bmatrix} = \tilde{C}\tilde{d} \tag{148}$$

where

$$\tilde{A} = P^{-1}AP \qquad \tilde{C} = CP. \tag{149}$$

The relative performance of IEKF1 compared to IEKF2 and matrix IEKF varies depending on how 'close' the transformed $\tilde{A}$ and $\tilde{C}$ are to the form of (86)–(87). If the transformed system is still relatively close to the required form, IEKF2 and matrix IEKF will still have good performance. The graphs in Figure 6 demonstrate such a situation. IEKF2 and matrix IEKF have better transient performance than IEKF1, which has large initial errors. However, once IEKF1 converges to its lower bound it remains, where IEKF2 and matrix IEKF have growing errors as time progresses. The EKF was left out of these graphs since it had diverged.

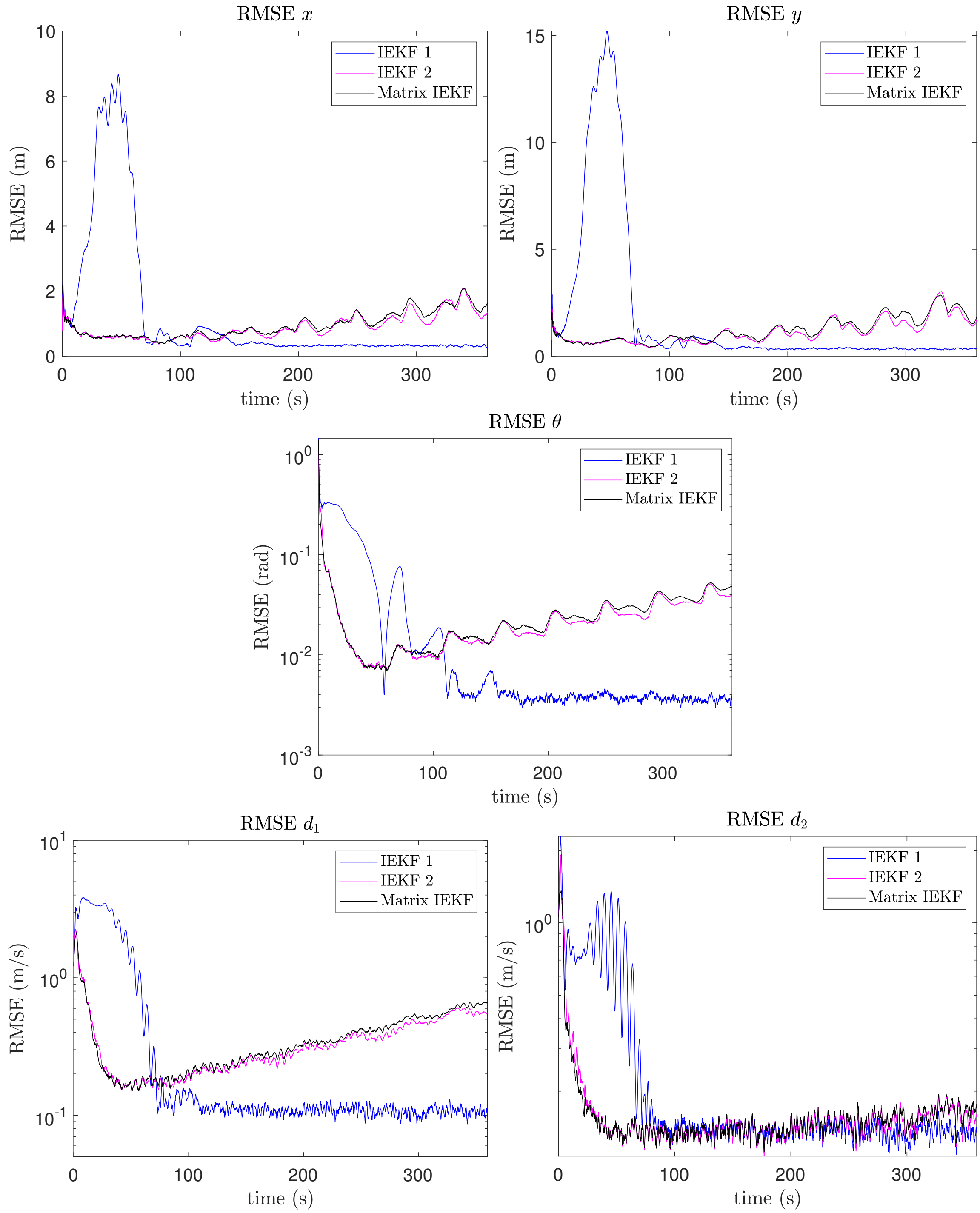

Figure 6: RMSE comparison of EKF, IEKF1, IEKF2 and Matrix IEKF when the disturbance conditions are not satisfied.

If the transformed system is ‘far’ enough from the required form of (86)–(87), IEKF1 will still have better or comparable performance to the EKF, when IEKF2 and matrix IEKF may diverge completely. Figure 7 demonstrates a case when this is true. The graphs in Figure 7 were generated using a different disturbance model than shown in (139)–(140). However, like the graphs in Figure 6, the original disturbance model was in the required form before the similarity transform with $\epsilon = 0.3$. However in this case we see that IEKF2 and matrix IEKF diverge, while IEKF1 still estimates all the states with better performance that the EKF. The differences between, these two scenarios are quite small, but with different outcomes. This points to a subtle connection between the disturbance model and the stability of IEKF2 and matrix IEKF that is still to be determined. Nonetheless, this demonstrates one example of a situation where IEKF2 and matrix IEKF are not applicable, but IEKF1 still is. It is worth noting, that for IEKF2 and matrix IEKF, the large errors in RMSE seen here are not the result of a few bad trials. Instead, every trial for both filters diverged, pointing to an inability to converge for the given disturbance model.

In summary, for the unicycle robot experiencing dynamic additive disturbances knowledge of the disturbance model will dictate which IEKF design should be used. IEKF1 is more consistent with less sensitivity to the specific form of the disturbance model. Therefore, if the disturbance model is not accurately known or known to not satisfy the conditions of Proposition 5, IEKF1 should be considered. However, if the disturbance is more accurately known to satisfy the specific conditions, then better estimation performance can be achieved by using IEKF2 or the matrix IEKF.

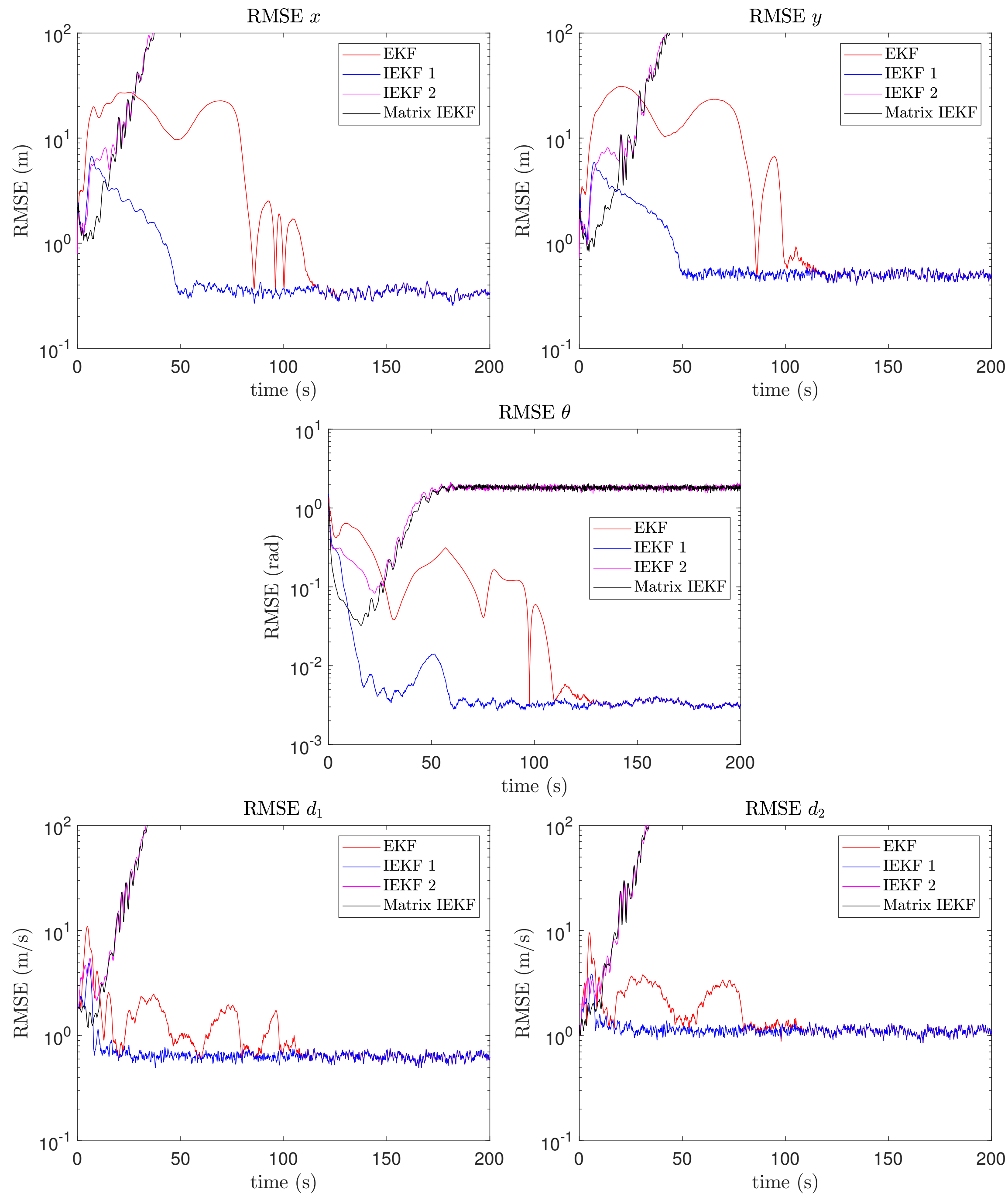

Figure 7: RMSE comparison of EKF, IEKF1, IEKF2 and Matrix IEKF when the disturbance conditions are not satisfied.

CHAPTER V

CONCLUSIONS AND FUTURE WORK

This thesis has investigated the topic of invariant nonlinear systems and how to design observers for these systems even when disturbances are applied. We have extended the theory of invariant systems by analyzing the requirements for invariant systems to remain invariant when dynamic additive disturbances are applied. This resulted in two sets of sufficient conditions that apply to general invariant systems when disturbances are applied that preserve the system invariance, allowing for symmetry preserving observers to be designed. We also proposed a first order approximation of the standard filtering covariance matrices that more accurately represents the uncertainties needed by the IEKF. Using the two developed sufficient conditions and the proposed approximation, we design two IEKFs for a unicycle robot with disturbances produced from a linear system. We applied additional theoretical concepts from the literature to design a third IEKF in a matrix Lie group framework, and showed that this design possesses an underlying error coordinate that has truly linear dynamics. Finally, through Monte Carlo simulations we demonstrated the performance improvement of all three designs when compared with the traditional EKF and the contribution of the proposed first order approximation.

This is still a developing field of research, both in theory and in application. Hence there are many opportunities for further research. Future work may include applying these concepts to design an invariant observer for a six degree of freedom quadcopter model experiencing disturbances. Additionally, effort could be spent working to make these design approaches applicable to a more general set of nonlinear systems. Lastly, an interesting prospect would be studying the potential connections between constant spatial varying disturbances with time varying spatially independent disturbances.

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