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〔Article〕

Sufficient conditions for log-concave conjecture on all-terminal reliability polynomial of a network

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Abstract

Consider a graph G that is simple, undirected, and connected, and has n vertices and m edges, and let $N_i(G)$ denote the number of connected spanning i-edge-subgraphs in a graph G for an integer i $(n-1 \leqslant i \leqslant m)$. For a graph G and all integers i's $(n \leqslant i \leqslant m-1)$, it is well-known that the problem of computing all $N_i(G)$'s is $\#P$-complete (see e.g., [3, 7, 14, 31]), and that log-concave conjecture (see e.g., [3, 14, 37]), that is, $N_i(G)^2 \geqslant N_{i-1}(G)N_{i+1}(G)$ holds, is still open. In this paper, by introducing new methods of partitioning N_i into a sequence of part integers, and by investigating properties of the sequence, we propose sufficient conditions to ensure the validity of log-concavity of sequence $N_{n-1}(G), N_n(G), \cdots, N_m(G)$.

Keyword: Network Reliability, Reliability Polynomial, Log-Concave Conjecture, Graph Theory, Connected Spanning Subgraph

ネットワークの全節点間信頼性多項式における対数的凹形予想の十分条件

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要　旨

n個の節点とm本の辺を持つ単純な連結無向グラフGを考える。$N_i(G)$はGのi本の辺からなる連結全域部分グラフの個数を表す。$N_{n-1}(G)$，$N_n(G)$，…，$N_m(G)$が全節点間のネットワーク信頼性評価において信頼性多項式の係数として使われている。また，$N_{n-1}(G)$，$N_n(G)$，…，$N_m(G)$を計算する問題は#P-完全な問題であると知られている一方，$N_{n-1}(G)$，$N_n(G)$，…，$N_m(G)$における対数的凹形予想（log-concave conjecture），つまり，$N_i^2 \geq N_{i-1}N_{i+1}(n \leq i \leq m-1)$が成り立つことも予想されている。本稿では，$N_i(G)$をより小さい整数からなる系列に分割し，そして分割した整数系列の性質を用いて対数的凹形予想が成り立つための十分条件を示す。

キーワード：ネットワーク信頼性，信頼性多項式，対数的凹形予想，グラフ理論，連結全域部分グラフ

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1 Introduction

In daily life, there are many systems utilized such as computer networks (Internet), biological networks, utility infrastructure (for transport of energy, water, gas, waste, etc.), social networks, and so on. Such complex systems can be modeled as various kinds of graphs, and the problems that exit in complex systems can be also considered as problems of graph theory (see. e.g., [1, 3, 6, 13, 15, 17, 18, 19, 26, 29, 38]). As one of the well-known problems in network analysis, the problem of network reliability analysis has been vigorously studied, and a lots of research results (see e.g. [3, 4, 7, 13, 20, 21, 23, 24, 25, 27, 28, 32]) have been reported.

In this paper we use graph terminologies in [19] unless otherwise declaration. A graph $G=(V, E)$ is considered to be simple, connected, and undirected, and consisting of $|V|=n$ vertices and $|E|=m$ edges. Let $N_i(G)$ for an integer i $(n-1 \leqslant i \leqslant m)$ denote the number of all possible connected spanning subgraphs with i edges in G.

In all-terminal reliability analysis of a network, $N_{n-1}(G)$, $N_n(G)$, $\cdots$, $N_m(G)$ are usually employed as the coefficients of reliability polynomial $Rel_{all}(G, \rho)$, defined as follows:

$$Rel_{all}(G,\rho) \quad = \quad \sum_{i=n-1}^{m} N_i(G)\rho^i(1-\rho)^{m-i}, \qquad (1)$$

where a network(, namely, probabilistic graph) (G, ρ) is consisting of the vertices that have no failure, and the edges that operate statistically independent probability ρ $(0 \leqslant \rho \leqslant 1)$. Namely, $Rel_{all}(G, \rho)$ is the probability that, if some edges fail with probability ρ independently, the remaining graph is connected.

It has been shown in [31] that the problem of computing all $N_i(G)$'s is #P-complete, even if G is a bipartite planar graph as well (see e.g., [10, 30, 36]). In particular, it is still unsolved whether there is a polynomial time algorithm to compute $N_n(G)$ for a graph G (see e.g., [7]), even if $N_{n-1}(G)$ is efficiently computed by the well-known Matrix-Tree theorem (see e.g., [19]).

In addition, explicit formulas in terms of n to count $N_n(G)$ and $N_{n+1}(G)$ have been obtained for some special cases: $G = K_n$ in [9]; $G = K_n - e$, $K_n \cdot e$, $K_n^{P_r}$, $K_n^{P_r} - e$, $K_n^{P_r} \cdot e$ in [10]; $G = K_{p.q}$ in [11]. Moreover, the known results with respect to computational complexity on various kinds of network reliability measurement can be found in e.g., [2, 3, 4, 7, 13, 14, 27, 35].

However, it remains open whether there is an algorithm for efficiently computing $N_n(G)$ for a graph G except for the above special cases. This means that the problem of efficiently computing $N_i(G)$'s is very hard in a point of view of computational complexity theory (see e.g., [16]) as complexity grows exponentially with the size n of G. Thus, it is important for network reliability analysis to find some algorithms of approximately computing $N_n(G)$'s. In addition, the known results in studying approximation algorithms can be also found in e.g., [21, 23].

A sequence of real numbers a_0, a_1, $\cdots$, a_m is said to be *log-concave* if $a_i^2 \geqslant a_{i-1}a_{i+1}$ for all indices i's ($1 \leqslant i \leqslant m-1$). Studies on log-concavity of coefficient sequences of various kinds of polynomials can be found in e.g., [5, 22, 27, 33, 34, 37]. Note that log-concavity on a sequence implies that all elements a_i's are approximately computed by starting at any two a_i, a_{i+1}.

For all-terminal reliability polynomial, it was posed as *log-concave conjecture* (see e.g., [14, 37]) that sequence $N_{n-1}(G)$, $N_n(G)$, $\cdots$, $N_m(G)$ is log-concave. We showed in [12] that $N_i(G)^2 \geqslant N_{i-1}(G)N_{i+1}(G)$ holds for indices i's satisfying $i \geqslant \left\lceil \frac{3-2\sqrt{2}}{2}n^2 - \frac{1}{2}n - \frac{7-2\sqrt{2}}{4\sqrt{2}} \right\rceil$. However, when index i is less than $\left\lceil \frac{3-2\sqrt{2}}{2}n^2 - \frac{1}{2}n - \frac{7-2\sqrt{2}}{4\sqrt{2}} \right\rceil$, even if $i = n$, little is known about results in proving $N_n(G)^2 \geqslant N_{n-1}(G)N_{n+1}(G)$, except for several special cases: G has at most 7 vertices (namely, $n \leqslant 7$) in [8]; G is a multigraph with a pair of vertices having at least $\lceil \frac{2}{3}(m-n) \rceil + 1$ multiple edges in [8]; G is K_n in [9], $K_n - e$, $K_n \cdot e$, $K_n^{P_r}$, $K_n^{P_r} - e$, $K_n^{P_r} \cdot e$ in [10], and $K_{p,q}$ in [11].

This paper mainly focuses on the problem of finding some sufficient conditions with respect to log-concavity of sequence $N_{n-1}(G)$, $N_n(G)$, $\cdots$, $N_m(G)$ for a graph G with n vertices and m edges. In order to find it, we introduce notation $N_i(G; b)$ for an integer b ($0 \leqslant b \leqslant n-1$) to denote the number of connected spanning i-edge b-bridge subgraphs in a graph G, and notation β_i to denote the average value on these numbers of bridges in all connected spanning i-edge subgraphs. Furthermore, we introduce notation β_i^*, defined in Section 4, which is a value more than β_i. We propose the sufficient conditions as follows. For a graph G and an integer i ($n \leqslant i \leqslant m-1$),

- if $\beta_i(G) + 1 \geqslant \beta_{i+1}(G)$ then $N_i^2(G) \geqslant N_{i-1}(G)N_{i+1}(G)$, and
- if $\beta_i(G) + 1 < \beta_{i+1}(G)$ and $i \geqslant \left(1 - \frac{1}{\beta_{i+1}-\beta_i}\right)m + \frac{\beta_{i+1}-1}{\beta_{i+1}-\beta_i}$ then $N_i^2(G) \geqslant N_{i-1}(G)N_{i+1}(G)$.

In addition, the following inequality is obtained to express a relationship between β_i, β_{i+1}, and β_{i+1}^*.

$$\beta_{i+1} \leqslant \beta_i + \frac{\beta_{i+1}}{i+1}(\beta_{i+1}^* - \beta_i - 1)$$

The remainder of this paper is organized as follows. In Section 2, we clarify the basic terminologies used in this paper, and introduce notations $B_i(G; b)$ for $0 \leqslant b \leqslant n-1$, and $T_i(G; t)$ for $0 \leqslant t \leqslant n-1$, respectively, such that $N_i(G)$ is respectively partitioned into both $B_i(G; b)$, and $T_i(G; t)$. In addition, a fundamental relationship between $B_i(G; b)$ and $T_i(G; t)$ is also shown by formulas. In Section 3, fundamental relationships between $N_i(G)$ and $B_i(G; b)$ is expressed by formulas. In section 4, by introducing the average values β of these numbers of bridges in connected spanning i-edge-subgraphs, we propose sufficient conditions to ensure that $N_i^2(G) \geqslant N_{i-1}(G)N_{i+1}(G)$ holds. In Section 5, some remarks with respect to the results of this paper will be given, and some interesting subjects as future research will be presented.

2 Preliminaries

Throughout this paper, a graph $G = (V, E)$ is considered to be consisting of *vertex-set* V and *edge-set* E, and to be simple, undirected, and connected. Furthermore, we always assume that G has n vertices and m edges, namely, $|V| = n$ and $|E| = m$. Figure 1 depicts an example of the graphs considered in this paper, where G is simple, undirected, and connected.

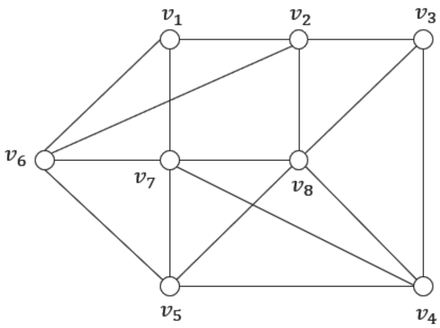

Figure 1 A graph G with 8 vertices and 16 edges, namely, $n = 8$ and $m = 16$.

For two sets X and Y, let $X - Y$ denote the set obtained from X by removing all elements of Y. For an edge-subset $U (\subseteq E)$, let $G_{\backslash U}$ denote the spanning subgraph obtained by removing all edges of U from G, namely, $G_{\backslash U} = (V, E - U)$. A graph having i edges is also called *i-edge-graph.* Figure 2 depicts four spanning 10–edge-subgraphs of G shown in Figure 1.

Each of the spanning subgraphs in Figure 2 is obtained by removing exactly 6 edges from G of Figure 1, namely, each of the subgraphs has exactly 10 edges. Clearly, the spanning subgraphs are either disconnected (see Figure 2(a)) nor connected (see Figure 2 (b), (c), (d)). In fact, $\binom{16}{6} =$ 8008 spanning subgraphs can be obtained by respectively removing exactly 6 edges from G of Figure 1.

For a graph $G = (V, E)$, an edge-subset $U (\subseteq E)$ is called *edge-cut* of G, if $G_{\backslash U}$ becomes a disconnected spanning subgraph. For example, for $G = (V, E)$ shown in Figure 1,

$$U = \{(v_1, v_2), (v_2, v_6), (v_7, v_8), (v_4, v_7), (v_5, v_8), (v_4, v_5\} \subseteq E$$

is an edge-cut of G, as $G_{\backslash U}$ (see Figure 2(a)) is disconnected. However,

$$U' = \{(v_1, v_2), (v_2, v_6), (v_1, v_6), (v_4, v_6), (v_5, v_8), (v_4, v_5\} \subseteq E$$

is not an edge-cut of G, as $G_{\backslash U'}$ (see Figure 2(b)) is connected as well.

Furthermore, if an edge-subset $U = \{e\}$ consisting of only one $e \in E$ is an edge-cut, then the edge e is said to be *bridge* of G. For example, edge $e = (v_7, v_8)$ is a bridge of the graph of Figure

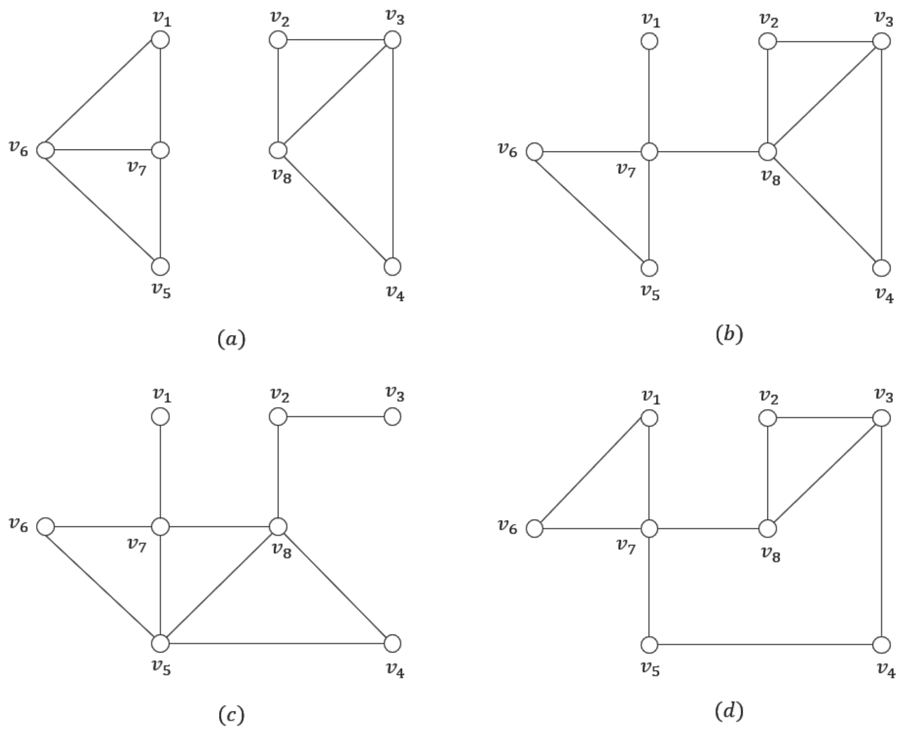

Figure 2 Four spanning 10-edge-subgraphs of G of Figure 1.

2(b). However, edge $e = (v_7, v_8)$ of the graph of Figure 1 is not a bridge. This means that an edge not being a bridge in G may be contained as a bridge in some connected spanning subgraphs of G.

Let $brg(G)$ denote the number of bridges in a graph G. Clearly, $brg(G) = n - 1$ iff G is a tree, and $brg(G) = 0$ for some graphs (see e.g. Figure 1, Figure 2(d)). Thus, $0 \leqslant brg(G) \leqslant n - 1$ holds for a graph G. Furthermore, it is not hard to verify that, in general, $brg(G) = brg(G')$ may not hold for two graphs G, G' with n vertices and m edges. Note that, for given two integers n, m $(\geqslant n-1)$, the maximum number of bridges for all connected graphs consisting of n vertices and m edges is invariant.

Let $\max_\beta(n,\ m)$ denote the number of bridges in the graph that has the maximum number of bridges among all connected graphs consisting of n vertices and m edges. Both two graphs in Figure 3 have the maximum number of bridges for all connected graphs consisting of 8 vertices and 11 edges.

Given two integers n and m $(\geqslant n)$, we can show the following formula to find $\max_\beta(n,\ m)$, where $\lceil x \rceil$ denotes the least integer more than or equal to x.

$$\max_\beta(n, m) = n - \left\lceil \frac{3 + \sqrt{9 + 8(m-n)}}{2} \right\rceil \tag{2}$$

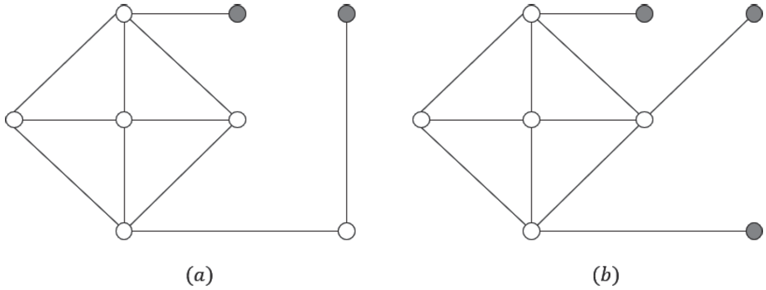

Figure 3　Two graphs with 8 vertices and 11 edges.

By formula (2), for the graphs of Figure 3 with $n=8$ and $m=11$, $\max_\beta(8,11)=8-\left\lceil\frac{3+\sqrt{9+8(11-8)}}{2}\right\rceil = 3$, which implies that the graph with the maximum number of bridges is not unique.

For an integer i $(n-1 \leqslant i \leqslant m)$, let $\mathcal{N}_i(G)$ denote the set of all possible connected spanning i-edge-subgraphs in G. In addition, let $N_i(G)=|\mathcal{N}_i(G)|$.

Note that $|V|=n$ and $|E|=m$ for a graph $G=(V, E)$. We introduce a new notation $\mathcal{B}_i(G;b)$ for an integer b $(0 \leqslant b \leqslant \max_\beta(n,i))$, which is defined as follows.

Definition 1. For a graph G and two integers i, b $(n-1 \leqslant i \leqslant m, 0 \leqslant b \leqslant \max_\beta(n,i))$, let $\mathcal{B}_i(G;b)$ denote the set of all possible connected spanning i-edge-subgraphs of G, each of which has exactly b bridges. In addition, let $B_i(G;b)=|\mathcal{B}_i(G;b)|$.

Note that we have $B_i(G;b)=0$ for an integer b satisfying $b \geqslant \max_\beta(n,i)+1$. Consequently, $\mathcal{N}_i(G)$ is partitioned into $\mathcal{B}_i(G;0)$, $\mathcal{B}_i(G;1)$, $\cdots$, $\mathcal{B}_i(G;\max_\beta(n,i))$, namely,

$$N_i(G)=\sum_{b=0}^{\max_\beta(n,i)} B_i(G;b). \tag{3}$$

Formula (3) implies that $N_i(G)$ is obtained by computing $B_i(G;0), B_i(G;1), \cdots, B_i(G;\max_\beta(n,i))$. Consequently, the problem of computing all $B_i(G;b)$'s for $0 \leqslant b \leqslant \max_\beta(n,i)$ is $\#P$-complete.

The *degree* of a vertex $v \in V$, denoted by $deg(v)$, is the number of edges incident on v. Clearly, $0 \leqslant deg(v) \leqslant n-1$ holds for any $v \in V$ by $|V|=n$. A vertex v with $deg(v)=1$ is called *terminal* of G. Let $trm(G)$ denote the number of terminals in G.

Clearly, the only one edge incident on a terminal v (namely, $deg(v)=1$) must be a bridge. This means that for a graph G we have

$$trm(G) \leqslant brg(G). \tag{4}$$

For example, three subgraphs in Figure 2(b), (c), (d) respectively have one terminal v_1, two terminals v_1, v_3, and no terminal. A complete bipartite graph $K_{1,n-1}$ is the unique graph with $n-1$ terminals. In general, $0 \leqslant trm(G) \leqslant n-1$.

Let $\max_\tau(n,m)$ denote the number of terminals in a graph that has the maximum number of terminals among all connected graphs consisting of n vertices and m edges. Note that, for the graph G in Figure 3(b), $trm(G) = brg(G)$. In fact, we can also verify that the maximum number of terminals is equal to the maximum number of bridges, namely,

$$\max_{\tau}(n, m) = \max_{\beta}(n, m). \tag{5}$$

Definition 2. For a graph G and two integers i, t $(n-1 \leqslant i \leqslant m, 0 \leqslant t \leqslant \max_{\tau}(n, i))$, let $\mathcal{T}_i(G; t)$ denote the set of all possible connected spanning i-edge-subgraphs of G, each of which has exactly t terminals. In addition, let $T_i(G; t) = |\mathcal{T}_i(G; t)|$.

By definitions, we have $T_i(G; t) = 0$ for $t \geqslant \max_{\tau}(n, i) + 1$. Consequently, $\mathcal{N}_i(G)$ is justly partitioned into $\mathcal{T}_i(G; 0)$, $\mathcal{T}_i(G; 1)$, $\cdots$, $\mathcal{T}_i(G; \max_{\tau}(n, i))$, namely,

$$N_i(G) = \sum_{t=0}^{\max_{\tau}(n,i)} T_i(G; t). \tag{6}$$

Formula (6) implies that we can obtain $N_i(G)$ by computing $T_i(G; 0), T_i(G; 1), \cdots,$ $T_i(G; \max_{\tau}(n, i))$. Consequently, the problem of computing all $T_i(G; b)$'s is #P-complete. Next, we introduce new notations $T_i^{\geqslant x}(G)$ and $B_i^{\geqslant x}(G)$, respectively, defined as follows:

$$T_i^{\geqslant x}(G) \equiv \sum_{t=x}^{\max_{\tau}(n,i)} T_i(G; t) \tag{7}$$

$$B_i^{\geqslant x}(G) \equiv \sum_{b=x}^{\max_{\beta}(n,i)} B_i(G; b) \tag{8}$$

It is not difficult to see that $T_i^{\geqslant x}(G)$ and $B_i^{\geqslant x}(G)$ respectively represent the number of connected spanning i-edge-subgraphs of G, each of which has at least x terminals, and bridges, respectively.

Now, we give the following theorems to reveal a fundamental relationship between $B_i(G; b)$ and $T_i(G; t)$.

Theorem 1. *For a graph G and two integers i, x $(n-1 \leqslant i \leqslant m, 0 \leqslant x \leqslant max_{\tau}(n, i))$, we have*

$$T_i^{\geqslant x}(G) \leqslant B_i^{\geqslant x}(G). \tag{9}$$

Proof. By formula (4), a graph with t terminals has at least t bridges, as one terminal justly corresponds to one bridge. On the other hand, it is not hard to see that a graph with b bridges has at most b terminals by definitions.

For every subgraph $H \in \mathcal{N}_i(G)$, it is easy to see that if H is counted one time by $T_i^{\geqslant x}(G)$ then it is also counted one time by $B_i^{\geqslant x}(G)$. Thus, we obtain the validity of inequality (9). □

Similarly, we also introduce notations $T_i^{\leqslant x}(G)$ and $B_i^{\leqslant x}(G)$, respectively, defined as follows:

$$T_i^{\leqslant x}(G) \equiv \sum_{t=0}^{x} T_i(G;t) \tag{10}$$

$$B_i^{\leqslant x}(G) \equiv \sum_{b=0}^{x} B_i(G;b) \tag{11}$$

Clearly, $T_i^{\leqslant x}(G)$ and $B_i^{\leqslant x}(G)$ respectively represent the number of connected spanning i-edge-subgraphs of G, each of which has at most x terminals, and bridges, respectively.

Theorem 2. *For a graph G and two integers i, x $(n-1 \leqslant i \leqslant m, 0 \leqslant x \leqslant max_\tau(n,i))$, we have*

$$T_i^{\leqslant x}(G) \geqslant B_i^{\leqslant x}(G). \tag{12}$$

Proof. By definitions, for a given x $(0 \leqslant x \leqslant \max_\tau(n,i))$, we have

$$N_i(G) = \sum_{t=0}^{x} T_i(G;t) + \sum_{t=x+1}^{\max_\tau(n,i)} T_i(G;t) = T_i^{\leqslant x}(G) + T_i^{\geqslant x+1}(G)$$

and

$$N_i(G) = \sum_{b=0}^{x} B_i(G;b) + \sum_{b=x+1}^{\max_\beta(n,i)} B_i(G;b) = B_i^{\leqslant x}(G) + B_i^{\geqslant x+1}(G).$$

Note that $\max_\beta(n,i) = \max_\tau(n,i)$. Immediately, formula (12) follows formula (9). □

By definitions, it is obvious that

$$N_i(G) = B_i^{\geqslant 0}(G) = B_i^{\leqslant \max_\beta(n,i)}(G) \tag{13}$$

and

$$N_i(G) = T_i^{\geqslant 0}(G) = T_i^{\leqslant \max_\tau(n,i)}(G). \tag{14}$$

In the following discussions, for shorting notations, when the graph G is clearly specified, $\mathcal{N}_i(G)$, $N_i(G)$, $\mathcal{B}_i(G;b)$, $B_i(G;b)$ are always abbreviated to $\mathcal{N}_i,\ N_i,\ \mathcal{B}_i(b),\ B_i(b)$, respectively.

3 Formulas for Expressing Relationships between N_i and $B_i(b)$

This section aims to show some formulas for specifying relationships between N_i and $B_i(b)$. In order to do it, we need new notations.

In order to clarify affiliation without confusion, we also employ E_G to denote the edge-set of a graph G. Note that $brg(G)$ denotes the number of bridges in G. By definitions, $|E_H| = i$ and

$brg(H) = b$ for a subgraph $H \in \mathcal{B}_i(b)$.

For a subgraph $H \in \mathcal{B}_i(b)$ and an edge $e \in E_G - E_H$, let $H+e$ denote the graph obtained by adding e into H. Namely, $H + e = (V, E_H \cup \{e\})$. Let $H-e$ denote the graph obtained by removing e from H. Namely, $H - e = (V, E_H - \{e\})$.

By definitions, the sum of the numbers $brg(H)$'s for all $H \in \mathcal{N}_i$ is expressed as follows:

$$\sum_{H \in \mathcal{N}_i} brg(H) = \sum_{b=0}^{\max_\beta(n,i)} bB_i(b). \tag{15}$$

We introduce the average value on the numbers $brg(H)$'s for all $H \in \mathcal{N}_i$, denoted by $\beta_i(G)$ abbreviated to β_i, to be defined as follows.

$$\beta_i \equiv \frac{\sum_{b=0}^{\max_\beta(n,i)} bB_i(b)}{\sum_{b=0}^{\max_\beta(n,i)} B_i(b)} = \frac{1}{N_i} \sum_{b=0}^{\max_\beta(n,i)} bB_i(b). \tag{16}$$

Clearly, $\beta_{n-1} = n - 1$.

Lemma 1. *For a graph G and an integer (also called index) i $(n-1 \leqslant i \leqslant m)$,*

$$\beta_i \leqslant max_\beta(n,i).$$

Proof. As $b \leqslant \max_\beta(n,i)$ in formula (16), it is trivial. □

By setting $m = i$ into formula (2), we have

$$\max_\beta(n,i) = n - \left\lceil \frac{3 + \sqrt{9 + 8(i-n)}}{2} \right\rceil. \tag{17}$$

The following lemma establishes a fundamental relationship between N_i and N_{i+1} by employing β_{i+1}.

Lemma 2. *For a graph G and an integer (also called an index) i $(n-1 \leqslant i < m)$,*

$$(m-i)N_i = (i + 1 - \beta_{i+1})N_{i+1}. \tag{18}$$

Proof. By definitions, $H = (V, E_H) \in \mathcal{N}_i$ is a connected spanning i-edge-subgraph of G. Thus, for every $e \in E_G - E_H$, we can obtain $H + e$ that is a connected spanning $(i+1)$-edge-subgraph of G. Namely, $H + e \in \mathcal{N}_{i+1}$. Note that $|E_G| - |E_H| = m - i$ as $|E_G| = m$ and $|E_H| = i$. Consequently, from every $H \in \mathcal{N}_i$, we can obtain the number $m - i$ of connected spanning $(i+1)$-edge-subgraphs in $\mathcal{N}_{i+1}$ by adding every $e \in E_G - E_H$ into H.

On the other hand, $F \in \mathcal{N}_{i+1}$ is a connected spanning $(i+1)$-edge-subgraph of G. Thus, by removing an edge $e \in E_F$ from F, where e is not a bridge of F, we can obtain one subgraph $F - e \in \mathcal{N}_i$. This means that every $F \in \mathcal{N}_{i+1}$ are obtained from the number $i + 1 - brg(F)$ of different connected spanning i-edge-subgraphs in $\mathcal{N}_i$.

Combining with the above discussions, we obtain

$$(m-i)N_i = \sum_{H\in\mathcal{N}i}(m-i) = \sum_{F\in\mathcal{N}_{i+1}}(i+1-brg(F)) = \sum_{b=0}^{\max_\beta(n,i+1)}(i+1-b)B_{i+1}(b),$$

which has proven the validity of this lemma. □

Lemma 2 implies that the following theorem is true.

Theorem 3. *For a graph G and an index $i(n \leqslant i \leqslant m-1)$,*

$$\frac{{N_i}^2}{N_{i-1}N_{i+1}} = \frac{m-i+1}{m-i}\frac{i+1-\beta_{i+1}}{i-\beta_i}. \tag{19}$$

Proof. It is straightforward from formula (18). □

In Theorem 3 a very useful fact has been described, that is, it is possible to prove the validity of $N_i^2 \geqslant N_{i-1}N_{i+1}$ by investigating the property on β_i and β_{i+1}. In the next section, we will do it.

4 Sufficient Conditions for Satisfying $N_i^2 \geqslant N_{i-1}N_{i+1}$

In this section, we will show sufficient conditions such that the validity of log-concavity of sequence $N_{n-1}, N_n, \cdots, N_m$ is true.

Theorem 4. *For a graph G, the inequality ${N_i}^2 \geqslant N_{i-1}N_{i+1}$ holds if an index i $(n \leqslant i \leqslant m-1)$ satisfies one of the following conditions*:

$$\begin{aligned}&(i)\ \ \beta_i + 1 \geqslant \beta_{i+1}\\ &(ii)\ \beta_i + 1 < \beta_{i+1} \quad and \quad i \geqslant \left(1-\tfrac{1}{\beta_{i+1}-\beta_i}\right)m + \tfrac{\beta_{i+1}-1}{\beta_{i+1}-\beta_i}\end{aligned}$$

Proof. By Theorem 3, it is sufficient to find the condition satisfying the following inequality:

$$\frac{m-i+1}{m-i}\frac{i+1-\beta_{i+1}}{i-\beta_i} \geqslant 1.$$

Furthermore, the above inequality is rewritten as follows:

$$(m-i)(\beta_i+1-\beta_{i+1}) + (i+1-\beta_{i+1}) \geqslant 0. \tag{20}$$

Note that $i+1 \geqslant n \geqslant \max_\beta(n, i+1) \geqslant \beta_{i+1}$, namely, $i+1-\beta_{i+1} > 0$. When $\beta_i+1-\beta_{i+1} \geqslant 0$, inequality (20) holds immediately. This means that (i) can be considered as a sufficient condition such that ${N_i}^2 \geqslant N_{i-1}N_{i+1}$ holds.

Next, we assume $\beta_i + 1 < \beta_{i+1}$, and show the condition on an index i such that inequality (20) holds. We further write inequality (20) as follows:

$$
\begin{aligned}
i &\geqslant \frac{(\beta_{i+1}-1-\beta_i)m+\beta_{i+1}-1}{\beta_{i+1}-\beta_i} \\
&= \Big(1-\frac{1}{\beta_{i+1}-\beta_i}\Big)m+\frac{\beta_{i+1}-1}{\beta_{i+1}-\beta_i},
\end{aligned} \tag{21}
$$

which implies that (ii) can be considered as a sufficient condition such that ${N_i}^2 \geqslant N_{i-1}N_{i+1}$ holds. □

Next, we further investigate some properties on a graph such that the sufficient conditions in Theorem 4 hold. In order to do it, we need new notations. For an integer $x(0\leqslant x\leqslant \max_\beta(n,i))$, we recall $B_i^{\leqslant x}$ and $B_i^{\geqslant x}$, written as follows:

$$
B_i^{\leqslant x} = \textstyle\sum_{b=0}^{x} B_i(b), \qquad B_i^{\geqslant x} = \sum_{b=x}^{\max_\beta(n,i)} B_i(b).
$$

We introduce notations $\beta_i^{\leqslant x}$ and $\beta_i^{\geqslant x}$, respectively, defined as follows:

$$
\beta_i^{\leqslant x} \equiv \frac{\sum_{b=0}^{x} bB_i(b)}{\sum_{b=0}^{x} B_i(b)} = \frac{1}{B_i^{\leqslant x}}\sum_{b=0}^{x} bB_i(b) \leqslant x, \tag{22}
$$

$$
\beta_i^{\geqslant x} \equiv \frac{\sum_{b=x}^{\max_\beta(n,i)} bB_i(b)}{\sum_{b=x}^{\max_\beta(n,i)} B_i(b)} = \frac{1}{B_i^{\geqslant x}}\sum_{b=x}^{\max_\beta(n,i)} bB_i(b) \geqslant x. \tag{23}
$$

Lemma 3. *For a graph G and an index i $(n-1 \leqslant i \leqslant m)$, we have the following inequalities with respect to an integer x $(0 \leqslant x \leqslant max_\beta(n,i))$.*

$$
(i) \quad \beta_i^{\leqslant x} \leqslant x \qquad (ii) \quad \beta_i^{\geqslant x} \geqslant x
$$

Proof. It is trivial by definitions. □

Clearly, $\beta_i^{\leqslant x} \leqslant \beta_i^{\geqslant x}$ by Lemma 3. We further give the following lemma to present more strict inequalities.

Lemma 4. *For a graph G and an index i $(n-1 \leqslant i \leqslant m)$, we have the following inequalities with respect to an integer x $(0 \leqslant x \leqslant max_\beta(n,i)-1)$.*

$$
(i) \quad \beta_i^{\leqslant x} \leqslant \beta_i^{\leqslant x+1} \leqslant \beta_i \qquad (ii) \quad \beta_i \leqslant \beta_i^{\geqslant x} \leqslant \beta_i^{\geqslant x+1}
$$

Proof. (i) By definitions, it is sufficient to show the validity of the following inequality.

$$
\beta_i^{\leqslant x} = \frac{\sum_{b=0}^{x} bB_i(b)}{\sum_{b=0}^{x} B_i(b)} \leqslant \frac{\sum_{b=0}^{x+1} bB_i(b)}{\sum_{b=0}^{x+1} B_i(b)} = \beta_i^{\leqslant x+1} \tag{24}
$$

Note that

$$\frac{\sum_{b=0}^{x+1} bB_i(b)}{\sum_{b=0}^{x+1} B_i(b)} = \frac{\sum_{b=0}^{x} bB_i(b) + (x+1)B_i(x+1)}{\sum_{b=0}^{x} B_i(b) + B_i(x+1)}.$$

Clearly, the validity of inequality (24) is true with equivalent iff $B_i(x+1) = 0$. Next, we assume $B_i(x+1) > 0$, and show the validity of inequality (24).

By Lemma 3 and definitions, we have

$$\frac{\sum_{b=0}^{x} bB_i(b)}{\sum_{b=0}^{x} B_i(b)} \leqslant x.$$

Therefore,

$$\frac{\sum_{b=0}^{x} bB_i(b)}{\sum_{b=0}^{x} B_i(b)} < x+1 = \frac{(x+1)B_i(x+1)}{B_i(x+1)},$$

which is rewritten as follows:

$$\Big(\sum_{b=0}^{x} bB_i(b)\Big)B_i(x+1) < \Big(\sum_{b=0}^{x} B_i(b)\Big)(x+1)B_i(x+1).$$

Thus, we can obtain inequality (24) by adding the term $(x+1)B_i(x+1)^2$ into both hand-sides of the above inequality.

By definition, it is not hard to see that $\beta_i^{\leqslant \max_\beta(n,i)} = \beta_i$. Hence, the validity of (i) has been shown.

(ii) We can also show the validity of (ii) by employing the method similar to that of (i). $\square$

By Lemma 4, we obtain

$$0 = \beta_0^{\leqslant 0} \leqslant \beta_i^{\leqslant 1} \leqslant \cdots \leqslant \beta_i^{\leqslant \max_\beta(n,i)} = \beta_i = \beta_i^{\geqslant 0} \leqslant \beta_i^{\geqslant 1} \leqslant \cdots \leqslant \beta_i^{\geqslant \max_\beta(n,i)} = \max_\beta(n,i).$$

Lemma 5. *For a graph G and an index i $(n-1 \leqslant i \leqslant m)$, we have the following inequalities with respect to an integer x $(0 \leqslant x \leqslant max_\beta(n,i))$.*

$$(i) \quad (m-i)\sum_{b=0}^{x} B_i(b) \leqslant \sum_{b=0}^{x}(i+1-b)B_{i+1}(b)$$

$$(ii) \quad (m-i)\sum_{b=x}^{max_\beta(n,i)} B_i(b) \geqslant \sum_{b=x}^{max_\beta(n,i)}(i+1-b)B_{i+1}(b)$$

Proof. Let $H = (V, E_H) \in \mathcal{N}_i$ with at most x bridges, namely, $brg(H) \leqslant x$. It is easy to verify that, for every $e \in E_G - E_H$, $H + e$ has at most x bridges. This means that $H + e$ is in $\mathcal{N}_{i+1}$, and has at most x bridges, namely, $brg(H+e) \leqslant x$. Note that $|E_G| - |E_H| = m - i$ as $|E_G| = m$ and $|E_H| = i$. Hence, the number $m - i$ of connected spanning $(i+1)$-edge-subgraphs, each of which has at most x bridges, are obtained by adding every $e \in E_G - E_H$ into H.

Conversely, for every $F \in \mathcal{N}_{i+1}$ with $brg(F) \leqslant x$ and an edge $e \in E_F$ where e is not a bridge of F, $F - e$ has at least $brg(F)$ bridges, namely, $brg(F-e) \geqslant x$. This means that there may be a subgraph F in $\mathcal{N}_{i+1}$, where $brg(F) \leqslant x$, and an edge e in E_F, where e is not bridge of F, such that $brg(F-e) > x$.

Based on the above discussion, the validity of (i) has been shown.

By employing the method similar with that of (i), the validity of (ii) can be shown. □

By Lemma 5 and definitions, we immediately obtain the following inequalities.

$$(m-i)B_i^{\leqslant x} \leqslant (i+1-\beta_{i+1}^{\leqslant x})B_{i+1}^{\leqslant x} \tag{25}$$

$$(m-i)B_i^{\geqslant x} \geqslant (i+1-\beta_{i+1}^{\geqslant x})B_{i+1}^{\geqslant x} \tag{26}$$

The following formulas are driven by the definitions of $B_i^{\leqslant x}$, $B_i^{\geqslant x}$, and β_i.

$$\begin{aligned}\sum_{x=0}^{\max_\beta(n,i)} B_i^{\leqslant x} &= \sum_{x=0}^{\max_\beta(n,i)} \Big(\sum_{b=0}^{x} B_i(b)\Big) \\ &= \sum_{b=0}^{\max_\beta(n,i)} \big(\max_\beta(n,i)+1-b\big)B_i(b) \\ &= \big(\max_\beta(n,i)+1-\beta_i\big)N_i \end{aligned}\tag{27}$$

$$\begin{aligned}\sum_{x=0}^{\max_\beta(n,i)} B_i^{\geqslant x} &= \sum_{x=0}^{\max_\beta(n,i)} \Big(\sum_{b=x}^{\max_\beta(n,i)} B_i(b)\Big) \\ &= \sum_{b=0}^{\max_\beta(n,i)} (b+1)B_i(b) \\ &= (1+\beta_i)N_i \end{aligned}\tag{28}$$

We introduce notation β_i^* to be defined as follows:

$$\begin{aligned}\sum_{b=0}^{\max_\beta(n,i)} b^2 B_i(b) &= \beta_i^* \sum_{b=0}^{\max_\beta(n,i)} bB_i(b) \\ &= \beta_i^* \beta_i N_i \end{aligned}\tag{29}$$

It is not hard to verify that $\beta_i^* \leqslant \beta_i$ by definitions. Namely,

$$(\beta_i^* - \beta_i)\beta_i = \frac{1}{N_i^2} \sum_{0 \leqslant a < b \leqslant \max_\beta(n,i)} (b-a)^2 B_i(a)B_i(b) \ \geqslant\ 0.$$

By the definition of $\beta_i^{\leqslant x}$, we also have

$$\beta_i^{\leqslant x} B_i^{\leqslant x} = \sum_{b=0}^{x} bB_i(b) \tag{30}$$

Thus,

$$\begin{aligned}\sum_{x=0}^{\max_\beta(n,i)} \beta_i^{\leqslant x} B_i^{\leqslant x} &= \sum_{x=0}^{\max_\beta(n,i)} \Big(\sum_{b=0}^{x} bB_i(b)\Big) \\ &= \sum_{x=0}^{\max_\beta(n,i)} (\max_\beta(n,i)+1-b)bB_i(b) \\ &= \big(\max_\beta(n,i)+1-\beta_i^*\big)\beta_i N_i \end{aligned} \tag{31}$$

By formula (2), it is easy to verify that

$$\max_\beta(n,i) \geqslant \max_\beta(n,i+1) \geqslant \max_\beta(n,i)+1.$$

In particular, there are many integers i's such that $\max_\beta(n,i) = \max_\beta(n,i+1)$.

Now, we consider an integer i of the case: $\max_\beta(n,i) = \max_\beta(n,i+1)$. We take the sum on both hand-sides of inequality (25) over all x's ($0 \leqslant x \leqslant \max_\beta(n,i)$), and obtain

$$(m-i)\sum_{x=0}^{\max_\beta(n,i)} B_i^{\leqslant x} \leqslant \sum_{x=0}^{\max_\beta(n,i+1)} (i+1-\beta_{i+1}^{\leqslant x})B_{i+1}^{\leqslant x}. \tag{32}$$

Note that $\max_\beta(n,i) = \max_\beta(n,i+1)$. By applying the above formulas, from the above inequality, we can obtain

$$\begin{aligned}&(m-i)(\max_\beta(n,i)+1-\beta_i)N_i \\ \leqslant\ &\big((i+1)(\max_\beta(n,i+1)+1-\beta_{i+1}) - (\max_\beta(n,i+1)+1-\beta_{i+1}^*)\beta_{i+1}\big)N_{i+1}\end{aligned}$$

By formula (18), the above inequality is written as follows:

$$(i+1-\beta_{i+1})(\max_\beta(n,i)+1-\beta_i) \leqslant (i+1-\beta_{i+1})(\max_\beta(n,i)+1) - (i+2-\beta_{i+1}^*)\beta_{i+1},$$

which is further rewritten as follows:

$$(i+2-\beta_{i+1}^*)\beta_{i+1} \leqslant (i+1-\beta_{i+1})\beta_i. \tag{33}$$

Concluding the above discussions, we have obtained the following theorem.

Theorem 5. *For a graph G and an index $i\,(n-1 \leqslant i \leqslant m-1)$ with $max_\beta(n,i) = max_\beta(n,i+1)$, we have*

$$\beta_{i+1} \leqslant \beta_i + \frac{\beta_{i+1}}{i+1}(\beta^*_{i+1} - \beta_i - 1). \qquad (34)$$

Proof. Concluding the above discussions, it follows inequality (33). □

For integers i's satisfying $\max_\beta(n,i) = 1 + \max_\beta(n, i+1)$, we can also obtain the result similar to inequality (34) by the method above used.

In particular, by Theorem 4(i), if

$$\frac{\beta_{i+1}}{i+1}(\beta^*_{i+1} - \beta_i - 1) \leqslant 1,$$

equivalently,

$$(\beta^*_{i+1} - \beta_i - 1)\beta_{i+1} \leqslant i + 1, \qquad (35)$$

then

$$N_i^2 \geqslant N_{i-1}N_{i+1}.$$

This means that $N_i^2 \geqslant N_{i-1}N_{i+1}$ holds for the larger integers that are larger than $(\beta^*_{i+1} - \beta_i - 1)$ β_{i+1}. Investigating properties on the graphs satisfying inequality (35) is also interesting as a future research subject.

5 Concluding Remarks

Based on the number of bridges in a graph, we propose a new method of partitioning the set $\mathcal{N}_i(G)$ of the all possible connected spanning i-edge-subgraphs of G into the subsets: $\mathcal{B}_i(G;0)$, $\mathcal{B}_i(G;1)$, $\cdots$, $\mathcal{B}_i(G;\max_\beta(n,i))$, where $\mathcal{B}_i(G;b)$ represents the set of all possible connected spanning i-edge-subgraphs of G having b bridges. Similarly, by employing the number of terminals in a graph, we can also partition $\mathcal{N}_i(G)$ into the subsets, $\mathcal{T}_i(G;0)$, $\mathcal{T}_i(G;1)$, $\cdots$, $\mathcal{T}_i(G;\max_\beta(n,i))$ where $\mathcal{T}_i(G;t)$ represents the set of all possible connected spanning i-edge-subgraphs of G having t terminals. In particular, inequalities have been shown in Theorems 1, 2 to express fundamental relationships between $|\mathcal{B}_i(G;b)|$ and $|\mathcal{T}_i(G;t)|$.

Furthermore, we introduce notation β_i, defined in formula (16), to represent the average value of bridges with respect to all possible connected spanning i-edge-subgraphs. By applying β_i, we have obtained a formula (19) to express a fundamental relationship between $N_i(G)$ and $N_{i+1}(G)$, shown in Theorem 3. Consequently, we have obtained sufficient conditions, shown in Theorem 4, to ensure that sequence $N_{n-1}(G), N_n(G), \cdots, N_m(G)$ is log-concave, namely, $N_i^2(G) \geqslant N_{i-1}(G)N_{i+1}$ (G) holds for all integers i's $(n \leqslant i \leqslant m-1)$. Note that both β_i, β_{i+1} are contained in the sufficient conditions.

In order to implicate some relationships between β_i and β_{i+1}, we introduce notation β^*_i to establish some inequalities for expressing some relationships between β_i and β_{i+1}, e.g., see Theorems 5. In particular, by Theorems 4, 5, we have obtained that sequence $N_{n-1}(G)$, $N_n(G)$, $\cdots$, $N_m(G)$ for a graph G is log-concave if the following inequality holds for all integers i's $(n-1 \leqslant i \leqslant m-1)$.

$$(\beta^*_{i+1} - \beta_i - 1)\beta_{i+1} \leqslant i+1$$

This means that if we can prove inequality $(\beta^*_{i+1} - \beta_i - 1)\beta_{i+1} \leqslant n$ then the validity of log-concave conjecture on sequence $N_{n-1}(G)$, $N_n(G)$, $\cdots$, $N_m(G)$ will be obtained. However, it is open whither, $(\beta^*_{i+1} - \beta_i - 1)\beta_{i+1} \leqslant n$ holds or not.

Then, it seems to be interesting to find properties of a graph G such that $(\beta^*_{i+1}-\beta_i-1)$ $\beta_{i+1} \leqslant n$ holds by further investigating properties of β_i, β^*_i. In addition, little is known about results in investigating them.

On the other hand, by using the method similar to that of investigating $|\mathcal{B}_i(G;b)|$'s in this paper, we can also obtain formulas on $|\mathcal{T}_i(G;t)|$'s, and may obtain some results desired for proving log-concavity of sequence $N_{n-1}(G)$, $N_n(G)$, $\cdots$, $N_m(G)$ by using the formulas. Then, it seems to be an interesting subject as future research.

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