Estimation of the Jump Activity Index in the Presence of Random Observation Times

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Abstract

This work studies the estimation of the *jump activity index* of Itô semimartingales in a setting of high frequency observations with a fixed time horizon and random observation times.

We give a quick overview over the underlying theory and briefly review already existing literature connected to the estimation of *jump activity index* in various settings.

We then prove a central limit theorem based on the *empirical characteristic function* whose value is in our case codetermined by the (possibly unknown) structure of the underlying observation scheme. To bypass this problem we employ an approach, that is new to existing literature, using a Taylor expansion of the natural logarithm and the exponential function to develop a consistent estimator for the *jump activity index*. Yet again, the connected central limit theorem (CLT) depends on the setting of the observation scheme and is therefore not directly applicable in most situations. Hence, we develop a further CLT that works without any prior knowledge of the underlying structures.

Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit der Schätzung des *jump activity index* von Itô-Semimartingalen in einem Szenario von hochfrequenten Beobachtungen mit fixem Zeithorizont und zufälligen Beobachtungszeiten.
Zuerst geben wir einen kurzen Überblick über die zugrunde liegende Theorie und besprechen bereits vorhandene Literatur zur Schätzung des *jump activity index* unter verschiedenen Annahmen.
Dann beweisen wir einen zentralen Grenzwertsatz basierend auf der *empirischen charakteristischen Funktion*, deren Wert, in unserem Fall, von der (gegebenenfalls unbekannten) Struktur des zugrunde liegenden Beobachtungsschemas abhängt. Um dieses Problem zu umgehen, verwenden wir einen bis dato noch nicht benutzten Ansatz, basierend auf einer Taylor-Entwicklung des natürlichen Logarithmus und der Exponentialfunktion, um einen konsistenten Schätzer für den *jump activity index* zu konstruieren. Jedoch ist auch in diesem Fall der zugehörige zentrale Grenzwertsatz abhängig von der Struktur des Beobachtungsschemas und somit in vielen Situationen nicht direkt anwendbar. Deswegen entwickeln wir einen weiteren zentralen Grenzwertsatz, der ohne vorheriges Wissen über den Aufbau des Beobachtungschemas auskommt.

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Chapter 1

Motivation and Content of the Work at Hand

Classical financial mathematic models with a time continuous setting (like the famous Black-Scholes model) often involve general forms of stochastic integrals with respect to a Brownian motion but very rarely include jumps at all.
Yet the heavy tails of financial asset returns and other properties of financial data suggest the existence of jumps. General semimartingales that occur for example as the solution of stochastic differential equations driven by Lévy processes with jumps offer a great deal of flexibility when it comes to modeling asset prices, e.g. when modeling electricity prices ([GKM11]).
However, fitting these models to real data is often more involved, as the ingredients governing the jumps of the process have to be estimated as well. A key factor here is the *jump activity index*, a measure for the intensity or rate with which jumps occur. In the past decade much work has been done on the estimation of this index in various statistical settings though often under quite restrictive assumptions. The work at hand builds upon the existing literature while trying to fill some of the gaps where only few work has been done until now, namely by finding estimators for the jump activity index when the process is observed at randomly chosen time points in contrast to equidistant spaced time points. For many applications this seems like a natural (and much needed) generalization although full generality does not seem to be achievable with present techniques.
While our estimator builds upon ideas already developed in a setting of equidistant observations, some concepts cannot be directly applied in the case of random observation times and have to be fitted to our specific setting. In particular, the main concept of our estimator, the *empirical characteristic function* is in our setting dependent on the (possibly unknown) structure of the observation scheme and this is why the evaluation of

those is notably harder than in the equidistant case. To solve this problem, our estimator evaluates the *empirical characteristic function* in points converging to zero and then uses a Taylor expansion of the natural logarithm and exponential function.

This work is structured in the following way: It starts with two introductory chapters, the first one being a basic introduction to semimartingales where fundamental terms like *Itô semimartingales, jump measures* and their *compensators* are briefly explained. The second chapter deals with the topic of how jumps of Lévy processes or in general semimartingales can be characterized, here the *Blumenthal-Getoor index* is introduced and likewise his semimartingale counterpart, the *jump activity index.* The chapter ends with an overview of recent developments in the estimation of the jump activity index in the statistical setting of high frequency statistics, i.e. when the mesh of observation points gets finer while keeping a finite time horizon.
The next two chapters are the main parts of this work. In Chapter 4 we prepare basic estimates for Itô semimartingales and apply a localization procedure to our exact setting of random observation times in order to strengthen general assumptions to more useful stronger ones. This establishes the foundation for Chapter 5 where we introduce our actual estimator for the jump activity index. Furthermore, we give a heuristic explanation of how and why our adaption of the concepts for equidistant time points works in our specific setting and finally prove an associated central limit result.
In the last chapter we provide a numerical assessment of our estimator. For this purpose we simulate an underlying process that is observed at a realistic number of random observation times and investigate how the asymptotic properties from the previous chapter perform for a finite sample. Furthermore, as the limiting distribution in the CLT from the previous chapter contains moments depending on the structure of the observation scheme and therefore direct application, e.g. for finding confidence intervals, is usually not feasible, we find a consistent estimator for that variance and upon this build a CLT that works without any prior knowledge of the observation scheme. In particular, the estimator for these unknown moments is a small result in itself and may be used in other applications as well.

Chapter 2

Itô Semimartingales and their Basic Properties

The following chapter is an introduction to the terms associated with the analysis and estimation of the jumps of a Lévy process or more general of an Itô semimartingale, it is adapted from the introductions found in the standard textbooks on stochastic processes and their estimation, i.e. Chapter 2.1 in [JP12], Chapters 1.4-2.2 in [JS87], Chapter 1.4 in [ASJ14] or Chapter 3 in [EK19]. Rather than providing extensive proof we only present the basic definitions and results that are needed to understand the following chapters. However, rigorous proofs can especially be found in [JS87].

We enter this section by introducing the term (Itô) semimartingale, starting with the characteristics and the definition of a general semimartingale. In this opening we consider (like in our references) d-dimensional semimartingales although later on we only work with one-dimensional ones.

What follows now is the fundamental class of processes with which we deal throughout this work.

Definition 2.1. *An $\mathbb{R}^d$-valued process X on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is called a semimartingale if*

1. *X is adapted;*
2. *X has càdlàg paths;*
3. *$X = X_0 + A + L$ where L is a local martingale and A is a process of finite variation with $A_0 = L_0 = 0$.*

This decomposition can be made more precise by splitting up L into a continuous local martingale X^c and a *purely discontinuous local martingale* M, the latter meaning that the product MN is a local martingale for any continuous local martingale N. The decomposition of X then reads as

$$X_t = X_0 + A_t + X_t^c + M_t, \tag{2.1}$$

again with $A_0 = X_0^c = M_0 = 0$.

Nevertheless the decomposition above is not unique. Therefore one wants to employ a strengthened version of the definition above:

Definition 2.2. *A semimartingale Y is called a special semimartingale if $Y = Y_0 + A' + L'$ where L' is a local martingale and A' is a predictable process of finite variation. A' is called the compensator of Y.*

In this case the decomposition is unique and can be seen as a more general version of the Doob-Meyer decomposition.

Theorem 2.1. *For a process X the following properties are equivalent:*

1. *X is a special semimartingale.*
2. *X is a locally integrable semimartingale.*

As semimartingales with bounded jumps are at least locally integrable, e.g. by using the localizing sequence of stopping times $\tau_n := \inf\{t : X_t \geq n\} \nearrow \infty$, we have the following implication.

Remark 2.1. *A semimartingale X is a special semimartingale if its jumps are bounded.*

Using the definition of special semimartingales we can find a more detailed decomposition of a general semimartingale X. For the jumps $\Delta X_t := X_t - \lim\limits_{s \nearrow t} X_s$ of X we set $J_t = \sum_{0 \leq s \leq t} \Delta X_s \mathbb{1}_{\{||\Delta X_s|| \geq 1\}}$ (where $|| \cdot ||$ is the Euclidean vector norm in $\mathbb{R}^d$) and look at the special semimartingale Y defined by

$$Y_t := X_t - J_t = X_0 + B_t + X_t^c + M_t'$$

which now has a unique decomposition into a predictable process of finite variation B and a continuous X^c and purely discontinuous martingale part M'. As X^c is continuous and therefore does not contain jumps it does not depend on the decomposition by J and is the same as in (2.1). In total this yields

$$X_t = X_0 + B_t + X_t^c + M_t' + \sum_{0 \leq s \leq t} \Delta X_s \mathbb{1}_{\{||\Delta X_s|| \geq 1\}}. \tag{2.2}$$

That means depending on where we set the cutoff for the jumps of $Y = X - J$ we find a unique decomposition of X, or more precise we find a unique decomposition for each truncation function $\kappa : \mathbb{R}^d \to \mathbb{R}^d$, i.e. κ is a bounded measurable function with $\kappa(x) = x$ in a neighborhood around 0, in the following way: Setting $J_t = \sum_{0\leq s\leq t} \kappa'(\Delta X_s)$ for $\kappa'(x) = x - \kappa(x)$ we find a decomposition of X as

$$X_t = X_0 + B_t^\kappa + X_t^c + M_t^\kappa + \sum_{0\leq s\leq t} \kappa'(\Delta X_s), \tag{2.3}$$

for a unique predictable process of finite variation B^κ and a unique completely discontinuous local martingale M^κ.

If the process X in (2.3) were to be a Lévy process with *characteristic triplet* (b^κ, c, F) w.r.t. κ where $b^\kappa \in \mathbb{R}^d$ is the *drift vector*, $c \in \mathbb{R}^{d\times d}$ is a symmetric, nonnegative definite *diffusion matrix* and $F(dx)$ is a *Lévy measure* on $\mathbb{R}^d$ then $B_t^\kappa = b^\kappa t$ and $X_t^c = c^{1/2} W_t$, where $c^{1/2} \in \mathbb{R}^{d\times d}$ with $\left(c^{1/2} c^{1/2}\right)^T = c$ and W a standard Brownian motion on $\mathbb{R}^d$. The remaining part $M^\kappa + J$ then contains the jumps of the process and is completely characterized by the Lévy measure $F(dx)$. How this part can be constructed from $F(dx)$ is laid out below.

Now Itô semimartingales can be described as a subclass of general semimartingales such that for an infinitesimal small time period these behave like a Lévy process, meaning that for each time point $s \in [0, T]$ there exists a characteristic triplet (b_s^κ, c_s, F_s) (making b_s^κ, c_s, F_s processes, though with very different spaces to which they map) and the set of these triplets for all $s \in [0, T]$ characterizes the behavior of the process up to time T completely. For an Itô semimartingale we have that

$$B_t^\kappa = \int_0^t b_s^\kappa ds \qquad \text{and} \qquad X_t^c = \int_0^t c_s^{1/2} dW_s.$$

In order to understand how the remaining parts consisting of the jumps can be constructed from F_s one has to understand the concept of *random measures* and its *compensators.* Let $D(X) = \{(\omega, t) : \Delta X_t(\omega) \neq 0\}$ then the *jump measure* μ^X of X is defined as

$$\mu^X(\omega; dt, dx) = \sum_{(\omega,s)\in D(X)} \epsilon_{(s,\Delta X_s(\omega))}(dt, dx) \tag{2.4}$$

where ϵ_a is the Dirac measure with mass 1 in $a \in \mathbb{R}_+ \times \mathbb{R}^d$. μ^X is then a *random measure* meaning that for each that $\omega, \mu^X(\omega; \cdot)$ is an integer valued measure on $\mathbb{R}_+ \times \mathbb{R}^d$. Furthermore for each Borel subset A of $\mathbb{R}^d$ we define

$$\begin{aligned} \mathbb{1}_A \star \mu_t^X(\omega) &:= \mu^X(\omega; (0,t] \times A) \\ &= \sum_{s\leq t} \mathbb{1}_A(\Delta X_s(\omega)) = |\{(s,x) \in (0,t] \times A : \Delta X_s(\omega) = x\}|. \end{aligned} \tag{2.5}$$

Now $\mathbb{1}_A \star \mu^X$ can be seen as a non-decreasing and adapted process which is finite valued if $\inf\{||x|| : x \in A\} > 0$ as we only have a finite number of jumps bigger than $\inf\{||x|| : x \in A\}$ on any interval $(0,t]$. If $\inf\{||x|| : x \in A\} > 0$ the process $\mathbb{1}_A \star \mu^X$ admits a predictable compensator and one can find a positive valued *random measure* $\nu^X(\omega; dt, dx)$ on $\mathbb{R}_+ \times \mathbb{R}^d$ such that the process defined via

$$\mathbb{1}_A \star \nu_t^X(\omega) = \nu^X(\omega; (0,t] \times A) \tag{2.6}$$

is the compensator of the process $\mathbb{1}_A \star \mu^X$. The random measure ν^X is then called the *(predictable) compensator* of μ^X. One may extend the notation of (2.5) and (2.6) to more general functions of the form $\delta : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}, (\omega, t, x) \mapsto \delta(\omega, t, x)$, by defining "$\omega$ - wise"

$$\delta \star \mu_t^X(\omega) = \int_{[0,t]\times\mathbb{R}^d} \delta(\omega, s, x)\mu^X(\omega; ds, dx)$$
$$\delta \star \nu_t^X(\omega) = \int_{[0,t]\times\mathbb{R}^d} \delta(\omega, s, x)\nu^X(\omega; ds, dx)$$

whenever the right hand sides make sense, i.e. when for $t \geq 0$

$$\int_{[0,t]\times\mathbb{R}^d} |\delta(\omega, s, x)|\mu^X(\omega; ds, dx) < \infty \quad \text{or} \quad \int_{[0,t]\times\mathbb{R}^d} |\delta(\omega, s, x)|\nu^X(\omega; ds, dx) < \infty. \tag{2.7}$$

Here it should be noted that the second condition in (2.7) implies the first one (c.f. [EK19], Theorem 3.36) and that it is customary to use a shorthand notation for some functions, e.g. $\mathbb{1}_A(\omega, t, x) = \mathbb{1}_A(x)$ (which we already used above) and $x(\omega, t, y) = y$. As a very prominent example we have

$$x\mathbb{1}_{\{||x||\geq 1\}} \star \mu_t^X = \int_{[0,t]\times\mathbb{R}^d} x\mathbb{1}_{\{||x||\geq 1\}}\mu^X(ds, dx) = \sum_{s\leq t} \Delta X_s \mathbb{1}_{\{||\Delta X_s||\geq 1\}}.$$

Again notation may be found a little bit misleading as x stands for two different things in the line above, on the one hand it is a function and on the other an integration variable. Also in contrast to (2.7), ω is suppressed in line with general notation for stochastic processes. Now we can finally define the *characteristics* of a general semimartingale X also known as "predictable characteristics" or "integrated characteristics" as being the triplet (B^κ, C, ν^X), for a truncation function κ, where

- $B^\kappa = ((B^\kappa)^i)_{1\leq i\leq d}$, the predictable process of locally finite variation with $B_0^\kappa = 0$, occuring in (2.3) when truncating with $\kappa(x)$,
- $C = (C^{ij})_{1\leq i,j\leq d}$, where $C^{ij} = \langle (X^c)^i, (X^c)^j \rangle$ (for a definition see either p.28 in [JP12] or the end of this chapter),

- ν^X is the compensator of the jump measure μ^X as defined above.

As Lévy processes are semimartingales the characteristic triplet (b^κ, c, F) of a Lévy process and their semimartingale characteristics are directly linked in the following way, already partly mentioned above.

Remark 2.2. *A d-dimensional $(\mathcal{F}_t)$-semimartingale X is an $(\mathcal{F}_t)$-Lévy process if and only if $X_0 = 0$ (depending on the definition) and its characteristics are of the form*

$$B_t^\kappa(\omega) = b^\kappa t, \qquad C_t(\omega) = ct, \qquad \nu^X(\omega; dt, dx) = dt \otimes F(dx).$$

So these characteristics are non-random and are furthermore linear over time. As illustrated above Itô semimartingales can be seen as time-varying Lévy processes whose behavior at a certain time point s is characterized by a characteristic Lévy triplet (b_s^κ, c_s, F_s) giving rise to the following definition:

Definition 2.3. *A d-dimensional semimartingale X is an Itô semimartingale if its characteristics (B^κ, C, ν^X) are absolutely continuous with respect to the Lebesgue measure, that is*

$$B_t^\kappa(\omega) = \int_0^t b_s^\kappa(\omega) ds, \qquad C_t(\omega) = \int_0^t c_s(\omega) ds, \qquad \nu^X(\omega; [0,t] \times A) = \int_0^t F_s(\omega, A) ds,$$

where $(b_t^\kappa)_{t \geq 0}$ is a $\mathbb{R}^d$-valued process, $(c_t)_{t \geq 0}$ is a process in the space of symmetric, non-negative definite matrices, and $F_t = F_t(\omega, dx)$ is for each (ω, t) a measure on $\mathbb{R}^d$.

These b^κ, c and F_t have to fulfill additional measurability properties that ensure that the definitions above make sense and fit into the definition of general semimartingale characteristics. In addition one can always find a version of $F_t(\cdot, dx)$ that fulfills, similar to standard Lévy measures, for each (ω, t):

$$\int \left(||x||^2 \wedge 1 \right) F_t(\omega, dx) < \infty.$$

To round things up we want to write the discontinuous martingale part M^κ in (2.3) in terms of the jump measure μ^X and its compensator ν^X. As for each Borel subset A of $\mathbb{R}^d$ with $\inf \{||x|| : x \in A\} > 0$ we have that $\mathbb{1}_A \star \mu^X - \mathbb{1}_A \star \nu^X$ is a local martingale and for an Itô semimartingale it holds that $\kappa(\Delta M^\kappa) = \kappa(\Delta X)$ we would like to define $M^\kappa = \kappa \star \mu^X - \kappa \star \nu^X$. The problem that arises here is that, due to $\kappa(x) = x$ in a neighborhood around 0, $\kappa \star \mu^X < \infty$ $a.s$ is equivalent to X being of finite variation which is too restrictive. To bypass this problem one can define the term of a *predictable function*

δ, i.e. $\delta : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ that is measurable w.r.t. the σ-field $\mathcal{P} \otimes \mathcal{R}^d$, where $\mathcal{P}$ is the predictable σ-field on $\Omega \times \mathbb{R}_+$ and $\mathcal{R}^d$ is the Borel σ-field on $\mathbb{R}^d$. If it additionally fulfills

$$\left(|\delta| \wedge \delta^2\right) \star \nu_t^X < \infty, \forall t > 0, \tag{2.8}$$

there exists a unique purely discontinuous local martingale whose jumps are given by

$$\int_{\mathbb{R}^d} \delta(t, dx)(\mu^X - \nu^X)(\{t\}, dx) = \delta(t, \Delta X_t) - \int_{\mathbb{R}^d} \delta(t, x)\nu^X(\{t\}, dx). \tag{2.9}$$

The unique purely discontinuous local martingale with jumps as in (2.9) is called *stochastic integral* of δ with respect to $\mu - \nu$ and is denoted by

$$\int_0^t \int_{\mathbb{R}^d} \delta(s, x)(\mu^X - \nu^X)(ds, dx) \qquad \text{or} \qquad \delta \star (\mu^X - \nu^X)_t.$$

Moreover this coincides with the jumps of $\delta \star \mu^X - \delta \star \nu^X$ if $\delta \star \nu^X$ is well-defined. We note that the function $\delta(\omega, t, x) = \kappa(x)$ is predictable and that when we decompose, for some $\epsilon > 0$, $\kappa(x) = x \mathbb{1}_{\{||x|| < \epsilon\}} + \kappa(x)\mathbb{1}_{\{||x|| \geq \epsilon\}}$ we have that $\kappa^2 \star \mu_t^X = \sum_{s \leq t} \kappa(\Delta X_s)^2 < \infty$ due to the fact that $\sum_{s \leq t} (\Delta X_s)^2 < \infty$, $\kappa(x)$ is bounded and we have almost surely only finitely many jumps bigger than ϵ. Furthermore, again due to the boundedness of κ, $\kappa^2 \star \mu^X$ has bounded jumps, is therefore locally integrable and allows for the compensator $\kappa^2 \star \nu^X$ to exist, therefore condition (2.8) is fulfilled. Finally putting all the components together we arrive at the *Lévy-Itô decomposition* of a semimartingale

$$X = X_0 + B^\kappa + X^c + \kappa \star (\mu^X - \nu^X) + (x - \kappa) \star \mu^X,$$

where $\kappa \star (\mu^X - \nu^X)$ and $(x - \kappa) \star \mu^X$ should be read component by component if $d > 1$. In the case of an Itô semimartingale we have more specifically

$$\begin{aligned} X_t = X_0 + \int_0^t b_s^\kappa ds + \int_0^t c_s^{1/2} dW_s + \int_{(0,t] \times \mathbb{R}^d} \kappa(x)(\mu^X - \nu^X)(ds, dx) \\ + \int_{(0,t] \times \mathbb{R}^d} (x - \kappa(x))\mu^X(ds, dx). \end{aligned} \tag{2.10}$$

Usually in applications Itô semimartingales are the objects considered and for estimating their rate of growth we need this specific form of time continuous characteristics. Therefore in the sequel (as in most literature regarding the subject of statistics on semimartingales) we only look at the class of Itô semimartingales. There is one last addition to make, namely that not only there exists a decomposition in the form of (2.10) but also that every d-dimensional Itô semimartingale can be written with respect to the same Brownian motion and (compensated) random measure μ, $\mu - \nu$. This is then called the *Grigelionis decomposition* of semimartingale. For this matter we need to be able to define *random* *measures* that are not directly associated to a process but are rather defined by a single (non random) measure λ on some space E. To be more precise, $(E, \mathcal{E})$ is an arbitrary Polish space endowed with its Borel σ-field $\mathcal{E}$ and λ is a σ-finite measure. Then a random measure $\mu = \mu(\omega; dt, dx)$ on $\mathbb{R}_+ \times E$ is called $(\mathcal{F}_t)$-*Poisson random measure* if it is the sum of Dirac masses, no two such masses lie on the same "vertical" line $\{t\} \times E$ and that for all $A \in \mathcal{E}$ with $\lambda(A) < \infty$ we have (again using the shorthand $\mathbb{1}_A(\omega, t, x) = \mathbb{1}_A(x)$):

- $\mathbb{1}_A \star \mu_t = \mu([0,t] \times A)$ is an $(\mathcal{F}_t)$-Lévy process,
- $\mathbb{E}\left[\mathbb{1}_A \star \mu_t\right] = t\lambda(A)$.

When $\lambda(A) < \infty$ we notice that $\mathbb{1}_A \star \mu_t$ is an ordinary Poisson process with parameter $\lambda(A)$. Setting $\nu(\omega; dt, dx) = dt \otimes \lambda(dx)$ we find that for all $B \in \mathcal{R}_+ \otimes \mathcal{E}$ with $\nu(B) < \infty$ $\mathbb{1}_B \star \nu_t$ is the compensator of $\mathbb{1}_B \star \mu_t$ and therefore ν is the (non random and hence predictable) compensator of μ. Comparing this with Remark 2.2, it is no surprise that the jump measure μ^X of an $(\mathcal{F}_t)$-Lévy process is indeed a $(\mathcal{F}_t)$-Poisson random measure with $E = \mathbb{R}^d$ and therefore when defining μ as above the measure λ is often called the *Lévy measure* of ν. For all predictable (i.e. measurable w.r.t. $\mathcal{P} \otimes \mathcal{E}$) functions δ on $\Omega \times \mathbb{R}_+ \times E$ which satisfy (2.8) with ν^X replaced by ν one may then define $\delta \star (\mu - \nu)_t$ as in (2.9) and similarly one can generalize the definitions of $\delta \star \mu_t, \delta \star \nu_t$ and the other concepts presented previously.
Now follows the Grigelionis decomposition, let $d' \geq d$ and E be an arbitrary Polish space with a σ-finite measure λ with $\lambda(E) = \infty$ having not atoms.

Theorem 2.2 (Thm. 2.1.2 in [JP12]). *Let X be a d-dimensional Itô semimartingale on the space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbb{P})$, with characteristics (B, C, ν^X) given as in Definition 2.3. Then there exists a very good filtered extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}})_{t\geq 0}, \widetilde{\mathbb{P}})$(definition p.36, [JP12]), on which there are defined a d'-dimensional Brownian motion W and a Poisson random measure μ on $\mathbb{R}_+ \times E$ with compensator ν, such that*

$$X_t = X_0 + \int_0^t b_s^\kappa ds + \int_0^t \eta_s dW_s + \int_{(0,t]\times E} \kappa(\delta^X(s,x))(\mu - \nu)(ds, dx) \tag{2.11}$$
$$+ \int_{(0,t]\times E} (\delta^X(s,x) - \kappa(\delta^X(s,x)))\mu(ds, dx),$$

where η_t is an $\mathbb{R}^d \otimes \mathbb{R}^{d'}$- valued predictable process on $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbb{P})$ and δ^X is a predictable $\mathbb{R}^d$-valued function on $\Omega \times \mathbb{R}_+ \times E$.
Additionally outside a null set one has $\eta_t \eta_t^T = c_t$ and $F_t(\omega, A) = \lambda(\{x : \delta(\omega, t, x) \in A\})$ for each $A \in \mathcal{R}^d$ with $0 \notin \overline{A}$, where $\overline{A}$ is the closure of A.
Conversely, even if X is defined via (2.11) with b^κ, η, δ defined on the extension it is still an Itô semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbb{P})$ if it is further adapted to $(\mathcal{F}_t)_{t\geq 0}$.

There is much freedom in choosing the number of Brownian motions d', the space E and the measure λ. A canonical choice would be $E = \mathbb{R}^d$ with λ being the Lebesgue measure but $E = \mathbb{R}$ is also always possible even if the semimartingale X has more than one dimension. The important thing here is that even countably many semimartingales $X, Y, Z, \ldots$ can be represented at the same time by the same random measure μ by using a different function $\delta^X, \delta^Y, \delta^Z, \ldots$ for each process to be represented. So basically when comparing two processes X, Y with a representation as in (2.11) all information about the jumps is encapsulated in the functions δ^X, δ^Y, e.g. if δ^X is bounded the jumps of X are bounded.
(2.11) is how most papers on statistics for semimartingales are set up and we continue in the same manner when we come to Chapter 4 where we give upper bounds for $\mathbb{E}\left|X_{t_i} - X_{t_{i-1}}\right|$ dependent on the *coefficients* b^κ, η, δ^X of X.

At last we want to briefly present the *quadratic variation* of a process X, a concept which is very important to the world of stochastic calculus in general and for us of relevance when we want to calculate the aforementioned upper bounds. If Y is a continuous local martingale the local submartingale Y^2 allows, by the Doob-Meyer decomposition, for an unique increasing adapted continuous process with $Y_0 = 0$, and denoted by $< Y, Y >$, such that $Y^2 - < Y, Y >$ is a local martingale. For a one-dimensional semimartingale X we define

$$[X, X]_t = \langle X^c, X^c \rangle_t + \sum_{s \leq t} (\Delta X_s)^2$$

as the *quadratic variation process of* X. If X is in addition an Itô semimartingale we find that this reads as

$$[X, X]_t = \int_0^t c_s^2 \, ds + x^2 \star \mu_t^X.$$

The last two formulas will play a leading role when we want to find the aforementioned upper bounds for $\mathbb{E}\left|X_{t_i} - X_{t_{i-1}}\right|$ with the help of the Burkholder-Davis-Gundy inequality, cf. Theorem 4.1.
For two one-dimensional process X, Y this concept can be generalized to the *quadratic covariation process of* X *and* Y, denoted as $[X, Y]$ and if $[X, Y]$ is locally integrable it admits a (predictable) compensator denoted by $< X, Y >$. Furthermore the quadratic covariation gives rise to a definition of $[X, X]$ and $[X, Y]$ when X, Y are d-dimensional. However, this case is of no further relevance for the rest of the work and hence we point to any of the references mentioned at the beginning of this chapter.

Chapter 3

The Blumenthal-Getoor and the Jump Activity Index

3.1 The Blumenthal-Getoor Index

We now take one step back from the general setting of semimartingales and look at Lévy processes, in particular the properties of their jumps and how these can be characterized. As the results presented in this section are much closer related to the quantities that we want to estimate in Chapter 5 and as such closer to the actual topic of this work, we employ a more rigorous approach than in the previous chapter.
We assume that X is a one-dimensional Lévy process (although the concepts presented here may be lifted easily to more than one dimension) on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with a characteristic triplet in the Lévy process sense (b^κ, c, F) with respect to some truncation function $\kappa(x)$. Then, as already mentioned, the jump measure $\mu = \mu^X$ is a Poisson random measure with compensator

$$\nu(dt, dx) = dt \otimes F(dx).$$

In this introductory chapter we fix a setting of equidistant observation times with a finite time horizon $0 < T < \infty$, i.e. for $\Delta_n = \frac{1}{n}$ we observe $X_0, X_{\Delta_n}, X_{2\Delta_n}, \ldots, X_T$ while $\Delta_n \to 0$. The question that arises now, is which parts of the measure F can be estimated from a single path of the process X.
In Lemma 3.3 we prove that for any Borel subset $A \subset \mathbb{R}$ we either have almost surely infinitely many jumps on any time interval $(t, t+s]$ iff $F(A) = \infty$ and almost surely finitely many iff $F(A) < \infty$. Choosing some $\epsilon > 0$ we have $F((\epsilon, \infty)) < \infty$ and therefore a single path of the process may or may not have a jump of size ϵ or bigger at all (this is true not only for Lévy processes but for all Itô semimartingales in general). Therefore

we cannot estimate F on its complete domain as even if the complete information of this single path would be available to us we would not be able to estimate $F((\epsilon,\infty))$. While testing for $F(\mathbb{R})<\infty$ or $F(\mathbb{R})=\infty$ is always possible, we want to infer the behavior of the jumps around zero, e.g. $F((x,\infty))$ when $x\searrow 0$. This is only feasible when the mass of these sets (x,∞) is increasing to infinity, which is equivalent to $F(\mathbb{R})=\infty$, and this is why we usually restrict ourselves to this case of infinite jump activity.

To describe the behavior of $F(dx)$ around zero and the related path behavior of X Blumenthal and Getoor [BG61] introduced the *Blumenthal-Getoor(BG) index* for Lévy processes

$$\beta := \inf(I) \quad \text{where} \quad I=\left\{p\geq 0: \int_{\{|x|\leq 1\}} |x|^p F(dx)<\infty\right\}. \tag{3.1}$$

Some key properties of β or the set I are rather obvious. First the set is always of the form $I=(\beta,\infty)$ or $I=[\beta,\infty)$ and since it must hold for all Lévy measures that $\int(|x|^2\wedge 1)F(dx)<\infty$ we have that $2\in I$ and in particular $\beta\in[0,2]$. Referring to basic properties of the process itself, X has finite jump activity if and only if $F(\mathbb{R})<\infty$ or $0\in I$ and $F(\mathbb{R})=\infty$. Furthermore we have $\beta>0$ in the case of infinite jumps. A more precise connection is stated later on.

As outlined above another way to describe the behavior of $F(dx)$ around 0 is to work with the (double sided) *tail function* of F namely

$$\overline{F}(r)=F(\{y:|y|\geq r\}). \tag{3.2}$$

Now we find an alternative definition/characterization of β in terms of $\overline{F}(x)$.

Theorem 3.1 (Theorem 2.1 in [BG61]). *It holds that*

$$\beta=\inf\left\{\alpha>0:\lim_{r\to 0} r^\alpha\overline{F}(r)=0\right\}.$$

Proof. First we note that for the (signed) measure $\mu_{\overline{F}}(dx)$ induced by the decreasing function $\overline{F}$ we have for $x>\epsilon>0$

$$\begin{aligned}\mu_{\overline{F}}([\epsilon,x]) &= \lim_{\widetilde{x}_n\searrow x}\overline{F}(\widetilde{x}_n)-\lim_{\widetilde{y}_n\nearrow\epsilon}\overline{F}(\widetilde{y}_n)\\ &= F((-\infty,-x)\cup(x,\infty))-F((-\infty,-\epsilon]\cup[\epsilon,\infty))\\ &= -F([-x,-\epsilon]\cup[\epsilon,x]),\end{aligned}$$

using the σ-continuity of F and the fact that $\{y:|y|\geq\widetilde{x}_n\}\nearrow(-\infty,-x)\cup(x,\infty)$ when $\widetilde{x}_n\searrow x$ and $\{y:|y|\geq\widetilde{y}_n\}\searrow(-\infty,\epsilon]\cup[\epsilon,\infty)$ when $\widetilde{y}_n\nearrow\epsilon$. Set $\gamma=\inf\left\{\alpha>0:\lim_{r\to0}r^\alpha\overline{F}(r)=0\right\}$ and choose some δ,δ' with $\delta>\delta'>\gamma$. Then for all $\epsilon>0$

$$\int_{\epsilon\leq|x|\leq1}|x|^\delta F(dx)=-\int_\epsilon^1 r^\delta d\overline{F}(r)=\epsilon^\delta\overline{F}(\epsilon)-\overline{F}(1)+\delta\int_\epsilon^1 r^{\delta-\delta'-1}r^{\delta'}\overline{F}(r)dr \tag{3.3}$$

where we used the partial integration rule for Stieltjes integrals in the last step. And because $r^{\delta'}\overline{F}(r) \to 0$ as $r \to 0$ we have $r^{\delta'}\overline{F}(r)\mathbb{1}_{[0,1]}(r) \leq K\mathbb{1}_{[0,1]}(r)$ for some constant $K > 0$ and as such for another finite number $M < \infty$

$$\int_{\epsilon}^{1} r^{\delta-\delta'-1} r^{\delta'}\overline{F}(r)dr \leq K \int_{\epsilon}^{1} r^{\delta-\delta'-1}dr \to M \quad \text{as} \quad \epsilon \to 0.$$

Considering the limit when $\epsilon \to 0$ in (3.3) we have that $\delta \geq \beta$ and hence $\gamma \geq \beta$.
To prove the other direction we take an $\delta > \beta$ and have for all $\eta > \epsilon > 0$

$$\int_{0\leq|x|\leq\eta} |x|^{\delta}F(dx) = -\int_{0}^{\eta} r^{\delta}d\overline{F}(r) \geq -\int_{\epsilon}^{\eta} r^{\delta}d\overline{F}(r) \geq \epsilon^{\delta}\left[\overline{F}(\epsilon) - \overline{F}(\eta)\right]$$

where the last two steps hold because $\mu_{\overline{F}}(dx)$ is a negative valued measure. Therefore

$$\limsup_{\epsilon\to 0} \epsilon^{\delta}\overline{F}(\epsilon) \leq \int_{0\leq|x|\leq\eta} |x|^{\delta}F(dx)$$

and since $\delta > \beta$ the right hand side converges to 0 when $\eta \to 0$ ($F(dx)$ has no mass in 0). It follows that $r^{\delta}\overline{F}(r) \to 0$ as $r \to 0$ and as such $\delta \geq \gamma$ and $\beta \geq \gamma$. □

3.2 The Blumenthal-Getoor Index and Basic Path Properties

The Blumenthal-Getoor index indicates the behavior of the process paths in numerous ways though we will only point out a few here. In our case the most relevant feature is the relation between the BG index and whether the jumps of the paths are *p-summable*, i.e. whether the sums

$$A(p)_t := \sum_{s\leq t} |\Delta X_s|^p \tag{3.4}$$

are a.s. finite or not for some $p > 0$.

The following two results are needed for the proofs of Lemma 3.3 and 3.4.

Lemma 3.1. *Let Y be a nonnegative random variable then it holds that*

$$Y < \infty \ a.s. \iff \mathbb{E}\left[e^{-\lambda Y}\right] \to 1 \quad when \quad \lambda \searrow 0, \tag{3.5}$$

$$Y = \infty \ a.s. \iff \mathbb{E}\left[e^{-\lambda Y}\right] = 0 \quad for\ all \quad \lambda > 0. \tag{3.6}$$

Proof. We start with the proof of (3.5). Let Y be almost surely finite then $e^{-\lambda Y} \xrightarrow{\mathbb{P}} 1$ when $\lambda \searrow 0$ and as $\left|e^{-\lambda Y}\right| = e^{-\lambda Y} \leq 1$ we have with dominated convergence that $\mathbb{E}\left[e^{-\lambda Y}\right] \to 1$.

For the converse assume that $\mathbb{E}\left[e^{-\lambda Y}\right] \to 1$ when $\lambda \searrow 0$. Then this implies $e^{-\lambda Y} \xrightarrow{\mathbb{P}} 1$ when $\lambda \searrow 0$ and by passing unto a subsequence we even find $e^{-\lambda Y} \xrightarrow{a.s.} 1$ which is equivalent to $Y < \infty$ *a.s.*
To prove (3.6) assume that $Y = \infty$ *a.s.* then it follows that $e^{-\lambda Y} = 0$ *a.s.* for all $\lambda > 0$ and therefore $\mathbb{E}\left[e^{-\lambda Y}\right] = 0$. For the converse assume that $Y = \infty$ *a.s.* does not hold. Then there exists $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$ and $Y(\omega) < \infty$ for all $\omega \in A$. Therefore $e^{-\lambda Y(\omega)} > 0$ for all $\lambda > 0, \omega \in A$ and as $e^{-\lambda Y} \geq 0$ we can conclude $\mathbb{E}\left[e^{-\lambda Y}\right] > 0$. □

Lemma 3.2. *Let $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a nonnegative Borel measurable function and μ be the jump measure and ν its compensator of some Lévy process X. Then it holds*

$$\mathbb{E}\left[\exp\left(-\int f(r,x)\mu(dr,dx)\right)\right] = \exp\left(-\int \left(1 - \exp(-f(r,x))\right)\nu(dr,dx)\right).$$

Proof. As f is nonnegative there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}} \nearrow f$ where each function is of the form

$$f_n = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}, \qquad n \in \mathbb{N}$$

for some A_i which are disjoint Borel measurable subsets of $\mathbb{R}_+ \times \mathbb{R}$ and $a_i \geq 0$. For each A_i we have that $\mu(A_i)$ is a Poisson distributed random variable with mean $\nu(A_i)$, i.e. for all $n \in \mathbb{N}$

$$\mathbb{P}(\mu(A_i) = n) = \exp(-\nu(A_i))\frac{\nu(A_i)^n}{n!}.$$

Therefore we have

$$\mathbb{E}\left[\exp\left(-a_i\mu(A_i)\right)\right] = \sum_{n\geq 0} \exp(-a_i n)\exp(-\nu(A_i))\frac{\nu(A_i)^n}{n} = \exp\left(-(1-\exp(-a_i))\nu(A_i)\right).$$

Furthermore, as the A_i are disjoint, the random variables $\nu(A_1), \dots, \nu(A_n)$ are independent and hence

$$\begin{aligned}
\mathbb{E}\left[\exp\left(-\int f_n(r,x)\mu(dr,dx)\right)\right] &= \mathbb{E}\left[\exp\left(-\sum_{i=1}^{n} a_i\mu(A_i)\right)\right] \\
&= \prod_{i=1}^{n} \mathbb{E}\left[\exp\left(-a_i\mu(A_i)\right)\right] \\
&= \prod_{i=1}^{n} \exp\left(-(1-\exp(-a_i))\nu(A_i)\right) \\
&= \exp\left(-\sum_{i=1}^{n}(1-\exp(a_i))\nu(A_i)\right) \\
&= \exp\left(-\int (1-\exp(-f_n(r,x)))\nu(dr,dx)\right).
\end{aligned}$$

Using monotone convergence for the measures $\mu(dr,dx),\nu(dr,dx)$ and dominated convergence for the expected value we have

$$\mathbb{E}\left[\exp\left(-\int f(r,x)\mu(dr,dx)\right)\right] = \lim_{n\to\infty}\mathbb{E}\left[\exp\left(-\int f_n(r,x)\mu(dr,dx)\right)\right]$$
$$= \lim_{n\to\infty}\exp\left(-\int(1-\exp(-f_n(r,x)))\nu(dr,dx)\right) = \exp\left(-\int(1-\exp(-f(r,x)))\nu(dr,dx)\right).$$

□

Now follows the result that proves the properties used in the introduction of this chapter.

Lemma 3.3. *Let $A\in\mathcal{R}$ where $\mathcal{R}$ is the Borel σ-algebra on $\mathbb{R}$. Then we have that*

$$F(A)<\infty \iff \sum_{s\le t}\mathbb{1}_A(\Delta X_s)<\infty,\ \ a.s.\ \forall t>0, \tag{3.7}$$

$$F(A)=\infty \iff \sum_{t<r\le t+s}\mathbb{1}_A(\Delta X_r)=\infty,\ \ a.s.\ \forall t\ge 0, s>0. \tag{3.8}$$

Proof. Set $g(x)=\mathbb{1}_A(x)$. We prove (3.8) by defining the function $f(r,x)=\lambda(g(x)\wedge 1)\mathbb{1}_{(t,t+s]}(r)$ for some $\lambda>0$ and set

$$\begin{aligned}Y &= \sum_{t<r\le t+s}\mathbb{1}_A(\Delta X_r)\\ &= \sum_{t<r\le t+s}\left(g(\Delta X_r)\mathbb{1}_{[0,1]}(g(\Delta X_r))+\mathbb{1}_{(1,\infty)}(g(\Delta X_r))\right) = \int_{\mathbb{R}_+\times\mathbb{R}}\frac{1}{\lambda}f(r,x)\mu(dr,dx).\end{aligned}$$

We then have with Lemma 3.2

$$\mathbb{E}\left[\exp(-\lambda Y)\right]=\mathbb{E}\left[\exp\left(-\int f(r,x)\mu(dr,dx)\right)\right]=\exp\left(-\int\left(1-\exp(-f(r,x))\right)\nu(dr,dx)\right).$$

As

$$1-\exp(-f(r,x))=\exp(\epsilon_{r,x})f(r,x)$$

for some $\epsilon_{r,x}\in[-f(r,x),0]$ and $f(r,x)\le\lambda$ we have

$$\exp(-\lambda)f(r,x)\le 1-\exp(-f(r,x))\le f(r,x)$$

which results in

$$\begin{aligned}-\int\left(1-\exp(-f(r,x))\right)\nu(dr,dx) &\le \exp(-\lambda)\int -f(r,x)\nu(dr,dx)\\ &= \exp(-\lambda)\int_{\mathbb{R}_+}\int_{\mathbb{R}}-\lambda(g(x)\wedge 1)\mathbb{1}_{(t,t+s]}(r)F(dx)dr\\ &= -\exp(-\lambda)\lambda\int_t^{t+s}\int_{\mathbb{R}}(g(x)\wedge 1)F(dx)dr\\ &= -\exp(-\lambda)\lambda\int_t^{t+s}F(A)dr \qquad (3.9)\end{aligned}$$

and

$$-\int \left(1-\exp(-f(r,x))\right)\nu(dr,dx) \geq -\lambda \int_t^{t+s} \int_{\mathbb{R}} (g(x)\wedge 1)F(dx)dr = -\lambda \int_t^{t+s} F(A)dr. \tag{3.10}$$

The two inequalities above together with the fact that by Lemma 3.1 Y is a.s. infinite if and only if $\mathbb{E}\left[e^{-\lambda Y}\right] = 0$ for all $\lambda > 0$ gives (3.8).
To prove (3.7) we define the function $f(r,x) = \lambda(g(x)\wedge 1)\mathbb{1}_{[0,t]}(r)$ and use that by Lemma 3.1 a nonnegative random variable Y is a.s. finite if and only if $\mathbb{E}\left[e^{-\lambda Y}\right] \to 1$ as $\lambda \searrow 0$. Then replacing the domain of integration for the slightly altered $f(r,x)$ in (3.9) and (3.10) yields (3.7). □

The following Lemma is a generalization of the previous one and is used to state the connection between the BG index and *p-summability* in the next corollary.

Lemma 3.4 (cf. p.31 in [ASJ14])**.** *Let $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative Borel measurable function with $g(0) = 0$. Then we have that*

$$\int (g(x)\wedge 1)F(dx) < \infty \iff \sum_{s\leq t} g(\Delta X_s) < \infty, \ \ a.s.\ \forall t > 0, \tag{3.11}$$

$$\int (g(x)\wedge 1)F(dx) = \infty \iff \sum_{t<r\leq t+s} g(\Delta X_r) = \infty, \ \ a.s.\ \forall t \geq 0, s > 0. \tag{3.12}$$

Proof. We start by discussing the case of $F\left(g^{-1}([1,\infty))\right) = \infty$, we then have that the left hand side of (3.12) is true. Furthermore by (3.8) we have infinitely many jumps ΔX_r on any interval $r \in (t, t+s]$ with $g(\Delta X_r) \geq 1$ yielding the right hand side of (3.12). Therefore (3.11) and (3.12) are then fulfilled trivially.
For the rest of the proof we may now assume that $F\left(g^{-1}([1,\infty))\right) < \infty$. Then by (3.7) for any interval $r \in (t, t+s]$ there exists only a finite number of jumps ΔX_r with $g(\Delta X_r) \geq 1$ and hence

$$\sum_{s\leq t} g(\Delta X_s) < \infty \iff \sum_{s\leq t} \left(g(\Delta X_s)\mathbb{1}_{[0,1]}(g(\Delta X_s)) + \mathbb{1}_{(1,\infty)}(g(\Delta X_s))\right) < \infty,$$
$$\sum_{t<r\leq t+s} g(\Delta X_r) < \infty \iff \sum_{t<r\leq t+s} \left(g(\Delta X_r)\mathbb{1}_{[0,1]}(g(\Delta X_r)) + \mathbb{1}_{(1,\infty)}(g(\Delta X_r))\right) < \infty.$$

To show (3.12) we define as in the previous proof $f(r,x) = \lambda(g(x)\wedge 1)\mathbb{1}_{(t,t+s]}(r)$ for some $\lambda > 0$, set

$$Y = \sum_{t<r\leq t+s} \left(g(\Delta X_r)\mathbb{1}_{[0,1]}(g(\Delta X_r)) + \mathbb{1}_{(1,\infty)}(g(\Delta X_r))\right) = \int_{\mathbb{R}_+\times\mathbb{R}} \frac{1}{\lambda} f(r,x)\mu(dr,dx)$$

and from now on follow exactly the proof of Lemma 3.3 omitting the (not needed) identity $\int_{\mathbb{R}}(g(x)\wedge 1)F(dx)dr = F(A)$. For (3.11) we define $f(r,x) = \lambda(g(x)\wedge 1)\mathbb{1}_{[0,t]}(r)$ and then likewise use the arguments in the proof of Lemma 3.3. □

Corollary 3.1. *Applying Lemma 3.4 to the function $g(x) = |x|^p$ yields the well known result*

$$\int (|x|^p \wedge 1) F(dx) < \infty \iff \sum_{s \le t} |\Delta X_s|^p < \infty, \ \ a.s.\ \forall t > 0, \tag{3.13}$$

$$\int (|x|^p \wedge 1) F(dx) = \infty \iff \sum_{t < r \le t+s} |\Delta X_r|^p = \infty, \ \ a.s.\ \forall t \ge 0, s > 0. \tag{3.14}$$

Using the above corollary we see that the process has finite variation iff $1 \in I$.

Furthermore in [BG61] more connections between the BG index and other key properties of the process are made. Although these will not play a role in the sequel we name few here. First the BG index is connected to the "scalability" of a process namely

Theorem 3.2 (Theorem 3.1 in [BG61])**.** *If $\alpha > \beta$ then*

$$t^{-1/\alpha} X_t \to 0 \text{ for } t \to 0 \ \ a.s.$$

Theorem 3.3 (Theorem 3.3 in [BG61])**.** *If $\alpha < \beta$ then*

$$\limsup_{t \to 0} t^{-1/\alpha} |X_t| = \infty \ \ a.s.$$

We will encounter a much stronger version of this scaling property in the class of stable processes featured in the section below.
Other results in [BG61] link the BG index to the Hausdorff dimension of the image of a process. Depending on the conditions one can achieve lower or upper bounds.

3.3 Stable Processes and Related Processes

In this section we give basic examples of Lévy processes in reference to their BG index. The most prominent one being the class of stable processes which plays a major role in Chapter 5, being one of the building blocks of the observed process.

3.3.1 Stable Processes

Stable distributions were originally introduced by Lévy as an example for infinitely divisible distributions and are well known as the limiting objects of central limit theorems when second moment conditions are missing.
A random variable Y is *strictly stable* distributed when for independent copies $Y_1, \dots, Y_n, n \in \mathbb{N}$, of Y it holds that for some $a_n \in \mathbb{R}$

$$Y_1 + \ldots + Y_n \sim a_n Y$$

where the equality is in distribution, so this is a very special case of infinite divisibility. It turns out, if Y is not constant 0, that it must hold $a_n = n^{1/\beta}$ for some $\beta \in (0, 2]$ which is called the *stability index* of a stable distribution. The *characteristic exponent* $\Psi(\phi) = -\log\left(\mathbb{E}\left[\exp\left(-i\phi Y\right)\right]\right)$ of such a distribution is given by

$$\Psi(\phi) = \begin{cases} c^\beta |\phi|^\beta (1 - i\alpha \operatorname{sgn}(\phi) \tan\left(\frac{\pi\beta}{2}\right)), & \text{if } \beta \neq 1 \\ c|\phi|, & \text{if } \beta = 1 \end{cases} \tag{3.15}$$

where $\alpha \in [-1, 1]$ and $c > 0$. As Y has an infinitely divisible distribution there exists a Lévy process S whose characteristic exponent $-\log\left(\mathbb{E}\left[\exp\left(-i\phi S_1\right)\right]\right)$ equals (3.15) (p.5 in [Kyp14]). We call these Lévy processes the class of *strictly stable processes*. Here two special cases are included in this class of processes. The case $\beta = 1$ contains, besides the Cauchy process, a linear (non random) process starting in 0 and $\beta = 2$ refers to a Brownian motion. When $\beta \in (0, 1) \cup (1, 2)$ the characteristic triplet of S with respect to a truncation function κ is then of the form $(b^\kappa, 0, F)$ for some drift $b^\kappa \in \mathbb{R}$ and Lévy measure satisfying

$$F(dx) = \left(\frac{a^{(+)}}{|x|^{1+\beta}} \mathbb{1}_{\{x>0\}} + \frac{a^{(-)}}{|x|^{1+\beta}} \mathbb{1}_{\{x<0\}} \right) dx$$

with $a^{(+)}, a^{(-)} \geq 0$ and $a^{(+)} + a^{(-)} > 0$ and b^κ fulfills in addition

$$b^\kappa = \begin{cases} 0, & \text{if } \beta \in (0, 1) \text{ for } \kappa(x) = 0 \\ 0, & \text{if } \beta \in (1, 2) \text{ for } \kappa(x) = x. \end{cases}$$

In the context of Chapter 2 the choice of $\kappa(x)$ tells us that S is a special semimartingale if $\beta > 1$ and a process of finite variation if $\beta < 1$. Note that $\kappa(x) = x$ is not a "real" truncation function, as it is not bounded, but in the case of a special semimartingale and S being not a Brownian motion it still gives us a valid decomposition of S of the form

$$S_t = \int_{(0,t] \times \mathbb{R}^d} x \; (\mu - \nu)(ds, dx)$$

where μ is a jump measure with compensator $\nu(dt, dx) = dt \otimes F(dx)$. Furthermore we note that there is no conflicting notation here, i.e. the stability index of the process coincides with its BG index. β controls the rate with which F diverges near 0: the higher the value of β, the faster F diverges and therefore we have a higher concentration of small jumps by Corollary 3.1. When $\beta \to 2$ the jumps become "so dense" that the limiting object has continuous (albeit still not differentiable) paths and is a Brownian motion. In the case of stable processes the parameter β also governs the behavior of the big jumps and using the fact that for all $p > 0, t > 0$

$$\mathbb{E}\left[|X_t|^p\right] < \infty \iff \int_{\{|x|>1\}} |x|^p F(dx) < \infty, \tag{3.16}$$

(c.f. Theorem 2.19.1 in [EK19]) we see that if S is a stable process we have

$$\mathbb{E}\left[|S_t|^p\right] < \infty \quad \text{for} \quad 0 \le p < \beta \qquad \text{and} \qquad \mathbb{E}\left[|S_t|^p\right] = \infty \quad \text{otherwise.}$$

The last key property of stable processes is their *self-similarity*. For all $n \in \mathbb{N}$ it holds that

$$S_n = S_1 + (S_2 - S_1) + \ldots + S_n - S_{n-1} \sim n^{1/\beta} S_1$$

or more generally for all $\lambda > 0$ we see that $\{S_{\lambda t} : t \ge 0\}$ has the same law as $\{\lambda^{1/\beta} S_t : t \ge 0\}$.

In general the density of a stable distribution/process is unknown, though, except for cases $\beta = 1, 2$ mentioned above. In the sequel we will work with the characteristic function of strictly stable processes in particular if these are symmetric, i.e. $a^{(+)} = a^{(-)}$ or $\alpha = 0$, then the Lévy measure and characteristic function reduce to

$$F(dx) = \frac{A}{|x|^{1+\beta}} dx \quad \text{and} \quad \mathbb{E}\left[\exp(-iuS_1)\right] = \exp(-A_\beta |u|^\beta) \tag{3.17}$$

for some constants $A, A_\beta > 0$.

3.3.2 Tempered Stable Process

A tempered stable process of index $\beta \in (0, 2)$ is a Lévy process whose characteristic triplet is $(b, 0, F)$, where $b \in \mathbb{R}$ and F is

$$F(dx) = \left(\frac{a^{(+)} \exp(-B_+|x|)}{|x|^{1+\beta}} \mathbb{1}_{\{x>0\}} + \frac{a^{(-)} \exp(-B_-|x|)}{|x|^{1+\beta}} \mathbb{1}_{\{x<0\}} \right) dx,$$

for some $a^{(+)}, a^{(-)} \ge 0$ with $a^{(+)} + a^{(-)} > 0$, and $B_-, B_+ \ge 0$. The reason for introducing tempered stable processes is that since $\exp(-B_+|x|), \exp(-B_-|x|) \to 1$ as $x \to 0$ their small jumps behave similar to stable processes and as a result their BG index equal to β. But in contrast to stable processes applying (3.16) yields that moments of all orders exist, as $\exp(-|x|)|x|^p \mathbb{1}_{(1,\infty)}(|x|)$ is integrable for all $p \in \mathbb{R}_+$. Tempered stable processes are featured in financial applications and can also be incorporated as an underlying process for the estimator presented in Chapter 5.

The well known inverse Gaussian process is a prominent example of a tempered stable process with $a^{(-)} = 0$ (i.e. it is a subordinator) and index equal to $\frac{1}{2}$.

3.4 The Jump Activity Index and Basic Models

We now lift the concept of the BG index to general semimartingales. (3.4) is defined for semimartingales in the same manner and again the question arises when these objects are finite and when not. We now assume that X is an Itô semimartingale with characteristics (b_t, c_t, v) where the compensator of the jump measure is of the form

$$v(dt, dx) = dt \otimes F_t(dx).$$

As the *spot Lévy measure* $F_t(dx)$ might vary over time for semimartingales we set up a definition alternative to (3.1)

$$I_t := \{p \geq 0 : \int (|x|^p \wedge 1) F_t(dx) < \infty\}, \qquad \beta_t := \inf I_t, \tag{3.18}$$

$$J_t := \{p \geq 0 : (|x|^p \wedge 1) \star v_t < \infty\}, \qquad \gamma_t := \inf J_t \tag{3.19}$$

with $(|x|^p \wedge 1) \star v_t = \int_{[0,t]\times\mathbb{R}} (|x|^p \wedge 1)\ \nu^X(ds, dx)$ as defined in Chapter 2. The form of these random sets is similar to the set I in (3.1) and again both γ_t and β_t take values in $[0, 2]$. It is obvious that when t increases the set J_t decreases. One can interpret γ_t as the "global" Blumenthal-Getoor index on the interval $[0, t]$ and β_t is the "spot" index at time t. When X is a Lévy process there is no need for this distinction as the Lévy measure is then non-random and $\beta_t = \gamma_t = \beta$ for all $t > 0$. Though a non constant β_t may occur in a general semimartingale setting our estimator presented in Chapter 5 only works in a setting where β_t and γ_t are assumed to be constant over time and non-random (and therefore equal) and we call this number *jump activity index*. This nomenclature allows us to still have a distinction between the Lévy case and the case of semimartingales who may have a time changing $F_t(dx)$ but a non varying jump activity index β. Most estimators from recent papers work in such a setting of a constant index. One of the very few exceptions is the paper [Tod17] by Todorov where a test is developed to check whether the instantaneous index β_t stays constant or varies over time.
Again compared to the Lévy case it does not surprise that we have a result similar to Corollary 3.1.

Lemma 3.5 (cf. Lemma 3.2.1 in [JP12]). *For all $2 > p, t > 0$ it holds that*

$$A(p)_t < \infty \iff p \in J_t.$$

The reason for estimating β is simple, many models in finance stopped using only processes with continuous paths for the underlying structure but instead use processes with jumps instead. The advantage of including jumps into financial models is that these

can be used to reproduce various stylized fact of asset prices that cannot be explained in classical models such as heavy tailed distribution and "big" jumps. These models include simple compound Poisson-based models ($\beta = 0$), normal inverse Gaussian models ($\beta = 0.5$), variance gamma models ($\beta = 0$) or hyperbolic/generalized hyperbolic models and the CGMY model of Carr et al. [CG02] where β is a free parameter. Therefore estimation procedures may be used to fit real data to a model or, if they include a central limit theorem, to come up with statistical tests for model assumptions.
To lift the setting of Lévy processes to a bigger class of semimartingales one can use the following model assumption on the observed process X:

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \eta_s dW_s + \int_0^t H_{s-} dL_s, \tag{3.20}$$

where L is Lévy process whose BG-index is known, e.g. a stable process, b a locally bounded, η a càdlàg and H a locally bounded process. Then under general conditions the activity index of X equals the one of L (c.f. Example 11.4 in [Kyp14]).

3.5 Estimation procedures in Recent Literature

Most papers in the recent past use an underlying process of the form (3.20) for their estimators while the assumptions regarding the Lévy measure of the process L may vary slightly but follow the same underlying principles. They all describe the behavior of the Lévy measure of L such that in a neighborhood around 0 it can be related to the behavior of a stable process. The first example is taken from [TT11].

Assumption A. *The density of the Lévy measure of L is given by*

$$\nu(x) = \frac{A}{|x|^{1+\beta}} + \nu'(x), \qquad \beta \in (0, 2), \tag{3.21}$$

where $A > 0$ and there exists some $x_0 > 0$ with $|\nu'(x)| \leq \frac{C}{|x|^{1+\beta'}}$ for some $\beta' < \beta$ and all $|x| < x_0$.

The following assumption is from Chapter 11 in [ASJ14] using the tail function (3.2) but generalized to semimartingales (exclusion of the point x in this definition in contrast to (3.2) is of no real relevance here)

$$\overline{F}_t(x) = F_t((-\infty, x) \cup (x, \infty)), \qquad t > 0, x > 0.$$

Assumption B. *There exist constants $0 \leq \beta' < \beta < 2$ such that for all $t > 0, x \in (0, 1]$:*

$$|x^\beta \overline{F}_t(x) - a_t| \leq M_t x^{\beta - \beta'}, \tag{3.22}$$

where a_t, M_t are nonnegative predictable and locally bounded processes.

Like already mentioned both assumptions work with a setting that assumes a constant jump activity index equal to β though one must mention that, because we allow for $a_t = 0$, (3.22) allows for times when the process does not jump at all or only with intensity lower than β'. Both convey that the behavior of the jumps of L around 0 resemble those of a stable process and restrict the possible forms of Lévy measures quite a bit. Nevertheless nearly all models proposed for financial application fulfill (3.21) and (3.22). The most notable differences in the model assumptions occur when X is allowed to contain a diffusion part like $\int_0^t \eta_s dW_s$ or not. In the following we will present estimators for both those situations. Before that we introduce the probably most used tool when it comes to the analysis of integrated volatility or activity estimation: the *power variation* of the process X

$$V_t(X, p, \Delta_n) = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} |\Delta_i^n X|^p, \quad p > 0,\ t > 0, \tag{3.23}$$

where $\Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n}$ is the difference of the regularly spaced observations at times $0, \Delta_n, 2\Delta_n, \ldots, \lfloor t/\Delta_n \rfloor \Delta_n$. Results for convergence in probability of the power variation for this class of underlying process are stated as early as 2003 by [Woe03b],[Woe03a], [BNS03] and continue with adjoined central limit theorems until [TT11]. From the latter one we cite a few results omitting some technical assumptions

Theorem 3.4 (cf. Theorem 3.2 in [TT11]). *1. Suppose X is given by (3.20), L is a Lévy process with characteristic triple $(0, 0, \nu)$ w.r.t. to some truncation function $\kappa(x)$, where ν is given by (3.21) for some $\beta < 2$ and $|\eta_s|, |\eta_{s-}| > 0$ a.s. Then for a fixed $T > 0$ we have*

$$\Delta_n^{1-p/2} V_T(X, p, \Delta_n) \xrightarrow{\mathbb{P}} \mu_p(2) \int_0^T |\eta_s|^p ds \tag{3.24}$$

locally uniformly in $p \in (0, 2)$, with $\mu_p(2) = \mathbb{E}\left[|Z|^p\right]$ where Z is a standard normal distributed random variable.

2. Suppose that for a fixed $T > 0$ X is defined by (3.20) with $\eta_s = 0$ for all $s \leq T$ a.s. and again L is a Lévy process with characteristic triple $(0, 0, \nu)$ w.r.t. to some truncation function $\kappa(x)$, where ν is given by (3.21) for some $\beta \in (0, 2)$. Further assume that if $\beta \leq 1$ then $b_s - H_{s-} \int_{\mathbb{R}} \kappa(x) \nu(dx)$ is identically zero on $[0, T]$, then it holds that:

$$\Delta_n^{1-p/\beta} V_T(X, p, \Delta_n) \xrightarrow{\mathbb{P}} \mu_p(\beta) K_\beta \int_0^T |H_s|^p ds \tag{3.25}$$

locally uniformly in $p \in (0,\beta)$, with $\mu_p(\beta) = \mathbb{E}[|Z|^p]$ where Z is a symmetrical stable random variable with stability index β and $K_\beta > 0$ some constant only depending on β.

The last theorem clearly shows the different cases where each time a different part of the underlying process determines the limit of the power variation. We briefly explain the reason for this limit behavior when $\Delta_n \to 0$.
For a Brownian motion W we always have that

$$\Delta_n^{-p/2} \left| \int_{(i-1)\Delta_n}^{i\Delta_n} \eta_s dW_s \right|^p \approx \Delta_n^{-p/2} \left| \eta_{(i-1)\Delta_n} (W_{i\Delta_n} - W_{(i-1)\Delta_n}) \right|^p \tag{3.26}$$
$$= |\eta_{(i-1)\Delta_n}|^p \left| \frac{W_{i\Delta_n} - W_{(i-1)\Delta_n}}{\sqrt{\Delta_n}} \right|^p \sim |\eta_{(i-1)\Delta_n}|^p |Z|^p, \qquad p > 0,$$

where Z is a standard normal distributed random variable. On the other hand if S is a stable process with stability index β we have from the self scaling property that

$$\Delta_n^{-p/\beta} \left| \int_{(i-1)\Delta_n}^{i\Delta_n} H_{s-} dS_s \right|^p \approx \Delta_n^{-p/\beta} |H_{i\Delta_n -}(S_{i\Delta_n} - S_{(i-1)\Delta_n})|^p \tag{3.27}$$
$$= |H_{i\Delta_n -}|^p \left| \frac{S_{i\Delta_n} - S_{(i-1)\Delta_n}}{\Delta_n^{1/\beta}} \right|^p \sim |H_{i\Delta_n -}|^p |S_1|^p, \qquad p \in [0,\beta).$$

For the drift part it is obvious that

$$\Delta_n^{-p} \left| \int_{(i-1)\Delta_n}^{i\Delta_n} b_s ds \right|^p \approx |b_{(i-1)\Delta_n}|^p, \qquad p > 0.$$

Putting these things together we can see the overall behavior outlined in Theorem 3.4, namely (assuming $p = 1$ for simplicity)

- If a diffusion part is present it dominates the jump part with activity $\beta < 2$ and the drift part alike. The latter parts converge to zero faster than the diffusion and scaling with a coefficient smaller than $\Delta_n^{-1/2}$ would yield a degenerated limit.
- If no diffusion is present and $1 < \beta < 2$ the jumps driven by the stable process have to be scaled with $\Delta_n^{-1/\beta}$ and the drift part converges faster to zero.
- If $\beta < 1$ we have $\Delta_n^{-1/\beta} \leq \Delta_n^{-1}$ and the drift determines the limit behavior. The only possibility to still infer something about the jumps is when the drift is essentially zero which boils down to the condition $b_s - H_{s-} \int_{\mathbb{R}} \kappa(x)\nu(dx) = 0$.

As with simple (non truncated) power variations only the part of the process can be inferred that converges the slowest towards zero, we have that most of the estimators for

β restrict X to have no diffusion part. In this case of $\eta_s = 0$ and some smaller additional conditions we find with (3.25) that $\frac{(2\Delta_n)^{1-p/\beta} V_T(X,p,2\Delta_n)}{\Delta_n^{1-p/\beta} V_T(X,p,\Delta_n)} \xrightarrow{\mathbb{P}} 1$ and from here one can build a basic version of an estimator for such a setting of no diffusion:

$$\hat{\beta}(p) = \frac{p\log(2)}{\log(2V_T(X,p,2\Delta_n)/V_T(X,p,2\Delta_n))}.$$

This concept can be extended for example by using the difference of increments, i.e. $\Delta_i^n X - \Delta_{i-1}^n X$ instead of $\Delta_i^n X$, as carried out in [Tod13], to obtain better convergence rates. A major advantage of this method is that the influence of the drift part is diminished and we will use this concept when we construct our estimator in Chapter 5.

When the underlying process may contain a diffusion, the problem is that the "small jumps" determining β are now contaminated by the small increments of the diffusion. The key idea to disentangling the jumps from the diffusion part is that increments of the diffusion behave approximately like $\sqrt{\Delta_n}$ times a constant, see (3.26), and therefore to only take into account the increments that are bigger than some threshold $u_n \approx \Delta_n^{\varpi}$ for some $\varpi \in (0, \frac{1}{2})$. For an X of the form (3.20) this leads to functionals of the form

$$J(\Delta_n, u_n)_t = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{1}_{\{|\Delta_i^n X| > u_n\}}, \qquad t > 0$$

and under the assumption of (3.22) Ait-Sahalia and Jacod show in [ASJ09]

$$u_n^{\beta} J(\Delta_n, u_n)_t \xrightarrow{\mathbb{P}} A_t := \int_0^t a_s ds, \qquad t > 0, \tag{3.28}$$

which leads to an estimator of β even if a diffusion part is present:

$$\hat{\beta}_n(\gamma, u_n) := \frac{\log(J(\Delta_n, u_n)_t / J(\Delta_n, \gamma u_n)_t)}{\log(\gamma)} \mathbb{1}_{\{J(\Delta_n, \gamma u_n)_t > 0\}}, \qquad t > 0,$$

for some $\gamma > 1$. This estimator can be improved by using test functions $g(x)$ as smooth approximations for the plain indicator function used in $J(\Delta_n, u_n)_t$, e.g. carried out in Chapter 11.2 of [ASJ14] or [JKLM12]. The main drawback of this concept is that depending on the parameter ϖ one might omit most of the observations which hinders the accuracy when only a small sample is available.

We now present an estimator for β taken from [Tod15] which is based on the empirical characteristic function of a process. The following can also be seen as an introduction to Chapter 5 as a large portion of the concepts used there is inspired by this estimator. The main idea is that if X is defined according to (3.20) with no diffusion part and Lévy measure as in (3.21), that in view of (3.27) and (3.17) it seems reasonable to assume, as long as $\beta > 1$, that:

$$\Delta_n \sum_{i=1}^{n} \cos\left(u\Delta_n^{-1/\beta}(\Delta_i^n X - \Delta_{i-1}^n X)\right) \xrightarrow{\mathbb{P}} \int_0^1 e^{-u^\beta |H_s|^\beta A_\beta} ds, \quad u \in \mathbb{R}_+, \tag{3.29}$$

for some $A_\beta > 0$. The difference of increments again diminishes the influence of the drift and furthermore makes the Lévy measure symmetric around zero. Note that for a symmetric stable process S we have

$$\mathbb{E}\left[\exp(iuS_1)\right] = \mathbb{E}\left[\cos(uS_1) + i\sin(uS_1)\right] = \mathbb{E}\left[\cos(uS_1)\right]$$

which motivates the definition in the first place. The problem is that if one wants to estimate β from the limit above, it is not only part of the correct scaling but also of the limit. Therefore it does not directly untangle from the integral above when we evaluate the above functional for different values of u. A way out is to use power variations for a local estimator of H and use it to scale the increments appropriately. This leads to the estimator proposed in [Tod15]:

$$\hat{L}^n(p,u) = \frac{1}{n - k_n - 2} \sum_{i=k_n+3}^{n} \cos\left(u \frac{\Delta_i^n X - \Delta_{i-1}^n X}{(V_i^n(p))^{1/p}}\right), \quad u \in \mathbb{R}_+, p > 0,$$

with $(V_i^n(p))^{1/p}$ being the local estimator for H_s scaled by a constant:

$$V_i^n(p) := \frac{1}{k_n} \sum_{j=i-k_n-1}^{i-2} |\Delta_i^n X - \Delta_{i-1}^n X|^p, \quad i = k_n + 3, \ldots, n,$$

built out of $k_n \to \infty$, $\frac{k_n}{n} \to 0$ intervals prior to the two increments forming the characteristic function. Note that the scaling needed in (3.29) is gone as the scalings in the nominator and denominator cancel. The law of large numbers (LLN) result is then

Theorem 3.5 ((Theorem 1 in [Tod15]). *Assume $\beta \in (1,2)$. Let k_n be a sequence with $k_n \asymp n^{\varpi}$ with a $\varpi \in (0,1)$. Then we have for $0 < p < \beta$*

$$\hat{L}^n(p,u) \xrightarrow{\mathbb{P}} L(p,u,\beta) := e^{-C_{p,\beta} u^\beta} \quad \textit{for } n \to \infty$$

with $C_{p,\beta} > 0$.

We will return to this estimator in Chapter 5 as a base for our own estimator and discuss the associated central limit theorem (CLT) there.

At last we want to mention a very recent publication by Jacod and Todorov [JT18] in which the authors work with a related kind of functionals, namely *local empirical* *characteristic exponents*. Here $h_n, k_n \in \mathbb{N}, u_n \in \mathbb{R}_+$ are tuning parameters of the estimator which all behave asymptotically like n^x for different $x \in (0,1)$. We define

$$\widehat{c}_i^n(y) := \log\left(L(y)_i^n \vee \frac{1}{h_n}\right), \tag{3.30}$$

where

$$L(y)_i^n := \frac{1}{k_n}\sum_{l=0}^{k_n-1}\cos\left(u_n y\left(\widetilde{Y}_{i+2lh_n}^n - \widetilde{Y}_{i+2(l+1)h_n}^n\right)\right) \quad \text{with} \quad \widetilde{Y}_i^n := \sum_{j=1}^{h_n-1} g(j/h_n)\Delta_{i+j}^n Y,$$

for some function $g(x)$ with support on $(0,1)$. The process Y is basically the process X from (3.20) (with a possible diffusion part) plus some added noise using (3.22) to define the behavior of the jumps. The $\widetilde{Y}_i^n$ are *pre-averaged increments* over h_n intervals and one takes the difference of k_n of those $\widetilde{Y}_i^n$ to build $L(y)_i^n$ respectively $\widehat{c}_i^n(y)$, meaning that it needs in total $w_n = 2h_nk_n$ increments to form $L(y)_i^n$. The final estimator then consists of $\lfloor N_t^n/w_n\rfloor - 1$ of the $\widehat{c}_i^n(y)$:

$$\widehat{C}(y)_t^n = \sum_{j=0}^{\lfloor N_t^n/w_n\rfloor-1}\left(\widehat{c}_{jw_n}^n(y) - \frac{1}{2k_n}f(\widehat{c}_{jw_n}^n(y), \widehat{c}_{jw_n}^n(2j) - \frac{1}{2h_n}\overline{\phi}_n y^2 u_n^2\right),$$

with $f(x,y) = \frac{1}{2}\left(\exp(2x-y) + \exp(2x) - 2\right)$ and $\overline{\phi}_n = h_n\sum_{i\in\mathbb{N}}(g(\frac{i+1}{h_n}) - g(\frac{i}{h_n}))^2$. N_t^n is the integer valued random variable that indicates the number observation times of Y smaller than t. The second and last expression in the bracket are only needed for bias corrections due to the noise and the nonlinear transformation in (3.30) and are of no further interest here. Again $h_n, k_n \to \infty$ with $\frac{h_n}{n}, \frac{k_n}{n} \to 0$ so we really form a local version of the characteristic exponent. The centered process is of the form

$$Z(y)_t^n = \widehat{C}(y)_t^n - \frac{y^2u_n^2\phi_n}{2k_n}C_t - \frac{2}{k_n}|y|\beta u_n^\beta\widetilde{\phi}_n^\beta\chi(\beta)A_t$$

where $\phi_n, \widetilde{\phi}_n$ are non-random values depending on the function $g(x)$ used for the pre-averaging, $\chi(\beta)$ is a constant, A_t as in (3.28) and $C_t = \int_0^t \eta_s^2 ds$. The authors show that for a normalizing sequence u_n of positive numbers and any $t \geq 0, y \neq 0$ under many restrictions on the tuning parameters u_n, k_n, h_n it holds $Z(y)_t^n \xrightarrow{\mathbb{P}} 0$ (cf. Theorem 1 in [JT18]).

With the result it is possible to construct various estimators for the integrated volatility C_t or β and the associated value A_t, one example being:

$$\hat{\beta}_t^{n,1} = f^{-1}\left(\frac{\widehat{C}(4)_t^n - 16\widehat{C}(1)_t^n}{\widehat{C}(2)_t^n - 4\widehat{C}(1)_t^n}\right),$$

where f^{-1} is the reciprocal function to $f(x) = \frac{4^x-16}{2^x-4}$. For more details see sections 4.3.1 and 4.3.2 in [JT18]. The reason we mention this paper in particular, is that it is the only publication which can deal with underlying semimartingale in nearly full generality. That means a process that may contain a diffusion in addition to a jump part with added noise. Furthermore their estimator, similar to our estimator from Chapter 5, works in a setting of (exogenous) random observation times which none of other articles mentioned in this section does.

Chapter 4

Estimates for Itô Semimartingales and the Localization Procedure

With this chapter we now leave the introductory part of this work. While the first chapter is mainly concerned with the basic definitions of semimartingales and the second one is essentially a long motivation for why it is actually important to have estimators for the jump activity index, this chapter finally provides results that have direct applications in the following proofs. Furthermore, whilst in the previous chapters we only repeated results and in some case added a rigorous proof, many of the results in this part are adapted to the setting of random observation times, presented in subsection 4.2.1, and are therefore, in this form, not included in the standard textbooks.
Throughout the proofs in Chapter 5 we will repeatedly use upper bounds for stochastic processes to determine the rate of convergence for the different parts of Itô semimartingales when the distance Δ_n between observations $X_{\Delta_n}, X_{2\Delta_n}, \ldots$ goes to zero. These estimates are used commonly by authors working in a semimartingale setting and by nearly all papers mentioned in the previous chapter. Usually they consist of two things. First, upper bounds for different parts of an Itô semimartingale stated in a very general manner, sometimes these bounds may be infinite (and therefore render uninformative) depending on the conditions. Second, a localization procedure that makes the aforementioned bounds applicable to semimartingales that only fulfill very general conditions. The localization procedure strengthens the weak/general assumptions such that the earlier stated bounds that may have been uninformative before are now actually finite. Section 4.1 states estimates for different parts of a semimartingale and Section 4.2 then deals with the localization of semimartingales.

4.1 Basic Estimates for Itô semimartingales

The following estimates are all stated in Section 2.1.5 of [JP12] and will play a major role in the upcoming proofs of Chapter 5. As already mentioned the results are phrased in a slightly generalized version that is fitted to our setting of irregular observation times. That means in contrast to [JP12] we allow for both U and τ to be (bounded) stopping times. For the sake of completeness we include some of the proofs which can be found in the Appendix of [JP12].

The motivation for the following lemmas is that we consider a one-dimensional semimartingale in the form of Theorem 2.2, i.e.

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \eta_s dW_s + \int_{(0,t]\times\mathbb{R}^d} \kappa(\delta^X(s,x))(\mu-\nu)(ds,dx) \\ + \int_{(0,t]\times\mathbb{R}^d} (\delta^X(s,x) - \kappa(\delta^X(s,x)))\mu(ds,dx), \tag{4.1}$$

where the compensator of μ is given by $\nu(dt,dx) = dt \otimes \lambda(dx)$ for a σ-finite measure λ on some polish space E and the other parts of X likewise fulfill the conditions of Theorem 2.2 with $d = d' = 1$. We start off with an upper bound for the drift part.

Lemma 4.1. *(cf. p.40 in [JP12]). Let $U > \tau$ stopping times where U is bounded then we have for $p \geq 0$*

$$\sup_{0\leq u\leq (U-\tau)} \left| \int_\tau^{\tau+u} b_r dr \right|^p \leq (U-\tau)^p \left(\sup_{\tau\leq u\leq U} |b_u| \right)^p. \tag{4.2}$$

Proof.

$$\sup_{0\leq u\leq (U-\tau)} \left| \int_\tau^{\tau+u} b_r dr \right|^p \leq \left(\sup_{0\leq u\leq (U-\tau)} \int_\tau^{\tau+u} |b_r| dr \right)^p \\ \leq \left((U-\tau) \sup_{\tau\leq u\leq U} |b_u| \right)^p \leq (U-\tau)^p \left(\sup_{\tau\leq u\leq U} |b_u| \right)^p.$$

□

The idea of this and the following estimates is always that τ and U are two random time points where we observe our process and that $\mathbb{E}_\tau[U-\tau]$ is of the order Δ_n. When the distance between observations goes to 0, i.e. $\Delta_n \to 0$, the rate of the drift term is Δ_n^p if the process b is bounded. In general this is too strong to be an assumption but in Section 4.2 we show that it suffices that b is locally bounded to assume by localization that it is bounded.

Before we start with the estimates involving stochastic integrals we need to cite the famous Burkholder-Davis-Gundy (BDG) inequality which will contribute to many of the upcoming proofs.

Theorem 4.1 (Burkholder-Davis-Gundy inequality)**.** *For each real $p \geq 1$ there exist constants $0 < c_p < C_p < \infty$ such that for any local martingale M with $M_0 = 0$ and any two stopping times $\tau \leq T$, we have*

$$c_p \mathbb{E}\left[([M,M]_T - [M,M]_\tau)^{p/2} \,|\mathcal{F}_\tau\right] \leq \mathbb{E}\left[\sup_{t\in\mathbb{R}_+:\tau<t\leq T} |M_t - M_\tau|^p|\mathcal{F}_\tau\right] \leq C_p \mathbb{E}\left[([M,M]_T - [M,M]_\tau)^{p/2} \,|\mathcal{F}_\tau\right].$$

From now on and for the rest of this work we use for any stopping time τ the following notation $\mathbb{E}_\tau[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_\tau]$. Furthermore, if not explicitly mentioned $K > 0$ is a constant that may change from line to and $K_q > 0$ is likewise a constant that may depend on a parameter $q \in \mathbb{R}$ that is non-random and usually known or fixed.
Now direct application of the BDG inequality gives us an estimate for the continuous martingale part.

Lemma 4.2. *(cf. p.40 in [JP12]). Let $U > \tau$ stopping times where U is bounded then for $p \geq 1$:*

$$\mathbb{E}_\tau\left[\sup_{0\leq u\leq(U-\tau)} \left|\int_\tau^{\tau+u} \eta_z dW_z\right|^p\right] \leq K_p \mathbb{E}_\tau\left[(U-\tau)^{p/2}\left(\frac{1}{(U-\tau)}\int_\tau^U |\eta_z|^2 dz\right)^{p/2}\right]$$

and for $p \leq 1$ we have:

$$\mathbb{E}_\tau\left[\sup_{0\leq u\leq(U-\tau)} \left|\int_\tau^{\tau+u} \eta_s dW_s\right|^p\right] \leq \left(K_1^p \mathbb{E}_\tau\left[(U-\tau)^{1/2}\left(\frac{1}{(U-\tau)}\int_\tau^U |\eta_z|^2 dz\right)^{1/2}\right]\right)^p.$$

Proof. For $p \geq 1$:

$$\begin{aligned}\mathbb{E}_\tau\left[\sup_{0\leq u\leq(U-\tau)} \left|\int_\tau^{\tau+u} \eta_z dW_z\right|^p\right] &\leq K_p \mathbb{E}_\tau\left[\left(\int_\tau^U |\eta_z|^2 dz\right)^{p/2}\right] \\ &\leq K_p \mathbb{E}_\tau\left[(U-\tau)^{p/2}\left(\frac{1}{(U-\tau)}\int_\tau^U |\eta_z|^2 dz\right)^{p/2}\right]\end{aligned}$$

and for $p \leq 1$

$$\begin{aligned}\mathbb{E}_\tau\left[\sup_{0\leq u\leq(U-\tau)} \left|\int_\tau^{\tau+u} \eta_s dW_s\right|^p\right] &\leq \mathbb{E}_\tau\left[\sup_{0\leq u\leq(U-\tau)} \left|\int_\tau^{\tau+u} \eta_s dW_s\right|\right]^p \\ &\leq \left(K_1^p \mathbb{E}_\tau\left[(U-\tau)^{1/2}\left(\frac{1}{(U-\tau)}\int_\tau^U |\eta_z|^2 dz\right)^{1/2}\right]\right)^p.\end{aligned}$$

□

Again the right hand sides of the inequalities do not have a specific order of convergence when $(U-\tau)\to 0$ in general, but only under additional assumptions usually obtained through localization.

Next we turn our attention to estimates for purely discontinuous martingales. The following Lemma is 2.1.5 of [JP12] in our slightly more general version for stopping times. Here we define for a predictable function δ on $\Omega\times\mathbb{R}_+\times E$ and $s>0$:

$$\widehat{\delta}(q)_{\tau,s} := \frac{1}{s}\int_{\tau}^{\tau+s}\int_E |\delta(u,z)|^q\lambda(dz)du. \tag{4.3}$$

Lemma 4.3. *Suppose that* $\int_0^t\int_E|\delta(s,z)|^2\lambda(dz)ds<\infty$ *for all* $t>0$. *Let* $U>\tau$ *be stopping times where* U *is bounded then for* $Y=\delta\star(\mu-\nu)$, $q\in[1,2]$,

$$\mathbb{E}_\tau\left[\sup_{0<u<(U-\tau)}|Y_{\tau+u}-Y_\tau|^q\right]\le K_q\mathbb{E}_\tau\left[(U-\tau)\widehat{\delta}(q)_{\tau,(U-\tau)}\right]$$

and for $q\ge 2$*:*

$$\mathbb{E}_\tau\left[\sup_{0<u<(U-\tau)}|Y_{\tau+u}-Y_\tau|^q\right]$$
$$\le K_q\left(\mathbb{E}_\tau\left[(U-\tau)\widehat{\delta}(q)_{\tau,(U-\tau)}\right]+\mathbb{E}_\tau\left[(U-\tau)^{p/2}\widehat{\delta}(2)^{q/2}_{\tau,(U-\tau)}\right]\right).$$

Proof. The proof is conducted by checking that the arguments in [JP12] still hold in this slightly more general setting.

Let $q\in[1,2]$ and define for all $w\ge 0$ the processes

$$Z(w)=(|\delta|^w\mathbb{1}_{(\tau,\infty)})\star\mu \quad \textit{and} \quad \widetilde{Z}(w)=(|\delta|^w\mathbb{1}_{(\tau,\infty)})\star\nu,$$

noting that $\widetilde{Z}(w)_U=(U-\tau)\widehat{\delta}(w)_{\tau,(U-\tau)}$.

By the BDG-inequality we have

$$\mathbb{E}_\tau\left[\sup_{0\le s\le(U-\tau)}|Y_{\tau+s}-Y_\tau|^q\right]=\mathbb{E}_\tau\left[\sup_{\tau\le s\le U}|Y_s-Y_\tau|^q\right]\le K_q\mathbb{E}_\tau\left[Z(2)_U^{q/2}\right].$$

Using

$$\left|\sum_i|a_i|\right|^p\le\sum_i|a_i|^p \qquad \forall p\in(0,1] \text{ and all real valued } \{a_i\}_{i>1} \tag{4.4}$$

we have $Z(2)^{q/2}\le Z(q)$ and as such

$$\mathbb{E}_\tau\left[\sup_{0\le s\le(U-\tau)}|Y_{\tau+s}-Y_\tau|^q\right]\le K_q\mathbb{E}_\tau\left[Z(q)_U\right]=K_q\mathbb{E}\left[\widetilde{Z}(q)_U\right],$$

where we used the Optional Stopping Theorem for bounded stopping times in the last step.
The proof in the case of $q \geq 2$ and U being non random is more lengthy and can be found on p.566 in [JP12]. However it may be generalized in the same way. □

Note that Lemma 2.1.7 b) of [JP12] can also be proven under the same conditions, that is:

Lemma 4.4. *Suppose that $\int_0^t \int_E |\delta(s,z)|\lambda(dz)ds < \infty$ for all $t > 0$. Let $U > \tau$ be stopping times where U is bounded then the process $Y = \delta \star \mu$ is of locally integrable variation and for $p \in (0,1]$ we have*

$$\mathbb{E}_\tau\left[\sup_{0\leq u\leq(U-\tau)}\left|\int_\tau^{\tau+u}\int_E \delta(s,x)\mu(ds,dx)\right|^p\right] \leq K_p \mathbb{E}_\tau[(U-\tau)\widehat{\delta}(p)_{\tau,(U-\tau)}]$$

and for $p > 1$

$$\mathbb{E}_\tau\left[\sup_{0\leq u\leq(U-\tau)}\left|\int_\tau^{\tau+u}\int_E \delta(s,x)\mu(ds,dx)\right|^p\right]$$
$$\leq K_p\left(\mathbb{E}_\tau[(U-\tau)^p\widehat{\delta}(1)^p_{\tau,(U-\tau)}] + \mathbb{E}_\tau[(U-\tau)\widehat{\delta}(p)_{\tau,(U-\tau)}]\right).$$

Again the above estimates are in general of no further use as it is not clear why $\mathbb{E}_\tau[\widehat{\delta}(p)_{\tau,(U-\tau)}]$ should be finite in the setting above. The necessary assumptions to guarantee the finiteness are now covered in the next section whereas section 4.3 puts these strengthened assumptions together with the previous estimates.

4.2 Localization Procedure

We start this section by introducing the general assumptions that we impose on our observed semimartingale X used as the underlying process for the estimator presented in the next chapter:

$$X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \sigma_{s-} dL_s + dY_t, \qquad t > 0, \tag{4.5}$$

where the precise assumptions of the processes L, α, σ, Y are specified in Assumptions A and B below. The way we represent X in (4.5) differs from (4.1) because we do not need an estimate on something like $\mathbb{E}_\tau\left[\sup_{0\leq s\leq(U-\tau)}|X_{\tau+s} - X_\tau|^q\right]$ directly but only on its components and rather emphasize how X would be written in modeling applications. Nevertheless the processes occurring as components of X are represented as in (4.1).
We assume that:

Assumption A. *The process X is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and $\kappa(x)$ is a truncation function, i.e. it is the identity in a neighbourhood around zero, odd, bounded and equals zero for large values of x and we set $\kappa(x)' = x - \kappa(x)$.*

1. *L is a Lévy process with characteristic triple $(-\int_{\mathbb{R}} \kappa'(x) \frac{A}{|x|^{\beta+1}} dx, 0, F)$ with respect to the truncation function $\kappa(x)$, where the Lebesgue density of the Lévy measure $F(dx)$ is given by*

$$h(x) = \frac{A}{|x|^{1+\beta}} + \tilde{h}(x) \text{ with a } \beta \in (1,2), A > 0;$$

and for $\tilde{h}(x)$ there exist $x_0 > 0$ with $|\tilde{h}(x)| \leq \frac{C}{|x|^{1+\beta'}}$ for all $|x| \leq x_0$ and some fixed $\beta' < 1$.

According to the appendix of [TT12] we can find (with a suitable extension of the probability space) a decomposition as follows:

$$L_t = S_t + \acute{S}_t - \grave{S}_t \tag{4.6}$$

where S is a Lévy process with characteristic triplet $(-\int_{\mathbb{R}} \kappa'(x) \frac{A}{|x|^{\beta+1}} dx, 0, \frac{A}{|x|^{1+\beta}} dx)$, $\acute{S}$ and $\grave{S}$ are pure-jump Lévy processes with the first two characteristics zero (with respect to the truncation function κ) and densities of the Lévy measure $|\tilde{h}(x)|$ and $2|\tilde{h}(x)|\mathbb{1}_{\{\tilde{h}(x)<0\}}$. This means in particular that S is a strictly β-stable Lévy process. We denote the associated jump measures of $S, \acute{S}$ and $\grave{S}$ with μ, μ_1 and μ_2.

Assumption B. *The processes α, σ and Y are Itô semimartingales of the form*

$$\begin{aligned}
\alpha_t &= \alpha_0 + \int_0^t b_s^\alpha ds + \int_0^t \eta_s^\alpha dW_s + \int_0^t \tilde{\eta}_s^\alpha d\widetilde{W}_s + \int_0^t \int_E \kappa(\delta^\alpha(s,x)) \underline{\widetilde{\mu}}(ds,dx) \\
&\quad + \int_0^t \int_E \kappa'(\delta^\alpha(s,x)) \underline{\mu}(ds,dx), \\
\sigma_t &= \sigma_0 + \int_0^t b_s^\sigma ds + \int_0^t \eta_s^\sigma dW_s + \int_0^t \tilde{\eta}_s^\sigma d\widetilde{W}_s + \int_0^t \int_E \kappa(\delta^\sigma(s,x)) \underline{\widetilde{\mu}}(ds,dx) \\
&\quad + \int_0^t \int_E \kappa'(\delta^\sigma(s,x)) \underline{\mu}(ds,dx), \\
Y_t &= \int_0^t \int_{\mathbb{R}} x \mu^Y(ds,dx),
\end{aligned}$$

where

1. *$|\sigma_t|$ and $|\sigma_{t-}|$ are strictly positive;*

2. *W and $\widetilde{W}$ are independent Brownian motions; $\underline{\mu}$ is a Poisson random measure on $\mathbb{R}_+ \times E$ having arbitrary dependence with the jump measure of L, with compensator $dt \otimes \lambda(dx)$ for some σ-finite measure λ on E. $\underline{\widetilde{\mu}}$ is the compensated jump measure;*

3. *$\delta^\alpha(t,x)$ and $\delta^\sigma(t,x)$ are predictable with $|\delta^\alpha(t,x)| + |\delta^\sigma(t,x)| \le \gamma_k(x)$ for all $t \le T_k$, where $\gamma_k(x)$ is a deterministic function on $\mathbb{R}$ with $\int_E (|\gamma_k(x)|^r \wedge 1)\,\lambda(dx) < \infty$ for some $0 \le r < 2$ and T_k is a sequence of stopping times increasing to $+\infty$;*

4. *b^α, b^σ are locally bounded;*

5. *$\eta^\alpha, \eta^\sigma, \tilde{\eta}^\alpha$ and $\tilde{\eta}^\alpha$ are processes with càdlàg paths;*

6. *μ^Y is an integer-valued random measure on $\mathbb{R}_+ \times \mathbb{R}$ with jump compensator $dt \otimes \nu_t^Y(dx)$ such that the process $\left(\int_{\mathbb{R}} \left(|x|^{\beta'} \wedge 1\right) \nu_t^Y(dx)\right)_{t\ge 0}$ is locally bounded for β' from Assumption A.*

Following are the stronger assumptions that allow us to use the estimates of the previous section:

Assumption SB. *In addition to Assumptions A and B we have*

1. *$|\sigma_t|$ and $|\sigma_t|^{-1}$ are uniformly bounded;*

2. *$|\delta^\alpha(t,x)| + |\delta^\sigma(t,x)| \le \gamma(x)$ for all $t > 0$, where $\gamma(x)$ is a deterministic bounded function on $\mathbb{R}$ with $\int_E |\gamma(x)|^r \lambda(dx) < \infty$ for some $0 \le r < 2$;*

3. *$b^\alpha, b^\sigma, \eta^\alpha, \eta^\sigma, \tilde{\eta}^\alpha$ and $\tilde{\eta}^\alpha$ are bounded;*

4. *the process $\left(\int_{\mathbb{R}} \left(|x|^{\beta'} \wedge 1\right) \nu_t^Y(dx)\right)_{t\ge 0}$ is bounded and the jumps of Y are bounded;*

5. *the jumps of $\acute{S}$ and $\grave{S}$ are bounded;*

We now argue how to strengthen Assumption B meaning why we may assume the stronger Assumption SB instead. Section 4.4.3 in [JP12] discusses the *localization procedure* in very great detail. Here we will roughly outline the path discussed there and only give detailed account when deviations occur.

In a simplified version Jacod and Protter say that the *localization procedure* applies from (SB) to (B) if it holds that: If some sequence of functionals $U^n(X)_t$, depending on the underlying process X, converges for all $t > 0$ stably in law under a strong assumption on the process X (in our case (SB)) towards some limit $U(X)_t$, then it also converges under a weaker assumption (in or case (B)).

The proof that this actually holds true for various sets of strong and weak assumptions is conducted in Lemma 4.4.9 of [JP12] and needs some prerequisites which we will check in the following. As for $t > 0$ the functionals $U^n(X)_t$ considered in our case are build solely out of the increments of the underlying process up to the time t and furthermore, as we will see in Chapter 5, the limiting object $U(X)_t$ does not depend on X at all, we certainly have condition (4.4.2) in [JP12], that is: If X and X' are two semimartingales and τ is a stopping time then they are subject to the following condition:

If $X_t = X'_t$ a.s. $\forall t < \tau$ then:

- for $t < \tau \Rightarrow U^n(X)_t = U^n(X')_t$ a.s.,
- the $\mathcal{F}-$ conditional laws of $(U(X)_t)_{t<\tau}$ and $(U(X')_t)_{t<\tau}$ are a.s. equal.

The proof of Lemma 4.4.9 now consists of two steps. The first step is to show that for a semimartingale X satisfying Assumptions A and B there exists a localizing sequence of stopping times, $F_p \nearrow \infty$ a.s. when $p \to \infty$, such that for each $p > 0$ there exists a semimartingale $X(p)$ with

- for all $t < F_p \Rightarrow X(p)_t = X_t$ a.s.,
- each $X(p)$ satisfies Assumption SB.

The second step is to show that if we have a localizing sequence as above and convergence $U^n(X(p))_t \xrightarrow{\mathcal{L}-s} U(X(p))_t$ then it also holds that $U^n(X)_t \xrightarrow{\mathcal{L}-s} U(X)_t$. The second step is generic and independent of the assumptions placed upon the processes X and $X(p)$. Therefore we omit this step which consists of part 1) and 2) of the proof of Theorem 4.4.9 in [JP12] and only show that a localizing sequence as above exists.

Lemma 4.5. *Let X be a process fulfilling Assumption A where the components of X fulfill Assumption B. Then for each $p > 0$ there exists a stopping time F_p and a process $X(p)$ such that $X(p)$ and its components, $\alpha(p), \sigma^{(p)}$ and $Y(p)$, fulfill Assumption SB and it holds that $X(p)_t = X_t$ for all $t < F_p$ while $F_p \nearrow \infty$ when $p \to \infty$.*

Proof. The proof follows in great parts and notation the proof of Theorem 4.4.9 3) in [JP12]. We start with the assumptions on the process σ and omit the superscript in $b^\sigma, \eta^\sigma, \tilde{\eta}^\sigma, \delta^\sigma$ in the following. By (B) b is locally bounded therefore we have a localizing sequence of stopping times $V_p \nearrow \infty$ such that $|b_t| \leq p$ if $0 \leq t \leq V_p$. Then we define the stopping times $U_p := \inf\{t : |\sigma_t| + |\eta_t| + |\tilde{\eta}_t| \geq p\}$ and $L_p := \inf\{t : |\sigma_t| \leq \frac{1}{p}\}$. As $\sigma, \eta, \tilde{\eta}$ are assumed to be càdlàg it is a well known fact that $U_p \nearrow \infty$. We now prove that $L_p \nearrow \infty$ by showing that for any $t > 0, \omega \in \Omega$ there exists some constant $M(\omega) > 0$ with $|\sigma_s(\omega)| > M(\omega)$ on the interval $s \in [0, t]$. Then we can find a $p' \in \mathbb{N}$ with $1/p' < M(\omega)$ and have $L_p(\omega) > t$ for all $p \geq p'$. Assume such an $M(\omega)$ does not exists then there exists a sequence $x_n \to x$ in $[0, t]$ such that $\lim\limits_{x_n \to x} \sigma_{x_n}(\omega) = 0$. By passing onto a subsequence of

x_n we either have that $\lim_{x_n \nearrow x} \sigma_{x_n}(\omega) = 0$ or $\lim_{x_n \searrow x} \sigma_{x_n}(\omega) = 0$. The first is a contradiction to the assumption that $|\sigma_-|$ is strictly positive and the second one is a contradiction to the assumption that $|\sigma|$ is strictly positive and a càdlàg process. For $p \in \mathbb{N}$ let T_p be the stopping time from Assumption B.3. Set

$$E_p := V_p \wedge U_p \wedge L_p \wedge T_p \tag{4.7}$$

and

$$\begin{aligned}
b_t^{(p)} &= b_{t\wedge E_p},\\
\eta_t^{(p)} &= \eta_{t\wedge E_p}\mathbb{1}_{\{|\eta_{t\wedge E_p}|\le p\}},\\
\tilde{\eta}_t^{(p)} &= \tilde{\eta}_{t\wedge E_p}\mathbb{1}_{\{|\tilde{\eta}_{t\wedge E_p}|\le p\}},\\
\delta^{(p)}(t,z) &= \delta(t\wedge E_p, z)\mathbb{1}_{\{|\delta(t\wedge E_p,z)|\le 2p\}}.
\end{aligned}$$

By construction we have $|b^{(p)}| \le p$, $|\eta^{(p)}| \le p$, $|\tilde{\eta}^{(p)}| \le p$ (note that $\eta_{t\wedge E_p}, \tilde{\eta}_{t\wedge E_p}$ would not be bounded in general) and $\eta^{(p)}, \tilde{\eta}^{(p)}$ are càdlàg, $\delta^{(p)}(t,z)$ predictable. Furthermore, it holds that $|\delta^{(p)}(t,z)| \le \gamma^{(p)}(z)$, where $\gamma^{(p)}(z) = \gamma_p \wedge 2p$ and γ_p is the associated function to T_p from Assumption B. Due to $\int_E (|\gamma_k(x)|^r \wedge 1)\,\lambda(dx) < \infty$ we find that $\int_E \left(|\gamma^{(p)}(x)|^r\right)\lambda(dx) < \infty$. The process $\sigma^{(p)}$ defined as

$$\sigma_t^{(p)} = \begin{cases} p, & \text{if } t \ge E_p,\\ \sigma_0 + \int_0^t b_s^{(p)} ds + \int_0^t \eta_s^{(p)} dW_s + \int_0^t \tilde{\eta}_s^{(p)} d\widetilde{W}_s & \\ \quad + \int_0^t \int_E \kappa(\delta^{(p)}(s,x))\underline{\widetilde{\mu}}(ds,dx) + \int_0^t \int_E \kappa'(\delta^{(p)}(s,x))\underline{\mu}(ds,dx), & \text{if } t < E_p, \end{cases} \tag{4.8}$$

then satisfies Assumptions SB.1, SB.2, SB.3 as $\min\left(|\frac{1}{p} - 2p|, \frac{1}{p}\right) = \frac{1}{p} \le |\sigma_t^{(p)}| \le 3p$ and $E_p \nearrow \infty$. The only thing that remains to show is that $\sigma_t = \sigma_t^{(p)}$ a.s. when $t < E_p$. The proof and arguments here are exactly the same is in the proof of Theorem 4.4.9 3) in [JP12] therefore we omit it here.

The same methods can then be applied to the process α to give us a localized version $\alpha(p)$ fulfilling Assumption SB with localizing sequence $A_p \nearrow \infty$.

Moving on to the process $Y_t = \int_0^t \int_{\mathbb{R}} x\mu^Y(ds,dx)$ we let B_p be the localizing sequence from Assumption B.6 such that $\int_{\mathbb{R}} \left(|x|^{\beta'} \wedge 1\right)\nu_t^Y(dx) \le p$ for all $t \le B_p$. We set $Z_p := \inf\{t : Y_t \ge p\}$ and

$$Y_t^{(p)} = \int_0^{t\wedge B_p \wedge Z_p} \int_{\mathbb{R}} x\mathbb{1}_{\{|x|\le 2p\}}\mu^Y(ds,dx).$$

We see that the process $Y^{(p)}$ fulfills Assumption SB.4 as its jumps are bounded and furthermore $Y^{(p)}_t = Y_t$ for all $t < Z_p$. Moving on to the process $\acute{S}$ from (4.6) we proceed in the same manner as with the process Y, meaning we set $\widetilde{Z}_p := \inf\{t : \acute{S}_t \geq p\}$ and

$$\acute{S}^{(p)}_t = \int_0^{t\wedge\widetilde{Z}_p} \int_{\mathbb{R}} x \mathbb{1}_{\{|x|\leq 2p\}} \mu_1(ds, dx).$$

Likewise we construct a process $\grave{S}^{(p)}$ and stopping time $\widehat{Z}_p$ for the process $\grave{S}$. We note again that the jumps of $\acute{S}^{(p)}$ and $\grave{S}^{(p)}$ are bounded and therefore satisfy Assumption SB.5. The proof that for all $t < \widetilde{Z}_p$ we have $\int_0^{\widetilde{Z}_p\wedge t} \sigma^{(p)}_{s-} d\acute{S}^{(p)}_s = \int_0^{\widetilde{Z}_p\wedge t} \sigma_{s-} d\acute{S}_s$ (and similar for $\grave{S}^{(p)}$) is again part of the proof of Theorem 4.4.9 3) in [JP12].
Finally we define $F_p = E_p \wedge A_p \wedge B_p \wedge Z_p \wedge \widetilde{Z}_p \wedge \widehat{Z}_p$ and

$$\begin{aligned} X(p)_t = X_0 + \int_0^{F_p\wedge t} \alpha^{(p)}_s ds + \int_0^{F_p\wedge t} \sigma^{(p)}_{s-} dS_s + \int_0^{F_p\wedge t} \sigma^{(p)}_{s-} d\acute{S}^{(p)}_s \\ - \int_0^{F_p\wedge t} \sigma^{(p)}_{s-} d\grave{S}^{(p)}_s + \left(Y^{(p)}_{F_p\wedge t}\right), \end{aligned}$$

which fulfills Assumption SB, $X(p)_t = X_t$ a.s. for all $t < F_p$ and $F_p \nearrow \infty$.

□

4.2.1 Localization Procedure for Random Discretization Schemes

We now introduce the specific scheme of observation times how the process X is actually observed. This is a simple case of the way "restricted discretization schemes" are introduced in chapter 14.1 of [JP12]. For this matter we assume that the probability measure $\mathbb{P}$ is defined on a σ-field $\mathcal{G}$ bigger than $\mathcal{F}$.

Assumption C. *For each $n \in \mathbb{N}$ we observe the process X at stopping times $0 = \tau^n_0 < \tau^n_1 < \tau^n_2 < \ldots$ with:*
$\tau^n_0 = 0, \tau^n_1 = \Delta_n \phi^n_1$ *and*

$$\tau^n_i = \tau^n_{i-1} + \Delta_n \phi^n_i \lambda_{\tau^n_{i-2}} \text{ for all } 2 \leq i.$$

For all $t > 0$ the random variable $N_n(t)$ is the number of observation times smaller than t and can be written as

$$N_n(t) = \sum_{i\geq 1} \mathbb{1}_{\{\tau^n_i \leq t\}}.$$

Furthermore we assume:

1. *The process λ_t is a strictly positive semimartingale w.r.t. the filtration $(\mathcal{F}_t)_{t\geq 0}$ and in addition fulfills the same structural conditions as the process σ_t stated in Assumption B.*

2. *$(\phi_i^n)_{i\geq 1}$ is a family of random variables with respect to the σ-field $\mathcal{G}$ and independent of $\mathcal{F}$;*

3. *$\phi_i^n \sim \phi$ for a strictly positive random variable ϕ with $\mathbb{E}[\phi] = 1$. We assume that for all $p \in (-2, \infty)$ the moments $\mathbb{E}[\phi^p]$ exist;*

This Assumption is build such that there is infinite number of observations while in applications we usually have only a finite number of observations up to fixed time point $T > 0$, i.e. the number of observations is $N_n(T)$. This does not pose a problem as the estimator from the next chapter only uses the values of the process X up to the $N_n(T)$-th observation and all observations after that are only used to conduct the proofs but have no impact on the estimator (or its limit) whatsoever.
For the proofs in Chapter 5 it is necessary that the process λ_t, driving the observation times τ_i^n, is bounded from below and above. So again we need additional assumptions that are stronger than Assumption C and can be derived in a similar way than before. These stronger assumptions then are:

Assumption SC. *In addition to assumption C there exists a constant $C > 1$ such that*

1. *The process λ fulfills the stronger assumptions for σ in Assumption SB, in particular for all $t > 0$:*

$$\frac{1}{C} \leq \lambda_t \leq C.$$

2. *The final number of observation times is bounded from above by n times some constant, i.e.*

$$N_n(T) \leq CnT.$$

The reason why we cannot directly employ the previous localization procedure is that the process λ_t is part of the discretization scheme but not embodied in an actual class of processes like before. Therefore we have the following Lemma that proves Assumption SC.1. It is formulated in a general way for an arbitrary sequence of random variables $(F_n)_{n\in\mathbb{N}}$ that are dependent on the process X, the discretization scheme $\{\tau_i^n : i \geq 0\}$, $N_n(T)$ and the process λ, and likewise a possible limit in distribution F of F_n dependent on the same factors and realized on an extension $(\widetilde{\Omega}, \widetilde{\mathcal{G}}, \widetilde{\mathbb{P}})$ of the original probability space $(\Omega, \mathcal{G}, \mathbb{P})$. Here F_n takes the role of our (appropriately scaled) estimator for the jump activity index from Chapter 5 and F is then the random variable it converges to in the central limit theorem we want to prove. However, in the proof of this CLT we need estimates for expressions like $\mathbb{E}\left[\left|\lambda_{\tau_i^n} - \lambda_{\tau_{i-1}^n}\right|\right]$ and one way to get these estimates is to assume the same structure on λ as we did for σ and to stop the process appropriately. Hence, for $C > 1$ let E_C be the stopping time defined by (4.7) with C replacing p and the process λ and its components replacing σ. Furthermore define $\lambda_t^{(C)}$ according to (4.8) again with the components of λ replacing those of σ everywhere. In particular it now holds for the stopped process $\lambda^{(C)}$ that $\frac{1}{C} \leq \lambda_t^{(C)} \leq C$ and that we have the analogues of Assumption SB. Now we proof that a localization for our observation scheme is actually possible:

Lemma 4.6. *Assume that Assumption C holds and construct, for each $C > 1$, stopping time E_C and process $\lambda^{(C)}$, a new discretization scheme, i.e. new stopping times $\{\tau_i^{n,C} : i \geq 0\}$ and a new $N_n^C(T)$ as in Condition C but with the process $\lambda^{(C)}$ instead of λ. Define a sequence of associated random variables $F_n(C)$ similar to F_n as well but with the process $\lambda^{(C)}$ replacing λ, $\{\tau_i^{n,C} : i \geq 0\}$ replacing $\{\tau_i^n : i \geq 0\}$ and $N_n^C(T)$ replacing $N_n(T)$, and likewise for $F(C)$ on $(\widetilde{\Omega}, \widetilde{\mathcal{G}}, \widetilde{\mathbb{P}})$. If for each $C > 1$ it holds that*

$$F_n(C) \xrightarrow{\mathcal{L}-s} F(C) \tag{4.9}$$

and if furthermore

$$F_n(C)\mathbb{1}_{\{E_C > T\}} = F_n \mathbb{1}_{\{E_C > T\}} \quad \text{and} \quad F(C)\mathbb{1}_{\{E_C > T\}} = F\mathbb{1}_{\{E_C > T\}} \tag{4.10}$$

then we have $F_n \xrightarrow{\mathcal{L}-s} F$.

Proof. Let $\widetilde{\mathbb{E}}$ be the expectation w.r.t. $\widetilde{\mathbb{P}}$. We clearly need to prove

$$\begin{aligned}
&\limsup_{n\to\infty} \left|\mathbb{E}\left[Yf(F_n)\right] - \widetilde{\mathbb{E}}\left[Yf(F)\right]\right| = \limsup_{C\to\infty} \limsup_{n\to\infty} \left|\mathbb{E}\left[Yf(F_n)\right] - \widetilde{\mathbb{E}}\left[Yf(F)\right]\right| \\
&\leq \limsup_{C\to\infty} \limsup_{n\to\infty} \left|\mathbb{E}\left[Yf(F_n)\right] - \mathbb{E}\left[Yf(F_n(C))\right]\right| \\
&+ \limsup_{C\to\infty} \limsup_{n\to\infty} \left|\mathbb{E}\left[Yf(F_n(C))\right] - \widetilde{\mathbb{E}}\left[Yf(F(C))\right]\right| \\
&+ \limsup_{C\to\infty} \limsup_{n\to\infty} \left|\widetilde{\mathbb{E}}\left[Yf(F(C))\right] - \widetilde{\mathbb{E}}\left[Yf(F)\right]\right| = 0
\end{aligned}$$

where Y is any bounded random variable on $(\Omega, \mathcal{G})$ and f is any bounded continuous function, and we show the claim for each of the three terms above separately. For the first one, by boundedness of Y and f and using (4.10), it is obvious that

$$\left|\mathbb{E}\left[Y(f(F_n) - f(F_n(C)))\right]\right| = \left|\mathbb{E}\left[Y(f(F_n) - f(F_n(C)))\mathbb{1}_{\{E_C \leq T\}}\right]\right| \leq K\mathbb{P}\left(E_C \leq T\right)$$

for some constant $K > 0$. Thus

$$\limsup_{C\to\infty}\limsup_{n\to\infty}|\mathbb{E}[Yf(F_n)] - \mathbb{E}[Yf(F_n(C))]| \leq K \limsup_{C\to\infty}\mathbb{P}(E_C \leq T) = 0,$$

and the same proof applies for the third term. Finally, note that

$$\limsup_{n\to\infty}\left|\mathbb{E}[Yf(F_n(C))] - \widetilde{\mathbb{E}}[Yf(F(C))]\right| = 0$$

for each fixed C is an immediate consequence of (4.9). □

Remark 4.1. *By construction λ_t and $\lambda_t^{(C)}$ coincide on the set $\{E_C \leq T\}$ for all $0 \leq t \leq T$. As the estimator from Chapter 5 will only look at observations up to a fixed time horizon T (in our specific case the convenient but arbitrary $T = 1$) and therefore the values of $X_t, \lambda_t^{(C)}$ for $t > T$ are irrelevant to the estimator and its limit, we have that condition* (4.10) *is met. Therefore we may assume that for the following proofs Assumption SC.1 is in force.*

In order to show that we can deduct Assumption SC.2 from Assumption C one again has to construct a discretization scheme with the desired properties and find an appropriate way of localizing it. Here we reference to part 2) of the proof of Lemma 9 in [JT18] where this procedure is carried out in great detail.
For further information on random discretization schemes one can consult Section 14.1 in [JP12] where a slightly different version of Lemma 4.6 and other important properties of objects connected to these schemes are proven. We want to name one of those properties in particular because we will use it repeatedly in the following chapters: (cf. (14.1.10) in [JP12]) For all $t \geq 0$ we have

$$\frac{N_n(1)}{n} \xrightarrow{\mathbb{P}} \int_0^1 \frac{1}{\lambda_s} ds. \tag{4.11}$$

4.3 Estimates for Itô semimartingales under Strengthened Assumptions

We start off by using the strengthened assumptions to show the finiteness of $\widehat{\kappa(\delta^\alpha)}, \widehat{\kappa'(\delta^\alpha)}$ which is (4.3) applied to the functions $\kappa(\delta^\alpha), \kappa'(\delta^\alpha)$.

Lemma 4.7. *As long as $(U - \tau)$ is bounded, we have under Assumption SB that*

$$\widehat{\kappa(\delta^\alpha)}(q)_{\tau,(U-\tau)} < \infty \quad \text{for } q \in [r, \infty),$$
$$\widehat{\kappa'(\delta^\alpha)}(q)_{\tau,(U-\tau)} < \infty \quad \text{for } q > 0$$

and likewise results for δ^σ.

Proof. Because κ is a truncation function and as such equals the identity in a neighbourhood around zero and is bounded, we can split it up with $\epsilon > 0, K > 0$

$$|\kappa(x)| = |x|\mathbb{1}_{\{|x|<\epsilon\}} + |\kappa(x)|\mathbb{1}_{\{|x|\geq\epsilon\}} \leq |x|\mathbb{1}_{\{|x|<\epsilon\}} + K\mathbb{1}_{\{|x|\geq\epsilon\}}.$$

Therefore by Assumption SB.2

$$\begin{aligned} \frac{|\kappa(\delta^\alpha(u,z))|}{\gamma(z)} &\leq \frac{|\delta^\alpha(u,z)|}{\gamma(z)}\mathbb{1}_{\{|\delta^\alpha(u,z)|<\epsilon\}} + \frac{K}{\gamma(z)}\mathbb{1}_{\{|\delta^\alpha(u,z)|\geq\epsilon\}} \\ &\leq \mathbb{1}_{\{|\delta^\alpha(u,z)|<\epsilon\}} + \frac{K}{\epsilon}\mathbb{1}_{\{|\delta^\alpha(u,z)|\geq\epsilon\}} \end{aligned} \tag{4.12}$$

and

$$\int_E |\kappa(\delta^\alpha(u,z))|^q\lambda(dz) \leq \left(1+\frac{K}{\epsilon}\right)^q \int_E \gamma(z)^q\lambda(dz) < \infty \text{ for all } q \in [r,\infty), u \geq 0. \tag{4.13}$$

Therefore with some constant $K_q > 0$

$$\begin{aligned} \widehat{\kappa(\delta^\alpha)}(q)_{\tau,(U-\tau)} &= \frac{1}{U-\tau}\int_\tau^U \int_E |\kappa(\delta^\alpha(u,z))|^q\lambda(dz)du \\ &\leq \frac{1}{U-\tau}\int_\tau^U K_q du \leq K_q. \end{aligned}$$

Moving on to $\widehat{\kappa'(\delta^\alpha)}(q)_{\tau,s}$ we have

$$\begin{aligned} |\kappa'(x)| = |x-\kappa(x)| &= |x - (x\mathbb{1}_{\{|x|<\epsilon\}} + \kappa(x)\mathbb{1}_{\{|x|\geq\epsilon\}})| \\ &= (|x| + |\kappa(x)|)\mathbb{1}_{\{|x|\geq\epsilon\}} \end{aligned}$$

and therefore because $|\delta^\alpha(u,z)| \leq \gamma(z)$ for some constant $K > 0$

$$|\kappa'(\delta^\alpha(u,z))| \leq (|\kappa(\delta^\alpha(u,z))| + |\delta^\alpha(u,z)|)\,\mathbb{1}_{\{\gamma(z)\geq\epsilon\}} \leq (K+\gamma(z))\,\mathbb{1}_{\{\gamma(z)\geq\epsilon\}}.$$

Due to $\int_E |\gamma(x)|^r\lambda(dx) < \infty$ we have $\lambda(\{\gamma(z) \geq \epsilon\}) < \infty$ and because $\gamma(z)$ is bounded we find that

$$\int_E |\kappa'(\delta^\alpha(u,z))|^q\lambda(dz) < \int_{\{\gamma(z)\geq\epsilon\}} (\gamma(z)+K)^q\lambda(dz) < \infty \text{ for all } q > 0, u \geq 0 \tag{4.14}$$

and as such $\widehat{\kappa'(\delta^\alpha)}(q)_{\tau,s} < \infty$ for all $q > 0$. □

Lemma 4.8. *For any $q \in (0,\infty)$, stopping times $U > \tau$ such that U is bounded then it holds under Assumption SB*

$$\mathbb{E}_\tau[\sup_{0\le u\le (U-\tau)} |\alpha_{\tau+u}-\alpha_\tau|^q] \le K_q \Big(\mathbb{E}_\tau[(U-\tau)^q] + \mathbb{E}_\tau\left[(U-\tau)^{q/2}\right] + \mathbb{E}_\tau\left[(U-\tau)^{1/2}\right]^q$$
$$+\mathbb{E}_\tau\left[(U-\tau)\right]^{q/(r\vee 1)} + \mathbb{E}_\tau[(U-\tau)]\Big) \tag{4.15}$$

and likewise

$$\mathbb{E}_\tau[\sup_{0\le u\le (U-\tau)} |\sigma_{\tau+u}-\sigma_\tau|^q] \le K_q \left(\mathbb{E}_\tau[(U-\tau)^q] + \mathbb{E}_\tau[(U-\tau)^{1/2}]^q + \mathbb{E}_\tau[(U-\tau)^{q/2}]\right.$$
$$\left.+\mathbb{E}_\tau\left[(U-\tau)\right]^{q/(r\vee 1)} + \mathbb{E}_\tau[(U-\tau)]\right).$$

In general these estimates hold true for any process fulfilling the structural assumptions for either the process α or σ in Assumption SB, in particular this holds for the process λ.

Proof. We prove Lemma 4.8 by breaking down the process α into its components and proving the estimates one at a time.
For the drift term we have with (4.2) and the strengthened assumptions from the localization

$$\sup_{0\le u\le (U-\tau)} \left|\int_\tau^{\tau+u} b_s^\alpha ds\right|^q \le (U-\tau)^q \left(\sup_{\tau\le u\le U} |b_u^\alpha|\right)^q \le K(U-\tau)^q.$$

For the continuous martingale part we use Lemma 4.2 for $q \ge 1$ and with some constant $M > 0$

$$\mathbb{E}_\tau\left[\sup_{0\le u\le (U-\tau)}\left|\int_\tau^{\tau+u} \eta_z^\alpha dW_z\right|^q\right] \le K_q\mathbb{E}_\tau\left[(U-\tau)^{q/2}\left(\frac{1}{(U-\tau)}\int_\tau^U |\eta_z^\alpha|^2 dz\right)^{q/2}\right]$$
$$\le K_q\mathbb{E}_\tau\left[(U-\tau)^{q/2}M^{q/2}\right] \le K_q\mathbb{E}_\tau\left[(U-\tau)^{q/2}\right].$$

And likewise with Lemma 4.2 for $q \le 1$

$$\mathbb{E}_\tau\left[\sup_{0\le u\le (U-\tau)}\left|\int_\tau^{\tau+u} \eta_s^\alpha dW_s\right|^q\right] \le K_q\mathbb{E}_\tau\left[(U-\tau)^{1/2}\right]^q.$$

Moving on to the jump components we apply Lemma 4.3 together with Lemma 4.7 to get for $q \in [r\vee 1, 2]$

$$\mathbb{E}_\tau\left[\sup_{0\le u\le (U-\tau)}\left|\int_\tau^{\tau+u}\int_E \kappa(\delta^\alpha(s,x))\underline{\widetilde{\mu}}(ds,dx)\right|^q\right] \le K_q\mathbb{E}_\tau\left[(U-\tau)\widehat{\kappa(\delta^\alpha)}(q)_{\tau,(U-\tau)}\right]$$
$$\le K_q\mathbb{E}_\tau\left[(U-\tau)\right]$$

and for $q > 2$

$$\mathbb{E}_\tau\left[\sup_{0\le u\le (U-\tau)}\left|\int_\tau^{\tau+u}\int_E \kappa(\delta^\alpha(s,x))\underline{\widetilde{\mu}}(ds,dx)\right|^q\right]$$
$$\le K_q\left(\mathbb{E}_\tau[(U-\tau)\widehat{\kappa(\delta^\alpha)}(q)_{\tau,(U-\tau)}] + \mathbb{E}_\tau[(U-\tau)^{q/2}\widehat{\kappa(\delta^\alpha)}(2)^{q/2}_{\tau,(U-\tau)}]\right)$$
$$\le K_q\left(\mathbb{E}_\tau[(U-\tau)] + \mathbb{E}_\tau[(U-\tau)^{q/2}]\right).$$

In the case of $q \in (0, r\vee 1]$ we apply Jensen inequality and have

$$\mathbb{E}_\tau\left[\sup_{0\le u\le (U-\tau)}\left|\int_\tau^{\tau+u}\int_E \kappa(\delta^\alpha(s,x))\underline{\widetilde{\mu}}(ds,dx)\right|^q\right]$$
$$\le \mathbb{E}_\tau\left[\sup_{0\le u\le (U-\tau)}\left|\int_\tau^{\tau+u}\int_E \kappa(\delta^\alpha(s,x))\underline{\widetilde{\mu}}(ds,dx)\right|^{(r\vee 1)}\right]^{q/(r\vee 1)}$$
$$\le K_q\left(\mathbb{E}_\tau[(U-\tau)\widehat{\kappa(\delta^\alpha)}(r\vee 1)_{\tau,(U-\tau)}]\right)^{q/(r\vee 1)} \le K_q\mathbb{E}_\tau[(U-\tau)]^{q/(r\vee 1)}.$$

Moving on to the "big jumps" applying Lemma 4.4 gives for $q \in (0,1]$

$$\mathbb{E}_\tau\left[\sup_{0\le u\le (U-\tau)}\left|\int_\tau^{\tau+u}\int_E \kappa'(\delta^\alpha(s,x))\underline{\mu}(ds,dx)\right|^q\right] \le K_q\mathbb{E}_\tau[(U-\tau)\widehat{\kappa'(\delta^\alpha)}(q)_{\tau,(U-\tau)}]$$
$$\le K_q\mathbb{E}_\tau[(U-\tau)]$$

and for $q > 1$

$$\mathbb{E}_\tau\left[\sup_{0\le u\le (U-\tau)}\left|\int_\tau^{\tau+u}\int_E \kappa'(\delta^\alpha(s,x))\underline{\mu}(ds,dx)\right|^q\right]$$
$$\le K_q\left(\mathbb{E}_\tau[(U-\tau)^q\widehat{\kappa'(\delta^\alpha)}(1)_{\tau,s}] + \mathbb{E}_\tau[(U-\tau)\widehat{\kappa'(\delta^\alpha)}(q)_{\tau,s}]\right)$$
$$\le K_q\left(\mathbb{E}_\tau[(U-\tau)^q] + \mathbb{E}_\tau[(U-\tau)]\right).$$

□

Using the previous lemma we can now derive the asymptotic behavior in our specific setting. We set $(\mathcal{F}^n_t)_{t\ge 0}$ as the smallest filtration containing $(\mathcal{F}_t)_{t\ge 0}$ and for which all τ^n_i, $i \ge 1$, are stopping times. Furthermore in accordance with the previous notation for conditional expectations we set $\mathbb{E}^n_i[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_{\tau^n_i}]$. It should be noted that $q \in (0,\infty)$ appearing in the next lemma and in the lemmas above will be in our application a fixed and known number. In order to make the next result more readable we suppress the dependency on q (or a, b appearing in the proof below) of the constant $K_q > 0$ in the following.

Corollary 4.1. *For $q \in (0,\infty)$, $i,j \in \mathbb{N}_0$ with $i+j \le N_n(1)$ it holds*

$$\mathbb{E}_i^n[\sup_{0\le u\le(\tau_{i+1}^n-\tau_i^n)} |\sigma_{\tau_i^n+u}-\sigma_{\tau_i^n}|^q] \le K\Delta_n^{(q/2)\wedge 1},$$
$$\mathbb{E}_i^n[\sup_{0\le u\le(\tau_{i+j}^n-\tau_i^n)} |\sigma_{\tau_i^n+u}-\sigma_{\tau_i^n}|^q] \le K(j\Delta_n)^{(q/2)\wedge 1}. \tag{4.16}$$

With similar results for the processes α and λ.

Proof. First we note that due to $\tau_{N_n(1)}^n \le 1$ in Assumption C we have that $\tau_i \le 1$ for all $0 \le i \le N_n(1)$ and therefore Lemma 4.8 is applicable. We note that the assumptions on $\tau_{i+1}^n - \tau_i^n$ give us that for $a,b \ge 0$

$$\mathbb{E}_i^n\left[(\tau_{i+1}^n-\tau_i^n)^a\right]^b \le \mathbb{E}_i^n\left[\left(\Delta_n\lambda_{\tau_{i-2}^n}\phi_i^n\right)^a\right]^b \le K\Delta_n^{ab}$$

using the boundedness of λ and that moments of all powers a for ϕ_i^n exist. More generally we have if $a \le 1$ by applying Jensen inequality

$$\begin{aligned}\mathbb{E}_i^n\left[(\tau_{i+j}^n-\tau_i^n)^a\right]^b &= \mathbb{E}_i^n\left[\left(\sum_{k=i}^{i+j-1}(\tau_{k+1}^n-\tau_k^n)\right)^a\right]^b\\ &\le \mathbb{E}_i^n\left[\sum_{k=i}^{i+j-1}(\tau_{k+1}^n-\tau_k^n)\right]^{ab} \le K\left(j\Delta_n\right)^{ab}\end{aligned}$$

and if $a > 1$

$$\begin{aligned}\mathbb{E}_i^n\left[(\tau_{i+j}^n-\tau_i^n)^a\right]^b &= \mathbb{E}_i^n\left[j^a\left(\frac{1}{j}\sum_{k=i}^{i+j-1}(\tau_{k+1}^n-\tau_k^n)\right)^a\right]^b\\ &\le \mathbb{E}_i^n\left[\frac{j^a}{j}\sum_{k=i}^{i+j-1}(\tau_{k+1}^n-\tau_k^n)^a\right]^b \le K\frac{j^{ab}}{j^b}\left(j\Delta_n^a\right)^b = \left(j\Delta_n\right)^{ab}.\end{aligned}$$

Applying the lines above to Lemma 4.8 and comparing the rates for Δ_n gives

$$\begin{aligned}&\mathbb{E}_i^n[\sup_{0\le u\le(\tau_{i+j}^n-\tau_i^n)} |\sigma_{\tau_i^n+u}-\sigma_{\tau_i^n}|^q]\\ &\le K\left(\mathbb{E}_i^n[(\tau_{i+j}^n-\tau_i^n)^q] + \mathbb{E}_i^n[(\tau_{i+j}^n-\tau_i^n)^{1/2}]^q + \mathbb{E}_i^n[(\tau_{i+j}^n-\tau_i^n)^{q/2}]\right.\\ &\qquad \left.+\mathbb{E}_i^n[(\tau_{i+j}^n-\tau_i^n)] + \mathbb{E}_i^n[(\tau_{i+j}^n-\tau_i^n)]^{q/(r\vee 1)}\right)\\ &\le K\left((j\Delta_n)^q + (j\Delta_n)^{q/2} + (j\Delta_n) + (j\Delta_n)^{q/(r\vee 1)}\right)\\ &\le K(j\Delta_n)^{(q/2)\wedge 1}.\end{aligned}$$

$\square$

Remark 4.2. *For some proofs of the next chapter the previous results are not specific enough yet. Instead of taking conditional expectations with respect to $\mathcal{F}^n_{\tau^n_i}$ as in Corollary 4.1 we would like to take them with respect to a slightly bigger σ-algebra, e.g. $\mathcal{A} = \mathcal{F}^n_{\tau^n_i} \bigvee \sigma(\phi^n_{i+2}, \phi^n_{i+1})$. As the $(\phi^n_i)_{i\geq 1}$ are independent from $\mathcal{F}$ by Assumption C we note that all the processes from Assumptions A and B retain their properties when we take them as processes w.r.t. the filtration $(\mathcal{F}^n_t \bigvee \sigma(\phi^n_{i+2}, \phi^n_{i+1}))_{t\geq 0}$ and similar do the $(\tau^n_i)_{i\geq 1}$ remain stopping times. Therefore the Lemmas from Section 4.1 are still applicable in this case. For example Corollary 4.1 then reads as:*
Let $q \in (0,\infty)$, $i, j \in \mathbb{N}_0$ with $i+j \leq N_n(1)$ and $\mathcal{A} = \mathcal{F}^n_{\tau^n_i} \bigvee \sigma(\phi^n_{i+j}, \dots, \phi^n_{i+1})$ then it holds

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq u\leq (\tau^n_{i+j}-\tau^n_i)} |\sigma_{\tau^n_i+u} - \sigma_{\tau^n_i}|^q \middle| \mathcal{A}\right] \\
&\quad \leq K_q \Big(\mathbb{E}[(\tau^n_{i+j} - \tau^n_i)^q | \mathcal{A}] + \mathbb{E}\left[(\tau^n_{i+j} - \tau^n_i)^{1/2} \middle| \mathcal{A}\right]^q + \mathbb{E}\left[(\tau^n_{i+j} - \tau^n_i)^{q/2} \middle| \mathcal{A}\right] \\
&\qquad\qquad + \mathbb{E}\left[(\tau^n_{i+j} - \tau^n_i) \middle| \mathcal{A}\right]^{q/(r\vee 1)} + \mathbb{E}[(\tau^n_{i+j} - \tau^n_i) | \mathcal{A}]\Big) \\
&\quad \leq K_q (\tau^n_{i+j} - \tau^n_i)^{(q/2)\wedge 1},
\end{aligned}$$

using the boundedness of $(\tau^n_{i+j} - \tau^n_i)$. Other definitions of $\mathcal{A}$ are possible as well as long as the "added information" does not change the properties of the processes considered.

Chapter 5

Estimating the Jump Activity Index in the Presence of Random Observation Times

This chapter can be seen as the main part of this work. Here we motivate and construct our estimator for the jump activity index of a semimartingale defined as in (4.5) and then use the results from the previous chapter to prove an associated central limit theorem. It will become apparent why the previous localization procedure is so important for us, as many proofs rely on the boundedness of the processes involved and therefore most of the following results would not be feasible without use of the strengthened Assumptions SB and SC.

5.1 Basics and Preliminaries

As already mentioned, we look at the following class of pure-jump semimartingales as defined by (4.5), i.e.

$$X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \sigma_{s-} dL_s + dY_t,$$

where L is a pure-jump Lévy process that can be decomposed as follows

$$L_t = S_t + \acute{S}_t - \grave{S}_t, \tag{5.1}$$

where S is a Lévy process with characteristic triplet $(-\int_{\mathbb{R}} \kappa'(x) \frac{A}{|x|^{\beta+1}} dx, 0, \frac{A}{|x|^{1+\beta}} dx)$, $\acute{S}$ and $\grave{S}$ are pure-jump Lévy processes with the first two characteristics zero (with respect to the truncation function κ) and densities of the Lévy measure $|\tilde{h}(x)|$ and $2|\tilde{h}(x)|\mathbb{1}_{\{\tilde{h}(x)<0\}}$.

Y is a pure-jump process whose jump behavior at high frequencies is dominated by S, see Assumptions A and B for the exact definition of S, Y and their components. Our aim is to estimate the jump activity index

$$\beta = \inf\left\{p : \sum_{t\leq 1} |\Delta X_t|^p < \infty\right\}$$

where the process is observed only at discrete random time points τ_i^n with the time between two observations $\sup_{i\in\mathbb{N}} |\tau_i^n - \tau_{i-1}^n| \to 0$ and the exact behavior of our observation times made precise by Assumptions C and SC.

The key ingredients towards estimation of the activity index are, like already mentioned, that it coincides with the β from the definition of the jump measure and that we know the form of the characteristic function of the strictly stable process S, namely

$$\mathbb{E}[\cos(uS_t)] = \mathbb{E}[\exp(iuS_t)] = \exp(-A_\beta u^\beta t) \text{ with a constant } A_\beta > 0, u \in \mathbb{R}_+.$$

We define $\Delta_i^n X := X_{\tau_i^n} - X_{\tau_{i-1}^n}$ and as before $\Delta_n := \frac{1}{n}$. The estimator for β in a setting of n equidistant observations, i.e. when $\tau_i^n - \tau_{i-1}^n = \Delta_n$, proposed in [Tod15] is based on the "empirical characteristic function",

$$L^n(p,u) := \frac{1}{n-k_n-2}\sum_{i=k_n+3}^{n} \cos\left(u\frac{\Delta_i^n X - \Delta_{i-1}^n X}{(V_i^n(p))^{1/p}}\right), \ u \in \mathbb{R}_+, \qquad \textit{with}$$

$$V_i^n(p) := \frac{1}{k_n}\sum_{j=i-k_n-1}^{i-2} |\Delta_j^n X - \Delta_{j-1}^n X|^p, \quad i = k_n+3,\dots,n, \ p > 0,$$

for some $k_n \asymp n^\varrho, \varrho \in (0,1)$. In the setting of equidistant observations $(\Delta_n^{-1/\beta} V_i^n(p))$ can be a seen as the localized version of (3.25) on the time-interval $[(i-k_n-1)\Delta_n, (i-2)\Delta_n]$ and hence is a local estimator for $|\sigma_{\tau_{i-2}^n}|^p$ multiplied by $\mathbb{E}[|S_1|^p]$.
These estimators make use of the fact that in the equidistant case the difference of the drift terms of $\Delta_i^n X$ and $\Delta_{i-1}^n X$ have a higher rate of convergence than the drift term of just the single increment $\Delta_i^n X$. This concept does not apply in the presence of random observation times. Therefore we propose a modified version of the estimator above, namely by scaling $\Delta_i^n X$ with its corresponding interval length. Therefore we introduce

$$\widetilde{\Delta_i^n X} := \frac{\Delta_n}{\tau_i^n - \tau_{i-1}^n}\left(X_{\tau_i^n} - X_{\tau_{i-1}^n}\right) = \frac{\Delta_n}{\tau_i^n - \tau_{i-1}^n}\Delta_i^n X \quad \text{and} \quad \widetilde{\Delta_i^n S} := \frac{\Delta_n}{\tau_i^n - \tau_{i-1}^n}\Delta_i^n S$$

and our modified version of the estimator above becomes

$$\widetilde{L}^n(p,u) := \frac{1}{N_n(1)-k_n-2}\sum_{i=k_n+3}^{N_n(1)} \cos\left(u\frac{\widetilde{\Delta_i^n X} - \widetilde{\Delta_{i-1}^n X}}{(\widetilde{V}_i^n(p))^{1/p}}\right), \ u \in \mathbb{R}_+, \ \text{with}$$

$$\widetilde{V}_i^n(p) := \frac{1}{k_n}\sum_{j=i-k_n-1}^{i-2} |\widetilde{\Delta_j^n X} - \widetilde{\Delta_{j-1}^n X}|^p, \quad i = k_n+3,\dots,N_n(1), \ p > 0.$$

We note that the possibly unknown Δ_n which is used to scale the $\Delta_i^n X$ in the numerator and denominator cancels such that $\widetilde{L}^n(p,u)$ “simplifies” to

$$\widetilde{L}^n(p,u) = \frac{1}{N_n(1)-k_n-2}\sum_{i=k_n+3}^{N_n(1)} \cos\left(u\frac{\frac{\Delta_i^n X}{\tau_i^n-\tau_{i-1}^n}-\frac{\Delta_{i-1}^n X}{\tau_{i-1}^n-\tau_{i-2}^n}}{\left(\frac{1}{k_n}\sum_{j=i-k_n-1}^{i-2}\left|\frac{\Delta_j^n X}{\tau_j-\tau_{j-1}}-\frac{\Delta_{j-1}^n X}{\tau_{j-1}-\tau_{j-2}}\right|^p\right)^{1/p}}\right), \quad u\in\mathbb{R}_+. \tag{5.2}$$

In order to state the limit of the estimator above we introduce ϕ' as an independent copy of the variable ϕ from Assumption C governing the behavior of the stopping times. With the constants

$$\mu_{p,\beta} := \mathbb{E}[|S_1|^p]^{\frac{\beta}{p}}, \quad \kappa_{p,\beta} := \mathbb{E}[(\phi^{1-\beta}+(\phi')^{1-\beta})^{\frac{p}{\beta}}]^{\frac{\beta}{p}}, \quad C_{p,\beta} := \frac{A_\beta}{\mu_{p,\beta}\kappa_{p,\beta}} > 0,$$

we can show that the limit of the estimator $\hat{L}^n(p,u)$ will be:

$$L(p,u,\beta) := \mathbb{E}[\exp(-u^\beta C_{p,\beta}(\phi^{1-\beta}+(\phi')^{1-\beta}))].$$

The problem here is that we cannot directly interfere the parameter β because unlike in the setting with fixed observation times we have to evaluate some expectation based on the unknown distribution of the observation times. To bypass this problem we will let $u\to 0$ to use the linearity of the exponential function for values around zero, meaning

$$\exp(x) = 1+x+o(x) \text{ for } x\to 0.$$

Intuitively we have for "small" u

$$\mathbb{E}[\exp(-u^\beta C_{p,\beta}(\phi^{1-\beta}+(\phi')^{1-\beta}))] \approx 1+\mathbb{E}\left[-u^\beta C_{p,\beta}(\phi^{1-\beta}+(\phi')^{1-\beta})\right] = 1-u^\beta C_{p,\beta}\kappa_{\beta,\beta},$$

from which we can interfere β by evaluating our estimator at different points u, v:

$$\hat{\beta}(p,u,v) := \frac{\log(-(\widetilde{L}^n(p,u)-1))-\log(-(\widetilde{L}^n(p,v)-1))}{\log(u/v)}, \tag{5.3}$$

see Section 5.3 for more details. We state the following definitions in dependence of u but whenever rates of convergence matter we will use u_n instead of u, assuming that $u_n\to 0$ with some rate made precise below.

5.1.1 Preliminary Results

Throughout all the proofs of this chapter we assume that Assumptions SB and SC (which implies Assumptions A,B and C as well) are in force.

Lemma 5.1. *Let A be a semimartingale with*

$$\left|\mathbb{E}_i^n\left[A_{\tau_{i+j}^n}-A_{\tau_i^n}\right]\right| \leq Kj\Delta_n,$$
$$\mathbb{E}_i^n\left[\left|A_{\tau_{i+j}^n}-A_{\tau_i^n}\right|^q\right] \leq K_q(j\Delta_n)^{q/2\wedge 1}$$

for all $i,j \in \mathbb{N}_0$ with $i+j \leq N_n(1), q \in (0,\infty)$, and let $|A_t|$ be in addition bounded from below. Then it holds for $0<p<1$, $y>0$ and some constant K

$$\left|\mathbb{E}_i^n\left[|A_{\tau_{i+j}^n}|^p-|A_{\tau_i^n}|^p\right]\right| \leq Kj\Delta_n,$$
$$\mathbb{E}_i^n\left[\left||A_{\tau_{i+j}^n}|^p-|A_{\tau_i^n}|^p\right|^y\right] \leq K_y(j\Delta_n)^{y/2\wedge 1}.$$

Proof. For $a,b\in\mathbb{R}, a\neq 0, 0<p<1$ with some constant K_p we cite the inequality (cf. [Tod15])

$$\left||a+b|^p-|a|^p-p\operatorname{sign}(a)|a|^{p-1}b\right| \leq K_p|a|^{p-2}|b|^2. \tag{5.4}$$

Using (5.4) together with the boundedness of $|A_t|^{-1}$, the triangular inequality and the assumptions on the speed of convergence of the process A_t we get

$$\begin{aligned}
&\mathbb{E}_i\left[\left||A_{\tau_{i+j}^n}|^p-|A_{\tau_i^n}|^p\right|^y\right] = \mathbb{E}_i^n\left[\left||A_{\tau_i^n}+A_{\tau_{i+j}^n}-A_{\tau_i^n}|^p-|A_{\tau_i^n}|^p\right|^y\right]\\
&\leq K_y\mathbb{E}_i^n\left[\left|p\operatorname{sign}(A_{\tau_i^n})|A_{\tau_i^n}|^{p-1}(A_{\tau_{i+j}^n}-A_{\tau_i^n})\right|^y + K_p\left||A_{\tau_i^n}|^{p-2}(A_{\tau_{i+j}^n}-A_{\tau_i^n})^2\right|^y\right]\\
&\leq K_{p,y}\mathbb{E}_i^n\left[\left|A_{\tau_{i+j}^n}-A_{\tau_i^n}\right|^y \vee \left|A_{\tau_{i+j}^n}-A_{\tau_i^n}\right|^{2y}\right]\\
&\leq K_{p,y}(j\Delta_n)^{y/2\wedge 1},
\end{aligned}$$

where we use $|a+b|^y \leq |a|^y+|b|^y$ if $y\leq 1$ and $|a+b|^y \leq 2^{y-1}\left(|a|^y+|b|^y\right)$ if $y>1$. A small calculation yields that from (5.4) it follows that

$$-K_p|a|^{p-2}|b|^2+p\operatorname{sign}(a)|a|^{p-1}b \leq |a+b|^p-|a|^p \leq K_p|a|^{p-2}|b|^2+p\operatorname{sign}(a)|a|^{p-1}b,$$

from where we can deduce that

$$\begin{aligned}
&\left|\mathbb{E}_i^n\left[|A_{\tau_{i+j}^n}|^p-|A_{\tau_i^n}|^p\right]\right|\\
&=\left|\mathbb{E}_i^n\left[|A_{\tau_i^n}+A_{\tau_{i+j}^n}-A_{\tau_i^n}|^p-|A_{\tau_i^n}|^p\right]\right|\\
&\leq \max\Big\{\left|\mathbb{E}_i^n\left[-K_p|A_{\tau_i^n}|^{p-2}(A_{\tau_{i+j}^n}-A_{\tau_i^n})^2+p\operatorname{sign}(A_{\tau_i^n})|A_{\tau_i^n}|^{p-1}(A_{\tau_{i+j}^n}-A_{\tau_i^n})\right]\right|,\\
&\qquad\left|\mathbb{E}_i^n\left[K_p|A_{\tau_i^n}|^{p-2}(A_{\tau_{i+j}^n}-A_{\tau_i^n})^2+p\operatorname{sign}(A_{\tau_i^n})|A_{\tau_i^n}|^{p-1}(A_{\tau_{i+j}^n}-A_{\tau_i^n})\right]\right|\Big\}\\
&\leq \left|\mathbb{E}_i^n\left[p\operatorname{sign}(A_{\tau_i^n})|A_{\tau_i^n}|^{p-1}(A_{\tau_{i+j}^n}-A_{\tau_i^n})\right]\right| + K_p\left|\mathbb{E}_i^n\left[|A_{\tau_i^n}|^{p-2}(A_{\tau_{i+j}^n}-A_{\tau_i^n})^2\right]\right|\\
&= p|A_{\tau_i^n}|^{p-1}\left|\mathbb{E}_i^n\left[(A_{\tau_{i+j}^n}-A_{\tau_i^n})\right]\right| + |A_{\tau_i^n}|^{p-2}K_p\left|\mathbb{E}_i^n\left[(A_{\tau_{i+j}^n}-A_{\tau_i^n})^2\right]\right|\\
&\leq Kj\Delta_n.
\end{aligned}$$

□

Remark 5.1. *Using the same arguments as in Lemma 4.8, because the components of σ_t not appearing in the representation below are martingales, we have*

$$\left|\mathbb{E}_i^n\left[\sigma_{\tau_{i+j}^n} - \sigma_{\tau_i^n}\right]\right| = \left|\mathbb{E}_i^n\left[\int_{\tau_i^n}^{\tau_{i+j}^n} b_s^\sigma ds + \int_{\tau_i^n}^{\tau_{i+j}^n}\int_E \kappa'(\delta^\sigma(s,x))\underline{\mu}(ds,dx)\right]\right| \leq Kj\Delta_n \quad (5.5)$$

and likewise for the process λ. Combined with Corollary 4.1 we find that Lemma 5.1 is applicable to the processes σ and λ or any other process that fulfills the equivalent of Assumption SB replacing σ.

The following inequality is needed in the proof of the subsequent lemma.

Lemma 5.2. *Let $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be an increasing function in C^2 with $f''(x) \leq 0$ and let $a, b, c \in \mathbb{R}_+$. Then it holds that*

$$|f(a+c) - f(a+b)| \leq |f(c) - f(b)|. \quad (5.6)$$

This holds in particular for the function $|\cdot|^p : \mathbb{R}_+ \to \mathbb{R}_+$ with $0 < p < 1$ so we have

$$|(a+c)^p - (a+b)^p| \leq |c^p - b^p|$$

for all $a, b, c \in \mathbb{R}_+$.

Proof. We may assume that $c \geq b$ because otherwise we simply switch positions inside the absolute values on both sides of (5.6). There exist $\epsilon_1 \in [b,c], \epsilon_2 \in [a+b, a+c]$ with

$$f'(\epsilon_1) = \frac{f(c) - f(b)}{c - b}$$
$$f'(\epsilon_2) = \frac{f(a+c) - f(a+b)}{(a+c) - (a+b)}$$

and due to $f''(x) \leq 0$ we have that

$$f(a+c) - f(a+b) = f'(\epsilon_2)(c-b) \leq f'(\epsilon_1)(c-b) = f(c) - f(b).$$

As f is also increasing we have that $f(c) \geq f(b)$ and $f(a+c) \geq f(a+b)$ and therefore

$$|f(a+c) - f(a+b)| = f(a+c) - f(a+b) \leq f(c) - f(b) = |f(c) - f(b)|.$$

□

Lemma 5.3. *Under the previous assumptions it holds that for $p \in (-1, \beta)$ and for some constant $M > 0$*

$$\mathbb{E}_{i-2}^n\left|\Delta_n^{-1/\beta}(\widetilde{\Delta_i^n S} - \widetilde{\Delta_{i-1}^n S})\right|^p < M, \quad (5.7)$$

$$\mathbb{E}_{i-2}^n\left|\Delta_n^{-1/\beta}(\lambda_{\tau_{i-2}^n}^{1-1/\beta}\widetilde{\Delta_i^n S} - \lambda_{\tau_{i-3}^n}^{1-1/\beta}\widetilde{\Delta_{i-1}^n S})\right|^p = \kappa_{p,\beta}^{p/\beta}\mu_{p,\beta}^{p/\beta}. \quad (5.8)$$

and furthermore for $p \in (0, \beta)$

$$\left|\mathbb{E}_{i-2}^n\left|\Delta_n^{-1/\beta}(\widetilde{\Delta_i^n S} - \widetilde{\Delta_{i-1}^n S})\right|^p - \lambda_{\tau_{i-2}^n}^{p/\beta - p}\kappa_{p,\beta}^{p/\beta}\mu_{p,\beta}^{p/\beta}\right| \leq K\Delta_n^{1/2}$$

Proof. Let S_1', S_1'' be r.v. with the same distribution as S_1 which are independent from $(\phi_i^n)_{i\geq 1}$ and in addition (contrary to S_1) independent from $\mathcal{F}$,that means in particular from the process λ. Using standard properties of stable processes (see Section 1.2 in [ST94] as a reference) we have that for constants $\sigma_1, \sigma_2 \in \mathbb{R}$

$$\sigma_1 S_1 + \sigma_2 S_1' \sim (|\sigma_1|^\beta + |\sigma_2|^\beta)^{1/\beta} S_1'',$$

and due to the self similarity of stable processes:

$$(S_t - S_{t-\Delta_n}) \sim \Delta_n^{1/\beta} S_1 \text{ for all } t \geq \Delta_n.$$

Because the increments of the process $(S_t)_{t\geq \tau_{i-2}^n}$ are independent of the difference of stopping times $(\tau_i^n - \tau_{i-1}^n) = \Delta_n \phi_i^n \lambda_{\tau_{i-2}^n}$ it holds that

$$\mathbb{E}\left[\Delta_i^n S | (\tau_i^n - \tau_{i-1}^n) = a\right] = \mathbb{E}\left[(\tau_i^n - \tau_{i-1}^n)^{1/\beta} S_1' | (\tau_i^n - \tau_{i-1}^n) = a\right] = \mathbb{E}\left[a^{1/\beta} S_1'\right].$$

Using the last line it holds for all Borel sets M (note that the following calculation is possible because the moments of $(\phi_i^n)^q$ for $q \in (-2, 0)$ exist and $1/\beta - 1 > -1$)

$$\begin{aligned}
\mathbb{E}\left[\mathbb{1}_M\left(\frac{\Delta_i^n S}{\tau_i^n - \tau_{i-1}^n}\right)\right] &= \int_{\mathbb{R}} \mathbb{E}\left[\mathbb{1}_M\left(\frac{\Delta_i^n S}{\tau_i^n - \tau_{i-1}^n}\right) \middle| (\tau_i^n - \tau_{i-1}^n) = a\right] \mathbb{P}^{(\tau_i^n - \tau_{i-1}^n)}(da) \\
&= \int_{\mathbb{R}} \mathbb{E}\left[\mathbb{1}_M\left(a^{1/\beta - 1} S_1'\right)\right] \mathbb{P}^{(\tau_i^n - \tau_{i-1}^n)}(da) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}_M\left(a^{1/\beta-1} b\right) \mathbb{P}^{S_1'}(db) \mathbb{P}^{(\tau_i^n - \tau_{i-1}^n)}(da) \\
&= \mathbb{E}\left[\mathbb{1}_M\left((\tau_i^n - \tau_{i-1}^n)^{1/\beta - 1} S_1'\right)\right]
\end{aligned}$$

or put differently

$$\begin{aligned}
\Delta_n^{-1/\beta} \widetilde{\Delta_i^n S} &\sim \Delta_n^{1-1/\beta} (\tau_i^n - \tau_{i-1}^n)^{1/\beta - 1} S_1' \sim (\lambda_{\tau_{i-2}^n} \phi_i^n)^{1/\beta - 1} S_1' \\
\Delta_n^{-1/\beta} \widetilde{\Delta_{i-1}^n S} &\sim \Delta_n^{1-1/\beta} (\tau_{i-1}^n - \tau_{i-2}^n)^{1/\beta - 1} S_1'' \sim (\lambda_{\tau_{i-3}^n} \phi_{i-1}^n)^{1/\beta - 1} S_1''
\end{aligned}$$

where $\phi_{i-1}^n, \phi_i^n, S_1', S_1''$ are independent of $\mathcal{F}_{\tau_{i-2}^n}$ and of each other. Taking conditional expectation then yields for $p < \beta$

$$\begin{aligned}
\mathbb{E}_{i-2}^n \left|\Delta_n^{-1/\beta} (\widetilde{\Delta_i^n S} - \widetilde{\Delta_{i-1}^n S})\right|^p &= \mathbb{E}_{i-2}^n \left|\Delta_n^{1-1/\beta} ((\Delta_n \lambda_{\tau_{i-2}^n} \phi_i^n)^{1/\beta - 1} S_1' - (\Delta_n \lambda_{\tau_{i-3}^n} \phi_{i-1}^n)^{1/\beta-1} S_1'')\right|^p \\
&= \mathbb{E}_{i-2}^n \left[\mathbb{E}\left[\left|\Delta_n^{1-1/\beta} ((\Delta_n \lambda_{\tau_{i-2}^n} \phi_i^n)^{1/\beta - 1} S_1' - (\Delta_n \lambda_{\tau_{i-3}^n} \phi_{i-1}^n)^{1/\beta-1} S_1'')\right|^p \middle| \mathcal{F}_{\tau_{i-2}^n}^n, \phi_i^n, \phi_{i-1}^n\right]\right] \\
&= \mathbb{E}_{i-2}^n \left|\Delta_n^{1-1/\beta} ((\Delta_n \lambda_{\tau_{i-2}^n} \phi_i^n)^{1-\beta} + (\Delta_n \lambda_{\tau_{i-3}^n} \phi_{i-1}^n)^{1-\beta})^{1/\beta} S_1'')\right|^p \\
&= \mathbb{E}_{i-2}^n \left|((\lambda_{\tau_{i-2}^n} \phi_i^n)^{1-\beta} + (\lambda_{\tau_{i-3}^n} \phi_{i-1}^n)^{1-\beta})^{1/\beta} S_1'')\right|^p \\
&= \mathbb{E}_{i-2} \left[((\lambda_{\tau_{i-2}^n} \phi_i^n)^{1-\beta} + (\lambda_{\tau_{i-3}^n} \phi_{i-1}^n)^{1-\beta})^{p/\beta}\right] \mathbb{E}\left|S_1'\right|^p < M
\end{aligned} \tag{5.9}$$

using the boundedness of λ in the last step. Again moments of the stopping times exist by Assumption C (which is implied by Assumption SC) and by $(1-\beta)p/\beta > -1$ due to $p < \beta$ and $\beta < 2$. Because the density of a β-stable random variable exists and is continuous (cf. p.9 in ST94) it is bounded in a neighborhood of zero and we have that if f is the density of S_1, for a $q \in (-1,0)$, $t > 0$ and some constant M

$$\int_0^t f(x)x^{-q}dx \leq \int_0^t Mx^{-q}dx < \infty$$
$$\int_t^\infty f(x)x^{-q}dx \leq \int_t^\infty f(x)t^{-q}dx < \infty$$

and as such the expectation $\mathbb{E}|S_1|^q$ is finite. Using the same arguments as above we can calculate

$$\begin{aligned}
&\mathbb{E}^n_{i-2}\left|\Delta_n^{-1/\beta}(\lambda_{\tau^n_{i-2}}^{1-1/\beta}\widetilde{\Delta^n_i S} - \lambda_{\tau^n_{i-3}}^{1-1/\beta}\widetilde{\Delta^n_{i-1} S})\right|^p \\
&\quad = \mathbb{E}^n_{i-2}\left|\Delta_n^{1-1/\beta}(\lambda_{\tau^n_{i-2}}^{1-1/\beta}(\Delta_n\lambda_{\tau^n_{i-2}}\phi^n_i)^{1/\beta-1}S_1' - \lambda_{\tau^n_{i-3}}^{1-1/\beta}(\Delta_n\lambda_{\tau^n_{i-3}}\phi^n_{i-1})^{1/\beta-1}S_1'')\right|^p \\
&\quad = \mathbb{E}^n_{i-2}\left[\mathbb{E}\left[\left|\Delta_n^{1-1/\beta}((\Delta_n\phi^n_i)^{1/\beta-1}S_1' - (\Delta_n\phi^n_{i-1})^{1/\beta-1}S_1'')\right|^p \left|\mathcal{F}^n_{\tau^n_{i-2}}, \phi^n_i, \phi^n_{i-1}\right.\right]\right] \\
&\quad = \mathbb{E}_{i-2}\left[((\phi^n_i)^{1-\beta} + (\phi^n_{i-1})^{1-\beta})^{p/\beta}\right]\mathbb{E}\left|S_1'\right|^p = \kappa_{p,\beta}^{p/\beta}\mu_{p,\beta}^{p/\beta}.
\end{aligned}$$

If we restrict $p \in (0,\beta)$ we can with the lines above approximate the conditional expectation of the difference of increments $\widetilde{\Delta^n_i S} - \widetilde{\Delta^n_{i-1} S}$ to the part of the stopping times that is $\mathcal{F}_{\tau^n_{i-2}}$-measurable (i.e. $\lambda_{\tau^n_{i-2}}$):

$$\begin{aligned}
&\left|\mathbb{E}^n_{i-2}\left|\Delta_n^{-1/\beta}(\widetilde{\Delta^n_i S} - \widetilde{\Delta^n_{i-1} S})\right|^p - \lambda_{\tau^n_{i-2}}^{p/\beta-p}\kappa_{p,\beta}^{p/\beta}\mu_{p,\beta}^{p/\beta}\right| \\
&\quad = \mu_{p,\beta}^{p/\beta}\left|\mathbb{E}^n_{i-2}\left[((\lambda_{\tau^n_{i-2}}\phi^n_i)^{1-\beta} + (\lambda_{\tau^n_{i-3}}\phi^n_{i-1})^{1-\beta})^{p/\beta}\right] - \lambda_{\tau^n_{i-2}}^{p/\beta-p}\mathbb{E}^n_{i-2}\left[((\phi^n_i)^{1-\beta} + (\phi^n_{i-1})^{1-\beta})^{p/\beta}\right]\right| \\
&\quad = \mu_{p,\beta}^{p/\beta}\left|\mathbb{E}^n_{i-2}\left[((\lambda_{\tau^n_{i-2}}\phi^n_i)^{1-\beta} + (\lambda_{\tau^n_{i-3}}\phi^n_{i-1})^{1-\beta})^{p/\beta} - ((\lambda_{\tau^n_{i-2}}\phi^n_i)^{1-\beta} + (\lambda_{\tau^n_{i-2}}\phi^n_{i-1})^{1-\beta})^{p/\beta}\right]\right| \\
&\quad \leq K\left|\mathbb{E}^n_{i-2}\left[(\lambda_{\tau^n_{i-3}}\phi^n_{i-1})^{p/\beta-p} - (\lambda_{\tau^n_{i-2}}\phi^n_{i-1})^{p/\beta-p}\right]\right| \\
&\quad \leq K\mathbb{E}\left[(\phi^n_{i-1})^{p/\beta-p}\right]\mathbb{E}^n_{i-2}\left[\left|(\lambda_{\tau^n_{i-2}})^{p/\beta-p} - (\lambda_{\tau^n_{i-3}})^{p/\beta-p}\right|\right] \\
&\quad \leq K\Delta_n^{1/2},
\end{aligned}$$

where we used Lemma 5.2 in the third step and Lemma 5.1 (first apply Lemma 5.7 on the process λ and the function $f(x) = x^{\frac{1}{\beta}-1}$, see the proof of Lemma 5.9) and the independence from ϕ^n_{i-1} of $\mathcal{F}^n_{i-2}$ in the second to last step.

□

Later on we will show that $\Delta_n^{-1/\beta}\widetilde{V^n_i}(p) \xrightarrow{\mathbb{P}} \mu_{p,\beta}^{1/\beta}\kappa_{p,\beta}^{1/\beta}|\sigma_{\tau^n_{i-2}}|^p|\lambda_{\tau^n_{i-2}}|^{\frac{p}{\beta}-p}$. Keeping this in mind when we look at the definition of $\widetilde{L}^n(p,u)$ we can motivate the definition of $L(p,u,\beta)$ with the following lemma.

Lemma 5.4. *It holds that*

$$\mathbb{E}^n_{i-2}\left[\cos\left(u\frac{\lambda^{1-1/\beta}_{\tau^n_{i-2}}\widetilde{\Delta^n_i S}-\lambda^{1-1/\beta}_{\tau^n_{i-3}}\widetilde{\Delta^n_{i-1}S}}{\Delta_n^{1/\beta}\mu_{p,\beta}^{1/\beta}\kappa_{p,\beta}^{1/\beta}}\right)\right]=L(p,u,\beta).$$

Proof. Using the same arguments as in equation (5.9) we have

$$\begin{aligned}
&\mathbb{E}^n_{i-2}\left[\cos\left(u\frac{\lambda^{1-1/\beta}_{\tau^n_{i-2}}\widetilde{\Delta^n_i S}-\lambda^{1-1/\beta}_{\tau^n_{i-3}}\widetilde{\Delta^n_{i-1}S}}{\Delta_n^{1/\beta}\mu_{p,\beta}^{1/\beta}\kappa_{p,\beta}^{1/\beta}}\right)\right]\\
&\quad=\mathbb{E}^n_{i-2}\left[\cos\left(u\frac{((\phi^n_i)^{1-\beta}+(\phi^n_{i-1})^{1-\beta})^{1/\beta}S'_1)}{\mu_{p,\beta}^{1/\beta}\kappa_{p,\beta}^{1/\beta}}\right)\right]\\
&\quad=\mathbb{E}\left[\mathbb{E}\left[\cos\left(u\frac{((\phi^n_i)^{1-\beta}+(\phi^n_{i-1})^{1-\beta})^{1/\beta}S'_1)}{\mu_{p,\beta}^{1/\beta}\kappa_{p,\beta}^{1/\beta}}\right)\Bigg|\phi_i,\phi_{i-1}\right]\right]\\
&\quad=\mathbb{E}\left[\exp\left(-u^\beta\frac{A_\beta}{\mu_{p,\beta}\kappa_{p,\beta}}((\phi^n_i)^{1-\beta}+(\phi^n_{i-1})^{1-\beta})\right)\right]\\
&\quad=\mathbb{E}\left[\exp\left(-u^\beta C_{p,\beta}((\phi^n_i)^{1-\beta}+(\phi^n_{i-1})^{1-\beta})\right)\right]=L(p,u,\beta).
\end{aligned}$$

□

Furthermore we need throughout the following proofs some basic inequalities which we prove now.

Lemma 5.5. *It holds that*

$$|\cos(x)-\cos(y)|\le 2|x-y|^p \text{ for all } x,y\in\mathbb{R} \text{ and } p\in(0,1], \tag{5.10}$$

$$|\cos(x)-\cos(y)|^2\le 4|x-y|^p \text{ for all } x,y\in\mathbb{R} \text{ and } p\in(0,2], \tag{5.11}$$

$$|\exp(-x)-\exp(-y)|^2\le |x-y|^p \text{ for } x,y\in\mathbb{R}_+ \text{ and } p\in(0,2]. \tag{5.12}$$

Proof. We start with (5.10). Let $x,y\in\mathbb{R}, p\in(0,1]$ then it holds that for some ϵ between x and y

$$\cos(x)-\cos(y)=-\sin(\epsilon)(x-y).$$

Using the last line we can distinguish two cases

$$|\cos(x)-\cos(y)|\begin{cases}\le 2\le 2|x-y|^p & ,\text{ if } |x-y|\ge 1\\ \le|\sin(\epsilon)||x-y|\le 2|x-y|^p & ,\text{ if } |x-y|\le 1\end{cases}.$$

Let now $p\in(0,2]$. Then by (5.10)

$$|\cos(x)-\cos(y)|\le 2|x-y|^{p/2}$$

and as a result

$$|\cos(x) - \cos(y)|^2 \leq 4|x - y|^p$$

which proves (5.11). The proof of (5.12) is similar with the difference that

$$|\exp(-x) - \exp(-y)| \leq 1 \text{ for all } x, y \in \mathbb{R}_+.$$

□

5.2 A Central Limit Theorem for the Empirical Characteristic Function $\widetilde{L}^n(p, u)$

In the following the term ι always refers to an arbitrarily small number greater than zero which might change from line to line. The same holds for $K > 0$ but without limitations on the size.

In order to prove a CLT for $\hat{\beta}(p, u, v) - \beta$ we first need to prove a CLT for $\widetilde{L}^n(p, u) - L(p, u, \beta)$. For this purpose we decompose the latter difference into five terms and look at their limiting behavior separately

$$\widetilde{L}^n(p, u) - L(p, u, \beta) = \frac{1}{N_n(1) - k_n - 2} \sum_{i=k_n+3}^{N_n(1)} [R_1^n + R_2^n + Z^n + R_3^n + R_4^n]$$

with the term driving the limiting behavior

$$\begin{aligned} z_i(u) := \cos &\left(u \frac{\sigma_{\tau_{i-2}^n} \left(\widetilde{\Delta_i^n S} - \left(\frac{\lambda_{\tau_{i-2}^n}}{\lambda_{\tau_{i-3}^n}} \right)^{\frac{1}{\beta}-1} \widetilde{\Delta_{i-1}^n S} \right)}{\widetilde{V}_i^n(p)^{1/p}} \right) \\ &- \mathbb{E}_{i-2}^n \left[\exp\left(- \frac{A_\beta u^\beta |\sigma_{\tau_{i-2}^n}|^\beta |\lambda_{\tau_{i-2}^n}|^{1-\beta} ((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})}{\Delta_n^{-1} \widetilde{V}_i^n(p)^{\beta/p}} \right) \right], \\ Z^n := &\sum_{i=k_n+3}^{N_n(1)} z_i(u) \end{aligned}$$

and the residual terms being

$$r_i^1(u) = \cos\left(u\frac{\widetilde{\Delta_i^n X - \Delta_{i-1}^n X}}{\widetilde{V}_i^n(p)^{1/p}}\right) - \cos\left(u\frac{\sigma_{\tau_{i-2}^n}(\widetilde{\Delta_i^n S - \Delta_{i-1}^n S})}{\widetilde{V}_i^n(p)^{1/p}}\right),$$

$$r_i^2(u) = \cos\left(u\frac{\sigma_{\tau_{i-2}^n}(\widetilde{\Delta_i^n S - \Delta_{i-1}^n S})}{\widetilde{V}_i^n(p)^{1/p}}\right) - \cos\left(u\frac{\sigma_{\tau_{i-2}^n}\left(\widetilde{\Delta_i^n S} - \left(\frac{\lambda_{\tau_{i-2}^n}}{\lambda_{\tau_{i-3}^n}}\right)^{\frac{1}{\beta}-1}\widetilde{\Delta_{i-1}^n S}\right)}{\widetilde{V}_i^n(p)^{1/p}}\right)$$

$$r_i^3(u) = \mathbb{E}_{i-2}^n\left[\exp\left(-\frac{A_\beta u^\beta |\sigma_{\tau_{i-2}^n}|^\beta |\lambda_{\tau_{i-2}^n}|^{1-\beta}((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})}{\Delta_n^{-1}\widetilde{V}_i^n(p)^{\beta/p}}\right)\right]$$
$$- \mathbb{E}_{i-2}^n\left[\exp\left(-\frac{C_{p,\beta} u^\beta |\sigma_{\tau_{i-2}^n}|^\beta |\lambda_{\tau_{i-2}^n}|^{1-\beta}((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})}{(|\overline{\sigma\lambda}|_i^p)^{\beta/p}}\right)\right],$$

$$r_i^4(u) = \mathbb{E}_{i-2}^n\left[\exp\left(-\frac{C_{p,\beta} u^\beta |\sigma_{\tau_{i-2}^n}|^\beta |\lambda_{\tau_{i-2}^n}|^{1-\beta}((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})}{(|\overline{\sigma\lambda}|_i^p)^{\beta/p}}\right)\right] - L(p,u,\beta),$$

$$R_j^n = \sum_{i=k_n+3}^{N_n(1)} r_i^j(u) \text{ for } j \in \{1,2,3,4\},$$

where

$$|\overline{\sigma\lambda}|_i^p := \frac{1}{k_n}\sum_{j=i-k_n-1}^{i-2} |\sigma_{\tau_{j-2}^n}|^p |\lambda_{\tau_{j-2}^n}|^{\frac{p}{\beta}-p}.$$

In order to determine the limit of $\frac{1}{N_n(1)-k_n-2}Z^n$ we approximate the summands z_i via:

$$\overline{z}_i(u) := \cos\left(u\frac{\lambda_{\tau_{i-2}^n}^{1-1/\beta}\widetilde{\Delta_i^n S} - \lambda_{\tau_{i-3}^n}^{1-1/\beta}\widetilde{\Delta_{i-1}^n S}}{\Delta_n^{1/\beta}\mu_{p,\beta}^{1/\beta}\kappa_{p,\beta}^{1/\beta}}\right) - L(p,u,\beta) \text{ and } \overline{Z}^n := \sum_{i=k_n+3}^{N_n(1)} \overline{z}_i(u).$$

It should be noted, that in particular how the decomposition above is structured is adapted from the repeatedly mentioned paper [Tod15]. Furthermore, many concepts used in the proofs of this chapter, most notably those of Lemma 5.1, 5.10, 5.11, 5.16 and 5.17, are part of [Tod15] or [Tod17].

For the rest of the proof we need some auxiliary notation

$$\overline{V_i^n}(p) := \frac{1}{k_n}\sum_{j=i-k_n-1}^{i-2} \mathbb{E}_{j-2}^n|\widetilde{\Delta_j^n X - \Delta_{j-1}^n X}|^p$$

and the following random function

$$f_{i,u}(x) = \exp\left(-\frac{A_\beta u^\beta |\sigma_{\tau_{i-2}^n}|^\beta |\lambda_{\tau_{j-2}^n}|^{1-\beta}((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})}{x^{\beta/p}}\right), \tag{5.13}$$

with pointwise first and second order derivatives

$$f'_{i,u}(x) = \frac{\beta}{p} f_{i,u}(x) \frac{A_\beta u^\beta |\sigma_{\tau^n_{i-2}}|^\beta |\lambda_{\tau^n_{j-2}}|^{1-\beta}((\phi^n_i)^{1-\beta} + (\phi^n_{i-1})^{1-\beta})}{x^{\beta/p+1}}, \tag{5.14}$$

$$\begin{aligned} f''_{i,u}(x) = f_{i,u}(x) &\left(\frac{\beta}{p} \frac{A_\beta u^\beta |\sigma_{\tau^n_{i-2}}|^\beta |\lambda_{\tau^n_{j-2}}|^{1-\beta}((\phi^n_i)^{1-\beta} + (\phi^n_{i-1})^{1-\beta})}{x^{\beta/p+1}} \right)^2 \\ &- f_{i,u}(x) \frac{\beta}{p} \left(\frac{\beta}{p} + 1 \right) \frac{A_\beta u^\beta |\sigma_{\tau^n_{i-2}}|^\beta |\lambda_{\tau^n_{j-2}}|^{1-\beta}((\phi^n_i)^{1-\beta} + (\phi^n_{i-1})^{1-\beta})}{x^{\beta/p+2}}. \end{aligned} \tag{5.15}$$

For the all the following proofs keep in mind that Assumptions SB and SC are in force.

Lemma 5.6. *If for some $2 \le i \le N_n(1)$, X is a positive $\mathcal{F}^n_{i-2}$-measurable random variable it holds for the (random) terms $f'_{i,u_n}(X), f''_{i,u_n}(X)$ that*

$$\mathbb{E}^n_{i-2}\left[|f'_{i,u_n}(X)| \right] \le K \frac{u_n^\beta}{X^{p/\beta+1}} \quad \text{and} \quad \mathbb{E}^n_{i-2}\left[|f''_{i,u_n}(X)| \right] \le K \frac{u_n^\beta}{X^{p/\beta+2}}$$

If X is also bounded from below and above we have that

$$\mathbb{E}^n_{i-2}\left[|f'_{i,u_n}(x)| \right] \le K u_n^\beta \quad \text{and} \quad \mathbb{E}^n_{i-2}\left[|f''_{i,u_n}(x)| \right] \le K u_n^\beta$$

Proof. We have by Assumption C that $\mathbb{E}\left[(\phi^n_i)^{1-\beta} + (\phi^n_{i-1})^{1-\beta} \right] < \infty$ and therefore

$$\begin{aligned} &\mathbb{E}^n_{i-2}\left[\left| f_{i,u_n}(X) \frac{A_\beta u_n^\beta |\sigma_{\tau^n_{i-2}}|^\beta |\lambda_{\tau^n_{j-2}}|^{1-\beta}((\phi^n_i)^{1-\beta} + (\phi^n_{i-1})^{1-\beta})}{X^{\beta/p+2}} \right| \right] \\ \le &\frac{A_\beta u_n^\beta |\sigma_{\tau^n_{i-2}}|^\beta |\lambda_{\tau^n_{j-2}}|^{1-\beta}}{X^{\beta/p+2}} \mathbb{E}_{i-2}\left[f_{i,u_n}(X)((\phi^n_i)^{1-\beta} + (\phi^n_{i-1})^{1-\beta}) \right] \le K u_n^\beta \frac{1}{X^{\beta/p+2}}, \end{aligned}$$

where in the last step we use that $f_{i,u_n}(x)$ is bounded by 1 for positive values of x and that $|\sigma_t|, |\sigma_t|^{-1}$ are uniformly bounded by Assumption SB and therefore $|\sigma_{\tau^n_{i-2}}|$ is bounded from above and below as is $|\lambda_{\tau^n_{j-2}}|^{1-\beta}$ by the same arguments. Also all components involved are positive so we can omit the outer absolute value. Similarly we get

$$\mathbb{E}^n_{i-2}\left[f'_{i,u_n}(X) \right] \le K u_n^\beta \frac{1}{X^{\beta/p+1}}.$$

In order to deal with the first term in (5.15) we have that the function

$$y \mapsto |\exp(-y)y| \quad \text{with } y \in \mathbb{R}_+$$

is bounded. Using this result with $y = \frac{A_\beta u_n^\beta |\sigma_{\tau^n_{i-2}}|^\beta |\lambda_{\tau^n_{j-2}}|^{1-\beta}((\phi^n_i)^{1-\beta} + (\phi^n_{i-1})^{1-\beta})}{X^{\beta/p}}$ in the second step we have that

$$\begin{aligned}
&\mathbb{E}^n_{i-2}\left[\left|f_{i,u_n}(X)\left(\frac{\beta}{p}\frac{u_n^\beta A_\beta|\sigma_{\tau^n_{i-2}}|^\beta|\lambda_{\tau^n_{j-2}}|^{1-\beta}((\phi^n_i)^{1-\beta}+(\phi^n_{i-1})^{1-\beta})}{X^{\beta/p+1}}\right)^2\right|\right]\\
&=\left(\frac{\beta}{p}\right)^2\mathbb{E}_{i-2}\left[f_{i,u_n}(X)\frac{u_n^\beta A_\beta|\sigma_{\tau^n_{i-2}}|^\beta|\lambda_{\tau^n_{j-2}}|^{1-\beta}((\phi^n_i)^{1-\beta}+(\phi^n_{i-1})^{1-\beta})}{X^{\beta/p}}\right.\\
&\qquad\left.\frac{u_n^\beta A_\beta|\sigma_{\tau^n_{i-2}}|^\beta|\lambda_{\tau^n_{j-2}}|^{1-\beta}((\phi^n_i)^{1-\beta}+(\phi^n_{i-1})^{1-\beta})}{X^{\beta/p+2}}\right]\\
&\le K\frac{A_\beta u_n^\beta|\sigma_{\tau^n_{i-2}}|^\beta|\lambda_{\tau^n_{j-2}}|^{1-\beta}}{X^{\beta/p+2}}\mathbb{E}_{i-2}\left[(\phi^n_i)^{1-\beta}+(\phi^n_{i-1})^{1-\beta}\right]\le Ku_n^\beta\frac{1}{X^{\beta/p+2}}
\end{aligned}$$

again using the boundedness of $|\sigma_{\tau^n_{i-2}}|, |\lambda_{\tau^n_{i-2}}|$ in the last step. Finally

$$\begin{aligned}
\mathbb{E}^n_{i-2}\left[|f''_{i,u_n}(X)|\right]&\le\mathbb{E}^n_{i-2}\left[\left|f_{i,u}(X)\left(\frac{\beta}{p}\frac{A_\beta u^\beta|\sigma_{\tau^n_{i-2}}|^\beta|\lambda_{\tau^n_{j-2}}|^{1-\beta}((\phi^n_i)^{1-\beta}+(\phi^n_{i-1})^{1-\beta})}{X^{\beta/p+1}}\right)^2\right|\right]\\
&\quad+\mathbb{E}^n_{i-2}\left[\left|f_{i,u}(X)\frac{\beta}{p}\left(\frac{\beta}{p}+1\right)\frac{A_\beta u^\beta|\sigma_{\tau^n_{i-2}}|^\beta|\lambda_{\tau^n_{j-2}}|^{1-\beta}((\phi^n_i)^{1-\beta}+(\phi^n_{i-1})^{1-\beta})}{X^{\beta/p+2}}\right|\right]\\
&\le K\frac{u_n^\beta}{X^{p/\beta+2}}.
\end{aligned}$$

□

5.2.1 Auxiliary Results

The purpose of the two following Lemmas is solely to prove Lemma 5.9.

Lemma 5.7. *Let*

$$A_t = A_0+\int_0^t b^A_s ds+\int_0^t \eta^A_s dW_s+\int_0^t\int_E \kappa(\delta^A(s,x))\widetilde{\underline{\mu}}(ds,dx)+\int_0^t\int_E\kappa'(\delta^A(s,x))\underline{\mu}(ds,dx) \tag{5.16}$$

be a semimartingale where Assumption SB holds true with A replacing σ. Let $f(x)$ be a C^2-function on an open interval including the domain of A. Then the process $f(A)$ equally fulfills Assumption SB.

Proof. As Assumption SB allows us to assume that A is bounded so is the process $f(A)$. In particular both are special semimartingales by Remark 2.1 and we may write

$$A_t = A_0+\int_0^t\left(b^A_s+\int_E\kappa'(\delta^A(s,x))\lambda(dx)\right)ds+\int_0^t\eta^A_s dW_s+\int_0^t\int_E\delta^A(s,x)\widetilde{\underline{\mu}}(ds,dx).$$

We start by using Itô's formula (cf. (3.53) in [EK19]), let μ^A, ν^A denote the jump measure of A respectively its compensator, then:

$$\begin{aligned} f(A_t) = f(A_0) &+ \int_0^t \left(f'(A_{s-}) \left(b_s^A + \int_E \kappa'(\delta^A(s,x))\lambda(dx) \right) + \frac{1}{2} f''(A_{s-})(\eta_s^A)^2 \right) ds \\ &+ \int_0^t f'(A_{s-})\eta_s^A dW_s + \int_0^t \int_{\mathbb{R}} (f(A_{s-}+x) - f(A_{s-}))\ (\mu^A - \nu^A)(ds,dx) \\ &+ \int_0^t \int_{\mathbb{R}} (f(A_{s-}+x) - f(A_{s-}) - f'(A_{s-})x)\ \nu^A(ds,dx). \end{aligned} \tag{5.17}$$

Let Δ be an additional point outside E,´then there exists an $E \cup \{\Delta\}$-valued optional process θ_t such that

$$\mu(dt,dz) = \sum_{s:\theta_s(\omega)\in E} \epsilon_{(s,\theta_s(\omega))}(dt,dz), \tag{5.18}$$

where ϵ_a is the Dirac mass sitting at the point $a \in \mathbb{R}_+ \times E$. Setting $\delta(\omega,t,\Delta) = 0$, then outside a $\mathbb{P}$-null set we have $\Delta X_s = \delta^A(s,\theta_s)$ (cf. p. 119 in [JP12]).
Because $\int_0^t \int_{\mathbb{R}} (f(A_{s-}+x) - f(A_{s-}) - f'(A_{s-})x)\ \nu^A(ds,dx)$ exists we may write

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}} (f(A_{s-}+x) - f(A_{s-}) - f'(A_{s-})x)\ \mu^A(ds,dx) \\ &\quad= \sum_{s\le t} (f(A_{s-}+\Delta X_s) - f(A_{s-}) - f'(A_{s-})\Delta X_s) \\ &\quad= \sum_{s\le t} \left(f(A_{s-}+\delta^A(s,\theta_s)) - f(A_{s-}) - f'(A_{s-})\delta^A(s,\theta_s)\right) \\ &\quad= \int_0^t \int_E \left(f(A_{s-}+\delta^A(s,x)) - f(A_{s-}) - f'(A_{s-})\delta^A(s,x)\right) \mu(ds,dx) \end{aligned}$$

and taking compensators on both sides yields

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}} (f(A_{s-}+x) - f(A_{s-}) - f'(A_{s-})x)\ \nu^A(ds,dx) \\ &\quad= \int_0^t \int_E \left(f(A_{s-}+\delta^A(s,x)) - f(A_{s-}) - f'(A_{s-})\delta^A(s,x)\right) \lambda(dx)\ ds. \end{aligned} \tag{5.19}$$

Similarly we get

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}} (f(A_{s-}+x) - f(A_{s-}))\ (\mu^A - \nu^A)(ds,dx) \\ &\quad= \int_0^t \int_E \left(f(A_{s-}+\delta^A(s,x)) - f(A_{s-})\right)\ \underline{\widetilde{\mu}}(ds,dx) \end{aligned} \tag{5.20}$$

arguing that both side are completely discontinuous martingales with the same jumps and therefore coincide. Plugging (5.19) and (5.20) into (5.17) leaves us with

$$\begin{aligned} f(A_t) = f(A_0) + \int_0^t \Bigg(f'(A_{s-}) \left(b_s^A + \int_E \kappa'(\delta^A(s,x))\lambda(dx) \right) + \frac{1}{2} f''(A_{s-})(\eta_s^A)^2 \\ + \int_E \left(f(A_{s-} + \delta^A(s,x)) - f(A_{s-}) - f'(A_{s-})\delta^A(s,x) \right) \ \lambda(dx) \Bigg) \, ds \\ + \int_0^t f'(A_{s-})\eta_s^A dW_s + \int_0^t \int_E \left(f(A_{s-} + \delta^A(s,x)) - f(A_{s-}) \right) \ \underline{\widetilde{\mu}}(ds, dx). \end{aligned}$$

The coefficients of $f(A)$ then are (b^f, η^f, δ^f), where

$$\begin{aligned} b_s^f &= \Bigg(f'(A_{s-}) \left(b_s^A + \int_E \kappa'(\delta^A(s,x))\lambda(dx) \right) + \frac{1}{2} f''(A_{s-})(\eta_s^A)^2 \\ &\quad + \int_E \left(f(A_{s-} + \delta^A(s,x)) - f(A_{s-}) - f'(A_{s-})\delta^A(s,x) \right) \ \lambda(dx) \Bigg), \\ \eta_s^f &= f'(A_{s-})\eta_s^A, \\ \delta^f(s,x) &= f(A_{s-} + \delta^A(s,x)) - f(A_{s-}). \end{aligned}$$

We are left with to check whether these fulfill Assumption SB. According to (4.14) we have that the (random) integral $\int_E \kappa'(\delta^A(s,x))\lambda(dx)$ is bounded. Using $|\delta^A(s,x)| \le \gamma(x)$, where $\gamma(x)$ is the bounded function from Assumption SB, and a second order Taylor expansion there exists for each $(\omega, s, x) \in \Omega \times \mathbb{R}_+ \times E$ an $\epsilon_{(\omega,s,x)} \in [A_{s-}(\omega), A_{s-}(\omega) + \delta^A(\omega, s, x)]$ with

$$\begin{aligned} &\left| f(A_{s-}(\omega) + \delta^A(\omega,s,x)) - f(A_{s-}(\omega)) - f'(A_{s-}(\omega))\delta^A(\omega,s,x) \right| \\ &\quad \le \frac{|f''(\epsilon_{(\omega,s,x)})\delta^A(\omega,s,x)^2|}{2} \le K\gamma(x)^2 \end{aligned}$$

resulting with Assumption SB.2 in the boundedness of $\int_E \left(f(A_{s-} + \delta^A(s,x)) - f(A_{s-}) - f'(A_{s-})\delta^A(s,x) \right) \ \lambda(dx)$. In the last step we used that the process A is bounded and $|\delta(s,x)| \le \gamma(x)$ which leads to $\epsilon_{(\omega,s,x)}$ being bounded as well. By the boundedness of b_s^A, η_s^A and A_{s-} we then have boundedness of b_s^f and η_s^f. Finally we see, as previously, with a Taylor expansion that for each $(\omega, s, x) \in \Omega \times \mathbb{R}_+ \times E$ there exists $\epsilon'_{(\omega,s,x)} \in [A_{s-}(\omega), A_{s-}(\omega) + \delta^A(\omega,s,x)]$ with

$$|\delta^f(\omega,s,x)| = |f(A_{s-}(\omega) + \delta^A(\omega,s,x)) - f(A_{s-}(\omega))| = |f'(\epsilon'_{(\omega,s,x)})\delta^A(\omega,s,x))| \le K\gamma(x). \tag{5.21}$$

Concluding, due to (5.21) $\delta^f(s,x)$ fulfills Assumption SB.2 for a modified function $\gamma'(x) = K\gamma(x)$ having the same properties as $\gamma(x)$. □

Lemma 5.8. *Let A, B be two semimartingales of the form* (5.16) *where Assumption SB holds true with A, B replacing σ. Then AB likewise fulfills Assumption SB.*

Proof. Again we may assume that A, B are bounded so AB is bounded as well and in particular a special semimartingale. Defining

$$\begin{aligned} a_s^A &= b_s^A + \int_E \kappa'(\delta^A(s,x))\lambda(dx), \\ a_s^B &= b_s^B + \int_E \kappa'(\delta^B(s,x))\lambda(dx). \end{aligned}$$

we may write

$$\begin{aligned} A_t &= A_0 + \int_0^t a_s^A ds + \int_0^t \eta_s^A dW_s + \int_0^t \int_E \delta^A(s,x)\underline{\widetilde{\mu}}(ds,dx), \\ B_t &= B_0 + \int_0^t a_s^B ds + \int_0^t \eta_s^B dW_s + \int_0^t \int_E \delta^B(s,x)\underline{\widetilde{\mu}}(ds,dx). \end{aligned}$$

By the integration by parts rule for semimartingales we have

$$\begin{aligned} AB_t &= A_0B_0 + \int_0^t A_{s-}dB_s + \int_0^t B_{s-}dA_s + [A,B]_t \\ &= A_0B_0 + \int_0^t A_{s-}a_s^B ds + \int_0^t A_{s-}\eta_s^B dW_s + \int_0^t \int_E A_{s-}\delta^B(s,x)\underline{\widetilde{\mu}}(ds,dx) \\ &\quad + \int_0^t B_{s-}a_s^A ds + \int_0^t B_{s-}\eta_s^A dW_s + \int_0^t \int_E B_{s-}\delta^A(s,x)\underline{\widetilde{\mu}}(ds,dx) \\ &\quad + \int_0^t \eta_s^A \eta_s^B ds + \sum_{s\le t} \Delta A_s \Delta B_s \end{aligned}$$

Proceeding as in the previous Lemma we have due to (5.18)

$$\begin{aligned} \sum_{s\le t} \Delta A_s \Delta B_s &= \sum_{s\le t} \delta^A(s,\theta_s)\delta^B(s,\theta_s) \\ &= \int_0^t \int_E \delta^A(s,x)\delta^B(s,x)\underline{\mu}(ds,dx). \end{aligned}$$

And because $\delta^A(s,x)\delta^B(s,x) \le \gamma(x)^2$ we find with Assumption SB that

$$\int_E \delta^A(s,x)\delta^B(s,x)\ \lambda(dx) < M \tag{5.22}$$

for some constant $M > 0$, which leaves us with

$$\begin{aligned} AB_t = A_0B_0 &+ \int_0^t \left(A_{s-}a_s^B + B_{s-}a_s^A + \eta_s^A\eta_s^B + \int_E \delta^A(s,x)\delta^B(s,x)\ \lambda(dx) \right) ds \\ &+ \int_0^t \left(A_{s-}\eta_s^B + B_{s-}\eta_s^A \right) dW_s \\ &+ \int_0^t \int_E \left(A_{s-}\delta^B(s,x) + B_{s-}\delta^A(s,x) + \delta^A(s,x)\delta^B(s,x) \right) \underline{\widetilde{\mu}}(ds,dx). \end{aligned}$$

Similar to the proof of Lemma 5.7 we find that a_s^A, a_s^B are bounded and so are $A_{s-}, B_{s-}, \eta_s^A, \eta_s^B$ by Assumption SB. Considering (5.22) we find that the coefficients of the semimartingale AB fulfill Assumption SB, because

$$|A_{s-}\delta^B(s,x) + B_{s-}\delta^A(s,x) + \delta^A(s,x)\delta^B(s,x)| \leq K\gamma(x).$$

$\square$

Lemma 5.9. *Let $k_n \asymp n^{\varpi}$, $\varpi \in (0,1)$, then it holds that for $k_n + 3 \leq i \leq N_n(1), 0 < p < \beta/2$ and $y > 1$*

$$\left|\mathbb{E}^n_{i-k_n-3}[|\overline{\sigma\lambda}|_i^p - |\sigma_{\tau^n_{i-2}}|^p |\lambda_{\tau^n_{i-2}}|^{\frac{p}{\beta}-p}]\right| \leq K k_n \Delta_n, \tag{5.23}$$

$$\mathbb{E}_{i-k_n-3}\left[\left||\overline{\sigma\lambda}|_i^p - |\sigma_{\tau^n_{i-2}}|^p |\lambda_{\tau^n_{i-2}}|^{\frac{p}{\beta}-p}\right|^y\right] \leq K(k_n\Delta_n)^{y/2\wedge 1}. \tag{5.24}$$

Proof. Using Lemma 5.7 on the process λ and the function $f(x) = x^{\frac{1}{\beta}-1}$ we have that $\lambda^{\frac{1}{\beta}-1}$ is again an Itô semimartingale fulfilling Assumption SB. Then applying Lemma 5.8 to the processes σ and $\lambda^{\frac{1}{\beta}-1}$ yields that $\sigma\lambda^{\frac{1}{\beta}-1}$ fulfills the same ssumption. Finally applying Lemma 5.1 (note Remark 5.1) we get:

$$\left|\mathbb{E}^n_i\left[|\sigma_{\tau^n_{i+j}}|^p |\lambda_{\tau^n_{i+j}}|^{\frac{p}{\beta}-p} - |\sigma_{\tau^n_i}|^p |\lambda_{\tau^n_i}|^{\frac{p}{\beta}-p}\right]\right| \leq K j\Delta_n \tag{5.25}$$

and likewise

$$\mathbb{E}^n_i\left[\left||\sigma_{\tau^n_{i+j}}|^p |\lambda_{\tau^n_{i+j}}|^{\frac{p}{\beta}-p} - |\sigma_{\tau^n_i}|^p |\lambda_{\tau^n_i}|^{\frac{p}{\beta}-p}\right|^y\right] \leq K(j\Delta_n)^{\frac{y}{2}\wedge 1}. \tag{5.26}$$

(5.25) is sufficient to prove (5.23):

$$\begin{aligned}
&\left|\mathbb{E}^n_{i-k_n-3}[|\overline{\sigma\lambda}|_i^p - |\sigma_{\tau^n_{i-2}}|^p |\lambda_{\tau^n_{i-2}}|^{\frac{p}{\beta}-p}]\right| \\
&\quad= \left|\mathbb{E}^n_{i-k_n-3}\left[\frac{1}{k_n}\sum_{j=i-k_n-1}^{i-2}(|\sigma_{\tau^n_{j-2}}|^p |\lambda_{\tau^n_{j-2}}|^{\frac{p}{\beta}-p} - |\sigma_{\tau^n_{i-2}}|^p |\lambda_{\tau^n_{i-2}}|^{\frac{p}{\beta}-p})\right]\right| \\
&\quad\leq \frac{1}{k_n}\sum_{j=i-k_n-1}^{i-2}\left|\mathbb{E}^n_{i-k_n-3}\left[|\sigma_{\tau^n_{j-2}}|^p |\lambda_{\tau^n_{j-2}}|^{\frac{p}{\beta}-p} - |\sigma_{\tau^n_{i-2}}|^p |\lambda_{\tau^n_{i-2}}|^{\frac{p}{\beta}-p}\right]\right| \\
&\quad\leq \frac{1}{k_n}\sum_{j=i-k_n-1}^{i-2} K(i-j)\Delta_n \leq K k_n \Delta_n.
\end{aligned}$$

and likewise (5.24) can be proven with (5.26) due to $x \mapsto x^y$ being a convex function on

$\mathbb{R}_+$

$$\mathbb{E}^n_{i-k_n-3}\left[\left||\overline{\sigma\lambda}|^p_i - |\sigma_{\tau^n_{i-2}}|^p|\lambda_{\tau^n_{i-2}}|^{\frac{p}{\beta}-p}\right|^y\right]$$
$$= \mathbb{E}^n_{i-k_n-3}\left[\left|\frac{1}{k_n}\sum_{j=i-k_n-1}^{i-2}(|\sigma_{\tau^n_{j-2}}|^p|\lambda_{\tau^n_{j-2}}|^{\frac{p}{\beta}-p} - |\sigma_{\tau^n_{i-2}}|^p|\lambda_{\tau^n_{i-2}}|^{\frac{p}{\beta}-p})\right|^y\right]$$
$$\leq \frac{1}{k_n}\sum_{j=i-k_n-1}^{i-2}\mathbb{E}^n_{i-k_n-3}\left[\left||\sigma_{\tau^n_{j-2}}|^p|\lambda_{\tau^n_{j-2}}|^{\frac{p}{\beta}-p} - |\sigma_{\tau^n_{i-2}}|^p|\lambda_{\tau^n_{i-2}}|^{\frac{p}{\beta}-p}\right|^y\right].$$

□

Lemma 5.10. *For $2 \leq i \leq N_n(1), 0 < p < \frac{\beta}{2}$ and an arbitrarily small constant $\iota > 0$ it holds that*

$$\Delta_n^{-p/\beta}\mathbb{E}^n_{i-2}\left[\left||\widetilde{\Delta^n_i X} - \widetilde{\Delta^n_{i-1} X}|^p - |\sigma_{\tau^n_{i-2}}|^p|\widetilde{\Delta^n_i S} - \widetilde{\Delta^n_{i-1} S}|^p\right|\right] \leq K\alpha_n \tag{5.27}$$

with $\alpha_n = \Delta_n^{\frac{\beta}{2}\frac{p+1}{\beta+1}\wedge((\frac{p}{\beta'}\wedge 1)-\frac{p}{\beta})\wedge\frac{1}{2}-\iota}$.

Proof. We decompose $S_t = S^{(1)}_t + S^{(2)}_t + S^{(3)}_t$ where $S^{(1)}_t = \int_0^t\int_{\mathbb{R}}\kappa(x)\widetilde{\mu}(ds,dx)$, with $\widetilde{\mu}(ds,dx)$ being the compensated jump measure of S, $S^{(2)}_t = \int_0^t\int_{\mathbb{R}}\kappa'(x)\mu(ds,dx)$ and $S^{(3)}_t := -\int_0^t\int_{\mathbb{R}}\kappa'(x)\frac{A}{|x|^{\beta+1}}dx\,ds$. Then we have with $\mathcal{A} = \mathcal{F}^n_{\tau^n_{i-2}}\bigvee\sigma(\phi^n_i,\phi^n_{i-1})$

$$\mathbb{E}^n_{i-2}\left|\frac{\Delta_n}{\tau^n_{i-1+l}-\tau^n_{i-2+l}}\int_{\tau^n_{i-2+l}}^{\tau^n_{i-1+l}}(\sigma_{u-}-\sigma_{\tau^n_{i-2}})dS^{(1)}_u\right|^q$$
$$= \mathbb{E}^n_{i-2}\left[\left(\frac{\Delta_n}{\tau^n_{i-1+l}-\tau^n_{i-2+l}}\right)^q\mathbb{E}\left[\left|\int_{\tau^n_{i-2+l}}^{\tau^n_{i-1+l}}(\sigma_{u-}-\sigma_{\tau^n_{i-2}})dS^{(1)}_u\right|^q\middle|\mathcal{A}\right]\right]$$
$$= \mathbb{E}^n_{i-2}\left[\left(\frac{\Delta_n}{\tau^n_{i-1+l}-\tau^n_{i-2+l}}\right)^q\mathbb{E}\left[\left|\int_{\tau^n_{i-2+l}}^{\tau^n_{i-1+l}}\int_{\mathbb{R}}(\sigma_{u-}-\sigma_{\tau^n_{i-2}})\kappa(x)\widetilde{\mu}(du,dx)\right|^q\middle|\mathcal{A}\right]\right], \tag{5.28}$$

where we used Proposition 3.37 in [EK19] for the last step. Then applying Remark 4.2 in the first step one has for $q \in (\beta, 2], l = 0, 1$

$$(5.28) \leq \mathbb{E}^n_{i-2}\left[\left(\frac{\Delta_n}{\tau^n_{i-1+l}-\tau^n_{i-2+l}}\right)^q\mathbb{E}\left[\int_{\tau^n_{i-2+l}}^{\tau^n_{i-1+l}}\int_{\mathbb{R}}|(\sigma_{u-}-\sigma_{\tau^n_{i-2}})\kappa(x)|^q\frac{A}{|x|^{\beta+1}}dx\,du\middle|\mathcal{A}\right]\right]$$
$$\leq K\mathbb{E}^n_{i-2}\left[\left(\frac{\Delta_n}{\tau^n_{i-1+l}-\tau^n_{i-2+l}}\right)^q\mathbb{E}\left[\int_{\tau^n_{i-2+l}}^{\tau^n_{i-1+l}}|\sigma_{u-}-\sigma_{\tau^n_{i-2}}|^q du\middle|\mathcal{A}\right]\right]$$
$$\leq K\mathbb{E}^n_{i-2}\left[\left(\frac{\Delta_n}{\tau^n_{i-1+l}-\tau^n_{i-2+l}}\right)^q\mathbb{E}\left[(\tau^n_{i-1+l}-\tau^n_{i-2+l})\sup_{u\in[\tau^n_{i-2+l},\tau^n_{i-1+l}]}|\sigma_{u-}-\sigma_{\tau^n_{i-2}}|^q\middle|\mathcal{A}\right]\right]$$
$$\leq \Delta_n^q\mathbb{E}^n_{i-2}\left[(\tau^n_{i-1+l}-\tau^n_{i-2+l})^{1-q}\mathbb{E}\left[\sup_{u\in[\tau^n_{i-2+l},\tau^n_{i-1+l}]}|\sigma_{u-}-\sigma_{\tau^n_{i-2}}|^q\middle|\mathcal{A}\right]\right]$$
$$\leq \Delta_n^q\mathbb{E}^n_{i-2}\left[(\tau^n_{i-1+l}-\tau^n_{i-2+l})^{1-q}\Delta_n^{q/2}\right] \leq K\Delta_n^{q/2+1},$$

where the second step holds by

$$\int_{\mathbb{R}} |\kappa(x)|^q \frac{A}{|x|^{\beta+1}} dx < \infty,$$

using the boundedness of $\kappa(x)$ and the fact that around 0 we have $|\kappa(x)|^q = |x|^q$ with $q > \beta$. In the second to last step we applied one more time Remark 4.2 to Corollary 4.1. Applying Jensen inequality for $q \in (0, \beta]$:

$$\mathbb{E}^n_{i-2} \left| \frac{\Delta_n}{\tau^n_{i-2+l} - \tau^n_{i-1+l}} \int_{\tau^n_{i-2+l}}^{\tau^n_{i-1+l}} (\sigma_{u-} - \sigma_{\tau^n_{i-2}}) dS^{(1)}_u \right|^q$$
$$\leq \left(\mathbb{E}^n_{i-2} \left| \frac{\Delta_n}{\tau^n_{i-2+l} - \tau^n_{i-1+l}} \int_{\tau^n_{i-2+l}}^{\tau^n_{i-1+l}} (\sigma_{u-} - \sigma_{\tau^n_{i-2}}) dS^{(1)}_u \right|^{\beta+\iota} \right)^{q/(\beta+\iota)} \leq K \Delta_n^{q/2+q/\beta-\iota}.$$

Combining the last two estimates yields for $q \in (0, 2]$

$$\mathbb{E}^n_{i-2} \left| \frac{\Delta_n}{\tau^n_{i-2+l} - \tau^n_{i-1+l}} \int_{\tau^n_{i-2+l}}^{\tau^n_{i-1+l}} (\sigma_{u-} - \sigma_{\tau^n_{i-2}}) dS^{(1)}_u \right|^q \leq K \Delta_n^{q/2+q/\beta\wedge 1-\iota}.$$

Setting $M := \int_{\mathbb{R}} \kappa'(x) \frac{A}{|x|^{\beta+1}} dx < \infty$ we find that similarly to the previous calculations for $q \in [1, 2]$ with Corollary 4.1

$$\mathbb{E}^n_{i-2} \left| \frac{\Delta_n}{\tau^n_{i-1+l} - \tau^n_{i-2+l}} \int_{\tau^n_{i-2+l}}^{\tau^n_{i-1+l}} (\sigma_{u-} - \sigma_{\tau^n_{i-2}}) dS^{(3)}_u \right|^q$$
$$= \mathbb{E}^n_{i-2} \left| \frac{\Delta_n}{\tau^n_{i-1+l} - \tau^n_{i-2+l}} \int_{\tau^n_{i-2+l}}^{\tau^n_{i-1+l}} (\sigma_{u-} - \sigma_{\tau^n_{i-2}}) M du \right|^q$$
$$\leq K \Delta_n^q \mathbb{E}^n_{i-2} \left[\sup_{u \in [\tau^n_{i-2+l}, \tau^n_{i-1+l}]} |\sigma_{u-} - \sigma_{\tau^n_{i-2}}|^q \right] \leq K \Delta_n^{3q/2}$$

and applying with Jensen inequality in the case $q < 1$ we have for $q \in [0, 2]$:

$$\mathbb{E}^n_{i-2} \left| \frac{\Delta_n}{\tau^n_{i-1+l} - \tau^n_{i-2+l}} \int_{\tau^n_{i-2+l}}^{\tau^n_{i-1+l}} (\sigma_{u-} - \sigma_{\tau^n_{i-2}}) dS^{(3)}_u \right|^q \leq K \Delta_n^{3q/2}.$$

Because by Assumption SB.1 σ_t is bounded, the process $\int_0^t (\sigma_{u-} - \sigma_{\tau^n_{i-2}}) dS^{(2)}_u$ is of finite variation, then conditioning first on $\mathcal{A}$ as above and afterwards using Lemma 4.4 with Remark 4.2 one gets likewise for $q \in (0, 1]$

$$\mathbb{E}^n_{i-2} \left| \frac{\Delta_n}{\tau^n_{i-2+l} - \tau^n_{i-1+l}} \int_{\tau^n_{i-2+l}}^{\tau^n_{i-1+l}} (\sigma_{u-} - \sigma_{\tau^n_{i-2}}) dS^{(2)}_u \right|^q$$
$$\leq K \mathbb{E}^n_{i-2} \left[\left(\frac{\Delta_n}{\tau^n_{i-1+l} - \tau^n_{i-2+l}} \right)^q \mathbb{E} \left[\int_{\tau^n_{i-2+l}}^{\tau^n_{i-1+l}} \int_{\mathbb{R}} |(\sigma_{u-} - \sigma_{\tau^n_{i-2}}) \kappa'(x)|^q \frac{A}{|x|^{\beta+1}} dx \, du \middle| \mathcal{A} \right] \right]$$
$$\leq K \Delta_n^{q/2+1}$$

and for $q \in (1,\beta)$ with Jensen inequality

$$\begin{aligned}
&\mathbb{E}^n_{i-2}\left|\frac{\Delta_n}{\tau^n_{i-2+l}-\tau^n_{i-1+l}}\int_{\tau^n_{i-2+l}}^{\tau^n_{i-1+l}}(\sigma_{u-}-\sigma_{\tau^n_{i-2}})dS^{(2)}_u\right|^q\\
&\leq K\mathbb{E}^n_{i-2}\left[\left(\frac{\Delta_n}{\tau^n_{i-1+l}-\tau^n_{i-2+l}}\right)^q\left(\mathbb{E}\left[\left(\int_{\tau^n_{i-2+l}}^{\tau^n_{i-1+l}}\int_{\mathbb{R}}|(\sigma_{u-}-\sigma_{\tau^n_{i-2}})\kappa'(x)|^1\frac{A}{|x|^{\beta+1}}dx\; du\right)^q\middle|\mathcal{A}\right]\right.\right.\\
&\qquad\left.\left.+\mathbb{E}\left[\int_{\tau^n_{i-2+l}}^{\tau^n_{i-1+l}}\int_{\mathbb{R}}|(\sigma_{u-}-\sigma_{\tau^n_{i-2}})\kappa'(x)|^q\frac{A}{|x|^{\beta+1}}dx\; du\middle|\mathcal{A}\right]\right)\right]\\
&\leq K\Delta_n^{q/2+1}.
\end{aligned}$$

For the difference of the drift parts we have that

$$\begin{aligned}
&\frac{\Delta_n}{\tau^n_i-\tau^n_{i-1}}\int_{\tau^n_{i-1}}^{\tau^n_i}\alpha_u du-\frac{\Delta_n}{\tau^n_{i-1}-\tau^n_{i-2}}\int_{\tau^n_{i-2}}^{\tau^n_{i-1}}\alpha_u du\\
&=\Delta_n\left(\frac{1}{\tau^n_i-\tau^n_{i-1}}\int_{\tau^n_{i-1}}^{\tau^n_i}(\alpha_u-\alpha_{\tau^n_{i-2}})du+\alpha_{\tau^n_{i-2}}-\left(\frac{1}{\tau^n_{i-1}-\tau^n_{i-2}}\int_{\tau^n_{i-2}}^{\tau^n_{i-1}}(\alpha_u-\alpha_{\tau^n_{i-2}})du+\alpha_{\tau^n_{i-2}}\right)\right)\\
&=\Delta_n\left(\frac{1}{\tau^n_i-\tau^n_{i-1}}\int_{\tau^n_{i-1}}^{\tau^n_i}(\alpha_u-\alpha_{\tau^n_{i-2}})du-\frac{1}{\tau^n_{i-1}-\tau^n_{i-2}}\int_{\tau^n_{i-2}}^{\tau^n_{i-1}}(\alpha_u-\alpha_{\tau^n_{i-2}})du\right).
\end{aligned}$$

Furthermore we have for $q \in [1,2]$ using Hölder inequality

$$\begin{aligned}
&\mathbb{E}^n_{i-2}\left|\Delta_n\left(\frac{1}{\tau^n_i-\tau^n_{i-1}}\int_{\tau^n_{i-1}}^{\tau^n_i}(\alpha_u-\alpha_{\tau^n_{i-2}})du-\frac{1}{\tau^n_{i-1}-\tau^n_{i-2}}\int_{\tau^n_{i-2}}^{\tau^n_{i-1}}(\alpha_u-\alpha_{\tau^n_{i-2}})du\right)\right|^q\\
&\leq K\Delta_n^q\mathbb{E}^n_{i-2}\left[(\tau^n_i-\tau^n_{i-1})^{-1}\int_{\tau^n_{i-1}}^{\tau^n_i}|\alpha_u-\alpha_{\tau^n_{i-2}}|^q du+(\tau^n_{i-1}-\tau^n_{i-2})^{-1}\int_{\tau^n_{i-2}}^{\tau^n_{i-1}}|\alpha_u-\alpha_{\tau^n_{i-2}}|^q du\right]
\end{aligned}$$

and as above for $l=0,1$ with Corollary 4.1

$$\begin{aligned}
\Delta_n^q\mathbb{E}^n_{i-2}&\left[(\tau^n_{i-1+l}-\tau^n_{i-2+l})^{-1}\int_{\tau^n_{i-2+l}}^{\tau^n_{i-1+l}}|\alpha_u-\alpha_{\tau^n_{i-2}}|^q du\right]\\
&\leq\Delta_n^q\mathbb{E}^n_{i-2}\left[\sup_{u\in[\tau^n_{i-2+l},\tau^n_{i-1+l}]}|\alpha_u-\alpha_{\tau^n_{i-2}}|^q\right]\leq\Delta_n^{3q/2}.
\end{aligned}$$

Using the steps above and applying Jensen inequality in the case of $q<1$ we have for $q \in [0,2]$

$$\mathbb{E}^n_{i-2}\left|\frac{\Delta_n}{\tau^n_i-\tau^n_{i-1}}\int_{\tau^n_{i-1}}^{\tau^n_i}\alpha_u du-\frac{\Delta_n}{\tau^n_{i-1}-\tau^n_{i-2}}\int_{\tau^n_{i-2}}^{\tau^n_{i-1}}\alpha_u du\right|^q\leq\Delta_n^{3q/2}.$$

In order to bound $\frac{\Delta_n}{\tau^n_i-\tau^n_{i-1}}\Delta^n_i Y$ we make use of Assumption SB.4. By the boundedness of $\int_{\mathbb{R}}(|x|^{\beta'}\wedge 1)\nu^Y_t(dx)$ and $\beta'<1$ we have that $\int_{\mathbb{R}}(|x|^q\wedge 1)\nu^Y_t(dx)$ is bounded too for all

$q \geq \beta'$. Because the jumps of Y are also bounded we conclude that for all $t > 0, q \geq \beta'$

$$\int_{\mathbb{R}} |x|^q \nu_t^Y(dx) < \infty.$$

As $\beta' < \beta/2 < 1$ the process $Y_t = \int_0^t \int_{\mathbb{R}} x\mu^Y(ds, dx)$ then has locally integrable variation and furthermore $\mathbb{E}[Y_t^q] < \infty$ for all $q \geq 0, t \geq 0$ (see the next few lines). Using (4.4) and the fact that the jumps of Y are bounded by Assumption SB we get for some constant $M > 0, \mathcal{A}_l := \mathcal{F}^n_{\tau^n_{i-2+l}} \bigvee \sigma(\phi^n_{i-1+l})$ and $\beta' \leq q \leq 1$, $l = 0, 1$

$$\begin{aligned}
\mathbb{E}\left[\left|\Delta^n_{i-1+l}Y\right|^q \middle| \mathcal{A}_l\right] &\leq \mathbb{E}\left[\left|\sum_{\tau_{i-2+l} \leq s \leq \tau^n_{i-1+l}} |\Delta Y_s|\right|^q \middle| \mathcal{A}_l\right] && (5.29)\\
&\leq \mathbb{E}\left[\sum_{\tau^n_{i-2+l} \leq s \leq \tau^n_{i-1+l}} |\Delta Y_s|^q \middle| \mathcal{A}_l\right]\\
&= \mathbb{E}\left[\int_{\tau^n_{i-2+l}}^{\tau^n_{i-1+l}} \int_{\mathbb{R}} |x|^q \mu_s^Y(dx)ds \middle| \mathcal{A}_l\right]\\
&= \mathbb{E}\left[\int_{\tau^n_{i-2+l}}^{\tau^n_{i-1+l}} \int_{\mathbb{R}} (|x|^q \wedge M)\, \mu_s^Y(dx)ds \middle| \mathcal{A}_l\right]\\
&= \mathbb{E}\left[\int_{\tau^n_{i-2+l}}^{\tau^n_{i-1+l}} \int_{\mathbb{R}} (|x|^q \wedge M)\, \nu_s^Y(dx)ds \middle| \mathcal{A}_l\right] \leq K(\tau^n_{i-1+l} - \tau^n_{i-2+l}).
\end{aligned}$$

using Remark 4.2 in the second to last step and the fact that by Assumption SB the process $\left(\int_{\mathbb{R}} (|x|^q \wedge 1)\, \nu_t^Y(dx)\right)_{t \geq 0}$ is bounded in the last step. In the case of $q \geq 1$ we again apply Remark 4.2 on Lemma 4.4 and get

$$\mathbb{E}\left[\left|\Delta^n_{i-1+l}Y\right|^q \middle| \mathcal{A}_l\right] \leq K_q\left((\tau^n_{i-1+l} - \tau^n_{i-2+l}) + (\tau^n_{i-1+l} - \tau^n_{i-2+l})^q\right). \tag{5.30}$$

With a final application of Jensen inequality in the case of $0 < q < \beta'$ and iterated expectations we can then conclude for all $q > 0$

$$\mathbb{E}^n_{i-2}\left[\left|\Delta^n_{i-1+l}Y\right|^q\right] \leq K\Delta_n^{(q/\beta')\wedge 1} \quad \text{and likewise} \quad \mathbb{E}^n_{i-2}\left|\frac{\Delta_n}{\tau^n_{i-1+l} - \tau^n_{i-2+l}}\Delta^n_{i-1+l}Y\right|^q \leq K\Delta_n^{(q/\beta')\wedge 1}.$$

We proceed with bounds for $\frac{\Delta_n}{\tau_i^n - \tau_{i-1}^n}\int_{\tau_{i-1}^n}^{\tau_i^n} \sigma_{u-} d\acute{S}_u - \frac{\Delta_n}{\tau_{i-1}^n - \tau_{i-2}^n}\int_{\tau_{i-2}^n}^{\tau_{i-1}^n} \sigma_{u-} d\acute{S}_u$. Let $\acute{F}$ denote the Lévy measure of $\acute{S}$ then we have with $|\tilde{h}(x)| \leq \frac{C}{|x|^{1+\beta'}}$ for all $|x| \leq x_0$ from Assumption A that $\int_{\mathbb{R}} (|x|^q \wedge 1)\acute{F}(dx) < \infty <$ for all $q > \beta'$. As the jumps of $\acute{S}$ are also bounded by Assumption SB.5 we have similarly to Y

$$\int_{\mathbb{R}} |x|^q \acute{F}(dx) < \infty,$$

for all $q \geq \beta'$. Due to its finite variation we can write

$$\begin{aligned}\acute{S}_t &= \int_0^t \int_{\mathbb{R}} \kappa(x)\widetilde{\mu_1}(ds,dx) + \int_0^t \int_{\mathbb{R}} \kappa'(x)\mu_1(ds,dx) \\ &= \int_0^t \int_{\mathbb{R}} x\mu_1(ds,dx) - \int_0^t \int_{\mathbb{R}} \kappa'(x)\acute{F}(dx)\ ds\end{aligned}$$

and note that

$$\frac{\Delta_n}{\tau_i^n - \tau_{i-1}^n} \int_{\tau_{i-1}^n}^{\tau_i^n} \int_{\mathbb{R}} \kappa(x)\acute{F}(dx)\ ds - \frac{\Delta_n}{\tau_{i-1}^n - \tau_{i-2}^n} \int_{\tau_{i-2}^n}^{\tau_{i-1}^n} \int_{\mathbb{R}} \kappa(x)\acute{F}(dx)\ ds = 0. \tag{5.31}$$

Proceeding as in (5.29) and (5.30) and using the subsequent arguments we find that for $q > 0$ and $l = 0, 1$

$$\mathbb{E}_{i-2+l}^n \left[\left| \int_{\tau_{i-2+l}^n}^{\tau_{i-1+l}^n} \int_{\mathbb{R}} x\mu_1(ds,dx) \right|^q \right] \leq K\Delta_n^{(q/\beta')\wedge 1-\iota}.$$

Due to the boundedness of σ_{t-} we have likewise

$$\begin{aligned}&\mathbb{E}_{i-2+l}^n \left[\left| \frac{\Delta_n}{\tau_{i-1+l}^n - \tau_{i-2+l}^n} \int_{\tau_{i-2+l}^n}^{\tau_{i-1+l}^n} \sigma_{u-} d\acute{S}_u \right|^q \right] \\ &\qquad = \mathbb{E}_{i-2+l}^n \left[\left| \frac{\Delta_n}{\tau_{i-1+l}^n - \tau_{i-2+l}^n} \int_{\tau_{i-2+l}^n}^{\tau_{i-1+l}^n} \int_{\mathbb{R}} \sigma_{u-} x\mu_1(ds,dx) \right|^q \right] \\ &\qquad \leq K\Delta_n^{(q/\beta')\wedge 1-\iota}.\end{aligned}$$

Combining the last line with (5.31) we have for all $q > 0$

$$\mathbb{E}_{i-2}^n \left[\left| \frac{\Delta_n}{\tau_i^n - \tau_{i-1}^n} \int_{\tau_{i-1}^n}^{\tau_i^n} \sigma_{u-} d\acute{S}_u - \frac{\Delta_n}{\tau_{i-1}^n - \tau_{i-2}^n} \int_{\tau_{i-2}^n}^{\tau_{i-1}^n} \sigma_{u-} d\acute{S}_u \right|^q \right] \leq K\Delta_n^{(q/\beta')\wedge 1-\iota}$$

and a similar result for $\frac{\Delta_n}{\tau_i^n - \tau_{i-1}^n} \int_{\tau_{i-1}^n}^{\tau_i^n} \sigma_{u-} d\grave{S}_u - \frac{\Delta_n}{\tau_{i-1}^n - \tau_{i-2}^n} \int_{\tau_{i-2}^n}^{\tau_{i-1}^n} \sigma_{u-} d\grave{S}_u$.

Now we introduce χ_1, χ_2, χ_3 with $\widetilde{X_i} - \widetilde{X_{i-1}} = \chi_1 + \chi_2 + \chi_3$ where

$$\begin{aligned}
\chi_1 &= \sigma_{\tau^n_{i-2}} \left(\widetilde{S_i} - \widetilde{S_{i-1}} \right), \\
\chi_2 &= \frac{\Delta_n}{\tau^n_i - \tau^n_{i-1}} \int_{\tau^n_{i-1}}^{\tau^n_i} (\sigma_{u-} - \sigma_{\tau^n_{i-2}}) dS^{(1)}_u - \frac{\Delta_n}{\tau^n_{i-1} - \tau^n_{i-2}} \int_{\tau^n_{i-2}}^{\tau^n_{i-1}} (\sigma_{u-} - \sigma_{\tau^n_{i-2}}) dS^{(1)}_u \\
&\quad + \frac{\Delta_n}{\tau^n_i - \tau^n_{i-1}} \int_{\tau^n_{i-1}}^{\tau^n_i} (\sigma_{u-} - \sigma_{\tau^n_{i-2}}) dS^{(3)}_u - \frac{\Delta_n}{\tau^n_{i-1} - \tau^n_{i-2}} \int_{\tau^n_{i-2}}^{\tau^n_{i-1}} (\sigma_{u-} - \sigma_{\tau^n_{i-2}}) dS^{(3)}_u \\
&\quad + \frac{\Delta_n}{\tau^n_i - \tau^n_{i-1}} \int_{\tau^n_{i-1}}^{\tau^n_i} \alpha_u du - \frac{\Delta_n}{\tau^n_{i-1} - \tau^n_{i-2}} \int_{\tau^n_{i-2}}^{\tau^n_{i-1}} \alpha_u du, \\
\chi_3 &= \frac{\Delta_n}{\tau^n_i - \tau^n_{i-1}} \int_{\tau^n_{i-1}}^{\tau^n_i} (\sigma_{u-} - \sigma_{\tau^n_{i-2}}) dS^{(2)}_u - \frac{\Delta_n}{\tau^n_{i-1} - \tau^n_{i-2}} \int_{\tau^n_{i-2}}^{\tau^n_{i-1}} (\sigma_{u-} - \sigma_{\tau^n_{i-2}}) dS^{(2)}_u \\
&\quad + \frac{\Delta_n}{\tau^n_i - \tau^n_{i-1}} \Delta^n_i Y - \frac{\Delta_n}{\tau^n_{i-1} - \tau^n_{i-2}} \Delta^n_{i-1} Y \\
&\quad + \frac{\Delta_n}{\tau^n_i - \tau^n_{i-1}} \int_{\tau^n_{i-1}}^{\tau^n_i} \sigma_{u-} d\acute{S}_u - \frac{\Delta_n}{\tau^n_{i-1} - \tau^n_{i-2}} \int_{\tau^n_{i-2}}^{\tau^n_{i-1}} \sigma_{u-} d\acute{S}_u \\
&\quad - \frac{\Delta_n}{\tau^n_i - \tau^n_{i-1}} \int_{\tau^n_{i-1}}^{\tau^n_i} \sigma_{u-} d\grave{S}_u - \frac{\Delta_n}{\tau^n_{i-1} - \tau^n_{i-2}} \int_{\tau^n_{i-2}}^{\tau^n_{i-1}} \sigma_{u-} d\grave{S}_u,
\end{aligned}$$

remembering that $L_t = S^{(1)}_t + S^{(2)}_t + S^{(3)}_t + \acute{S}_t - \grave{S}_t$. As a result of the inequalities above and noting the obvious $3q/2 \geq q/2 + q/\beta \wedge 1$ and $q/2 + 1 \geq q/\beta' \wedge 1$ we can determine the rate of convergence for χ_2, χ_3

$$\mathbb{E}^n_{i-2} |\chi_2|^q \leq K \Delta_n^{q/2 + q/\beta \wedge 1 - \iota} \text{ for all } q \in [0, 2], \tag{5.32}$$

$$\begin{aligned}
\mathbb{E}^n_{i-2} |\chi_3|^q \leq K \Bigg(& \mathbb{E}^n_{i-2} \left| \frac{\Delta_n}{\tau^n_i - \tau^n_{i-1}} \int_{\tau^n_{i-1}}^{\tau^n_i} (\sigma_{u-} - \sigma_{\tau^n_{i-2}}) dS^{(2)}_u \right|^q \\
& + \mathbb{E}^n_{i-2} \left| \frac{\Delta_n}{\tau^n_{i-1} - \tau^n_{i-2}} \int_{\tau^n_{i-2}}^{\tau^n_{i-1}} (\sigma_{u-} - \sigma_{\tau^n_{i-2}}) dS^{(2)}_u \right|^q \\
& + \mathbb{E}^n_{i-2} \left[\mathbb{E}^n_{i-1} \left| \frac{\Delta_n}{\tau^n_i - \tau^n_{i-1}} \Delta^n_i Y \right|^q \right] + \mathbb{E}^n_{i-2} \left| \frac{\Delta_n}{\tau^n_{i-1} - \tau^n_{i-2}} \Delta^n_{i-1} Y \right|^q \\
& + \mathbb{E}^n_{i-2} \left[\left| \frac{\Delta_n}{\tau^n_i - \tau^n_{i-1}} \int_{\tau^n_{i-1}}^{\tau^n_i} \sigma_{u-} d\acute{S}_u - \frac{\Delta_n}{\tau^n_{i-1} - \tau^n_{i-2}} \int_{\tau^n_{i-2}}^{\tau^n_{i-1}} \sigma_{u-} d\acute{S}_u \right|^q \right] \\
& + \mathbb{E}^n_{i-2} \left[\left| \frac{\Delta_n}{\tau^n_i - \tau^n_{i-1}} \int_{\tau^n_{i-1}}^{\tau^n_i} \sigma_{u-} d\grave{S}_u - \frac{\Delta_n}{\tau^n_{i-1} - \tau^n_{i-2}} \int_{\tau^n_{i-2}}^{\tau^n_{i-1}} \sigma_{u-} d\grave{S}_u \right|^q \right] \Bigg) \\
\leq{} & K \Delta_n^{q/\beta' \wedge 1 - \iota} \text{ for all } q \in (0, \beta).
\end{aligned} \tag{5.33}$$

Furthermore $\mathbb{E}^n_{i-2} |\Delta_n^{-1/\beta} \chi_1|^p$ is a constant for all $p \in (-1, \beta)$ by Lemma 5.3.
For proving (5.27) we use the shorthand $\widetilde{\chi_i} = \Delta_n^{-1/\beta} \chi_i$ for $i = 1, 2, 3$ and see that using

the estimates (5.32),(5.33) above on χ_2, χ_3 we have

$$\mathbb{E}^n_{i-2} |\widetilde{\chi_2}|^q \leq K\Delta_n^{-q/\beta}\Delta_n^{q/2+q/\beta\wedge 1-\iota} = K\Delta_n^{q/2+q/\beta\wedge 1-q/\beta-\iota} \text{ for all } q \in [0,2], \tag{5.34}$$
$$\mathbb{E}^n_{i-2} |\widetilde{\chi_3}|^q \leq K\Delta_n^{q/\beta'\wedge 1-q/\beta-\iota} \text{ for all } q \in (0,\beta), \tag{5.35}$$

furthermore for all $q \in (0,\beta)$ we have $\mathbb{E}^n_{i-2} |\widetilde{\chi_1}|^q < M$ for some constant $M > 0$. Then

$$\begin{aligned}(5.27) =& \mathbb{E}^n_{i-2} \left||\widetilde{\chi}_1 + \widetilde{\chi}_2 + \widetilde{\chi}_3|^p - |\widetilde{\chi}_1|^p\right| \\ \leq& \mathbb{E}^n_{i-2} \left||\widetilde{\chi}_1 + \widetilde{\chi}_2 + \widetilde{\chi}_3|^p - |\widetilde{\chi}_1 + \widetilde{\chi}_2|^p\right| + \mathbb{E}^n_{i-2} \left||\widetilde{\chi}_1 + \widetilde{\chi}_2|^p - |\widetilde{\chi}_1|^p\right|.\end{aligned}$$

We note, because $p < \beta/2 < 1$ and therefore $|\cdot|^p$ is subadditive, that

$$\mathbb{E}^n_{i-2} \left||\widetilde{\chi}_1 + \widetilde{\chi}_2 + \widetilde{\chi}_3|^p - |\widetilde{\chi}_1 + \widetilde{\chi}_2|^p\right| \leq \mathbb{E}^n_{i-2}|\widetilde{\chi}_3|^p \leq K\Delta_n^{p/\beta'\wedge 1-p/\beta-\iota}.$$

For the remaining term we use the algebraic inequality

$$\left||\widetilde{\chi}_1 + \widetilde{\chi}_2|^p - |\widetilde{\chi}_1|^p\right| \leq K|\widetilde{\chi}_1|^{p-1}|\widetilde{\chi}_2|\mathbb{1}_{\{|\widetilde{\chi}_1|>\epsilon, |\widetilde{\chi}_2|<\frac{1}{2}\epsilon\}} + |\widetilde{\chi}_2|^p(\mathbb{1}_{\{|\widetilde{\chi}_1|\leq\epsilon\}} + \mathbb{1}_{\{|\widetilde{\chi}_2|>\frac{1}{2}\epsilon\}}),$$

which holds for any $\epsilon > 0$ and $p \in (0,1]$ and a constant K that does not depend on ϵ. Using (5.34), (5.35) plus Markov and Hölder inequality then yields

$$\begin{aligned}\mathbb{E}^n_{i-2}\left[|\widetilde{\chi}_1|^{p-1}|\widetilde{\chi}_2|\mathbb{1}_{\{|\widetilde{\chi}_1|>\epsilon, |\widetilde{\chi}_2|<\frac{1}{2}\epsilon\}}\right] &\leq \left(\mathbb{E}^n_{i-2}\left[|\widetilde{\chi}_1|^{\frac{(p-1)\beta}{\beta-1}}\mathbb{1}_{\{|\widetilde{\chi}_1|>\epsilon, |\widetilde{\chi}_2|<\frac{1}{2}\epsilon\}}\right]\right)^{1-\frac{1}{\beta}}\left(\mathbb{E}^n_{i-2}|\widetilde{\chi}_2|^\beta\right)^{\frac{1}{\beta}} \\ &\leq (\Delta_n^{\beta/2-\iota})^{\frac{1}{\beta}} = \Delta_n^{1/2-\iota}, \\ \mathbb{E}^n_{i-2}\left[|\widetilde{\chi}_2|^p\mathbb{1}_{\{|\widetilde{\chi}_1|\leq\epsilon\}}\right] &\leq \left(\mathbb{E}^n_{i-2}\left[\mathbb{1}_{\{|\widetilde{\chi}_1|\leq\epsilon\}}\right]\right)^{1-\frac{p}{\beta}}\left(\mathbb{E}^n_{i-2}\left[|\widetilde{\chi}_2|^\beta\right]\right)^{\frac{p}{\beta}} \\ &\leq K\left(\mathbb{E}^n_{i-2}\left[|\widetilde{\chi}_1|^{-1+\iota}\right]\epsilon^{1-\iota}\right)^{1-\frac{p}{\beta}}(\Delta_n^{\beta/2-\iota})^{p/\beta} \qquad (5.36) \\ &\leq \epsilon^{1-p/\beta-\iota}\Delta_n^{p/2-\iota}, \\ \mathbb{E}^n_{i-2}\left[|\widetilde{\chi}_2|^p\mathbb{1}_{\{|\widetilde{\chi}_2|>\frac{1}{2}\epsilon\}}\right] &\leq \left(\mathbb{E}^n_{i-2}\left[\mathbb{1}_{\{|\widetilde{\chi}_2|>\frac{1}{2}\epsilon\}}\right]\right)^{1-\frac{p}{\beta}}\left(\mathbb{E}^n_{i-2}\left[|\widetilde{\chi}_2|^\beta\right]\right)^{\frac{p}{\beta}} \\ &\leq \left(\frac{\mathbb{E}^n_{i-2}\left[|\widetilde{\chi}_2|^\beta\right]}{(\frac{1}{2}\epsilon)^\beta}\right)^{1-\frac{p}{\beta}}\Delta_n^{p/2-\iota} \qquad (5.37) \\ &\leq K\Delta_n^{\frac{\beta}{2}\frac{\beta-p}{\beta}}\Delta_n^{p/2-\iota}\epsilon^{-(\beta-p)} \leq K\epsilon^{-(\beta-p)}\Delta_n^{\beta/2-\iota},\end{aligned}$$

where the last inequality only holds for an $\epsilon \leq 1$. Setting $\epsilon = \Delta_n^{\frac{1}{2}\frac{\beta}{\beta+1}}$ gives the same orders both in (5.36) and (5.37) which yields the result. □

Corollary 5.1. *For all $2 \le i \le N_n(1)$ and $0 < q < \beta$ it holds that for some constant M*

$$\mathbb{E}^n_{i-2}\left[\Delta_n^{-q/\beta}\left|\widetilde{\Delta^n_i X_i} - \widetilde{\Delta^n_{i-1} X_{i-1}}\right|^q\right] \le M.$$

Proof.

$$\begin{aligned}\Delta_n^{-q/\beta}\mathbb{E}^n_{i-2}\left|\left|\widetilde{\Delta^n_i X_i} - \widetilde{\Delta^n_{i-1} X_{i-1}}\right|^q\right| &= \mathbb{E}^n_{i-2}\,||\widetilde{\chi}_1 + \widetilde{\chi}_2 + \widetilde{\chi}_3|^q| \\ &\le K_q\left(\mathbb{E}^n_{i-2}\,|\widetilde{\chi}_1|^q + \mathbb{E}^n_{i-2}\,|\widetilde{\chi}_2|^q + \mathbb{E}^n_{i-2}\,|\widetilde{\chi}_3|^q\right)\end{aligned}$$

and according to proof of the previous Lemma all expectations in the last line are bounded. This yields the result. □

Lemma 5.11. *Let $k_n \asymp n^{\varpi}$ with an $\varpi \in (0,1)$. Then we have for $k_n+3 \le i \le N_n(1), 0 < p < \frac{\beta}{2}$ and $1 \le x < \frac{\beta}{p}$*

$$\Delta_n^{-xp/\beta}\mathbb{E}[|\widetilde{V}^n_i(p) - \overline{V}^n_i(p)|^x] \le K_x k_n^{-x/2} \tag{5.38}$$

Proof. Using the notation

$$\zeta^n_j := \Delta_n^{-p/\beta}\left|\widetilde{\Delta^n_j X} - \widetilde{\Delta^n_{j-1} X}\right|^p - \Delta_n^{-p/\beta}\mathbb{E}^n_{j-2}\left|\widetilde{\Delta^n_j X} - \widetilde{\Delta^n_{j-1} X}\right|^p$$

we have

$$\begin{aligned}\Delta_n^{-p/\beta}(\widetilde{V}^n_i(p) - \overline{V}^n_i(p)) &= \frac{1}{k_n}\sum_{j=i-k_n-1}^{i-2}\zeta_j \\ &= \frac{1}{k_n}\sum_{j=0}^{\lfloor\frac{k_n-1}{2}\rfloor}\zeta_{i-k_n-1+2j} + \frac{1}{k_n}\sum_{j=0}^{\lfloor\frac{k_n-2}{2}\rfloor}\zeta_{i-k_n+2j}\end{aligned}$$

and

$$\begin{aligned}&\Delta_n^{-xp/\beta}\mathbb{E}[|\widetilde{V}^n_i(p) - \overline{V}^n_i(p)|^x] \\ &\quad\le K\left(\mathbb{E}\left[\left|\frac{1}{k_n}\sum_{j=0}^{\lfloor\frac{k_n-1}{2}\rfloor}\zeta_{i-k_n-1+2j}\right|^x\right] + \mathbb{E}\left[\left|\frac{1}{k_n}\sum_{j=0}^{\lfloor\frac{k_n-2}{2}\rfloor}\zeta_{i-k_n+2j}\right|^x\right]\right).\end{aligned} \tag{5.39}$$

Each of the sums can be seen as a discrete martingale w.r.t. to its own filtration with $k_n/2$ (or $k_n/2-1$) jumps and we subsequently use the BDG inequality to bound each of the sums individually. The corollary above gives us that the x-th conditional moment of ζ_j is bounded, i.e.

$$\mathbb{E}^n_{j-2}|\zeta_j|^x \le K \tag{5.40}$$

and so is the unconditional moment. Due to $p < \frac{\beta}{2}$ in particular the second (conditional) moment exists. Therefore applying the BDG inequality yields:

$$\mathbb{E}\left[\left|\frac{1}{k_n}\sum_{j=0}^{\lfloor\frac{k_n-1}{2}\rfloor}\zeta_{i-k_n-1+2j}\right|^x\right] \leq K\mathbb{E}\left[\left|\frac{1}{k_n^2}\sum_{j=0}^{\lfloor\frac{k_n-1}{2}\rfloor}\left|\zeta_{i-k_n-1+2j}\right|^2\right|^{x/2}\right], \tag{5.41}$$

and for an $x \leq 2$ we may use Jensen inequality:

$$(5.41) \leq K\frac{1}{k_n^x}\left(\mathbb{E}\left[\left|\sum_{j=0}^{\lfloor\frac{k_n-1}{2}\rfloor}\left|\zeta_{i-k_n-1+2j}\right|^2\right|\right]\right)^{x/2} \leq Kk_n^{-x/2}.$$

In the case of $2 < x \leq 4$ we use (4.4) and that the function $y \mapsto y^{x/2}$ is convex, and therefore one gets again with BDG inequality, (5.40), (4.4) and $k(j) := i - k_n - 1 + 2j$

$$\begin{aligned}
(5.41) &= K\mathbb{E}\left[\left|\frac{1}{k_n^2}\sum_{j=0}^{\lfloor\frac{k_n-1}{2}\rfloor}\left(\left|\zeta_{k(j)}\right|^2 - \mathbb{E}_{k(j)-2}\left[\zeta_{k(j)}^2\right]\right) + \frac{1}{k_n^2}\sum_{j=0}^{\lfloor\frac{k_n-1}{2}\rfloor}\mathbb{E}_{k(j)-2}\left[\zeta_{k(j)}^2\right]\right|^{x/2}\right] \\
&\leq K\frac{1}{k_n^x}\mathbb{E}\left[\left|\sum_{j=0}^{\lfloor\frac{k_n-1}{2}\rfloor}\left(\left|\zeta_{k(j)}\right|^2 - \mathbb{E}_{k(j)-2}\left[\zeta_{k(j)}^2\right]\right)\right|^{x/2}\right] + Kk_n^{-x/2} \\
&\leq K\frac{1}{k_n^x}\mathbb{E}\left[\left|\sum_{j=0}^{\lfloor\frac{k_n-1}{2}\rfloor}\left(\left|\zeta_{k(j)}\right|^2 - \mathbb{E}_{k(j)-2}\left[\zeta_{k(j)}^2\right]\right)^2\right|^{x/4}\right] + Kk_n^{-x/2} \\
&\leq K\frac{1}{k_n^x}\mathbb{E}\left[\sum_{j=0}^{\lfloor\frac{k_n-1}{2}\rfloor}\left|\left|\zeta_{k(j)}\right|^2 - \mathbb{E}_{k(j)-2}\left[\zeta_{k(j)}^2\right]\right|^{x/2}\right] + Kk_n^{-x/2} \\
&\leq Kk_n^{1-x} + Kk_n^{-x/2} \leq Kk_n^{-x/2}.
\end{aligned}$$

If $x > 4$ one repeats the previous steps (for $x > 8$ more than once). In the case of

$4 < x \leq 8$ (5.41) reads as

$$K\mathbb{E}\left[\left|\frac{1}{k_n^2}\sum_{j=0}^{\lfloor\frac{k_n-1}{2}\rfloor}\left(\left|\zeta_{k(j)}\right|^2-\mathbb{E}_{k(j)-2}\left[\zeta_{k(j)}^2\right]\right)+\frac{1}{k_n^2}\sum_{j=0}^{\lfloor\frac{k_n-1}{2}\rfloor}\mathbb{E}_{k(j)-2}\left[\zeta_{k(j)}^2\right]\right|^{x/2}\right]$$
$$\leq K\frac{1}{k_n^x}\mathbb{E}\left[\left|\sum_{j=0}^{\lfloor\frac{k_n-1}{2}\rfloor}\left(\left|\zeta_{k(j)}\right|^2-\mathbb{E}_{k(j)-2}\left[\zeta_{k(j)}^2\right]\right)\right|^{x/2}\right]+Kk_n^{-x/2}$$
$$\leq K\frac{1}{k_n^x}\mathbb{E}\left[\left|\sum_{j=0}^{\lfloor\frac{k_n-1}{2}\rfloor}\left(\left|\zeta_{k(j)}\right|^2-\mathbb{E}_{k(j)-2}\left[\zeta_{k(j)}^2\right]\right)^2\right|^{x/4}\right]+Kk_n^{-x/2}$$
$$\leq K\frac{1}{k_n^x}\mathbb{E}\left[\left|\sum_{j=0}^{\lfloor\frac{k_n-1}{2}\rfloor}\left(\left(\left|\zeta_{k(j)}\right|^2-\mathbb{E}_{k(j)-2}\left[\zeta_{k(j)}^2\right]\right)^2-\mathbb{E}_{k(j)-2}\left[\left(\left|\zeta_{k(j)}\right|^2-\mathbb{E}_{k(j)-2}\left[\zeta_{k(j)}^2\right]\right)^2\right]\right)\right|^{x/4}\right]$$
$$+\frac{1}{k_n^x}\mathbb{E}\left[\left|\sum_{j=0}^{\lfloor\frac{k_n-1}{2}\rfloor}\mathbb{E}_{k(j)-2}\left[\left(\left|\zeta_{k(j)}\right|^2-\mathbb{E}_{k(j)-2}\left[\zeta_{k(j)}^2\right]\right)^2\right]\right|^{x/4}\right]+Kk_n^{-x/2}$$
$$\leq K\frac{1}{k_n^x}\mathbb{E}\left[\left|\sum_{j=0}^{\lfloor\frac{k_n-1}{2}\rfloor}\left(\left(\left|\zeta_{k(j)}\right|^2-\mathbb{E}_{k(j)-2}\left[\zeta_{k(j)}^2\right]\right)^2-\mathbb{E}_{k(j)-2}\left[\left(\left|\zeta_{k(j)}\right|^2-\mathbb{E}_{k(j)-2}\left[\zeta_{k(j)}^2\right]\right)^2\right]\right)^2\right|^{x/8}\right]$$
$$+Kk_n^{-x3/4}+Kk_n^{-x/2}$$
$$\leq Kk_n^{1-x}+Kk_n^{-x/2}\leq Kk_n^{-x/2}.$$

□

Lemma 5.12. *Let $k_n+3\leq i\leq N_n(1), 0<p<\frac{\beta}{2}$ and $k_n \asymp n^{\varpi}$ with a $\varpi\in(0,1)$. Then it holds for the set $\mathcal{C}_i^n := \{|\Delta_n^{-p/\beta}\widetilde{V}_i^n(p)-|\overline{\sigma\lambda}|_i^p\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}|>\frac{1}{2}|\overline{\sigma\lambda}|_i^p\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}\}$ that*

$$\mathbb{P}(\mathcal{C}_i^n)\leq K_\iota k_n^{-\beta/2p+\iota}. \tag{5.42}$$

Proof. Using Lemma 5.3 and Lemma 5.10 it follows that

$$\left|\Delta_n^{-p/\beta}\mathbb{E}_{i-2}^n\left|\widetilde{\Delta_i^n X}-\widetilde{\Delta_{i-1}^n X}\right|^p-|\sigma_{\tau_{i-2}^n}|^p|\lambda_{\tau_{j-2}^n}|^{\frac{p}{\beta}-p}\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}\right|$$
$$\leq\Delta_n^{-p/\beta}\mathbb{E}_{i-2}^n\left|\left|\widetilde{\Delta_i^n X}-\widetilde{\Delta_{i-1}^n X}\right|^p-|\sigma_{\tau_{i-2}^n}|^p\left|\widetilde{\Delta_i^n S}-\widetilde{\Delta_{i-1}^n S}\right|^p\right|$$
$$+\left|\mathbb{E}_{i-2}\left|\Delta_n^{-p/\beta}|\sigma_{\tau_{i-2}^n}|^p\left|\widetilde{\Delta_i^n S}-\widetilde{\Delta_{i-1}^n S}\right|^p\right|-|\sigma_{\tau_{i-2}^n}|^p|\lambda_{\tau_{j-2}^n}|^{\frac{p}{\beta}-p}\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}\right|$$
$$\leq K\alpha_n+K'|\sigma_{\tau_{i-2}^n}|^p\Delta_n^{1/2}$$

with α_n being defined as in Lemma 5.10 and therefore

$$
\begin{aligned}
&|\Delta_n^{-p/\beta}\overline{V_i^n}(p) - |\overline{\sigma\lambda}|_i^p \mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}| && (5.43)\\
&= \left| \frac{1}{k_n} \sum_{j=i-k_n-1}^{i-2} \left(\Delta_n^{-p/\beta} \mathbb{E}_{j-2}^n \left| \widetilde{\Delta_j^n X} - \widetilde{\Delta_{j-1}^n X} \right|^p - |\sigma_{\tau_{j-2}^n}|^p |\lambda_{\tau_{j-2}^n}|^{\frac{p}{\beta}-p} \mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta} \right) \right| && (5.44)\\
&\le \frac{1}{k_n} \sum_{j=i-k_n-1}^{i-2} \left| \Delta_n^{-p/\beta} \mathbb{E}_{j-2}^n \left| \widetilde{\Delta_j^n X} - \widetilde{\Delta_{j-1}^n X} \right|^p - |\sigma_{\tau_{j-2}^n}|^p |\lambda_{\tau_{j-2}^n}|^{\frac{p}{\beta}-p} \mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta} \right|\\
&\le K\alpha_n + K'|\sigma_{\tau_{j-2}^n}|^p \Delta_n^{1/2}.
\end{aligned}
$$

Due to the boundedness from below in (SB) it holds that $|\overline{\sigma\lambda}|_i^p, |\sigma_{\tau_{i-2}^n}|^p > M$ and therefore as $\alpha_n \to 0$ and $\Delta_n^{1/2} \to 0$ an $n_0 \in \mathbb{N}$ exists with $K\alpha_n + K'|\sigma_{\tau_{i-2}^n}|^p \Delta_n^{1/2} < \frac{1}{4}|\overline{\sigma\lambda}|_i^p \mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}$ for all $n \ge n_0$ and as such

$$
\mathbb{P}(|\Delta_n^{-p/\beta}\overline{V_i^n}(p) - |\overline{\sigma\lambda}|_i^p \mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}| > \frac{1}{4}|\overline{\sigma\lambda}|_i^p \mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}) = 0 \text{ for all } n \ge n_0.
$$

For $\mathcal{C}_i^n$ it now follows with Lemma 5.11

$$
\begin{aligned}
\mathbb{P}(\mathcal{C}_i^n) &\le \mathbb{P}(\Delta_n^{-p/\beta}|\widetilde{V}_i^n(p) - \overline{V_i^n}(p)| > \frac{1}{4}|\overline{\sigma\lambda}|_i^p \mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta})\\
&\quad + \mathbb{P}(|\Delta_n^{-p/\beta}\overline{V_i^n}(p) - |\overline{\sigma\lambda}|_i^p \mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}| > \frac{1}{4}|\overline{\sigma\lambda}|_i^p \mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta})\\
&\le K_\iota k_n^{-\beta/2p+\iota}
\end{aligned}
$$

for all $n \ge n_0$. □

Corollary 5.2. *As a result of the proof above we have that for some $K > 0$ large enough*

$$
|\Delta_n^{-p/\beta}\overline{V_i^n}(p) - |\overline{\sigma\lambda}|_i^p \mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}| \le K(\alpha_n + \Delta_n^{1/2}) \text{ a.s.} \qquad (5.45)
$$

5.2.2 Bounding the Residual Terms

Lemma 5.13. *Let $0 < p < \frac{\beta}{2}$ and $k_n \asymp n^{\varpi}$ with a $\varpi \in (0,1)$. Then it holds that*

$$
\frac{1}{n-k_n-2}\mathbb{E}\,|R_1^n(u_n)| \le K_\iota \left(k_n^{-\beta/2p+\iota} \vee u_n^{\beta'} \Delta_n^{(\beta-\beta')/\beta} \vee u_n \Delta_n^{1/2-\iota} \right).
$$

Proof. Because we need to bound the term $\widetilde{V}_i^n(p)$ from below we decompose $r_i^1(u_n) = r_i^1(u_n)\mathbb{1}_{\mathcal{C}_i^n} + r_i^1(u_n)\mathbb{1}_{(\mathcal{C}_i^n)^C}$ and note that because $\cos(x)$ is bounded we have for $k_n + 3 \le i \le N_n(1)$ by (5.42)

$$
\mathbb{E}_{i-2}^n \left| r_i^1(u_n)\mathbb{1}_{\mathcal{C}_i^n} \right| \le K\mathbb{P}(\mathcal{C}_i^n) \le K_\iota k_n^{-\beta/2p+\iota}
$$

and as a direct consequence for all $k_n + 3 \le i$

$$\mathbb{E}|\mathbb{1}_{\{N_n(1)\ge i\}} r_i^1(u)\mathbb{1}_{\mathcal{C}_i^n}| \le K_\iota k_n^{-\beta/2p+\iota}.$$

Applying this we get due to $N_n(1) \le Cn$:

$$\begin{aligned}
\frac{1}{n-k_n-2}\mathbb{E}\left|R_1^n(u_n)\mathbb{1}_{\mathcal{C}_i^n}\right| &\le \frac{1}{n-k_n-2}\mathbb{E}\left[\sum_{i=k_n+3}^{N_n(1)}\left|r_i^1(u)\mathbb{1}_{\mathcal{C}_i^n}\right|\right] \\
&= \frac{1}{n-k_n-2}\mathbb{E}\left[\sum_{i=k_n+3}^{nC}\left|\mathbb{1}_{\{N_n(1)\ge i\}} r_i^1(u)\mathbb{1}_{\mathcal{C}_i^n}\right|\right] \\
&\le \frac{1}{n-k_n-2}\sum_{i=k_n+3}^{nC}\mathbb{E}|\mathbb{1}_{\{N_n(1)\ge i\}} r_i^1(u)\mathbb{1}_{\mathcal{C}_i^n}| \le K_\iota k_n^{-\beta/2p+\iota}.
\end{aligned} \tag{5.46}$$

From the definition of the set it follows that on $(\mathcal{C}_i^n)^C$ it holds that for $k_n+3 \le i \le N_n(1)$

$$|\Delta_n^{-p/\beta}\widetilde{V}_i^n(p) - |\overline{\sigma\lambda}|_i^p \mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}| \le \frac{1}{2}|\overline{\sigma\lambda}|_i^p \mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}$$

and as a result:

$$\Delta_n^{-p/\beta}\widetilde{V}_i^n(p) \ge \frac{1}{2}|\overline{\sigma\lambda}|_i^p \mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta} \text{ and } \Delta_n^{-p/\beta}\widetilde{V}_i^n(p) \le \frac{3}{2}|\overline{\sigma\lambda}|_i^p \mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}.$$

Because by (SB) $|\overline{\sigma\lambda}|_i^p$ is bounded from above and below, $\Delta_n^{-p/\beta}\widetilde{V}_i^n(p)$ is now likewise. Using the notation from Lemma 5.10 we have that

$$\begin{aligned}
\Delta_n^{-1/\beta}(\widetilde{\Delta_i^n X} - \widetilde{\Delta_{i-1}^n X}) &= \widetilde{\chi_1} + \widetilde{\chi_2} + \widetilde{\chi_3}, \\
\Delta_n^{-1/\beta}\sigma_{\tau_{i-2}^n}(\widetilde{\Delta_i^n S} - \widetilde{\Delta_{i-1}^n S}) &= \widetilde{\chi_1}.
\end{aligned}$$

Using the boundedness of $\Delta_n^{-p/\beta}\widetilde{V}_i^n(p)$ and the inequality $|\cos(x)-\cos(y)| \le 2|x-y|^p$ for all $x, y \in \mathbb{R}$ and $p \in (0,1]$ we have

$$\begin{aligned}
\mathbb{E}_{i-2}^n\left|r_i^1(u_n)\mathbb{1}_{(\mathcal{C}_i^n)^C}\right| &\le \mathbb{E}_{i-2}^n\left|\cos\left(u_n\frac{\widetilde{\chi_1}+\widetilde{\chi_2}+\widetilde{\chi_3}}{\Delta_n^{-1/\beta}(\widetilde{V}_i^n(p))^{1/p}}\right) - \cos\left(u_n\frac{\widetilde{\chi_1}+\widetilde{\chi_2}}{\Delta_n^{-1/\beta}(\widetilde{V}_i^n(p))^{1/p}}\right)\right|\mathbb{1}_{(\mathcal{C}_i^n)^C} \\
&\quad + \mathbb{E}_{i-2}^n\left|\cos\left(u_n\frac{\widetilde{\chi_1}+\widetilde{\chi_2}}{\Delta_n^{-1/\beta}(\widetilde{V}_i^n(p))^{1/p}}\right) - \cos\left(u_n\frac{\widetilde{\chi_1}}{\Delta_n^{-1/\beta}(\widetilde{V}_i^n(p))^{1/p}}\right)\right|\mathbb{1}_{(\mathcal{C}_i^n)^C} \\
&\le K(\mathbb{E}_{i-2}^n\left|u_n\widetilde{\chi_3}\right|^{\beta'} + \mathbb{E}_{i-2}^n\left|u_n\widetilde{\chi_2}\right|)
\end{aligned}$$

and likewise

$$\mathbb{E}_{i-2}^n\left|r_i^1(u_n)\mathbb{1}_{(\mathcal{C}_i^n)^C}\mathbb{1}_{\{N_n(1)\ge i\}}\right| \le K(u_n^{\beta'}\mathbb{E}_{i-2}^n\left|\mathbb{1}_{\{N_n(1)\ge i\}}\widetilde{\chi_3}\right|^{\beta'} + u_n\mathbb{E}_{i-2}^n\left|\mathbb{1}_{\{N_n(1)\ge i\}}\widetilde{\chi_2}\right|). \tag{5.47}$$

For the rate of convergence we get

$$\mathbb{E}_{i-2}^n \left|\mathbb{1}_{\{N_n(1)\geq i\}}\widetilde{\chi 2}\right| \leq K\Delta_n^{1/2-\iota},$$

$$\mathbb{E}_{i-2}^n \left|\mathbb{1}_{\{N_n(1)\geq i\}}\widetilde{\chi 3}\right|^{\beta'} \leq K\Delta_n^{\frac{\beta-\beta'}{\beta}}$$

as on the set $\{N_n(1) \geq i\}$ we can apply (5.34) and (5.35) and otherwise $\mathbb{1}_{\{N_n(1)\geq i\}}\widetilde{\chi 2}$, $\mathbb{1}_{\{N_n(1)\geq i\}}\widetilde{\chi 3}$ are zero. By inserting these into (5.47) and proceeding as in (5.46) we get

$$\frac{1}{n-k_n-2}\mathbb{E}\left|R_1^n(u_n)\mathbb{1}_{(\mathcal{C}_i^n)^C}\right| \leq K\left(u_n^{\beta'}\Delta_n^{(\beta-\beta')/\beta} \vee u_n\Delta_n^{1/2-\iota}\right).$$

□

In the following Lemma we will, for easier readability, usually omit the condition $i \leq N_n(1)$ which is needed to guarantee that τ_i^n is indeed a bounded stopping time in order to make the estimates from the previous chapter applicable. However, to proceed as in (5.46) we are looking for estimates of the type $\mathbb{E}|\mathbb{1}_{\{N_n(1)\geq i\}}r_i^1(u)\mathbb{1}_{\mathcal{C}_i^n}| \leq K_\iota k_n^{-\beta/2p+\iota}$ and here the indicator $\mathbb{1}_{\{N_n(1)\geq i\}}$ suspends the restriction on i. Besides that, we usually need $i \geq k_n+3$ but this can be easily deducted from the specific equation.

Lemma 5.14. *Let $0 < p < \frac{\beta}{2}$ and $k_n \asymp n^{\varpi}$ with a $\varpi \in (0,1)$. Then it holds that*

$$\frac{1}{n-k_n-2}\mathbb{E}|R_2^n(u_n)| \leq K(k_n^{-\beta/2p+\iota} \vee u_n\Delta_n^{1/2}).$$

Proof. As with the previous term we bound $\widetilde{V}_i^n(p)$ by decomposing $r_i^2(u_n) = r_i^2(u_n)\mathbb{1}_{\mathcal{C}_i^n} + r_i^2(u_n)\mathbb{1}_{(\mathcal{C}_i^n)^C}$ and note that because $\cos(x)$ is bounded we have as in the previous lemma using (5.42)

$$\mathbb{E}_{i-2}^n\left|r_i^2(u_n)\mathbb{1}_{\mathcal{C}_i^n}\right| \leq K\mathbb{P}(\mathcal{C}_i^n) \leq Kk_n^{-\beta/2p+\iota} \quad \text{and} \quad \frac{1}{n-k_n-2}\mathbb{E}_{i-2}^n\left|R_2^n(u_n)\mathbb{1}_{\mathcal{C}_i^n}\right| \leq Kk_n^{-\beta/2p+\iota},$$

where we proceed as in (5.46). Using the boundedness of $\Delta_n^{-p/\beta}\widetilde{V}_i^n(p)$ and the inequality $|\cos(x)-\cos(y)| \leq 2|x-y|^p$ for all $x, y \in \mathbb{R}$ and $p \in (0,1]$ we have

$$\begin{aligned}
&\mathbb{E}_{i-2}^n\left|r_i^2(u_n)\mathbb{1}_{(\mathcal{C}_i^n)^C}\right| \\
&= \mathbb{1}_{(\mathcal{C}_i^n)^C}\mathbb{E}_{i-2}^n\left|\cos\left(u_n\frac{\sigma_{\tau_{i-2}^n}(\widetilde{\Delta_i^n S} - \widetilde{\Delta_{i-1}^n S})}{\widetilde{V}_i^n(p)^{1/p}}\right) - \cos\left(u_n\frac{\sigma_{\tau_{i-2}^n}\left(\widetilde{\Delta_i^n S} - \left(\frac{\lambda_{\tau_{i-2}^n}}{\lambda_{\tau_{i-3}^n}}\right)^{\frac{1}{\beta}-1}\widetilde{\Delta_{i-1}^n S}\right)}{\widetilde{V}_i^n(p)^{1/p}}\right)\right| \\
&\leq K\mathbb{1}_{(\mathcal{C}_i^n)^C}u_n\left|\frac{\sigma_{\tau_{i-2}^n}}{\Delta_n^{-1/\beta}\widetilde{V}_i^n(p)^{1/p}}\right|\mathbb{E}_{i-2}^n\left|\Delta_n^{-1/\beta}\widetilde{\Delta_{i-1}^n S} - \left(\frac{\lambda_{\tau_{i-2}^n}}{\lambda_{\tau_{i-3}^n}}\right)^{\frac{1}{\beta}-1}\Delta_n^{-1/\beta}\widetilde{\Delta_{i-1}^n S}\right| \\
&\leq Ku_n\left|\frac{\left(\lambda_{\tau_{i-3}^n}\right)^{\frac{1}{\beta}-1} - \left(\lambda_{\tau_{i-2}^n}\right)^{\frac{1}{\beta}-1}}{\left(\lambda_{\tau_{i-3}^n}\right)^{\frac{1}{\beta}-1}}\right|.
\end{aligned}$$

Now as in the proof of Lemma 5.9 we apply Lemma 5.7 on the process λ and the function $f(x) = x^{\frac{1}{\beta}-1}$. Then the boundedness from below of λ_t and Lemma 5.1 yield

$$\mathbb{E}\left|r_i^2(u_n)\mathbb{1}_{(\mathcal{C}_i^n)^C}\right| \leq K u_n \Delta_n^{1/2}.$$

Proceeding as in (5.46) we get

$$\frac{1}{n-k_n-2}\mathbb{E}\left|R_2^n(u_n)\mathbb{1}_{(\mathcal{C}_i^n)^C}\right| \leq K u_n \Delta_n^{1/2}.$$

□

Lemma 5.15. *Let $0 < p < \frac{\beta}{2}$ and $k_n \asymp n^{\varpi}$ with a $\varpi \in (0,1)$ then it holds that*

$$\frac{1}{n-k_n-2}\mathbb{E}|R_3^n(u_n)| \leq K(u_n^\beta \alpha_n \vee u_n^\beta k_n^{-1/2} \vee k_n^{-\beta/2p+\iota}).$$

Proof. As with the previous term we bound $\widetilde{V}_i^n(p)$ by decomposing $r_i^3(u_n) = r_i^3(u_n)\mathbb{1}_{\mathcal{C}_i^n} + r_i^3(u_n)\mathbb{1}_{(\mathcal{C}_i^n)^C}$ and note that because $\exp(-x)$ is bounded for all $x \geq 0$ we have again by (5.42)

$$\mathbb{E}_{i-2}^n\left|r_i^3(u_n)\mathbb{1}_{\mathcal{C}_i^n}\right| \leq K\mathbb{P}(\mathcal{C}_i^n) \leq K k_n^{-\beta/2p+\iota} \quad \text{and} \quad \frac{1}{n-k_n-2}\mathbb{E}_{i-2}^n\left|R_3^n(u_n)\mathbb{1}_{\mathcal{C}_i^n}\right| \leq K k_n^{-\beta/2p+\iota}.$$

In order to deal with the term including $\mathbb{1}_{(\mathcal{C}_i^n)^C}$ we use a first order Taylor expansion of (5.13) and get that

$$\begin{aligned}
&\mathbb{E}_{i-2}^n\left[\exp\left(-\frac{A_\beta u_n^\beta|\sigma_{\tau_{i-2}^n}|^\beta|\lambda_{\tau_{j-2}^n}|^{1-\beta}((\phi_i^n)^{1-\beta}+(\phi_{i-1}^n)^{1-\beta})}{\Delta_n^{-1}\widetilde{V}_i^n(p)^{\beta/p}}\right)\right] \\
&\qquad - \mathbb{E}_{i-2}^n\left[\exp\left(-\frac{C_{p,\beta} u_n^\beta|\sigma_{\tau_{i-2}^n}|^\beta|\lambda_{\tau_{j-2}^n}|^{1-\beta}((\phi_i^n)^{1-\beta}+(\phi_{i-1}^n)^{1-\beta})}{(|\overline{\sigma\lambda}|_i^p)^{\beta/p}}\right)\right] \\
&= \left(\Delta_n^{-p/\beta}\widetilde{V}_i^n(p) - |\overline{\sigma\lambda}|_i^p \mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}\right)\mathbb{E}_{i-2}\left[f'_{i,u_n}(\epsilon_i)\right]
\end{aligned}$$

for some ϵ_i between $\Delta_n^{-p/\beta}\widetilde{V}_i^n(p)$ and $|\overline{\sigma\lambda}|_i^p \mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}$. Again by the conditions on the set $(\mathcal{C}_i^n)^C$, $\Delta_n^{-p/\beta}\widetilde{V}_i^n$ is bounded from below and above, as is $|\sigma_{\tau_{i-2}^n}||\lambda_{\tau_{i-2}^n}|$, $|\overline{\sigma\lambda}|_i^p$ by combining the Assumptions SB and SC. Then using Lemma 5.6 we have

$$\begin{aligned}
\mathbb{E}|r_i^3(u_n)\mathbb{1}_{(\mathcal{C}_i^n)^C}| &\leq K u_n^\beta \mathbb{E}\left|\Delta_n^{-p/\beta}\widetilde{V}_i^n(p) - |\overline{\sigma\lambda}|_i^p \mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}\right| \\
&\leq K u_n^\beta \mathbb{E}\left[\left|\Delta_n^{-p/\beta}(\widetilde{V}_i^n(p) - \overline{V}_i^n(p))\right| + \left|\Delta_n^{-p/\beta}\overline{V}_i^n(p) - |\overline{\sigma\lambda}|_i^p \mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}\right|\right] \\
&\leq K u_n^\beta (k_n^{-1/2} \vee \alpha_n \vee \Delta_n^{1/2}) \leq K u_n^\beta (k_n^{-1/2} \vee \alpha_n)
\end{aligned}$$

where the last line holds by (5.38) and (5.45). □

Lemma 5.16. *Let* $0 < p < \frac{\beta}{2}$*,* $k_n \asymp n^{\varpi}$ *with a* $\varpi \in (0,1)$ *and* $u_n \asymp n^{-\varrho}$ *for a* $\varrho \in (0,1)$ *then it holds that*

$$\frac{1}{n-k_n-2}\mathbb{E}[|R_4^n(u_n)] \leq K u_n^{\beta}\Delta_n k_n$$

Proof. We prove Lemma 5.16 with a Taylor expansion of second order of the function $f_{i,u}(x)$. For this purpose we start by decomposing with $\tilde{r}_i^4 = (|\overline{\sigma\lambda}|_i^p - |\sigma_{\tau_{i-2}^n}|^p|\lambda_{\tau_{i-2}^n}|^{\frac{p}{\beta}-p})$

$$\mathbb{E}\,|R_4^n(u_n)| \leq \mathbb{E}\left|R_4^n(u_n) - \mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}\sum_{i=k_n+3}^{N_n(1)}\mathbb{E}_{i-2}^n\left[f'_{i,u_n}(\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}|\sigma_{\tau_{i-2}^n}|^p|\lambda_{\tau_{i-2}^n}|^{\frac{p}{\beta}-p})\right]\tilde{r}_i^4\right| \tag{5.48}$$

$$+\,\mathbb{E}\left|\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}\sum_{i=k_n+3}^{N_n(1)}(\tilde{r}_i^4 - \mathbb{E}_{i-k_n-3}^n\left[\tilde{r}_i^4\right])\mathbb{E}_{i-2}^n\left[f'_{i,u_n}(\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}|\sigma_{\tau_{i-2}^n}|^p|\lambda_{\tau_{i-2}^n}|^{\frac{p}{\beta}-p})\right]\right| \tag{5.49}$$

$$+\,\mathbb{E}\left|\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}\sum_{i=k_n+3}^{N_n(1)}\mathbb{E}_{i-2}^n\left[f'_{i,u_n}(\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}|\sigma_{\tau_{i-2}^n}|^p|\lambda_{\tau_{i-2}^n}|^{\frac{p}{\beta}-p})\right]\mathbb{E}_{i-k_n-3}\left[\tilde{r}_i^4\right]\right|. \tag{5.50}$$

In the sequel prove the same rate of convergence for all three terms on the right hand side. Starting with (5.48), from the definition of r_i^4 we have with the function (5.13)

$$\begin{aligned}
r_i^4(u_n) &= \mathbb{E}_{i-2}^n\left[\exp\left(-\frac{C_{p,\beta}u_n^{\beta}|\sigma_{\tau_{i-2}^n}|^{\beta}|\lambda_{\tau_{i-2}^n}|^{1-\beta}|((\phi_i^n)^{1-\beta}+(\phi_{i-1}^n)^{1-\beta})}{(|\overline{\sigma\lambda}|_i^p)^{\beta/p}}\right)\right] \\
&\quad - \mathbb{E}[\exp(-u_n^{\beta}C_{p,\beta}((\phi_i^n)^{1-\beta}+(\phi_{i-1}^n)^{1-\beta}))] \\
&= \mathbb{E}_{i-2}^n\left[f_{i,u_n}(\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}|\overline{\sigma\lambda}|_i^p) - f_{i,u_n}(\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}|\sigma_{\tau_{i-2}^n}|^p|\lambda_{\tau_{i-2}^n}|^{\frac{p}{\beta}-p})\right].
\end{aligned}$$

Using Taylor expansion we get for some ϵ_i between $\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}|\overline{\sigma\lambda}|_i^p$ and $\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}|\sigma_{\tau_{i-2}^n}|^p|\lambda_{\tau_{i-2}^n}|^{\frac{p}{\beta}-p}$ that

$$\begin{aligned}
&\mathbb{E}\left|R_4^n(u_n) - \sum_{i=k_n+3}^{N_n(1)}\tilde{r}_i^4\mathbb{E}_{i-2}^n\left[f'_{i,u_n}(\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}|\sigma_{\tau_{i-2}^n}|^p|\lambda_{\tau_{i-2}^n}|^{\frac{p}{\beta}-p})\right]\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}\right| \\
&\quad = \mathbb{E}\left|\sum_{i=k_n+3}^{N_n(1)}\frac{1}{2}(\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta})^2u_n^{\beta}(\tilde{r}_i^4)^2\mathbb{E}_{i-2}^n\left[\frac{f''_{i,u_n}(\epsilon_i)}{u_n^{\beta}}\right]\right| \leq Ku_n^{\beta}k_n,
\end{aligned}$$

proceeding as in (5.46) and using that with (5.24) we have

$$\mathbb{E}\left[\left(\tilde{r}_i^4\right)^2\right] \leq k_n\Delta_n$$

and furthermore that $\left|\mathbb{E}_{i-2}\left[\frac{f''_{i,u_n}(\epsilon_i)}{u_n^\beta}\right]\right|$ is bounded because of Lemma 5.6 and the boundedness of $|\sigma_t|^{-1}, \lambda_t^{-1}$. For (5.50) we note again that because $\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}|\sigma_{\tau_{i-2}^n}|^p|\lambda_{\tau_{i-2}^n}|^{\frac{p}{\beta}-p}$ is bounded from above and below, $\left|\mathbb{E}_{i-2}\left[f'_{i,u_n}(\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}|\sigma_{\tau_{i-2}^n}|^p|\lambda_{\tau_{i-2}^n}|^{\frac{p}{\beta}-p})\right]\right| \leq Ku_n^\beta$ by Lemma 5.6. Furthermore we obtain with (5.23)

$$\left|\mathbb{E}_{i-k_n-3}\left[\tilde{r}_i^4\right]\right| \leq Kk_n\Delta_n \tag{5.51}$$

and therefore (again proceeding as in (5.46)):

$$\begin{aligned}
&\mathbb{E}\left|\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}\sum_{i=k_n+3}^{N_n(1)}\mathbb{E}_{i-2}^n\left[f'_{i,u_n}(\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}|\sigma_{\tau_{i-2}^n}|^p|\lambda_{\tau_{i-2}^n}|^{\frac{p}{\beta}-p})\right]\mathbb{E}_{i-k_n-3}^n\left[\tilde{r}_i^4\right]\right|\\
&\mathbb{E}\left[\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}\sum_{i=k_n+3}^{N_n(1)}\left|\mathbb{E}_{i-2}^n\left[f'_{i,u_n}(\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}|\sigma_{\tau_{i-2}^n}|^p|\lambda_{\tau_{i-2}^n}|^{\frac{p}{\beta}-p})\right]\right|\left|\mathbb{E}_{i-k_n-3}^n\left[\tilde{r}_i^4\right]\right|\right]\\
&\leq Ku_n^\beta k_n.
\end{aligned}$$

For (5.49) it remains to prove similar rates of convergence for the sum of differences $\Xi_i = \tilde{r}_i^4 - \mathbb{E}_{i-k_n-3}[\tilde{r}_i^4]$. Applying (5.24) to $\tilde{r}_i^4$ we have

$$\mathbb{E}\left|\tilde{r}_i^4\right|^q \leq K(k_n\Delta_n)^{q/2\wedge 1} \text{ for all } q \in [0,2]$$

and therefore with (5.51)

$$\mathbb{E}\left|\Xi_i\right|^q \leq K(k_n\Delta_n)^{q/2\wedge 1} \text{ for all } q \in [0,2]. \tag{5.52}$$

To get the right order of convergence we have to apply (5.52) with $q = 2$. We achieve this by applying the BDG inequality. As Ξ_i is the difference of $\tilde{r}_i^4$ and its conditional expectation with respect to $\mathcal{F}_{\tau_{i-k_n-3}^n}^n$, the sums over Ξ_i spaced by k_n+1 steps are discrete martingales, meaning

$$\mathbb{E}_{(j-1)+(l-1)(k_n+1)}^n\left[\sum_{i=1}^{l}\Xi_{k_n+3+(j-1)+(i-1)(k_n+1)}\right] = \sum_{i=1}^{l-1}\Xi_{k_n+3+(j-1)+(i-1)(k_n+1)},$$
$$\text{for all } j = 1,\dots,k_n+1 \text{ and } l = 1,\dots,\lfloor(N_n(1)-k_n-2)/(k_n+1)\rfloor.$$

Therefore our discrete martingales are

$$A_j = \sum_{i=1}^{\lfloor(N_n(1)-k_n-2)/(k_n+1)\rfloor}\Xi_{k_n+3+(j-1)+(i-1)(k_n+1)}, \qquad j = 1,\dots,k_n+1.$$

Because these objects have $\lfloor (N_n(1)-k_n-2)/(k_n+1)\rfloor \leq \lfloor N_n(1)/k_n\rfloor \leq \lfloor Cn/k_n\rfloor$ elements we get with (5.52) and BDG inequality

$$\begin{aligned}\mathbb{E}\,|A_j| &\leq K\mathbb{E}\left|\sum_{i=1}^{\lfloor (N_n(1)-k_n-2)/(k_n+1)\rfloor} \left(\Xi_{k_n+3+(j-1)+(i-1)(k_n+1)}\right)^2\right|^{1/2} \\ &\leq K\left(\lfloor Cn/k_n\rfloor (\,k_n\Delta_n)\right)^{1/2} \leq K.\end{aligned}$$

Because $k_n\Delta_n \to 0$ we may assume $N_n(1)$ to be large in enough in relation to k_n, i.e. $N_n(1) \geq 2k_n+3$ (which also yields $\lfloor (N_n(1)-k_n-2)/(k_n+1)\rfloor \geq 1$), such that we can decompose the whole sum of the Ξ_i as

$$\sum_{i=k_n+3}^{N_n(1)} \Xi_i = \sum_{j=1}^{k_n+1} A_j + \sum_{i=2k_n+4+(\lfloor (N_n(1)-k_n-2)/(k_n+1)\rfloor -1)(k_n+1)}^{N_n(1)} \Xi_i, \tag{5.53}$$

As the second sum in (5.53) has at most k_n elements we have that

$$\frac{1}{n-k_n-2}\mathbb{E}\left|\sum_{i=k_n+3}^{N_n(1)} \Xi_i\right| \leq K(\Delta_n k_n).$$

□

Lemma 5.17. *Let $k_n \asymp n^{\varpi}$ with $\varpi \in (0,1)$ and $u_n \asymp n^{-\varrho}$ for $\varrho \in (0,1)$, then it holds for $0 < p < \beta/2, \iota > 0$,*

$$\frac{1}{n-k_n-2}\mathbb{E}|Z^n(u_n) - \overline{Z}^n(u_n)| \leq K\left(k_n^{-\beta/2p+\iota} \vee \Delta_n^{1/2} u_n^{\beta/2-\iota}\left(k_n^{-1/2} \vee \alpha_n \vee (k_n\Delta_n)^{1/2}\right)^{1/2}\right). \tag{5.54}$$

Proof. Like before we decompose $Z^n(u_n) - \overline{Z}^n(u_n) = E_1^n(u_n) + E_2^n(u_n)$ with $E_1^n(u_n) = \sum_{i=k_n+1}^{N_n(1)}(z_i(u_n) - \overline{z}_i(u_n))\mathbb{1}_{\mathcal{C}_i^n}$, $E_2^n(u_n) = \sum_{i=k_n+1}^{N_n(1)}(z_i(u_n) - \overline{z}_i(u_n))\mathbb{1}_{(\mathcal{C}_i^n)^c}$. As usual $x \mapsto \cos(x)$ and $x \mapsto \exp(-x)$ are bounded functions and therefore with (5.42) it holds that

$$\frac{1}{n-k_n-2}\mathbb{E}\left[|E_1^n(u_n)|\right] \leq K\mathbb{P}(\mathcal{C}_i^n) \leq Kk_n^{-\beta/2p+\iota}.$$

Recalling the notation of $L(p,u,\beta)$ we have

$$L(p,u_n,\beta) = \mathbb{E}_{i-2}^n\left[\exp(-u_n^\beta C_{p,\beta}((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})\right].$$

Using the inequalities (5.11), (5.12), $2xy \leq x^2+y^2$ for $x, y \in \mathbb{R}_+, p \in (0,2]$, $\cos(x) =$

$\cos(|x|)$ and the line above we get for arbitrarily small $\iota > 0$

$$\begin{aligned}
&\mathbb{E}^n_{i-2}\left[|(z_i(u_n) - \overline{z}_i(u_n)\mathbb{1}_{(\mathcal{C}^n_i)^C}|^2\right] \\
&\leq 2\mathbb{E}^n_{i-2}\left[\left|\cos\left(u_n\frac{\sigma_{\tau^n_{i-2}}\left(\widetilde{\Delta^n_i S} - \left(\frac{\lambda_{\tau^n_{i-2}}}{\lambda_{\tau^n_{i-3}}}\right)^{\frac{1}{\beta}-1}\widetilde{\Delta^n_{i-1} S}\right)}{\widetilde{V}^n_i(p)^{1/p}}\right)\right.\right. \\
&\qquad\left.\left. - \cos\left(u_n\frac{\lambda^{1-1/\beta}_{\tau^n_{i-2}}\widetilde{\Delta^n_i S} - \lambda^{1-1/\beta}_{\tau^n_{i-3}}\widetilde{\Delta^n_{i-1} S}}{\Delta_n^{1/\beta}\mu^{1/\beta}_{p,\beta}\kappa^{1/\beta}_{p,\beta}}\right)\right|^2\mathbb{1}_{(\mathcal{C}^n_i)^c}\right] \\
&+ 2\mathbb{E}^n_{i-2}\left[\left|\mathbb{E}^n_{i-2}\left[\exp\left(-\frac{A_\beta u_n^\beta|\sigma_{\tau^n_{i-2}}|^\beta|\lambda_{\tau^n_{i-2}}|^{1-\beta}((\phi^n_i)^{1-\beta} + (\phi^n_{i-1})^{1-\beta})}{\Delta_n^{-1}\widetilde{V}^n_i(p)^{\beta/p}}\right)\right]\right.\right. \\
&\qquad\left.\left. - \mathbb{E}^n_{i-2}\left[\exp(-u_n^\beta C_{p,\beta}((\phi^n_i)^{1-\beta} + (\phi^n_{i-1})^{1-\beta})\right]\right|^2\mathbb{1}_{(\mathcal{C}^n_i)^C}\right] \\
&\leq K u_n^{\beta-\iota}\mathbb{E}^n_{i-2}\left[\left|\frac{|\sigma_{\tau^n_{i-2}}||\lambda_{\tau^n_{i-2}}|^{\frac{1}{\beta}-1}}{\widetilde{V}^n_i(p)^{1/p}} - \frac{1}{\Delta_n^{1/\beta}\mu^{1/\beta}_{p,\beta}\kappa^{1/\beta}_{p,\beta}}\right|^{\beta-\iota}\mathbb{1}_{(\mathcal{C}^n_i)^C}|\lambda^{1-1/\beta}_{\tau^n_{i-2}}\widetilde{\Delta^n_i S} - \lambda^{1-1/\beta}_{\tau^n_{i-3}}\widetilde{\Delta^n_{i-1} S}|^{\beta-\iota}\right] \\
&\qquad + K u_n^\beta\mathbb{E}^n_{i-2}\left|\left(\frac{A_\beta|\sigma_{\tau^n_{i-2}}|^\beta|\lambda_{\tau^n_{i-2}}|^{1-\beta}}{\Delta_n^{-1}\widetilde{V}^n_i(p)^{\beta/p}} - \frac{A_\beta}{\mu_{p,\beta}\kappa_{p,\beta}}\right)\mathbb{1}_{(\mathcal{C}^n_i)^C}|(\phi^n_i)^{1-\beta} + (\phi^n_{i-1})^{1-\beta}|\right| \qquad (5.55)
\end{aligned}$$

Then together with the fact that by Lemma 5.3

$$\Delta_n^{(-1/\beta)(\beta-\iota)}\mathbb{E}^n_{i-2}[|\lambda^{1-1/\beta}_{\tau^n_{i-2}}\widetilde{\Delta^n_i S} - \lambda^{1-1/\beta}_{\tau^n_{i-3}}\widetilde{\Delta^n_{i-1} S}|^{\beta-\iota}] = \kappa^{(\beta-\iota)/\beta}_{\beta-\iota,\beta}\mu^{(\beta-\iota)/\beta}_{\beta-\iota,\beta}, \qquad (5.56)$$

is a constant, $\mathbb{E}\left|(\phi^n_i)^{1-\beta} + (\phi^n_{i-1})^{1-\beta}\right| < \infty$ and the measurability of the other terms we have

$$\begin{aligned}
(5.55) \leq & K_\iota u_n^{\beta-\iota}\left|\frac{|\sigma_{\tau^n_{i-2}}||\lambda_{\tau^n_{i-2}}|^{\frac{1}{\beta}-1}\mu^{1/\beta}_{p,\beta}\kappa^{1/\beta}_{p,\beta} - \Delta_n^{-1/\beta}\widetilde{V}^n_i(p)^{1/p}}{\Delta_n^{-1/\beta}\widetilde{V}^n_i(p)^{1/p}\mu^{1/\beta}_{p,\beta}\kappa^{1/\beta}_{p,\beta}}\right|^{\beta-\iota}\mathbb{1}_{(\mathcal{C}^n_i)^C} \\
&+ K u_n^\beta\left|\frac{|\sigma_{\tau^n_{i-2}}|^\beta|\lambda_{\tau^n_{i-2}}|^{1-\beta}\mu_{p,\beta}\kappa_{p,\beta} - \Delta_n^{-1}\widetilde{V}^n_i(p)^{\beta/p}}{\Delta_n^{-1}\widetilde{V}^n_i(p)^{\beta/p}\mu_{p,\beta}\kappa_{p,\beta}}\right|\mathbb{1}_{(\mathcal{C}^n_i)^C}.
\end{aligned}$$

Like before we get the boundedness from above and below for $\Delta_n^{-p/\beta}\widetilde{V}^n_i$ via the conditions on the set $\mathcal{C}^n_i$ and for $|\sigma_{\tau^n_{i-2}}|, |\overline{\sigma\lambda}|^p_i$ by (SB). Using the inequality for $x, y \in \mathbb{R}_+$ and some $\epsilon \in [x, y]$

$$\begin{aligned}
|x^q - y^q| &= q|\epsilon^{q-1}|\,|x-y| \qquad (5.57) \\
&\leq \begin{cases} q|\min(x,y)^{q-1}||x-y|, & \text{if } q < 1 \\ q|\max(x,y)^{q-1}||x-y|, & \text{if } q \geq 1 \end{cases}
\end{aligned}$$

it follows that

$$\left|\frac{|\sigma_{\tau_{i-2}^n}||\lambda_{\tau_{i-2}^n}|^{\frac{1}{\beta}-1}\mu_{p,\beta}^{1/\beta}\kappa_{p,\beta}^{1/\beta}-\Delta_n^{-1/\beta}\widetilde{V}_i^n(p)^{1/p}}{\Delta_n^{-1/\beta}\widetilde{V}_i^n(p)^{1/p}\mu_{p,\beta}^{1/\beta}\kappa_{p,\beta}^{1/\beta}}\right|^{\beta-\iota}\mathbb{1}_{(\mathcal{C}_i^n)^C}$$
$$\leq K_\iota\left|\frac{|\sigma_{\tau_{i-2}^n}||\lambda_{\tau_{i-2}^n}|^{\frac{1}{\beta}-1}\mu_{p,\beta}^{1/\beta}\kappa_{p,\beta}^{1/\beta}-\Delta_n^{-1/\beta}\widetilde{V}_i^n(p)^{1/p}}{\left(|\overline{\sigma\lambda}|_i^p\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}\right)^{1/p}\mu_{p,\beta}^{1/\beta}\kappa_{p,\beta}^{1/\beta}}\right|^{\beta-\iota}\mathbb{1}_{(\mathcal{C}_i^n)^C}$$
$$\stackrel{(5.57)}{\leq} K_\iota\left|\frac{1}{p}\max\left(|\sigma_{\tau_{i-2}^n}|^p|\lambda_{\tau_{i-2}^n}|^{\frac{p}{\beta}-p}\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta},\Delta_n^{-p/\beta}\widetilde{V}_i^n(p)\right)^{1/p-1}\mathbb{1}_{(\mathcal{C}_i^n)^C}\right.$$
$$\left.\left|\Delta_n^{-p/\beta}\widetilde{V}_i^n(p)-|\sigma_{\tau_{i-2}^n}|^p|\lambda_{\tau_{i-2}^n}|^{\frac{p}{\beta}-p}\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}\right|\right|^{\beta-\iota}$$
$$\leq K_\iota\left|\Delta_n^{-p/\beta}\widetilde{V}_i^n(p)-|\sigma_{\tau_{i-2}^n}|^p|\lambda_{\tau_{i-2}^n}|^{\frac{p}{\beta}-p}\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}\right|^{\beta-\iota}$$

and likewise

$$\left|\frac{|\sigma_{\tau_{i-2}^n}|^\beta|\lambda_{\tau_{i-2}^n}|^{1-\beta}\mu_{p,\beta}\kappa_{p,\beta}-\Delta_n^{-1}\widetilde{V}_i^n(p)^{\beta/p}}{\Delta_n^{-1}\widetilde{V}_i^n(p)^{\beta/p}\mu_{p,\beta}\kappa_{p,\beta}}\right|\mathbb{1}_{(\mathcal{C}_i^n)^C}$$
$$\leq K\left|\Delta_n^{-p/\beta}\widetilde{V}_i^n(p)-|\sigma_{\tau_{i-2}^n}|^p|\lambda_{\tau_{i-2}^n}|^{\frac{p}{\beta}-p}\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}\right|.$$

Together we have

$$(5.55)\leq\mathbb{1}_{(\mathcal{C}_i^n)^C}u_n^{\beta-\iota}\left(K_\iota\left|\Delta_n^{-p/\beta}\widetilde{V}_i^n(p)-|\sigma_{\tau_{i-2}^n}|^p|\lambda_{\tau_{i-2}^n}|^{\frac{p}{\beta}-p}\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}\right|^{\beta-\iota}\right.$$
$$\left.+K\left|\Delta_n^{-p/\beta}\widetilde{V}_i^n(p)-|\sigma_{\tau_{i-2}^n}|^p|\lambda_{\tau_{i-2}^n}|^{\frac{p}{\beta}-p}\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}\right|\right).$$

Because $\beta-\iota>1$ (for ι chosen small enough) and we may, with a possibly modified version of $\mathcal{C}_i^n$, assume that on the set $(\mathcal{C}_i^n)^C$ it holds $|\Delta_n^{-p/\beta}\widetilde{V}_i^n(p)-|\sigma_{\tau_{i-2}^n}|^p|\lambda_{\tau_{i-2}^n}|^{\frac{p}{\beta}-p}\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}|\leq 1$ we have

$$\mathbb{E}_{i-2}^n\left[|(z_i(u_n)-\overline{z}_i(u_n)\mathbb{1}_{(\mathcal{C}_i^n)^C}|^2\right]\leq Ku_n^{\beta-\iota}\mathbb{E}_{i-2}\left|\Delta_n^{-p/\beta}\widetilde{V}_i^n(p)-|\sigma_{\tau_{i-2}^n}|^p|\lambda_{\tau_{i-2}^n}|^{\frac{p}{\beta}-p}\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}\right|. \tag{5.58}$$

In order to bound $E_2^n(u_n)$ we note that by Lemma 5.4 and using the same arguments for $z_i(u_n)$

$$\mathbb{E}_{i-2}^n\left[z_i(u_n)\right]=0=\mathbb{E}_{i-2}^n\left[\overline{z}_i(u_n)\right].$$

As a result it holds that for all $i,j\in\mathbb{N}$ where $|i-j|\geq 2$ (assuming here w.l.o.g. $j>i$)

$$\mathbb{E}\left[(z_i(u_n)-\overline{z}_i(u_n))(z_j(u_n)-\overline{z}_j(u_n))\mathbb{1}_{(\mathcal{C}_i^n)^C}\mathbb{1}_{(\mathcal{C}_j^n)^C}\mathbb{1}_{\{N_n(1)+3\geq j\}}\mathbb{1}_{\{N_n(1)+3\geq i\}}\right]$$
$$=\mathbb{E}\left[\mathbb{1}_{(\mathcal{C}_i^n)^C}\mathbb{1}_{(\mathcal{C}_j^n)^C}\mathbb{1}_{\{N_n(1)+3\geq j\}}\mathbb{1}_{\{N_n(1)+3\geq i\}}(z_i(u_n)-\overline{z}_i(u_n))\mathbb{E}_{j-2}^n\left[(z_j(u_n)-\overline{z}_j(u_n))\right]\right]=0,$$

using that $\mathbb{1}_{\{N_n(1)+3\geq j\}}$, $\mathbb{1}_{\{N_n(1)+3\geq i\}}$, $\mathbb{1}_{(\mathcal{C}_i^n)^C}$ and $\mathbb{1}_{(\mathcal{C}_j^n)^C}$ are $\mathcal{F}^n_{\tau^n_{j-2}}$-measurable. We note that due to the boundedness of $z_i(u_n)$ and $\overline{z}_i(u_n)$ we have that $\frac{1}{n-k_n-2}\left(\sum_{i=k_n+3}^{N_n(1)+3}(z_i(u_n)-\overline{z}_i(u_n))\mathbb{1}_{(\mathcal{C}_i^n)^c}\right)$ and $\frac{1}{n-k_n-2}E_2^n(u_n)$ are asymptotically equivalent. With the last line, $2|xy|\leq x^2+y^2$ and (5.58) we get that

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i=k_n+3}^{N_n(1)+3}(z_i(u_n)-\overline{z}_i(u_n))\mathbb{1}_{(\mathcal{C}_i^n)^c}\right)^2\right]\\
&=\mathbb{E}\left[\sum_{i=k_n+3}^{nC+3}\Big((z_i(u_n)-\overline{z}_i(u_n))\mathbb{1}_{(\mathcal{C}_i^n)^C}\mathbb{1}_{\{N_n(1)+3\geq i\}}\Big((z_i(u_n)-\overline{z}_i(u_n))\mathbb{1}_{(\mathcal{C}_i^n)^C}\mathbb{1}_{\{N_n(1)+3\geq i\}}\right]\\
&+(z_{i-1}(u_n)-\overline{z}_{i-1}(u_n))\mathbb{1}_{(\mathcal{C}_{i-1}^n)^C}\mathbb{1}_{\{N_n(1)+3\geq i-1\}}+(z_{i+1}(u_n)-\overline{z}_{i+1}(u_n))\mathbb{1}_{(\mathcal{C}_{i+1}^n)^C}\mathbb{1}_{\{N_n(1)+3\geq i+1\}}\Big)\Big)\\
&+\mathbb{E}\left[\sum_{\substack{k_n+3\leq i,j\leq nC+3\\|i-j|\geq 2}}(z_i(u_n)-\overline{z}_i(u_n))(z_j(u_n)-\overline{z}_j^1(u_n))\mathbb{1}_{(\mathcal{C}_i^n)^C}\mathbb{1}_{(\mathcal{C}_j^n)^C}\mathbb{1}_{\{N_n(1)+3\geq j\}}\mathbb{1}_{\{N_n(1)+3\geq i\}}\right]\\
&\leq\mathbb{E}\left[\sum_{i=k_n+3}^{nC+3}3(z_i(u_n)-\overline{z}_i(u_n))^2\mathbb{1}_{(\mathcal{C}_i^n)^C}\mathbb{1}_{\{N_n(1)+3\geq i\}}\right]\\
&\leq K\mathbb{E}\left[\sum_{i=k_n+3}^{nC+3}(z_i(u_n)-\overline{z}_i(u_n))^2\mathbb{1}_{(\mathcal{C}_i^n)^C}\right]\\
&\leq Ku_n^{\beta-\iota}\sum_{i=k_n+3}^{nC+3}\mathbb{E}\left[\left|\Delta_n^{-p/\beta}\widetilde{V}_i^n(p)-|\sigma_{\tau^n_{i-2}}|^p|\lambda_{\tau^n_{i-2}}|^{\frac{p}{\beta}-p}\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}\right|\right]\\
&\leq Ku_n^{\beta-\iota}\sum_{i=k_n+3}^{nC+3}\mathbb{E}\Big[\left|\Delta_n^{-p/\beta}\widetilde{V}_i^n(p)-\Delta_n^{-p/\beta}\overline{V}_i^n(p)\right|+\left|\Delta_n^{-p/\beta}\overline{V}_i^n(p)-|\overline{\sigma\lambda}|_i^p\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}\right|\\
&\qquad+\left||\sigma_{\tau^n_{i-2}}|^p|\lambda_{\tau^n_{i-2}}|^{\frac{p}{\beta}-p}\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}-|\overline{\sigma\lambda}|_i^p\mu_{p,\beta}^{p/\beta}\kappa_{p,\beta}^{p/\beta}\right|\Big]\\
&\leq Ku_n^{\beta-\iota}(nC-k_n+1)\left(k_n^{-1/2}\vee\alpha_n\vee\Delta_n^{1/2}\vee(k_n\Delta_n)^{1/2}\right),
\end{aligned}
$$

where the last line results from (5.38), (5.45) and (5.24). Using this we get

$$
\begin{aligned}
&\frac{1}{n-k_n-2}\mathbb{E}\left[\sum_{i=k_n+3}^{N_n(1)+3}(z_i(u_n)-\overline{z}_i(u_n))\mathbb{1}_{(\mathcal{C}_i^n)^c}\right]\\
&\leq\frac{1}{n-k_n-2}\mathbb{E}\left[\left(\sum_{i=k_n+3}^{N_n(1)+3}(z_i(u_n)-\overline{z}_i(u_n))\mathbb{1}_{(\mathcal{C}_i^n)^c}\right)^2\right]^{1/2}\\
&\leq Ku_n^{\beta/2-\iota}\frac{(nC-k_n+1)^{1/2}}{n-k_n-2}\left(k_n^{-1/2}\vee\alpha_n\vee(k_n\Delta_n)^{1/2}\right)^{1/2},
\end{aligned}
$$

which yields the result. □

Corollary 5.3. *As a result of the previous Lemmas 5.13, 5.14, 5.15, 5.16 and 5.17, we have that if* $u_n \asymp n^{-\varrho}$ *with a* $\varrho \in (0,1)$ *and* $k_n \asymp n^{\varpi}, \varpi \in (0,1)$ *under the conditions*

$$\beta' < \frac{\beta}{2}, \quad \frac{1}{3} \vee \frac{1}{8\varrho} < p < \frac{\beta}{2}, \quad \varpi \geq \frac{2}{3}, \quad \frac{1}{3\beta} < \varrho < \frac{1}{\beta} \text{ and in addition} \tag{5.59}$$

$$\frac{1}{\beta} < \frac{\varpi}{p} - \varrho, \tag{5.60}$$

$$2\varpi - \varrho\beta < 1 \tag{5.61}$$

it holds:

$$\frac{\sqrt{n}}{u_n^{\beta/2}} \frac{1}{n-k_n-2} |Z^n(u_n) - \overline{Z}^n(u_n)| \xrightarrow{\mathbb{P}} 0, \tag{5.62}$$

$$\frac{\sqrt{n}}{u_n^{\beta/2}} \frac{1}{n-k_n-2} |R_1^n(u_n)| \xrightarrow{\mathbb{P}} 0, \tag{5.63}$$

$$\frac{\sqrt{n}}{u_n^{\beta/2}} \frac{1}{n-k_n-2} |R_2^n(u_n)| \xrightarrow{\mathbb{P}} 0, \tag{5.64}$$

$$\frac{\sqrt{n}}{u_n^{\beta/2}} \frac{1}{n-k_n-2} |R_3^n(u_n)| \xrightarrow{\mathbb{P}} 0, \tag{5.65}$$

$$\frac{\sqrt{n}}{u_n^{\beta/2}} \frac{1}{n-k_n-2} |R_4^n(u_n)| \xrightarrow{\mathbb{P}} 0. \tag{5.66}$$

Proof. In order to apply Lemmas 5.13, 5.14, 5.15 and 5.17 we need that

$$\frac{\sqrt{n}}{u_n^{\beta/2}} K_\iota k_n^{-\beta/2p+\iota} \asymp n^{1/2} n^{\varrho\beta/2} n^{-\varpi(\beta/2p+\iota)}$$

goes to zero. Because we can choose an arbitrarily small but fixed $\iota > 0$ the dependency of K_ι is irrelvant and the last line is implied by

$$\frac{1}{2} + \varrho\frac{\beta}{2} - \varpi(\frac{\beta}{2p} - \iota) < 0 \Leftarrow \frac{1}{2} + \varrho\frac{\beta}{2} - \varpi\frac{\beta}{2p} < 0 \Leftrightarrow (5.60).$$

Furthermore we have to show that for Lemma 5.13

$$\frac{\sqrt{n}}{u_n^{\beta/2}} u_n^{\beta'} \Delta_n^{\frac{\beta-\beta'}{\beta}} \to 0, \tag{5.67}$$

$$\frac{\sqrt{n}}{u_n^{\beta/2}} u_n \Delta_n^{1/2-\iota} \to 0. \tag{5.68}$$

The last condition is immediately fulfilled due to $\beta/2 < 1$ and due to the same reason the condition of Lemma 5.14

$$\frac{\sqrt{n}}{u_n^{\beta/2}} u_n \Delta_n^{1/2} \to 0$$

is fulfilled likewise. For Lemma 5.15 we split up $\alpha_n = \Delta_n^{\frac{\beta}{2}\frac{p+1}{\beta+1}\wedge(\frac{p}{\beta'}\wedge 1-\frac{p}{\beta})\wedge\frac{1}{2}-\iota}$ into

$$\sqrt{n}u_n^{\beta/2}\Delta_n^{\frac{\beta}{2}\frac{p+1}{\beta+1}-\iota} \to 0, \tag{5.69}$$

$$\sqrt{n}u_n^{\beta/2}\Delta_n^{(\frac{p}{\beta'}\wedge 1-\frac{p}{\beta})-\iota} \to 0, \tag{5.70}$$

$$\sqrt{n}u_n^{\beta/2}\Delta_n^{\frac{1}{2}-\iota} \to 0, \tag{5.71}$$

$$\sqrt{n}u_n^{\beta/2}k_n^{-1/2} \to 0, \tag{5.72}$$

and finally for Lemma 5.16

$$\sqrt{n}u_n^{\beta/2}\Delta_n k_n \to 0. \tag{5.73}$$

The second condition for Lemma 5.17

$$\frac{\sqrt{n}}{u_n^{\beta/2}}\Delta_n^{1/2}u_n^{\beta/2-\iota}\left(k_n^{-1/2} \vee \alpha_n^{\beta} \vee (k_n\Delta_n)^{\beta/2}\right)^{1/2} \to 0 \tag{5.74}$$

is fulfilled because ι can be chosen arbitrarily small and therefore (remember $\alpha_n = \Delta_n^{\frac{\beta}{2}\frac{p+1}{\beta+1}\wedge((\frac{p}{\beta'}\wedge 1)-\frac{p}{\beta})\wedge\frac{1}{2}-\iota}$)

$$u_n^{-\iota}\left(k_n^{-1/2} \vee \alpha_n^{\beta} \vee (k_n\Delta_n)^{\beta/2}\right)^{1/2} \to 0.$$

Starting with (5.67)

$$(5.67) \Leftarrow \frac{1}{2} + \varrho\left(\frac{\beta}{2} - \beta'\right) - \frac{\beta-\beta'}{\beta} < 0 \Leftrightarrow \frac{1}{2} + \varrho\left(\frac{\beta}{2} - \beta'\right) - \frac{1}{\beta}\left(\frac{\beta}{2} - \beta'\right) - \frac{1}{2} < 0$$
$$\Leftrightarrow \left(\varrho - \frac{1}{\beta}\right)\left(\frac{\beta}{2} - \beta'\right) < 0$$

which is true by (5.59).
Continuing with (5.69) we have

$$(5.69) \Leftarrow 1 < \varrho\beta + \frac{\beta}{\beta+1}(p+1)$$

which is, because of $\varrho > \frac{1}{3\beta}$ and $\frac{\beta+1}{\beta} < 2$, fulfilled if

$$\frac{4}{3} < p+1 \Leftrightarrow p > \frac{1}{3}.$$

Continuing with the second term from α_n, in the case of $p \leq \beta'$

$$(5.70) \Leftarrow 1 < 2p\frac{\beta-\beta'}{\beta\beta'} + \varrho\beta.$$

Because of $\beta' < \frac{\beta}{2}$ this is true if

$$1 < \frac{2p}{\beta} + \varrho\beta.$$

As the function $\beta \mapsto \frac{2p}{\beta} + \varrho\beta$ achieves its minimum at $(\beta^*)^2 = \frac{2p}{\varrho}$ this is true if

$$\beta^* < 2p + \varrho(\beta^*)^2 \Leftrightarrow \frac{2p}{\varrho} < 16p^2 \Leftrightarrow p > \frac{1}{8\varrho} \Leftarrow (5.59).$$

In the case of $p > \beta'$ (5.70) is true because

$$p < \frac{\beta}{2} \Leftrightarrow 1 - \frac{p}{\beta} > \frac{1}{2}.$$

(5.71) is trivial because, for ι small enough, $\Delta_n^{-\iota} u_n^{\beta/2} \to 0$. For (5.72) we have

$$(5.72) \Leftrightarrow \frac{1}{2} - \varrho\frac{\beta}{2} - \varpi\frac{1}{2} < 0 \Leftrightarrow 1 < \varrho\beta + \varpi$$

which is fulfilled if

$$1 \leq \frac{1}{3} + \varpi \Leftrightarrow \varpi \geq \frac{2}{3} \Leftarrow (5.59)$$

as $\varrho > \frac{1}{3\beta}$ by (5.59). For the last condition to hold true we have

$$(5.73) \Leftarrow \frac{1}{2} - \varrho\frac{\beta}{2} - (1 - \varpi) < 0 \Leftrightarrow (5.61).$$

□

Remark 5.2. *The above choice of parameters ϱ, ϖ and p is feasible if we do not know β. One possible choice for example could be $\varrho = \frac{1}{3}, \varpi = \frac{2}{3}$ and any $p \in (\frac{3}{8}, \frac{1}{2})$.*

5.2.3 Limiting Behavior of $\overline{Z}^n$

Lemma 5.18. *Let $u_n \asymp n^{-\varrho}$, $\varrho \in (0, 1/2)$ and $v_n = \rho u_n$ with $0 < \rho \leq 1$ then it holds that for all $2 < i < n$ when $n \to \infty$*

$$\frac{1}{u_n^{\beta/2} v_n^{\beta/2}} \mathbb{E}\left[\overline{z}_i(u_n)\overline{z}_i(v_n)\right] \to C_{p,\beta}\kappa_{\beta,\beta} \frac{2 + 2\rho^\beta - (1-\rho)^\beta - (1+\rho)^\beta}{2\rho^{\beta/2}},$$
$$\frac{1}{u_n^{\beta/2} v_n^{\beta/2}} \mathbb{E}\left[\overline{z}_i(u_n)\overline{z}_{i-1}(v_n)\right] \to C_{p,\beta}\kappa_{\beta,\beta} \frac{2 + 2\rho^\beta - (1-\rho)^\beta - (1+\rho)^\beta}{4\rho^{\beta/2}}$$

and if $\rho \geq 1$

$$\frac{1}{u_n^{\beta/2} v_n^{\beta/2}} \mathbb{E}\left[\overline{z}_i(u_n)\overline{z}_i(v_n)\right] \to C_{p,\beta}\kappa_{\beta,\beta} \frac{2 + 2\rho^\beta - (\rho-1)^\beta - (1+\rho)^\beta}{2\rho^{\beta/2}},$$
$$\frac{1}{u_n^{\beta/2} v_n^{\beta/2}} \mathbb{E}\left[\overline{z}_i(u_n)\overline{z}_{i-1}(v_n)\right] \to C_{p,\beta}\kappa_{\beta,\beta} \frac{2 + 2\rho^\beta - (\rho-1)^\beta - (1+\rho)^\beta}{4\rho^{\beta/2}},$$

with the same results when we exchange positions of u_n, v_n.

Proof. Throughout the proof we assume $0 < \rho \leq 1$ and discuss the case $\rho \geq 1$ in the end. Recalling the definition of $\overline{z}_i(u_n)$ we have

$$\overline{z}_i(u_n)\overline{z}_i(v_n) = \left(\cos\left(u_n \frac{\lambda_{\tau_{i-2}^n}^{-1/\beta+1}\widetilde{\Delta_i^n S} - \lambda_{\tau_{i-3}^n}^{-1/\beta+1}\widetilde{\Delta_{i-1}^n S}}{\Delta_n^{1/\beta}\mu_{p,\beta}^{1/\beta}\kappa_{p,\beta}^{1/\beta}}\right) - L(p,u_n,\beta)\right)$$
$$\left(\cos\left(v_n \frac{\lambda_{\tau_{i-2}^n}^{-1/\beta+1}\widetilde{\Delta_i^n S} - \lambda_{\tau_{i-3}^n}^{-1/\beta+1}\widetilde{\Delta_{i-1}^n S}}{\Delta_n^{1/\beta}\mu_{p,\beta}^{1/\beta}\kappa_{p,\beta}^{1/\beta}}\right) - L(p,v_n,\beta)\right).$$

Using the shorthand notation $\widehat{\Delta_i^n S} = \lambda_{\tau_{i-2}^n}^{-1/\beta+1}\widetilde{\Delta_i^n S}$ and the equality $\cos(x)\cos(y) = \frac{1}{2}\left(\cos(x-y) + \cos(x+y)\right)$ we have that conditionally on $\mathcal{F}_{\tau_{i-2}^n}$

$$\cos\left(u_n \frac{\widehat{\Delta_i^n S} - \widehat{\Delta_{i-1}^n S}}{\Delta_n^{1/\beta}\mu_{p,\beta}^{1/\beta}\kappa_{p,\beta}^{1/\beta}}\right)\cos\left(v_n \frac{\widehat{\Delta_i^n S} - \widehat{\Delta_{i-1}^n S}}{\Delta_n^{1/\beta}\mu_{p,\beta}^{1/\beta}\kappa_{p,\beta}^{1/\beta}}\right)$$
$$= \frac{1}{2}\left(\cos\left(\frac{(u_n - v_n)\widehat{\Delta_i^n S} + (-u_n + v_n)\widehat{\Delta_{i-1}^n S}}{\Delta_n^{1/\beta}\mu_{p,\beta}^{1/\beta}\kappa_{p,\beta}^{1/\beta}}\right) + \cos\left(\frac{(u_n + v_n)\widehat{\Delta_i^n S} + (-u_n - v_n)\widehat{\Delta_{i-1}^n S}}{\Delta_n^{1/\beta}\mu_{p,\beta}^{1/\beta}\kappa_{p,\beta}^{1/\beta}}\right)\right),$$

$$\cos\left(u_n \frac{\widehat{\Delta_i^n S} - \widehat{\Delta_{i-1}^n S}}{\Delta_n^{1/\beta}\mu_{p,\beta}^{1/\beta}\kappa_{p,\beta}^{1/\beta}}\right)\cos\left(v_n \frac{\widehat{\Delta_{i-1}^n S} - \widehat{\Delta_{i-2}^n S}}{\Delta_n^{1/\beta}\mu_{p,\beta}^{1/\beta}\kappa_{p,\beta}^{1/\beta}}\right)$$
$$= \frac{1}{2}\left(\cos\left(\frac{u_n\widehat{\Delta_i^n S} + (-u_n - v_n)\widehat{\Delta_{i-1}^n S} + v_n\widehat{\Delta_{i-2}^n S}}{\Delta_n^{1/\beta}\mu_{p,\beta}^{1/\beta}\kappa_{p,\beta}^{1/\beta}}\right)\right.$$
$$\left. + \cos\left(\frac{u_n\widehat{\Delta_i^n S} + (-u_n + v_n)\widehat{\Delta_{i-1}^n S} - v_n\widehat{\Delta_{i-2}^n S}}{\Delta_n^{1/\beta}\mu_{p,\beta}^{1/\beta}\kappa_{p,\beta}^{1/\beta}}\right)\right).$$

With the same arguments and notation as in Lemma 5.3 and $0 < \rho \leq 1$, conditionally on $\mathcal{F}_{\tau_{i-2}^n}$ we can calculate the exact distributions of the random variables above, namely:

$$\begin{aligned}
&(u_n - v_n)\Delta_n^{-1/\beta}\widehat{\Delta_i^n S} + (-u_n + v_n)\Delta_n^{-1/\beta}\widehat{\Delta_{i-1}^n S} \\
&\quad\sim u_n(1-\rho)\lambda_{\tau_{i-2}^n}^{-1/\beta+1}((\phi_i^n\lambda_{\tau_{i-2}^n})^{1-\beta})^{1/\beta}S_1' + u_n(\rho-1)\lambda_{\tau_{i-3}^n}^{-1/\beta+1}(((\phi_{i-1}^n\lambda_{\tau_{i-3}^n})^{1-\beta})^{1/\beta}S_1'' \\
&\quad\sim \left(u_n^\beta(1-\rho)^\beta((\phi_i^n)^{1-\beta}) + u_n^\beta(1-\rho)^\beta((\phi_{i-1}^n)^{1-\beta})\right)^{1/\beta} S_1', \\
&\quad\sim u_n(1-\rho)S_1'((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})^{1/\beta} \qquad (5.75)
\end{aligned}$$

and in the same manner, again conditionally on $\mathcal{F}_{\tau_{i-2}^n}$

$$\begin{aligned}
&(u_n + v_n)\Delta_n^{-1/\beta}\widehat{\Delta_i^n S} + (-u_n - v_n)\Delta_n^{-1/\beta}\widehat{\Delta_{i-1}^n S} \sim u_n(1+\rho)S_1'((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})^{1/\beta} \\
&u_n\Delta_n^{-1/\beta}\widehat{\Delta_i^n S} + (-u_n - v_n)\Delta_n^{-1/\beta}\widehat{\Delta_{i-1}^n S} + v_n\Delta_n^{-1/\beta}\widehat{\Delta_{i-2}^n S} \\
&\qquad\sim u_n S_1'((\phi_i^n)^{1-\beta} + (1+\rho)^\beta(\phi_{i-1}^n)^{1-\beta} + \rho^\beta(\phi_{i-2}^n)^{1-\beta})^{1/\beta}, \\
&u_n\Delta_n^{-1/\beta}\widehat{\Delta_i^n S} + (-u_n + v_n)\Delta_n^{-1/\beta}\widehat{\Delta_{i-1}^n S} - v_n\Delta_n^{-1/\beta}\widehat{\Delta_{i-2}^n S} \\
&\qquad\sim u_n S_1'((\phi_i^n)^{1-\beta} + (1-\rho)^\beta(\phi_{i-1}^n)^{1-\beta} + \rho^\beta(\phi_{i-2}^n)^{1-\beta})^{1/\beta}. \qquad (5.76)
\end{aligned}$$

In the previous calculations we can see that exchanging roles of u_n and v_n is irrelevant to the distributions as only the absolute value of the factors in front of $\widehat{\Delta_i^n S}, \widehat{\Delta_{i-1}^n S}$ and $\widehat{\Delta_{i-2}^n S}$ is relevant. Using these distributions and the equalities on cos from above we can explicitly compute the expectations using the result of Lemma 5.4 and its proof

$$
\begin{aligned}
\frac{1}{u_n^{\beta/2} v_n^{\beta/2}} \mathbb{E}\left[\overline{z}_i(u_n)\overline{z}_i(v_n)\right] = \frac{1}{u_n^{\beta/2} v_n^{\beta/2}} \mathbb{E}\Bigg[& \frac{1}{2}\cos\left(\frac{u_n(1-\rho)S_1'((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})^{1/\beta}}{\mu_{p,\beta}^{1/\beta}\kappa_{p,\beta}^{1/\beta}}\right) + \\
& \frac{1}{2}\cos\left(\frac{u_n(1+\rho)S_1'((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})^{1/\beta}}{\mu_{p,\beta}^{1/\beta}\kappa_{p,\beta}^{1/\beta}}\right) - L(p,u_n,\beta)L(p,v_n,\beta)\Bigg] \\
= \frac{1}{2u_n^{\beta}\rho^{\beta/2}} \mathbb{E}\Big[& \exp\left(-C_{p,\beta}u_n^{\beta}(1-\rho)^{\beta}((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})\right) \\
& + \exp\left(-C_{p,\beta}u_n^{\beta}(1+\rho)^{\beta}((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})\right)\Big] \\
& - \frac{1}{u_n^{\beta}\rho^{\beta/2}}\mathbb{E}[\exp(-u_n^{\beta}C_{p,\beta}((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta}))]\mathbb{E}[\exp(-u_n^{\beta}\rho^{\beta}C_{p,\beta}((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta}))],
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{u_n^{\beta/2} v_n^{\beta/2}} \mathbb{E}\left[\overline{z}_i(u_n)\overline{z}_{i-1}(v_n)\right] \\
& = \frac{1}{u_n^{\beta/2} v_n^{\beta/2}} \mathbb{E}\Bigg[\frac{1}{2}\cos\left(\frac{u_n S_1'((\phi_i^n)^{1-\beta} + (1+\rho)^{\beta}(\phi_{i-1}^n)^{1-\beta} + \rho^{\beta}(\phi_{i-2}^n)^{1-\beta})^{1/\beta}}{\mu_{p,\beta}^{1/\beta}\kappa_{p,\beta}^{1/\beta}}\right) \\
& \quad + \frac{1}{2}\cos\left(\frac{u_n S_1'((\phi_i^n)^{1-\beta} + (1-\rho)^{\beta}(\phi_{i-1}^n)^{1-\beta} + \rho^{\beta}(\phi_{i-2}^n)^{1-\beta})^{1/\beta}}{\mu_{p,\beta}^{1/\beta}\kappa_{p,\beta}^{1/\beta}}\right) - L(p,u_n,\beta)L(p,v_n,\beta)\Bigg] \\
& = \frac{1}{2u_n^{\beta/2} v_n^{\beta/2}} \mathbb{E}\Big[\exp\left(-C_{p,\beta}u_n^{\beta}((\phi_i^n)^{1-\beta} + (1+\rho)^{\beta}(\phi_{i-1}^n)^{1-\beta} + \rho^{\beta}(\phi_{i-2}^n)^{1-\beta})\right) \\
& \quad + \exp\left(-C_{p,\beta}u_n^{\beta}((\phi_i^n)^{1-\beta} + (1-\rho)^{\beta}(\phi_{i-1}^n)^{1-\beta} + \rho^{\beta}(\phi_{i-2}^n)^{1-\beta})\right)\Big] \\
& \quad - \frac{1}{u_n^{\beta/2} v_n^{\beta/2}}\mathbb{E}[\exp(-u_n^{\beta}C_{p,\beta}((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta}))]\mathbb{E}[\exp(-u_n^{\beta}\rho^{\beta}C_{p,\beta}((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta}))].
\end{aligned}
$$

With $\epsilon_{1,i} \in [0, u_n^{\beta}(1-\rho)^{\beta}C_{p,\beta}((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})], \epsilon_{2,i} \in [0, u_n^{\beta}(1+\rho)^{\beta}C_{p,\beta}((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})]$,
$\epsilon_{3,i} \in [0, u_n^{\beta}C_{p,\beta}((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})]$ and $\epsilon_{4,i} \in [0, u_n^{\beta}\rho^{\beta}C_{p,\beta}((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})]$ we have

that

$$\frac{1}{u_n^{\beta/2}v_n^{\beta/2}}\mathbb{E}\left[\overline{z}_i(u_n)\overline{z}_i(v_n)\right] = \frac{-C_{p,\beta}u_n^{\beta}(1-\rho)^{\beta}\mathbb{E}\left[\exp(-\epsilon_{1,i})((\phi_i^n)^{1-\beta}+(\phi_{i-1}^n)^{1-\beta})\right]+1}{2u_n^{\beta}\rho^{\beta/2}}$$
$$+\frac{-C_{p,\beta}u_n^{\beta}(1+\rho)^{\beta}\mathbb{E}\left[\exp(-\epsilon_{2,i})((\phi_i^n)^{1-\beta}+(\phi_{i-1}^n)^{1-\beta})\right]+1}{2u_n^{\beta}\rho^{\beta/2}}$$
$$-\frac{2\left(-u_n^{\beta}C_{p,\beta}\mathbb{E}\left[\exp(-\epsilon_{3,i})((\phi_i^n)^{1-\beta}+(\phi_{i-1}^n)^{1-\beta})\right](-v_n^{\beta}C_{p,\beta})\mathbb{E}\left[\exp(-\epsilon_{4,i})((\phi_i^n)^{1-\beta}+(\phi_{i-1}^n)^{1-\beta})\right]\right)}{2u_n^{\beta}\rho^{\beta/2}}$$
$$-\frac{2\left(-u_n^{\beta}C_{p,\beta}\mathbb{E}\left[\exp(-\epsilon_{3,i})((\phi_i^n)^{1-\beta}+(\phi_{i-1}^n)^{1-\beta})\right]-u_n^{\beta}\rho^{\beta}C_{p,\beta}\mathbb{E}\left[\exp(-\epsilon_{4,i})((\phi_i^n)^{1-\beta}+(\phi_{i-1}^n)^{1-\beta})\right]\right)}{2u_n^{\beta}\rho^{\beta/2}}$$
$$-\frac{2}{2u_n^{\beta}\rho^{\beta/2}}.$$

Because $\mathbb{E}\left[(\phi_i^n)^{1-\beta}\right], \mathbb{E}\left[(\phi_i^n)^{1-\beta}\right], \mathbb{E}\left[(\phi_i^n)^{1-\beta}\right] < M$ for some constant M we have that for any $\epsilon > 0$

$$\mathbb{P}\left(u_n^{\beta}(\phi_i^n)^{1-\beta} > \epsilon\right) = \mathbb{P}\left((\phi_i^n)^{1-\beta} > \frac{\epsilon}{u_n^{\beta}}\right)$$
$$\leq u_n^{\beta}\frac{\mathbb{E}\left[(\phi_i^n)^{1-\beta}\right]}{\epsilon} \to 0 \text{ when } u_n \to 0$$

and therefore with a similar result on $u_n^{\beta}(\phi_{i-1}^n)^{1-\beta}$:

$$\epsilon_{1,i},\ldots,\epsilon_{4,i} \xrightarrow{\mathbb{P}} 0 \text{ when } u_n \to 0.$$

Because $x \mapsto \exp(-x)$ is bounded by 1 for $x \in \mathbb{R}_+$ and $\epsilon_{1,i},\ldots,\epsilon_{4,i} \geq 0$ we find that with dominated convergence

$$\mathbb{E}\left[\exp(-\epsilon_l)((\phi_i^n)^{1-\beta}+(\phi_{i-1}^n)^{1-\beta})\right] \to \mathbb{E}\left[((\phi_i^n)^{1-\beta}+(\phi_{i-1}^n)^{1-\beta})\right] = \kappa_{\beta,\beta}$$

when $u_n \to 0$ for $l \in 1,\ldots,4$. As a result we have

$$\frac{2\left(-u_n^{\beta}C_{p,\beta}\mathbb{E}\left[\exp(-\epsilon_{3,i})((\phi_i^n)^{1-\beta}+(\phi_{i-1}^n)^{1-\beta})\right](-v_n^{\beta}C_{p,\beta})\mathbb{E}\left[\exp(-\epsilon_{4,i})((\phi_i^n)^{1-\beta}+(\phi_{i-1}^n)^{1-\beta})\right]\right)}{2u_n^{\beta}\rho^{\beta/2}} \to 0$$

and therefore

$$\frac{1}{u_n^{\beta/2}v_n^{\beta/2}}\mathbb{E}\left[\overline{z}_i(u_n)\overline{z}_i(v_n)\right]$$
$$\to \frac{-C_{p,\beta}(1-\rho)^{\beta}\kappa_{\beta,\beta}+1-C_{p,\beta}(1+\rho)^{\beta}\kappa_{\beta,\beta}+1-2\left(-C_{p,\beta}\kappa_{\beta,\beta}-\rho^{\beta}C_{p,\beta}\kappa_{\beta,\beta}+1\right)}{2\rho^{\beta/2}}$$
$$= C_{p,\beta}\kappa_{\beta,\beta}\frac{2+2\rho^{\beta}-(1-\rho)^{\beta}-(1+\rho)^{\beta}}{2\rho^{\beta/2}}.$$

We treat the second term alike and have with $\epsilon_{1,i} \in [0, u_n^\beta C_{p,\beta}((\phi_i^n)^{1-\beta}+(1+\rho)^\beta(\phi_{i-1}^n)^{1-\beta}+\rho^\beta(\phi_{i-2}^n)^{1-\beta})]$, $\epsilon_{2,i} \in [0, u_n^\beta C_{p,\beta}((\phi_i^n)^{1-\beta} + (1-\rho)^\beta(\phi_{i-1}^n)^{1-\beta} + \rho^\beta(\phi_{i-2}^n)^{1-\beta})]$, $\epsilon_{3,i} \in [0, u_n^\beta C_{p,\beta}((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})]$ and $\epsilon_{4,i} \in [0, u_n^\beta \rho^\beta C_{p,\beta}((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})]$ that

$$\begin{aligned}
&\frac{1}{u_n^{\beta/2} v_n^{\beta/2}} \mathbb{E}\left[\overline{z}_i(u_n)\overline{z}_{i-1}(v_n)\right] \\
&= \frac{-C_{p,\beta} u_n^\beta \mathbb{E}\left[\exp(-\epsilon_{1,i})((\phi_i^n)^{1-\beta} + (1+\rho)^\beta(\phi_{i-1}^n)^{1-\beta} + \rho^\beta(\phi_{i-2}^n)^{1-\beta})\right] + 1}{2u_n^\beta \rho^{\beta/2}} \\
&+ \frac{-C_{p,\beta} u_n^\beta \mathbb{E}\left[\exp(-\epsilon_{2,i})((\phi_i^n)^{1-\beta} + (1-\rho)^\beta(\phi_{i-1}^n)^{1-\beta} + \rho^\beta(\phi_{i-2}^n)^{1-\beta})\right] + 1}{2u_n^\beta \rho^{\beta/2}} \\
&- \frac{2\left(-u_n^\beta C_{p,\beta}\mathbb{E}\left[\exp(-\epsilon_{3,i})((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})\right](-v_n^\beta C_{p,\beta})\mathbb{E}\left[\exp(-\epsilon_{4,i})((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})\right]\right)}{2u_n^\beta \rho^{\beta/2}} \\
&- \frac{2\left(-u_n^\beta C_{p,\beta}\mathbb{E}\left[\exp(-\epsilon_{3,i})((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})\right]\right)}{2u_n^\beta \rho^{\beta/2}} \\
&- \frac{2\left(-u_n^\beta \rho^\beta C_{p,\beta}\mathbb{E}\left[\exp(-\epsilon_{4,i})((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})\right] + 1\right)}{2u_n^\beta \rho^{\beta/2}}.
\end{aligned}$$

Using the same arguments as above we have when $u_n \to 0$ (note: $\mathbb{E}\left[(\phi_i^n)^{1-\beta}\right]) = \frac{\kappa_{\beta,\beta}}{2}$)

$$\begin{aligned}
\mathbb{E}\left[\exp(-\epsilon_{1,i})((\phi_i^n)^{1-\beta} + (1+\rho)^\beta(\phi_{i-1}^n)^{1-\beta} + \rho^\beta(\phi_{i-2}^n)^{1-\beta})\right] &\to \frac{\kappa_{\beta,\beta}}{2}(1 + (1+\rho)^\beta + \rho^\beta) \\
\mathbb{E}\left[\exp(-\epsilon_{2,i})((\phi_i^n)^{1-\beta} + (1-\rho)^\beta(\phi_{i-1}^n)^{1-\beta} + \rho^\beta(\phi_{i-2}^n)^{1-\beta})\right] &\to \frac{\kappa_{\beta,\beta}}{2}(1 + (1-\rho)^\beta + \rho^\beta)
\end{aligned}$$

and as a result

$$\begin{aligned}
\frac{1}{u_n^{\beta/2} v_n^{\beta/2}} \mathbb{E}\left[\overline{z}_i(u_n)\overline{z}_{i-1}(v_n)\right] &\xrightarrow{\mathbb{P}} C_{p,\beta}\frac{\kappa_{\beta,\beta}}{2} \frac{4 + 4\rho^\beta - (1 + (1+\rho)^\beta + \rho^\beta) - (1 + (1-\rho)^\beta + \rho^\beta)}{2\rho^{\beta/2}} \\
&= C_{p,\beta}\kappa_{\beta,\beta} \frac{2 + 2\rho^\beta - (1-\rho)^\beta - (1+\rho)^\beta}{4\rho^{\beta/2}}.
\end{aligned}$$

We now discuss the case of $\rho \geq 1$ and see that the only lines where this condition is relevant are (5.75), (5.76). As in these calculations only the absolute value of $(1-\rho)$ is decisive, we only need to exchange $(1-\rho)^\beta$ for $(\rho-1)^\beta$ here and in all the subsequent calculations if $\rho \geq 1$. □

Lemma 5.19. *Let $u_n \asymp n^{-\varrho}$, $\varrho \in (0, 1/\beta)$ and $v_n = \rho u_n$ with $0 < \rho < 1$ then it holds that*

$$\left(\frac{\sqrt{n}}{u_n^{\beta/2}} \frac{1}{n - k_n - 2}\overline{Z}^n(u_n), \frac{\sqrt{n}}{v_n^{\beta/2}} \frac{1}{n - k_n - 2}\overline{Z}^n(v_n)\right) \xrightarrow{\mathcal{L}} (X, Y), \tag{5.77}$$

where X, Y are normal distributed random variables with mean 0 and covariance matrix $\mathcal{C}$ with

$$\begin{aligned}
\mathcal{C}_{11} = \mathcal{C}_{22} &= \int_0^1 \frac{1}{\lambda_s} ds \; C_{p,\beta}\kappa_{\beta,\beta}(4 - 2^\beta), \\
\mathcal{C}_{12} = \mathcal{C}_{21} &= \int_0^1 \frac{1}{\lambda_s} ds \; C_{p,\beta}\kappa_{\beta,\beta} \frac{2 + 2\rho^\beta - (1+\rho)^\beta - (1-\rho)^\beta}{\rho^{\beta/2}},
\end{aligned}$$

if $0 < \rho \leq 1$ *and*

$$\mathcal{C}_{11} = \mathcal{C}_{22} = \int_0^1 \frac{1}{\lambda_s} ds \; C_{p,\beta} \kappa_{\beta,\beta} (4 - 2^\beta),$$
$$\mathcal{C}_{12} = \mathcal{C}_{21} = \int_0^1 \frac{1}{\lambda_s} ds \; C_{p,\beta} \kappa_{\beta,\beta} \frac{2 + 2\rho^\beta - (1+\rho)^\beta - (\rho - 1)^\beta}{\rho^{\beta/2}},$$

if $\rho \geq 1$.

Proof. To prove the theorem we define

$$\zeta_i^n = \frac{\sqrt{n}}{u_n^{\beta/2}} \frac{1}{n} \left(\overline{z}_i(u_n), \overline{z}_i(v_n)\right)$$
$$= \frac{1}{\sqrt{n}} \left(\frac{1}{u_n^{\beta/2}} \left(\cos\left(u_n \frac{\widehat{\Delta_i^n S} - \widehat{\Delta_{i-1}^n S}}{\Delta_n^{1/\beta} \mu_{p,\beta}^{1/\beta} \kappa_{p,\beta}^{1/\beta}} \right) - L(p, u_n, \beta) \right), \frac{1}{v_n^{\beta/2}} \left(\cos\left(v_n \frac{\widehat{\Delta_i^n S} - \widehat{\Delta_{i-1}^n S}}{\Delta_n^{1/\beta} \mu_{p,\beta}^{1/\beta} \kappa_{p,\beta}^{1/\beta}} \right) - L(p, v_n, \beta) \right) \right).$$

Furthermore we decompose

$$\sum_{i=k_n+3}^{N_n(1)} \zeta_i^n = \sum_{i=k_n+3}^{N_n(1)} \left(\zeta_i^n - \mathbb{E}_{i-1}^n \left[\zeta_i^n\right]\right) + \sum_{i=k_n+2}^{N_n(1)-1} \mathbb{E}_i^n \left[\zeta_{i+1}^n\right]$$
$$= \left(\zeta_{N_n(1)}^n - \mathbb{E}_{N_n(1)-1}^n \left[\zeta_{N_n(1)}^n\right]\right) + \mathbb{E}_{k_n+2}^n \left[\zeta_{k_n+3}^n\right] + \sum_{i=k_n+3}^{N_n(1)-1} \left(\zeta_i^n - \mathbb{E}_{i-1}^n \left[\zeta_i^n\right] + \mathbb{E}_i^n \left[\zeta_{i+1}^n\right]\right).$$

Because $\overline{z}_i(u_n), \overline{z}_i(v_n)$ are bounded by the boundedness of $x \mapsto \cos(x)$ and $\mathbb{R}_+ \to \mathbb{R}_+, x \mapsto \exp(-x)$, and because of $n u_n^\beta \asymp n^{1-\beta\varrho} \to \infty$ we see that

$$\sum_{i=k_n+3}^{N_n(1)} \zeta_i^n \text{ and } \sum_{i=k_n+3}^{N_n(1)-1} \left(\zeta_i^n - \mathbb{E}_{i-1}^n \left[\zeta_i^n\right] + \mathbb{E}_i^n \left[\zeta_{i+1}^n\right]\right) \text{ and } \sum_{i=k_n+3}^{N_n(1)+1} \left(\zeta_i^n - \mathbb{E}_{i-1}^n \left[\zeta_i^n\right] + \mathbb{E}_i^n \left[\zeta_{i+1}^n\right]\right)$$

are asymptotically equivalent. We note that $N_n(1)+1$ is a $(\mathcal{F}_{\tau_i^n})_{i\geq 1}$-stopping time and therefore in order to apply Theorem 2.2.13 in [JP12] it is sufficient to show that for $q := \frac{2}{1-\varrho\beta} + 2 > 2$, $\eta_i^n := \zeta_i^n - \mathbb{E}_{i-1}^n \left[\zeta_i^n\right] + \mathbb{E}_i^n \left[\zeta_{i+1}^n\right]$

$$\sum_{i=k_n+3}^{N_n(1)+1} \mathbb{E}_{i-1}^n \left[\eta_i^n\right] \xrightarrow{\mathbb{P}} (0,0), \tag{5.78}$$
$$\sum_{i=k_n+3}^{N_n(1)+1} \left(\mathbb{E}_{i-1}^n \left[\eta_i^{n,j} \eta_i^{n,k}\right] - \mathbb{E}_{i-1}^n \left[\eta_i^{n,j}\right] \mathbb{E}_{i-1}^n \left[\eta_i^{n,k}\right]\right) \xrightarrow{\mathbb{P}} \mathcal{C}_{jk}, \tag{5.79}$$
$$\sum_{i=k_n+3}^{N_n(1)+1} \mathbb{E}_{i-1}^n \left[\|\zeta_i^n\|^q\right] \xrightarrow{\mathbb{P}} 0. \tag{5.80}$$

We first note that because of Lemma 5.4 we have

$$\mathbb{E}_{i-1}^n \left[\zeta_{i+1}^n\right] = (0,0) \tag{5.81}$$

and therefore

$$\mathbb{E}^n_{i-1}\left[\eta^n_i\right] = \mathbb{E}^n_{i-1}\left[\zeta^n_i - \mathbb{E}^n_{i-1}\left[\zeta^n_i\right] + \mathbb{E}^n_i\left[\zeta^n_{i+1}\right]\right] = (0,0) \tag{5.82}$$

and by this (5.78) holds.
By the boundedness of $\overline{z}_i(u_n), \overline{z}_i(v_n)$, $\varrho < \frac{1}{\beta}$, $N_n(1) \le Cn$ due to Assumption SC and

$$\varrho\beta\frac{1+1-\varrho\beta}{1-\varrho\beta} - \frac{1}{1-\varrho\beta} = \frac{-(1-\varrho\beta)^2}{1-\varrho\beta} < 0$$

we have

$$\sum_{i=k_n+3}^{N_n(1)+1} \mathbb{E}^n_{i-1}\left[\left|\frac{1}{\sqrt{nu_n^\beta}}\overline{z}_i(u_n)\right|^q\right] \le N_n(1)Ku_n^{-\beta\frac{1+1-\varrho\beta}{1-\varrho\beta}} n^{-\frac{1}{1-\varrho\beta}-1} \le Ku_n^{-\beta\frac{1+1-\varrho\beta}{1-\varrho\beta}} n^{-\frac{1}{1-\varrho\beta}} \longrightarrow 0,$$

which proves (5.80). To show (5.79) we first note that $\mathbb{E}^n_{i-1}\left[\eta^{n,j}_i\right] = 0$ and $\mathbb{E}^n_{i-1}\left[\eta^{n,k}_i\right] = 0$ due to (5.82) and furthermore make use of (5.81) to get

$$\begin{aligned}
&\mathbb{E}^n_{i-1}\left[\left(\zeta^{n,j}_i - \mathbb{E}^n_{i-1}\left[\zeta^{n,j}_i\right] + \mathbb{E}^n_i\left[\zeta^{n,j}_{i+1}\right]\right)\left(\zeta^{n,k}_i - \mathbb{E}^n_{i-1}\left[\zeta^{n,k}_i\right] + \mathbb{E}^n_i\left[\zeta^{n,k}_{i+1}\right]\right)\right] \\
&= \mathbb{E}^n_{i-1}\left[(\zeta^{n,j}_i)(\zeta^{n,k}_i) - \mathbb{E}^n_{i-1}\left[\zeta^{n,j}_i\right]\zeta^{n,k}_i - \mathbb{E}^n_{i-1}\left[\zeta^{n,k}_i\right]\zeta^{n,j}_i\right] \\
&\quad + \mathbb{E}^n_{i-1}\left[\zeta^{n,j}_i\right]\mathbb{E}^n_{i-1}\left[\zeta^{n,k}_i\right] + \mathbb{E}^n_{i-1}\left[\zeta^{n,j}_i\mathbb{E}^n_i\left[\zeta^{n,k}_{i+1}\right]\right] + \mathbb{E}^n_{i-1}\left[\zeta^{n,k}_i\mathbb{E}^n_i\left[\zeta^{n,j}_{i+1}\right]\right] \\
&\quad - \mathbb{E}^n_{i-1}\left[\mathbb{E}^n_{i-1}\left[\zeta^{n,j}_i\right]\mathbb{E}^n_i\left[\zeta^{n,k}_{i+1}\right]\right] - \mathbb{E}^n_{i-1}\left[\mathbb{E}^n_{i-1}\left[\zeta^{n,k}_i\right]\mathbb{E}^n_i\left[\zeta^{n,j}_{i+1}\right]\right] + \mathbb{E}^n_{i-1}\left[\mathbb{E}^n_i\left[\zeta^{n,j}_{i+1}\right]\mathbb{E}^n_i\left[\zeta^{n,k}_{i+1}\right]\right] \\
&= \mathbb{E}^n_{i-1}\left[\zeta^{n,j}_i\zeta^{n,k}_i\right] - \mathbb{E}^n_{i-1}\left[\zeta^{n,j}_i\right]\mathbb{E}^n_{i-1}\left[\zeta^{n,k}_i\right] + \mathbb{E}^n_{i-1}\left[\zeta^{n,j}_i\zeta^{n,k}_{i+1}\right] + \mathbb{E}^n_{i-1}\left[\zeta^{n,k}_i\zeta^{n,j}_{i+1}\right] \\
&\quad + \mathbb{E}\left[\mathbb{E}^n_i\left[\zeta^{n,j}_{i+1}\right]\mathbb{E}^n_i\left[\zeta^{n,k}_{i+1}\right]\right],
\end{aligned}$$

using in the last step that, by the appropriate scaling inside $\widehat{\Delta^n_i S} = \lambda_{\tau^n_{i-2}}^{-1/\beta+1}\widetilde{\Delta^n_i S}$, the distribution $\mathbb{E}^n_i\left[\zeta^{n,j}_{i+1}\right], \mathbb{E}^n_i\left[\zeta^{n,k}_{i+1}\right]$ is independent of $\mathcal{F}_{\tau^n_{i-1}}$(cf. (5.86)). We note that $\mathbb{E}^n_{i-1}\left[\zeta^{n,j}_i\zeta^{n,k}_i\right], \mathbb{E}^n_{i-1}\left[\zeta^{n,j}_i\right]\mathbb{E}^n_{i-1}\left[\zeta^{n,k}_i\right], \mathbb{E}^n_{i-1}\left[\zeta^{n,j}_i\zeta^{n,k}_{i+1}\right], \mathbb{E}^n_{i-1}\left[\zeta^{n,k}_i\zeta^{n,j}_{i+1}\right]$ build each on its own a triangular array of random variables. We want to show that

$$\sum_{i=k_n+3}^{N_n(1)} \mathbb{E}^n_{i-1}\left[\zeta^{n,j}_i\zeta^{n,k}_i\right] \xrightarrow{\mathbb{P}} \int_0^1 \frac{1}{\lambda_s}ds \lim_{n\to\infty} n\mathbb{E}\left[\zeta^{n,j}_i\zeta^{n,k}_i\right],$$

$$\sum_{i=k_n+3}^{N_n(1)} \mathbb{E}^n_{i-1}\left[\zeta^{n,j}_i\right]\mathbb{E}^n_{i-1}\left[\zeta^{n,k}_i\right] \xrightarrow{\mathbb{P}} \int_0^1 \frac{1}{\lambda_s}ds \lim_{n\to\infty} n\mathbb{E}\left[\mathbb{E}^n_{i-1}\left[\zeta^{n,j}_i\right]\mathbb{E}^n_{i-1}\left[\zeta^{n,k}_i\right]\right],$$

$$\sum_{i=k_n+3}^{N_n(1)} \mathbb{E}^n_{i-1}\left[\zeta^{n,j}_i\zeta^{n,k}_{i+1}\right] \xrightarrow{\mathbb{P}} \int_0^1 \frac{1}{\lambda_s}ds \lim_{n\to\infty} n\mathbb{E}\left[\zeta^{n,j}_i\zeta^{n,k}_{i+1}\right],$$

$$\sum_{i=k_n+3}^{N_n(1)} \mathbb{E}^n_{i-1}\left[\zeta^{n,k}_i\zeta^{n,j}_{i+1}\right] \xrightarrow{\mathbb{P}} \int_0^1 \frac{1}{\lambda_s}ds \lim_{n\to\infty} n\mathbb{E}\left[\zeta^{n,k}_i\zeta^{n,j}_{i+1}\right]$$

and that the limits right-hand side exist. This would result in

$$\sum_{i=k_n+3}^{N_n(1)} \left(\zeta_i^n - \mathbb{E}_{i-1}^n\left[\zeta_i^n\right] + \mathbb{E}_i^n\left[\zeta_{i+1}^n\right]\right)^T \left(\zeta_i^n - \mathbb{E}_{i-1}^n\left[\zeta_i^n\right] + \mathbb{E}_i^n\left[\zeta_{i+1}^n\right]\right) \tag{5.83}$$
$$\xrightarrow{\mathbb{P}} \int_0^1 \frac{1}{\lambda_s} ds \lim_{n\to\infty} n\left(\mathbb{E}\left[(\zeta_m^n)^T\zeta_m^n\right] + \mathbb{E}\left[(\zeta_m^n)^T\zeta_{m+1}^n\right] + \mathbb{E}\left[(\zeta_{m+1}^n)^T\zeta_m^n\right]\right),$$

everything for an arbitrary m.
In order to prove the results above we would like to make use of Lemma 2.2.12 in [JP12], setting $X_i^{n,1} = \mathbb{E}_i^n\left[(\zeta_{i+1}^n)^T(\zeta_{i+1}^n)\right]$, $X_i^{n,2} = \mathbb{E}_i^n\left[\zeta_{i+1}^n\right]^T \mathbb{E}_i^n\left[\zeta_{i+1}^n\right]$, $X_i^{n,3} = \mathbb{E}_i^n\left[(\zeta_{i+1}^n)^T\zeta_{i+2}^n\right]$ and $X_i^{n,4} = \mathbb{E}_i^n\left[(\zeta_{i+2}^n)^T\zeta_{i+1}^n\right]$ and show that for $l = 1,2,3,4$

$$\sum_{i=k_n+2}^{N_n(1)-1} \mathbb{E}_{i-1}^n\left[X_i^{n,l}\right] \xrightarrow{\mathbb{P}} \int_0^1 \frac{1}{\lambda_s} ds \lim_{n\to\infty} n\mathbb{E}\left[X_m^{n,l}\right]$$

and that the array $\left(\mathbb{E}_{i-1}^n\left[(X_i^{n,l})^2\right]\right)$ is asymptotically negligible meaning that

$$\sum_{i=k_n+2}^{N_n(1)-1} \mathbb{E}_{i-1}^n\left[\left(\left(X_i^{n,l}\right)_{jk}\right)^2\right] \xrightarrow{\mathbb{P}} 0 \text{ for } j,k = 1,2.$$

Noting that $N_n(1) - 1$ is not a $(\mathcal{F}_{\tau_i^n})_{i\geq 1}$-stopping time in general we again use asymptotic equivalence and show instead

$$\sum_{i=k_n+2}^{N_n(1)+1} \mathbb{E}_{i-1}^n\left[X_i^{n,l}\right] \xrightarrow{\mathbb{P}} \int_0^1 \frac{1}{\lambda_s} ds \lim_{n\to\infty} n\mathbb{E}\left[X_m^{n,l}\right], \tag{5.84}$$
$$\sum_{i=k_n+2}^{N_n(1)+1} \mathbb{E}_{i-1}^n\left[\left(\left(X_i^{n,l}\right)_{jk}\right)^2\right] \xrightarrow{\mathbb{P}} 0 \text{ for } j,k = 1,2. \tag{5.85}$$

To show (5.84) we note that the distribution of the $X_i^{n,l}$ does not depend on the process λ_t anymore but only on ϕ_i^n and the increments of the process S after τ_{i-1}^n which are independent of $\mathcal{F}_{\tau_{i-1}^n}$ and therefore

$$\mathbb{E}_{i-1}^n\left[X_i^{n,l}\right] = \mathbb{E}\left[X_i^{n,l}\right] \text{ and } \mathbb{E}_{i-1}^n\left[(X_i^{n,l})^2\right] = \mathbb{E}\left[(X_i^{n,l})^2\right]. \tag{5.86}$$

Using Lemma 5.18 we have the convergences for $j \neq k$ and $0 < \rho \leq 1$

$$n\mathbb{E}\left[\left((\zeta_m^n)^T\zeta_m^n\right)_{jk}\right] = n\mathbb{E}\left[\zeta_m^{n,j}\zeta_m^{n,k}\right] \to C_{p,\beta}\kappa_{\beta,\beta}\frac{2+2\rho^\beta-(1-\rho)^\beta-(1+\rho)^\beta}{2\rho^{\beta/2}}, \tag{5.87}$$
$$n\mathbb{E}\left[(\zeta_m^n\zeta_{m+1}^n)_{jk}\right] = n\mathbb{E}\left[\zeta_m^{n,j}\zeta_{m+1}^{n,k}\right] \to C_{p,\beta}\kappa_{\beta,\beta}\frac{2+2\rho^\beta-(1-\rho)^\beta-(1+\rho)^\beta}{4\rho^{\beta/2}} \tag{5.88}$$

and, as u_n, v_n can also be equal in Lemma 5.18, for $j = k$

$$n\mathbb{E}\left[\left((\zeta_m^n)^T\zeta_m^n\right)_{jk}\right] = n\mathbb{E}\left[\zeta_m^{n,j}\zeta_m^{n,k}\right] \to C_{p,\beta}\kappa_{\beta,\beta}\frac{4-2^\beta}{2},$$
$$n\mathbb{E}\left[(\zeta_m^n\zeta_{m+1}^n)_{jk}\right] = n\mathbb{E}\left[\zeta_m^{n,j}\zeta_{m+1}^{n,k}\right] \to C_{p,\beta}\kappa_{\beta,\beta}\frac{4-2^\beta}{4}.$$

With the previous lines and (4.11) we can prove (5.84):

$$\sum_{i=k_n+2}^{N_n(1)+1}\mathbb{E}_{i-1}^n\left[X_i^{n,l}\right] = n\mathbb{E}\left[X_i^{n,l}\right]\frac{1}{n}(N_n(1)+1)$$
$$\xrightarrow{\mathbb{P}} \int_0^1 \frac{1}{\lambda_s}ds \lim_{n\to\infty} n\mathbb{E}\left[X_m^{n,l}\right].$$

Using Jensen inequality we have

$$\begin{aligned}
\mathbb{E}\left[((X_i^{n,1})_{jk})^2\right] &= \mathbb{E}\left[\left(\mathbb{E}_i^n\left[\zeta_{i+1}^{n,j}\zeta_{i+1}^{n,k}\right]\right)^2\right] \leq \mathbb{E}\left[\mathbb{E}_i^n\left[\left(\zeta_{i+1}^{n,j}\right)^2\left(\zeta_{i+1}^{n,k}\right)^2\right]\right]\\
&\leq \mathbb{E}\left[\mathbb{E}_i^n\left[\left(\zeta_{i+1}^{n,j}\right)^2\right]\right]\left\|\left(\zeta_{i+1}^{n,k}\right)^2\right\|_\infty,\\
\mathbb{E}\left[((X_i^{n,2})_{jk})^2\right] &= \mathbb{E}\left[\left(\mathbb{E}_i^n\left[\zeta_{i+1}^{n,j}\right]\mathbb{E}_i^n\left[\zeta_{i+1}^{n,k}\right]\right)^2\right] = \mathbb{E}\left[\left(\mathbb{E}_i^n\left[\zeta_{i+1}^{n,j}\right]\right)^2\left(\mathbb{E}_i^n\left[\zeta_{i+1}^{n,k}\right]\right)^2\right]\\
&\leq \mathbb{E}\left[\left(\mathbb{E}_i^n\left[\zeta_{i+1}^{n,j}\right]\right)^2\right]\left\|\left(\mathbb{E}_i^n\left[\zeta_{i+1}^{n,k}\right]\right)^2\right\|_\infty\\
&\leq \mathbb{E}\left[\mathbb{E}_i^n\left[\left(\zeta_{i+1}^{n,j}\right)^2\right]\right]\left\|\left(\mathbb{E}_i^n\left[\zeta_{i+1}^{n,k}\right]\right)^2\right\|_\infty,\\
\mathbb{E}\left[((X_i^{n,3})_{jk})^2\right] &= \mathbb{E}\left[\left(\mathbb{E}_i^n\left[\zeta_{i+1}^{n,j}\zeta_{i+2}^{n,k}\right]\right)^2\right] \leq \mathbb{E}\left[\mathbb{E}_i^n\left[\left(\zeta_{i+1}^{n,j}\right)^2\left(\zeta_{i+2}^{n,k}\right)^2\right]\right]\\
&\leq \mathbb{E}\left[\mathbb{E}_i^n\left[\left(\zeta_{i+1}^{n,j}\right)^2\right]\right]\left\|\left(\zeta_{i+2}^{n,k}\right)^2\right\|_\infty,\\
\mathbb{E}\left[((X_i^{n,4})_{jk})^2\right] &= \mathbb{E}\left[\left(\mathbb{E}_i^n\left[\zeta_{i+2}^{n,j}\zeta_{i+1}^{n,k}\right]\right)^2\right] \leq \mathbb{E}\left[\mathbb{E}_i^n\left[\left(\zeta_{i+2}^{n,j}\right)^2\right]\right]\left\|\left(\zeta_{i+1}^{n,k}\right)^2\right\|_\infty.
\end{aligned}$$

Again using the fact that $N_n(1) \leq Cn$ and $\overline{z}_i(u_n), \overline{z}_i(v_n)$ are bounded we can conclude that for $j,k = 1,2$

$$\sum_{i=k_n+2}^{N_n(1)+1}\mathbb{E}\left[((X_i^{n,1})_{jk})^2\right] \leq K\sum_{i=k_n+2}^{N_n(1)+1}\frac{1}{nu_n^\beta}\mathbb{E}\left[\left(\zeta_{i+1}^{n,j}\right)^2\right]$$
$$\leq K(N_n(1)+1)\frac{1}{nu_n^\beta}\mathbb{E}\left[\left(\zeta_{i+1}^{n,j}\right)^2\right] \xrightarrow{n\to\infty} 0,$$

using (5.87) and $nu_n^\beta \to \infty$ in the last step. Proceeding likewise for the sums with $((X_i^{n,2})_{j,k})^2, ((X_i^{n,3})_{j,k})^2, ((X_i^{n,4})_{j,k})^2$ yields a similar result.

Finally combining (5.83) with (5.87) and (5.88) we have for $0 < \rho \leq 1$

$$\begin{aligned}\mathcal{C}_{12} = \mathcal{C}_{21} &= \int_0^1 \frac{1}{\lambda_s} ds \; C_{p,\beta}\kappa_{\beta,\beta} \frac{2 + 2\rho^\beta - (1-\rho)^\beta - (1+\rho)^\beta}{2\rho^{\beta/2}} \\ &+ \int_0^1 \frac{1}{\lambda_s} ds \; 2C_{p,\beta}\kappa_{\beta,\beta} \frac{2 + 2\rho^\beta - (1-\rho)^\beta - (1+\rho)^\beta}{4\rho^{\beta/2}} \\ &= \int_0^1 \frac{1}{\lambda_s} ds \; C_{p,\beta}\kappa_{\beta,\beta} \frac{2 + 2\rho^\beta - (1-\rho)^\beta - (1+\rho)^\beta}{\rho^{\beta/2}}.\end{aligned}$$

The calculations for $\mathcal{C}_{11}$ and $\mathcal{C}_{22}$ follow along the same lines. In both cases the results change only by setting ρ to one. The case of $\rho \geq 1$ works similar, only exchanging the results in (5.87) and (5.88) according to Lemma 5.18. □

Theorem 5.1. *Under the same conditions and for the same variables X, Y as in Lemma 5.19 we have that*

$$\left(\frac{\sqrt{n}}{u_n^{\beta/2}} \frac{1}{n - k_n - 2} \overline{Z}^n(u_n), \frac{\sqrt{n}}{v_n^{\beta/2}} \frac{1}{n - k_n - 2} \overline{Z}^n(v_n) \right) \xrightarrow{\mathcal{L}-s} (X, Y), \tag{5.89}$$

meaning that the convergence is not only in law, but stably in law. As a result we have that

$$\sqrt{\frac{n}{N_n(1)}} \left(\frac{\sqrt{n}}{u_n^{\beta/2}} \frac{1}{n - k_n - 2} \overline{Z}^n(u_n), \frac{\sqrt{n}}{v_n^{\beta/2}} \frac{1}{n - k_n - 2} \overline{Z}^n(v_n) \right) \xrightarrow{\mathcal{L}-s} (X', Y'), \tag{5.90}$$

where X', Y' are normal distributed random variables with mean 0 and covariance matrix $\mathcal{C}'$ where $\mathcal{C}'_{11} = \mathcal{C}'_{22} = C_{p,\beta}\kappa_{\beta,\beta}(4 - 2^\beta)$ and $\mathcal{C}'_{12} = \mathcal{C}'_{21} = C_{p,\beta}\kappa_{\beta,\beta}\frac{2+2\rho^\beta-(1+\rho)^\beta-(1-\rho)^\beta}{\rho^{\beta/2}}$ if $0 < \rho \leq 1$ and $\mathcal{C}'_{12} = \mathcal{C}'_{21} = C_{p,\beta}\kappa_{\beta,\beta}\frac{2+2\rho^\beta-(1+\rho)^\beta-(\rho-1)^\beta}{\rho^{\beta/2}}$ if $\rho \geq 1$.
Furthermore under the conditions of Corollary 5.3 we have for the estimator $\widetilde{L}^n(p, u)$:

$$\left(\frac{\sqrt{N_n(1)}}{u_n^{\beta/2}} (\widetilde{L}^n(p, u_n) - L(p, u_n, \beta)), \frac{\sqrt{N_n(1)}}{v_n^{\beta/2}} (\widetilde{L}^n(p, v_n) - L(p, v_n, \beta)) \right) \xrightarrow{\mathcal{L}-s} (X', Y'). \tag{5.91}$$

Proof. Using Theorem 2.2.15 in [JP12] we have to show that in addition to Lemma 5.19 it holds that

$$\sum_{i=k_n+3}^{N_n(1)} \mathbb{E}_{i-1}^n \left[\zeta_i^n (M_{\tau_i} - M_{\tau_{i-1}}) \right] \xrightarrow{\mathbb{P}} 0 \tag{5.92}$$

wherever M is either one of the Brownian motions $W, \tilde{W}$ or a bounded martingale orthogonal to W.
Following the idea of the proof of Lemma 26 in [JT18] it is sufficient to show the line above only for M equal to a continuous bounded martingale or $M = W, M = \tilde{W}$. To

prove (5.92) we use Theorem 4.34 in Chapter III of [JS87]. We set for $k_n+3 \leq i \leq N_n(1)$ and $t \geq \tau_{i-2}^n$:

$$\mathcal{H} := \mathcal{F}_{\tau_{i-2}^n} \qquad \text{and} \qquad \mathcal{H}_t := \mathcal{H} \bigvee \sigma\left(S_r : r \geq \tau_{i-2}^n\right),$$

i.e. $(\mathcal{H}_t)_{t\geq\tau_{i-2}^n}$ is the filtration generated by $\mathcal{H}$ and $\sigma\left(S_r : r \geq \tau_{i-2}^n\right)$. Now $(S_t)_{t\geq\tau_{i-2}^n}$ is a process with independent increments w.r.t. to $\sigma\left(S_r : r \geq \tau_{i-2}^n\right)$. For all $t \geq \tau_{i-2}^n$ we set $K_t := \mathbb{E}\left[\zeta_i|\mathcal{H}_t\right]$ and note that $K_{\tau_i^n} = \zeta_i$ due to ζ_i being $\mathcal{H}_{\tau_i^n}$-measurable. Then with the aforementioned Theorem 4.34 we have

$$\zeta_i = K_{\tau_i^n} = K_{\tau_{i-2}^n} + \int_{\tau_{i-2}}^{\tau_i} H_s dS_s,$$

where $(H_t)_{t\geq\tau_{i-2}^n}$ is a predictable process. Then

$$\begin{aligned}\mathbb{E}_{i-1}^n\left[\zeta_i^n(M_{\tau_i} - M_{\tau_{i-1}})\right] &= \left(K_{\tau_{i-2}^n} + \int_{\tau_{i-2}}^{\tau_{i-1}} H_s dS_s\right)\mathbb{E}_{i-1}^n\left[M_{\tau_i} - M_{\tau_{i-1}}\right] \\ &\quad + \mathbb{E}_{i-1}^n\left[\int_{\tau_{i-1}}^{\tau_i} H_s dS_s (M_{\tau_i} - M_{\tau_{i-1}})\right] \\ &= 0,\end{aligned}$$

where we used that the martingale $(S_t)_{t\geq 0}$ is orthogonal to M in all cases.

We remember that $\frac{N_n(1)}{n} \xrightarrow{\mathbb{P}} \int_0^1 \frac{1}{\lambda_s} ds$ and as such

$$\sqrt{\frac{n}{N_n(1)}} \xrightarrow{\mathbb{P}} \left(\int_0^1 \frac{1}{\lambda_s} ds\right)^{-1/2}, \tag{5.93}$$

which gives us (5.90). For the statement (5.91) concerning the actual estimator $\widetilde{L}^n(p, u_n)$, we remember that by Corollary 5.3 we have in addition to (5.89) for $i = 1, 2, 3, 4$

$$\frac{\sqrt{n}}{u_n^{\beta/2}} \frac{1}{n - k_n - 2} |R_i^n(u_n)| \xrightarrow{\mathbb{P}} 0, \tag{5.94}$$

$$\frac{\sqrt{n}}{u_n^{\beta/2}} \frac{1}{n - k_n - 2} |Z^n(u_n) - \overline{Z}^n(u_n)| \xrightarrow{\mathbb{P}} 0. \tag{5.95}$$

which gives us in addition to (5.90) with (5.93)

$$\sqrt{\frac{n}{N_n(1)}} \frac{\sqrt{n}}{u_n^{\beta/2}} \frac{1}{n - k_n - 2} |R_i^n(u_n)| \xrightarrow{\mathbb{P}} 0, \tag{5.96}$$

$$\sqrt{\frac{n}{N_n(1)}} \frac{\sqrt{n}}{u_n^{\beta/2}} \frac{1}{n - k_n - 2} |Z^n(u_n) - \overline{Z}^n(u_n)| \xrightarrow{\mathbb{P}} 0. \tag{5.97}$$

Calculating

$$\frac{\sqrt{N_n(1)}}{u_n^{\beta/2}(N_n(1)-k_n-2)} - \frac{n}{u_n^{\beta/2}\sqrt{N_n(1)}(n-k_n-2)}$$
$$= \frac{(k_n+2)(n-N_n(1))}{u_n^{\beta/2}(N_n(1)-k_n-2)\sqrt{n}(n-k_n-2)}$$
$$= \frac{(k_n+2)}{(N_n(1)-k_n-2)}\frac{n-N_n(1)}{n}\frac{\sqrt{n}}{u_n^{\beta/2}}\frac{1}{n-k_n-2}$$

and noting that $\frac{(k_n-2)}{(N_n(1)-k_n-2)} \xrightarrow{\mathbb{P}} 0$, $|\frac{n-N_n(1)}{n}| < K$ we have with (5.89),(5.94),(5.95) that

$$\frac{\sqrt{N_n(1)}}{u_n^{\beta/2}}\frac{1}{N_n(1)-k_n-2}|R_i^n(u_n)| - \sqrt{\frac{n}{N_n(1)}}\frac{\sqrt{n}}{u_n^{\beta/2}}\frac{1}{n-k_n-2}|R_i^n(u_n)| \xrightarrow{\mathbb{P}} 0,$$
$$\frac{\sqrt{N_n(1)}}{u_n^{\beta/2}}\frac{1}{N_n(1)-k_n-2}|Z^n(u_n)-\overline{Z}^n(u_n)| - \sqrt{\frac{n}{N_n(1)}}\frac{\sqrt{n}}{u_n^{\beta/2}}\frac{1}{n-k_n-2}|Z^n(u_n)-\overline{Z}^n(u_n)| \xrightarrow{\mathbb{P}} 0,$$
$$\frac{\sqrt{N_n(1)}}{u_n^{\beta/2}}\frac{1}{N_n(1)-k_n-2}\overline{Z}^n(u_n) - \sqrt{\frac{n}{N_n(1)}}\frac{\sqrt{n}}{u_n^{\beta/2}}\frac{1}{n-k_n-2}\overline{Z}^n(u_n) \xrightarrow{\mathbb{P}} 0.$$

This results by (5.96),(5.97) and (5.90) in

$$\frac{\sqrt{N_n(1)}}{u_n^{\beta/2}}\frac{1}{N_n(1)-k_n-2}|R_i^n(u_n)| \xrightarrow{\mathbb{P}} 0,$$
$$\frac{\sqrt{N_n(1)}}{u_n^{\beta/2}}\frac{1}{N_n(1)-k_n-2}|Z^n(u_n)-\overline{Z}^n(u_n)| \xrightarrow{\mathbb{P}} 0,$$
$$\frac{\sqrt{N_n(1)}}{u_n^{\beta/2}}\frac{1}{N_n(1)-k_n-2}\overline{Z}^n(u_n) \xrightarrow{\mathcal{L}-s} X'$$

and finally

$$\frac{\sqrt{N_n(1)}}{u_n^{\beta/2}}(\widetilde{L}^n(p,u_n) - L(p,u_n,\beta))$$
$$= \frac{\sqrt{N_n(1)}}{u_n^{\beta/2}}\frac{1}{N_n(1)-k_n-2}\sum_{i=k_n+3}^{N_n(1)}[R_1^n + R_2^n + Z^n + R_3^n + R_4^n] \xrightarrow{\mathcal{L}-s} X'.$$

Applying similar calculations to the tuple
$\left(\frac{\sqrt{N_n(1)}}{u_n^{\beta/2}}(\widetilde{L}^n(p,u_n) - L(p,u_n,\beta)), \frac{\sqrt{N_n(1)}}{v_n^{\beta/2}}(\widetilde{L}^n(p,v_n) - L(p,v_n,\beta))\right)$ we get (5.91). □

5.3 A Central Limit Theorem for the Estimator of β

Theorem 5.2. *Under the conditions of Corollary* 5.3, $\varrho < 1/\beta$ *and* $v_n = \rho u_n$ *we have for the estimator of* β

$$\hat{\beta}(p,u_n,v_n) = \frac{\log(-(\widetilde{L}^n(p,u_n)-1)) - \log(-(\widetilde{L}^n(p,v_n)-1))}{\log(u_n/v_n)} \tag{5.98}$$

that the convergence

$$u_n^{\beta/2}\sqrt{N_n(1)}(\hat{\beta}(p,u_n,v_n)-\beta) \xrightarrow{\mathcal{L}-s} X \tag{5.99}$$

holds, where X is a normal distributed random variable with mean 0 and variance

$$\frac{(\rho^\beta+1)(4-2^\beta)-2(2+2\rho^\beta-(1+\rho)^\beta-(1-\rho)^\beta)}{\kappa_{\beta,\beta}\rho^\beta\log(1/\rho)^2C_{p,\beta}}, \quad \text{if} \quad 0<\rho<1$$

and

$$\frac{(\rho^\beta+1)(4-2^\beta)-2(2+2\rho^\beta-(1+\rho)^\beta-(\rho-1)^\beta)}{\kappa_{\beta,\beta}\rho^\beta\log(1/\rho)^2C_{p,\beta}}, \quad \text{if} \quad \rho>1.$$

Proof. Using a two dimensional Taylor expansion of the function

$$(x,y)\mapsto \frac{\log(-(x-1))-\log(-(y-1))}{\log(u_n/v_n)}$$

with gradient

$$(g_1(x),g_2(y))=\left(\frac{1}{\log(u_n/v_n)(x-1)},\frac{1}{\log(u_n/v_n)(1-y)}\right)$$

around the point $(L(p,u_n,\beta),L(p,v_n,\beta))$ it holds that

$$u_n^{\beta/2}\sqrt{N_n(1)}(\hat{\beta}(p,u_n,v_n)-\beta)=$$

$$u_n^{\beta/2}\sqrt{N_n(1)}\left(\frac{\log(-(L(p,u_n,\beta)-1))-\log(-(L(p,v_n,\beta)-1))}{\log(u_n/v_n)}-\beta\right) \tag{5.100}$$

$$+\frac{1}{\log(u_n/v_n)}\frac{u_n^\beta}{\mathbb{E}[\exp(-u_n^\beta C_{p,\beta}((\phi_i^n)^{1-\beta}+(\phi_{i-1}^n)^{1-\beta})]-1}\frac{\sqrt{N_n(1)}}{u_n^{\beta/2}}(\widetilde{L}^n(p,u_n)-L(p,u_n,\beta)) \tag{5.101}$$

$$+\frac{1}{\log(u_n/v_n)}\frac{1}{\rho^{\beta/2}}\frac{v_n^\beta}{1-\mathbb{E}[\exp(-v_n^\beta C_{p,\beta}((\phi_i^n)^{1-\beta}+(\phi_{i-1}^n)^{1-\beta}))]}\frac{\sqrt{N_n(1)}}{v_n^{\beta/2}}(\widetilde{L}^n(p,v_n)-L(p,v_n,\beta)) \tag{5.102}$$

$$+u_n^\beta(g_1(\eta_1)-g_1(L(p,u_n,\beta)))\frac{\sqrt{N_n(1)}}{u_n^{\beta/2}}(\widetilde{L}^n(p,u_n)-L(p,u_n,\beta))$$

$$+v_n^\beta\frac{1}{\rho^{\beta/2}}(g_2(\eta_2)-g_2(L(p,v_n,\beta)))\frac{\sqrt{N_n(1)}}{v_n^{\beta/2}}(\widetilde{L}^n(p,v_n)-L(p,v_n,\beta)), \tag{5.103}$$

for some η_1 between $\widetilde{L}^n(p,u_n),L(p,u_n,\beta)$ and η_2 between $\widetilde{L}^n(p,v_n),L(p,v_n,\beta)$.
As $\widetilde{L}^n(p,u_n),\widetilde{L}^n(p,v_n)\in(-1,1)$ a.s. and g_1,g_2 are continuous on $(-1,1)$ we have that by (5.91) $g_1(\eta_1)\xrightarrow{\mathbb{P}}g_1(L(p,u_n,\beta))$ and $g_2(\eta_2)\xrightarrow{\mathbb{P}}g_2(L(p,v_n,\beta))$ and as a result together with the convergence in (5.91):

$$u_n^\beta(g_1(\eta_1)-g_1(L(p,u_n,\beta)))\frac{\sqrt{N_n(1)}}{u_n^{\beta/2}}(\widetilde{L}^n(p,u_n)-L(p,u_n,\beta))\xrightarrow{\mathbb{P}}0$$

$$v_n^\beta\frac{1}{\rho^{\beta/2}}(g_2(\eta_2)-g_2(L(p,v_n,\beta)))\frac{\sqrt{N_n(1)}}{v_n^{\beta/2}}(\widetilde{L}^n(p,v_n)-L(p,v_n,\beta))\xrightarrow{\mathbb{P}}0$$

We now prove the convergence of the bias term (5.100) towards zero.
With $\epsilon_{1,i} \in [0, C_{p,\beta}u_n^\beta((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})]$, $\epsilon_{2,i} \in [\mathbb{E}[\exp(-\epsilon_{1,i})((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})], \kappa_{\beta,\beta}]$ it holds that:

$$\mathbb{E}[\exp(-C_{p,\beta}u_n^\beta((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta}))] - 1 = \mathbb{E}[\exp(-\epsilon_{1,i})(-C_{p,\beta}u_n^\beta((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta}))]$$

and as such

$$\begin{aligned}\log(-(L(u_n,p,\beta)-1)) &= \log(u_n^\beta C_{p,\beta}) + \log(\mathbb{E}[\exp(-\epsilon_{1,i})((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})]) \\ &= \log(u_n^\beta C_{p,\beta}) + \frac{1}{\epsilon_{2,i}}(\mathbb{E}[\exp(-\epsilon_{1,i})((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})] - \kappa_{\beta,\beta}) \\ &\quad + \log(\kappa_{\beta,\beta}).\end{aligned}$$

As $\epsilon_{1,i} > 0$ we have with dominated convergence $\mathbb{E}[\exp(-\epsilon_{1,i})((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})] \to \kappa_{\beta,\beta}$ and as such $\epsilon_{2,i} \to \kappa_{\beta,\beta}$ for $u_n \to 0$. With $\epsilon_3 \in [0, \epsilon_{1,i}]$ it holds for $\iota > 0$

$$\begin{aligned}&\frac{1}{u_n^{\beta-\iota}\epsilon_{2,i}}\left(\mathbb{E}[\exp(-\epsilon_{1,i})((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})] - \kappa_{\beta,\beta}\right) \\ &= \frac{1}{u_n^{\beta-\iota}\epsilon_{2,i}}\left(\mathbb{E}[(\exp(-\epsilon_3)(-\epsilon_{1,i}) + 1)((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})] - \kappa_{\beta,\beta}\right) \\ &= \frac{1}{\epsilon_{2,i}}\mathbb{E}[\exp(-\epsilon_3)\frac{(-\epsilon_{1,i})}{u_n^{\beta-\iota}}((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})] \to 0,\end{aligned}$$

as $\epsilon_{1,i}((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta}) \le u_n^\beta C_{p,\beta}((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta})^2$ again with dominated convergence and Assumption C.3. The same arguments hold for $\log(-(L(v_n,p,\beta)-1))$, therefore we get:

$$\begin{aligned}&\frac{1}{u_n^{\beta-\iota}}\left(\frac{\log(-(L(p,u_n,\beta)-1)) - \log(-(L(p,v_n,\beta)-1))}{\log(u_n/v_n)} - \beta\right) \\ &= \frac{1}{u_n^{\beta-\iota}}\left(\frac{\log(u_n^\beta C_{p,\beta}) + \log(\kappa_{\beta,\beta}) - (\log(v_n^\beta C_{p,\beta}) + \log(\kappa_{\beta,\beta}))}{\log(u_n/v_n)} - \beta\right) + o_p(1) = 0 + o_p(1)\end{aligned}$$

and as a result, because $u_n^{\frac{3}{2}\beta-\iota}\sqrt{N_n(1)} \to 0$ due to $\frac{1}{3\beta} < \varrho$ under Corollary 5.3 and $N_n(1) \le Cn$,

$$(5.100) = u_n^{\frac{3}{2}\beta-\iota}\sqrt{N_n(1)}\frac{1}{u_n^{\beta-\iota}}\left(\frac{\log(-(L(p,u_n,\beta)-1)) - \log(-(L(p,v_n,\beta)-1))}{\log(u_n/v_n)} - \beta\right) = o_p(1).$$

Again using a Taylor expansion of $\exp(x)$ and $\mathbb{E}\left[(\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta}\right] = \kappa_{\beta,\beta}$ we have

$$\frac{u_n^\beta}{1 - \mathbb{E}\left[\exp(-u_n^\beta C_{p,\beta}((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta}))\right]} \to \frac{1}{\kappa_{\beta,\beta}C_{p,\beta}},$$

$$\frac{v_n^\beta}{\mathbb{E}\left[\exp(-v_n^\beta C_{p,\beta}((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta}))\right] - 1} \to -\frac{1}{\kappa_{\beta,\beta}C_{p,\beta}}$$

and therefore get for the variance of X by (5.91) if $0 < \rho < 1$:

$$\begin{aligned} Var(X) &= \frac{1}{(\log(1/\rho)\kappa_{\beta,\beta}C_{p,\beta})^2}C_{p,\beta}\kappa_{\beta,\beta}(4-2^\beta) + \frac{1}{(\rho^{\beta/2}\log(1/\rho)\kappa_{\beta,\beta}C_{p,\beta})^2}C_{p,\beta}\kappa_{\beta,\beta}(4-2^\beta) \\ &\quad - \frac{2}{\kappa_{\beta,\beta}^2 C_{p,\beta}^2 \log(1/\rho)^2\rho^{\beta/2}}C_{p,\beta}\kappa_{\beta,\beta}\frac{2+2\rho^\beta-(1+\rho)^\beta-(1-\rho)^\beta}{\rho^{\beta/2}} \\ &= \frac{(\rho^\beta+1)(4-2^\beta)-2(2+2\rho^\beta-(1+\rho)^\beta-(1-\rho)^\beta)}{\kappa_{\beta,\beta}\rho^\beta\log(1/\rho)^2 C_{p,\beta}} \end{aligned}$$

and the respective result in the case of $\rho > 1$. $\square$

Chapter 6

Numerical Assessment

In this chapter we use a numerical implementation of a setting fulfilling Assumptions A, B, C and an implementation of the estimator $\hat{\beta}(p, u_n, v_n)$ defined by (5.98) to gauge its finite sample quality. However, as the variance of the limiting object in Theorem 5.2 is dependent on the (probably unknown) $\kappa_{\beta,\beta}, C_{p,\beta}$ and β itself, it is not possible, apart from the consistency of $\hat{\beta}(p, u_n, v_n)$, to use Theorem 5.2 in applications, e.g. to construct confidence intervals. Therefore Section 6.4 deals with the problem of finding a CLT for $\hat{\beta}(p, u_n, v_n) - \beta$ where the limiting object is not determined by unknown variables.

6.1 Setting

For the underlying process X in (4.5) we define for all $t > 0$

$$\alpha_t = \int_0^t 2(1-\alpha_s)ds + 2\int_0^t dW_s,$$
$$\sigma_t = \int_0^t \alpha_s dW_s,$$
$$Y_t = 0$$

and

$$X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \sigma_{s-} dL_s,$$

with L being a symmetrical stable process, i.e. its Lévy measure is given via the density

$$h(x) = \frac{1}{|x|^{1+\beta}},$$

for some $\beta \in (1,2)$. For the observation scheme we assume

$$\lambda_t = \int_0^t (5 - \lambda_s) ds + \int_0^t d\widetilde{W}_s, \tag{6.1}$$

$$\phi = \frac{\phi' \vee 0.1}{\mathbb{E}\left[\phi' \vee 0.1\right]}, \qquad \text{where} \qquad \phi' \sim Exp(1), \tag{6.2}$$

with starting values of the processes being $\alpha_0 = \sigma_0 = X_0 = \lambda_0 = 1$. Again, $W, \widetilde{W}$ are two independent Brownian motions. The purpose of the minimum in the definition of ϕ in (6.2) is to ensure the (negative) moment condition of ϕ in Assumption C.3. Furthermore numerical analysis has shown that large values for the negative moments of ϕ, i.e. $\mathbb{E}\left[\phi^{1-\beta}\right]$ when β is close to 2 worsen the asymptotic quality of the estimator strongly. The choice of the processes has no specific application in mind and could easily replaced by different (and more complex) variants. It was done to underline the possibilities of the model assumptions, while simulation should remain a feasible task. It should be noted that the choice of $\lambda_0 = 1$ combined with the tendency of λ to return to 5 leads to an irregular change in observation times over the course of time.

6.2 Numerical Approximation and Simulation

To approximate the processes $\alpha, \sigma, \lambda, X$, we use a simple Euler scheme. We start with the approximation for the observation scheme, i.e. for $N_n(1) > i \geq 1$ we set recursively

$$\begin{aligned} \lambda_{\tau_i^n} &\approx \lambda_{\tau_{i-1}^n} + (5 - \lambda_{\tau_{i-1}^n})(\tau_i^n - \tau_{i-1}^n) + (\widetilde{W}_{\tau_i^n} - \widetilde{W}_{\tau_{i-1}^n}), \\ \tau_{i+1}^n &= \tau_i^n + \Delta_n \phi_i^n \lambda_{\tau_{i-1}^n} \end{aligned} \tag{6.3}$$

and $\tau_0^n = 0, \tau_1^n = \Delta_n \phi_1^n$. For the remaining processes we set for $N_n(1) > i \geq 0$

$$\begin{aligned} \alpha_{\tau_{i+1}^n} &\approx \alpha_{\tau_i^n} + 2(1 - \alpha_{\tau_i^n})(\tau_{i+1}^n - \tau_i^n) + 2(W_{\tau_{i+1}^n} - W_{\tau_i^n}), \\ \sigma_{\tau_{i+1}^n} &\approx \sigma_{\tau_i^n} + \alpha_{\tau_i^n}(W_{\tau_{i+1}^n} - W_{\tau_i^n}), \\ X_{\tau_{i+1}^n} &\approx X_{\tau_i^n} + \alpha_{\tau_i^n}(\tau_{i+1}^n - \tau_i^n) + \sigma_{\tau_i^n}(L_{\tau_{i+1}^n} - L_{\tau_i^n}). \end{aligned} \tag{6.4}$$

For the purpose of simulation we note that

$$\widetilde{W}_{\tau_{i+1}^n} - \widetilde{W}_{\tau_i^n} \sim \sqrt{\tau_{i+1}^n - \tau_i^n} \times N,$$

where $N \sim \mathcal{N}(0,1)$ is a standard normally distributed random variable and that all occurrences of $W, \widetilde{W}$ in the approximation above are either independent from each other or exactly the same. Therefore it is sufficient for the simulation of $\widetilde{W}_{\tau_i^n} - \widetilde{W}_{\tau_{i-1}^n}, W_{\tau_{i+1}^n} - W_{\tau_i^n}$, etc. that we are able to simulate independent standard normal random variables. For the increments of the stable process we note that

$$L_{\tau_{i+1}^n} - L_{\tau_i^n} \sim (\tau_{i+1}^n - \tau_i^n)^{1/\beta} S,$$

where S is a symmetrical stable random variable with characteristic function given by

$$\mathbb{E}\left[\exp(iuS)\right] = \exp(-|u|^\beta), \qquad u \in \mathbb{R}. \tag{6.5}$$

To simulate S we use the following result.

Theorem 6.1 (cf. Proposition 1.7.1 in [ST94])**.** *Let $\beta \in (0,2]$, γ be uniform on $(-\pi/2, \pi/2)$ and let Q be exponential with mean 1. Assume γ and Q are independent. Then*

$$X = \frac{\sin(\beta\gamma)}{\cos(\gamma)^{1/\beta}} \left(\frac{\cos((1-\beta)\gamma)}{Q} \right)^{(1-\beta)\beta}$$

has characteristic function given by (6.5)*.*

Assuming that the software/programming language used for the implementation offers the possibility to simulate normally and exponentially distributed random variables we have all the tools to proceed with the implementation.
In the appendix we provide an implementation in Python which follows these steps:

1. First we select the model parameter $\beta \in (1,2)$, the last time point $T = 1$ for the observation, the number N of paths that we simulate, the approximate number n of observations and the parameters for our estimator p, q.
2. Implementation of $\widetilde{L}^n(p,u)$ according to (5.2) and $\hat{\beta}(p,u,v)$ according to (5.98) as functions of $p, u_n, v_n, k_n, \{\tau_i : 0 \le i \le N_n(1)\}$ and $\{\Delta_i^n X : 0 \le i \le N_n(1)\}$.
3. The main loop running N times with these steps:
   (a) For $1 \le i \le N_n(1)$ simulate iteratively $\lambda_{\tau_i^n}, \tau_{i+1}^n, \alpha_{\tau_{i+1}^n}, \sigma_{\tau_{i+1}^n}, X_{\tau_{i+1}^n}$ according to (6.3) and (6.4) with $\lambda_0 = X_0 = \alpha_0 = \sigma_0 = 1$.
   (b) Choose $u_n = N_n(1)^{-1/3}, k_n = N_n(1)^{2/3}, p = 1/2$ in accordance with Remark 5.2.
   (c) Choose $v_n = \rho u_n$ for $\rho \in \{1/2, 2\}$. More on the choice of ρ follows in the next section.
   (d) Apply the implementation of $\hat{\beta}(p,u,v)$ to the simulation from (a) and save the result in an array.
   (e) To determine the quality of the approximation to a normal distribution in Theorem 5.2 we save $u_n^{\beta/2}\sqrt{N_n(1)}(\hat{\beta}(p,u,v) - \beta)$ to a second array.

6.3 Results

We use the implementation presented above to derive results for $\beta \in \{1.1, 1.3, 1.5, 1.7, 1.9\}$ and $\rho \in \{1/2, 2\}$. We choose $N = 1000$ and $n = 1000$ which yields for (6.1) and (6.2) roughly $N_n(1) \approx 520$ observations before the terminal time $T = 1$. Displayed are results for $\rho = 1/2$ and in brackets the results for $\rho = 2$. Here "Mean" is the empirical mean of the $N = 1000$ samples of $\hat{\beta}(p, u_n, v_n)$, "Empirical variance" is the empirical variance of the 1000 samples of $u_n^{\beta/2}\sqrt{N_n(1)}(\hat{\beta}(p, u_n, v_n) - \beta)$ and "Theoretical variance" is the asymptotic variance from Theorem 5.2. In order to build this asymptotic variance the values for $\kappa_{p,\beta}$ and $\kappa_{\beta,\beta}$ are calculated via a separate Monte-Carlo estimate with a large sample size and can be assumed to be sufficiently accurate. More on how the variance can be calculated follows in the next section.

β	Mean of $\hat{\beta}(p, u, v)$	Empirical Variance	Theoretical Variance
1.1	1.1181 (1.0455)	7.2689 (3.2672)	7.2457 (3.3802)
1.3	1.3123 (1.2356)	5.2131 (2.2777)	5.4853 (2.2274)
1.5	1.4923 (1.4421)	3.2153 (1.3201)	3.907 (1.3817)
1.7	1.7173 (1.6086)	1.6501 (0.73274)	2.354 (0.7245)
1.9	1.8849 (1.7759)	0.425 (0.2716)	0.7852 (0.2107)

We can see that the larger choice of ρ directly effects the error of the estimator for small sample sizes in a negative way while it reduces the variance. Analysis of the small sample error is quite delicate here and complete understanding seems to be a non feasible task. However, using the Taylor approximation in Theorem 5.2 and analyzing the bias term (5.100) yields that many of the estimates there depend on the size of u_n respectively v_n and large values worsen the convergence towards zero. Furthermore numerical analysis supports this claim as we have for $n = 1000, \rho = 0.5, \beta = 1.9$ that (5.100)≈ -0.1807 while for $\rho = 2$ we have (5.100)≈ -0.641. As this does not account for the complete difference in the sample error, one factor that one may consider additionally is the normal approximation of (5.102), Here large values of ρ respectively v_n worsen the approximation as well and may additionally contribute to the error. We note that the variance in Theorem 5.2 is monotone decreasing in ρ for $\rho > 1$. Therefore we have in this range a direct trade off between variance and bias.

We follow this discussion with a table in the same manner as above for $n = 10000$ which roughly yields $N_n(1) \approx 5200$. We see that the sample error diminishes for larger values of n in the case of $\rho = 2$ while the sample size of appears $N = 1000$ to be not sufficient large enough for further analysis of the already small error in the case $\rho = 0.5$. Nevertheless, we can see that the approximated variance for $\beta \in \{1.5, 1.7, 1.9\}$ is much closer to the theoretical one.

β	Mean of $\hat{\beta}(p,u,v)$	Empirical Variance	Theoretical Variance
1.1	1.1048 (1.0852)	7.451 (3.3392)	7.2458 (3.3802)
1.3	1.3067 (1.2763)	5.257 (2.229)	5.4853 (2.2274)
1.5	1.5197 (1.4772)	3.6036 (1.3626)	3.907 (1.3817)
1.7	1.7088 (1.6743)	2.1925 (0.7034)	2.354 (0.7245)
1.9	1.914 (1.8726)	0.4864 (0.21937)	0.7852 (0.2107)

Before we delve into the analysis of QQ-plots we have a small intermediate result that can contribute to the discussion.

Lemma 6.1. *Let $\rho = 1/2$ and $v_n = \rho u_n$ then it holds that*

$$\hat{\beta}(p,u_n,v_n) = \frac{\log(-(\widetilde{L}^n(p,u_n)-1)) - \log(-(\widetilde{L}^n(p,v_n)-1))}{\log(u_n/v_n)} \leq 2. \tag{6.6}$$

Proof. Using the definition of $\widetilde{L}^n(p,u_n), \widetilde{L}^n(p,v_n)$ we see with $a_i := \frac{\widetilde{\Delta_i^n X - \Delta_{i-1}^n X}}{(\widetilde{V}_i^n(p))^{1/p}}$ that (6.6) is equivalent to

$$\frac{\log(-(\frac{1}{N_n(1)-k_n-2}\sum_{i=k_n+3}^{N_n(1)}\cos(u_n a_i)-1)) - \log(-(\frac{1}{N_n(1)-k_n-2}\sum_{i=k_n+3}^{N_n(1)}\cos(v_n a_i)-1))}{\log(1/\rho)} \leq 2$$

$$\Leftrightarrow \log\left(\frac{\frac{1}{N_n(1)-k_n-2}\sum_{i=k_n+3}^{N_n(1)}(1-\cos(u_n a_i))}{\frac{1}{N_n(1)-k_n-2}\sum_{i=k_n+3}^{N_n(1)}(1-\cos(\rho u_n a_i))}\right) \leq \log\left(\frac{1}{\rho^2}\right)$$

$$\Leftrightarrow \frac{1}{N_n(1)-k_n-2}\sum_{i=k_n+3}^{N_n(1)}\rho^2(1-\cos(u_n a_i)) \leq \frac{1}{N_n(1)-k_n-2}\sum_{i=k_n+3}^{N_n(1)}(1-\cos(\rho u_n a_i)). \tag{6.7}$$

For (6.7) to hold, it is sufficient that

$$\rho^2(1-\cos(b)) \leq 1-\cos(\rho b) \quad \text{for all } b \in \mathbb{R}$$

which is equivalent to

$$g_\rho(b) := 1-\cos(\rho b) - \rho^2(1-\cos(b)) \geq 0 \quad \text{for all } b \in \mathbb{R}. \tag{6.8}$$

Using properties of the cosine and inserting $\rho = 1/2$ we note $g_{\frac{1}{2}}(b) = g_{\frac{1}{2}}(-b)$ and $g_{\frac{1}{2}}(b) = g_{\frac{1}{2}}(b+4\pi)$. For (6.8) to hold it then suffices to show $g_{\frac{1}{2}}(b) \geq 0$ for all $b \in [0,2\pi]$. Let $x \in [0,2\pi]$ then

$$\begin{aligned} g'_{\frac{1}{2}}(x) &= \frac{1}{2}\sin\left(\frac{x}{2}\right) - \frac{1}{4}\sin(x) \\ &= \frac{1}{2}\sin\left(\frac{x}{2}\right) - \frac{1}{2}\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = \frac{1}{2}\sin\left(\frac{x}{2}\right)\left(1-\cos\left(\frac{x}{2}\right)\right) \geq 0, \end{aligned}$$

by properties of the trigonometric functions. Together with $g_{\frac{1}{2}}(0) = 0$ the last line yields $g_{\frac{1}{2}}(b) \geq 0$ for all $b \in [0, 2\pi]$. □

Noting that $\hat{\beta}(p, u_n, v_n) = \hat{\beta}(p, v_n, u_n)$, this result also holds in the case of $\rho = 2$. Furthermore, analyzing (6.8) for different values of ρ numerically one can find that

$$g_\rho(b) \geq 0 \quad \text{for} \quad 0 < \rho \leq 0.5, \; b \in [-4\pi, 4\pi],$$
$$g_\rho(b) \geq 0 \quad \text{for} \quad 0 < \rho \leq 1, \; b \in [-2\pi, 2\pi].$$

Because the a_i, as defined in the previous lemma, converge to a non degenerate distribution, small values of u_n will eventually lead to $\hat{\beta}(p, u_n, v_n)$ attaining values above 2 very rarely, in particular when $\rho \leq 0.5$. From the symmetry of $\hat{\beta}(p, u_n, v_n)$ the same can be said for $\rho \in (1, 2]$ or $\rho > 1$ in general.
The following QQ-plots of $u_n^{\beta/2}\sqrt{N_n(1)}(\hat{\beta}(p, u_n, v_n) - \beta)$ against a standard-normal distribution with variance equal to the theoretical variance use the same configuration of parameters as discussed earlier for $\rho \in \{0.5, 2\}$ and $n \in \{1000, 10000\}$. They clearly display the aforementioned boundedness of $\hat{\beta}(p, u_n, v_n)$ for both choices of ρ. However, when $\rho = 2$ the smaller variance makes the boundedness less noticeable and therefore we have different qualities in the approximation towards a normal distribution for the two different choices of ρ, in particular when β attains values closer to 2. It should be noted that the distributional approximation quality increases visibly with the higher sample-size in both cases. This becomes more apparent for values of β close to 2 as $u_n^{\beta/2}\sqrt{N_n(1)}$ is relatively small for our choice of $u_n = N_n(1)^{-1/3}$ and therefore the asymptotic normal distribution becomes visible only for larger $N_n(1)$ respectively n.

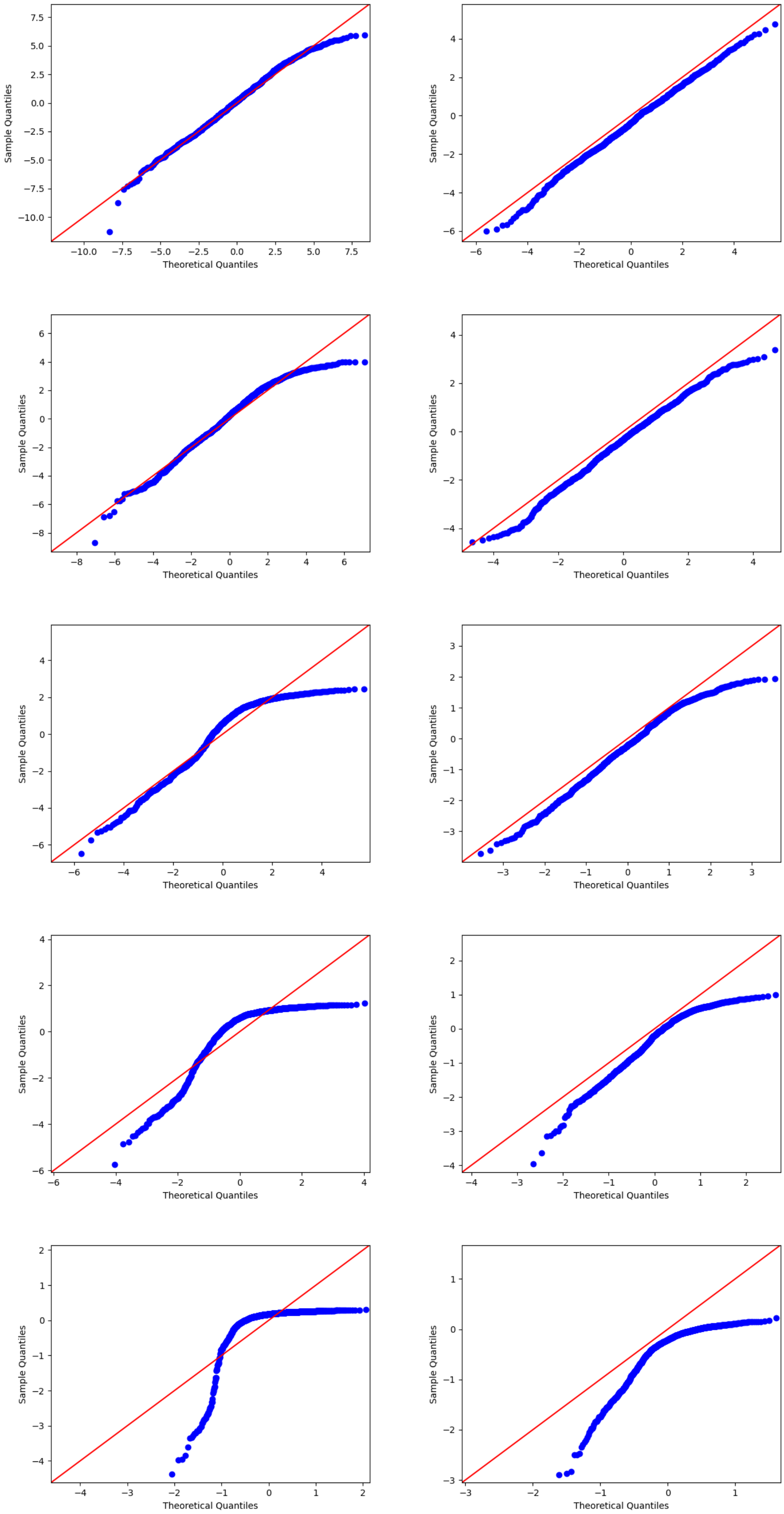

Figure 6.1: $N = 1000, n = 1000, \beta \in \{1.1, 1.3, 1.5, 1.7, 1.9\}$
left side $\rho = 0.5$, right side $\rho = 2$

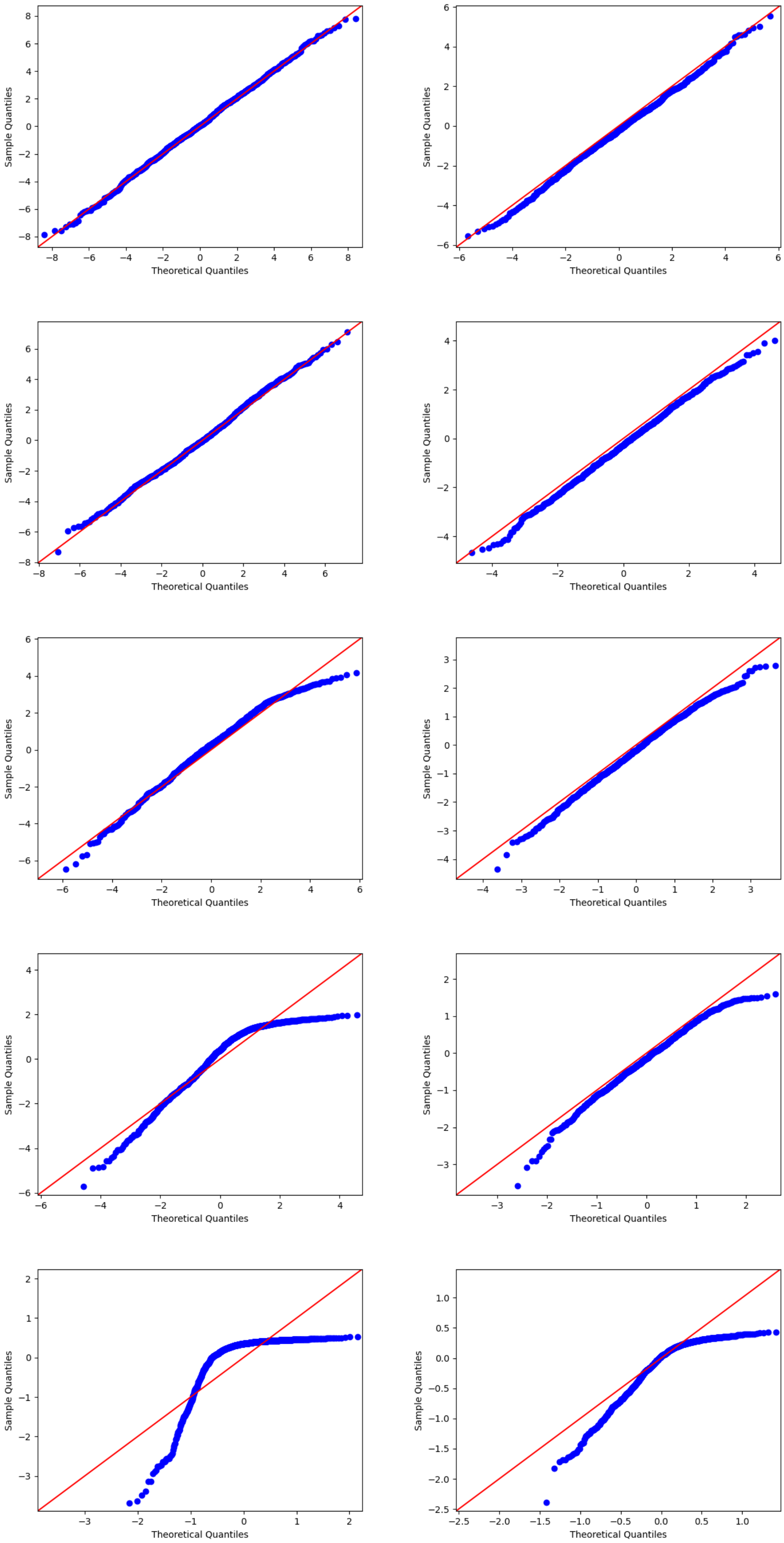

Figure 6.2: $N = 1000, n = 10000, \beta \in \{1.1, 1.3, 1.5, 1.7, 1.9\}$
left side $\rho = 0.5$, right side $\rho = 2$

6.4 A Consistent Estimator for the Variance and a Further CLT

As already addressed in the introduction, the major problem that arises when we want to apply the estimator $\hat{\beta}(p, u_n, v_n)$ from the previous sections in testing procedures is that the asymptotic variance of
$u_n^{\beta/2}\sqrt{N_n(1)}(\hat{\beta}(p, u_n, v_n) - \beta)$ is

$$\frac{(\rho^\beta + 1)(4 - 2^\beta) - 2(2 + 2\rho^\beta - (1+\rho)^\beta - (1-\rho)^\beta)}{\kappa_{\beta,\beta}\rho^\beta \log(1/\rho)^2 C_{p,\beta}}, \tag{6.9}$$

(for simplicity we assume in this section that $0 < \rho < 1$ while the case $\rho > 1$ follows in a completely similar manner) where

$$\mu_{p,\beta} := \mathbb{E}[|S_1|^p]^{\frac{\beta}{p}}, \quad \kappa_{p,\beta} := \mathbb{E}[(\phi^{1-\beta} + (\phi')^{1-\beta})^{\frac{p}{\beta}}]^{\frac{\beta}{p}}, \quad C_{p,\beta} := \frac{A_\beta}{\mu_{p,\beta}\kappa_{p,\beta}},$$
$$\text{with} \quad \exp(-A_\beta u^\beta t) = \mathbb{E}[\exp(iuS_t)], \ u \in \mathbb{R}_+.$$

Thus it is determined by many non observable model parameters, in particular it depends on β itself. To bypass this problem, one may hope that inserting the estimator $\hat{\beta}(p, u_n, v_n)$ whenever β is needed will yield the correct result. However the moments $\kappa_{p,\beta}, \kappa_{\beta,\beta}$ have to be derived from the random variables ϕ_i^n that are not directly observable as they are intertwined with values of the process λ when building our observation scheme. Furthermore, the normalization $u_n^{\beta/2}$ in $u_n^{\beta/2}\sqrt{N_n(1)}(\hat{\beta}(p, u_n, v_n) - \beta)$ is dependent on β as well. This section now deals with the problem of finding a consistent estimator for (6.9) and then applying it to find a normalization that works without the use of unknown model variables or parameters.

Theorem 6.2. *Let $r_n \in \mathbb{N}$ with $1 \le r_n \le N_n(1) - 3, r_n \asymp n^\Psi$ for some $\Psi \in (0,1)$, $0 < p < \beta/2$. Let $\hat{\beta}_n$ be a consistent estimator for β such that there exists a $\varsigma > 0$ with*

$$\left|\beta - \hat{\beta}_n\right| n^\varsigma \xrightarrow{\mathbb{P}} 0 \tag{6.10}$$

and furthermore assume that ϕ from Assumption C fulfills $M < \phi$ for some $0 < M < 1$. Setting

$$\chi_i = \left(\left(\frac{r_n}{\tau_{i-2}^n - \tau_{i-2-r_n}^n}\right)^{1-\hat{\beta}_n}\left(\left(\tau_i^n - \tau_{i-1}^n\right)^{1-\hat{\beta}_n} + \left(\tau_{i-1}^n - \tau_{i-2}^n\right)^{1-\hat{\beta}_n}\right)\right)^{p/\hat{\beta}_n},$$

we have that

$$\widehat{\kappa}_n^p := \frac{1}{N_n(1) - r_n - 2}\sum_{i=r_n+3}^{N_n(1)} \chi_i \xrightarrow{\mathbb{P}} \mathbb{E}[(\phi^{1-\beta} + (\phi')^{1-\beta})^{\frac{p}{\beta}}] = \kappa_{p,\beta}^{p/\beta}. \tag{6.11}$$

Proof. Before we start with the actual proof we note that $N_n(\tau_i^n) = i$ and thus for all $i > r_n + 3$ we can write

$$r_n = \sum_{j=N_n(\tau_{i-1-r_n}^n)}^{N_n(\tau_{i-2}^n)} 1 = \sum_{j \geq i-1-r_n} \mathbb{1}_{\{\tau_j^n \leq \tau_{i-2}^n\}} = \sum_{j \geq 1} \mathbb{1}_{\{\tau_{i-1-r_n}^n \leq \tau_j^n \leq \tau_{i-2}^n\}}$$

which can be seen as a version of $N_n(t) = \sum_{i \geq 1} \mathbb{1}_{\{\tau_i^n \leq t\}}$ restricted to the time interval $(\tau_{i-2-r_n}^n, \tau_{i-2}^n]$. Throughout the last chapters we repeatedly used that $\Delta_n N_n(1) \xrightarrow{\mathbb{P}} \int_0^t \frac{1}{\lambda_s} ds$ therefore our motivation for the estimator in (6.11) is the idea (which is proven later on) that $\frac{\Delta_n r_n}{\tau_{i-2}^n - \tau_{i-2-r_n}^n} \xrightarrow{\mathbb{P}} \frac{1}{\tau_{i-2}^n - \tau_{i-2-r_n}^n} \int_{\tau_{i-2-r_n}^n}^{\tau_{i-2}^n} \frac{1}{\lambda_s} ds \xrightarrow{\mathbb{P}} \frac{1}{\lambda_{\tau_{i-2}^n}}$.

The additional notation needed for the proof of (6.11) is as following:

$$\begin{aligned}
e_i^1 &= \left(\left(\frac{r_n}{\tau_{i-2}^n - \tau_{i-2-r_n}^n} \right)^{1-\beta} \left(\left(\tau_i^n - \tau_{i-1}^n \right)^{1-\beta} + \left(\tau_{i-1}^n - \tau_{i-2}^n \right)^{1-\beta} \right) \right)^{p/\beta}, \\
e_i^2 &= \left(\left(\frac{1}{\tau_{i-2}^n - \tau_{i-2-r_n}^n} \int_{\tau_{i-2-r_n}^n}^{\tau_{i-2}^n} \frac{1}{\lambda_s} ds \right)^{1-\beta} \left(\left(\lambda_{\tau_{i-2}^n} \phi_i^n \right)^{1-\beta} + \left(\lambda_{\tau_{i-3}^n} \phi_{i-1}^n \right)^{1-\beta} \right) \right)^{p/\beta}, \\
e_i^3 &= \left((\phi_i^n)^{1-\beta} + \left(\frac{\lambda_{\tau_{i-3}^n}}{\lambda_{\tau_{i-2}^n}} \phi_{i-1}^n \right)^{1-\beta} \right)^{p/\beta}, \\
\zeta_i &= \left((\phi_i^n)^{1-\beta} + \left(\phi_{i-1}^n \right)^{1-\beta} \right)^{p/\beta}
\end{aligned}$$

with corresponding sums

$$\begin{aligned}
E_n^1 &= \frac{1}{N_n(1) - r_n - 2} \sum_{i=r_n+3}^{N_n(1)} e_i^1, \\
E_n^2 &= \frac{1}{N_n(1) - r_n - 2} \sum_{i=r_n+3}^{N_n(1)} e_i^2, \\
E_n^3 &= \frac{1}{N_n(1) - r_n - 2} \sum_{i=r_n+3}^{N_n(1)} e_i^3, \\
Z_n &= \frac{1}{N_n(1) - r_n - 2} \sum_{i=r_n+3}^{N_n(1)} \zeta_i
\end{aligned}$$

and the following decomposition:

$$\widehat{\kappa}_n^p - \kappa_{p,\beta}^{p/\beta} = \left(\widehat{\kappa}_n^p - E_n^1 \right) + \left(E_n^1 - E_n^2 \right) + \left(E_n^2 - E_n^3 \right) + \left(E_n^3 - Z_n \right) + \left(Z_n - \kappa_{p,\beta}^{p/\beta} \right). \quad (6.12)$$

As the final step we prove $Z_n \xrightarrow{\mathbb{P}} \mathbb{E}[(\phi^{1-\beta} + (\phi')^{1-\beta})^{\frac{p}{\beta}}] = \kappa_{p,\beta}^{p/\beta}$. Therefore we aim to show that $\widehat{\kappa}_n^p - Z_n \xrightarrow{\mathbb{P}} 0$.

In order to show $\widehat{\kappa}_n^p - E_n^1 \xrightarrow{\mathbb{P}} 0$ we define the set $A_n := \left\{|\beta - \hat{\beta}_n| > \frac{\beta-1}{2}\right\}$ and note that for all $\epsilon > 0$

$$\mathbb{P}\left(\left|\widehat{\kappa}_n^p - E_n^1\right| \mathbb{1}_{A_n} > \epsilon\right) \leq \mathbb{P}\left(\mathbb{1}_{A_n} > \epsilon\right) = \mathbb{P}\left(\left|\beta - \hat{\beta}_n\right| > \frac{\beta - 1}{2}\right) \to 0 \tag{6.13}$$

as $\hat{\beta}_n$ is a consistent estimator and $\beta > 1$.
And in contrast on the set A_n^C, due to $\beta < 2$:

$$\hat{\beta}_n - 1 = \hat{\beta}_n - \beta + \frac{\beta - 1}{2} + \frac{\beta - 1}{2} \geq \frac{\beta - 1}{2} > 0,$$
$$3 - \hat{\beta}_n = 3 - \beta + \beta - \hat{\beta}_n > 1 + \beta - \hat{\beta}_n > \frac{\beta - 1}{2} + \beta - \hat{\beta}_n \geq 0$$

or put differently $\hat{\beta}_n \in (1, 3)$. For the sake of easier notation we define $a_i = \frac{\Delta_n r_n}{\tau_{i-2}^n - \tau_{i-2-r_n}^n}$, $b_i = \frac{\tau_i^n - \tau_{i-1}^n}{\Delta_n}$, $c_i = \frac{\tau_{i-1}^n - \tau_{i-2}^n}{\Delta_n}$ and furthermore the (random) function

$$f_{a_i,b_i,c_i,p}(x) = \left((a_i)^{1-x}\left((b_i)^{1-x} + (c_i)^{1-x}\right)\right)^{p/x}.$$

We show that $(\widehat{\kappa}_n^p - E_n^1)\, \mathbb{1}_{A_n^C} \xrightarrow{\mathbb{P}} 0$ via looking at the differences $(\chi_i - e_i^1)\mathbb{1}_{A_n^C}$ and note that for some $\epsilon_n \in [\hat{\beta}_n, \beta] \subset (1, 3)$:

$$\begin{aligned}|\chi_i - e_i^1|\mathbb{1}_{A_n^C} &= \left|f_{a_i,b_i,c_i,p}(\hat{\beta}_n) - f_{a_i,b_i,c_i,p}(\beta)\right| \mathbb{1}_{A_n^C} \\ &= \mathbb{1}_{A_n^C}|f'_{a_i,b_i,c_i,p}(\epsilon_n)||\beta - \hat{\beta}_n|,\end{aligned} \tag{6.14}$$

with the derivative

$$\begin{aligned}&f'_{a_i,b_i,c_i,p}(x) \\ &= f_{a_i,b_i,c_i,p}(x)p\left(\frac{-b_i^{1-x}\log(b_i) - c_i^{1-x}\log(c_i) - \log(a_i)(b_i^{1-x} + c_i^{1-x})}{x(b_i^{1-x} + c_i^{1-x})} - \frac{\log(a_i^{1-x}(b_i^{1-x} + c_i^{1-x}))}{x^2}\right).\end{aligned}$$

Our goal is to show that we have $|f'_{a_i,b_i,c_i,p}(x)||\beta - \hat{\beta}_n| \xrightarrow{\mathbb{P}} 0$ uniformly over $x \in (1, 3)$.
We note, because $a_i, b_i, c_i > 0$, that for $x > 0$:

$$\begin{aligned}&|f'_{a_i,b_i,c_i,p}(x)| \\ &\leq f_{a_i,b_i,c_i,p}(x)p\left(\frac{b_i^{1-x}|\log(b_i)| + c_i^{1-x}|\log(c_i)| + |\log(a_i)|(b_i^{1-x} + c_i^{1-x})}{x(b_i^{1-x} + c_i^{1-x})}\right. \\ &\qquad\left. + \frac{|\log(a_i^{1-x})| + |\log(b_i^{1-x} + c_i^{1-x})|}{x^2}\right).\end{aligned} \tag{6.15}$$

To continue with our calculations, we need a further localization of the observation scheme (cf. p.435 in [JP12]) that allows us to assume

$$\phi_i \leq n^\gamma, \tag{6.16}$$

for an arbitrarily (but a priori) small chosen $\gamma > 0$. We remember that due to Assumption SC we have for some $C > 1$

$$\frac{1}{C} \leq \lambda \leq C. \tag{6.17}$$

Combining (6.16) and (6.17) gives

$$CM\Delta_n \leq \tau_i^n - \tau_{i-1}^n = \Delta_n \phi_i^n \lambda_{\tau_{i-2}^n} \leq C\Delta_n n^\gamma$$
$$CMr_n\Delta_n \leq \tau_{i-2}^n - \tau_{i-2-r_n}^n = \sum_{j=i-1-r_n}^{i-2} \tau_j^n - \tau_{j-1}^n \leq \sum_{j=i-1-r_n}^{i-2} C\Delta_n n^\gamma \leq Cr_n\Delta_n n^\gamma,$$

which yields

$$(CM)^q \leq (b_i)^q = \left(\frac{\tau_i^n - \tau_{i-1}^n}{\Delta_n}\right)^q \leq C^q n^{\gamma q} \quad \text{for } \; q > 0, \tag{6.18}$$
$$C^q n^{\gamma q} \leq (b_i)^q = \left(\frac{\tau_i^n - \tau_{i-1}^n}{\Delta_n}\right)^q \leq (CM)^q \quad \text{for } \; q < 0.$$

We note that for all bounded sets $B \subset \mathbb{R}$ we have $\sup\{C^q : q \in B\}$, $\sup\{(CM)^q : q \in B\} < \infty$ and similarly $\inf\{C^q : q \in B\}$, $\inf\{(CM)^q : q \in B\} > 0$. In particular we have for $3 > x > 1$

$$\inf\{C^q : q \in (0,-2)\} n^{-2\gamma} \leq (b_i)^{1-x} \leq \sup\{(CM)^q : q \in (0,-2)\} \tag{6.19}$$

and likewise for c_i. As in a_i we have $\tau_{i-2}^n - \tau_{i-2-r_n}^n$ in the denominator we find similar results to (6.18) and (6.19):

$$C^q n^{-\gamma q} \leq (a_i)^q = \left(\frac{\Delta_n r_n}{\tau_{i-2}^n - \tau_{i-2-r_n}^n}\right)^q \leq (CM)^q \quad \text{for } \; q > 0, \tag{6.20}$$
$$(CM)^q \leq (a_i)^q = \left(\frac{\Delta_n r_n}{\tau_{i-2}^n - \tau_{i-2-r_n}^n}\right)^q \leq C^q n^{-\gamma q} \quad \text{for } \; q < 0,$$

which results for $3 > x > 1$ in

$$\inf\{(CM)^q : q \in (0,-2)\} \leq (a_i)^{1-x} \leq \max\{C^q : q \in (0,-2)\} n^{2\gamma}. \tag{6.21}$$

Combining (6.19) and (6.21) we get for all $3 > x > 1$

$$f_{a_i,b_i,c_i,p}(x) \leq K n^{2\gamma(p-p/x)} \leq K n^{\frac{4}{3}\gamma p}. \tag{6.22}$$

Using (6.18) - (6.21) we find that for $3 > x > 1$

$$|\log(a_i^{1-x})|, |\log(b_i^{1-x} + c_i^{1-x})| \leq K|\log(n^{-2\gamma})|,$$
$$|\log(a_i)|, |\log(b_i)|, |\log(c_i)| \leq K|\log(n^{-\gamma})|$$

which yields

$$\frac{b_i^{1-x}|\log(b_i)| + c_i^{1-x}|\log(c_i)| + |\log(a_i)|(b_i^{1-x} + c_i^{1-x})}{x(b_i^{1-x} + c_i^{1-x})} \leq K|\log(n^{-\gamma})| n^{2\gamma},$$
$$\frac{|\log(a_i^{1-x})| + |\log(b_i^{1-x} + c_i^{1-x})|}{x^2} \leq K|\log(n^{-2\gamma})|.$$

Finally linking the last two inequalities with (6.22) and (6.15) we get for all $3 > x > 1$

$$f'_{a_i,b_i,c_i,p}(x) \le K n^{\frac{4}{3}\gamma p} \left(|\log(n^{-\gamma})| n^{2\gamma} + K |\log(n^{-2\gamma})| \right) \tag{6.23}$$

and therefore with (6.14)

$$\begin{aligned}\frac{1}{n - r_n - 2} \sum_{i=r_n+3}^{N_n(1)} |\chi_i - e_i^1| \mathbb{1}_{A_n^C} &\le \frac{1}{n - r_n - 2} \sum_{i=r_n+3}^{Cn} |\chi_i - e_i^1| \mathbb{1}_{A_n^C} \\ &\le K \frac{Cn}{n - r_n - 2} n^{\frac{4}{3}\gamma p} \left(|\log(n^{-\gamma})| n^{2\gamma} + K |\log(n^{-2\gamma})| \right) |\beta - \hat{\beta}_n| \xrightarrow{\mathbb{P}} 0,\end{aligned}$$

due to (6.10), $|\log(n^{-q}) n^{-r}| \to 0$ for all $q, r > 0$ and the possibility to choose γ (in dependence of the a priori known ς and p) sufficiently small. Using $\Delta_n N_n(1) \xrightarrow{\mathbb{P}} \int_0^t \frac{1}{\lambda_s} ds$ one more time we have

$$\begin{aligned}\left| \widehat{\kappa}_n^p - E_n^1 \right| \mathbb{1}_{A_n^C} &\le \frac{1}{N_n(1) - r_n - 2} \sum_{i=r_n+3}^{N_n(1)} |\chi_i - e_i^1| \mathbb{1}_{A_n^C} \\ &= \frac{n - r_n - 2}{N_n(1) - r_n - 2} \frac{1}{n - r_n - 2} \sum_{i=r_n+3}^{N_n(1)} |\chi_i - e_i^1| \mathbb{1}_{A_n^C} \xrightarrow{\mathbb{P}} 0\end{aligned}$$

and with (6.13) we have proven $\widehat{\kappa}_n^p - E_n^1 \xrightarrow{\mathbb{P}} 0$.
To prove that $|E_n^1 - E_n^2| \xrightarrow{\mathbb{P}}$ we need a few preliminaries. Using Lemma 5.7 on the process λ and the function $f(x) = x^{-1}$ together with Lemma 4.8 yields that for $1 \le j \le N_n(1) - i$

$$\mathbb{E}\left[\sup_{\tau_i^n \le s \le \tau_{i+j}^n} \left| \frac{1}{\lambda_s} - \frac{1}{\lambda_{\tau_i^n}} \right| \right] \le (j\Delta_n)^{1/2}. \tag{6.24}$$

For $1 \leq j \leq N_n(1)$ we then define $\eta_j = \frac{\phi_j^n}{r_n}$ and look at the following difference

$$
\begin{aligned}
&\mathbb{E}\left[\left|\sum_{j=i-1-r_n}^{i-2} \eta_j - \frac{1}{\Delta_n r_n}\int_{\tau^n_{i-2-r_n}}^{\tau^n_{i-2}} \frac{1}{\lambda_s} ds\right|\right] \\
&= \frac{1}{\Delta_n r_n}\mathbb{E}\left[\left|\sum_{j=i-1-r_n}^{i-2} \Delta_n \phi_j^n \lambda_{\tau^n_{j-2}} \frac{1}{\lambda_{\tau^n_{j-2}}} - \int_{\tau^n_{i-2-r_n}}^{\tau^n_{i-2}} \frac{1}{\lambda_s} ds\right|\right] \\
&= \frac{1}{\Delta_n r_n}\mathbb{E}\left[\left|\sum_{j=i-1-r_n}^{i-2} \int_{\tau^n_{j-1}}^{\tau^n_j} \left(\frac{1}{\lambda_s} - \frac{1}{\lambda_{\tau^n_{j-2}}}\right) ds\right|\right] \\
&\leq \frac{1}{\Delta_n r_n} \sum_{j=i-1-r_n}^{i-2} \mathbb{E}\left[\int_{\tau^n_{j-1}}^{\tau^n_j} \left|\frac{1}{\lambda_s} - \frac{1}{\lambda_{\tau^n_{j-2}}}\right| ds\right] \\
&\leq \frac{1}{\Delta_n r_n} \sum_{j=i-1-r_n}^{i-2} \mathbb{E}\left[(\tau^n_j - \tau^n_{j-1}) \sup_{\tau^n_{j-1} \leq s \leq \tau^n_j} \left|\frac{1}{\lambda_s} - \frac{1}{\lambda_{\tau^n_{j-2}}}\right|\right] \\
&\leq \frac{1}{\Delta_n r_n} \sum_{j=i-1-r_n}^{i-2} \mathbb{E}\left[(\tau^n_j - \tau^n_{j-1}) \left(\sup_{\tau^n_{j-1} \leq s \leq \tau^n_j} \left|\frac{1}{\lambda_s} - \frac{1}{\lambda_{\tau^n_{j-1}}}\right| + \left|\frac{1}{\lambda_{\tau^n_{j-1}}} - \frac{1}{\lambda_{\tau^n_{j-2}}}\right|\right)\right] \\
&\leq \frac{1}{\Delta_n r_n} \sum_{j=i-1-r_n}^{i-2} \left(\mathbb{E}\left[(\tau^n_j - \tau^n_{j-1})^2\right]^{1/2} \mathbb{E}\left[\left(\sup_{\tau^n_{j-1} \leq s \leq \tau^n_j} \left|\frac{1}{\lambda_s} - \frac{1}{\lambda_{\tau^n_{j-1}}}\right|\right)^2\right]^{1/2}\right. \\
&\qquad\qquad \left. + \mathbb{E}\left[(\tau^n_j - \tau^n_{j-1})^2\right]^{1/2} \mathbb{E}\left[\left(\frac{1}{\lambda_{\tau^n_{j-1}}} - \frac{1}{\lambda_{\tau^n_{j-2}}}\right)^2\right]^{1/2}\right) \\
&\leq \frac{1}{\Delta_n r_n} \sum_{j=i-1-r_n}^{i-2} K\Delta_n \left(\Delta_n^{1/2} + \Delta_n^{1/2}\right) \leq K\Delta_n^{1/2},
\end{aligned}
$$

using the Cauchy-Schwarz inequality in the third to last to second to last line and (6.24) in the step afterwards and remembering that for $q \geq 0$, due to Assumption SC:

$$
\mathbb{E}\left[(\tau^n_j - \tau^n_{j-1})^q\right] \leq K\Delta_n^q.
$$

Proceeding, we find that $M_t = \sum_{i=1}^{N_n(t)} \left(\eta_i - \mathbb{E}^n_{i-1}[\eta_i]\right)$ is a square-integrable martingale w.r.t. the filtration $(\mathcal{F}_{\tau^n_{N_n(t)}})_{t\geq 0}$ (cf. p. 578 in [JP12]) and therefore using the BDG-inequality we have with $\mathbb{E}^n_{i-1}[\eta_i] = \frac{1}{r_n}$ that

$$\begin{aligned}
\mathbb{E}\left[\left|1-\sum_{j=i-1-r_n}^{i-2}\eta_j\right|^2\right] &= \mathbb{E}\left[\left|\sum_{j=i-1-r_n}^{i-2}\left(\eta_j-\frac{1}{r_n}\right)\right|^2\right] \\
&= \mathbb{E}\left[\left|\sum_{j=N_n(\tau^n_{i-1-r_n})}^{N_n(\tau^n_{i-2})}\left(\eta_j-\mathbb{E}^n_{j-1}[\eta_j]\right)\right|^2\right] \\
&= \mathbb{E}\left[\left|M_{\tau^n_{i-2}}-M_{\tau^n_{i-1-r_n}}\right|^2\right] \\
&\leq \mathbb{E}\left[[M,M]_{\tau^n_{i-2}}-[M,M]_{\tau^n_{i-1-r_n}}\right] \\
&= \mathbb{E}\left[\sum_{j=N_n(\tau^n_{i-1-r_n})}^{N_n(\tau^n_{i-2})}\left(\eta_j-\mathbb{E}^n_{j-1}[\eta_j]\right)^2\right] \\
&= \mathbb{E}\left[\sum_{j=N_n(\tau^n_{i-1-r_n})}^{N_n(\tau^n_{i-2})}\left(\frac{1}{r_n}(\phi_j-1)\right)^2\right] \leq \frac{1}{r_n}(1-n^\gamma)^2.
\end{aligned}$$

The two previous calculations then result in

$$\mathbb{E}\left[\left|1-\frac{1}{\Delta_n r_n}\int_{\tau^n_{i-2-r_n}}^{\tau^n_{i-2}}\frac{1}{\lambda_s}ds\right|\right] \leq K\left(\Delta_n^{1/2}+\frac{n^\gamma}{r_n^{1/2}}\right). \tag{6.25}$$

Finally, using the boundedness from below of ϕ_i in the assumption of this theorem we have that due to $\beta < 1$:

$$\left((\lambda_{\tau^n_{i-2}}\phi^n_i)^{1-\beta}+(\lambda_{\tau^n_{i-3}}\phi^n_{i-1})^{1-\beta}\right)^{p/\beta} < K. \tag{6.26}$$

Combing the last line and a Taylor expansion we can proceed with our calculations for all $i \geq r_n+3$:

$$\begin{aligned}
&|e^1_i-e^2_i| \\
&= \left((\lambda_{\tau^n_{i-2}}\phi^n_i)^{1-\beta}+(\lambda_{\tau^n_{i-3}}\phi^n_{i-1})^{1-\beta}\right)^{p/\beta}\left|\left(\frac{\Delta_n r_n}{\tau^n_{i-2}-\tau^n_{i-2-r_n}}\right)^{p/\beta-p}-\left(\frac{1}{\tau^n_{i-2}-\tau^n_{i-2-r_n}}\int_{\tau^n_{i-2-r_n}}^{\tau^n_{i-2}}\frac{1}{\lambda_s}ds\right)^{p/\beta-p}\right| \\
&= \left((\lambda_{\tau^n_{i-2}}\phi^n_i)^{1-\beta}+(\lambda_{\tau^n_{i-3}}\phi^n_{i-1})^{1-\beta}\right)^{p/\beta}|p/\beta-p|\left|\frac{\Delta_n r_n}{\tau^n_{i-2}-\tau^n_{i-2-r_n}}\right|^{p/\beta-p}\left|\epsilon_{i,n}^{p/\beta-p-1}\left(1-\frac{1}{\Delta_n r_n}\int_{\tau^n_{i-2-r_n}}^{\tau^n_{i-2}}\frac{1}{\lambda_s}ds\right)\right|,
\end{aligned}$$

for some $\epsilon_{i,n}$ between 1 and $\frac{1}{\Delta_n r_n}\int_{\tau^n_{i-2-r_n}}^{\tau^n_{i-2}}\frac{1}{\lambda_s}ds$. For an upper bound on $|\epsilon_{i,n}^{p/\beta-p-1}|$ we note that with (6.20):

$$\frac{1}{\Delta_n r_n}\int_{\tau^n_{i-2-r_n}}^{\tau^n_{i-2}}\frac{1}{\lambda_s}ds \geq \left(\frac{\tau^n_{i-2}-\tau^n_{i-2-r_n}}{\Delta_n r_n}\right)\inf_{\tau^n_{i-2-r_n}\leq s\leq\tau^n_{i-3}}\frac{1}{\lambda_s} \geq K \tag{6.27}$$

and therefore $|\epsilon_{i,n}^{p/\beta-p-1}| \leq K$. Combining the last bound on $|\epsilon_{i,n}^{p/\beta-p-1}|$ with (6.18), (6.25), (6.26) and (6.27) yields for $r_n+3 \leq i \leq N_n(1)$

$$\mathbb{E}|e^1_i-e^2_i| \leq Kn^{\gamma(p-p/\beta)}\left(\Delta_n^{1/2}+\frac{n^\gamma}{r_n^{1/2}}\right).$$

As $\gamma > 0$ can be chosen arbitrarily small, this finally results in

$$\frac{1}{n-r_n-2}\mathbb{E}\left[\sum_{i=r_n+3}^{N_n(1)} |e_i^1 - e_i^2|\right] \le \frac{1}{n-r_n-2}\sum_{i=r_n+3}^{nC} \mathbb{E}|e_i^1 - e_i^2|\mathbb{1}_{\{i\le N_n(1)\}} \to 0,$$

which, with the usual procedure, yields $E_n^1 - E_n^2 \xrightarrow{\mathbb{P}} 0$. For the difference $E_n^2 - E_n^3$ we again use a Taylor expansion and (6.26):

$$\begin{aligned}
&\mathbb{E}\left[|e_i^2 - e_i^3|\right]\\
&= \mathbb{E}\left[\left|\left(\frac{1}{\tau_{i-2}^n - \tau_{i-2-r_n}^n}\int_{\tau_{i-2-r_n}^n}^{\tau_{i-2}^n} \frac{1}{\lambda_s} ds\right)^{p/\beta - p} - \left(\frac{1}{\lambda_{\tau_{i-2}^n}}\right)^{p/\beta-p}\right| \left((\lambda_{\tau_{i-2}^n}\phi_i^n)^{1-\beta} + (\lambda_{\tau_{i-3}^n}\phi_{i-1}^n)^{1-\beta}\right)^{p/\beta}\right]\\
&\le K\,|p/\beta - p|\,\mathbb{E}\left[\left|\epsilon_{i,n}^{p/\beta-p-1}\left(\frac{1}{\tau_{i-2}^n - \tau_{i-2-r_n}^n}\int_{\tau_{i-2-r_n}^n}^{\tau_{i-2}^n} \frac{1}{\lambda_s} ds - \frac{1}{\lambda_{\tau_{i-2}^n}}\right)\right|\right]\\
&\le K\mathbb{E}\left[\left|\frac{1}{\tau_{i-2}^n - \tau_{i-2-r_n}^n}\int_{\tau_{i-2-r_n}^n}^{\tau_{i-2}^n}\left(\frac{1}{\lambda_s} - \frac{1}{\lambda_{\tau_{i-2}^n}}\right) ds\right|\right]\\
&\le K\mathbb{E}\left[\sup_{\tau_{i-r_n-2}^n < s < \tau_{i-3}^n}\left|\frac{1}{\lambda_s} - \frac{1}{\lambda_{\tau_{i-2}^n}}\right|\right], \qquad (6.28)
\end{aligned}$$

for some $\epsilon_{i,n}$ between $\frac{1}{\tau_{i-2}^n - \tau_{i-2-r_n}^n}\int_{\tau_{i-2-r_n}^n}^{\tau_{i-2}^n}\frac{1}{\lambda_s}ds$ and $\frac{1}{\lambda_{\tau_{i-2}^n}}$ which is bounded from above and below due to (6.17).

Applying (6.24) for (6.28) yields that for $r_n + 3 \le i \le N_n(1)$

$$\mathbb{E}\left[|e_i^2 - e_i^3|\right] \le (\Delta_n r_n)^{1/2},$$

which in return yields $E_n^2 - E_n^3 \xrightarrow{\mathbb{P}} 0$. Moving on to the difference $E_n^3 - Z_n$ we have that due to $p/\beta < 1$ and $|\cdot|^{p/\beta}$ being a norm then and the reverse triangular inequality for all $i \ge r_n + 3$

$$\begin{aligned}
|e_i^3 - \zeta_i| &= \left|\left((\phi_i^n)^{1-\beta} + \left(\frac{\lambda_{\tau_{i-3}^n}}{\lambda_{\tau_{i-2}^n}}\phi_{i-1}^n\right)^{1-\beta}\right)^{p/\beta} - \left((\phi_i^n)^{1-\beta} + (\phi_{i-1}^n)^{1-\beta}\right)^{p/\beta}\right|\\
&\le \left|\left(\frac{\lambda_{\tau_{i-3}^n}}{\lambda_{\tau_{i-2}^n}}\phi_{i-1}^n\right)^{1-\beta} - (\phi_{i-1}^n)^{1-\beta}\right|^{p/\beta}\\
&\le \left|\frac{|\lambda_{\tau_{i-3}^n}|^{1-\beta} - |\lambda_{\tau_{i-2}^n}|^{1-\beta}}{|\lambda_{\tau_{i-2}^n}|^{1-\beta}}\right|^{p/\beta}\\
&\le \left||\lambda_{\tau_{i-3}^n}|^{1-\beta} - |\lambda_{\tau_{i-2}^n}|^{1-\beta}\right|^{p/\beta}.
\end{aligned}$$

Applying Lemma 5.9 on the function $f(x) = x^{1-\beta}$ and the process λ (see proof of Lemma 5.9) and then using Lemma 5.1 we can conclude that for $r_n + 3 \le i \le N_n(1)$

$$\mathbb{E}\left[|e_i^3 - \zeta_i|\right] \le K\Delta_n^{p/(2\beta)}$$

which again leads to $E_n^3 - Z_n \xrightarrow{\mathbb{P}} 0$. It is left to show that $Z_n \xrightarrow{\mathbb{P}} \kappa_{p,\beta}^{p/\beta}$. Due to the assumed boundedness from below of ϕ_i we have $\zeta_i < K$ and therefore

$$\frac{1}{(n-r_n-2)^2}\sum_{i=r_n+3}^{N_n(1)+1}\mathbb{E}_{i-1}^n\left[\zeta_i^2\right] \xrightarrow{\mathbb{P}} 0,$$
$$\frac{1}{(n-r_n-2)^2}\sum_{i=r_n+3}^{N_n(1)+1}\mathbb{E}_{i-2}^n\left[\mathbb{E}_{i-1}^n\left[\zeta_i\right]^2\right] \xrightarrow{\mathbb{P}} 0.$$

Finally, applying Lemma 2.2.11 a) in [JP12] twice yields that for $\epsilon > 0$

$$\mathbb{P}\left(\left|\frac{1}{n-r_n-2}\sum_{i=r_n+3}^{N_n(1)+1}\zeta_i - \frac{1}{n-r_n-2}\sum_{i=r_n+3}^{N_n(1)+1}\mathbb{E}_{i-1}^n\left[\zeta_i\right]\right| > \epsilon\right) \to 0$$
$$\mathbb{P}\left(\left|\frac{1}{n-r_n-2}\sum_{i=r_n+3}^{N_n(1)+1}\mathbb{E}_{i-1}^n\left[\zeta_i\right] - \frac{1}{n-r_n-2}\sum_{i=r_n+3}^{N_n(1)+1}\mathbb{E}_{i-2}^n\left[\mathbb{E}_{i-1}^n\left[\zeta_i\right]\right]\right| > \epsilon\right) \to 0.$$

Using that $\frac{1}{n-r_n-2}\sum_{i=r_n+3}^{N_n(1)+1}\mathbb{E}_{i-2}^n\left[\mathbb{E}_{i-1}^n\left[\zeta_i\right]\right] = \frac{N_n(1)-r_n-1}{n-r_n-2}\kappa_{p,\beta}^{p/\beta}$ we get

$$Z_n = \frac{n-r_n-2}{N_n(1)-r_n-2}\frac{1}{n-r_n-2}\sum_{i=r_n+3}^{N_n(1)+1}\zeta_i \xrightarrow{\mathbb{P}} \kappa_{p,\beta}^{p/\beta}.$$

□

Using exactly the same arguments as in the previous proof and omitting E_n^1 in the decomposition (6.12) we get the following result:

Corollary 6.1. *Assuming that ϕ from Assumption C fulfills $M < \phi$ for some $0 < M < 1$, we have for*

$$\chi_i = \left(\left(\frac{r_n}{\tau_{i-2}^n - \tau_{i-2-r_n}^n}\right)^{1-\hat{\beta}_n}\left(\left(\tau_i^n - \tau_{i-1}^n\right)^{1-\hat{\beta}_n} + \left(\tau_{i-1}^n - \tau_{i-2}^n\right)^{1-\hat{\beta}_n}\right)\right)$$

that

$$\widehat{\kappa}_n := \frac{1}{N_n(1)-r_n-2}\sum_{i=r_n+3}^{N_n(1)}\chi_i \xrightarrow{\mathbb{P}} \mathbb{E}[\phi^{1-\beta} + (\phi')^{1-\beta}] = \kappa_{\beta,\beta}. \tag{6.29}$$

Lemma 6.2. *Let $p > 0$, $\hat{\beta}_n$ be a consistent estimator for β and $\widehat{\kappa}_n^p$ be consistent estimator for $\kappa_{p,\beta}^{p/\beta} > 0$. Then*

$$(\widehat{\kappa}_n^p)^{\hat{\beta}_n/p} \xrightarrow{\mathbb{P}} \kappa_{p,\beta}. \tag{6.30}$$

Proof. We start the proof by noting that

$$(\widehat{\kappa}_n^p)^{\hat{\beta}_n/p} - \kappa_{p,\beta} = (\widehat{\kappa}_n^p)^{\hat{\beta}_n/p} - (\widehat{\kappa}_n^p)^{\beta/p} + (\widehat{\kappa}_n^p)^{\beta/p} - \left(\kappa_{p,\beta}^{p/\beta}\right)^{\beta/p}.$$

For the first difference we define the set $A_n := \left\{\left|\widehat{\kappa}_n^p - \kappa_{p,\beta}^{p/\beta}\right| > \frac{\kappa_{p,\beta}^{p/\beta}}{2}\right\} \cup \left\{|\beta - \hat{\beta}_n| > \frac{\beta-1}{2}\right\}$ and split accordingly

$$\left|(\widehat{\kappa}_n^p)^{\hat{\beta}_n/p} - (\widehat{\kappa}_n^p)^{\beta/p}\right| = \left|(\widehat{\kappa}_n^p)^{\hat{\beta}_n/p} - (\widehat{\kappa}_n^p)^{\beta/p}\right| \mathbb{1}_{A_n} + \left|(\widehat{\kappa}_n^p)^{\hat{\beta}_n/p} - (\widehat{\kappa}_n^p)^{\beta/p}\right| \mathbb{1}_{A_n^C}.$$

We find that

$$\begin{aligned}\left|(\widehat{\kappa}_n^p)^{\hat{\beta}_n/p} - (\widehat{\kappa}_n^p)^{\beta/p}\right| \mathbb{1}_{A_n} \leq &\left|(\widehat{\kappa}_n^p)^{\hat{\beta}_n/p} - (\widehat{\kappa}_n^p)^{\beta/p}\right| \mathbb{1}_{\left\{\left|\widehat{\kappa}_n^p - \kappa_{p,\beta}^{p/\beta}\right| > \frac{\kappa_{p,\beta}^{p/\beta}}{2}\right\}} \\ &+ \left|(\widehat{\kappa}_n^p)^{\hat{\beta}_n/p} - (\widehat{\kappa}_n^p)^{\beta/p}\right| \mathbb{1}_{\left\{|\beta-\hat{\beta}_n| > \frac{\beta-1}{2}\right\}}\end{aligned}$$

and similar to (6.13) we have for all $\epsilon > 0$

$$\mathbb{P}\left(\left|(\widehat{\kappa}_n^p)^{\hat{\beta}_n/p} - (\widehat{\kappa}_n^p)^{\beta/p}\right| \mathbb{1}_{\left\{\left|\widehat{\kappa}_n^p - \kappa_{p,\beta}^{p/\beta}\right| > \frac{\kappa_{p,\beta}^{p/\beta}}{2}\right\}} > \epsilon\right) \leq \mathbb{P}\left(\left\{\left|\widehat{\kappa}_n^p - \kappa_{p,\beta}^{p/\beta}\right| > \frac{\kappa_{p,\beta}^{p/\beta}}{2}\right\} > \epsilon\right) \to 0,$$

$$\mathbb{P}\left(\left|(\widehat{\kappa}_n^p)^{\hat{\beta}_n/p} - (\widehat{\kappa}_n^p)^{\beta/p}\right| \mathbb{1}_{\left\{|\beta-\hat{\beta}_n| > \frac{\beta-1}{2}\right\}} > \epsilon\right) \xrightarrow{\mathbb{P}} 0,$$

which yields $\left|(\widehat{\kappa}_n^p)^{\hat{\beta}_n/p} - (\widehat{\kappa}_n^p)^{\beta/p}\right| \mathbb{1}_{A_n} \xrightarrow{\mathbb{P}} 0$. For $\left|(\widehat{\kappa}_n^p)^{\hat{\beta}_n/p} - \left(\kappa_{p,\beta}^{p/\beta}\right)^{\beta/p}\right| \mathbb{1}_{A_n^C}$ we define the function $f_{\widehat{\kappa}_n^p,p}(x) = (\widehat{\kappa}_n^p)^{x/p}$ with derivative $f'_{\widehat{\kappa}_n^p,p}(x) = \frac{\log(\widehat{\kappa}_n^p)(\widehat{\kappa}_n^p)^{x/p}}{p}$ and see that with a Taylor-Expansion

$$\begin{aligned}(\widehat{\kappa}_n^p)^{\hat{\beta}_n/p} - (\widehat{\kappa}_n^p)^{\beta/p} &= f_{\widehat{\kappa}_n^p,p}(\hat{\beta}_n) - f_{\widehat{\kappa}_n^p,p}(\beta) \\ &= \frac{\log(\widehat{\kappa}_n^p)(\widehat{\kappa}_n^p)^{\epsilon_n/p}}{p}(\hat{\beta}_n - \beta)\end{aligned} \tag{6.31}$$

for some $\epsilon_n \in (\hat{\beta}_n, \beta)$. On A_n^C we have as in the proof of Theorem 6.2 that $\hat{\beta}_n \in (1,3)$ and additionally $\widehat{\kappa}_n^p \in \left[\frac{\kappa_{p,\beta}^{p/\beta}}{2}, \frac{3}{2}\kappa_{p,\beta}^{p/\beta}\right]$. Therefore, we see with (6.31) that

$$\left|(\widehat{\kappa}_n^p)^{\hat{\beta}_n/p} - (\widehat{\kappa}_n^p)^{\beta/p}\right| \mathbb{1}_{A_n^C} = \left|\frac{\log(\widehat{\kappa}_n^p)(\widehat{\kappa}_n^p)^{\epsilon_n/p}}{p}(\hat{\beta}_n - \beta)\right| \mathbb{1}_{A_n^C} \leq K|\hat{\beta}_n - \beta| \xrightarrow{\mathbb{P}} 0.$$

For the difference $\left|(\widehat{\kappa}_n^p)^{\beta/p} - \left(\kappa_{p,\beta}^{p/\beta}\right)^{\beta/p}\right|$ we proceed similarly with $B_n := \left\{\left|\widehat{\kappa}_n^p - \kappa_{p,\beta}^{p/\beta}\right| > \frac{\kappa_{p,\beta}^{p/\beta}}{2}\right\}$ and like before we have $\left|(\widehat{\kappa}_n^p)^{\beta/p} - \left(\kappa_{p,\beta}^{p/\beta}\right)^{\beta/p}\right| \mathbb{1}_{B_n} \xrightarrow{\mathbb{P}} 0$. Finally due to $\widehat{\kappa}_n^p$ being consistent and $x \mapsto x^{\beta/p}$ being a continuous function on $\left[\frac{\kappa_{p,\beta}^{p/\beta}}{2}, \frac{3}{2}\kappa_{p,\beta}^{p/\beta}\right]$ we find that $\left|(\widehat{\kappa}_n^p)^{\beta/p} - \left(\kappa_{p,\beta}^{p/\beta}\right)^{\beta/p}\right| \mathbb{1}_{B_n^C} \xrightarrow{\mathbb{P}} 0$, which finishes the proof. □

The final piece that is missing to provide a normalization without prior knowledge of β is the following Lemma.

Lemma 6.3. *Let $\hat\beta_n$ be a consistent estimator for β such that there exists a $\varsigma > 0$ with*

$$\left|\beta - \hat\beta_n\right| n^{\varsigma} \xrightarrow{\mathbb{P}} 0 \tag{6.32}$$

and $u_n \asymp n^{-\varrho}$ with a $\varrho \in (0,1)$. Then we have that

$$(u_n)^{\hat\beta_n/2} - (u_n)^{\beta/2} \xrightarrow{\mathbb{P}} 0.$$

Proof. As before we define $A_n := \left\{|\beta - \hat\beta_n| > \frac{\beta-1}{2}\right\}$ and split up

$$\left|(u_n)^{\hat\beta_n/2} - (u_n)^{\beta/2}\right| = \left|(u_n)^{\hat\beta_n/2} - (u_n)^{\beta/2}\right| \mathbb{1}_{A_n} + \left|(u_n)^{\hat\beta_n/2} - (u_n)^{\beta/2}\right| \mathbb{1}_{A_n^C},$$

where we already know from the previous proofs that $\left|(u_n)^{\hat\beta_n/2} - (u_n)^{\beta/2}\right| \mathbb{1}_{A_n} \xrightarrow{\mathbb{P}} 0$. For $\left|(u_n)^{\hat\beta_n/2} - (u_n)^{\beta/2}\right| \mathbb{1}_{A_n^C}$ we proceed as in the former proof and define the function $f_{u_n}(x) = (u_n)^{x/2}$ with derivative $f'_{u_n}(x) = \frac{\log(u_n)(u_n)^{x/2}}{2}$ and see that with a Taylor expansion

$$\left|(u_n)^{\hat\beta_n/2} - (u_n)^{\beta/2}\right| \mathbb{1}_{A_n^C} = \left|\frac{\log(u_n)(u_n)^{\epsilon_n/2}}{2}\right| \left|\hat\beta_n - \beta\right| \mathbb{1}_{A_n^C} \tag{6.33}$$

for some $\epsilon_n \in (1,3)$. We note that $(u_n)^{\epsilon_n/2} < 1$ and

$$|\log(u_n)| \left|\hat\beta_n - \beta\right| = |\log(u_n)|\, u_n^{-\varsigma} u_n^{\varsigma} \left|\hat\beta_n - \beta\right| \xrightarrow{\mathbb{P}} 0$$

due to (6.32) and $|\log(u_n)|u_n^{-\varsigma} \to 0$ when $u_n \to 0$. The last equation then yields $\left|(u_n)^{\hat\beta_n/2} - (u_n)^{\beta/2}\right| \mathbb{1}_{A_n^C}$ which concludes the proof. □

The previous theorem and lemmas now finally culminate in a central limit theorem that works without prior knowledge of any (unknown) model specific parameters.

Theorem 6.3. *Under the conditions of Corollary 5.3, $\varrho < 1/\beta$ and $v_n = \rho u_n$ with $0 < \rho < 1$ we have for the estimator of $\hat\beta(p, u_n, v_n)$ from (5.98), $\widehat{\kappa}_n^p$ from (6.11) and $\widehat{\kappa}_n$ from (6.29), both using $\hat\beta(p, u_n, v_n)$ as the estimator for β, that with*

$$Var_{p,\rho}(\beta, \kappa_{\beta,\beta}, \kappa_{p,\beta}) = \frac{(\rho^\beta + 1)(4 - 2^\beta) - 2(2 + 2\rho^\beta - (1+\rho)^\beta - (1-\rho)^\beta)}{\kappa_{\beta,\beta}\rho^\beta \log(1/\rho)^2 \left(\frac{2^p \Gamma((1+p)/2)\Gamma(1-p/\beta)}{\sqrt{\pi}\Gamma(1-p/2)}\right)^{-\beta/p} \kappa_{p,\beta}^{-1}}$$

we have the convergence

$$\frac{u_n^{\hat\beta(p,u_n,v_n)/2}\sqrt{N_n(1)}}{\sqrt{Var_{p,\rho}(\hat\beta(p,u_n,v_n), \widehat{\kappa}_n, (\widehat{\kappa}_n^p)^{\hat\beta(p,u_n,v_n)/p})}}(\hat\beta(p,u_n,v_n) - \beta) \xrightarrow{\mathcal{L}} X, \tag{6.34}$$

where X is a normal distributed random variable with mean 0 and variance 1.

Proof. The idea of the proof is to combine the previous results to find that $Var_{p,\rho}(\hat{\beta}(p,u_n,v_n),\widehat{\kappa}_n,(\widehat{\kappa}_n^p)^{\hat{\beta}(p,u_n,v_n)/p})$ is indeed a consistent estimator for $\frac{(\rho^\beta+1)(4-2^\beta)-2(2+2\rho^\beta-(1+\rho)^\beta-(1-\rho)^\beta)}{\kappa_{\beta,\beta}\rho^\beta\log(1/\rho)^2C_{p,\beta}}$ and then use that the convergence in (5.99) is stably in law and combine it with the consistency of $Var_{p,\rho}(\hat{\beta}(p,u_n,v_n),\widehat{\kappa}_n,(\widehat{\kappa}_n^p)^{\hat{\beta}(p,u_n,v_n)/p})$ and Lemma 6.3.

In order to calculate $C_{p,\beta}=\frac{A_\beta}{\mu_{p,\beta}\kappa_{p,\beta}}$ we find that according to p.6 in [Tod15] we have the following formula

$$\frac{A_\beta}{\mu_{p,\beta}}=\left(\frac{2^p\Gamma((1+p)/2)\Gamma(1-p/\beta)}{\sqrt{\pi}\Gamma(1-p/2)}\right)^{-\beta/p}. \tag{6.35}$$

We note that for $0<p<\beta/2$ and $\beta\in(1,3)$ the right hand side of (6.35) is continuously differentiable in β and using the same techniques as before we find that for our consistent estimator $\hat{\beta}(p,u_n,v_n)$ we have

$$\left(\frac{2^p\Gamma((1+p)/2)\Gamma(1-p/\hat{\beta}(p,u_n,v_n))}{\sqrt{\pi}\Gamma(1-p/2)}\right)^{-\hat{\beta}(p,u_n,v_n)/p}\xrightarrow{\mathbb{P}}\frac{A_\beta}{\mu_{p,\beta}}. \tag{6.36}$$

With similar arguments we also find that

$$\begin{aligned}&\frac{(\rho^{\hat{\beta}(p,u_n,v_n)}+1)(4-2^{\hat{\beta}(p,u_n,v_n)})-2(2+2\rho^{\hat{\beta}(p,u_n,v_n)}-(1+\rho)^{\hat{\beta}(p,u_n,v_n)}-(1-\rho)^{\hat{\beta}(p,u_n,v_n)})}{\rho^{\hat{\beta}(p,u_n,v_n)}}\\&\xrightarrow{\mathbb{P}}\frac{(\rho^\beta+1)(4-2^\beta)-2(2+2\rho^\beta-(1+\rho)^\beta-(1-\rho)^\beta)}{\rho^\beta}.\end{aligned} \tag{6.37}$$

Combining Lemma 6.2 with Theorem 6.2 and (6.36) we have

$$\left(\frac{2^p\Gamma((1+p)/2)\Gamma(1-p/\hat{\beta}(p,u_n,v_n))}{\sqrt{\pi}\Gamma(1-p/2)}\right)^{-\hat{\beta}(p,u_n,v_n)/p}\frac{1}{(\widehat{\kappa}_n^p)^{\hat{\beta}(p,u_n,v_n)/p}}\xrightarrow{\mathbb{P}}C_{p,\beta}.$$

Together with the last line (6.37) and Corollary 6.1 we finally get that

$$Var_{p,\rho}(\hat{\beta}(p,u_n,v_n),\widehat{\kappa}_n,(\widehat{\kappa}_n^p)^{\hat{\beta}(p,u_n,v_n)/p})\xrightarrow{\mathbb{P}}Var_{p,\rho}(\beta,\kappa_{\beta,\beta},\kappa_{p,\beta})$$

which, in conjunction with Lemma 6.3, finishes the proof. □

We finish this section by applying Theorem 6.3 to our simulation routine from the previous sections. That is we additionally implement the estimators $\widehat{\kappa}_n^p,\widehat{\kappa}_n$ from Theorem 6.2 and Corollary 6.1 to build the estimator $Var_{p,\rho}(\hat{\beta}(p,u_n,v_n),\widehat{\kappa}_n,(\widehat{\kappa}_n^p)^{\hat{\beta}(p,u_n,v_n)/p})$ which we then use to build the normalization from Theorem 6.3.

We start the discussion by singling out results for the estimator $(\widehat{\kappa}_n^p)^{\hat{\beta}_n/p}$ and compare it with $\kappa_{p,\beta}:=\mathbb{E}[(\phi^{1-\beta}+(\phi')^{1-\beta})^{\frac{p}{\beta}}]^{\frac{\beta}{p}}$ for the same set of parameters as in Section 6.3, i.e. $p = 1/2, \beta \in \{1.1, 1.3, 1.5, 1.7, 1.9\}, \rho \in \{1/2, 2\}, N = 1000$ and $n = 1000$ with again roughly $N_n(1) \approx 520$ observations before the terminal time $T = 1$. There is much freedom in choosing $r_n \asymp n^{\Psi}$ from Theorem 6.2. However, a choice of $r_n = N_n(1)^{0.8}$ (large enough for a good estimate of $\frac{1}{\lambda_s}$, small enough for a sufficient number of summands in $\frac{1}{N_n(1)-r_n-2}\sum_{i=r_n+3}^{N_n(1)} \chi_i$) seems to provide adequate results in our setting. Similar to the previous section we collect the results in a table where outside the brackets we have results for $\rho = 1/2$ and inside the brackets for $\rho = 2$.

β	Mean of $(\widehat{\kappa}_n^p)^{\hat{\beta}(p,u_n,v_n)/p}$	Empirical Variance	Theoretical Value of $\kappa_{p,\beta}$
1.1	2.2574 (2.0919)	0.2172 (0.0416)	2.108
1.3	2.5161 (2.2971)	0.3811 (0.1136)	2.3864
1.5	2.8959 (2.5961)	0.4695 (0.2039)	2.7769
1.7	3.2834 (2.9131)	0.3179 (0.2343)	3.3066
1.9	3.6522 (3.2987)	0.425 (0.2315)	4.0041

We note that already for this limited number of observations the estimated values are relatively close to the theoretical ones and for this reason we omit a second table with $n = 10000$. Furthermore, as $Var_{p,\rho}(\beta, \kappa_{\beta,\beta}, \kappa_{p,\beta})$ is not linear in β, we do not expect the empirical mean of $(\widehat{\kappa}_n^p)^{\hat{\beta}(p,u_n,v_n)/p}$ to be exactly the theoretical value of $\kappa_{p,\beta}$. Finally $\kappa_{p,\beta}$ does not depend on ρ. Its choice only affects the quality of the estimator $\hat{\beta}(p, u_n, v_n)$ and through this the estimates in the first column.

At last, we present QQ-plots for the normalized estimator of the form (6.34) with again the same parameter configuration for $\rho \in \{1/2, 2\}, n \in \{1000, 10000\}$. The reference for the theoretical quantiles is a normal distribution with mean 0 and variance 1.

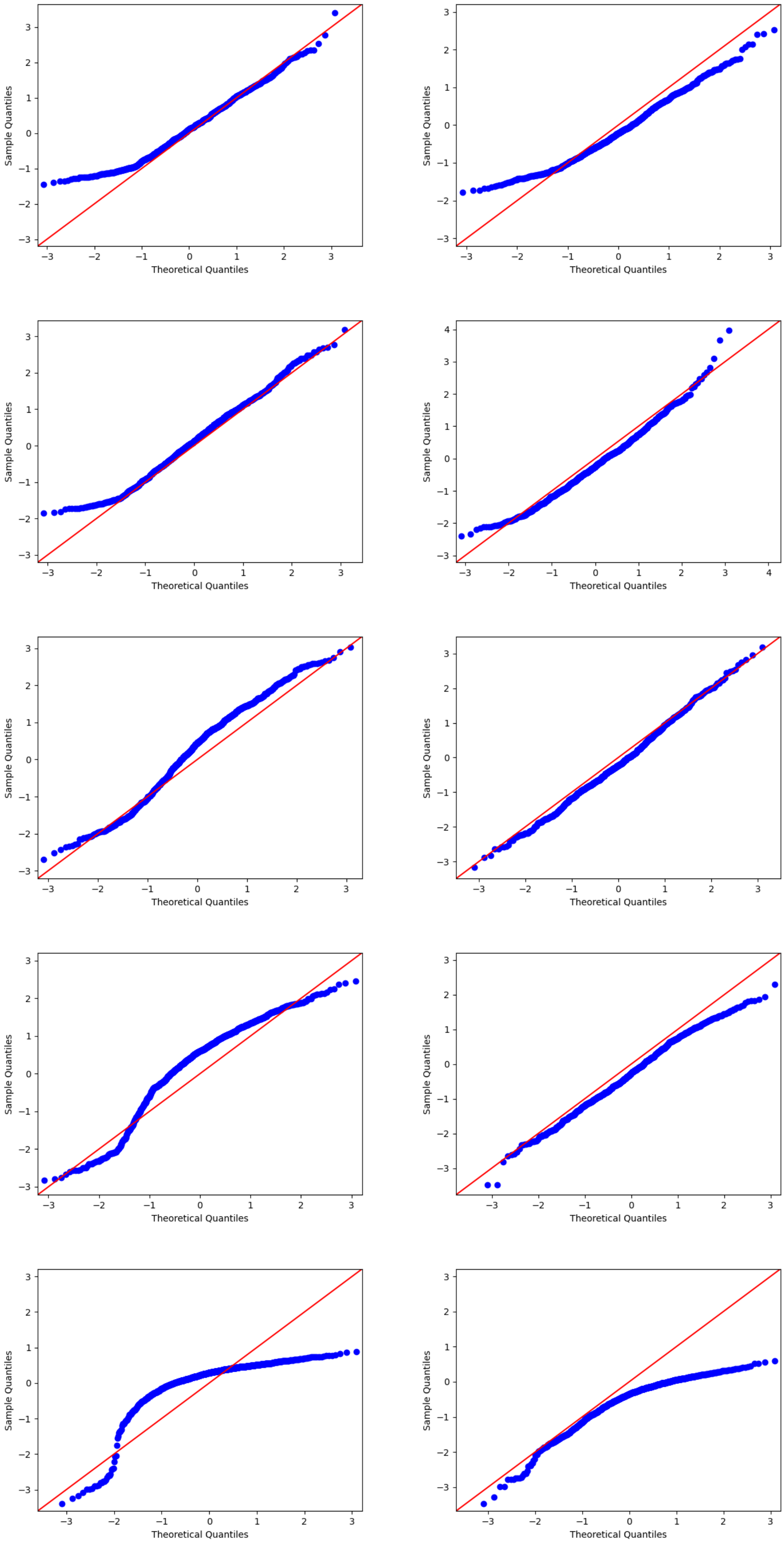

Figure 6.3: $N = 1000, n = 1000, \beta \in \{1.1, 1.3, 1.5, 1.7, 1.9\}$
left side $\rho = 0.5$, right side $\rho = 2$

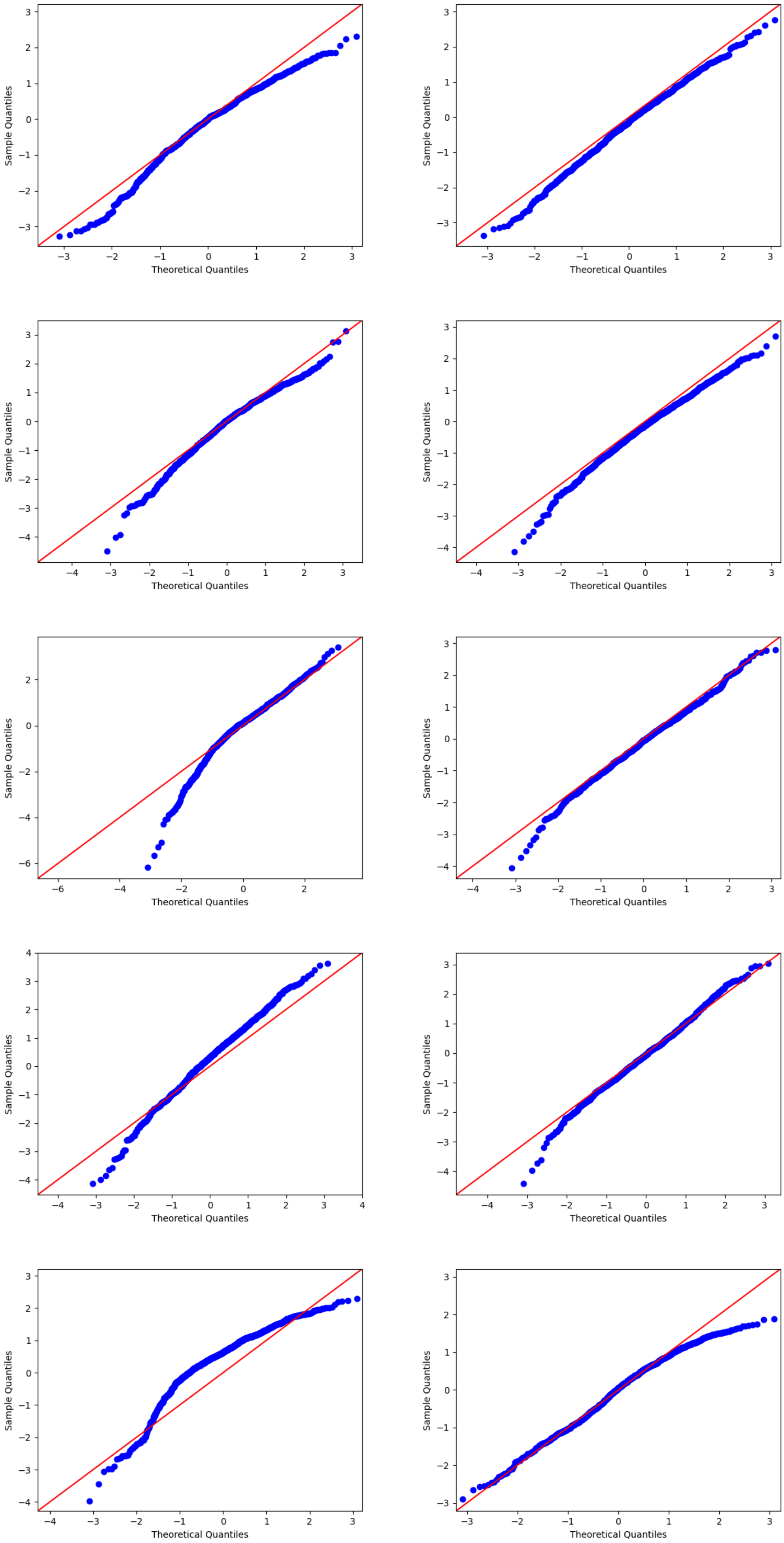

Figure 6.4: $N = 1000, n = 10000, \beta \in \{1.1, 1.3, 1.5, 1.7, 1.9\}$
left side $\rho = 0.5$, right side $\rho = 2$

The most notable difference between these QQ-plots and the previous ones is that, even though the variance is now estimated and therefore less accurate, in some instances the distributional shape looks closer a normal distribution than before. In the previous plots the upper quantiles/largest outcomes of $u_n^{\beta/2}\sqrt{N_n(1)}(\hat{\beta}(p,u_n,v_n)-\beta)$ were not big enough to fit the quantiles of a normal distribution due to the boundedness of $\hat{\beta}(p,u_n,v_n)$. There are now two converse effects that cause a change. On the one hand overestimation of β leads to smaller values for $u_n^{\hat{\beta}(p,u_n,v_n)/2}$, however, on the other hand $Var_{p,\rho}(\beta,\kappa_{\beta,\beta},\kappa_{p,\beta})$ is monotone decreasing in β and therefore overestimation of β leads to larger values of $\frac{1}{\sqrt{Var_{p,\rho}(\hat{\beta}(p,u_n,v_n),\widehat{\kappa}_n,(\widehat{\kappa}_n^p)^{\hat{\beta}(p,u_n,v_n)/p})}}$ both contributing to the size of

$$\frac{u_n^{\hat{\beta}(p,u_n,v_n)/2}\sqrt{N_n(1)}}{\sqrt{Var_{p,\rho}(\hat{\beta}(p,u_n,v_n),\widehat{\kappa}_n,(\widehat{\kappa}_n^p)^{\hat{\beta}(p,u_n,v_n)/p})}}(\hat{\beta}(p,u_n,v_n)-\beta).$$

Which effect dominates which is dependent on the choice of β and ρ but also on n as it determines the size of u_n. Therefore we have these very different looking plots across our choice of parameters without a clear pattern compared to the previous ones.

Appendix: Python Implementation

```python
import numpy
import math
import pylab
import statsmodels.api as sm
import scipy.stats

### Model parameters and paramters of the estimator that are not choosen
### in dependence of N_n(1)
N = 1000
n = 100000
beta = 1.9
T = 1
p = 0.5
rho = 2


### Sperate Monte Carlo simulation to determine values of kappa_{p,beta}
### and kappa_{beta,beta}
def kappa(sample, beta, p):
    m = math.floor(len(sample) / 2)
    a = numpy.sum(
        (numpy.power(sample[0:m], 1 - beta) + numpy.power(
            sample[m:len(sample)], 1 - beta)) ** (
                p / beta))
    return (a / m) ** (beta / p)


phi = numpy.random.exponential(1, 100000)
phi = numpy.maximum(0.1, phi)
K = numpy.mean(phi)

kappa_p = kappa(phi, beta, p)
kappa_beta = kappa(phi, beta, beta)
```

```
### simulation of one sample path of the model described in section 6.2
def sample_path(n, T, X0, alpha0, sigma0, lambdaa0, beta):
    delta_n = T / n

    ### Random numbers needed for the simulation
    ### of a stable random variable
    gamma = numpy.random.rand(n, 1) * math.pi - math.pi / 2
    W = numpy.random.exponential(1, n)

    ### Phi from the observation scheme
    phi = numpy.random.exponential(1, n)
    phi = numpy.maximum(0.1, phi)
    phi = phi / K

    ### seperate Brownian motions in alpha/sigma and lambda
    W_tilde = numpy.random.normal(0, 1, n)
    W_tilde2 = numpy.random.normal(0, 1, n)

    X = numpy.empty(n + 1, dtype=float)
    S = numpy.empty(n + 1, dtype=float)
    alpha = numpy.empty(n + 1, dtype=float)
    sigma = numpy.empty(n + 1, dtype=float)
    lambdaa = numpy.empty(n + 2, dtype=float)
    tau = numpy.empty(n + 2, dtype=float)
    X[0] = X0
    alpha[0] = alpha0
    sigma[0] = sigma0
    S[0] = 0
    tau[0] = 0
    tau[-1] = 0
    lambdaa[:] = 1
    lambdaa[0] = lambdaa0

    ### Euler scheme for sample path
    for i in range(0, n):
        tau[i + 1] = tau[i] + delta_n * phi[i] * lambdaa[i - 1]
        if tau[i + 1] < T:
            lambdaa[i] = lambdaa[i - 1] + (5 - lambdaa[i - 1]) * (
                            tau[i] - tau[i - 1]) + lambdaa[
                                i - 1] * W_tilde2[
                                i] * math.sqrt((tau[i] - tau[i - 1]))
```

```
            S[i + 1] = math.sin(beta * gamma[i]) / math.pow(
                math.cos(gamma[i]),
                1 / beta) * math.pow(
                math.cos((1 - beta) * gamma[i]) / W[i],
                ((1 - beta) / beta))
            alpha[i + 1] = alpha[i] + (2 - 2 * alpha[i]) * (
                        tau[i + 1] - tau[i]) + 2 * W_tilde[
                               i] * math.sqrt(
                (tau[i + 1] - tau[i]))
            sigma[i + 1] = sigma[i] + alpha[i] * W_tilde[
                i] * math.sqrt((tau[i + 1] - tau[i]))
            X[i + 1] = X[i] + alpha[i] * (tau[i + 1] - tau[i]) + \
                       sigma[i] * S[i + 1] * math.pow(
                (tau[i + 1] - tau[i]),
                1 / beta)
        else:
            X[i + 1] = None

    X = X[~numpy.isnan(X)]
    tau = tau[0:len(X)]
    return [X, tau]


### Quotient of A_beta/mu_{p,beta}
def C_p(p, beta):
    if beta > p:
        a = 2 ** p * math.gamma((1 + p) / 2) * math.gamma(
            1 - p / beta)
        b = math.sqrt(math.pi) * math.gamma(1 - p / 2)
        return math.pow(a / b, -beta / p)
    else:
        return numpy.nan


### Estimator for sigma_s scaled by mu_{p,beta}
def V_i(delta_X_tau, p, k, i):
    return numpy.sum(
        numpy.power(abs(
            delta_X_tau[(i - k - 1):(i - 1)] - delta_X_tau[
                                                           (i - k - 2):(i - 2)]),
                    p)) / k
```

```python
### Implementation of the impericial characteristic function
def L_n(p, u, delta_X_tau, k):
    a = 0
    Nn = len(delta_X_tau)
    for i in range(k + 2, Nn):
        a = a + math.cos(
            u * (delta_X_tau[i] - delta_X_tau[i - 1]) / math.pow(
                V_i(delta_X_tau, p, k, i), 1 / p))
    return a / (Nn - k - 2)


### Estimator for beta from Theorem 5.2
def betahat(p, u, v, delta_X, delta_tau, k):
    delta_X_tau = delta_X * (1 / delta_tau)
    a = math.log(-(L_n(p, u, delta_X_tau, k) - 1)) - math.log(
        -(L_n(p, v, delta_X_tau, k) - 1))
    return a / math.log(u / v)


### Estimator for kappa_{p,beta} from Theorem 6.2
def kappa_hat(delta_tau, tau, beta, p, r):
    a = 0
    b = len(delta_tau) - 1
    for i in range(r, b):
        lambda_est = r / (tau[i] - tau[i - r])
        a = a + math.pow(
            (lambda_est * delta_tau[i]) ** (1 - beta) + (
                        lambda_est * delta_tau[i + 1]) ** (
                    1 - beta),
            p / beta)
    return math.pow(a / (b - r), beta / p)


### Variance from Theorem 5.2
def Var_beta(rho, beta, kappa_beta, C_p):
    if rho < 1:
        a = (rho ** beta + 1) * (4 - 2 ** beta) - 2 * (
                2 + 2 * (rho ** beta) - (1 + rho) ** beta - (
                    1 - rho) ** beta)
    else:
        a = (rho ** beta + 1) * (4 - 2 ** beta) - 2 * (
                2 + 2 * (rho ** beta) - (1 + rho) ** beta - (
                rho - 1) ** beta)
```

```
    b = kappa_beta * (rho ** beta) * (math.log(1 / rho) ** 2) * C_p
    if beta < 2:
        return (a / b)
    else:
        return numpy.nan


result = numpy.empty(N, dtype=float)
resultnormal = numpy.empty(N, dtype=float)
resultnormal_self = numpy.empty(N, dtype=float)
result_Nn = numpy.empty(N, dtype=float)
kappa_sample = numpy.zeros(N)
var_self = numpy.zeros(N)

### loop over N samples
for j in range(0, N):
    [X, tau] = sample_path(n, T, 1, 1, 1, 1, beta)

    ### choose parameters of the estimator according to Corollary 5.3
    Nn = len(X)
    k = math.floor(math.pow(Nn, 0.6))
    u = math.pow(Nn, -0.33)
    v = rho * u
    r = math.floor(Nn ** 0.8)

    delta_X = numpy.diff(X)
    delta_tau = numpy.diff(tau)

    ### apply the estimator for beta and normalization from Theorem 5.2
    result[j] = betahat(p, u, v, delta_X, delta_tau, k)
    resultnormal[j] = (result[j] - beta) * u ** (
                beta / 2) * math.sqrt(Nn - k)
    result_Nn[j] = Nn

    ### Estimate kappa_{p,beta}
    kappa_sample[j] = kappa_hat(delta_tau, tau, result[j], p, r)
    ### Apply normalization from Theorem 6.3
    var_self[j] = Var_beta(rho, result[j],
                           kappa_hat(delta_tau, tau, result[j],
                                     result[j], r),
                           C_p(p, result[j]) / kappa_sample[j])
    resultnormal_self[j] = (result[j] - beta) * u ** (
                result[j] / 2) * math.sqrt(
```

```
        Nn - k) / math.sqrt(var_self[j])
    print(j)

resultnormal_self = resultnormal_self[~numpy.isnan(resultnormal_self)]
var_self = var_self[~numpy.isnan(var_self)]

print('Mean beta_hat: ' + str(numpy.mean(result)))
print('Empirical Variance: ' + str(numpy.var(result)))
print('Theoretical Variance: ' + str(
    Var_beta(rho, beta, kappa_beta, C_p(p, beta) / kappa_p)))
print('Average number of observations: ' + str(numpy.mean(result_Nn)))
print('kappa_{p,beta} theoretical: ' + str(
    kappa_p) + ' #### kappa_sample: ' + str(
    numpy.mean(kappa_sample[~numpy.isnan(
        kappa_sample)])) + ' ### kappa_variance: ' + str(
    numpy.var(kappa_sample[~numpy.isnan(kappa_sample)])))
print('Selfnormalized Variance: ' + str(numpy.var(resultnormal_self)))

sm.qqplot(resultnormal, line='45',
          scale=math.sqrt(Var_beta(rho, beta, kappa_beta,
                                   C_p(p, beta) / kappa_p)))
sm.qqplot(resultnormal_self, line='45', scale=1)
pylab.show()
```

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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit – abgesehen von der Beratung durch meinen Betreuer Herrn Prof. Dr. Mathias Vetter – nach Inhalt und Form eigenständig angefertigt und nur die angegebenen Hilfsmittel benutzt habe. Dabei habe ich die Regeln guter wissenschaftlicher Praxis der *Deutschen Forschungsgemeinschaft* eingehalten.
Die Arbeit hat weder ganz noch in Teilen an einer anderen Stelle im Rahmen eines Prüfungsverfahrens vorgelegen oder ist anderweitig zur Veröffentlichung eingereicht worden. Weiter ist mir kein akademischer Grad entzogen worden.

Kiel, den 20.08.2020