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On the GIT Stratification of Prehomogeneous Vector Spaces I

by

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Abstract. We determine the set which parametrizes the GIT stratification for four prehomogeneous vector spaces in this paper.

1. Introduction

This is part one of a series of four papers. Let k be a perfect field. In this series of papers, we determine the GIT (geometric invariant theory) stratification of the following prehomogeneous vector spaces over k.

(1) $G = \mathrm{GL}_3 \times \mathrm{GL}_3 \times \mathrm{GL}_2$, $V = \mathrm{Aff}^3 \otimes \mathrm{Aff}^3 \otimes \mathrm{Aff}^2$.
(2) $G = \mathrm{GL}_6 \times \mathrm{GL}_2$, $V = \wedge^2\mathrm{Aff}^6 \otimes \mathrm{Aff}^2$.
(3) $G = \mathrm{GL}_5 \times \mathrm{GL}_4$, $V = \wedge^2\mathrm{Aff}^5 \otimes \mathrm{Aff}^4$.
(4) $G = \mathrm{GL}_8$, $V = \wedge^3\mathrm{Aff}^8$.

If the base field is $\mathbb{C}$ then orbits of (1)–(4) have been determined in [7, pp. 385–387], [6, pp. 456, 457], [12], [2] (see [11, p. 19] also) respectively.

The notion of GIT stratification was established by Ness, Kempf and Kirwan in [5], [4], [10], [8]. This notion will be reviewed in Section 2. If the base field k is algebraically closed then the GIT stratification gives us the orbit decomposition. The advantage of the GIT stratification is that it answers the rationality question of orbits. For the rationality of the GIT stratification, see [16] (if the group is split, the rationality follows easily from [4]). For the prehomogeneous vector spaces (1)–(4), we determine all orbits rationally over k. Moreover, the inductive structure of strata is guaranteed. Some smaller prehomogeneous vector spaces has been considered in [3] by naive method.

We refer to parts of this series of papers as Part I–Part IV. The GIT stratification is parametrized by a certain finite set $\mathfrak{B}$ (see Section 2). This set $\mathfrak{B}$ is combinatorially defined and so it is possible to determine $\mathfrak{B}$ by computer computations. The purpose of this part is to carry out the computer computations to determine $\mathfrak{B}$ for (1)–(4).

The stratum corresponding to $\beta \in \mathfrak{B}$ could be the empty set. So it is important to determine which strata S_β are non-empty. We carry this out and determine rational orbits

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in S_β for (1), (2) in Part II [17], (3) in Part III [14] (ch$(k) \neq 2$ is assumed in [14] to determine rational orbits in S_β) and (4) in Part IV [15].

The cardinality of the set $\mathfrak{B}$ for (1)–(4) is given in the following theorem.

THEOREM 1.1. *The cardinality of the set $\mathfrak{B}$ for the prehomogeneous vector spaces (1)–(4) is* 49, 81, 292, 183 *respectively.*

We list elements of $\mathfrak{B}$ for the prehomogeneous vector spaces (1)–(4) in Sections 6, 7, 8, 9 respectively. The numbers of non-empty strata are 16, 13, 61 for the cases (1), (2), (3) respectively (see [17], [14] (no assumption on ch(k) for this part). Note that in [7], [6], V^{ss}, $\{0\}$ are counted and so the numbers of orbits are 18, 15, 63 respectively. We expect to have 21 non-empty strata for the case (4).

The prehomogeneous vector spaces (1)–(4) are rather important prehomogeneous vector spaces with interesting arithmetic interpretations of rational orbits (see [18], [19]). The determination of the GIT stratification may have applications to some fields in number theory such as the zeta function theory.

The organization of this part is as follows. We review the notion of GIT stratification in Section 2. We explain the outline of the computer program in Section 3. We shall use multiple arrays in the computer program. We have to be careful not to use too much memory space and we have to go back and forth between multiple arrays and a single array. We discuss the combinatorial problem regarding the lexicographical order of combinations in Section 4.

Roughly speaking, we consider the set of weights of the representations (1)–(4) and find the closest point to the origin from the convex hull of each finite subset of the set of weights. Since the Weyl group acts on the set of finite subsets of the set of weights, we first find a set of representatives. For this part, we do not have to worry about the possibility of overflow and we can use a computer language such as "C". After reducing the number of cases, we make a certain matrix and find the rref (reduced row echelon form) for each case. For this part, we have to use a computer language such as "MAPLE" with no restriction of digits. We explain some details of the computer programs in Section 5. We list outputs of our computer program in Sections 6–9.

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2. GIT stratification

In this section we briefly review the notion of GIT stratification. Let k be a perfect field and $\overline{k}$ its algebraic closure. If X is a finite set, then $\#X$ will denote its cardinality. The standard symbols $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$ and $\mathbb{N}$ will denote respectively the fields of rational, real, complex numbers, the ring of rational integers and the set of non-negative integers. Let $\mathfrak{S}_n$ be the permutation group of $\{1, \dots, n\}$.

We denote the space of $n \times m$ matrices by $\mathrm{M}_{m,n}$, $\mathrm{M}_n = \mathrm{M}_{n,n}$ and the group of $n \times n$ invertible matrices by GL_n. Obviously, M_n has an algebra structure. Let $\mathrm{SL}_n = \{g \in \mathrm{GL}_n \mid \det(g) = 1\}$. We denote the unit matrix of dimension n by I_n. We use the notation $\mathrm{diag}(g_1, \dots, g_m)$ for the block diagonal matrix whose diagonal blocks are $g_1, \dots, g_m$.

We are mainly interested in prehomogeneous vector spaces, but we first consider a more general situation.

Let G be a connected reductive group, V a finite dimensional representation of G both defined over k. Since we only consider split reductive groups in this paper, we assume that G is split. We assume that there is a connected split reductive subgroup G_1 of G, a split torus $T_0 \subset Z(G)$ (the center of G), such that $T_0 \cap G_1$ is finite and $G = T_0 G_1$ as algebraic groups. We assume that there is a rational character χ of T_0 such that the action of $t \in T_0$ is given by the scalar multiplication by $\chi(t)$.

Let $(T_0 \cap G_1) \subset T \subset G_1$ be a maximal split torus, $X_*(T)$, $X^*(T)$ be the groups of one parameter subgroups (abbreviated as 1PS from now on) and the group of rational characters respectively. We put

$$\mathfrak{t} = X_*(T) \otimes \mathbb{R},\ \mathfrak{t}_{\mathbb{Q}} = X_*(T) \otimes \mathbb{Q},\ \mathfrak{t}^* = X^*(T) \otimes \mathbb{R},\ \mathfrak{t}^*_{\mathbb{Q}} = X^*(T) \otimes \mathbb{Q}\,.$$

Let $\mathbb{W} = N_G(T)/T$ be the Weyl group of G. $\mathbb{W}$ acts on $\mathfrak{t}^*$ also.

There is a natural pairing $\langle\ ,\ \rangle_T : X^*(T) \times X_*(T) \to \mathbb{Z}$ defined by $t^{\langle \chi, \lambda \rangle_T} = \chi(\lambda(t))$ for $\chi \in X^*(T)$, $\lambda \in X_*(T)$. This is a perfect paring ([1, pp. 113–115]).

There exists an inner product $(\ ,\ )$ on $\mathfrak{t}$ which is invariant under the actions of $\mathbb{W}$ and the Galois group $\mathrm{Gal}(\overline{k}/k)$. We may assume that this inner product is rational, i.e., $(\lambda, \nu) \in \mathbb{Q}$ for all $\lambda, \nu \in \mathfrak{t}_{\mathbb{Q}}$. Let $\|\ \|$ be the norm on $\mathfrak{t}$ defined by $(\ ,\ )$. We choose a Weyl chamber $\mathfrak{t}_+ \subset \mathfrak{t}$ for the action of $\mathbb{W}$.

For $\lambda \in \mathfrak{t}$, let $\beta = \beta(\lambda)$ be the element of $\mathfrak{t}^*$ such that $\langle \beta, \nu \rangle = (\lambda, \nu)$ for all $\nu \in \mathfrak{t}$. The map $\lambda \mapsto \beta(\lambda)$ is a bijection and we denote the inverse map by $\lambda = \lambda(\beta)$. There is a unique positive rational number a such that $a\lambda(\beta) \in X_*(T)$ and is indivisible. We use the notation λ_β for $a\lambda(\beta)$.

Identifying $\mathfrak{t}$ with $\mathfrak{t}^*$ we have a $\mathbb{W}$-invariant inner product $(\ ,\ )_*$ on $\mathfrak{t}^*$, the norm $\|\ \|_*$ determined by $(\ ,\ )_*$ and a Weyl chamber $\mathfrak{t}^*_+$.

Let $N = \dim V$. We choose a coordinate system $v = (v_1, \ldots, v_N)$ on V by which T acts diagonally. Let $\gamma_i \in \mathfrak{t}^*$ and $\mathbb{a}_i$ be the weight and the coordinate vector which corresponds to i-th coordinate. Let $\Gamma = \{\gamma_1, \ldots, \gamma_N\}$. For a subset $\mathfrak{I} \subset \Gamma$, we denote the convex hull of $\mathfrak{I}$ by $\mathrm{Conv}\,\mathfrak{I}$. Let $\mathbb{P}(V)$ be the projective space associated with V and $\pi_V : V \setminus \{0\} \to \mathbb{P}(V)$ the natural map. For $\mathfrak{I} \subset \Gamma$ such that $0 \notin \mathrm{Conv}\,\mathfrak{I}$, let β be the closest point of $\mathrm{Conv}\,\mathfrak{I}$ to the origin. Then β lies in $\mathfrak{t}^*_{\mathbb{Q}}$. Let $\mathfrak{B}$ be the set of all such β which lies in $\mathfrak{t}^*_+$.

We define

$$Y_\beta = \mathrm{Span}\{\mathbb{a}_i \mid (\gamma_i, \beta)_* \geq (\beta, \beta)_*\}, \quad Z_\beta = \mathrm{Span}\{\mathbb{a}_i \mid (\gamma_i, \beta)_* = (\beta, \beta)_*\},$$
$$W_\beta = \mathrm{Span}\{\mathbb{a}_i \mid (\gamma_i, \beta)_* > (\beta, \beta)_*\}$$

where Span is the spanned subspace. Clearly $Y_\beta = Z_\beta \oplus W_\beta$.

If λ is a 1PS of G, we define

$$P(\lambda) = \left\{ p \in G \;\middle|\; \lim_{t \to 0} \lambda(t) p \lambda(t)^{-1} \text{ exists} \right\},\ M(\lambda) = Z_G(\lambda) \text{ (the centralizer)},$$

$$U(\lambda) = \left\{ p \in G \;\middle|\; \lim_{t \to 0} \lambda(t) p \lambda(t)^{-1} = 1 \right\}.$$

The group $P(\lambda)$ is a parabolic subgroup of G ([13, p. 148]) with Levi part $M(\lambda)$ and unipotent radical $U(\lambda)$. We put $P_\beta = P(\lambda_\beta)$, $M_\beta = Z_G(\lambda_\beta)$ and $U_\beta = U(\lambda_\beta)$.

Let χ_β be the indivisible rational character of M_β such that the restriction of χ_β^a to T coincides with $b\beta$ for some positive integers a, b. We define $G_\beta = \{g \in M_\beta \mid \chi_\beta(g) = 1\}^\circ$ (the identity component). Then G_β acts on Z_β. Note that M_β and G_β are defined over k, and since $\langle \chi_\beta, \lambda_\beta \rangle$ is a positive multiple of $\|\beta\|_*$, $M_\beta = G_\beta \mathrm{Im}(\lambda_\beta)$. Moreover, if ν is any rational 1PS in G_β, $(\nu, \lambda_\beta) = 0$.

Let $\mathbb{P}(Z_\beta)^{\mathrm{ss}}$ be the set of semi-stable points of $\mathbb{P}(Z_\beta)$ with respect to the action of $G_\beta^1 \stackrel{\mathrm{def}}{=} G_\beta \cap G_1$. Since there is a difference between Z_β and $\mathbb{P}(Z_\beta)$, we remove appropriate scalar directions from G_β to consider stability. For the notion of semi-stable points, see [9]. We regard $\mathbb{P}(Z_\beta)^{\mathrm{ss}}$ as a subset of $\mathbb{P}(V)$. Put

$$Z_\beta^{\mathrm{ss}} = \pi_V^{-1}(\mathbb{P}(Z_\beta)^{\mathrm{ss}}), \ \ Y_\beta^{\mathrm{ss}} = \{(z, w) \mid z \in Z_\beta^{\mathrm{ss}}, w \in W_\beta\}\,.$$

We define $S_\beta = GY_\beta^{\mathrm{ss}}$. Note that S_β can be the empty set. We denote the set of k-rational points of S_β, etc., by $S_{\beta\, k}$, etc.

The following theorem is COROLLARY 1.4 [16, p. 264].

THEOREM 2.1. *Suppose that k is a perfect field. Then we have*

$$V_k \setminus \{0\} = V_k^{\mathrm{ss}} \coprod \coprod_{\beta \in \mathfrak{B}} S_{\beta\, k}\,.$$

Moreover, $S_{\beta\, k} \cong G_k \times_{P_{\beta\, k}} Y_{\beta\, k}^{\mathrm{ss}}$.

We call this stratification *the GIT stratification*. The importance of the above theorem is the rationality of the inductive structure of S_β. Obviously, we can use computer to determine $\mathfrak{B}$.

3. Outline of the program

In this section we explain the idea of the programming to compute the set $\mathfrak{B}$.

We assume that arrays start from the index 1 in this paper. For actual programming, adjustments have to be made if arrays start from the index 0 for a computer language.

In order to compute $\mathfrak{B}$, we have to consider the set of finite subsets of the set of weights of V and find the closest point β to the origin from the convex hull.

We first explain how to reduce the number of cases. Let G, G_1, V be as in Section 2. Let $r = \dim \mathfrak{t}^*$. We remind the reader that $\Gamma = \{\gamma_1, \ldots, \gamma_N\}$ is the set of weights of coordinates of V. Let A_R be the set consisting of all subsets of cardinality R of Γ. If C is a convex polytope then it is a finite union of simplices. Therefore, we only have to consider $\mathfrak{I} \in A_R$ which satisfy the following condition.

CONDITION 3.1. (1) $R \le r$.

(2) *If* $\mathfrak{I} = \{\gamma_{j_1}, \ldots, \gamma_{j_R}\}$ *and* β *is the closest point of* $\mathrm{Conv}\, \mathfrak{I}$ *to the origin, then* $\{\gamma_{j_2} - \gamma_{j_1}, \ldots, \gamma_{j_R} - \gamma_{j_1}\}$ *is linearly independent and* β *is orthogonal to* $\gamma_{j_2} - \gamma_{j_1}, \ldots, \gamma_{j_R} - \gamma_{j_1}$.

(3) β *is an interior point of* C.

We used the capital letter R because this is the constant we shall use in algorithms. It will be easier this way to distinguish constants and variables in algorithms.

Note that since $\operatorname{Conv}\mathfrak{I}$ does not contain the origin, we only have to consider the face of $\operatorname{Conv}\mathfrak{I}$ which contains β. So we may assume that the dimension of $\operatorname{Conv}\mathfrak{I}$ is strictly less than r. Since an $(r-1)$-dimensional simplex is determined by r vectors, the properties (1), (2) follow. The reason why we may assume (3) is that β can be obtained from $\mathfrak{I}' \in A_{R'}$ for $R' < R$ if β belongs to the boundary of $\operatorname{Conv}\mathfrak{I}$.

Let B_R be the set of all $\mathfrak{I} \in A_R$ which satisfies Condition 3.1. Obviously $\mathbb{W}$ acts on B_R. Let $C_R \subset B_R$ be a set of representatives of $\mathbb{W}\backslash B_R$. Let $\mathfrak{I} \in C_R$ and β' be the closest point of $\operatorname{Conv}\mathfrak{I}$ to the origin. We choose an element $g \in \mathbb{W}$ so that $\beta = g\beta' \in \mathfrak{t}^*_+$. Let S_R be the set of such β.

PROPOSITION 3.2. $\mathfrak{B} = \bigcup_{R=1}^r S_R$.

Proof. It is enough to prove that $\mathfrak{B} \subset \bigcup_{R=1}^r S_R$. Suppose that $\beta \in \mathfrak{B}$ is obtained from $\mathfrak{I} \in B_R$. Then there exist $\mathfrak{J} \in C_R$ and $g \in \mathbb{W}$ such that $\mathfrak{J} = g\mathfrak{I}$. Let β' be the closest point of $\operatorname{Conv}\mathfrak{J}$ to the origin and $h \in \mathbb{W}$ is an element such that $h\beta' \in \mathfrak{t}^*_+$.

Since β is the closest point of $\operatorname{Conv}\mathfrak{I}$ to the origin, $g\beta = \beta'$. So $hg\beta \in \mathfrak{t}^*_+$, which implies that $h\beta' = hg\beta = \beta$. Therefore, $\beta \in S_R$. □

By the above proposition, it is enough to determine S_R for $R = 1, \ldots, r$ and remove duplication.

We explain the algorithm more explicitly for the prehomogeneous vector spaces (1)–(4) in the following. We choose products of SL's as G_1 in Section 2. For example, $G_1 = \mathrm{SL}_5 \times \mathrm{SL}_4$ for the case (3). Let $T_0 \subset G$ be the center of G. For example, $T_0 = \{(t_1 I_6, t_2 I_2) \mid t_1, t_2 \in \mathrm{GL}_1\}$ for the case (2). Let $T \subset G_1$ be the subgroup consisting of elements whose components are diagonal matrices. We choose T in Section 2 for the cases (1)–(4) as follows.

(1) $T = \left\{ (\mathrm{diag}(t_{11}, t_{12}, t_{13}), \mathrm{diag}(t_{21}, t_{22}, t_{23}), \mathrm{diag}(t_{31}, t_{32})) \,\middle|\, \begin{array}{l} t_{11}t_{12}t_{13} = t_{21}t_{22}t_{23} \\ = t_{31}t_{32} = 1 \end{array} \right\}$.

(2) $T = \{(\mathrm{diag}(t_{11}, \ldots, t_{16}), \mathrm{diag}(t_{21}, t_{22})) \mid t_{11} \cdots t_{16} = t_{21}t_{22} = 1\}$.

(3) $T = \{(\mathrm{diag}(t_{11}, \ldots, t_{15}), \mathrm{diag}(t_{21}, \ldots, t_{24})) \mid t_{11} \cdots t_{15} = t_{21} \cdots t_{24} = 1\}$.

(4) $T = \{\mathrm{diag}(t_1, \ldots, t_8) \mid t_1 \cdots t_8 = 1\}$.

Then we can describe $\mathfrak{t}^*$ as follows.

(1) $\mathfrak{t}^* = \left\{ (a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}) \in \mathbb{R}^8 \,\middle|\, \sum_{j=1}^{3} a_{1j} = \sum_{j=1}^{3} a_{2j} = \sum_{j=1}^{2} a_{3j} = 0 \right\}$.

(2) $\mathfrak{t}^* = \left\{ (a_{11}, \ldots, a_{16}, a_{21}, a_{22}) \in \mathbb{R}^8 \,\middle|\, \sum_{j=1}^{6} a_{1j} = \sum_{j=1}^{2} a_{2j} = 0 \right\}$.

(3) $\mathfrak{t}^* = \left\{ (a_{11}, \ldots, a_{15}, a_{21}, \ldots, a_{24}) \in \mathbb{R}^9 \,\middle|\, \sum_{j=1}^{5} a_{1j} = \sum_{j=1}^{4} a_{2j} = 0 \right\}$.

(4) $$\mathfrak{t}^* = \left\{ (a_1, \dots, a_8) \in \mathbb{R}^8 \;\middle|\; \sum_{j=1}^{8} a_j = 0 \right\}.$$

For the case (3), $a = (a_{11}, \dots, a_{15}, a_{21}, \dots, a_{24}) \in \mathbb{Z}^8$ can be regarded as a character of T so that for $t = (\mathrm{diag}(t_{11}, \dots, t_{15}), \mathrm{diag}(t_{21}, \dots, t_{24}))$, $t^a \stackrel{\mathrm{def}}{=} \prod_{i=1}^5 t_{1i}^{a_{1i}} \prod_{i=1}^4 t_{2i}^{a_{2i}}$. Other cases are similar.

The Weyl groups $\mathbb{W}$ for the cases (1)–(4) are $\mathfrak{S}_3 \times \mathfrak{S}_3 \times \mathfrak{S}_2$, $\mathfrak{S}_6 \times \mathfrak{S}_2$, $\mathfrak{S}_5 \times \mathfrak{S}_4$, $\mathfrak{S}_8$ respectively. To define a $\mathbb{W}$-invariant inner product on $\mathfrak{t}$ is equivalent to define a $\mathbb{W}$-invariant inner product on $\mathfrak{t}^*$. For the case (3), we define

$$(a, b)_* = \sum_{i=1}^{5} a_{1i} b_{1i} + \sum_{i=1}^{4} a_{2i} b_{2i}$$

for $a = (a_{11}, \dots, a_{15}, a_{21}, \dots, a_{24})$, $b = (b_{11}, \dots, b_{15}, b_{21}, \dots, b_{24})$. This inner product is $\mathbb{W}$-invariant. We define $(\ ,\ )_*$ for other cases similarly. We choose the Weyl chamber for the cases (1)–(4) as follows.

(1) $\mathfrak{t}^*_+ = \left\{ (a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}) \in \mathfrak{t}^* \;\middle|\; \begin{array}{c} a_{i1} \le a_{i2} \le a_{i3}\ (i = 1, 2)\,, \\ a_{31} \le a_{32} \end{array} \right\}$.
(2) $\mathfrak{t}^*_+ = \{(a_{11}, \dots, a_{16}, a_{21}, a_{22}) \in \mathfrak{t}^* \mid a_{11} \le \cdots \le a_{16}, a_{21} \le a_{22}\}$.
(3) $\mathfrak{t}^*_+ = \{(a_{11}, \dots, a_{15}, a_{21}, \dots, a_{24}) \in \mathfrak{t}^* \mid a_{11} \le \cdots \le a_{15}, a_{21} \le \cdots \le a_{24}\}$.
(4) $\mathfrak{t}^*_+ = \{(a_1, \dots, a_8) \in \mathfrak{t}^* \mid a_1 \le \cdots \le a_8\}$.

Let $\mathbb{p}_{n,1}, \dots, \mathbb{p}_{n,n}$ be the coordinate vectors of Aff^n. We put $p_{n,ij} = \mathbb{p}_{n,i} \wedge \mathbb{p}_{n,j}$, $p_{n,ijk} = \mathbb{p}_{n,i} \wedge \mathbb{p}_{n,j} \wedge \mathbb{p}_{n,k}$, $q_{n,ij} = \mathbb{p}_{n,i} \otimes \mathbb{p}_{n,j}$. We choose a basis of V for the cases (1)–(4) so that the coordinate vectors are as follows. Let $N = \dim V$. Note that $N = 18, 30, 40, 56$ for the cases (1)–(4) respectively.

(1) $\mathbb{a}_1 = q_{3,11} \otimes \mathbb{p}_{2,1},\ \mathbb{a}_2 = q_{3,12} \otimes \mathbb{p}_{2,1}, \dots, \mathbb{a}_9 = q_{3,33} \otimes \mathbb{p}_{2,1}, \dots, \mathbb{a}_{18} = q_{3,33} \otimes \mathbb{p}_{2,2}$.

(2) $\mathbb{a}_1 = p_{6,12} \otimes \mathbb{p}_{2,1},\ \mathbb{a}_{15} = p_{6,56} \otimes \mathbb{p}_{2,1},\ \mathbb{a}_{16} = p_{6,12} \otimes \mathbb{p}_{2,2}, \dots, \mathbb{a}_{30} = p_{6,56} \otimes \mathbb{p}_{2,2}$.

(3) $\mathbb{a}_1 = p_{5,12} \otimes \mathbb{p}_{4,1},\ \mathbb{a}_{10} = p_{5,45} \otimes \mathbb{p}_{4,1},\ \mathbb{a}_{11} = p_{5,12} \otimes \mathbb{p}_{4,2}, \dots, \mathbb{a}_{40} = p_{5,45} \otimes \mathbb{p}_{4,4}$.

(4) $\mathbb{a}_1 = p_{8,123},\ \mathbb{a}_2 = p_{8,124}, \dots, \mathbb{a}_{56} = p_{8,678}$

Let γ_i be the weight of $\mathbb{a}_i$. Then $\{\gamma_1, \dots, \gamma_N\}$ for the cases (1)–(4) are as follows.

(1) $\gamma_1 = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, -\frac{1}{2}), \dots, \gamma_{18} = (-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, -\frac{1}{2}, \frac{1}{2})$.
(2) $\gamma_1 = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, -\frac{1}{2}), \dots, \gamma_{30} = (-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, \frac{2}{3}, -\frac{1}{2}, \frac{1}{2})$.
(3) $\gamma_1 = \left(\frac{3}{5}, \frac{3}{5}, -\frac{2}{5}, -\frac{2}{5}, -\frac{2}{5}, \frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$
$, \dots, \gamma_{40} = \left(-\frac{2}{5}, -\frac{2}{5}, -\frac{2}{5}, \frac{3}{5}, \frac{3}{5}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{3}{4}\right)$.
(4) $\gamma_1 = (\frac{5}{8}, \frac{5}{8}, \frac{5}{8}, -\frac{3}{8}, \dots, -\frac{3}{8}), \dots, \gamma_{56} = (-\frac{3}{8}, \dots, -\frac{3}{8}, \frac{5}{8}, \frac{5}{8}, \dots, \frac{5}{8})$.

We fix $1 \leq R \leq r$ ($r = 5, 6, 7, 7$ for the cases (1)–(4) respectively). Let A_R be the set of all subsets Y of Γ such that $\#Y = R$. We identify A_R with the set of all sequences $v = (v_1, \ldots, v_R)$ such that $1 \leq v_1 < \cdots < v_R \leq N$.

Step 1. We find a set C'_R of representatives of $\mathbb{W}\backslash A_R$. For this purpose, we assign the lexicographical order to any element $\mathfrak{I} \in A_R$, say $L(\mathfrak{I})$. Note that $1 \leq L(\mathfrak{I}) \leq \binom{N}{R}$. For $1 \leq i \leq \binom{N}{R}$, let $\mathfrak{I}(i) \in A_R$ be the subset such that $L(\mathfrak{I}(i)) = i$.

Let $X_{R,i}$ be an array of integers such that $1 \leq i \leq \binom{N}{R}$. At first we assign $X_{R,i} = 0$ for all i. We change the value of $X_{R,1}$ to 1 and then for all $w \in \mathbb{W}$, change the value of $X_{R,L(w(\mathfrak{I}(1)))}$ to 2. Then we consider the first $i \geq 2$ such that $X_{R,i} = 0$, change the value of $X_{R,i}$ to 1 and for all $w \in \mathbb{W}$, change the value of $X_{R,L(w(\mathfrak{I}(i)))}$ to 2. We continue this process. Then $C'_R = \{\mathfrak{I}(i) \mid X_{R,i} = 1\}$ is a set of representatives for $\mathbb{W}\backslash A_R$.

Step 2. Suppose that $\mathfrak{I} = \{\gamma_{j_1}, \ldots, \gamma_{j_R}\} \in C'_R$. We would like to find the closest point β' of $\operatorname{Conv}\mathfrak{I}$ to the origin and see if it satisfies Condition 3.1. Such β' is in the form $\beta' = c_1\gamma_{j_1} + \cdots + c_i\gamma_{j_R}$ where $c_1, \ldots, c_R \in \mathbb{Q}$ and $0 < c_1, \ldots, c_R < 1, c_1 + \cdots + c_R = 1$.

Let $M = (m_{kl})$ be the $R \times R$ matrix such that $m_{kl} = (\gamma_{j_k} - \gamma_{j_1}, \gamma_{j_l})_*$ for $k = 2, \ldots, R, l = 1, \ldots, R$ and $m_{11} = \cdots = m_{1R} = 1$. We put $c = [c_1, \ldots, c_R]$. Then β' is orthogonal to $\gamma_{j_2} - \gamma_{j_1}, \ldots, \gamma_{j_R} - \gamma_{j_1}$ if and only if entries of Mc are 0 except for the first entry. The condition $c_1 + \cdots + c_R = 1$ means that the first entry of Mc is 1. So c has to satisfy the condition $Mc = [1, 0, \ldots, 0]$.

If $\{\gamma_{j_2} - \gamma_{j_1}, \ldots, \gamma_{j_R} - \gamma_{j_1}\}$ is linearly independent then c is unique and so M has to be non-singular. We put $b = [1, 0, \ldots, 0]$. We form the augmented matrix $M' = (M\ b)$ and find the reduced row echelon form of M', say $(M_0\ d)$ ($d = [d_1, \ldots, d_R]$). Then M is non-singular if and only if the (R, R)-entry of M_0 is 1. Let C_R be the set of $\mathfrak{I} \in C'_R$ such that the (R, R)-entry of M_0 is 1 and $d_1, \ldots, d_R > 0$. Then this C_R can be regarded as C_R described before Proposition 3.2. If $\mathfrak{I} \in C_R$ then we form $\beta' = d_1\gamma_{j_1} + \cdots + d_R\gamma_{j_R}$. By sorting entries of β', we obtain an element $\beta \in \mathfrak{t}^*_+$.

Step 3. We combine all β obtained in Step 2 for $R = 1, \ldots, r$. We remove duplication and the zero vector. Then the list obtained is the set $\mathfrak{B}$.

4. lexicographical order of combinations

Let G, G_1, V, N, Γ, r be as in Section 2.

To assign a multiple array such as $a_{j_1,\ldots,j_R}$ to the subset $\mathfrak{I} = \{\gamma_{j_1}, \ldots, \gamma_{j_R}\}$ is not a good idea, because it consumes unnecessary memory space. So we consider the lexicographical order to $\mathfrak{I}$ and assign a single array. To use this idea, we must have a way to go back and forth between such combinations and their lexicographical orders. The purpose of this section is to explain this correspondence explicitly.

Let $\binom{n}{m}$ be the binomial coefficient. We consider integers $n \geq m \geq 0$ except for $m = n + 1$, where we define $\binom{n}{n+1} = 0$. Note that $\binom{n}{n} = \binom{n}{0} = 1$.

Let $N \geq R \geq 1$ be integers. Let $A(N, R)$ be the set of sequences $c = (c_1, \ldots, c_R)$ of integers such that $1 \leq c_1 < c_2 < \cdots < c_R \leq N$ (this is A_R in the previous section).

For such c, let $L(N,R,c) \geq 1$ be its lexicographical order. For example, if $N=3$, $R=2$, $c_1=2$, $c_2=3$ then $L(3,2,c)=3$.

We would like to express $L(N,R,c)$ in terms of c and vice versa.

PROPOSITION 4.1. *If $N \geq R \geq 1$ and $c \in A(N,R)$ then*

$$L(N,R,c) = \sum_{i=1}^{R} \left(\binom{N-i}{R-i+1} - \binom{N-c_i}{R-i+1} \right) + 1 \tag{4.2}$$

Proof. Note that $c_i \leq N-R+i$. So $N-c_i \geq R-i$ and $N-c_i = R-i$ if and only if $c_i = N-R+i$. If $c_i = N-R+i$ then $\binom{N-c_i}{R-i+1} = \binom{R-i}{R-i+1} = 0$.

If $R=1$ then $L(N,R,c)=c_1$ and (4.2) is valid in this case.

Suppose that $R>1$. We put $d_1 = c_2-c_1, \ldots, d_{R-1} = c_R - c_1$. Then $d \in A(N-c_1, R-1)$. If $c_1=1$ then $L(N,R,c) = L(N-1,R-1,d)$. If $c_1>1$ then the number of $c' = (c'_1, \ldots, c'_R) \in A(N,R)$ such that $c'_1 < c_1$ is

$$\binom{N}{R} - \binom{N-c_1+1}{R}.$$

Note that the number of m such that $c_1 \leq m \leq N$ is $N-c_1+1$. So

$$L(N,R,c) = \binom{N}{R} - \binom{N-c_1+1}{R} + L(N-c_1, R-1, d). \tag{4.3}$$

This formula is valid in the case $c_1 = 1$ also.

This formula implies by induction that

$$L(N,R,c) = \binom{N}{R} - \binom{N-c_1+1}{R} + \sum_{i=1}^{R-1} \left(\binom{N-c_1-i}{R-i} - \binom{N-c_{i+1}}{R-i} \right) + 1. \tag{4.4}$$

This formula is valid for the case $R=1$ also.

Note that if $0 \leq m \leq N$ then

$$\binom{N}{m} = \sum_{i=0}^{m-1} \binom{N-i-1}{m-i} + 1.$$

So

$$\binom{N}{R} - \binom{N-c_1+1}{R} = \sum_{i=0}^{R-1} \left(\binom{N-i-1}{R-i} - \binom{N-c_1-i}{R-i} \right).$$

Since

$$\sum_{i=0}^{R-1} \binom{N-i-1}{R-i} = \sum_{i=1}^{R} \binom{N-i}{R-i+1}$$

and

$$-\sum_{i=0}^{R-1} \binom{N-c_1-i}{R-i} + \sum_{i=1}^{R-1} \left(\binom{N-c_1-i}{R-i} - \binom{N-c_{i+1}}{R-i} \right) = -\sum_{i=1}^{R} \binom{N-c_i}{R-i+1},$$

we obtain (4.2). □

We consider the opposite direction. Let $m > 0$ be an integer such that $m \le \binom{N}{R}$. We would like to find $c \in A(N, R)$ such that $L(N, R, c) = m$.

We put $m_1 = m$. For $i = 2, \dots, R$, we put

$$m_i = m - \sum_{l=1}^{i-1} \left(\binom{N-l}{R-l+1} - \binom{N-c_l}{R-l+1} \right).$$

Note that m_i does not depend on $c_i, \dots, c_R$.

PROPOSITION 4.5. *If $L(N, R, c) = m$ then c_i is characterized by the following condition:*

$$\binom{N-i+1}{R-i+1} - \binom{N-c_i+1}{R-i+1} < m_i \le \binom{N-i+1}{R-i+1} - \binom{N-c_i}{R-i+1}.$$

Proof. By the consideration of (4.3), $c_1 \ge j$ if and only if $m > \binom{N}{R} - \binom{N-j+1}{R}$. Therefore, c_1 is characterize by the following formula:

$$\binom{N}{R} - \binom{N-c_1+1}{R} < m \le \binom{N}{R} - \binom{N-c_1}{R}.$$

So the statement of the proposition holds for $i = 1$.

Let $d_1 = c_2 - c_1, \dots, d_{R-1} = c_R - c_1$. Then $d \in A(N - c_1, R - 1)$. We put $m'_2 = L(N - c_1, R - 1, d)$. By (4.3),

$$m'_2 = m - \left(\binom{N}{R} - \binom{N-c_1+1}{R} \right)$$

Since $c_1 + d_1 = c_2$, c_2 is characterized by the following formula:

$$\binom{N-c_1}{R-1} - \binom{N-c_2+1}{R-1} < m'_2 \le \binom{N-c_1}{R-1} - \binom{N-c_2}{R-1}.$$

By continuing this process, for $i = 2, \dots, R$, c_i is characterized by the following condition:

$$\binom{N-c_{i-1}}{R-i+1} - \binom{N-c_i+1}{R-i+1} < m'_i \le \binom{N-c_{i-1}}{R-i+1} - \binom{N-c_i}{R-i+1} \tag{4.6}$$

where for $i = 2, \dots, R$,

$$m'_i = m - \left(\binom{N}{R} - \binom{N-c_1+1}{R} \right) - \sum_{l=1}^{i-2} \left(\binom{N-c_l}{R-l} - \binom{N-c_{l+1}+1}{R-l} \right).$$

We put $m''_i = m'_i + \binom{N-i+1}{R-i+1} - \binom{N-c_{i-1}}{R-i+1}$. Since

$$\binom{N}{R} - \binom{N-i+1}{R-i+1} = \sum_{l=1}^{i-1} \binom{N-l}{R-l+1}$$

and

$$\binom{N-c_1+1}{R} - \sum_{l=1}^{i-2} \left(\binom{N-c_l}{R-l} - \binom{N-c_{l+1}+1}{R-l} \right) - \binom{N-c_{i-1}}{R-i+1}$$

$$= \binom{N-c_1}{R} + \binom{N-c_2}{R-1} + \cdots + \binom{N-c_{i-1}}{R-i+2} = \sum_{l=1}^{i-1} \binom{N-c_l}{R-l+1},$$

we have

$$m_i'' = m - \sum_{l=1}^{i-1} \left(\binom{N-l}{R-l+1} - \binom{N-c_l}{R-l+1} \right) = m_i .$$

This implies that the condition (4.6) is equivalent to the condition in the statement of this proposition. □

Let

$$a_{i,j} = \binom{N-i}{R-i+1} - \binom{N-j}{R-i+1} \tag{4.7}$$

for $i = 1, \ldots, R$, $j = i, \ldots, N-R+i$ and

$$b_{i,j} = \binom{N-i+1}{R-i+1} - \binom{N-j}{R-i+1} \tag{4.8}$$

for $i = 1, \ldots, R$, $j = i, \ldots, N-R+i$.

Proposition 4.1 implies that if $c \in A(N, R)$ and $m = L(N, R, c)$ then $m = \sum_{i=1}^{R} a_{i,c_i}$. Also Proposition 4.5 implies that if $m_1 = m$, $m_i = m - \sum_{l=1}^{i-1} a_{l,c_l}$ $(i = 2, \ldots, R)$ then c_i is the smallest integer $i \le j \le N-R+i$ such that $b_{i,j} \ge m_i$.

5. Algorithms

In this section, we describe some details of algorithms to find the set $\mathfrak{B}$ for the prehomogeneous vector spaces (1)–(4). We describe the algorithms so that they do not depend on particular computer languages here. As we stated in Section 3, we assume that arrays start from the index 1 even though arrays start from the index 0 in some computer languages. The actual computer programs are made public in the second author's home page (https://www.math.kyoto-u.ac.jp/~yukie/Strata-pub.zip). The letters we use in algorithms are different from those used in actual programs, since in actual programs, variables like `i1`, `i2` are used and it may be confusing to use such names to explain the algorithms.

We use the formulation of Sections 2, 3. For the prehomogeneous vector spaces (1)–(4), let G_1, T, T_0, T_1, $\mathfrak{t}^*$, $\mathfrak{t}^*_+$, $\mathbb{W}$ be as in Section 3.

We consider Steps 1–3 of Section 3.

5.1. Step 1

Let B_i be the set in Section 3 (see the paragraph above Proposition 3.2). We find a set of representatives for $\mathbb{W}\backslash B_i$ in this step.

It is fairly easy to generate permutations. We assume that elements of $\mathfrak{S}_2, \ldots, \mathfrak{S}_8$ have been generated and stored in a file as $P^{(2)}_{ij}, \ldots, P^{(8)}_{ij}$. For example,

$$P^{(3)}_{11} = 1,\ P^{(3)}_{12} = 2,\ P^{(3)}_{13} = 3, \ldots, P^{(3)}_{61} = 3,\ P^{(3)}_{62} = 2,\ P^{(3)}_{63} = 1 .$$

Let $N = 18, 30, 40, 56$ for the prehomogeneous vector spaces (1)–(4) respectively. Let $\mathbb{a}_1, \ldots, \mathbb{a}_N$ be the coordinate vectors defined in Section 3 and $\gamma_1, \ldots, \gamma_N \in \mathfrak{t}^*$ their weights. To consider a subset $\mathfrak{I} \subset \{\gamma_1, \ldots, \gamma_N\}$ such that $\#\mathfrak{I} = R$ is the same as to consider combinations of R numbers from N numbers. If $\mathfrak{I} = \{\gamma_{i_1}, \ldots, \gamma_{i_R}\}$ where $i_1 < \cdots < i_R$ then we assign the lexicographical order of $\{i_1, \ldots, i_R\}$ to $\mathfrak{I}$. As we stated in Introduction, we use the lexicographical order to keep track of $\mathfrak{I}$.

For each R, we carry out algorithms in Steps 1,2. So algorithms in these steps depend on R. In Step 3, we combine results of Steps 1,2 for all R, remove duplication and obtain necessary informations for each $\beta \in \mathfrak{B}$.

If $v = (v_1, \ldots, v_R)$ is an array of distinct integers then the algorithm to sort $v_1, \ldots, v_R$ is well-known and we leave the details to the reader. It returns an array $w_1 < \cdots < w_R$ obtained from changing the order of $v_1, \ldots, v_R$. Also, it is easy to compute binomial coefficients $\binom{n}{m}$ by Pascal's identity and we will not describe the details.

For an array of distinct R integers $v = (v_1, \ldots, v_R)$ such that $1 \le v_1, \ldots, v_R \le N$, let **CombN**$(v)$ be the function which sorts $v_1, \ldots, v_R$ so that $v_1 < \cdots < v_R$ and returns the lexicographical order of v. For $1 \le m \le \binom{N}{R}$, let **NComb**(m, v) be the function which makes v the sequence $(v_1, \ldots, v_R)$ such that $1 \le v_1 < \cdots < v_R \le N$ and that **CombN**$(v) = m$. When we use these functions, we assume that the values of N, R are set. These functions **CombN**(v), **NComb**(m, v) can be computed by Propositions 4.1, 4.5 as follows. The values of $a_{i,j}$ and $b_{i,j}$ in (4.7), (4.8) are heavily used. So the following values should be computed before other algorithms.

(1) $a_{i,j}$ in (4.7) for $i = 1, \ldots, R$, $j = i, \ldots, N - R + i$.
(2) $b_{i,j}$ in (4.8) for $i = 1, \ldots, R$, $j = i, \ldots, N - R + i$.

ALGORITHM 5.1.

(i) Name: **CombN**(v)
Require: $v = (v_1, \ldots, v_R)$: an array of elements of $\mathbb{N}$.
Description: It returns the lexicographical order of v after sorting as **CombN**(v).
Local variables: $i \in \mathbb{N}$.

1. Sort v so that $v_1 < \cdots < v_R$.
2. Return the value $\sum_{i=1}^{R} a_{i,v_i}$ as **CombN**(v).

(ii) Name: **NComb**(m, v)
Require: $m \in \mathbb{N}$, $v = (v_1, \ldots, v_R)$: an array of elements of $\mathbb{N}$.
Description: It makes v a sorted array whose lexicographical order is m.
Local variables: $i, j \in \mathbb{N}$, $l = (l_1, \ldots, l_R)$: an array of R elements of $\mathbb{N}$.

1. $l_1 \leftarrow m$ and $j \leftarrow 1$.
2. If $b_{1,j} < l_1$ then $j \leftarrow j + 1$ and repeat.
3. $v_1 \leftarrow j$.
4. For $i = 2, \ldots, R$ do the following.
   4-a. $l_i \leftarrow m - \sum_{l=1}^{i-1} a_{l,v_l}$ and $j \leftarrow i$.
   4-b. If $b_{i,j} < l_i$ then $j \leftarrow j + 1$ and repeat.

4-c. $v_i \leftarrow j$.

This finishes the algorithm. $\diamond$

Note that **NComb**(m, v) is a "void type" function with no returned value.

Now we consider the prehomogeneous vector spaces (1)–(4). We can make algorithms so that they are common for the cases (1)–(4) except for definitions of some constants and some subroutines. So we basically explain algorithms for the case (3). In the following, (G, V) is the prehomogeneous vector space (3). We consider Step 1 of Section 3.

We first have to describe the action of $\mathbb{W} \cong \mathfrak{S}_5 \times \mathfrak{S}_4$ on $\{\gamma_1, \dots, \gamma_{40}\}$. The order of $\mathbb{W}$ is 2880. Each element of $\mathbb{W}$ induces an element of $\mathfrak{S}_{40}$. So to describe the action of $\mathbb{W}$ on $\{\gamma_1, \dots, \gamma_{40}\}$, it is enough to assign an array of 40 integers.

Elements of $\mathbb{W}$ are pairs (t_1, t_2) of permutations $t_1 \in \mathfrak{S}_5$, $t_2 \in \mathfrak{S}_4$. They are arrays of 5, 4 integers. Let $t_1(i)$ $(i = 1, \dots, 5)$, $t_2(j)$ $(j = 1, \dots, 4)$ be the values of t_1, t_2. Since the coordinate system of V involves $\wedge^2 \mathrm{Aff}^5$, we have to consider combinations of 2 elements of $\{1, \dots, 5\}$. So even though we set $N := 40$, $R := 1, \dots, 7$ in the main algorithm of Step 1, to describe the action of $\mathbb{W}$ on $\{\gamma_1, \dots, \gamma_{40}\}$, we set $N := 5$, $R := 2$ to use the functions **CombN**(v), **NComb**(m, v).

We define some constants as follows.

$$
\begin{aligned}
&N := 5,\ R := 2,\ A := 5,\ B := 4,\ C := 40\,, \\
&L_1 := 120,\ L_2 := 24,\ L := 2880,\ N_1 := 10\,.
\end{aligned} \tag{5.2}
$$

We consider the lexicographical order of combinations of 2 numbers from $\{1, \dots, 5\}$. We order coordinates of V by associating the order $1, \dots, 10$ (resp. $11, \dots, 20$, etc.,) to coordinates whose second tensor factor is $[1, 0, 0, 0]$ (resp. $[0, 1, 0, 0]$, etc.,). Let $i = 1, \dots, 10$, $j = 1, \dots, 4$ and $v = (v_1, v_2)$ be the i-th combination. Then by (t_1, t_2), the $(N_1(j-1)+i)$-th coordinate is mapped to the $(N_1(t_2(j)-1)+k)$-th coordinate where k is the lexicographical order of the combination $(t_1(v_1), t_1(v_2))$ (after sorted).

ALGORITHM 5.3. Name: **makeweyl54**(t_1, t_2, t_3)

Require: $t_1 \in \mathfrak{S}_A$, $t_2 \in \mathfrak{S}_B$, $t_3 \in \mathfrak{S}_C$.
Description: It makes t_3 the result of the action of $(t_1, t_2) \in \mathbb{W}$ on $\{\gamma_1, \dots, \gamma_C\}$.
Local variables: $i, j, k \in \mathbb{N}$, $v = (v_1, v_2)$, $w = (w_1, w_2)$: arrays of elements of $\mathbb{N}$.

1. For $i = 1, \dots, N_1$, $j = 1, \dots, B$, do the following.

1-a. **NComb**(i, v).

1-b. $w_1 \leftarrow t_1(v_1)$, $w_2 \leftarrow t_1(v_2)$.

1-c. $k \leftarrow$ **CombN**(w), $t_3(N_1(j-1)+i) \leftarrow N_1(t_2(j)-1)+k$.

This is the end of the function **makeweyl54**(t_1, t_2, t_3). $\diamond$

ALGORITHM 5.4. Description: This algorithm lists the action of all elements of $\mathbb{W} = \mathfrak{S}_A \times \mathfrak{S}_B$ on $\{\gamma_1, \dots, \gamma_C\}$. Since it is not a function, it does not require any variable as an input. However, constants in (5.2) have to be defined and elements of $\mathfrak{S}_A, \mathfrak{S}_B$ have to be read from a file as $P^{(5)}_{ij}$ $(i = 1, \dots, L_1, j = 1, \dots, A)$, $P^{(4)}_{ij}$ $(i = 1, \dots, L_2, j = 1, \dots, B)$.

Local variables: $i, j, k \in \mathbb{N}$, $W_k \in \mathfrak{S}_C$ $(k = 1, \ldots, L)$.

1. Initialize $k \leftarrow 1$.

2. For $i = 1, \ldots, L_1$ and $j = 1, \ldots, L_2$, do the following.

   2-a. **makeweyl54**$(P^{(5)}_{i*}, P^{(4)}_{j*}, W_k)$.

   2-b. $k \leftarrow k + 1$ (every time 2-a is done for a pair (i, j)).

   Note that for fixed i, $P^{(5)}_{i*} \in \mathfrak{S}_A$ $(* = 1, \ldots, A)$. We regard $P^{(4)}_{j*} \in \mathfrak{S}_B$ similarly.

3. Record W_k $(k = 1, \ldots, L)$ in a file.

This finishes the algorithm. ◇

After this algorithm, we set $N := 40$, $R := 1, \ldots, 7$. Now we consider the algorithm to reduce the number of cases.

We have to consider simplices of dimensions $0, \ldots, 6$. Since the Weyl group acts transitively on the set of coordinates, B_1 (see Proposition 3.2) is a single $\mathbb{W}$-orbit. The weight of the last coordinate is in the Weyl chamber and so S_1 consists of the weight of the last coordinate.

Let R be the number of vectors which determine an $(R-1)$-dimensional simplex. We consider the cases $R = 2, \ldots, 7$.

ALGORITHM 5.5 (The main algorithm for Step 1). Description: This algorithm determines S_7 of Proposition 3.2. Constants A, B, C, L_1, L_2, L have to be defined as in (5.2). Define $R := 7, M := \binom{40}{7} = 18643560$. We set the environment so that we can use the functions **CombN**, **NComb** for $N := 40$, $R := 7$. The list of $W_k \in \mathfrak{S}_C$ $(k = 1, \ldots, L = 2880)$ have to be read from a file.

Local variables: (i) $i, j, k, l, m \in \mathbb{N}$.
(ii) $t_1 = (t_1(1), \ldots, t_1(R))$, $t_2 = (t_2(1), \ldots, t_2(R))$: arrays of elements of $\mathbb{N}$.
(iii) $X = (X_1, \ldots, X_M)$: an array of elements of $\mathbb{N}$. (X_i is $X_{R,i}$ of Section 3.)
(iv) $v = (v_{i,j})$: an $M \times R$ matrix with entries in $\mathbb{N}$.

1. Initialize $X_i \leftarrow 0$ for $i = 1, \ldots, M$.

2. Initialize $m \leftarrow 0$.

3. For $i = 1, \ldots, M$, if $X_i = 0$, do the following (if $X_i \neq 0$ then do nothing).

   3-a. $m \leftarrow m + 1$.

   3-b. For $j = 1, \ldots, R$, $v_{m,j} \leftarrow t_1(i)$.

   3-c. $X_i \leftarrow 1$.

   3-d. **NComb**(i, t_1).

   3-e. For $j = 1, \ldots, L$, do the following.

      3-e-1. For $k = 1, \ldots, R$, $t_2(k) \leftarrow W_j(t_1(k))$.

      3-e-2. $l =$ **CombN**(t_2) and if $l > i$, $X_l \leftarrow 2$.

4. Record v.

This finishes the algorithm. ◇

Note that in the step 3-e-1, t_2 is the result of the action of the Weyl group element W_j to t_1. Even though the size of v is $M \times R$, v_i is recorded only for i from 1 to the final value of m. It turns out that the final value of m is 7891 in this case.

For $R := 6, \dots, 2$, we simply change the value of M to $M := \binom{N}{R}$ and Algorithm 5.5 works.

For the prehomogeneous vector space (2), we can record the action of $\mathbb{W}$ in the same manner as in Algorithms 5.3, 5.4 after changing the constants as follows (but one has to use $P_{ij}^{(6)}$, $P_{ij}^{(2)}$ in Algorithm 5.4).

N	R	A	B	C	L_1	L_2	L	N_1
6	2	6	2	30	720	2	1440	15

The rank of the group is 6 and so for Step 1, we have to consider $R = 6, \dots, 2$ (the case $R = 1$ is obvious). To carry out Algorithm 5.5, we have to change the values of N to 30, $R := 6, \dots, 2$. For $R = 6$, we have to define $M := 2035800$ and Algorithm 5.5 works assuming that the action of $\mathbb{W}$ is recorded as W_j $(j = 1, \dots, L)$. The situation is similar for other values of R.

For the prehomogeneous vector space (1), we use the following constants to record the action of $\mathbb{W}$. Since all factors of V are standard representations, we do not have to use the functions **CombN**, **NComb** for Algorithms 5.3, 5.4.

A	B	C	L_1	L_2	L
3	2	18	6	2	72

ALGORITHM 5.6. Name: **makeweyl332**(t_1, t_2, t_3, t_4)

Require: $t_1, t_2 \in \mathfrak{S}_A, t_3 \in \mathfrak{S}_B, t_4 \in \mathfrak{S}_C$.
Description: It makes t_4 the result of the action of $(t_1, t_2, t_3) \in \mathbb{W}$ on $\{\gamma_1, \dots, \gamma_C\}$.
Local variables: $i, j, k \in \mathbb{N}$.

1. For $i = 1, \dots, B$, $j = 1, \dots, A$, $k = 1, \dots, A$,

$$t_4(9(i-1) + 3(j-1) + k) \leftarrow 9(t_3(i) - 1) + 3(t_2(j) - 1) + t_1(k).$$

This is the end of the function. ◇

It is easy to make an algorithm for the prehomogeneous vector space (1) similar to Algorithm 5.4 and so we do not provide the details.

The rank of the group is 5 and so for Step 1, we have to consider $R = 5, \dots, 2$. To carry out Algorithm 5.5, we have to change the value of N to 18, $R := 5, \dots, 2$. For $R = 5$, we have to define $M := 8568$ and Algorithm 5.5 works assuming that the action of $\mathbb{W}$ is recorded as W_j $(j = 1, \dots, L)$. The situation is similar for other values of R.

For the prehomogeneous vector space (4), we us the following constants to documents elements of $\mathbb{W}$.

N	R	A	C	L
8	3	8	56	40320

Note that we have to set $N := 8$, $R := 3$ to use the functions **CombN**, **NComb**.

ALGORITHM 5.7. Name: **makeweyltri8**(t_1, t_2)

Require: $t_1 \in \mathfrak{S}_A, t_2 \in \mathfrak{S}_C$.
Description: It makes t_2 the result of the action of $t_1 \in \mathbb{W}$ on $\{\gamma_1, \dots, \gamma_C\}$.
Local variables: (i) $i, j, k \in \mathbb{N}$,
(ii) $v = (v_1, v_2, v_3)$, $w = (w_1, w_2, w_3)$: arrays of elements of $\mathbb{N}$.

For $i = 1, \dots, C$, do the following.

1. **NComb**(i, v).
2. $w_1 \leftarrow t_1(v_1),\ w_2 \leftarrow t_1(v_2),\ w_3 \leftarrow t_1(v_3)$.
3. $j \leftarrow$ **CombN**(w), $t_2(i) \leftarrow j$.

This is the end of the function. ◇

It is easy to make an algorithm for the prehomogeneous vector space (4) similar to Algorithm 5.4 and so we do not provide the details.

The rank of the group is 7 and so for Step 1, we have to consider $R = 7, \dots, 2$. To carry out Algorithm 5.5, we have to change the value of N to 56, $R := 7, \dots, 2$. For $R = 7$, we have to define $M := 231917400$ and Algorithm 5.5 works assuming that the action of $\mathbb{W}$ is recorded as W_j $(j = 1, \dots, L)$. The situation is similar for other values of R.

5.2. Step 2

We now explain algorithms in Step 2 of Section 3. We now have to be sensitive to the number of digits of integers. So one has to use a computer language which allows an arbitrary number of digits.

We assume that $\gamma_1, \dots, \gamma_N$ (the weights of coordinates) are recorded in a file (see Section 3 for the values of $\gamma_1, \dots, \gamma_N$). Let D be the number of entries of elements of $\mathfrak{t}^*$. Explicitly, $D = 8, 8, 9, 8$ for the prehomogeneous vector spaces (1)–(4) respectively.

We list some basic easy functions used in Steps 2,3. We do not provide the details of the algorithms. As we stated in Step 1, we assume that the values of N, R have to be set before algorithms are carried out.

ALGORITHM 5.8 (Basic easy functions).

(i) Name: **sorder**(a)

Require: $a = (a_1, \dots, a_D)$: an array of elements of $\mathbb{Q}$.

Description: It sorts a to an element of $\mathfrak{t}^*_+$ and returns the result. For example, for the prehomogeneous vector spaces (1), for $a = (a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32})$, $b = (b_{11}, b_{12}, b_{13}, b_{21}, b_{22}, b_{23}, b_{31}, b_{32})$ is obtained by sorting (a_{11}, a_{12}, a_{13}), (a_{21}, a_{22}, a_{23}) and (a_{31}, a_{32}) so that $b_{11} \le b_{12} \le b_{13}$, $b_{21} \le b_{22} \le b_{23}$ and $b_{31} \le b_{32}$. Other cases are similar. This is a variation of the standard sorting algorithm.

(ii) Name: **veq**(a, b)

Require: $a = (a_1, \dots, a_D), b = (b_1, \dots, b_D)$: arrays of elements of $\mathbb{Q}$.

Description: It returns the value 0 if $a = b$ and 1 otherwise.

(iii) Name: **allpositive**(x)

Require: $x = (x_1, \dots, x_R)$: an array of elements of $\mathbb{Q}$.

Description: It returns the value 1 if $x_i > 0$ for all i and 0 otherwise. ◇

We denote the zero vector in $\mathfrak{t}^*$ by Z as follows:

$$Z = (\overbrace{0, \dots, 0}^{D}) \in \mathfrak{t}^* . \tag{5.9}$$

We shall use this vector in Step 3.

We now explain non-trivial functions needed in the main algorithm of Step 2. Let r be the rank of the group G and we fix $R = 2, \dots, r$. Let A_R be the set defined in Section 3. We assume that we found a set of representatives C'_R of $\mathbb{W}\backslash A_R$ in Step 1. Let $Q = \# C'_R$. For example, it turns out that $Q = 7891$ if $R = 7$ for the prehomogeneous vector space (3). We assume that C'_R is documented in a file. For $i = 1, \dots, Q$, let $v_i \in \mathfrak{t}^*$ be the i-th element of C'_R. Note that $v_i = (v_{i,j})$ is an array of R distinct integers from $\{1, \dots, N\}$ $(N = \dim V)$.

For each i, we find a point of of form $\beta' = c_1\gamma_{v_{i,1}} + \cdots + c_R\gamma_{v_{i,R}}$ where $c_1 + \cdots + c_R = 1, c_1, \dots, c_R > 0$ and β' is orthogonal to vectors $\gamma_{v_{i,j}} - \gamma_{v_{i,1}}$ $(j = 2, \dots, R)$. So if we put $m_{1,k} = 1$ for $k = 1, \dots, R$, $m_{j,k} = (\gamma_{v_{i,j}} - \gamma_{v_{i,1}}, \gamma_{v_{i,k}})_*$ for $j = 2, \dots, R, k = 1, \dots, R$ and $m = (m_{j,k})$, then $c_1, \dots, c_R$ have to satisfy the condition:

$$m \begin{pmatrix} c_1 \\ \vdots \\ c_R \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix} . \tag{5.10}$$

We augment m by the vector $[1, 0, \dots, 0]$. As we discussed in Section 3, we only have to consider β' such that the above matrix $(m_{j,k})$ is non-singular, i.e., the (R, R)-entry of the reduced row echelon form of this augmented matrix is 1. If so, the last column is $[c_1, \dots, c_R]$ except that the positivity of $c_1, \dots, c_R$ has to be checked later.

ALGORITHM 5.11 (Functions in the main algorithm of Step 2).

In the following functions, R, D have to be defined and $\gamma_1, \dots, \gamma_N$, $v_i = (v_{i,j})$ $(i = 1, \dots, Q)$ from Step 1 have to be read from a file.

(i) Name: **mforclosest**(m, a)

Require: (i) m: an $R \times (R + 1)$ matrix with entries in $\mathbb{Q}$.

(ii) $a = (a_1, \dots, a_R)$: an array of elements of $\mathbb{N}$.

Description: It makes m the matrix in (5.10) augmented by the right-hand side if a is substituted by v_i.

Local variables: $j, k, l \in \mathbb{N}$.

1. For $j = 2, \dots, R$ and $k = 1, \dots, R$, $m_{j,k} \leftarrow \sum_{l=1}^{D} (\gamma_{a_j,l} - \gamma_{a_1,l})\gamma_{a_k,l}$.
2. For $k = 1, \dots, R + 1$, $m_{1,k} \leftarrow 1$.

3. For $j = 2, \ldots, R$, $m_{j,R+1} \leftarrow 0$.

(ii) Name: **betacoefficient**(x, a)

Require: (i) $x = (x_1, \ldots, x_R)$: an array of elements of $\mathbb{Q}$.

(ii) $a = (a_1, \ldots, a_R)$: an array of elements of $\mathbb{N}$.

Description: If a is substituted by v_i then it makes x the unique solution to (5.10) if the matrix $(m_{j,k})_{1 \le j,k \le R}$ is non-singular and the zero vector otherwise.

Local variables: (i) $i \in \mathbb{N}$.

(ii) $y = (y_{j,k})$, $z = (z_{j,k})$: $R \times (R+1)$ matrices with entries in $\mathbb{Q}$.

1. **mforclosest**(y, a).

2. $z \leftarrow$ the reduced row echelon form of y.

3. If $z_{R,R} = 1$ then for $i = 1, \ldots, R$, $x_i \leftarrow z_{i,R+1}$. Otherwise for $i = 1, \ldots, R$, $x_i \leftarrow 0$.

This is the end of the functions. $\diamond$

With these preparations, we can now describe the main algorithm of Step 2. We consider the prehomogeneous vector space (3) and the case $R = 7$. We use the following constants

N	R	D	Q
40	7	9	7891

For other cases, these constants have to be changed appropriately.

It will be convenient to use a computer language which has a linear algebra package with capability of computing the reduced row echelon from (rref) of a matrix.

ALGORITHM 5.12 (The main algorithm of Step 2).

Description: We assume that elements of C'_R are read from a file as $v_i = (v_{i,j})$ ($i = 1, \ldots, Q$). For all v_i ($i = 1, \ldots, Q$), this algorithm finds the closest point β' of the convex hull of $\{\gamma_{v_{i,1}}, \ldots, \gamma_{v_{i,R}}\}$ to the origin if the convex hull is an $(R-1)$-dimensional simplex and assigns the zero vector otherwise. We assume that constants N, R, D, Q are defined as above and the function **sorder** in Algorithm 5.8 and functions in Algorithm 5.11 are defined before this algorithm.

Local variables: (i) $i, j, k, l, m \in \mathbb{N}$.

(ii) $a = (a_1, \ldots, a_R)$: an array of elements of $\mathbb{N}$.

(iii) $x = (x_1, \ldots, x_R)$: an array of elements of $\mathbb{Q}$.

(iv) $y = (y_1, \ldots, y_D)$, $z = (z_1, \ldots, z_D)$: arrays of elements of $\mathbb{Q}$.

(v) $B^{(1)} = (B^{(1)}_{i,j})$, $C = (C_{i,j})$: $Q \times (D + 2R)$ matrices with entries in $\mathbb{Q}$.

1. Initialize $m \leftarrow 0$.

2. For $i = 1, \ldots, Q$, do the following.

2-a. $a \leftarrow v_i$.

2-b. **betacoefficient**(x, a).

2-c. If **allpositive**$(x) = 1$ then do the following.

2-c-1. $m \leftarrow m + 1$.

2-c-2. For $j = 1, \dots, R$, $C_{m,D+j} \leftarrow x_j$, $C_{m,D+R+j} \leftarrow a_j$.

2-c-3. For $j = 1, \dots, D$, $C_{m,j} \leftarrow \sum_{l=1}^{R} C_{m,D+l}\gamma_{C_{m,D+R+l},j}$.

In this step, we find $\beta' = \sum_{k=1}^{R} x_k \gamma_{v_{i,k}}$ where $x_1 + \cdots + x_R = 1$ and β' is orthogonal to $\gamma_{v_{i,k}} - \gamma_{v_{i,1}}$ for $k = 2, \dots, R$ if such β' is unique. If moreover $x_1, \dots, x_R > 0$ then we record entries of β' as $C_{m,1}, \dots, C_{m,D}$, $x_1, \dots, x_R$ as $C_{m,D+1}, \dots, C_{m,D+R}$ and $v_{i,1}, \dots, v_{i,R}$ as $C_{m,D+R+1}, \dots, C_{m,D+2R}$. The variable m counts the number of vectors β' which are recorded. We record the last $2R$-entries of $C_{m,*}$ so that we know from which coordinates β' is made.

We now move β' to an element β of $\mathfrak{t}_+^*$ as follows. It turns out that $m = 343$ at this point.

3. For $i = 1, \dots, m$ do the following.

3-a. For $j = 1, \dots, D$, $y_j \leftarrow C_{i,j}$.

3-b. $z \leftarrow$ **sorder**(y).

3-c. For $j = 1, \dots, D$, $B_{i,j}^{(1)} \leftarrow z_j$.

3-d. For $j = 1, \dots, 2R$, $B_{i,D+j}^{(1)} \leftarrow C_{i,D+j}$.

4 Record the first m rows of $B^{(1)}$ in a file.

This finishes the algorithm for given R. ◇

When we consider the prehomogeneous vector space (3), we consider $R = 7, \dots, 2$. If we combine informations for all R, it is convenient to change the step 1 so that the data for $B^{(1)}$ are without interruption. For example, when we consider the case $R = 6$, we change the step 1 to $m \leftarrow 343$ since the number of β for $R = 7$ is 343. We add the obvious choice for the case $R = 1$ in the end. Even though the sizes of the columns are different for different R's, we only use the first D-columns from now on and so it will not cause any problem.

We remove duplication from $B^{(1)}$ in Step 3.

5.3. Step 3

We assume that vectors $\beta \in \mathfrak{t}_+^*$ from Step 2 are read from a file as $B^{(1)} = (B_{i,j}^{(1)})$. As we pointed out above, we only use the first D-columns. Note that $[B_{i,1}^{(1)}, \dots, B_{i,D}^{(1)}]$ is the i-th $\beta \in \mathfrak{t}_+^*$. We also assume that the weight vectors $\gamma_1, \dots, \gamma_N$ and functions in Step 2 are read from a file. Note that $Z = [0, \dots, 0]$ is the zero vector in $\mathfrak{t}^*$.

Let Q be the number of β (in other words i's in $B^{(1)}$). In the actual program, we removed the duplication in Step 2 and moved to Step 3 and so we do not know this value Q. However, including the process of removing duplication twice probably makes the reader slightly confusing. So we explain the process of removing the duplication only in

Step 3. We use the constants $N = \dim V$, D, Q. For example, $N = 40$, $D = 9$ for the prehomogeneous vector space (3).

ALGORITHM 5.13 (The main algorithm of Step 3).

Description: This algorithm removes duplication and the zero vector from the list of β. Then it finds coordinate vectors contained in the subspaces Z_β, W_β. Constants N, D, Q have to be defined as above. The function **veq** in Algorithm 5.8 and the zero vector Z (see (5.9)) have to be defined.

Local variables: (i) $i, j, k, l, m, p \in \mathbb{N}$.

(ii) $y = (y_1, \ldots, y_D)$, $z = (z_1, \ldots, z_D)$: arrays of elements of $\mathbb{Q}$.

(iii) $B^{(2)} = (B^{(2)}_{i,j})$: a $Q \times D$ matrix with entries in $\mathbb{Q}$.

(iv) $E = (E_{i,j})$: a $Q \times N$ matrix with entries in $\mathbb{N}$.

1. Initialize $m \leftarrow 0$.

2. For $i = 1, \ldots, Q$, do the following.

   2-a. $l \leftarrow 1$.

   2-b. For $k = 1, \ldots, D$, $y_k \leftarrow B^{(1)}_{i,k}$.

   2-c. For $j = 1, \ldots, i-1$, do the following.

      2-c-1. For $k = 1, \ldots, D$, $z_k \leftarrow B^{(1)}_{j,k}$.

      2-c-2. $l \leftarrow \mathbf{veq}(y, z)l$.

   2-d. $l \leftarrow \mathbf{veq}(y, Z)l$.

   2-e. If $l = 1$ then $m \leftarrow m + 1$ and for $k = 1, \ldots, D$, $B^{(2)}_{m,k} \leftarrow B^{(1)}_{i,k}$.

The steps 2-a,...,2-e remove the duplication and the zero vector. We now determine coordinate vectors contained in Z_β, W_β.

3. For $i = 1, \ldots, m$, do the following.

   3-a. For $j = 1, \ldots, N$, do the following.

      3-a-1. $k \leftarrow \sum_{p=1}^{D} B^{(2)}_{i,p} \gamma_{j,p}$.

      3-a-2. $l \leftarrow \sum_{p=1}^{D} (B^{(2)}_{i,p})^2$.

      3-a-3. If $k > l$ then $E_{i,j} \leftarrow 2$.

      3-a-4. If $k = l$ then $E_{i,j} \leftarrow 1$.

      3-a-5. If $k < l$ then $E_{i,j} \leftarrow 0$.

4. Record the first m rows of $B^{(2)}$ and E.

This finishes the algorithm. ◇

It turns out that after the step 2 above, $m = 49, 81, 292, 183$ for the prehomogeneous vector spaces (1)–(4) respectively. E is a matrix which determines the subspaces Z_β, W_β.

For $\beta \in \mathfrak{B}$, the j-th coordinate vector $\mathbb{a}_j$ belongs to Z_β (resp. W_β) if and only if

$$(\beta, \gamma_j)_* = (\beta, \beta)_* \quad (\text{resp. } (\beta, \gamma_j)_* > (\beta, \beta)_*)\,.$$

Note that Z_β, W_β are spanned by coordinate vectors contained in them. We substituted 2, 1 to $E_{i,j}$ according as $\mathbb{a}_j \in W_\beta, Z_\beta$ and 0 otherwise in the step 3.

Algorithms 5.12, 5.13 are the same for the prehomogeneous vector spaces (1)–(4). One just has to include correct informations such as $\gamma_1, \ldots, \gamma_N, N, R, D$, etc.

6. Output for the case (1)

In this section we list the output of our programming for the case (1). We made the program so that the output will be a tex file. Let $\mathfrak{t}_+^*$ be the Weyl chamber and $\mathbb{a}_1, \ldots, \mathbb{a}_{18}$ the coordinate vectors both defined in Section 3. The set $\mathfrak{B}$ consists of β_i's in the following table. Note that $\beta_i \in \mathfrak{t}_+^*$ for all i.

β	i such that $\mathbb{a}_i \in Z_\beta$	i such that $\mathbb{a}_i \in W_\beta$
$\beta_1 = \frac{1}{42}(-2,-2,4,0,0,0,-3,3)$	7, 8, 9, 10, 11, 12, 13, 14, 15	16, 17, 18
$\beta_2 = \frac{1}{22}(-4,2,2,-2,-2,4,-3,3)$	6, 9, 12, 13, 14, 16, 17	15, 18
$\beta_3 = \frac{1}{22}(-2,-2,4,-4,2,2,-3,3)$	8, 9, 11, 12, 14, 15, 16	17, 18
$\beta_4 = \frac{1}{6}(-2,1,1,0,0,0,0,0)$	4, 5, 6, 7, 8, 9, 13, 14, 15, 16, 17, 18	-
$\beta_5 = \frac{1}{6}(0,0,0,-2,1,1,0,0)$	2, 3, 5, 6, 8, 9, 11, 12, 14, 15, 17, 18	-
$\beta_6 = \frac{1}{66}(-4,2,2,-4,2,2,-3,3)$	5, 6, 8, 9, 11, 12, 13, 16	14, 15, 17, 18
$\beta_7 = \frac{1}{42}(0,0,0,-2,-2,4,-3,3)$	3, 6, 9, 10, 11, 13, 14, 16, 17	12, 15, 18
$\beta_8 = \frac{5}{66}(-2,-2,4,-2,-2,4,-3,3)$	9, 12, 15, 16, 17	18
$\beta_9 = \frac{1}{114}(-6,0,6,-2,-2,4,-3,3)$	6, 7, 8, 12, 13, 14	9, 15, 16, 17, 18
$\beta_{10} = \frac{1}{114}(-2,-2,4,-6,0,6,-3,3)$	3, 6, 8, 11, 14, 16	9, 12, 15, 17, 18
$\beta_{11} = \frac{1}{18}(-2,0,2,-2,0,2,-1,1)$	6, 8, 12, 14, 16	9, 15, 17, 18
$\beta_{12} = \frac{1}{6}(-2,1,1,0,0,0,-3,3)$	13, 14, 15, 16, 17, 18	-
$\beta_{13} = \frac{1}{6}(0,0,0,-2,1,1,-3,3)$	11, 12, 14, 15, 17, 18	-
$\beta_{14} = \frac{1}{42}(-14,7,7,-2,-2,4,-3,3)$	6, 9, 13, 14, 16, 17	15, 18
$\beta_{15} = \frac{1}{42}(-2,-2,4,-14,7,7,-3,3)$	8, 9, 11, 12, 14, 15	17, 18
$\beta_{16} = \frac{1}{12}(-1,-1,2,-1,-1,2,-6,6)$	12, 15, 16, 17	18
$\beta_{17} = \frac{1}{30}(-10,-1,11,-4,-4,8,-6,6)$	9, 15, 16, 17	18
$\beta_{18} = \frac{1}{78}(-14,4,10,-8,-8,16,-9,9)$	6, 12, 16, 17	9, 15, 18
$\beta_{19} = \frac{1}{3}(-1,-1,2,0,0,0,0,0)$	7, 8, 9, 16, 17, 18	-
$\beta_{20} = \frac{1}{21}(-7,2,5,-1,-1,2,0,0)$	6, 7, 8, 15, 16, 17	9, 18
$\beta_{21} = \frac{1}{78}(-5,-2,7,-2,1,1,-6,6)$	8, 9, 11, 12, 13	14, 15, 16, 17, 18
$\beta_{22} = \frac{1}{12}(-1,-1,2,-1,-1,2,0,0)$	3, 6, 7, 8, 12, 15, 16, 17	9, 18
$\beta_{23} = \frac{1}{30}(-4,-4,8,-10,-1,11,-6,6)$	9, 12, 15, 17	18
$\beta_{24} = \frac{1}{78}(-8,-8,16,-14,4,10,-9,9)$	8, 12, 15, 16	9, 17, 18
$\beta_{25} = \frac{1}{3}(0,0,0,-1,-1,2,0,0)$	3, 6, 9, 12, 15, 18	-
$\beta_{26} = \frac{1}{21}(-1,-1,2,-7,2,5,0,0)$	3, 6, 8, 12, 15, 17	9, 18
$\beta_{27} = \frac{1}{78}(-2,1,1,-5,-2,7,-6,6)$	6, 9, 11, 13, 16	12, 14, 15, 17, 18
$\beta_{28} = \frac{1}{6}(-1,0,1,-1,0,1,-1,1)$	9, 12, 14, 16	15, 17, 18
$\beta_{29} = \frac{1}{6}(-2,1,1,-2,1,1,0,0)$	5, 6, 8, 9, 14, 15, 17, 18	-
$\beta_{30} = \frac{1}{6}(-2,-2,4,0,0,0,-3,3)$	16, 17, 18	-
$\beta_{31} = \frac{1}{42}(-14,4,10,-2,-2,4,-21,21)$	15, 16, 17	18
$\beta_{32} = \frac{1}{42}(-14,-14,28,-2,-2,4,-3,3)$	9, 16, 17	18

β	i such that $\mathbb{a}_i \in Z_\beta$	i such that $\mathbb{a}_i \in W_\beta$
$\beta_{33} = \frac{1}{66}(-22, 8, 14, -4, -4, 8, -3, 3)$	6, 16, 17	9, 15, 18
$\beta_{34} = \frac{1}{6}(0, 0, 0, -2, -2, 4, -3, 3)$	12, 15, 18	-
$\beta_{35} = \frac{1}{42}(-2, -2, 4, -14, 4, 10, -21, 21)$	12, 15, 17	18
$\beta_{36} = \frac{1}{42}(-2, -2, 4, -14, -14, 28, -3, 3)$	9, 12, 15	18
$\beta_{37} = \frac{1}{66}(-4, -4, 8, -22, 8, 14, -3, 3)$	8, 12, 15	9, 17, 18
$\beta_{38} = \frac{1}{2}(0, 0, 0, 0, 0, 0, -1, 1)$	10, 11, 12, 13, 14, 15, 16, 17, 18	-
$\beta_{39} = \frac{1}{6}(-2, 0, 2, -2, 0, 2, -1, 1)$	9, 15, 17	18
$\beta_{40} = \frac{1}{30}(-10, 2, 8, -4, 2, 2, -3, 3)$	8, 9, 14, 15, 16	17, 18
$\beta_{41} = \frac{1}{30}(-4, 2, 2, -10, 2, 8, -3, 3)$	6, 9, 12, 14, 17	15, 18
$\beta_{42} = \frac{1}{30}(-1, -1, 2, -1, -1, 2, -3, 3)$	9, 10, 11, 13, 14	12, 15, 16, 17, 18
$\beta_{43} = \frac{1}{6}(-2, -2, 4, -2, 1, 1, -3, 3)$	17, 18	-
$\beta_{44} = \frac{1}{6}(-2, 1, 1, -2, -2, 4, -3, 3)$	15, 18	-
$\beta_{45} = \frac{1}{6}(-2, 1, 1, -2, 1, 1, -3, 3)$	14, 15, 17, 18	-
$\beta_{46} = \frac{1}{3}(-1, -1, 2, -1, -1, 2, 0, 0)$	9, 18	-
$\beta_{47} = \frac{1}{6}(-2, -2, 4, -2, 1, 1, 0, 0)$	8, 9, 17, 18	-
$\beta_{48} = \frac{1}{6}(-2, 1, 1, -2, -2, 4, 0, 0)$	6, 9, 15, 18	-
$\beta_{49} = \frac{1}{6}(-2, -2, 4, -2, -2, 4, -3, 3)$	18	-

7. Output for the case (2)

In this section we list the output of our programming for the case (2). Let $\mathfrak{t}^*_+$ be the Weyl chamber and $\mathbb{a}_1, \dots, \mathbb{a}_{30}$ the coordinate vectors both defined in Section 3. The set $\mathfrak{B}$ consists of β_i's in the following table.

β	i such that $\mathbb{a}_i \in Z_\beta$	i such that $\mathbb{a}_i \in W_\beta$
$\beta_1 = \frac{1}{48}(-1, -1, -1, -1, -1, 5, -3, 3)$	5, 9, 12, 14, 15, 16, 17, 18, 19, 21, 22, 23, 25, 26, 28	20, 24, 27, 29, 30
$\beta_2 = \frac{1}{8}(-1, -1, -1, 1, 1, 1, -1, 1)$	13, 14, 15, 18, 19, 20, 22, 23, 24, 25, 26, 27	28, 29, 30
$\beta_3 = \frac{1}{15}(-5, 1, 1, 1, 1, 1, 0, 0)$	6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30	-
$\beta_4 = \frac{1}{120}(-7, -1, -1, -1, 5, 5, -3, 3)$	8, 9, 11, 12, 13, 14, 19, 20, 21, 22, 25	15, 23, 24, 26, 27, 28, 29, 30
$\beta_5 = \frac{7}{138}(-4, -4, -4, 2, 2, 8, -3, 3)$	14, 15, 20, 24, 27, 28	29, 30
$\beta_6 = \frac{1}{138}(-4, -4, -4, 2, 2, 8, -3, 3)$	5, 9, 12, 13, 18, 19, 22, 23, 25, 26	14, 15, 20, 24, 27, 28, 29, 30
$\beta_7 = \frac{2}{15}(-1, -1, -1, -1, -1, 5, 0, 0)$	5, 9, 12, 14, 15, 20, 24, 27, 29, 30	-
$\beta_8 = \frac{1}{2}(0, 0, 0, 0, 0, 0, -1, 1)$	16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30	-
$\beta_9 = \frac{2}{87}(-7, -1, -1, -1, 5, 5, -6, 6)$	15, 19, 20, 21, 22, 25	23, 24, 26, 27, 28, 29, 30
$\beta_{10} = \frac{1}{10}(-2, 0, 0, 0, 0, 2, -1, 1)$	9, 12, 14, 15, 20, 21, 22, 23, 25, 26, 28	24, 27, 29, 30
$\beta_{11} = \frac{1}{210}(-10, -4, 2, 2, 2, 8, -3, 3)$	9, 10, 11, 13, 20, 21, 22, 23	12, 14, 15, 24, 25, 26, 27, 28, 29, 30
$\beta_{12} = \frac{1}{51}(-5, -5, 1, 1, 1, 7, 0, 0)$	5, 9, 10, 11, 13, 20, 24, 25, 26, 28	12, 14, 15, 27, 29, 30
$\beta_{13} = \frac{1}{66}(-4, -4, 2, 2, 2, 2, -3, 3)$	10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24	25, 26, 27, 28, 29, 30
$\beta_{14} = \frac{1}{24}(-3, -1, -1, 1, 1, 3, -1, 1)$	9, 12, 13, 20, 22, 23, 25, 26	14, 15, 24, 27, 28, 29, 30
$\beta_{15} = \frac{5}{282}(-8, -8, -2, 4, 4, 10, -3, 3)$	12, 13, 20, 24, 25, 26	14, 15, 27, 28, 29, 30
$\beta_{16} = \frac{1}{264}(-7, -7, -1, -1, 5, 11, -3, 3)$	5, 9, 11, 13, 19, 23, 25	12, 14, 15, 20, 24, 26, 27, 28, 29, 30

β	i such that $\mathbb{a}_i \in Z_\beta$	i such that $\mathbb{a}_i \in W_\beta$
$\beta_{17} = \frac{1}{42}(-4, -2, 0, 0, 2, 4, -1, 1)$	9, 11, 13, 20, 23, 25	12, 14, 15, 24, 26, 27, 28, 29, 30
$\beta_{18} = \frac{1}{12}(-1, -1, -1, -1, 2, 2, 0, 0)$	4, 5, 8, 9, 11, 12, 13, 14, 19, 20, 23, 24, 26, 27, 28, 29	15, 30
$\beta_{19} = \frac{1}{30}(-4, -4, -4, -4, -4, 20, -15, 15)$	20, 24, 27, 29, 30	-
$\beta_{20} = \frac{1}{78}(-14, -14, -14, -14, 4, 52, -9, 9)$	15, 20, 24, 27, 29	30
$\beta_{21} = \frac{1}{102}(-34, -4, -4, 14, 14, 14, -9, 9)$	13, 14, 15, 22, 23, 24, 25, 26, 27	28, 29, 30
$\beta_{22} = \frac{1}{30}(-4, -4, -4, 2, 5, 5, -3, 3)$	13, 14, 19, 20, 23, 24, 26, 27	15, 28, 29, 30
$\beta_{23} = \frac{1}{42}(-8, -8, -8, -2, 10, 16, -9, 9)$	15, 20, 24, 27, 28	29, 30
$\beta_{24} = \frac{1}{15}(-2, -2, -2, 1, 1, 4, 0, 0)$	5, 9, 12, 13, 20, 24, 27, 28	14, 15, 29, 30
$\beta_{25} = \frac{1}{102}(-16, -16, -16, -10, -10, 68, -3, 3)$	14, 15, 20, 24, 27	29, 30
$\beta_{26} = \frac{1}{42}(-2, -2, 0, 0, 0, 4, -3, 3)$	12, 14, 15, 17, 18, 19, 21, 22, 23	20, 24, 25, 26, 27, 28, 29, 30
$\beta_{27} = \frac{1}{102}(-10, -10, 2, 2, 2, 14, -51, 51)$	20, 24, 25, 26, 28	27, 29, 30
$\beta_{28} = \frac{1}{222}(-44, -2, -2, -2, 10, 40, -27, 27)$	15, 20, 21, 22, 25	23, 24, 26, 27, 28, 29, 30
$\beta_{29} = \frac{1}{78}(-26, 4, 4, 4, 4, 10, -3, 3)$	9, 12, 14, 15, 21, 22, 23, 25, 26, 28	24, 27, 29, 30
$\beta_{30} = \frac{1}{48}(-4, -1, -1, 2, 2, 2, -3, 3)$	13, 14, 15, 18, 19, 20, 21	22, 23, 24, 25, 26, 27, 28, 29, 30
$\beta_{31} = \frac{1}{174}(-16, -16, 2, 2, 2, 26, -3, 3)$	5, 9, 25, 26, 28	12, 14, 15, 20, 24, 27, 29, 30
$\beta_{32} = \frac{1}{30}(-10, 2, 2, 2, 2, 2, -15, 15)$	21, 22, 23, 24, 25, 26, 27, 28, 29, 30	-
$\beta_{33} = \frac{1}{102}(-34, -10, -10, -10, 32, 32, -21, 21)$	15, 23, 24, 26, 27, 28, 29	30
$\beta_{34} = \frac{1}{22}(-4, -2, -2, 2, 2, 4, -3, 3)$	14, 15, 20, 22, 23, 25, 26	24, 27, 28, 29, 30
$\beta_{35} = \frac{1}{6}(-2, 0, 0, 0, 1, 1, 0, 0)$	8, 9, 11, 12, 13, 14, 23, 24, 26, 27, 28, 29	15, 30
$\beta_{36} = \frac{1}{66}(-22, 2, 4, 4, 6, 6, -1, 1)$	11, 12, 13, 14, 23, 24, 25	15, 26, 27, 28, 29, 30
$\beta_{37} = \frac{5}{66}(-2, -2, -2, -2, 4, 4, -3, 3)$	15, 19, 20, 23, 24, 26, 27, 28, 29	30
$\beta_{38} = \frac{1}{246}(-34, -28, -28, 26, 32, 32, -33, 33)$	15, 19, 20, 22, 25	23, 24, 26, 27, 28, 29, 30
$\beta_{39} = \frac{1}{66}(-22, -10, -10, 8, 8, 26, -9, 9)$	14, 15, 24, 27, 28	29, 30
$\beta_{40} = \frac{1}{114}(-38, -2, 4, 10, 10, 16, -3, 3)$	12, 13, 24, 25, 26	14, 15, 27, 28, 29, 30
$\beta_{41} = \frac{1}{30}(-4, -4, -2, 0, 4, 6, -1, 1)$	12, 13, 20, 24, 26	14, 15, 27, 28, 29, 30
$\beta_{42} = \frac{1}{102}(-7, -1, -1, 2, 2, 5, -3, 3)$	9, 12, 13, 20, 21	14, 15, 22, 23, 24, 25, 26, 27, 28, 29, 30
$\beta_{43} = \frac{1}{114}(-14, -8, -2, 4, 4, 16, -3, 3)$	9, 13, 20, 25, 26	12, 14, 15, 24, 27, 28, 29, 30
$\beta_{44} = \frac{1}{42}(-8, -2, -2, -2, 4, 10, -3, 3)$	9, 12, 14, 20, 23, 26, 28	15, 24, 27, 29, 30
$\beta_{45} = \frac{1}{114}(-6, -2, -2, 0, 4, 6, -3, 3)$	9, 12, 13, 19, 22, 25	14, 15, 20, 23, 24, 26, 27, 28, 29, 30
$\beta_{46} = \frac{1}{18}(-2, -2, 0, 0, 2, 2, -1, 1)$	11, 12, 13, 14, 19, 20, 23, 24, 25	15, 26, 27, 28, 29, 30
$\beta_{47} = \frac{1}{12}(-4, -1, -1, -1, -1, 8, -6, 6)$	24, 27, 29, 30	-
$\beta_{48} = \frac{1}{30}(-4, -4, -4, 2, 2, 8, -15, 15)$	20, 24, 27, 28	29, 30
$\beta_{49} = \frac{1}{30}(-10, -4, -4, -4, 2, 20, -3, 3)$	15, 24, 27, 29	30
$\beta_{50} = \frac{1}{48}(-7, -7, -7, 5, 5, 11, -3, 3)$	13, 20, 24, 27	14, 15, 28, 29, 30
$\beta_{51} = \frac{1}{48}(-16, 2, 2, 2, 5, 5, -3, 3)$	15, 21, 22, 25	23, 24, 26, 27, 28, 29, 30
$\beta_{52} = \frac{1}{6}(-2, 0, 0, 0, 1, 1, -3, 3)$	23, 24, 26, 27, 28, 29	30
$\beta_{53} = \frac{1}{42}(-14, -2, -2, 4, 7, 7, -3, 3)$	13, 14, 23, 24, 26, 27	15, 28, 29, 30
$\beta_{54} = \frac{1}{12}(-1, -1, -1, -1, 2, 2, -6, 6)$	19, 20, 23, 24, 26, 27, 28, 29	30
$\beta_{55} = \frac{1}{30}(-10, -4, -4, -1, 8, 11, -6, 6)$	15, 24, 27, 28	29, 30
$\beta_{56} = \frac{1}{24}(-5, -5, 1, 1, 1, 7, -3, 3)$	12, 14, 15, 20, 24, 25, 26, 28	27, 29, 30

β	i such that $\mathbb{a}_i \in Z_\beta$	i such that $\mathbb{a}_i \in W_\beta$
$\beta_{57} = \frac{1}{24}(-8, -2, 1, 1, 4, 4, -3, 3)$	15, 23, 24, 25	26, 27, 28, 29, 30
$\beta_{58} = \frac{1}{78}(-14, -8, -8, 4, 10, 16, -9, 9)$	14, 20, 23, 26	15, 24, 27, 28, 29, 30
$\beta_{59} = \frac{1}{3}(-1, -1, 0, 0, 0, 2, 0, 0)$	12, 14, 15, 27, 29, 30	-
$\beta_{60} = \frac{1}{3}(-1, -1, -1, 1, 1, 1, 0, 0)$	13, 14, 15, 28, 29, 30	-
$\beta_{61} = \frac{1}{21}(-7, -1, -1, 2, 2, 5, 0, 0)$	9, 12, 13, 24, 27, 28	14, 15, 29, 30
$\beta_{62} = \frac{1}{12}(-4, -1, -1, -1, -1, 8, 0, 0)$	9, 12, 14, 15, 24, 27, 29, 30	-
$\beta_{63} = \frac{1}{30}(-10, -10, -1, 5, 5, 11, -3, 3)$	14, 15, 27, 28	29, 30
$\beta_{64} = \frac{1}{78}(-14, -8, -8, 4, 4, 22, -3, 3)$	9, 12, 20, 28	14, 15, 24, 27, 29, 30
$\beta_{65} = \frac{1}{78}(-5, -2, -2, 1, 1, 7, -6, 6)$	14, 15, 18, 19, 21	20, 22, 23, 24, 25, 26, 27, 28, 29, 30
$\beta_{66} = \frac{1}{6}(-1, -1, 0, 0, 1, 1, -1, 1)$	15, 19, 20, 23, 24, 25	26, 27, 28, 29, 30
$\beta_{67} = \frac{1}{6}(-2, -2, 1, 1, 1, 1, 0, 0)$	10, 11, 12, 13, 14, 15, 25, 26, 27, 28, 29, 30	-
$\beta_{68} = \frac{1}{6}(-2, -2, 0, 0, 0, 4, -3, 3)$	27, 29, 30	-
$\beta_{69} = \frac{1}{6}(-2, -2, -2, 2, 2, 2, -3, 3)$	28, 29, 30	-
$\beta_{70} = \frac{1}{42}(-14, -2, -2, 4, 4, 10, -21, 21)$	24, 27, 28	29, 30
$\beta_{71} = \frac{1}{42}(-14, -14, -2, -2, 4, 28, -3, 3)$	15, 27, 29	30
$\beta_{72} = \frac{1}{42}(-14, -14, -14, 10, 16, 16, -3, 3)$	15, 28, 29	30
$\beta_{73} = \frac{1}{66}(-22, -4, -4, 8, 8, 14, -3, 3)$	13, 24, 27	14, 15, 28, 29, 30
$\beta_{74} = \frac{1}{6}(-2, -2, 0, 0, 2, 2, -1, 1)$	15, 26, 27, 28, 29	30
$\beta_{75} = \frac{1}{30}(-10, -4, 2, 2, 2, 8, -3, 3)$	12, 14, 15, 24, 25, 26, 28	27, 29, 30
$\beta_{76} = \frac{1}{30}(-1, -1, -1, -1, 2, 2, -3, 3)$	15, 16, 17, 18, 21, 22, 25	19, 20, 23, 24, 26, 27, 28, 29, 30
$\beta_{77} = \frac{1}{6}(-2, -2, -2, 1, 1, 4, -3, 3)$	29, 30	-
$\beta_{78} = \frac{1}{6}(-2, -2, 1, 1, 1, 1, -3, 3)$	25, 26, 27, 28, 29, 30	-
$\beta_{79} = \frac{1}{3}(-1, -1, -1, -1, 2, 2, 0, 0)$	15, 30	-
$\beta_{80} = \frac{1}{6}(-2, -2, -2, 1, 1, 4, 0, 0)$	14, 15, 29, 30	-
$\beta_{81} = \frac{1}{6}(-2, -2, -2, -2, 4, 4, -3, 3)$	30	-

8. Output for the case (3)

In this section we list the output of our programming for the case (3). Let $\mathfrak{t}^*_+$ be the Weyl chamber and $\mathbb{a}_1, \ldots, \mathbb{a}_{40}$ the coordinate vectors both defined in Section 3. The set $\mathfrak{B}$ consists of β_i's in the following table.

β	i such that $\mathbb{a}_i \in Z_\beta$ / i such that $\mathbb{a}_i \in W_\beta$
$\beta_1 =$	11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40
$\frac{1}{12}(0, 0, 0, 0, 0, -3, 1, 1, 1)$	-
$\beta_2 =$	4, 7, 9, 10, 14, 17, 19, 20, 24, 27, 29, 30, 31, 32, 33, 35, 36, 38
$\frac{7}{620}(-4, -4, -4, -4, 16, -5, -5, -5, 15)$	34, 37, 39, 40
$\beta_3 =$	8, 9, 10, 12, 13, 14, 15, 16, 17, 22, 23, 24, 25, 26, 27, 32, 33, 34, 35, 36, 37
$\frac{1}{780}(-12, -12, 8, 8, 8, -15, 5, 5, 5)$	18, 19, 20, 28, 29, 30, 38, 39, 40
$\beta_4 =$	7, 9, 10, 17, 19, 20, 27, 29, 30, 34, 35, 36, 38
$\frac{3}{44}(-4, 0, 0, 0, 4, -1, -1, -1, 3)$	37, 39, 40

β	i such that $\mathbb{a}_i \in Z_\beta$ / i such that $\mathbb{a}_i \in W_\beta$
$\beta_5 =$	4, 7, 9, 10, 14, 17, 19, 20, 21, 22, 23, 25, 26, 28, 31, 32, 33, 35, 36, 38
$\frac{1}{180}(-2,-2,-2,-2,8,-5,-5,5,5)$	24, 27, 29, 30, 34, 37, 39, 40
$\beta_6 =$	7, 9, 10, 17, 19, 20, 24, 25, 26, 28, 34, 35, 36, 38
$\frac{1}{12}(-2,0,0,0,2,-1,-1,1,1)$	27, 29, 30, 37, 39, 40
$\beta_7 =$	7, 9, 10, 14, 15, 16, 18, 24, 25, 26, 28, 34, 35, 36, 38
$\frac{1}{44}(-4,0,0,0,4,-3,1,1,1)$	17, 19, 20, 27, 29, 30, 37, 39, 40
$\beta_8 =$	7, 9, 10, 17, 19, 20, 24, 25, 26, 28, 31, 32, 33
$\frac{1}{76}(-4,0,0,0,4,-3,-3,1,5)$	27, 29, 30, 34, 35, 36, 37, 38, 39, 40
$\beta_9 =$	5, 6, 7, 8, 9, 10, 15, 16, 17, 18, 19, 20, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34
$\frac{3}{620}(-16,4,4,4,4,-5,-5,-5,15)$	35, 36, 37, 38, 39, 40
$\beta_{10} =$	8, 9, 10, 18, 19, 20, 28, 29, 30, 32, 33, 34, 35, 36, 37
$\frac{11}{780}(-12,-12,8,8,8,-5,-5,-5,15)$	38, 39, 40
$\beta_{11} =$	8, 9, 10, 18, 19, 20, 22, 23, 24, 25, 26, 27, 31
$\frac{1}{1580}(-12,-12,8,8,8,-15,-15,5,25)$	28, 29, 30, 32, 33, 34, 35, 36, 37, 38, 39, 40
$\beta_{12} =$	10, 20, 23, 24, 26, 27, 28, 29, 31, 32, 35
$\frac{9}{1580}(-8,-8,-8,12,12,-15,-15,5,25)$	30, 33, 34, 36, 37, 38, 39, 40
$\beta_{13} =$	10, 20, 30, 33, 34, 36, 37, 38, 39
$\frac{19}{780}(-8,-8,-8,12,12,-5,-5,-5,15)$	40
$\beta_{14} =$	8, 9, 10, 18, 19, 20, 22, 23, 24, 25, 26, 27, 32, 33, 34, 35, 36, 37
$\frac{3}{220}(-6,-6,4,4,4,-5,-5,5,5)$	28, 29, 30, 38, 39, 40
$\beta_{15} =$	10, 13, 14, 16, 17, 18, 19, 23, 24, 26, 27, 28, 29, 31, 32, 35
$\frac{1}{80}(-2,-2,-2,3,3,-5,0,0,5)$	20, 30, 33, 34, 36, 37, 38, 39, 40
$\beta_{16} =$	3, 4, 6, 7, 8, 9, 13, 14, 16, 17, 18, 19, 23, 24, 26, 27, 28, 29, 33, 34, 36, 37, 38, 39
$\frac{1}{30}(-2,-2,-2,3,3,0,0,0,0)$	10, 20, 30, 40
$\beta_{17} =$	10, 20, 23, 24, 26, 27, 28, 29, 33, 34, 36, 37, 38, 39
$\frac{7}{220}(-4,-4,-4,6,6,-5,-5,5,5)$	30, 40
$\beta_{18} =$	6, 7, 8, 9, 16, 17, 18, 19, 26, 27, 28, 29, 33, 34, 35
$\frac{7}{1420}(-24,-4,-4,16,16,-5,-5,-5,15)$	10, 20, 30, 36, 37, 38, 39, 40
$\beta_{19} =$	4, 7, 8, 14, 17, 18, 24, 27, 28, 32, 33, 35, 36
$\frac{3}{1420}(-16,-16,4,4,24,-5,-5,-5,15)$	9, 10, 19, 20, 29, 30, 34, 37, 38, 39, 40
$\beta_{20} =$	9, 10, 14, 17, 18, 22, 23, 25, 26, 32, 33, 35, 36
$\frac{1}{740}(-16,-16,4,4,24,-25,-5,15,15)$	19, 20, 24, 27, 28, 29, 30, 34, 37, 38, 39, 40
$\beta_{21} =$	9, 10, 14, 17, 18, 24, 27, 28, 32, 33, 35, 36
$\frac{1}{60}(-4,-4,1,1,6,-5,0,0,5)$	19, 20, 29, 30, 34, 37, 38, 39, 40
$\beta_{22} =$	9, 10, 19, 20, 24, 27, 28, 32, 33, 35, 36
$\frac{13}{2220}(-16,-16,4,4,24,-15,-15,5,25)$	29, 30, 34, 37, 38, 39, 40
$\beta_{23} =$	10, 20, 26, 27, 28, 29, 33, 34, 35
$\frac{17}{2220}(-24,-4,-4,16,16,-15,-15,5,25)$	30, 36, 37, 38, 39, 40
$\beta_{24} =$	9, 10, 19, 20, 29, 30, 34, 37, 38
$\frac{23}{1420}(-16,-16,4,4,24,-5,-5,-5,15)$	39, 40

β	i such that $\mathbb{a}_i \in Z_\beta$ / i such that $\mathbb{a}_i \in W_\beta$
$\beta_{25} =$	10, 20, 23, 24, 25, 31, 32
$\frac{7}{3820}(-24, -4, -4, 16, 16, -25, -25, 15, 35)$	26, 27, 28, 29, 30, 33, 34, 35, 36, 37, 38, 39, 40
$\beta_{26} =$	9, 10, 14, 17, 18, 24, 27, 28, 31
$\frac{3}{3020}(-16, -16, 4, 4, 24, -25, -5, -5, 35)$	19, 20, 29, 30, 32, 33, 34, 35, 36, 37, 38, 39, 40
$\beta_{27} =$	7, 8, 17, 18, 27, 28, 34, 35, 36
$\frac{11}{2380}(-32, -12, 8, 8, 28, -5, -5, -5, 15)$	9, 10, 19, 20, 29, 30, 37, 38, 39, 40
$\beta_{28} =$	10, 20, 24, 26, 28, 33, 35
$\frac{19}{4780}(-28, -8, -8, 12, 32, -25, -25, 15, 35)$	27, 29, 30, 34, 36, 37, 38, 39, 40
$\beta_{29} =$	10, 14, 16, 18, 23, 25, 33, 35
$\frac{1}{280}(-7, -2, -2, 3, 8, -10, 0, 5, 5)$	17, 19, 20, 24, 26, 27, 28, 29, 30, 34, 36, 37, 38, 39, 40
$\beta_{30} =$	10, 14, 16, 18, 24, 26, 28, 33, 35
$\frac{9}{3980}(-28, -8, -8, 12, 32, -35, 5, 5, 25)$	17, 19, 20, 27, 29, 30, 34, 36, 37, 38, 39, 40
$\beta_{31} =$	7, 8, 17, 18, 24, 25, 26, 34, 35, 36
$\frac{3}{620}(-16, -6, 4, 4, 14, -5, -5, 5, 5)$	9, 10, 19, 20, 27, 28, 29, 30, 37, 38, 39, 40
$\beta_{32} =$	9, 10, 19, 20, 27, 28, 34, 35, 36
$\frac{7}{1060}(-32, -12, 8, 8, 28, -15, -15, 5, 25)$	29, 30, 37, 38, 39, 40
$\beta_{33} =$	9, 10, 17, 18, 24, 25, 26, 32, 33
$\frac{1}{340}(-16, -6, 4, 4, 14, -15, -5, 5, 15)$	19, 20, 27, 28, 29, 30, 34, 35, 36, 37, 38, 39, 40
$\beta_{34} =$	7, 9, 17, 19, 24, 26, 28, 33, 35
$\frac{3}{1060}(-28, -8, -8, 12, 32, -15, -15, 5, 25)$	10, 20, 27, 29, 30, 34, 36, 37, 38, 39, 40
$\beta_{35} =$	5, 6, 7, 8, 9, 10, 15, 16, 17, 18, 19, 20, 25, 26, 27, 28, 29, 30, 35, 36, 37, 38, 39, 40
$\frac{1}{10}(-4, 1, 1, 1, 1, 0, 0, 0, 0)$	-
$\beta_{36} =$	10, 16, 17, 18, 19, 26, 27, 28, 29, 33, 34, 35
$\frac{1}{40}(-6, -1, -1, 4, 4, -5, 0, 0, 5)$	20, 30, 36, 37, 38, 39, 40
$\beta_{37} =$	6, 7, 8, 9, 16, 17, 18, 19, 23, 24, 25, 33, 34, 35
$\frac{1}{380}(-12, -2, -2, 8, 8, -5, -5, 5, 5)$	10, 20, 26, 27, 28, 29, 30, 36, 37, 38, 39, 40
$\beta_{38} =$	9, 10, 14, 17, 18, 24, 27, 28, 34, 37, 38
$\frac{13}{1420}(-16, -16, 4, 4, 24, -15, 5, 5, 5)$	19, 20, 29, 30, 39, 40
$\beta_{39} =$	9, 10, 17, 18, 24, 25, 26, 34, 35, 36
$\frac{11}{3180}(-32, -12, 8, 8, 28, -25, -5, 15, 15)$	19, 20, 27, 28, 29, 30, 37, 38, 39, 40
$\beta_{40} =$	7, 9, 14, 16, 18, 24, 26, 28, 33, 35
$\frac{1}{180}(-7, -2, -2, 3, 8, -5, 0, 0, 5)$	10, 17, 19, 20, 27, 29, 30, 34, 36, 37, 38, 39, 40
$\beta_{41} =$	7, 8, 14, 15, 16, 24, 25, 26, 34, 35, 36
$\frac{1}{2380}(-32, -12, 8, 8, 28, -15, 5, 5, 5)$	9, 10, 17, 18, 19, 20, 27, 28, 29, 30, 37, 38, 39, 40
$\beta_{42} =$	10, 13, 14, 16, 17, 18, 19, 23, 24, 26, 27, 28, 29, 33, 34, 36, 37, 38, 39
$\frac{3}{260}(-8, -8, -8, 12, 12, -15, 5, 5, 5)$	20, 30, 40
$\beta_{43} =$	10, 16, 17, 18, 19, 23, 24, 25, 33, 34, 35
$\frac{7}{2220}(-24, -4, -4, 16, 16, -25, -5, 15, 15)$	20, 26, 27, 28, 29, 30, 36, 37, 38, 39, 40
$\beta_{44} =$	9, 10, 17, 18, 24, 25, 26, 31
$\frac{1}{5580}(-32, -12, 8, 8, 28, -35, -15, 5, 45)$	19, 20, 27, 28, 29, 30, 32, 33, 34, 35, 36, 37, 38, 39, 40

β	i such that $\mathbb{a}_i \in Z_\beta$	i such that $\mathbb{a}_i \in W_\beta$
$\beta_{45} = \frac{1}{780}(-12, -2, -2, 8, 8, -15, -5, 5, 15)$	10, 16, 17, 18, 19, 23, 24, 25, 31, 32	20, 26, 27, 28, 29, 30, 33, 34, 35, 36, 37, 38, 39, 40
$\beta_{46} = \frac{7}{1020}(-14, -4, -4, 6, 16, -15, -5, 5, 15)$	10, 17, 19, 24, 26, 28, 33, 35	20, 27, 29, 30, 34, 36, 37, 38, 39, 40
$\beta_{47} = \frac{1}{3180}(-32, -12, 8, 8, 28, -15, -15, 5, 25)$	7, 8, 17, 18, 24, 25, 26, 32, 33	9, 10, 19, 20, 27, 28, 29, 30, 34, 35, 36, 37, 38, 39, 40
$\beta_{48} = \frac{1}{204}(-8, -4, 0, 4, 8, -3, -3, 1, 5)$	7, 8, 17, 18, 24, 26, 33, 35	9, 10, 19, 20, 27, 28, 29, 30, 34, 36, 37, 38, 39, 40
$\beta_{49} = \frac{1}{100}(-8, -4, 0, 4, 8, -9, -1, 3, 7)$	10, 17, 18, 24, 26, 33, 35	19, 20, 27, 28, 29, 30, 34, 36, 37, 38, 39, 40
$\beta_{50} = \frac{1}{60}(-4, -2, 0, 2, 4, -3, -1, 1, 3)$	9, 17, 18, 24, 26, 33, 35	10, 19, 20, 27, 28, 29, 30, 34, 36, 37, 38, 39, 40
$\beta_{51} = \frac{1}{220}(-8, 2, 2, 2, 2, -55, 15, 15, 25)$	15, 16, 17, 18, 19, 20, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34	35, 36, 37, 38, 39, 40
$\beta_{52} = \frac{1}{4}(0, 0, 0, 0, 0, -1, -1, 1, 1)$	21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40	-
$\beta_{53} = \frac{1}{120}(-48, 7, 7, 7, 27, -5, -5, -5, 15)$	7, 9, 10, 17, 19, 20, 27, 29, 30, 35, 36, 38	37, 39, 40
$\beta_{54} = \frac{1}{15}(-6, -1, -1, -1, 9, 0, 0, 0, 0)$	7, 9, 10, 17, 19, 20, 27, 29, 30, 37, 39, 40	-
$\beta_{55} = \frac{1}{4}(-1, 0, 0, 0, 1, -1, 0, 0, 1)$	17, 19, 20, 27, 29, 30, 34, 35, 36, 38	37, 39, 40
$\beta_{56} = \frac{1}{60}(-14, -14, -14, 6, 36, -5, -5, -5, 15)$	10, 20, 30, 34, 37, 39	40
$\beta_{57} = \frac{1}{460}(-44, -44, -44, 36, 96, -75, 5, 5, 65)$	10, 14, 17, 19, 24, 27, 29, 33, 36, 38	20, 30, 34, 37, 39, 40
$\beta_{58} = \frac{1}{120}(-48, -3, 17, 17, 17, -5, -5, -5, 15)$	8, 9, 10, 18, 19, 20, 28, 29, 30, 35, 36, 37	38, 39, 40
$\beta_{59} = \frac{1}{60}(-4, -4, -4, 0, 12, -3, -3, -3, 9)$	4, 7, 9, 14, 17, 19, 24, 27, 29, 33, 36, 38	10, 20, 30, 34, 37, 39, 40
$\beta_{60} = \frac{1}{8}(-1, 0, 0, 0, 1, -2, 0, 1, 1)$	17, 19, 20, 24, 25, 26, 28, 34, 35, 36, 38	27, 29, 30, 37, 39, 40
$\beta_{61} = \frac{2}{15}(-3, -3, 2, 2, 2, 0, 0, 0, 0)$	8, 9, 10, 18, 19, 20, 28, 29, 30, 38, 39, 40	-
$\beta_{62} = \frac{1}{540}(-56, 4, 4, 4, 44, -35, 5, 5, 25)$	7, 9, 10, 15, 16, 18, 25, 26, 28, 34	17, 19, 20, 27, 29, 30, 35, 36, 37, 38, 39, 40
$\beta_{63} = \frac{1}{240}(-6, -6, -6, 9, 9, -60, 5, 20, 35)$	20, 23, 24, 26, 27, 28, 29, 31, 32, 35	30, 33, 34, 36, 37, 38, 39, 40
$\beta_{64} = \frac{1}{220}(-8, -8, -8, -8, 32, -55, 5, 5, 45)$	14, 17, 19, 20, 24, 27, 29, 30, 31, 32, 33, 35, 36, 38	34, 37, 39, 40

β	i such that $\mathbb{a}_i \in Z_\beta$ / i such that $\mathbb{a}_i \in W_\beta$
$\beta_{65} = \frac{1}{240}(-31, -6, -6, -6, 49, -20, -20, 5, 35)$	7, 9, 10, 17, 19, 20, 24, 35, 36, 38
	27, 29, 30, 34, 37, 39, 40
$\beta_{66} = \frac{1}{460}(-34, 6, 6, 6, 16, -15, -15, -5, 35)$	7, 9, 10, 17, 19, 20, 25, 26, 28, 31, 32, 33
	27, 29, 30, 34, 35, 36, 37, 38, 39, 40
$\beta_{67} = \frac{1}{540}(-26, -26, -26, -16, 94, -35, -35, -25, 95)$	10, 20, 24, 27, 29, 31, 32, 35
	30, 33, 34, 36, 37, 38, 39, 40
$\beta_{68} = \frac{1}{60}(-4, -4, -4, 6, 6, -5, -5, -5, 15)$	10, 20, 30, 31, 32, 35
	33, 34, 36, 37, 38, 39, 40
$\beta_{69} = \frac{1}{140}(-26, -26, -26, 39, 39, -35, -10, -10, 55)$	20, 30, 33, 34, 36, 37, 38, 39
	40
$\beta_{70} = \frac{1}{140}(-6, -6, 4, 4, 4, -35, 5, 15, 15)$	18, 19, 20, 22, 23, 24, 25, 26, 27, 32, 33, 34, 35, 36, 37
	28, 29, 30, 38, 39, 40
$\beta_{71} = \frac{1}{140}(-16, -16, -16, 24, 24, -35, -15, 25, 25)$	20, 23, 24, 26, 27, 28, 29, 33, 34, 36, 37, 38, 39
	30, 40
$\beta_{72} = \frac{1}{340}(-36, -16, -16, 34, 34, -5, -5, -5, 15)$	6, 7, 8, 9, 16, 17, 18, 19, 26, 27, 28, 29, 33, 34
	10, 20, 30, 36, 37, 38, 39, 40
$\beta_{73} = \frac{1}{260}(-4, -4, 1, 1, 6, -65, 20, 20, 25)$	14, 17, 18, 24, 27, 28, 32, 33, 35, 36
	19, 20, 29, 30, 34, 37, 38, 39, 40
$\beta_{74} = \frac{1}{120}(-18, -3, -3, 12, 12, -30, -5, 10, 25)$	20, 26, 27, 28, 29, 33, 34, 35
	30, 36, 37, 38, 39, 40
$\beta_{75} = \frac{1}{60}(-4, -4, 1, 1, 6, -15, 0, 5, 10)$	19, 20, 24, 27, 28, 32, 33, 35, 36
	29, 30, 34, 37, 38, 39, 40
$\beta_{76} = \frac{1}{260}(-44, -44, -14, 51, 51, -40, -40, 25, 55)$	10, 20, 28, 29, 33, 34, 36, 37
	30, 38, 39, 40
$\beta_{77} = \frac{1}{190}(-16, -16, 9, 9, 14, -15, -10, 10, 15)$	9, 10, 18, 24, 27, 32, 33, 35, 36
	19, 20, 28, 29, 30, 34, 37, 38, 39, 40
$\beta_{78} = \frac{1}{260}(-19, -19, -4, -4, 46, -20, -20, -5, 45)$	9, 10, 19, 20, 24, 27, 32, 33, 35, 36
	29, 30, 34, 37, 38, 39, 40
$\beta_{79} = \frac{1}{340}(-56, -56, 34, 34, 44, -25, -25, -25, 75)$	9, 10, 19, 20, 29, 30, 32, 33, 35, 36
	34, 37, 38, 39, 40
$\beta_{80} = \frac{1}{35}(-4, -4, 1, 1, 6, 0, 0, 0, 0)$	4, 7, 8, 14, 17, 18, 24, 27, 28, 34, 37, 38
	9, 10, 19, 20, 29, 30, 39, 40
$\beta_{81} = \frac{1}{95}(-8, -8, -3, 7, 12, -5, 0, 0, 5)$	9, 14, 17, 18, 24, 27, 28, 33, 36
	10, 19, 20, 29, 30, 34, 37, 38, 39, 40
$\beta_{82} = \frac{1}{130}(-22, -2, -2, 3, 23, -10, -10, 10, 10)$	7, 9, 17, 19, 24, 26, 28, 34, 36, 38
	10, 20, 27, 29, 30, 37, 39, 40
$\beta_{83} = \frac{1}{460}(-44, -24, -24, 36, 56, -15, 5, 5, 5)$	7, 9, 14, 16, 18, 24, 26, 28, 34, 36, 38
	10, 17, 19, 20, 27, 29, 30, 37, 39, 40
$\beta_{84} = \frac{1}{560}(-4, -4, 1, 1, 6, -140, 40, 45, 55)$	19, 20, 24, 27, 28, 31
	29, 30, 32, 33, 34, 35, 36, 37, 38, 39, 40

β	i such that $\mathbb{a}_i \in Z_\beta$ / i such that $\mathbb{a}_i \in W_\beta$
$\beta_{85} = \frac{1}{740}(-56, -16, -16, 24, 64, -185, -5, 75, 115)$	20, 24, 26, 28, 33, 35
	27, 29, 30, 34, 36, 37, 38, 39, 40
$\beta_{86} = \frac{1}{440}(-56, -21, 14, 14, 49, -110, 25, 25, 60)$	17, 18, 27, 28, 34, 35, 36
	19, 20, 29, 30, 37, 38, 39, 40
$\beta_{87} = \frac{1}{220}(-8, -3, 2, 2, 7, -55, 15, 20, 20)$	17, 18, 24, 25, 26, 34, 35, 36
	19, 20, 27, 28, 29, 30, 37, 38, 39, 40
$\beta_{88} = \frac{1}{460}(-104, -104, -64, 116, 156, -55, -55, -55, 165)$	10, 20, 30, 34, 37, 38
	39, 40
$\beta_{89} = \frac{1}{140}(-36, -36, 4, 14, 54, -15, -15, -5, 35)$	10, 20, 29, 34, 37, 38
	30, 39, 40
$\beta_{90} = \frac{1}{560}(-89, -4, -4, 6, 91, -50, -50, 45, 55)$	10, 20, 24, 26, 28, 35
	27, 29, 30, 34, 36, 37, 38, 39, 40
$\beta_{91} = \frac{1}{110}(-4, -4, 1, 1, 6, -5, -5, 0, 10)$	9, 10, 19, 20, 24, 27, 28, 31
	29, 30, 32, 33, 34, 35, 36, 37, 38, 39, 40
$\beta_{92} = \frac{1}{580}(-82, -42, 28, 28, 68, -25, -25, 5, 45)$	8, 18, 27, 34, 35, 36
	9, 10, 19, 20, 28, 29, 30, 37, 38, 39, 40
$\beta_{93} = \frac{1}{480}(-42, -2, -2, 3, 43, -35, 10, 10, 15)$	10, 14, 16, 18, 24, 26, 28, 35
	17, 19, 20, 27, 29, 30, 34, 36, 37, 38, 39, 40
$\beta_{94} = \frac{3}{380}(-34, -4, -4, 6, 36, -15, -5, -5, 25)$	10, 17, 19, 27, 29, 34, 36, 38
	20, 30, 37, 39, 40
$\beta_{95} = \frac{1}{620}(-28, -8, -8, 12, 32, -35, -15, 5, 45)$	10, 17, 19, 24, 26, 28, 31, 32
	20, 27, 29, 30, 33, 34, 35, 36, 37, 38, 39, 40
$\beta_{96} = \frac{1}{940}(-76, -56, -56, 84, 104, -15, -15, 5, 25)$	7, 9, 17, 19, 24, 26, 28, 33
	10, 20, 27, 29, 30, 34, 36, 37, 38, 39, 40
$\beta_{97} = \frac{11}{580}(-12, -2, -2, 8, 8, -5, -5, -5, 15)$	10, 20, 30, 33, 34, 35
	36, 37, 38, 39, 40
$\beta_{98} = \frac{3}{170}(-6, -1, -1, 4, 4, -5, -5, 5, 5)$	10, 20, 23, 24, 25, 33, 34, 35
	26, 27, 28, 29, 30, 36, 37, 38, 39, 40
$\beta_{99} = \frac{1}{120}(-23, -18, 12, 12, 17, -10, -10, -5, 25)$	9, 10, 19, 20, 28, 34, 35, 36
	29, 30, 37, 38, 39, 40
$\beta_{100} = \frac{1}{460}(-124, -24, 16, 16, 116, -35, -35, -35, 105)$	9, 10, 19, 20, 29, 30, 34, 35, 36
	37, 38, 39, 40
$\beta_{101} = \frac{1}{420}(-48, -28, 12, 32, 32, -25, -25, 15, 35)$	8, 9, 18, 19, 26, 27, 33, 34, 35
	10, 20, 28, 29, 30, 36, 37, 38, 39, 40
$\beta_{102} = \frac{1}{60}(-9, -9, -9, -9, 36, -15, 5, 5, 5)$	14, 17, 19, 20, 24, 27, 29, 30, 34, 37, 39, 40
	-
$\beta_{103} = \frac{1}{70}(-13, -13, -13, -3, 42, -5, -5, 5, 5)$	10, 20, 24, 27, 29, 34, 37, 39
	30, 40
$\beta_{104} = \frac{1}{130}(-2, -2, -2, 3, 3, -5, -5, 5, 5)$	10, 20, 21, 22, 25, 31, 32, 35
	23, 24, 26, 27, 28, 29, 30, 33, 34, 36, 37, 38, 39, 40

β	i such that $\mathbb{a}_i \in Z_\beta$ / i such that $\mathbb{a}_i \in W_\beta$
$\beta_{105} = \frac{1}{140}(-56, 4, 4, 24, 24, -15, 5, 5, 5)$	10, 16, 17, 18, 19, 26, 27, 28, 29, 36, 37, 38, 39
	20, 30, 40
$\beta_{106} = \frac{1}{260}(-24, -4, -4, 16, 16, -65, 15, 15, 35)$	16, 17, 18, 19, 26, 27, 28, 29, 33, 34, 35
	20, 30, 36, 37, 38, 39, 40
$\beta_{107} = \frac{1}{60}(-4, 0, 0, 2, 2, -3, -1, -1, 5)$	10, 16, 17, 18, 19, 26, 27, 28, 29, 31, 32
	20, 30, 33, 34, 35, 36, 37, 38, 39, 40
$\beta_{108} = \frac{1}{20}(-2, -2, 0, 2, 2, -1, -1, 1, 1)$	8, 9, 18, 19, 23, 24, 26, 27, 33, 34, 36, 37
	10, 20, 28, 29, 30, 38, 39, 40
$\beta_{109} = \frac{1}{260}(-44, -44, 11, 11, 66, -65, -15, 40, 40)$	19, 20, 24, 27, 28, 34, 37, 38
	29, 30, 39, 40
$\beta_{110} = \frac{1}{60}(-4, -4, -4, 6, 6, -15, 5, 5, 5)$	13, 14, 16, 17, 18, 19, 23, 24, 26, 27, 28, 29, 33, 34, 36, 37, 38, 39
	20, 30, 40
$\beta_{111} = \frac{1}{130}(-22, -22, -7, 18, 33, -20, -20, 20, 20)$	10, 20, 24, 27, 28, 34, 37, 38
	29, 30, 39, 40
$\beta_{112} = \frac{9}{380}(-8, -8, 2, 2, 12, -5, -5, 5, 5)$	9, 10, 19, 20, 24, 27, 28, 34, 37, 38
	29, 30, 39, 40
$\beta_{113} = \frac{1}{60}(-9, -4, -4, 6, 11, -10, 0, 5, 5)$	10, 17, 19, 24, 26, 28, 34, 36, 38
	20, 27, 29, 30, 37, 39, 40
$\beta_{114} = \frac{1}{460}(-64, -64, -4, 36, 96, -75, 25, 25, 25)$	10, 14, 17, 18, 24, 27, 28, 34, 37, 38
	19, 20, 29, 30, 39, 40
$\beta_{115} = \frac{1}{380}(-37, -27, 18, 18, 28, -30, -20, 25, 25)$	9, 10, 18, 24, 25, 26, 34, 35, 36
	19, 20, 27, 28, 29, 30, 37, 38, 39, 40
$\beta_{116} = \frac{1}{580}(-12, -2, -2, 8, 8, -15, 5, 5, 5)$	10, 13, 14, 15, 23, 24, 25, 33, 34, 35
	16, 17, 18, 19, 20, 26, 27, 28, 29, 30, 36, 37, 38, 39, 40
$\beta_{117} = \frac{1}{140}(-31, -31, -11, -11, 84, -5, -5, -5, 15)$	9, 10, 19, 20, 29, 30, 34, 37
	39, 40
$\beta_{118} = \frac{1}{220}(-88, -38, 22, 22, 82, -15, -15, -15, 45)$	9, 10, 19, 20, 29, 30, 37, 38
	39, 40
$\beta_{119} = \frac{1}{380}(-152, 18, 38, 38, 58, -15, -15, 5, 25)$	9, 10, 19, 20, 27, 28, 35, 36
	29, 30, 37, 38, 39, 40
$\beta_{120} = \frac{1}{940}(-116, -16, -16, 64, 84, -35, -15, -15, 65)$	7, 9, 16, 18, 26, 28, 34, 35
	10, 17, 19, 20, 27, 29, 30, 36, 37, 38, 39, 40
$\beta_{121} = \frac{1}{140}(-4, -4, 0, 0, 8, -3, -3, 1, 5)$	4, 7, 14, 17, 28, 32, 33, 35, 36
	9, 10, 19, 20, 24, 27, 29, 30, 34, 37, 38, 39, 40
$\beta_{122} = \frac{1}{140}(-26, -26, 14, 14, 24, -15, -5, -5, 25)$	9, 10, 18, 28, 34, 37
	19, 20, 29, 30, 38, 39, 40
$\beta_{123} = \frac{1}{220}(-88, 2, 22, 22, 42, -15, 5, 5, 5)$	9, 10, 17, 18, 27, 28, 37, 38
	19, 20, 29, 30, 39, 40
$\beta_{124} = \frac{1}{940}(-256, -16, 4, 4, 264, -75, -55, -55, 185)$	9, 10, 17, 27, 34, 38
	19, 20, 29, 30, 37, 39, 40

β	i such that $\mathbb{a}_i \in Z_\beta$ / i such that $\mathbb{a}_i \in W_\beta$
$\beta_{125} = \frac{1}{440}(-71, -56, 14, 14, 99, -60, 10, 25, 25)$	9, 10, 17, 24, 28, 34, 38
	19, 20, 27, 29, 30, 37, 39, 40
$\beta_{126} = \frac{1}{1660}(-124, -24, -24, 76, 96, -135, -15, 65, 85)$	10, 16, 18, 24, 33, 35
	17, 19, 20, 26, 27, 28, 29, 30, 34, 36, 37, 38, 39, 40
$\beta_{127} = \frac{3}{520}(-16, -1, 4, 4, 9, -5, -5, 0, 10)$	7, 8, 17, 18, 25, 26, 34
	9, 10, 19, 20, 27, 28, 29, 30, 35, 36, 37, 38, 39, 40
$\beta_{128} = \frac{1}{340}(-36, -16, 9, 9, 34, -30, -5, 15, 20)$	9, 10, 17, 18, 24, 35, 36
	19, 20, 27, 28, 29, 30, 34, 37, 38, 39, 40
$\beta_{129} = \frac{1}{310}(-9, -4, -4, 1, 16, -10, -5, 5, 10)$	7, 9, 14, 26, 28, 33, 35
	10, 17, 19, 20, 24, 27, 29, 30, 34, 36, 37, 38, 39, 40
$\beta_{130} = \frac{1}{1260}(-64, -4, -4, 16, 56, -55, -55, 25, 85)$	10, 20, 24, 25, 31, 32
	26, 27, 28, 29, 30, 33, 34, 35, 36, 37, 38, 39, 40
$\beta_{131} = \frac{1}{60}(-24, 6, 6, 6, 6, -15, 5, 5, 5)$	15, 16, 17, 18, 19, 20, 25, 26, 27, 28, 29, 30, 35, 36, 37, 38, 39, 40
	-
$\beta_{132} = \frac{1}{620}(-68, -68, -38, 72, 102, -105, 5, 35, 65)$	10, 19, 24, 27, 28, 33, 36
	20, 29, 30, 34, 37, 38, 39, 40
$\beta_{133} = \frac{1}{180}(-7, -2, -2, 3, 8, -45, 10, 15, 20)$	17, 19, 24, 26, 28, 33, 35
	20, 27, 29, 30, 34, 36, 37, 38, 39, 40
$\beta_{134} = \frac{1}{680}(-12, -7, 3, 8, 8, -15, 0, 5, 10)$	10, 16, 17, 23, 24, 25, 32
	18, 19, 20, 26, 27, 28, 29, 30, 33, 34, 35, 36, 37, 38, 39, 40
$\beta_{135} = \frac{1}{940}(-96, -96, 4, 44, 144, -75, -75, 25, 125)$	9, 19, 24, 27, 28, 33, 36
	10, 20, 29, 30, 34, 37, 38, 39, 40
$\beta_{136} = \frac{1}{480}(-42, -2, -2, 23, 23, -35, -10, 15, 30)$	10, 16, 17, 18, 19, 25, 33, 34
	20, 26, 27, 28, 29, 30, 35, 36, 37, 38, 39, 40
$\beta_{137} = \frac{3}{340}(-7, -7, -2, 8, 8, -10, -10, 5, 15)$	10, 20, 23, 24, 26, 27, 32, 35
	28, 29, 30, 33, 34, 36, 37, 38, 39, 40
$\beta_{138} = \frac{1}{220}(-38, -8, -8, 22, 32, -25, -25, 5, 45)$	10, 20, 27, 29, 33, 35
	30, 34, 36, 37, 38, 39, 40
$\beta_{139} = \frac{1}{90}(-16, -1, -1, 4, 14, -10, -5, 5, 10)$	10, 17, 19, 26, 28, 34, 35
	20, 27, 29, 30, 36, 37, 38, 39, 40
$\beta_{140} = \frac{1}{940}(-86, -6, -6, 24, 74, -75, 5, 35, 35)$	10, 16, 18, 24, 25, 34, 35
	17, 19, 20, 26, 27, 28, 29, 30, 36, 37, 38, 39, 40
$\beta_{141} = \frac{1}{220}(-33, -8, 2, 2, 37, -20, -20, 15, 25)$	9, 10, 19, 20, 24, 28, 35, 36
	27, 29, 30, 34, 37, 38, 39, 40
$\beta_{142} = \frac{1}{780}(-112, -52, 8, 48, 108, -15, -15, -15, 45)$	7, 8, 17, 18, 27, 28, 34, 36
	9, 10, 19, 20, 29, 30, 37, 38, 39, 40
$\beta_{143} = \frac{1}{620}(-23, -18, 2, 7, 32, -10, -5, -5, 20)$	7, 14, 18, 24, 28, 33, 35
	9, 10, 17, 19, 20, 27, 29, 30, 34, 36, 37, 38, 39, 40
$\beta_{144} = \frac{1}{620}(-88, -58, -28, 72, 102, -105, 25, 25, 55)$	10, 17, 18, 27, 28, 34, 36
	19, 20, 29, 30, 37, 38, 39, 40

β	i such that $\mathbb{a}_i \in Z_\beta$ / i such that $\mathbb{a}_i \in W_\beta$
$\beta_{145} =$	9, 17, 18, 27, 28, 33, 35
$\frac{1}{340}(-36, -16, -1, 19, 34, -20, -5, -5, 30)$	10, 19, 20, 29, 30, 34, 36, 37, 38, 39, 40
$\beta_{146} =$	9, 10, 17, 18, 27, 28, 34, 35, 36
$\frac{1}{45}(-8, -3, 2, 2, 7, -5, 0, 0, 5)$	19, 20, 29, 30, 37, 38, 39, 40
$\beta_{147} =$	10, 20, 26, 27, 28, 29, 36, 37, 38, 39
$\frac{1}{40}(-16, -1, -1, 9, 9, -5, -5, 5, 5)$	30, 40
$\beta_{148} =$	10, 20, 24, 28, 33, 35
$\frac{1}{780}(-82, -52, -2, 28, 108, -75, -75, 35, 115)$	27, 29, 30, 34, 36, 37, 38, 39, 40
$\beta_{149} =$	9, 17, 18, 26, 34, 35
$\frac{1}{220}(-28, -8, 2, 12, 22, -15, -5, 5, 15)$	10, 19, 20, 27, 28, 29, 30, 36, 37, 38, 39, 40
$\beta_{150} =$	10, 19, 27, 28, 33, 35
$\frac{1}{60}(-8, -4, 0, 4, 8, -7, -3, 1, 9)$	20, 29, 30, 34, 36, 37, 38, 39, 40
$\beta_{151} =$	10, 19, 24, 26, 33, 35
$\frac{1}{20}(-2, -1, 0, 1, 2, -2, -1, 1, 2)$	20, 27, 28, 29, 30, 34, 36, 37, 38, 39, 40
$\beta_{152} =$	10, 19, 27, 28, 34, 36
$\frac{1}{20}(-4, -2, 0, 2, 4, -3, -1, 1, 3)$	20, 29, 30, 37, 38, 39, 40
$\beta_{153} =$	10, 20, 27, 28, 34, 35
$\frac{1}{420}(-88, -28, 12, 32, 72, -45, -45, 15, 75)$	29, 30, 36, 37, 38, 39, 40
$\beta_{154} =$	28, 29, 30, 32, 33, 34, 35, 36, 37
$\frac{1}{340}(-36, -36, 24, 24, 24, -85, -85, 55, 115)$	38, 39, 40
$\beta_{155} =$	17, 19, 20, 27, 29, 30, 35, 36, 38
$\frac{1}{340}(-136, 24, 24, 24, 64, -85, 15, 15, 55)$	37, 39, 40
$\beta_{156} =$	20, 30, 34, 37, 39
$\frac{1}{340}(-76, -76, -76, 24, 204, -85, -5, -5, 95)$	40
$\beta_{157} =$	14, 17, 19, 24, 27, 29, 33, 36, 38
$\frac{1}{420}(-28, -28, -28, 12, 72, -105, 15, 15, 75)$	20, 30, 34, 37, 39, 40
$\beta_{158} =$	18, 19, 20, 28, 29, 30, 35, 36, 37
$\frac{1}{340}(-136, 4, 44, 44, 44, -85, 15, 15, 55)$	38, 39, 40
$\beta_{159} =$	24, 27, 29, 30, 31, 32, 33, 35, 36, 38
$\frac{1}{260}(-4, -4, -4, -4, 16, -65, -65, 55, 75)$	34, 37, 39, 40
$\beta_{160} =$	17, 19, 20, 24, 35, 36, 38
$\frac{1}{80}(-7, -2, -2, -2, 13, -20, 0, 5, 15)$	27, 29, 30, 34, 37, 39, 40
$\beta_{161} =$	20, 30, 31, 32, 35
$\frac{1}{580}(-32, -32, -32, 48, 48, -145, -5, -5, 155)$	33, 34, 36, 37, 38, 39, 40
$\beta_{162} =$	31, 32, 33, 34, 35, 36, 37, 38, 39, 40
$\frac{1}{4}(0, 0, 0, 0, 0, -1, -1, -1, 3)$	-
$\beta_{163} =$	30, 33, 34, 36, 37, 38, 39
$\frac{1}{340}(-56, -56, -56, 84, 84, -85, -85, 15, 155)$	40
$\beta_{164} =$	23, 24, 26, 27, 28, 29, 33, 34, 36, 37, 38, 39
$\frac{1}{60}(-4, -4, -4, 6, 6, -15, -15, 15, 15)$	30, 40

β	i such that $\mathbb{a}_i \in Z_\beta$ / i such that $\mathbb{a}_i \in W_\beta$
$\beta_{165} = \frac{1}{660}(-24, -4, -4, 16, 16, -165, -165, 155, 175)$	26, 27, 28, 29, 33, 34, 35
	30, 36, 37, 38, 39, 40
$\beta_{166} = \frac{1}{660}(-104, -104, -24, 116, 116, -165, -65, 75, 155)$	20, 28, 29, 33, 34, 36, 37
	30, 38, 39, 40
$\beta_{167} = \frac{1}{220}(-48, -48, 12, 12, 72, -55, -55, 25, 85)$	29, 30, 34, 37, 38
	39, 40
$\beta_{168} = \frac{1}{380}(-32, -12, 8, 8, 28, -95, -95, 85, 105)$	27, 28, 34, 35, 36
	29, 30, 37, 38, 39, 40
$\beta_{169} = \frac{1}{220}(-48, -48, -28, 52, 72, -55, -15, -15, 85)$	20, 30, 34, 37, 38
	39, 40
$\beta_{170} = \frac{1}{260}(-64, -64, 16, 16, 96, -65, -5, -5, 75)$	19, 20, 29, 30, 34, 37, 38
	39, 40
$\beta_{171} = \frac{1}{340}(-46, -6, 4, 4, 44, -85, 5, 35, 45)$	17, 28, 34, 35, 36
	19, 20, 27, 29, 30, 37, 38, 39, 40
$\beta_{172} = \frac{1}{820}(-168, -28, -28, 112, 112, -205, -25, -25, 255)$	20, 30, 33, 34, 35
	36, 37, 38, 39, 40
$\beta_{173} = \frac{1}{820}(-48, -8, -8, 32, 32, -205, 15, 95, 95)$	20, 23, 24, 25, 33, 34, 35
	26, 27, 28, 29, 30, 36, 37, 38, 39, 40
$\beta_{174} = \frac{1}{660}(-164, -24, 16, 16, 156, -165, -5, -5, 175)$	19, 20, 29, 30, 34, 35, 36
	37, 38, 39, 40
$\beta_{175} = \frac{1}{340}(-56, -56, -56, -36, 204, -85, 15, 35, 35)$	20, 24, 27, 29, 34, 37, 39
	30, 40
$\beta_{176} = \frac{1}{580}(-12, -12, 8, 8, 8, -145, 35, 35, 75)$	18, 19, 20, 28, 29, 30, 31
	32, 33, 34, 35, 36, 37, 38, 39, 40
$\beta_{177} = \frac{1}{660}(-104, -104, -24, 76, 156, -165, -65, 115, 115)$	20, 24, 27, 28, 34, 37, 38
	29, 30, 39, 40
$\beta_{178} = \frac{1}{220}(-28, -8, -8, 12, 32, -55, 5, 25, 25)$	17, 19, 24, 26, 28, 34, 36, 38
	20, 27, 29, 30, 37, 39, 40
$\beta_{179} = \frac{1}{420}(-48, -48, 12, 12, 72, -105, 35, 35, 35)$	14, 17, 18, 24, 27, 28, 34, 37, 38
	19, 20, 29, 30, 39, 40
$\beta_{180} = \frac{1}{20}(-8, 0, 2, 2, 4, -1, -1, -1, 3)$	9, 10, 19, 20, 29, 30, 35, 36
	37, 38, 39, 40
$\beta_{181} = \frac{1}{340}(-136, -36, -16, -16, 204, -5, -5, -5, 15)$	9, 10, 19, 20, 29, 30, 37
	39, 40
$\beta_{182} = \frac{1}{340}(-136, -136, 84, 84, 104, -5, -5, -5, 15)$	9, 10, 19, 20, 29, 30, 38
	39, 40
$\beta_{183} = \frac{1}{580}(-52, -52, -12, -12, 128, -25, -25, -25, 75)$	4, 7, 14, 17, 24, 27, 38
	9, 10, 19, 20, 29, 30, 34, 37, 39, 40
$\beta_{184} = \frac{1}{660}(-264, 16, 16, 76, 156, -65, -5, -5, 75)$	10, 17, 19, 27, 29, 36, 38
	20, 30, 37, 39, 40

β	i such that $\mathbb{a}_i \in Z_\beta$ / i such that $\mathbb{a}_i \in W_\beta$
$\beta_{185} = \frac{1}{660}(-264, -24, 56, 116, 116, -65, -5, -5, 75)$	10, 18, 19, 28, 29, 36, 37
	20, 30, 38, 39, 40
$\beta_{186} = \frac{1}{180}(-20, -8, -8, 16, 20, -5, -1, -1, 7)$	7, 9, 16, 18, 26, 28, 34
	10, 17, 19, 20, 27, 29, 30, 36, 37, 38, 39, 40
$\beta_{187} = \frac{1}{820}(-148, -148, 52, 52, 192, -205, -45, 95, 155)$	19, 20, 28, 34, 37
	29, 30, 38, 39, 40
$\beta_{188} = \frac{1}{380}(-92, -32, 8, 8, 108, -95, -15, 25, 85)$	19, 20, 27, 34, 38
	29, 30, 37, 39, 40
$\beta_{189} = \frac{1}{1460}(-64, -4, 16, 16, 36, -365, 95, 115, 155)$	17, 18, 25, 26, 34
	19, 20, 27, 28, 29, 30, 35, 36, 37, 38, 39, 40
$\beta_{190} = \frac{1}{1140}(-96, -96, -36, 84, 144, -285, 35, 95, 155)$	19, 24, 27, 28, 33, 36
	20, 29, 30, 34, 37, 38, 39, 40
$\beta_{191} = \frac{1}{820}(-28, -28, -8, 32, 32, -205, 15, 75, 115)$	20, 23, 24, 26, 27, 32, 35
	28, 29, 30, 33, 34, 36, 37, 38, 39, 40
$\beta_{192} = \frac{1}{340}(-136, -56, -56, 44, 204, -25, -25, -25, 75)$	10, 20, 30, 37, 39
	40
$\beta_{193} = \frac{1}{220}(-48, -48, -28, -8, 132, -15, -15, 5, 25)$	10, 20, 29, 34, 37
	30, 39, 40
$\beta_{194} = \frac{1}{820}(-328, 32, 52, 52, 192, -45, -25, -25, 95)$	9, 10, 17, 27, 38
	19, 20, 29, 30, 37, 39, 40
$\beta_{195} = \frac{1}{820}(-148, -148, -28, 132, 192, -125, -125, 95, 155)$	10, 20, 28, 34, 37
	29, 30, 38, 39, 40
$\beta_{196} = \frac{1}{340}(-46, -46, 4, 4, 84, -45, 5, 5, 35)$	9, 10, 14, 17, 24, 27, 38
	19, 20, 29, 30, 34, 37, 39, 40
$\beta_{197} = \frac{1}{60}(-12, -12, 4, 4, 16, -7, -7, 5, 9)$	9, 10, 19, 20, 28, 34, 37
	29, 30, 38, 39, 40
$\beta_{198} = \frac{1}{380}(-52, -12, -12, 28, 48, -95, -15, 25, 85)$	20, 27, 29, 33, 35
	30, 34, 36, 37, 38, 39, 40
$\beta_{199} = \frac{1}{820}(-328, -28, 112, 112, 132, -45, -25, -25, 95)$	9, 10, 18, 28, 37
	19, 20, 29, 30, 38, 39, 40
$\beta_{200} = \frac{1}{40}(-4, 0, 0, 1, 3, -3, 0, 1, 2)$	10, 16, 18, 25, 34
	17, 19, 20, 26, 27, 28, 29, 30, 35, 36, 37, 38, 39, 40
$\beta_{201} = \frac{1}{1140}(-136, -96, -96, 84, 244, -185, -5, 35, 155)$	10, 17, 19, 24, 36, 38
	20, 27, 29, 30, 34, 37, 39, 40
$\beta_{202} = \frac{1}{180}(-32, -32, 8, 28, 28, -25, -5, -5, 35)$	10, 18, 19, 28, 29, 33, 34, 36, 37
	20, 30, 38, 39, 40
$\beta_{203} = \frac{1}{1140}(-136, -16, 4, 4, 144, -285, -5, 135, 155)$	19, 20, 24, 28, 35, 36
	27, 29, 30, 34, 37, 38, 39, 40
$\beta_{204} = \frac{1}{340}(-136, -136, 24, 124, 124, -25, -25, -25, 75)$	10, 20, 30, 38, 39
	40

β	i such that $\mathbb{a}_i \in Z_\beta$ / i such that $\mathbb{a}_i \in W_\beta$
$\beta_{205} = \frac{1}{380}(-32, -12, -12, 28, 28, -35, -35, 25, 45)$	10, 20, 23, 24, 35
	26, 27, 28, 29, 30, 33, 34, 36, 37, 38, 39, 40
$\beta_{206} = \frac{1}{260}(-24, -20, 12, 12, 20, -21, -13, 15, 19)$	9, 10, 18, 24, 35, 36
	19, 20, 27, 28, 29, 30, 34, 37, 38, 39, 40
$\beta_{207} = \frac{1}{1220}(-88, 12, 12, 32, 32, -45, -45, -5, 95)$	10, 20, 25, 31, 32
	26, 27, 28, 29, 30, 33, 34, 35, 36, 37, 38, 39, 40
$\beta_{208} = \frac{1}{820}(-108, -28, -28, -8, 172, -65, -65, 15, 115)$	7, 9, 17, 19, 24, 36, 38
	10, 20, 27, 29, 30, 34, 37, 39, 40
$\beta_{209} = \frac{1}{220}(-18, -18, -8, 2, 42, -15, -15, -5, 35)$	9, 19, 24, 27, 33, 36
	10, 20, 29, 30, 34, 37, 38, 39, 40
$\beta_{210} = \frac{1}{580}(-92, -92, 48, 68, 68, -45, -45, -45, 135)$	10, 20, 30, 32, 35
	33, 34, 36, 37, 38, 39, 40
$\beta_{211} = \frac{1}{1140}(-136, -76, -16, 84, 144, -285, 75, 75, 135)$	17, 18, 27, 28, 34, 36
	19, 20, 29, 30, 37, 38, 39, 40
$\beta_{212} = \frac{1}{1140}(-176, -136, -16, 84, 244, -185, 35, 75, 75)$	10, 17, 24, 28, 34, 38
	19, 20, 27, 29, 30, 37, 39, 40
$\beta_{213} = \frac{1}{260}(-24, -20, -8, 20, 32, -13, -1, -1, 15)$	9, 17, 18, 27, 28, 33
	10, 19, 20, 29, 30, 34, 36, 37, 38, 39, 40
$\beta_{214} = \frac{1}{740}(-96, -76, 24, 24, 124, -5, -5, -5, 15)$	7, 8, 17, 18, 27, 28, 34
	9, 10, 19, 20, 29, 30, 37, 38, 39, 40
$\beta_{215} = \frac{1}{220}(-38, -8, 2, 2, 42, -15, -15, 15, 15)$	7, 17, 24, 28, 34, 38
	9, 10, 19, 20, 27, 29, 30, 37, 39, 40
$\beta_{216} = \frac{1}{20}(-8, 0, 0, 4, 4, -5, -1, 3, 3)$	20, 26, 27, 28, 29, 36, 37, 38, 39
	30, 40
$\beta_{217} = \frac{1}{180}(-32, -12, 8, 8, 28, -45, -5, 15, 35)$	19, 20, 27, 28, 34, 35, 36
	29, 30, 37, 38, 39, 40
$\beta_{218} = \frac{1}{140}(-56, -16, -16, 44, 44, -15, -15, -15, 45)$	10, 20, 30, 36, 37, 38, 39
	40
$\beta_{219} = \frac{1}{260}(-104, -44, 16, 36, 96, -25, -25, -5, 55)$	10, 20, 29, 37, 38
	30, 39, 40
$\beta_{220} = \frac{1}{380}(-92, -32, -32, 48, 108, -55, -55, 25, 85)$	10, 20, 27, 29, 34, 36, 38
	30, 37, 39, 40
$\beta_{221} = \frac{1}{60}(-24, 4, 4, 8, 8, -3, -3, 1, 5)$	10, 20, 26, 27, 28, 29, 35
	30, 36, 37, 38, 39, 40
$\beta_{222} = \frac{1}{1460}(-224, -64, -4, 36, 256, -125, -125, 95, 155)$	9, 19, 24, 28, 36
	10, 20, 27, 29, 30, 34, 37, 38, 39, 40
$\beta_{223} = \frac{1}{140}(-36, -16, 4, 4, 44, -15, -15, 5, 25)$	9, 10, 19, 20, 27, 34, 38
	29, 30, 37, 39, 40
$\beta_{224} = \frac{1}{580}(-32, -12, 8, 8, 28, -25, -25, -5, 55)$	9, 10, 19, 20, 27, 28, 31
	29, 30, 32, 33, 34, 35, 36, 37, 38, 39, 40

β	i such that $\mathbb{a}_i \in Z_\beta$ / i such that $\mathbb{a}_i \in W_\beta$
$\beta_{225} =$	10, 19, 29, 34, 36
$\frac{1}{60}(-16, -4, 0, 4, 16, -7, -3, -3, 13)$	20, 30, 37, 38, 39, 40
$\beta_{226} =$	9, 17, 18, 24, 26, 34, 36
$\frac{1}{420}(-48, -28, -8, 32, 52, -25, -5, 15, 15)$	10, 19, 20, 27, 28, 29, 30, 37, 38, 39, 40
$\beta_{227} =$	9, 10, 17, 28, 34, 35, 36
$\frac{1}{140}(-26, -6, 4, 4, 24, -15, -5, 5, 15)$	19, 20, 27, 29, 30, 37, 38, 39, 40
$\beta_{228} =$	8, 18, 27, 34, 36
$\frac{1}{380}(-52, -32, 8, 28, 48, -15, -15, 5, 25)$	9, 10, 19, 20, 28, 29, 30, 37, 38, 39, 40
$\beta_{229} =$	10, 20, 30, 34, 35
$\frac{1}{740}(-196, -16, -16, 64, 164, -65, -65, -65, 195)$	36, 37, 38, 39, 40
$\beta_{230} =$	10, 20, 24, 33, 35
$\frac{1}{1460}(-124, -64, -64, -4, 256, -125, -125, -5, 255)$	27, 29, 30, 34, 36, 37, 38, 39, 40
$\beta_{231} =$	10, 20, 28, 29, 33, 34, 35
$\frac{1}{380}(-72, -52, 28, 48, 48, -35, -35, -15, 85)$	30, 36, 37, 38, 39, 40
$\beta_{232} =$	10, 17, 19, 27, 29, 33, 35
$\frac{1}{140}(-16, -6, -6, 4, 24, -15, -5, -5, 25)$	20, 30, 34, 36, 37, 38, 39, 40
$\beta_{233} =$	34, 37, 39, 40
$\frac{1}{20}(-3, -3, -3, -3, 12, -5, -5, -5, 15)$	-
$\beta_{234} =$	30, 34, 37, 39
$\frac{1}{20}(-4, -4, -4, 0, 12, -5, -5, 3, 7)$	40
$\beta_{235} =$	30, 31, 32, 35
$\frac{1}{80}(-2, -2, -2, 3, 3, -20, -20, 15, 25)$	33, 34, 36, 37, 38, 39, 40
$\beta_{236} =$	33, 34, 36, 37, 38, 39
$\frac{1}{60}(-4, -4, -4, 6, 6, -15, -15, -15, 45)$	40
$\beta_{237} =$	28, 29, 33, 34, 36, 37
$\frac{1}{140}(-16, -16, 4, 14, 14, -35, -35, 25, 45)$	30, 38, 39, 40
$\beta_{238} =$	30, 34, 37, 38
$\frac{1}{20}(-4, -4, -2, 4, 6, -5, -5, 1, 9)$	39, 40
$\beta_{239} =$	30, 33, 34, 35
$\frac{1}{40}(-6, -1, -1, 4, 4, -10, -10, 5, 15)$	36, 37, 38, 39, 40
$\beta_{240} =$	27, 29, 30, 37, 39, 40
$\frac{1}{60}(-24, -4, -4, -4, 36, -15, -15, 15, 15)$	-
$\beta_{241} =$	28, 29, 30, 38, 39, 40
$\frac{1}{60}(-24, -24, 16, 16, 16, -15, -15, 15, 15)$	-
$\beta_{242} =$	24, 27, 28, 34, 37, 38
$\frac{1}{140}(-16, -16, 4, 4, 24, -35, -35, 35, 35)$	29, 30, 39, 40
$\beta_{243} =$	19, 20, 29, 30, 39, 40
$\frac{1}{60}(-24, -24, 6, 6, 36, -15, 5, 5, 5)$	-
$\beta_{244} =$	19, 20, 29, 30, 35, 36
$\frac{1}{140}(-56, 4, 14, 14, 24, -35, 5, 5, 25)$	37, 38, 39, 40

β	i such that $\mathbb{a}_i \in Z_\beta$	i such that $\mathbb{a}_i \in W_\beta$
$\beta_{245} = \frac{1}{20}(-3,-3,-3,-3,12,-5,-5,5,5)$	24, 27, 29, 30, 34, 37, 39, 40	-
$\beta_{246} = \frac{1}{20}(-8,-2,2,2,6,-5,-5,3,7)$	29, 30, 37, 38	39, 40
$\beta_{247} = \frac{1}{20}(-8,-3,-3,2,12,-5,0,0,5)$	20, 30, 37, 39	40
$\beta_{248} = \frac{1}{20}(-4,-4,-2,-2,12,-5,1,1,3)$	19, 20, 29, 30, 34, 37	39, 40
$\beta_{249} = \frac{1}{40}(-16,-1,4,4,9,-10,0,5,5)$	19, 20, 27, 28, 37, 38	29, 30, 39, 40
$\beta_{250} = \frac{1}{260}(-44,-44,-4,36,56,-65,-25,35,55)$	20, 28, 34, 37	29, 30, 38, 39, 40
$\beta_{251} = \frac{1}{20}(-8,-8,2,7,7,-5,0,0,5)$	20, 30, 38, 39	40
$\beta_{252} = \frac{1}{340}(-16,-6,-6,14,14,-85,5,35,45)$	20, 23, 24, 35	26, 27, 28, 29, 30, 33, 34, 36, 37, 38, 39, 40
$\beta_{253} = \frac{1}{260}(-24,-14,-14,6,46,-65,5,15,45)$	17, 19, 24, 36, 38	20, 27, 29, 30, 34, 37, 39, 40
$\beta_{254} = \frac{1}{20}(-3,-3,2,2,2,-5,0,0,5)$	18, 19, 20, 28, 29, 30, 32, 33, 34, 35, 36, 37	38, 39, 40
$\beta_{255} = \frac{1}{260}(-34,-24,6,6,46,-65,15,25,25)$	17, 24, 28, 34, 38	19, 20, 27, 29, 30, 37, 39, 40
$\beta_{256} = \frac{1}{20}(-8,2,2,2,2,-5,-5,5,5)$	25, 26, 27, 28, 29, 30, 35, 36, 37, 38, 39, 40	-
$\beta_{257} = \frac{1}{20}(-8,-2,-2,6,6,-5,-1,-1,7)$	20, 30, 36, 37, 38, 39	40
$\beta_{258} = \frac{1}{20}(-8,-3,2,2,7,-5,0,0,5)$	19, 20, 29, 30, 37, 38	39, 40
$\beta_{259} = \frac{1}{60}(-14,-4,-4,6,16,-15,-5,5,15)$	20, 27, 29, 34, 36, 38	30, 37, 39, 40
$\beta_{260} = \frac{1}{260}(-64,-4,-4,16,56,-65,-5,-5,75)$	20, 30, 34, 35	36, 37, 38, 39, 40
$\beta_{261} = \frac{1}{5}(-2,-2,-2,3,3,0,0,0,0)$	10, 20, 30, 40	-
$\beta_{262} = \frac{1}{20}(-8,-8,0,4,12,-1,-1,-1,3)$	10, 20, 30, 39	40
$\beta_{263} = \frac{1}{80}(-32,3,3,13,13,-5,-5,-5,15)$	10, 20, 30, 35	36, 37, 38, 39, 40
$\beta_{264} = \frac{1}{10}(-4,-4,1,1,6,0,0,0,0)$	9, 10, 19, 20, 29, 30, 39, 40	-

β	i such that $\mathbb{a}_i \in Z_\beta$	i such that $\mathbb{a}_i \in W_\beta$
$\beta_{265} = \frac{1}{20}(-8,-2,-2,0,12,-1,-1,1,1)$	10, 20, 27, 29, 37, 39	30, 40
$\beta_{266} = \frac{1}{20}(-8,-8,4,6,6,-1,-1,1,1)$	10, 20, 28, 29, 38, 39	30, 40
$\beta_{267} = \frac{1}{260}(-104,-4,16,36,56,-25,-5,-5,35)$	10, 19, 29, 36	20, 30, 37, 38, 39, 40
$\beta_{268} = \frac{1}{340}(-46,-16,-6,-6,74,-25,-25,5,45)$	7, 17, 24, 38	9, 10, 19, 20, 27, 29, 30, 34, 37, 39, 40
$\beta_{269} = \frac{1}{180}(-22,-22,8,8,28,-5,-5,5,5)$	8, 18, 24, 27, 34, 37	9, 10, 19, 20, 28, 29, 30, 38, 39, 40
$\beta_{270} = \frac{3}{20}(-1,-1,-1,-1,4,0,0,0,0)$	4, 7, 9, 10, 14, 17, 19, 20, 24, 27, 29, 30, 34, 37, 39, 40	-
$\beta_{271} = \frac{1}{60}(-24,-4,6,6,16,-5,-5,5,5)$	9, 10, 19, 20, 27, 28, 37, 38	29, 30, 39, 40
$\beta_{272} = \frac{1}{220}(-28,-28,-8,12,52,-35,5,5,25)$	10, 14, 17, 24, 27, 38	19, 20, 29, 30, 34, 37, 39, 40
$\beta_{273} = \frac{1}{140}(-6,-6,4,4,4,-5,-5,-5,15)$	8, 9, 10, 18, 19, 20, 28, 29, 30, 31	32, 33, 34, 35, 36, 37, 38, 39, 40
$\beta_{274} = \frac{1}{60}(-24,-4,-4,-4,36,-15,-15,-15,45)$	37, 39, 40	-
$\beta_{275} = \frac{1}{60}(-24,-24,16,16,16,-15,-15,-15,45)$	38, 39, 40	-
$\beta_{276} = \frac{1}{140}(-16,-16,4,4,24,-35,-35,-35,105)$	34, 37, 38	39, 40
$\beta_{277} = \frac{1}{140}(-56,-16,-16,4,84,-35,-35,25,45)$	30, 37, 39	40
$\beta_{278} = \frac{1}{140}(-56,-56,24,44,44,-35,-35,25,45)$	30, 38, 39	40
$\beta_{279} = \frac{1}{220}(-28,-28,12,12,32,-55,-55,45,65)$	28, 34, 37	29, 30, 38, 39, 40
$\beta_{280} = \frac{1}{60}(-24,-4,-4,16,16,-15,-15,5,25)$	30, 36, 37, 38, 39	40
$\beta_{281} = \frac{1}{20}(-4,0,0,0,4,-5,-5,3,7)$	27, 29, 30, 34, 35, 36, 38	37, 39, 40
$\beta_{282} = \frac{1}{60}(-24,-24,-24,36,36,-15,5,5,5)$	20, 30, 40	-
$\beta_{283} = \frac{1}{140}(-56,-56,4,24,84,-35,5,5,25)$	20, 30, 39	40
$\beta_{284} = \frac{1}{220}(-88,12,12,32,32,-55,5,5,45)$	20, 30, 35	36, 37, 38, 39, 40

β	i such that $\mathbb{a}_i \in Z_\beta$	i such that $\mathbb{a}_i \in W_\beta$
$\beta_{285} = \frac{1}{60}(-24, -4, -4, -4, 36, -15, 5, 5, 5)$	17, 19, 20, 27, 29, 30, 37, 39, 40	-
$\beta_{286} = \frac{1}{60}(-24, -24, 16, 16, 16, -15, 5, 5, 5)$	18, 19, 20, 28, 29, 30, 38, 39, 40	-
$\beta_{287} = \frac{1}{20}(-2, -2, 0, 0, 4, -5, 1, 1, 3)$	14, 17, 24, 27, 38	19, 20, 29, 30, 34, 37, 39, 40
$\beta_{288} = \frac{1}{20}(-8, -8, 2, 2, 12, -5, -5, -5, 15)$	39, 40	-
$\beta_{289} = \frac{1}{20}(-8, 2, 2, 2, 2, -5, -5, -5, 15)$	35, 36, 37, 38, 39, 40	-
$\beta_{290} = \frac{1}{20}(-8, -8, -8, 12, 12, -5, -5, 5, 5)$	30, 40	-
$\beta_{291} = \frac{1}{20}(-8, -8, 2, 2, 12, -5, -5, 5, 5)$	29, 30, 39, 40	-
$\beta_{292} = \frac{1}{20}(-8, -8, -8, 12, 12, -5, -5, -5, 15)$	40	-

9. Output for the case (4)

In this section we list the output of our programming for the case (4). Let $\mathfrak{t}^*_+$ be the Weyl chamber and $\mathbb{a}_1, \ldots, \mathbb{a}_{56}$ the coordinate vectors both defined in Section 3. The set $\mathfrak{B}$ consists of β_i's in the following table.

β	i such that $\mathbb{a}_i \in Z_\beta$	i such that $\mathbb{a}_i \in W_\beta$
$\beta_1 = \frac{1}{120}(-5, -5, -5, 3, 3, 3, 3, 3)$	12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46	47, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_2 = \frac{1}{376}(-13, -5, -5, 3, 3, 3, 3, 11)$	15, 18, 20, 21, 26, 27, 28, 29, 31, 32, 34, 37, 38, 39, 41, 42, 44	30, 33, 35, 36, 40, 43, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_3 = \frac{3}{184}(-7, -7, 1, 1, 1, 1, 1, 9)$	11, 15, 18, 20, 21, 26, 30, 33, 35, 36, 37, 38, 39, 41, 42, 44, 47, 48, 50, 53	40, 43, 45, 46, 49, 51, 52, 54, 55, 56
$\beta_4 = \frac{1}{56}(-5, -1, -1, -1, -1, 3, 3, 3)$	19, 20, 21, 24, 25, 26, 28, 29, 30, 31, 32, 33, 38, 39, 40, 41, 42, 43, 47, 48, 49	34, 35, 36, 44, 45, 46, 50, 51, 52, 53, 54, 55, 56
$\beta_5 = \frac{1}{248}(-5, -5, -5, -5, 3, 3, 3, 11)$	6, 11, 15, 16, 17, 19, 26, 30, 31, 32, 34, 40, 41, 42, 44, 47, 48, 50	18, 20, 21, 33, 35, 36, 43, 45, 46, 49, 51, 52, 53, 54, 55, 56
$\beta_6 = \frac{3}{56}(-3, -3, -3, 1, 1, 1, 1, 5)$	15, 18, 20, 21, 30, 33, 35, 36, 40, 43, 45, 46, 47, 48, 50, 53	49, 51, 52, 54, 55, 56
$\beta_7 = \frac{1}{24}(-1, -1, -1, -1, -1, -1, 3, 3)$	5, 6, 10, 11, 14, 15, 17, 18, 19, 20, 25, 26, 29, 30, 32, 33, 34, 35, 39, 40, 42, 43, 44, 45, 48, 49, 50, 51, 53, 54	21, 36, 46, 52, 55, 56
$\beta_8 = \frac{9}{248}(-5, -5, -5, -5, 3, 3, 3, 11)$	18, 20, 21, 33, 35, 36, 43, 45, 46, 49, 51, 52, 53	54, 55, 56

β	i such that $\mathbb{a}_i \in Z_\beta$
	i such that $\mathbb{a}_i \in W_\beta$
$\beta_9 =$	22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\frac{3}{56}(-7, 1, 1, 1, 1, 1, 1, 1)$	-
$\beta_{10} =$	19, 20, 21, 34, 35, 36, 44, 45, 46, 50, 51, 52, 53, 54, 55
$\frac{7}{120}(3, -3, -3, -3, -3, 5, 5, 5)$	56
$\beta_{11} =$	21, 25, 26, 29, 30, 32, 33, 34, 35, 39, 40, 42, 43, 44, 45, 48, 49, 50, 51, 53, 54
$\frac{5}{184}(-9, -1, -1, -1, -1, -1, 7, 7)$	36, 46, 52, 55, 56
$\beta_{12} =$	19, 20, 21, 34, 35, 36, 38, 39, 40, 41, 42, 43, 47, 48, 49
$\frac{5}{312}(-9, -9, -1, -1, -1, 7, 7, 7)$	44, 45, 46, 50, 51, 52, 53, 54, 55, 56
$\beta_{13} =$	18, 20, 21, 26, 30, 31, 32, 34, 40, 41, 42, 44, 47, 48, 50
$\frac{1}{24}(-3, -1, -1, -1, 1, 1, 1, 3)$	33, 35, 36, 43, 45, 46, 49, 51, 52, 53, 54, 55, 56
$\beta_{14} =$	11, 15, 18, 19, 26, 30, 33, 34, 38, 39, 41, 42, 47, 48
$\frac{1}{120}(-5, -5, -1, -1, -1, 3, 3, 7)$	20, 21, 35, 36, 40, 43, 44, 45, 46, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{15} =$	18, 20, 21, 26, 30, 40, 53
$\frac{9}{632}(-13, -5, -5, -5, 3, 3, 3, 19)$	33, 35, 36, 43, 45, 46, 49, 51, 52, 54, 55, 56
$\beta_{16} =$	21, 36, 46, 48, 49, 50, 51, 53, 54
$\frac{11}{312}(-7, -7, -7, 1, 1, 1, 9, 9)$	52, 55, 56
$\beta_{17} =$	20, 21, 26, 30, 33, 34, 37
$\frac{3}{1208}(-23, -15, 1, 1, 1, 9, 9, 17)$	35, 36, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{18} =$	18, 20, 21, 26, 30, 31, 32, 34, 37, 38, 39
$\frac{1}{184}(-9, -5, -1, -1, 3, 3, 3, 7)$	33, 35, 36, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{19} =$	15, 18, 19, 30, 33, 34, 40, 43, 44, 47, 48
$\frac{5}{568}(-9, -9, -9, -1, -1, 7, 7, 15)$	20, 21, 35, 36, 45, 46, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{20} =$	15, 18, 20, 26, 29, 32, 34, 39, 42, 44, 47
$\frac{1}{232}(-15, -7, -7, 1, 1, 1, 9, 17)$	21, 30, 33, 35, 36, 40, 43, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{21} =$	21, 25, 26, 29, 30, 32, 33, 34, 35, 37, 38, 41, 47
$\frac{3}{440}(-15, -7, 1, 1, 1, 1, 9, 9)$	36, 39, 40, 42, 43, 44, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{22} =$	20, 21, 26, 30, 33, 34, 40, 43, 44, 49, 50, 53
$\frac{7}{376}(-11, -3, -3, -3, -3, 5, 5, 13)$	35, 36, 45, 46, 51, 52, 54, 55, 56
$\beta_{23} =$	20, 21, 26, 30, 33, 34, 38, 39, 41, 42, 47, 48
$\frac{5}{696}(-17, -9, -1, -1, -1, 7, 7, 15)$	35, 36, 40, 43, 44, 45, 46, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{24} =$	17, 18, 19, 20, 25, 26, 29, 30, 31, 39, 40, 41, 47
$\frac{1}{504}(-13, -5, -5, -5, 3, 3, 11, 11)$	21, 32, 33, 34, 35, 36, 42, 43, 44, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{25} =$	18, 19, 26, 30, 31, 32, 40, 41, 42, 47, 48
$\frac{3}{824}(-15, -7, -7, -7, 1, 9, 9, 17)$	20, 21, 33, 34, 35, 36, 43, 44, 45, 46, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{26} =$	6, 11, 26, 47, 48, 50, 53
$\frac{3}{440}(-7, -7, -7, 1, 1, 1, 1, 17)$	15, 18, 20, 21, 30, 33, 35, 36, 40, 43, 45, 46, 49, 51, 52, 54, 55, 56
$\beta_{27} =$	6, 11, 15, 18, 20, 21, 26, 30, 33, 35, 36, 40, 43, 45, 46, 49, 51, 52, 54, 55, 56
$\frac{5}{56}(-1, -1, -1, -1, -1, -1, -1, 7)$	-
$\beta_{28} =$	21, 26, 29, 32, 34, 39, 42, 44, 47
$\frac{3}{248}(-11, -3, -3, 1, 1, 1, 5, 9)$	30, 33, 35, 36, 40, 43, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56

β	i such that $\mathbb{a}_i \in Z_\beta$	i such that $\mathbb{a}_i \in W_\beta$
$\beta_{29} = \frac{13}{568}(-9,-9,-9,-1,-1,7,7,15)$	20, 21, 35, 36, 45, 46, 49, 50, 53	51, 52, 54, 55, 56
$\beta_{30} = \frac{9}{632}(-13,-13,-5,3,3,3,11,11)$	21, 36, 39, 40, 42, 43, 44, 45, 47	46, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{31} = \frac{11}{952}(-15,-15,-7,1,1,9,9,17)$	20, 21, 35, 36, 40, 43, 44, 47, 48	45, 46, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{32} = \frac{1}{8}(-1,-1,-1,-1,1,1,1,1)$	16, 17, 18, 19, 20, 21, 31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52	53, 54, 55, 56
$\beta_{33} = \frac{1}{40}(-7,-3,-3,1,1,1,5,5)$	21, 29, 30, 32, 33, 34, 35, 39, 40, 42, 43, 44, 45, 47	36, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{34} = \frac{3}{424}(-21,-13,-5,-5,3,11,11,19)$	20, 21, 33, 34, 40, 41, 42, 47, 48	35, 36, 43, 44, 45, 46, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{35} = \frac{1}{312}(-9,-5,-5,-1,-1,3,7,11)$	15, 18, 19, 26, 29, 32, 39, 42, 47	20, 21, 30, 33, 34, 35, 36, 40, 43, 44, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{36} = \frac{1}{888}(-13,-13,-5,-5,3,3,11,19)$	11, 15, 17, 19, 26, 30, 32, 34, 39, 41, 47	18, 20, 21, 33, 35, 36, 40, 42, 43, 44, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{37} = \frac{1}{104}(-7,-7,-7,1,1,1,9,9)$	14, 15, 17, 18, 19, 20, 29, 30, 32, 33, 34, 35, 39, 40, 42, 43, 44, 45, 47	21, 36, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{38} = \frac{1}{760}(-21,-5,-5,3,3,3,11,11)$	21, 25, 26, 27, 28, 31, 37, 38, 41	29, 30, 32, 33, 34, 35, 36, 39, 40, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{39} = \frac{7}{888}(-19,-11,-3,-3,5,5,13,13)$	21, 32, 33, 34, 35, 39, 40, 41, 47	36, 42, 43, 44, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{40} = \frac{1}{40}(-3,-3,-1,-1,1,1,3,3)$	17, 18, 19, 20, 32, 33, 34, 35, 39, 40, 41, 47	21, 36, 42, 43, 44, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{41} = \frac{1}{1912}(-21,-13,-13,-5,3,3,19,27)$	15, 17, 19, 26, 29, 39, 47	18, 20, 21, 30, 32, 33, 34, 35, 36, 40, 42, 43, 44, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{42} = \frac{7}{1528}(-27,-11,-3,-3,5,5,13,21)$	21, 26, 30, 32, 34, 39, 41, 47	33, 35, 36, 40, 42, 43, 44, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{43} = \frac{1}{216}(-17,-9,-9,-1,-1,7,7,23)$	15, 18, 26, 34, 44, 47, 48	20, 21, 30, 33, 35, 36, 40, 43, 45, 46, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{44} = \frac{1}{1272}(-21,-13,-5,-5,3,11,11,19)$	18, 19, 26, 30, 31, 32, 38, 39	20, 21, 33, 34, 35, 36, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{45} = \frac{1}{616}(-23,-15,-7,-7,1,9,17,25)$	18, 19, 26, 30, 32, 39, 41, 47	20, 21, 33, 34, 35, 36, 40, 42, 43, 44, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{46} = \frac{5}{24}(-1,-1,-1,-1,-1,-1,3,3)$	21, 36, 46, 52, 55, 56	-
$\beta_{47} = \frac{3}{40}(-5,-1,-1,1,1,1,1,3)$	30, 33, 35, 36, 40, 43, 45, 46, 47, 48, 50, 53	49, 51, 52, 54, 55, 56

β	i such that $\mathbb{a}_i \in Z_\beta$ / i such that $\mathbb{a}_i \in W_\beta$
$\beta_{48} =$	21, 36, 46, 52, 53, 54
$\frac{7}{88}(-3,-3,-3,-3,1,1,5,5)$	55, 56
$\beta_{49} =$	16, 17, 18, 19, 20, 21, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40
$\frac{1}{88}(-5,-5,-1,-1,3,3,3,3)$	41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{50} =$	6, 11, 15, 18, 19, 26, 30, 33, 34, 40, 43, 44, 49, 50, 53
$\frac{1}{16}(-1,-1,-1,-1,-1,1,1,3)$	20, 21, 35, 36, 45, 46, 51, 52, 54, 55, 56
$\beta_{51} =$	18, 20, 21, 33, 35, 36, 43, 45, 46, 47, 48, 50
$\frac{1}{12}(-2,-2,-2,0,1,1,1,3)$	49, 51, 52, 53, 54, 55, 56
$\beta_{52} =$	18, 20, 21, 33, 35, 36, 43, 45, 46, 49, 51, 52
$\frac{1}{24}(-3,-3,-3,-3,-1,-1,-1,15)$	54, 55, 56
$\beta_{53} =$	20, 21, 35, 36, 45, 46, 51, 52, 54, 55
$\frac{1}{40}(-7,-7,-7,-7,-7,5,5,25)$	56
$\beta_{54} =$	21, 30, 33, 35, 40, 43, 45, 48, 50, 53
$\frac{1}{40}(-9,-5,-5,1,1,1,5,11)$	36, 46, 49, 51, 52, 54, 55, 56
$\beta_{55} =$	34, 35, 36, 38, 39, 40, 41, 42, 43, 47, 48, 49
$\frac{1}{24}(-9,0,1,1,1,2,2,2)$	44, 45, 46, 50, 51, 52, 53, 54, 55, 56
$\beta_{56} =$	19, 20, 21, 34, 35, 36, 44, 45, 46, 47, 48, 49
$\frac{1}{6}(-1,-1,-1,0,0,1,1,1)$	50, 51, 52, 53, 54, 55, 56
$\beta_{57} =$	25, 26, 29, 30, 32, 33, 34, 35, 39, 40, 42, 43, 44, 45, 48, 49, 50, 51, 53, 54
$\frac{1}{40}(-15,1,1,1,1,1,5,5)$	36, 46, 52, 55, 56
$\beta_{58} =$	18, 20, 26, 30, 32, 34, 40, 42, 44, 48, 50
$\frac{1}{20}(-2,-1,-1,-1,0,0,2,3)$	21, 33, 35, 36, 43, 45, 46, 49, 51, 52, 53, 54, 55, 56
$\beta_{59} =$	21, 36, 46, 49, 51, 54
$\frac{1}{24}(-5,-5,-5,-1,-1,-1,3,15)$	52, 55, 56
$\beta_{60} =$	21, 36, 46, 49, 51, 53
$\frac{1}{184}(-45,-45,-45,-5,11,11,51,67)$	52, 54, 55, 56
$\beta_{61} =$	15, 18, 19, 30, 33, 34, 40, 43, 44, 48
$\frac{1}{184}(-13,-13,-13,-5,-5,3,19,27)$	20, 21, 35, 36, 45, 46, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{62} =$	21, 25, 26, 29, 30, 31, 39, 40, 41, 47
$\frac{1}{48}(-5,-1,-1,-1,1,1,3,3)$	32, 33, 34, 35, 36, 42, 43, 44, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{63} =$	26, 30, 33, 35, 36, 40, 43, 45, 46, 49, 51, 52, 54, 55, 56
$\frac{1}{24}(-9,-1,-1,-1,-1,-1,-1,15)$	-
$\beta_{64} =$	20, 21, 26, 30, 40, 53
$\frac{1}{232}(-47,-15,-15,-15,-3,17,17,61)$	33, 35, 36, 43, 45, 46, 49, 51, 52, 54, 55, 56
$\beta_{65} =$	15, 18, 20, 21, 26, 47, 48, 50, 53
$\frac{3}{152}(-7,-3,-3,1,1,1,1,9)$	30, 33, 35, 36, 40, 43, 45, 46, 49, 51, 52, 54, 55, 56
$\beta_{66} =$	33, 35, 36, 40, 41, 42, 44, 47, 48, 50
$\frac{1}{40}(-15,-1,1,1,3,3,3,5)$	43, 45, 46, 49, 51, 52, 53, 54, 55, 56
$\beta_{67} =$	21, 26, 30, 33, 34, 40, 43, 44, 49, 50, 53
$\frac{1}{120}(-29,-5,-5,-5,-5,3,19,27)$	35, 36, 45, 46, 51, 52, 54, 55, 56

β	i such that $\mathbb{a}_i \in Z_\beta$	i such that $\mathbb{a}_i \in W_\beta$
$\beta_{68} = \frac{1}{280}(-25,-17,-9,-9,-9,-1,31,39)$	20, 26, 30, 33, 34, 39, 42, 48	21, 35, 36, 40, 43, 44, 45, 46, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{69} = \frac{3}{152}(-7,-7,1,1,1,1,5,5)$	21, 36, 37, 38, 41, 47	39, 40, 42, 43, 44, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{70} = \frac{1}{24}(-9,-3,-3,-3,-3,7,7,7)$	34, 35, 36, 44, 45, 46, 50, 51, 52, 53, 54, 55	56
$\beta_{71} = \frac{1}{40}(-5,-4,0,0,1,1,1,6)$	18, 20, 21, 26, 30, 41, 42, 44, 47, 48, 50	33, 35, 36, 40, 43, 45, 46, 49, 51, 52, 53, 54, 55, 56
$\beta_{72} = \frac{1}{136}(-19,-19,-3,-3,-3,13,17,17)$	19, 20, 34, 35, 39, 40, 42, 43, 48, 49	21, 36, 44, 45, 46, 50, 51, 52, 53, 54, 55, 56
$\beta_{73} = \frac{1}{48}(-5,-2,0,0,0,1,3,3)$	21, 25, 26, 29, 30, 32, 33, 38, 41, 47	34, 35, 36, 39, 40, 42, 43, 44, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{74} = \frac{1}{88}(-9,-9,-1,-1,-1,-1,7,15)$	11, 15, 18, 20, 26, 30, 33, 35, 39, 42, 44, 48, 50, 53	21, 36, 40, 43, 45, 46, 49, 51, 52, 54, 55, 56
$\beta_{75} = \frac{1}{216}(-5,-5,-5,-1,-1,3,3,11)$	6, 11, 19, 26, 34, 44, 47, 48	15, 18, 20, 21, 30, 33, 35, 36, 40, 43, 45, 46, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{76} = \frac{1}{16}(-3,-3,-1,-1,1,1,1,5)$	18, 20, 21, 33, 35, 36, 40, 53	43, 45, 46, 49, 51, 52, 54, 55, 56
$\beta_{77} = \frac{1}{88}(-17,-17,-17,-13,7,11,11,35)$	20, 21, 35, 36, 45, 46, 49, 53	51, 52, 54, 55, 56
$\beta_{78} = \frac{1}{16}(-6,-1,0,1,1,1,2,2)$	36, 39, 40, 42, 43, 44, 45, 47	46, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{79} = \frac{1}{56}(-11,-11,-9,-9,-9,15,17,17)$	21, 36, 44, 45, 50, 51, 53, 54	46, 52, 55, 56
$\beta_{80} = \frac{1}{104}(-19,-15,-15,3,3,7,7,29)$	20, 21, 30, 33, 40, 43, 50, 53	35, 36, 45, 46, 49, 51, 52, 54, 55, 56
$\beta_{81} = \frac{1}{56}(-13,-5,-5,-5,3,3,11,11)$	21, 32, 33, 34, 35, 42, 43, 44, 45, 48, 49, 50, 51	36, 46, 52, 53, 54, 55, 56
$\beta_{82} = \frac{1}{152}(-37,-9,-5,-5,-1,-1,27,31)$	21, 33, 35, 40, 42, 44, 48, 50	36, 43, 45, 46, 49, 51, 52, 53, 54, 55, 56
$\beta_{83} = \frac{1}{152}(-25,-25,-17,-9,11,19,19,27)$	20, 21, 35, 36, 43, 44, 47, 48	45, 46, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{84} = \frac{1}{12}(-2,-1,-1,0,0,1,1,2)$	20, 21, 30, 33, 34, 40, 43, 44, 47, 48	35, 36, 45, 46, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{85} = \frac{1}{56}(-21,-11,-11,3,3,3,17,17)$	36, 46, 48, 49, 50, 51, 53, 54	52, 55, 56
$\beta_{86} = \frac{1}{104}(-39,3,3,5,5,7,7,9)$	30, 33, 34, 40, 43, 44, 47, 48	35, 36, 45, 46, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{87} = \frac{1}{24}(-4,-3,-1,0,0,2,3,3)$	21, 34, 35, 39, 40, 42, 43, 47	36, 44, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56

β	i such that $\mathbb{a}_i \in Z_\beta$ / i such that $\mathbb{a}_i \in W_\beta$
$\beta_{88} =$	15, 18, 19, 30, 33, 34, 38, 39, 41, 42
$\frac{1}{104}(-5,-5,-3,1,1,3,3,5)$	20, 21, 35, 36, 40, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{89} =$	21, 36, 45, 49, 50, 53
$\frac{1}{56}(-13,-13,-9,-1,-1,7,11,19)$	46, 51, 52, 54, 55, 56
$\beta_{90} =$	21, 36, 40, 43, 44, 47
$\frac{1}{376}(-69,-69,-29,11,11,27,51,67)$	45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{91} =$	20, 21, 30, 33, 34, 37
$\frac{1}{112}(-6,-5,-3,2,2,3,3,4)$	35, 36, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{92} =$	18, 19, 33, 34, 40, 42, 48
$\frac{1}{44}(-3,-3,-2,-2,-1,0,5,6)$	20, 21, 35, 36, 43, 44, 45, 46, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{93} =$	20, 21, 33, 34, 43, 44, 49, 50
$\frac{1}{88}(-17,-9,-9,-9,3,11,11,19)$	35, 36, 45, 46, 51, 52, 53, 54, 55, 56
$\beta_{94} =$	20, 21, 26, 34, 44, 47, 48
$\frac{1}{22}(-3,-1,-1,0,0,1,1,3)$	30, 33, 35, 36, 40, 43, 45, 46, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{95} =$	21, 36, 39, 40, 42, 43, 44, 45, 48, 49, 50, 51, 53, 54
$\frac{1}{4}(-1,-1,0,0,0,0,1,1)$	46, 52, 55, 56
$\beta_{96} =$	21, 35, 40, 43, 44, 48
$\frac{1}{88}(-21,-9,-5,-1,-1,3,15,19)$	36, 45, 46, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{97} =$	21, 26, 34, 39, 42, 47
$\frac{1}{296}(-39,-15,-7,1,1,9,17,33)$	30, 33, 35, 36, 40, 43, 44, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{98} =$	37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\frac{1}{8}(-3,-3,1,1,1,1,1,1)$	-
$\beta_{99} =$	15, 18, 19, 26, 47, 48
$\frac{1}{112}(-5,-3,-3,-1,-1,3,3,7)$	20, 21, 30, 33, 34, 35, 36, 40, 43, 44, 45, 46, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{100} =$	21, 36, 39, 40, 41, 47
$\frac{1}{56}(-9,-9,-1,-1,3,3,7,7)$	42, 43, 44, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{101} =$	21, 33, 35, 39, 41, 47
$\frac{1}{88}(-13,-9,-1,-1,3,3,7,11)$	36, 40, 42, 43, 44, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{102} =$	18, 30, 34, 40, 44, 47, 48
$\frac{1}{88}(-9,-6,-6,-2,1,5,5,12)$	20, 21, 33, 35, 36, 43, 45, 46, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{103} =$	21, 25, 26, 29, 30, 31, 37, 38
$\frac{1}{312}(-13,-5,-1,-1,3,3,7,7)$	32, 33, 34, 35, 36, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{104} =$	18, 19, 30, 31, 32, 40, 41, 42
$\frac{1}{88}(-5,-3,-3,-1,1,3,3,5)$	20, 21, 33, 34, 35, 36, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{105} =$	20, 33, 34, 40, 42, 48
$\frac{1}{168}(-23,-15,-7,-7,1,9,17,25)$	21, 35, 36, 43, 44, 45, 46, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{106} =$	18, 19, 33, 34, 40, 41, 42, 47, 48
$\frac{3}{248}(-7,-7,-3,-3,1,5,5,9)$	20, 21, 35, 36, 43, 44, 45, 46, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{107} =$	20, 30, 33, 34, 39, 42, 47
$\frac{1}{248}(-29,-17,-5,-1,-1,11,15,27)$	21, 35, 36, 40, 43, 44, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56

β	i such that $\mathbb{a}_i \in Z_\beta$ / i such that $\mathbb{a}_i \in W_\beta$
$\beta_{108} = \frac{1}{24}(-9,-1,-1,-1,3,3,3,3)$	31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52
	53, 54, 55, 56
$\beta_{109} = \frac{1}{72}(-5,-3,-1,-1,1,1,3,5)$	18, 20, 26, 30, 32, 34, 39, 41, 47
	21, 33, 35, 36, 40, 42, 43, 44, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{110} = \frac{1}{40}(-15,-7,-7,-7,-7,-7,25,25)$	36, 46, 52, 55, 56
	-
$\beta_{111} = \frac{1}{104}(-23,-23,-23,-23,1,1,25,65)$	21, 36, 46, 52, 54
	55, 56
$\beta_{112} = \frac{1}{136}(-51,-11,-11,5,13,13,13,29)$	33, 35, 36, 43, 45, 46, 47, 48, 50
	49, 51, 52, 53, 54, 55, 56
$\beta_{113} = \frac{1}{40}(-7,-7,-7,1,1,5,5,9)$	20, 21, 35, 36, 45, 46, 47, 48
	49, 50, 51, 52, 53, 54, 55, 56
$\beta_{114} = \frac{1}{136}(-27,-27,-27,-19,-19,13,21,85)$	21, 36, 46, 51, 54
	52, 55, 56
$\beta_{115} = \frac{1}{136}(-51,-27,-27,-27,13,13,53,53)$	36, 46, 52, 53, 54
	55, 56
$\beta_{116} = \frac{1}{56}(-13,-13,-13,-5,-5,11,19,19)$	21, 36, 46, 50, 51, 53, 54
	52, 55, 56
$\beta_{117} = \frac{1}{88}(-33,-9,-1,-1,7,7,7,23)$	33, 35, 36, 40, 53
	43, 45, 46, 49, 51, 52, 54, 55, 56
$\beta_{118} = \frac{1}{168}(-15,-15,-7,-7,-7,9,9,33)$	11, 15, 18, 26, 30, 33, 44, 50, 53
	20, 21, 35, 36, 40, 43, 45, 46, 49, 51, 52, 54, 55, 56
$\beta_{119} = \frac{1}{136}(-51,-19,-19,-3,-3,-3,13,85)$	36, 46, 49, 51, 54
	52, 55, 56
$\beta_{120} = \frac{1}{296}(-71,-15,-15,-15,1,1,41,73)$	21, 26, 30, 40, 53
	33, 35, 36, 43, 45, 46, 49, 51, 52, 54, 55, 56
$\beta_{121} = \frac{1}{136}(-51,-11,-11,5,5,21,21,21)$	34, 35, 36, 44, 45, 46, 47, 48, 49
	50, 51, 52, 53, 54, 55, 56
$\beta_{122} = \frac{1}{104}(-39,1,1,1,1,9,9,17)$	26, 30, 33, 34, 40, 43, 44, 49, 50, 53
	35, 36, 45, 46, 51, 52, 54, 55, 56
$\beta_{123} = \frac{1}{16}(-2,-1,-1,0,0,0,1,3)$	15, 18, 20, 26, 48, 50, 53
	21, 30, 33, 35, 36, 40, 43, 45, 46, 49, 51, 52, 54, 55, 56
$\beta_{124} = \frac{1}{232}(-31,-31,-7,-7,-7,25,25,33)$	19, 34, 40, 43, 49
	20, 21, 35, 36, 44, 45, 46, 50, 51, 52, 53, 54, 55, 56
$\beta_{125} = \frac{1}{40}(-15,-15,1,1,1,1,1,25)$	40, 43, 45, 46, 49, 51, 52, 54, 55, 56
	-
$\beta_{126} = \frac{1}{136}(-19,-19,-19,-11,-11,-3,-3,85)$	20, 21, 35, 36, 45, 46, 49
	51, 52, 54, 55, 56
$\beta_{127} = \frac{3}{136}(-17,-17,-1,-1,7,7,7,15)$	43, 45, 46, 49, 51, 52, 53
	54, 55, 56

β	i such that $\mathbb{a}_i \in Z_\beta$ / i such that $\mathbb{a}_i \in W_\beta$
$\beta_{128} = \frac{1}{136}(-51, -19, -19, -19, -11, 37, 37, 45)$	35, 36, 45, 46, 51, 52, 53
	54, 55, 56
$\beta_{129} = \frac{1}{264}(-59, -35, -35, -3, 13, 13, 37, 69)$	21, 33, 35, 43, 45, 48, 50
	36, 46, 49, 51, 52, 53, 54, 55, 56
$\beta_{130} = \frac{3}{40}(-5, -5, -5, 3, 3, 3, 3, 3)$	47, 48, 49, 50, 51, 52, 53, 54, 55, 56
	-
$\beta_{131} = \frac{1}{136}(-51, -51, -11, 13, 13, 13, 37, 37)$	46, 48, 49, 50, 51, 53, 54
	52, 55, 56
$\beta_{132} = \frac{1}{232}(-15, -15, 1, 1, 1, 9, 9, 9)$	19, 20, 21, 34, 35, 36, 37
	38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{133} = \frac{1}{264}(-99, -3, 5, 5, 13, 13, 29, 37)$	33, 35, 40, 42, 44, 48, 50
	36, 43, 45, 46, 49, 51, 52, 53, 54, 55, 56
$\beta_{134} = \frac{1}{264}(-59, -35, -35, -3, -3, 29, 53, 53)$	21, 34, 35, 44, 45, 48, 49
	36, 46, 50, 51, 52, 53, 54, 55, 56
$\beta_{135} = \frac{1}{88}(-17, -17, -17, -17, -9, 23, 23, 31)$	20, 21, 35, 36, 45, 46, 51, 52, 53
	54, 55, 56
$\beta_{136} = \frac{1}{88}(-33, -17, -17, -1, 7, 7, 23, 31)$	36, 46, 49, 51, 53
	52, 54, 55, 56
$\beta_{137} = \frac{1}{328}(-59, -51, -51, -3, 21, 29, 29, 85)$	20, 21, 33, 43, 50
	35, 36, 45, 46, 49, 51, 52, 53, 54, 55, 56
$\beta_{138} = \frac{1}{104}(-23, -23, -15, -15, 9, 9, 17, 41)$	21, 36, 43, 45, 49, 51, 53
	46, 52, 54, 55, 56
$\beta_{139} = \frac{1}{152}(-33, -25, -9, -9, 7, 7, 15, 47)$	21, 33, 35, 40, 53
	36, 43, 45, 46, 49, 51, 52, 54, 55, 56
$\beta_{140} = \frac{1}{88}(-33, -33, -1, 7, 7, 15, 15, 23)$	45, 46, 49, 50, 53
	51, 52, 54, 55, 56
$\beta_{141} = \frac{1}{328}(-123, -3, 13, 13, 21, 21, 29, 29)$	36, 39, 40, 41, 47
	42, 43, 44, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{142} = \frac{1}{328}(-59, -51, -51, -3, -3, 53, 53, 61)$	20, 21, 34, 44, 49
	35, 36, 45, 46, 50, 51, 52, 53, 54, 55, 56
$\beta_{143} = \frac{1}{104}(-39, -15, -15, 1, 1, 17, 17, 33)$	35, 36, 45, 46, 49, 50, 53
	51, 52, 54, 55, 56
$\beta_{144} = \frac{1}{152}(-57, -9, -1, 7, 7, 15, 15, 23)$	35, 36, 40, 43, 44, 47, 48
	45, 46, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{145} = \frac{1}{136}(-15, -15, -3, 1, 1, 5, 5, 21)$	15, 18, 30, 33, 44, 47, 48
	20, 21, 35, 36, 40, 43, 45, 46, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{146} = \frac{1}{120}(-21, -21, -13, -5, 11, 11, 19, 19)$	21, 36, 42, 43, 44, 45, 47
	46, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{147} = \frac{1}{136}(-15, -7, -7, -3, -3, 1, 13, 21)$	20, 26, 34, 44, 48
	21, 30, 33, 35, 36, 40, 43, 45, 46, 49, 50, 51, 52, 53, 54, 55, 56

β	i such that $\mathbb{a}_i \in Z_\beta$	i such that $\mathbb{a}_i \in W_\beta$
$\beta_{148} = \frac{1}{584}(-35, -27, -11, -11, 13, 21, 21, 29)$	18, 19, 31, 32, 40	20, 21, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{149} = \frac{1}{328}(-35, -11, -3, -3, 5, 5, 21, 21)$	21, 25, 26, 29, 30, 41, 47	32, 33, 34, 35, 36, 39, 40, 42, 43, 44, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{150} = \frac{1}{152}(-25, -17, -9, -1, -1, 15, 15, 23)$	20, 21, 34, 40, 43, 47, 48	35, 36, 44, 45, 46, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{151} = \frac{1}{456}(-43, -27, -27, -19, -3, 5, 45, 69)$	18, 30, 34, 40, 44, 48	20, 21, 33, 35, 36, 43, 45, 46, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{152} = \frac{1}{72}(-11, -11, -3, -3, 5, 5, 5, 13)$	18, 20, 21, 33, 35, 36, 40, 41, 42, 44, 47, 48, 50	43, 45, 46, 49, 51, 52, 53, 54, 55, 56
$\beta_{153} = \frac{1}{456}(-59, -43, -3, 5, 5, 13, 13, 69)$	20, 21, 26, 44, 47, 48	30, 33, 35, 36, 40, 43, 45, 46, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{154} = \frac{1}{40}(-15, -7, 1, 1, 1, 1, 9, 9)$	36, 39, 40, 42, 43, 44, 45, 48, 49, 50, 51, 53, 54	46, 52, 55, 56
$\beta_{155} = \frac{1}{8}(-3, -3, -1, -1, -1, -1, 5, 5)$	46, 52, 55, 56	-
$\beta_{156} = \frac{1}{40}(-15, -7, -7, -7, 1, 1, 9, 25)$	36, 46, 52, 54	55, 56
$\beta_{157} = \frac{1}{16}(-3, -3, -3, 1, 1, 1, 3, 3)$	21, 36, 46, 47	48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{158} = \frac{3}{8}(-1, -1, -1, -1, 1, 1, 1, 1)$	53, 54, 55, 56	-
$\beta_{159} = \frac{1}{16}(-6, 0, 0, 0, 1, 1, 1, 3)$	26, 30, 40, 53	33, 35, 36, 43, 45, 46, 49, 51, 52, 54, 55, 56
$\beta_{160} = \frac{1}{24}(-9, -9, -1, -1, -1, 3, 3, 15)$	45, 46, 51, 52, 54, 55	56
$\beta_{161} = \frac{1}{24}(-9, -9, -9, 3, 3, 7, 7, 7)$	50, 51, 52, 53, 54, 55	56
$\beta_{162} = \frac{1}{56}(-21, -5, -5, 3, 3, 7, 7, 11)$	35, 36, 45, 46, 47, 48	49, 50, 51, 52, 53, 54, 55, 56
$\beta_{163} = \frac{1}{8}(-3, -1, -1, -1, -1, 1, 1, 5)$	35, 36, 45, 46, 51, 52, 54, 55	56
$\beta_{164} = \frac{1}{40}(-7, -7, -3, -3, -3, -3, 1, 25)$	21, 36, 40, 43, 45, 49, 51, 54	46, 52, 55, 56
$\beta_{165} = \frac{1}{8}(-3, -3, -1, -1, 1, 1, 3, 3)$	46, 52, 53, 54	55, 56
$\beta_{166} = \frac{1}{40}(-15, -7, -7, -3, -3, 9, 13, 13)$	36, 46, 50, 51, 53, 54	52, 55, 56
$\beta_{167} = \frac{1}{40}(-15, -15, -3, 1, 5, 5, 9, 13)$	46, 49, 51, 53	52, 54, 55, 56

β	i such that $\mathbb{a}_i \in Z_\beta$ / i such that $\mathbb{a}_i \in W_\beta$
$\beta_{168} = \frac{1}{8}(-3,-1,0,0,0,1,1,2)$	35, 36, 40, 43, 44, 49, 50, 53
	45, 46, 51, 52, 54, 55, 56
$\beta_{169} = \frac{1}{104}(-23,-15,-15,1,1,9,17,25)$	21, 35, 45, 48
	36, 46, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{170} = \frac{1}{136}(-15,-3,-3,1,1,1,9,9)$	21, 25, 26, 47
	29, 30, 32, 33, 34, 35, 36, 39, 40, 42, 43, 44, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{171} = \frac{1}{104}(-11,-7,-7,-3,-3,5,5,21)$	15, 18, 26, 50, 53
	20, 21, 30, 33, 35, 36, 40, 43, 45, 46, 49, 51, 52, 54, 55, 56
$\beta_{172} = \frac{1}{8}(-3,-1,-1,-1,1,1,1,3)$	33, 35, 36, 43, 45, 46, 49, 51, 52, 53
	54, 55, 56
$\beta_{173} = \frac{1}{24}(-5,-5,-1,-1,-1,3,3,7)$	20, 21, 35, 36, 40, 43, 44, 49, 50, 53
	45, 46, 51, 52, 54, 55, 56
$\beta_{174} = \frac{1}{24}(-9,-9,-9,-1,-1,-1,15,15)$	52, 55, 56
	-
$\beta_{175} = \frac{1}{24}(-9,-9,-9,-9,7,7,7,15)$	54, 55, 56
	-
$\beta_{176} = \frac{1}{56}(-21,-21,-5,-5,3,3,11,35)$	46, 52, 54
	55, 56
$\beta_{177} = \frac{1}{56}(-21,-21,-21,3,11,11,19,19)$	52, 53, 54
	55, 56
$\beta_{178} = \frac{1}{88}(-33,-9,-9,7,7,7,15,15)$	36, 46, 47
	48, 49, 50, 51, 52, 53, 54, 55, 56
$\beta_{179} = \frac{1}{24}(-9,-9,-1,-1,-1,7,7,7)$	44, 45, 46, 50, 51, 52, 53, 54, 55
	56
$\beta_{180} = \frac{1}{40}(-3,-3,-3,-3,1,1,1,9)$	6, 11, 15, 26, 30, 40, 53
	18, 20, 21, 33, 35, 36, 43, 45, 46, 49, 51, 52, 54, 55, 56
$\beta_{181} = \frac{1}{8}(-3,-3,-3,-3,1,1,5,5)$	55, 56
	-
$\beta_{182} = \frac{1}{8}(-3,-3,-3,1,1,1,1,5)$	49, 51, 52, 54, 55, 56
	-
$\beta_{183} = \frac{1}{8}(-3,-3,-3,-3,-3,5,5,5)$	56
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