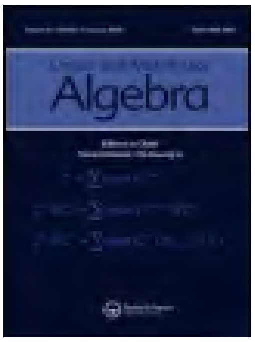

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ABSTRACT
We extend the classical associative PI-theory to Associative Pairs, and in doing so, we introduce related notions already present for algebras (and Jordan systems) as the ones of PI-element and PI-ideal, extended centroid and central closure.

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1. Introduction

Associative Pairs are the natural generalization of associative algebras in the context of Jordan pairs, which in turn, are related to Jordan and associative triple systems. As associative algebras do in the theory of Jordan algebras, they play a central role in the theory of Jordan Pairs. As a consequence, most of the questions on associative algebras that arise from Jordan theory, are also of great interest in the case of Associative Pairs (this is what could be named 'generalized Herstein Theory' after the line of research inaugurated by Herstein on the Jordan (and Lie) structures of associative algebras). In particular, that is the case with the theory of polynomial identities (PI-theory), which plays a central role in modern Jordan theory, after the sweeping work of Zelmanov on the subject.

In the present paper, we address that area for the case of Associative Pairs. Our objective is to extend most of the classical associative PI-theory. Namely, we deal with the classical theorems of that theory that are related to the structure theory, so to the notions of simplicity, primitivity or primeness, and their versions under the presence of an involution (again, a central ingredient for the associative systems appearing in Jordan -and Lie-theory) due to Kaplansky, Posner, Rowen, Amitsur and Martindale, among many other authors.

There are several features which are peculiar to Pairs in contrast to algebras. Apart from the structure theoretic particularities (often advantages, as in the study of the socle and the use of idempotents), from the viewpoint of PI-theory a feature which is always present is the possibility of moving to the theory of algebras by means of the so called standard

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imbeddings. Although this is often very useful (and, even necessary), we have preferred whenever possible, to adapt the algebra notions to Pairs (this is due to the fact that for the applications to Jordan theory we do not dispose of a construction as the standard imbedding, so using that for associative pairs arising in the Jordan context involves a detour which may obfuscate the arguments involved). On the other hand, some ideas which are natural in the Pair context, may shed light on developments of the associative theory. In the case of the theory of polynomial identities, this is quite notably the situation in the study of generalized polynomial identities as presented in [1].

Our study of polynomial identities on Associative Pairs has a peculiarity that deserves to be stressed. We do not really work with polynomial identities as such, although this could be done by resorting to the enveloping algebras. Instead, we adopt a more pair-theoretical approach as it is the use of homotope polynomial identities, and of PI-elements, in the line of what was done [2–4] for Jordan systems.

An additional remark, one to which we will not really devote much attention in this paper, is the fact that the standard imbedding is an associative superalgebra (of Morita type, with the terminology of [5]) so dealing with local algebras with a PI is in fact a particular case of dealing with graded generalized polynomial identities on those superalgebras. We will not delve in that connection, although it can be remarked that the use of local algebras with a PI is implicitly at the root of Chuang's approach to generalized polynomial identities [6], as adopted by Beidar, Martindale III, and Mikhalev [7].

This means, among other things, that we need to introduce the notion of extended centroid of an associative pair, and the corresponding scalar extension, its central closure. This raises many open problems that we do not address in the present paper, but whose solution would be doubtless quite interesting.

This paper is organized as follows. After this introductory section, in the first section we settle the basic notation, and recall some fundamental results on associative pairs mainly dealing with the relationship between associative pairs and their associative standard imbeddings. In the second section, we extend the construction of the extended centroid for semiprime associative algebras to semiprime associative pairs, following the approach of [8–10]. Accordingly, elements of the extended centroid of a semiprime associative pair are defined as equivalence classes of partially defined pair homomorphims (that is pair homomorphims defined over essential ideals, and not on the whole pair) commuting with all left, right and middle multiplication operators defined by pair elements. As proved in that section, and in a way similar to the case of associative algebras, the extended centroid of a semiprime associative pair is a commutative, unital, von Neumann regular ring, which in fact, is isomorphic to the extended centroid of the standard imbedding of the associative pair, so to the extended centroid of an associative algebra.

In the third section we consider the central closure of semiprime associative pairs, that is the natural scalar extension associated to the extended centroid, which is a tight scalar extension of the associative pair, and whose standard imbedding turns out to be isomorphic to the central closure of the standard imbedding of the original associative pair.

In section four we examine semiprime associative pairs endowed with polarized involutions. Such involutions have straightforward extensions to both the extended centroid and the central closure, so allowing the study of the $*$-extended centroid of a semiprime associative pair, that is the set of symmetric elements of the extended centroid under the extended involution, and of a new scalar extension, the $*$-central closure, that is the scalar extension linked to the *-extended centroid. Again these two constructions behave well with standard imbeddings, and it is not difficult to prove results analogous to those contained in the previous sections relating the *-extended centroid and the *-central closure of a semiprime associative pair with involution to those of their standard imbeddings.

Finally, in the fifth section, we deal with what is the central objective of the paper, namely the study of prime and primitive associative pairs having nonzero local algebras which satisfy polynomial identities. We introduce the notion of strongly primitive associative pair following [1,11], to be an associative pair with nonzero socle which is a dense subpair of the pair of homomorphims between two right vector spaces over a division PI-ring, and show that the strong primitivity of an associative pair is equivalent to that of its standard imbedding. Then analogous results to Amitsur, Kaplansky, Martindale and Posner Theorems are given for associative pairs, based on the existence of either local PI-algebras or on the fact that the associative pair satisfies some homotope polynomial identity.

2. Preliminaries

We will work with associative systems (algebras and pairs) over a unital commutative ring of scalars Φ that will be fixed throughout. We refer to [12,13] for notation, terminology and basic results. In this section, we recall some of those basic notation and results.

2.1. Multiplication operators

We denote operations $A^\sigma \times A^{-\sigma} \times A^\sigma \to A^\sigma$ of associative pairs $A = (A^+, A^-)$ over Φ by juxtaposition: $(x^\sigma, y^{-\sigma}, z^\sigma) \mapsto x^\sigma y^{-\sigma} z^\sigma$. We will also make use of the operator notation: $xyz = L(x, y)z = R(y, z)x = M(x, z)y$, where L, R and M are the left, right and middle multiplication operators respectively.

2.2. The standard imbedding of an associative pair

For any unital associative algebra $\mathcal{E}$ with an idempotent e, its associated Peirce decomposition $\mathcal{E} = \mathcal{E}_{11} \oplus \mathcal{E}_{12} \oplus \mathcal{E}_{21} \oplus \mathcal{E}_{22}$ gives rise to the associative pair $A = (\mathcal{E}_{12}, \mathcal{E}_{21})$ with operations inherited from the multiplication in $\mathcal{E}$.

Reciprocally, associative pairs are abstract off-diagonal Peirce spaces of associative algebras [12, p.92, 101]: given an associative pair $A = (A^+, A^-)$ we can construct a unital associative algebra $\mathcal{E}(A)$ (or simply $\mathcal{E}$, if A is understood, with a Peirce decomposition $\mathcal{E} = \mathcal{E}_{11} \oplus \mathcal{E}_{12} \oplus \mathcal{E}_{21} \oplus \mathcal{E}_{22}$, where $A = (\mathcal{E}_{12}, \mathcal{E}_{21})$. The Φ-module $\mathcal{E}_{ii}$ for $i = 1, 2$ is the subalgebra of $End_\Phi(A^\sigma) \times End_\Phi(A^{-\sigma})^{op}$, where $\sigma = +$ if $i = 1$, and $\sigma = -$ if $i = 2$, generated by the idempotent $e_i = (Id_{A^\sigma}, Id_{A^{-\sigma}})$ (hence $e_1 + e_2 = 1$), and all elements $x^\sigma y^{-\sigma} = (L(x^\sigma, y^{-\sigma}), R(y^{-\sigma}, x^\sigma))$.

It is clear then that $A^+ = \mathcal{E}_{12}$ is an $\mathcal{E}_{11} - \mathcal{E}_{22}$ bimodule, and $A^- = \mathcal{E}_{21}$ is an $\mathcal{E}_{22} - \mathcal{E}_{11}$ bimodule with the obvious actions (so that, in fact, $(\mathcal{E}_{11}, \mathcal{E}_{22}, A^+, A^-)$ is a Morita context, so $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$ with even part $\mathcal{E}_0 = \mathcal{E}_{11} \oplus \mathcal{E}_{22}$ and odd part $\mathcal{E}_1 = \mathcal{E}_{12} \oplus \mathcal{E}_{21}$ is a Morita superalgebra according to the definition introduced in [5, 1.4 (III)]).

The pair $(\mathcal{E}, e)$ (or simply, the associative algebra $\mathcal{E}$ if the idempotent e is understood) is termed the *standard imbedding* of the associative pair $A = (A^+, A^-)$. The *associative envelope* of an associative pair $A = (A^+, A^-)$ is the subalgebra $\mathcal{A}$ of its standard imbedding $\mathcal{E}$ generated by the odd part of the superalgebra $\mathcal{E}$.

The associative envelope $\mathcal{A}$ of A is an essential ideal of the standard imbedding $\mathcal{E}$ and $A = (\mathcal{A}_{12}, \mathcal{A}_{21})$, where $\mathcal{A}_{ij} = e_i\mathcal{A}e_j, i, j = 1, 2$. (Indeed the Peirce projections $\pi_{ij} : \mathcal{E} \to \mathcal{E}_{ij}$ can be restricted to $\mathcal{A} \to \mathcal{A}_{ij} = \mathcal{E}_{ij} \cap \mathcal{A}$, $i, j = 1, 2$.)

Remark 2.1: Notation and terminology for what we have referred to as the standard imbedding and the associative envelope of associative pairs have been rather interchangeably used in the literature. A careful review of the different references mentioned in the present paper should allow the reader to tackle this ambiguous usage. This will be, for instance, the case for the two references [14,15], where despite the used notation, the authors deal with the standard imbedding of associative pairs. See [14, p.2998] and [15, 3.3] for more details. In [16] standard imbedding and associative envelope appear as introduced in 2.2. Different notations are used for instance in [17] or [18]. We remark here that since $\mathcal{A}$ is an essential ideal of $\mathcal{E}$, all results proved in the paper will hold for both $\mathcal{E}$ and $\mathcal{A}$.

2.3. Involutions

An *involution* in an associative pair $A = (A^+, A^-)$ (sometimes named a polarized involution) is a pair of linear mappings $* : A^\sigma \to A^\sigma$ such that $(x^*)^* = x$ and $(xyz)^* = z^*y^*x^*$ for all $x, z \in A^\sigma$, $y \in A^{-\sigma}$, $\sigma = \pm$. Every (polarized) involution of an associative pair $A = (A^+, A^-)$ extends uniquely to an involution on its standard imbedding $\mathcal{E}$ which coincides with * on $\mathcal{E}_{12} = A^+$ and $\mathcal{E}_{21} = A^-$, and satisfies $e_1^* = e_2$ [15, 3.2].

2.4. (Semi)prime associative pairs

A *left ideal* of an associative pair $A = (A^+, A^-)$ is a Φ-module L of A^σ such that $A^\sigma A^{-\sigma} L \subseteq L$, $\sigma = \pm$. Right ideals are defined similarly. A *two-sided ideal* is simultaneously a left and a right ideal. A pair $I = (I^+, I^-)$ of two-sided ideals of A *is an* ideal if $A^\sigma I^{-\sigma} A^\sigma \subseteq I^\sigma$, $\sigma = \pm$. An associative pair $A = (A^+, A^-)$ is *semiprime* if and only if $I^\sigma A^{-\sigma} I^\sigma = 0, \sigma = \pm$, implies $I = 0$ and *prime* if $I^\sigma A^{-\sigma} J^\sigma = 0$, $\sigma = \pm$, implies $I = 0$ or $J = 0$, for any ideals I, J of A. If A is semiprime, then for any ideal $I = (I^+, I^-)$ of A it follows easily that $I^+ = 0$ if and only if $I^- = 0$ (see for instance [14, p.2992]). Primeness implies nondegenerancy ($a^\sigma A^{-\sigma} a^\sigma = 0$ implies $a^\sigma = 0$, $\sigma = \pm$) and semiprimeness is equivalent to nondegenerancy.

2.5. Vanishing conditions

Let A be an associative pair with standard imbedding $\mathcal{E}$. We denote by x_{ij} elements of $\mathcal{E}_{ij}$ for $i, j \in \{1, 2\}$. Then

$$x_{11}\mathcal{E}_{12} = \mathcal{E}_{21}x_{11} = 0 \Rightarrow x_{11} = 0,$$
$$x_{22}\mathcal{E}_{21} = \mathcal{E}_{12}x_{22} = 0 \Rightarrow x_{22} = 0.$$

If A is semiprime, the above conditions reduce to:

$$x_{11}\mathcal{E}_{12} = 0 \Rightarrow x_{11} = 0,$$
$$x_{22}\mathcal{E}_{21} = 0 \Rightarrow x_{22} = 0,$$

or, equivalently,

$$\mathcal{E}_{21}x_{11} = 0 \Rightarrow x_{11} = 0,$$
$$\mathcal{E}_{12}x_{22} = 0 \Rightarrow x_{22} = 0.$$

2.6. (Semi)prime standard imbeddings

(Semi)primeness of any associative pair is equivalent to that of its standard imbedding [14, Proposition 4.2]. This result stems from the correspondence between ideals of the associative pair $A = (A^+, A^-)$ and ideals of its standard imbedding $\mathcal{E}$ (see [14, Proposition 4.1]).

2.7. Annihilators

Let $A = (A^+, A^-)$ be an associative pair. For any subset $X \subseteq A^\sigma$, $\sigma = \pm$, the *left* and *right annihilators* of X in A are the sets

$$lann_A(X) = \{b \in A^{-\sigma} \mid bXA^{-\sigma} = A^\sigma bX = 0\},$$
$$rann_A(X) = \{b \in A^{-\sigma} \mid XbA^\sigma = A^{-\sigma}Xb = 0\},$$

and if A is semiprime, then

$$lann_A(X) = \{b \in A^{-\sigma} \mid A^\sigma bX = 0\},$$
$$rann_A(X) = \{b \in A^{-\sigma} \mid A^{-\sigma}Xb = 0\}.$$

which are left and right ideals of A respectively. The *annihilator* of X is $ann_A(X) = lann_A(X) \cap rann_A(X)$. If $I = (I^+, I^-)$ is an ideal of a semiprime associative pair A, then the *annihilator* $ann_A(I) = (ann_A(I^+), ann_A(I^-))$ of I is

$$ann_A(I^\sigma) = \{x \in A^{-\sigma} \mid xI^\sigma x = 0\}.$$

Moreover $ann_A(I)$ is itself an ideal of A, and $I^\sigma \cap ann_A(I^{-\sigma}) = 0$, $\sigma = \pm$ [14, Proposition 2.2].

2.8. Essential ideals

An ideal I of an associative pair A is *essential* if $I \cap J \neq 0$ for any nonzero ideal J of A. It follows from [14, Proposition 2.2] that essential ideals of semiprime associative pairs are exactly those ideals of A whose annihilator vanishes.

2.9. The socle of a semiprime associative pair

The *socle* of a semiprime associative pair A consists of the pair of subsets $Soc(A) = (Soc(A^+), Soc(A^-))$, where $Soc(A^\sigma)$, $\sigma = \pm$, is the sum of all minimal right ideals of A. $Soc(A)$ is an ideal of A, and if $Soc(A) \neq 0$, then it is a direct sum of simple ideals. If A is prime, $Soc(A)$ is a simple ideal contained in every nonzero ideal of A [19, Theorem 1].

Elements of the socle of semiprime associative algebras and pairs are von Neumann regular [12, Theorem 1]. We refer to [15,17,20] for descriptions of prime associative pairs with nonzero socle. Prime associative pairs with involution having nonzero socle together with their involutions were described in [15, Theorem 3.14].

2.10. The socle of the standard imbbeding

There is a good relation between the socle of a semiprime associative pair A and that of its standard imbedding $\mathcal{E}$: $Soc(A^+) = Soc(\mathcal{E}) \cap A^+ = e_1 Soc(\mathcal{E}) e_2$, and similarly $Soc(A^-) = Soc(\mathcal{E}) \cap A^- = e_2 Soc(\mathcal{E}) e_1$ [15, Proposition 3.4(4)]. The standard imbedding of $Soc(A)$ can be identified with the ideal $Soc(\mathcal{E})$ of the standard imbedding $\mathcal{E}$ of A.

2.11. Primitive associative pairs

A pair of Φ-modules $M = (M^+, M^-)$ is a *right A-module*, for an associative pair A, if M is endowed with a pair of Φ-bilinear maps

$$\begin{aligned} M^\sigma \times A^{-\sigma} &\to M^{-\sigma} \\ (m, x) &\mapsto mx \end{aligned}$$

satisfying $((mx)y)z = m(xyz)$ for all $m \in M^\sigma, x, z \in A^{-\sigma}, y \in A^\sigma, \sigma = \pm$. Left A-modules are defined similarly. A right A-module $M = (M^+, M^-)$ is *irreducible* if $M^{-\sigma} A^\sigma \neq 0, \sigma = \pm$, and it contains no proper submodules (different from 0 and M itself) and *faithful* if $M^{-\sigma} x = 0$ implies $x = 0$ for any $x \in A^\sigma, \sigma = \pm$. An associative pair $A = (A^+, A^-)$ is *right primitive* if it has a faithful irreducible right A-module. In [17, Theorem 1], it is proved that a Density Theorem holds for primitive associative pairs, and that an associative pair is primitive if and only if so is its standard imbedding.

2.12. Primitive associative pairs with nonzero socle

Primitive associative pairs are prime, and associative pairs with nonzero socle are primitive if and only if they are prime [17, 2.8]. A Structure Theorem for primitive associative pairs with nonzero socle was given in [17, Theorem 2]. See also [15, Theorem 3.9].

2.13. Local algebras

The *local algebra* of an associative pair A at an element $a \in A^{-\sigma}$ is the quotient algebra $A^\sigma_a = (A^\sigma)^{(a)}/Ker\, a$ of the *a-homotope algebra* $(A^\sigma)^{(a)}$ of A (the associative algebra over the Φ-module A^σ with product $x \cdot y = xay$ for all $x, y \in A^\sigma$) over the ideal $Ker\, a = \{x \in A^\sigma \mid axa = 0\}$ of $(A^\sigma)^{(a)}$ [2,13]. Local algebras of associative pairs interact well with standard imbeddings: $A_a \cong \mathcal{E}_a$ for all $a \in A^{-\sigma}$ [15, Proposition 3.4(3)].

As for regularity conditions and their interaction to local algebras, we recall from [14, Proposition 5.2] the following facts: Local algebras of semiprime associative pairs are semiprime associative algebras, an associative pair A is prime if and only if all its local algebras at nonzero elements are prime, and if A is simple, then so are all its local algebras at nonzero elements [14, Proposition 5.2].

2.14. Associative PI pairs

Associative pairs satisfying polynomial identities were studied in [18]. We denote by $FAP(X)$ the *free associative pair* over Φ on indeterminates $X = (X^+, X^-)$, which is the subpair of the pair $(FA(X^+ \cup X^-), FA(X^+ \cup X^-))$ obtained by doubling the free associative algebra $FA(X^+ \cup X^-)$, generated by (X^+, X^-). The universal property of $FAP(X)$ makes it possible to evaluate any pair polynomial $f_\sigma(x_1^+, \ldots x_n^+, x_1^-, \ldots, x_n^-)$ on an associative pair A, by assigning fixed values $x_i^\sigma = a_i^\sigma \in A^\sigma$. An associative polynomial $f_\sigma \in FAP(X)^\sigma$ is a *polynomial identity* of an associative pair A, if f_σ is monic (i.e. some of its leading monomials has coefficient 1) and all evaluations of f_σ on A vanish. Similarly we can consider **-polynomials* $p_\sigma(x_1^+, \ldots, x_n^+, (x_1^+)^*, \ldots, (x_n^+)^*, x_1^-, \ldots, x_n^-, (x_1^-)^*, \ldots, (x_n^-)^*)$ and **-polynomial* identities. We will say that an associative pair is *PI* (or that it is an *associative PI-pair*) if it satisfies a polynomial identity, and similarly one defines **-PI associative pairs*.

For a primitive pair, satisfying a (*-)polynomial identity ensures the existence of nonzero socle.

Proposition 2.1 ([18, Proposition 3.4, Theorem 3.6]): *Let A be a primitive associative pair.*

(i) *If A is PI, then A has nonzero socle.*
(ii) *If A has an involution *, and is *-PI, then A has nonzero socle.*

Moreover in either case, A is simple and has finite capacity.

Remark 2.2: The capacity of PI (or *-PI) primitive pairs is bounded by a constant depending only on the degree of the polynomial identity (see [18, Theorem 3.6]).

The following analogue of Amitsur's theorem for associative pairs with involution was also proved in [18].

Theorem 2.2 ([18, Theorem 3.9]): *Let A be an associative pair with involution *. If A has a *-polynomial identity of degree m, there exists a positive integer k such that every local algebra of A satisfies the polynomial identity S_{2m}^k. Moreover, if A is semiprime, every local algebra satisfies the standard identity S_{2m}.*

2.15. PI elements

The notion of PI-element for associative pairs was introduced in [2]: An element $a \in A^{-\sigma}$ of an associative pair A is a *PI-element* if the local algebra A_a^σ satisfies a polynomial identity. Then, the pair $PI(A) = (PI(A^+), PI(A^-))$, where $PI(A^\sigma)$ denotes the set of all PI-elements of A^σ, is an ideal of the associative pair A [2, Proposition 1.6].

Proposition 2.3: *Let A be a semiprime associative pair. Then $PI(A) = PI(\mathcal{E}) \cap A$.*

Proof: This follows from the relation equality 2.13 between the local algebras of the standard imbedding $\mathcal{E}$ of A at elements of the pair A, and the local algebras of the associative pair A. ■

2.16. The centroid of an associative pair

The centroid $\Gamma(A)$ of an associative pair $A = (A^+, A^-)$ is the set of all pairs $T = (T^+, T^-) \in End_\Phi(A^+) \times End_\Phi(A^-)$ satisfying:

$$T^\sigma(x^\sigma y^{-\sigma} z^\sigma) = T^\sigma(x^\sigma)y^{-\sigma}z^\sigma = x^\sigma T^{-\sigma}(y^{-\sigma})z^\sigma = x^\sigma y^{-\sigma} T^\sigma(z^\sigma)$$

for all $x^\sigma, z^\sigma \in A^\sigma$, $y^{-\sigma} \in A^{-\sigma}$, $\sigma = \pm$. The centroid of a semiprime associative pair is a commutative ring. If A is prime, then $\Gamma(A)$ is a domain acting faithfully on A, and it is a field if A is simple.

2.17. Extended centroid and central closure

The extended centroid and the central closure of associative rings were introduced by Martindale for prime rings [10], and generalized to semiprime rings by Amitsur [8]. The nonassociative case was considered in [21] and [9], and associative rings with involution were dealt with in [22] and [23]. In [3] the notions of extended centroid and the analogue to the central closure, called there extended central closure (since in that context there was already a notion of central closure) were introduced for Jordan systems (algebras, pairs and triple systems) on arbitrary rings of scalars.

2.18. The extended centroid of an associative algebra

We briefly recall now the definition of the extended centroid of an associative algebra.

Let I be an ideal of an associative algebra R. If $f : I \to R$ is a homomorphism of R-bimodules, and I is an essential ideal of R, then f will be called a *permissible map*. We will write it as the pair (f, I), since we will make use of the restrictions of f to smaller essential ideals, so it is convenient to have the domain explicitly displayed.

The extended centroid of a semiprime ring R is the direct limit $\mathcal{C}(R) = \varinjlim Hom_R(I, R)$ over the filter of essential ideals of R with the operations naturally inherited from R. Explicitly, as a set, $\mathcal{C}(R)$ consists of the equivalence classes of permissible maps $\overline{(f, I)}$ under the equivalence relation $(f, I) \sim (g, J)$ if $f_{|L} = g_{|L}$ for some essential ideal $L \subseteq I \cap J$.

The operations in $\mathcal{C}(R)$ are defined by $\overline{(f, I)} + \overline{(g, J)} = \overline{(f + g, I \cap J)}$, and $\overline{(f, I)} \cdot \overline{(g, J)} = \overline{(fg, g^{-1}(I))}$. As a result, $\mathcal{C}(R)$ becomes a commutative von Neumann regular unital ring [9, Theorem 2.5] which is called the *extended centroid* of the associative algebra R. The corresponding scalar extension $\mathcal{C}(R)R$, called the *central closure* of R, remains semiprime, is generated as a $\mathcal{C}(R)$-module by R, and is centrally closed [9, Theorem 2.15]. We also recall here that the extended centroid of a semiprime ring R is the center of its maximal right (and left) ring of quotients (also that of its symmetric ring of quotients) [7, Remark 2.3.1]. We also refer the reader to [24, Section 3] for further information on the construction of the central closure of a semiprime ring.

3. The extended centroid of semiprime associative pairs

In this section we first extend the construction of the extended centroid for semiprime algebras [8–10] to semiprime associative pairs, and then relate the extended centroid of a semiprime associative pair to that of its standard imbedding.

3.1. A-homomorphisms

Let $A = (A^+, A^-)$ be an associative pair and let $I = (I^+, I^-)$ be an ideal of A. Then $f : I \to A$ is an A-*homomorphism* if $f = (f^+, f^-)$ consists of a pair of Φ- linear maps $f^\sigma : I^\sigma \to A^\sigma$, $\sigma = \pm$, satisfying:

$$f^\sigma(x^\sigma y^{-\sigma} z^\sigma) = x^\sigma f^{-\sigma}(y^{-\sigma}) z^\sigma,$$
$$f^\sigma(y^\sigma x^{-\sigma} z^\sigma) = f^\sigma(y^\sigma) x^{-\sigma} z^\sigma,$$
$$f^\sigma(x^\sigma z^{-\sigma} y^\sigma) = x^\sigma z^{-\sigma} f^\sigma(y^\sigma),$$

for all $x^\sigma, z^\sigma \in A^\sigma, y^\sigma \in I^\sigma, \sigma = \pm$.

3.2. Permissible maps

Note that a pair of Φ- linear maps $f = (f^+, f^-)$ is an A-homomorphism if and only if it commutes with all left, right and middle multiplication operators defined by elements of A. We denote by $Hom_A(I, A)$, where $Hom_A(I, A) = (Hom_A(I^+, A^+), Hom_A(I^-, A^-))$, the set of all A-homomorphisms from I to A. A pair (f, I) where $f \in Hom_A(I, A)$ will be called a *permissible map* if I is an essential ideal of the associative pair A.

Theorem 3.1: *Let (f, I) and (g, J) be permissible maps of a semiprime associative pair A. Then*:

$$(f, I) \sim (g, J) \text{ if } f_{|K} = g_{|K} \text{ for some essential ideal } K \subseteq I \cap J,$$

defines an equivalence relation in the set of all A-permissible maps of A. Then the quotient set $\mathcal{C}(A)$, with the operations:

$$\overline{(f, I)} + \overline{(g, J)} = \overline{(f + g, I \cap J)},$$
$$\overline{(f, I)} \cdot \overline{(g, J)} = \overline{(fg, g^{-1}(I))},$$

is a commutative, von Neumann regular unital ring.

Proof: This is straightforward, arguing as for the corresponding results on algebras [9], mentioned before. ■

3.3. The extended centroid of a semiprime associative pair

We will refer to $\mathcal{C}(A)$ as the *extended centroid* of the semiprime associative pair A. Clearly $\mathcal{C}(A)$ contains a copy of the centroid $\Gamma(A)$ of A.

3.4. Notation

Our aim next is to relate the extended centroid of a semiprime associative pair to that of its standard imbedding. Since we will be simultaneously dealing with ideals of associative pairs and of their standard imbeddings, we will denote by $I, J, K, \ldots$ the associative pair ideals and by $\mathcal{I}, \mathcal{J}, \mathcal{K}, \ldots$ the algebra ideals. We will also denote by $id_{\mathcal{E}}(X)$ the ideal of $\mathcal{E}(A)$

generated by a pair subset $X = (X^+, X^-) \subseteq A$ or, equivalently, by the subset $X^+ \oplus X^- \subseteq \mathcal{E}(A)$.

Remark 3.1: It is obvious that if I is an essential ideal of an associative algebra R, the extended centroid $\mathcal{C}(I)$ can be identified with the extended centroid $\mathcal{C}(R)$ by considering the homomorphism induced by the restriction of permissible maps $(f, L) \mapsto (f_{|I\cap L}, L \cap I)$. Therefore it is immaterial whether we work with the standard imbedding or the associative envelope of associative pairs.

Lemma 3.2: *Let $\mathcal{E}$ be the standard imbedding of an associative pair $A = (A^+, A^-)$.*

(*i*) *If $\mathcal{I}$ is a nonzero ideal of $\mathcal{E}$, then $I = (\mathcal{I} \cap A^+, \mathcal{I} \cap A^-) = (\mathcal{I}_{12}, \mathcal{I}_{21})$ is a nonzero ideal of A.*
(*ii*) *If $I = (I^+, I^-)$ is a nonzero ideal of A, then, the ideal of $\mathcal{E}$ generated by I is*

$$\mathcal{I} = id_{\mathcal{E}}(I) = \left(I^+A^- + A^+I^-\right) \oplus I^+ \oplus I^- \oplus \left(I^-A^+ + A^-I^+\right)$$

which is nonzero. Moreover if A is semiprime and I is essential in A, then so is $\mathcal{I}$ in $\mathcal{E}$.

Proof: (i) is [14, Proposition 4.1(i)], and (ii) is straightforward from (i) and 2.2. ■

Lemma 3.3: *Let $\mathcal{E}$ be the standard imbedding of a semiprime associative pair $A = (A^+, A^-)$. If $\mathcal{I}$ is an essential ideal of $\mathcal{E}$, then $I = (\mathcal{I} \cap A^+, \mathcal{I} \cap A^-) = (\mathcal{I}_{12}, \mathcal{I}_{21})$ is essential in A.*

Proof: This easily follows from the previous Lemma. ■

Lemma 3.4: *Let $A = (A^+, A^-)$ be a semiprime associative pair. Then for any $\mathcal{E}$-homomorphism $(f, \mathcal{I})$ of its standard imbedding $\mathcal{E}$, we have $f\pi_{ij} = \pi_{ij}f$, for all i, $j = 1, 2$ (i.e. $\mathcal{E}$-homomorphisms are compatible with the Peirce decomposition of $\mathcal{E}$).*

Proof: Write $\mathcal{I} = \mathcal{I}_{11} \oplus \mathcal{I}_{12} \oplus \mathcal{I}_{21} \oplus \mathcal{I}_{22}$, where $\mathcal{I}_{ij} = e_i\mathcal{I}e_j = \mathcal{I} \cap \mathcal{E}_{ij}$, and take $x = x_{11} + x_{12} + x_{21} + x_{22} \in \mathcal{I}$. Then, since $x_{ij} \in \mathcal{I}_{ij} \subseteq \mathcal{I}$, we have

$$f\pi_{ij}(x) = f(x_{ij}) = f(e_ix_{ij}e_j) = f(e_ixe_j) = e_if(x)e_j = \pi_{ij}f(x).$$

Hence $f\pi_{ij}(x) = \pi_{ij}f(x)$ for all i, $j \in \{1, 2\}$, $x \in \mathcal{I}$. ■

Lemma 3.5: *Let A be a semiprime associative pair, and let $\lambda = \overline{(f, \mathcal{I})}$ and $\mu = \overline{(g, \mathcal{J})}$ be elements of the extended centroid $\mathcal{C}(\mathcal{E})$ of the standard imbedding $\mathcal{E}$ of A. If $f_{|\mathcal{I}\cap\mathcal{J}\cap A} = g_{|\mathcal{I}\cap\mathcal{J}\cap A}$, then $\lambda = \mu$ in $\mathcal{C}(\mathcal{E})$.*

Proof: We note first that replacing $\mathcal{I}$ and $\mathcal{J}$ by $\mathcal{I} \cap \mathcal{J}$, we can assume $\mathcal{I} = \mathcal{J}$ [9, Corollary 2.3]. Write then $\lambda = \overline{(f, \mathcal{I})}$ and $\mu = \overline{(g, \mathcal{I})}$ where $\mathcal{I} = \mathcal{I}_{11} \oplus \mathcal{I}_{12} \oplus \mathcal{I}_{21} \oplus \mathcal{I}_{22}$ is an essential ideal of $\mathcal{E}$ and assume $f(y_{ij}) = g(y_{ij})$ for all $y_{ij} \in \mathcal{I}_{ij}$ where $\{i, j\} = \{1, 2\}$.

Still with $\{i,j\} = \{1,2\}$, take now $y_{ii} \in \mathcal{I}_{ii}$ and $a_{ij} \in \mathcal{E}_{ij} = A^+$ or A^-. Then, since $y_{ii}a_{ij} \in \mathcal{I}_{ii}\mathcal{E}_{ij} \subseteq \mathcal{I}_{ij} = \mathcal{I} \cap A$, we have

$$\big(f(y_{ii}) - g(y_{ii})\big)a_{ij} = f(y_{ii})a_{ij} - g(y_{ii})a_{ij} = f(y_{ii}a_{ij}) - g(y_{ii}a_{ij}) = 0.$$

Hence $(f(y_{ii}) - g(y_{ii}))A^+ = 0$, where by Lemma 3.4, $f(y_{ii}), g(y_{ii}) \in \mathcal{E}_{ii}$. Therefore by the semiprimeness of A (see 2.5), this implies $f(y_{ii}) = g(y_{ii})$ for all $y_{ii} \in \mathcal{I}_{ii}$. Thus we have $f_{|\mathcal{I}_{ii}} = g_{|\mathcal{I}_{ii}}$. Hence $\lambda = \mu$. ■

Corollary 3.6: *Let A be a semiprime associative pair and let $\lambda = \overline{(f,\mathcal{I})}$ be an element of the extended centroid $\mathcal{C}(\mathcal{E})$ of the standard imbedding $\mathcal{E}$ of A. If $f_{|\mathcal{I}\cap A} = 0$ then $\lambda = 0$.*

Proof: This result is a particular case of Lemma 3.5. ■

Theorem 3.7: *Let A be a semiprime associative pair with standard imbedding $\mathcal{E}$. Then $\mathcal{C}(A) \cong \mathcal{C}(\mathcal{E})$.*

Proof: Take first an element $\lambda = \overline{(f,\mathcal{I})} \in \mathcal{C}(\mathcal{E})$, where $\mathcal{I}$ is an essential ideal of $\mathcal{E}$ and $f : \mathcal{I} \to \mathcal{E}$ is a permissible map. Write $I = (I^+, I^-) = (\mathcal{I}_{12}, \mathcal{I}_{21}) = \mathcal{I} \cap A$, which is an essential ideal of the associative pair by Lemma 3.3, and consider $g = (g^+, g^-)$ where $g^+ = f_{|\mathcal{I}_{12}}$ and $g^- = f_{|\mathcal{I}_{21}}$. By Lemma 3.4, $g = (g^+, g^-)$ consists of pair of linear maps with $g^\sigma : I^\sigma \to A^\sigma, \sigma = \pm$.

From the previous lemmas it follows that the mapping $\Psi : \mathcal{C}(\mathcal{E}) \to \mathcal{C}(A)$ given by $\Psi(\lambda) = \overline{(g,I)}$ is a well-defined ring homomorphism, which is injective by Corollary 3.6.

Conversely, take now $\mu = \overline{(g,I)} \in \mathcal{C}(A)$. By Lemma 3.2, $id_{\mathcal{E}}(I) = (I^+A^- + A^+I^-) \oplus I^+ \oplus I^- \oplus (I^-A^+ + A^-I^+)$ is an essential ideal of $\mathcal{E}$ which satisfies $\mathcal{I} \cap A = I$. We now define a linear map $\tilde{g} : id_{\mathcal{E}}(I) \to \mathcal{E}$ as follows:

(i) $\tilde{g}(u) = \sum_{i=1}^n g^+(y_i^+)a_i^- + \sum_{j=1}^m b_j^+ g^-(x_j^-)$ for any $u = \sum_{i=1}^n y_i^+ a_i^- + \sum_{j=1}^m b_j^+ x_j^- \in id_{\mathcal{E}}(I)_{11} = I^+A^- + A^+I^-, y_i^\sigma, x_j^\sigma \in I^\sigma, a_i^\sigma, b_j^\sigma \in A^\sigma, \sigma = \pm$,
(ii) $\tilde{g}(v) = g^\sigma(y^\sigma)$ for any $v = y^\sigma \in I^\sigma, \sigma = \pm$,
(iii) $\tilde{g}(w) = \sum_{k=1}^p g^-(z_k^-)c_k^+ + \sum_{l=1}^q d_l^- g^+(t_l^+)$ for any $w = \sum_{k=1}^p z_k^- c_k^+ + \sum_{l=1}^q d_l^- t_l^+ \in id_{\mathcal{E}}(I)_{22} = I^-A^+ + A^-I^+, z_k^\sigma, t_l^\sigma \in I^\sigma, c_k^\sigma, d_l^\sigma \in A^\sigma, \sigma = \pm$,

and extend it to $id_{\mathcal{E}}(I)$ by linearity.

A quite standard argument shows that the mapping $\tilde{g}$ is well defined: Indeed, if, for instance $0 = \sum_{i=1}^n y_i^+ a_i^- + \sum_{j=1}^m b_j^+ x_j^- \in id_{\mathcal{E}}(I)_{11} = I^+A^- + A^+I^-$, and we set $z = \sum_{i=1}^n g^+(y_i^+)a_i^- + \sum_{j=1}^m b_j^+ g^-(x_j^-)$, since (g,I) is permissible, so g is an A-homomorphism, we have $z\, id_{\mathcal{E}}(I) z = 0$. By Lemma 3.3, $id_{\mathcal{E}}(I)$ is essential, hence $z = 0$ by 2.8 and 2.7. The same argument applies to the remaining cases.

Next, if $(h,J) \in \overline{(g,I)}$, we can suppose $I = J$, and $h = g$ so that $id_{\mathcal{E}}(I) = id_{\mathcal{E}}(J)$, and the mapping $\Phi : \mathcal{C}(A) \to \mathcal{C}(\mathcal{E})$ given by $\Phi(\mu) = \overline{(\tilde{g}, id_{\mathcal{E}}(I))}$ is easily seen to be well-defined, and it is straightforward that it is an injective ring homomorphism.

Finally, the Theorem follows by noticing that Ψ and Φ are mutually inverses. ■

Even though the direct proof (in a pair environment) of the following result is also feasible, we obtain it here as consequence of Theorem 3.7.

Corollary 3.8: *The extended centroid $\mathcal{C}(A)$ of a prime associative pair A is a field.*

Proof: It follows from [21, Theorem 2.1] since, as noted in 2.6, an associative pair A is prime if and only if its standard imbedding is a prime associative algebra. ■

4. Closure of semiprime associative pairs

The extended centroid of a semiprime associative pair gives rise to a scalar extension that will be called the central closure. This section sketches the construction of the central closure for semiprime associative pairs. For more explicit details the reader is referred to [9,24]. As expected, central closures and standard imbeddings will be commuting constructions for semiprime associative pairs.

4.1. The central closure of a semiprime associative pair

As for associative (and Jordan) algebras, we define the *central closure* $\mathcal{C}(A)A$ of a semiprime associative pair A to be the quotient pair of the free scalar extension

$$\mathcal{C}(A) \otimes_\Phi A = (\mathcal{C}(A) \otimes_\Phi A^+, \mathcal{C}(A) \otimes_\Phi A^-)$$

by the pair ideal $R = (R^+, R^-)$, being R^σ the linear span of all elements of the form $\mu \otimes y^\sigma - 1 \otimes g^\sigma(y^\sigma)$, where $\mu \in \mathcal{C}(A)$, with $\mu = \overline{(g, I)} \in \mathcal{C}(A)$, $g = (g^+, g^-)$ and $y^\sigma \in I^\sigma$, $\sigma = \pm$. Then $\mathcal{C}(A)A = (\mathcal{C}(A)A^+, \mathcal{C}(A)A^-)$ and elements of $\mathcal{C}(A)A^\sigma$ will be written as $a^\sigma = \sum_{i=1}^n \lambda_i x_i^\sigma$, with $\lambda_i = \overline{(g_i, I_i)}$, being $g_i = (g_i^+, g_i^-)$ a pair of A-homomorphisms, $I_i = (I_i^+, I_i^-)$ an essential ideal of A and $x_i^\sigma \in A^\sigma$, $i = 1, \ldots, n$, $\sigma = \pm$.

Theorem 4.1: *The central closure $\mathcal{C}(A)A$ of a semiprime associative pair A is a tight scalar extension of A, and therefore it is a semiprime associative pair. Moreover, if A is prime, so is $\mathcal{C}(A)A$.*

Proof: This follows as the corresponding algebra result of [9], with the obvious changes for associative pairs. ■

4.2. The central closure of the standard imbedding

If A is an associative pair with standard imbedding $\mathcal{E}$, then the pair of orthogonal idempotents e_1 and e_2, with $e_1 + e_2 = 1$, and such that $A \cong (\mathcal{E}_{12}, \mathcal{E}_{21})$, also induces a Peirce decomposition on the central closure $\mathcal{C}(\mathcal{E})\mathcal{E}$ of the associative algebra $\mathcal{E}$. We next prove that the associative pair given by the off-diagonal Peirce components of $\mathcal{C}(\mathcal{E})\mathcal{E}$ corresponds to the central closure $\mathcal{C}(A)A$ of the associative pair A.

Theorem 4.2: *Let A be a semiprime associative pair with standard imbedding $\mathcal{E}$. Then the standard imbedding of the central closure $\mathcal{C}(A)A$ of A is isomorphic to the central closure $\mathcal{C}(\mathcal{E})\mathcal{E}$ of the standard imbedding $\mathcal{E}$ of A.*

Proof: It suffices to consider the induced Peirce decomposition of $\mathcal{C}(\mathcal{E})\mathcal{E}$ given by the pair of orthogonal idempotents e_1 and e_2 of $\mathcal{E}$, taking into account 2.2 and Theorem 3.7. ■

Corollary 4.3: *The central closure $\mathcal{C}(A)A$ of a semiprime associative pair A is closed over $\mathcal{C}(A)$.*

Proof: It follows from Theorem 4.2, Theorem 3.7 and [9, Theorem 2.15(c)]. ■

Remark 4.1: We note here that $\mathcal{C}(A)A = (e_1\mathcal{C}(\mathcal{E})\mathcal{E}e_2, e_2\mathcal{C}(\mathcal{E})\mathcal{E}e_1)$. We mention here as aside remark that this representation of the central closure $\mathcal{C}(A)A$ of a semiprime associative pair A as the off-diagonal Peirce spaces of the central closure of its standard imbedding $\mathcal{E}$ follows the approach given in [16, Definition 2.11] of the maximal left quotient pair of associative pairs without total zero divisors (and, in particular, for semiprime associative pairs). The same approach through the enveloping algebra (instead of the direct construction of the central closure followed here) can be indeed applied to other constructions of pairs of quotients of a semiprime associative pair A, as for instance, the maximal pair of symmetric quotients $Q_\sigma(A)$ or the Martindale symmetric ring of quotients $Q_s(A)$ of A [25].

5. Associative pairs with involution

In the present section we review the versions of the results on extended centroids and central closures for semiprime pairs with a (polarized) involution *. Recall (see 2.3) that such an involution extends uniquely to an involution also denoted by * on the standard imbedding $\mathcal{E}$ of A, with $e_1^* = e_2$.

5.1. The $*$-extended centroid of a semiprime ring

Involutions of a semiprime ring R extend easily to its extended centroid. Indeed, given $\lambda = \overline{(f, I)} \in \mathcal{C}(R)$, where f is an R-homomorphism and I an essential *-ideal of R, it suffices to define $\lambda^* = \overline{(f^*, I)}$ where $f^*(y) = (f(y^*))^*$ for all $y \in I$ [9, p.1125]. We recall here that for any semiprime ring with involution, the filter of essential ideals is equivalent to the filter of essential *-ideals. Then for a semiprime ring R the **-extended centroid* $\mathcal{C}_*(R)$ of R, defined as the set of all symmetric elements of $\mathcal{C}(R)$, is a unital commutative ring. Similarly, involutions of R also extend to the central closure $\mathcal{C}(R)R$ of R [22,23].

5.2. $*$-permissible maps

Although the subring of fixed elements of the extended centroid of a semiprime ring with involution was already considered in [9], the notion of **-extended centroid* was introduced in [22] for *-prime rings, and extended to semiprime rings with involution in [23, p.952], based on the set of equivalence classes of *-permissible maps defined on essential *-ideals (these are permissible maps $f : I \to R$ defined on an essential *-ideal: $I^* = I$, which commute with the involution: $f(y^*) = f(y)^*$).

5.3. The $*$-central closure of a semiprime ring

The **-extended centroid* $\mathcal{C}_*(R)$ of a semiprime ring with involution gives rise to a scalar extension $\mathcal{C}_*(R)R$, called the **-central closure* of R. Again, $\mathcal{C}_*(R)R$ is endowed with an involution * defined by $(\sum_{i=1}^n \lambda_i r_i)^* = \sum_{i=1}^n \lambda_i r_i^*$, for all $\lambda_i \in \mathcal{C}_*(R)$ and $r_i \in R$, $i = 1, \dots, n$.

If R is *-prime, $\mathcal{C}_*(R)$ is a field and $\mathcal{C}_*(R)R$ is a *-prime algebra over $\mathcal{C}_*(R)$ generated by R over $\mathcal{C}_*(R)$. Moreover $\mathcal{C}_*(\mathcal{C}_*(R)R) = \mathcal{C}_*(R)$, i.e. $\mathcal{C}_*(R)R$ is *-closed [22, Theorem 4]. (This can also be obtained through the symmetric ring of quotients $Q_s(R)$ of R [7, 2.3].)

Proposition 5.1: *Let A be a semiprime associative pair with involution * and let (f, I) be a permissible map of A. Then (f^*, I^*) given by $(f^\sigma)^*(y^\sigma) = (f^\sigma((y^\sigma)^*))^*$, for all $y^\sigma \in I^\sigma, \sigma = \pm$, is permissible and this defines an involution on the extended centroid $\mathcal{C}(A)$ of A.*

Proof: Take two permissible maps (f, I) and (g, J) of A, defined on essential ideals I and J of A and assume that $(f, I) \sim (g, J)$. Then it is straightforward that $(f^*, I^*) \sim (g^*, J^*)$. Moreover $(f^*)^* = f$. Hence, $\mathcal{C}(A)$ being a commutative ring, this defines an involution on $\mathcal{C}(A)$. ■

Theorem 5.2: *Let $A = (A^+, A^-)$ be a semiprime associative pair with an involution *. The set $\mathcal{C}_*(A)$ of all symmetric elements of the extended centroid $\mathcal{C}(A)$ of A with respect to the involution defined in Proposition 5.1 forms a commutative unital ring. Moreover $\mathcal{C}_*(A)$ is a field if A is *-prime.*

Proof: $\mathcal{C}_*(A)$ is a commutative unital ring as a result of Theorem 3.1 and Proposition 5.1. Note also that, since for any $\lambda = \overline{(f, I)} \in \mathcal{C}_*(A)$, both $Kerf = (Kerf^+, Kerf^-)$ and $Imf = (Imf^+, Imf^-)$ are *-ideals of A, if A is *-prime and $\lambda \neq 0$, then $Kerf$ vanishes. Then $\mu = \overline{(g, f(I))}$ given by $g^\sigma(f^\sigma(y^\sigma)) = y^\sigma$, $\sigma = \pm$, is an inverse of $\lambda = \overline{(f, I)}$ in $\mathcal{C}_*(A)$ (see [22, p.860]). ■

5.4. The $*$-extended centroid of a semiprime associative pair

We will refer to $\mathcal{C}_*(A)$ as the **-extended centroid* of the semiprime associative pair with involution A.

Remark 5.1: Given an essential *-ideal $I = (I^+, I^-)$ of a semiprime associative pair A with involution *, as for algebras, we say that a pair of A-homomorphisms $f = (f^+, f^-) \in Hom_A(I, A)$ is **-permissible* if $(f^\sigma)^*(y^\sigma) = f^\sigma(y^\sigma)$, for all $y^\sigma \in I^\sigma, \sigma = \pm$. The *-extended centroid $\mathcal{C}_*(A)$ of A can be therefore characterized as the set of all *-permissible maps in the extended centroid $\mathcal{C}(A)$ of A.

Theorem 5.3: *Let A be a semiprime associative pair with an involution * and standard imbedding $\mathcal{E}$. Then $\mathcal{C}_*(A) \cong \mathcal{C}_*(\mathcal{E})$.*

Proof: We first note that a direct check yields that the maps $\Psi : \mathcal{C}(\mathcal{E}) \to \mathcal{C}(A)$ and $\Phi : \mathcal{C}(\mathcal{E}) \to \mathcal{C}(A)$ given in Theorem 3.7 are ring *-homomorphisms, hence their restrictions define reciprocal isomorphisms between the *-extended centroids $\mathcal{C}_*(A)$ and $\mathcal{C}_*(\mathcal{E})$ of A and $\mathcal{E}$. ■

Remark 5.2: Let A be an associative pair and let $\mathcal{I}$ be a *-ideal of its standard imbedding $\mathcal{E}$. Then $I = \mathcal{I} \cap A = (\mathcal{I}_{12}, \mathcal{I}_{21})$ is a *-ideal of A, since clearly $(\mathcal{I}_{ij})^* = (e_i \mathcal{I} e_j)^* = e_j^* \mathcal{I}^* e_i^* = e_i \mathcal{I} e_j = \mathcal{I}_{ij}$, for $i \neq j$. We also note here that $(\mathcal{I}_{11})^* = \mathcal{I}_{22}$.

Lemma 5.4: *If $I = (I^+, I^-)$ is an *-ideal of an associative pair A with involution *, and $\mathcal{I}$ is the ideal of $\mathcal{E}$ generated by I as in Lemma 3.2(ii), then $\mathcal{I}$ is a *-ideal of the standard imbedding $\mathcal{E}$ of A. Moreover, if A is semiprime and I an essential *-ideal of A, then $\mathcal{I}$ is an essential *-ideal of $\mathcal{E}$.*

Proof: The first assertion is straightforward, and if A is semiprime and I is essential, the essentiality of $\mathcal{I}$ follows as in the proof of Lemma 3.2. ■

Proposition 5.5: *Let A be a semiprime associative pair with involution *. Then $(\sum_{i=1}^n \lambda_i a_i^\sigma)^* = \sum_{i=1}^n \lambda_i^*(a_i^\sigma)^*$, where $\lambda_i \in \mathcal{C}(A)$ and $a_i^\sigma \in A^\sigma, \sigma = \pm, i = 1, \ldots, n$, defines an involution on the central closure $\mathcal{C}(A)A$ of A, extending the one of A.*

Proof: Clearly the involution * of A, already extended to $\mathcal{C}(A)$ in Proposition 5.1, also extends to an involution on $\mathcal{C}(A) \otimes A$ given by $(\lambda \otimes a^\sigma)^* = \lambda^* \otimes (a^\sigma)^*$, $\sigma = \pm$. Let now $\mu = \overline{(g, I)} \in \mathcal{C}(A)$ and $y^\sigma \in I^\sigma$, $\sigma = \pm$. Then we have $(\mu \otimes y^\sigma - 1 \otimes g^\sigma(y^\sigma))^* = \mu^* \otimes (y^\sigma)^* - 1 \otimes (g^\sigma(y^\sigma))^*$. Thus since $\mu^* = \overline{(g^*, I^*)}$ with $g^*((y^\sigma)^*) = (g^\sigma((y^\sigma)^*)^*)^* = (g^\sigma(y^\sigma))^*$, this implies that the ideal $R = (R^+, R^-)$ defined in 4.1 is a *-ideal of A. Hence $\mathcal{C}(A)A$ inherits the involution, also denoted by *, given by $(\sum_{i=1}^n \lambda_i a_i^\sigma)^* = \sum_{i=1}^n \lambda_i^*(a_i^\sigma)^*$, where $\lambda_i \in \mathcal{C}(A)$ and $a_i^\sigma \in A^\sigma, \sigma = \pm, i = 1, \ldots, n$. ■

5.5. The $*$-central closure of a semiprime associative pair

As we did in the previous section, it is possible to define the scalar extension $\mathcal{C}_*(A)A$ of A, that will called the *-*central closure* of A. Then $\mathcal{C}_*(A)A$ is endowed with an involution $(\sum_{i=1}^n \lambda_i a_i^\sigma)^* = \sum_{i=1}^n \lambda_i(a_i^\sigma)^*$, where $\lambda_i \in \mathcal{C}_*(A)$ and $a_i^\sigma \in A^\sigma, \sigma = \pm, i = 1, \ldots, n$.

Theorem 5.6: *Let A be a semiprime associative pair with an involution * and standard imbedding $\mathcal{E}$. Then the standard imbedding of *-central closure $\mathcal{C}_*(A)A$ of the associative pair A is isomorphic to $\mathcal{C}_*(\mathcal{E})\mathcal{E}$.*

Proof: It follows from Theorem 4.2 considering Theorem 5.3. ■

6. Associative pairs with local PI-algebras

In this section we extend to associative pairs some of the main results on associative rings with (generalized) polynomial identities, such as the ones due to Amitsur, Kaplansky, Martindale and Posner Theorems. We refer the reader to [7,11,1,26] for quite complete expositions of the classical results of the associative theory of (generalized) polynomial identities.

6.1. Strongly primitive associative pairs

Borrowing the analogous notion from associative rings [1, p.48] (or [11, p.281]), we will say that an associative pair A is *strongly primitive* if $Soc(A) \neq 0$, and A is a dense subpair of $\mathcal{H} = (Hom_\Delta(M^-, M^+), Hom_\Delta(M^+, M^-))$ for a suitable pair of right vector spaces M^+ and M^- over a division PI-ring Δ.

Remark 6.1: Strongly primitive associative algebras are described in [11, 7.5, 7.6]. Here we limit ourselves to recall that an associative algebra R is strongly primitive if and only if R is primitive and has nonzero PI-ideal. As noted in [2, 1.3] in any strongly primitive associative algebra the socle and the PI-ideal coincide. (A similar characterization for Jordan systems was proved in [2, Theorem 4.6]. In [2] a Jordan system is called rationally primitive (and not strongly primitive) if it is primitive and has a nonzero PI-element. Motivation for that slightly different terminology is given in [2, 4.1].)

Theorem 6.1: *Let A be an associative pair. Then A is strongly primitive if and only if its standard imbedding $\mathcal{E}$ is strongly primitive.*

Proof: Let A be an associative pair, and assume that A is strongly primitive. Then $Soc(A)$ is nonzero, and therefore by 2.10 and Lemma 3.2, the socle $Soc(\mathcal{E})$ of its standard imbedding $\mathcal{E}$ is also nonzero. By primitivity of A, there are two right vector spaces M^+ and M^- over a division PI-ring Δ, such that A is a dense subpair of $\mathcal{H} = (Hom_\Delta(M^-, M^+), Hom_\Delta(M^+, M^-))$. Then it follows from [17, 2.3] that $\mathcal{E}$ is a primitive associative algebra with faithful irreducible right $\mathcal{E}$-module $M = M^- \oplus M^+$ (over the same division PI-ring $\Delta = End_{\mathcal{E}}(M^+ \oplus M^-)$). Hence $\mathcal{E}$ is strongly primitive.

Conversely, let A be an associative pair having a strongly primitive standard imbedding $\mathcal{E}$. Now we have $Soc(\mathcal{E}) \neq 0$, hence again by 2.10 and Lemma 3.2, A has nonzero socle. Suppose now that $\mathcal{E}$ is dense in $End(M_\Delta)(= End_\Delta(M))$ for a right vector space M over a division PI-ring Δ. As noted in [17, p.2598]), $M = M^- \oplus M^+$, with $M^+ = Me_2$ and $M^- = Me_1$, and then (M^+, M^-) is a faithful irreducible A-module over the same division PI-ring Δ. Therefore A is a strongly primitive associative pair. ■

We will apply now this result to obtain an analogue for associative pairs of Amitsur's theorem on primitive algebras with a GPI (see [11, 7.2.9, 7.4.6]). Here, as mentioned in the introduction, our GPIs will be nonzero local PI-algebras, so that our version of the GPI condition for a semiprime associative pair A will be the condition $PI(A) \neq 0$. This approach follows the one of [1], which in turn relies on the method of 'viewing' a generalized identity as a polynomial identity of a left (or right) ideal, a method which, according to Rowen [11, p.38], was initiated by Jain. This was also the approach followed in [3], where PI left ideals (which do not make sense in the Jordan theoretical context of that paper) are substituted by PI-elements.

Remark 6.2: The socle of a primitive ring R can be characterized as the set of all elements of finite rank [11, Theorem 7.1.13]. Indeed recall that the socle $Soc(R)$ of a semiprime ring R is defined (when is nonzero) to be the sum of all minimal left (equivalently right) ideals of R [11, Definition 1, Proposition 7.1.6], that is the sum of all left (or right) ideals generated by rank one elements [11, Lemma 7.1.11].

6.2. Rank of elements of primitive associative pairs

The main ideas given in [11, p.254–257] can also be applied to primitive associative pairs leading to similar results to those mentioned in Remark 6.2 above. Let $A = (A^+, A^-)$ be a primitive associative pair and suppose that up to isomorphism A is a dense subpair of a pair

$$\mathcal{H} = \left(Hom_\Delta(M^-, M^+), Hom_\Delta(M^+, M^-)\right),$$

for a faithful irreducible A-module (M^+, M^-).

Then $M = M^- \oplus M^+$ is a faithful irreducible module over the standard imbedding $\mathcal{E}$ of A [17, 2.3], and M (resp. (M^+, M^-)) is a left vector space (resp. a pair of left vector spaces) over the division algebra $\Delta = End(M_\mathcal{E})$. For any associative pair element $a^\sigma \in A^\sigma$, we define the *rank of* a^σ *in* A (also the (M^+, M^-)-rank of a^σ) to be:

$$rank(a^\sigma) = [M^{-\sigma} a^\sigma : \Delta] = [Ma^\sigma : \Delta], \quad \sigma = \pm.$$

Hence the rank of a^σ is the same independently of whether the element is considered an associative pair element or an element of the standard imbedding.

We can add the following remark that links the approach through the standard imbedding and the local algebra approach mentioned above, and whose proof is an easy exercise in associative theory: with the notations above, if $a^\sigma \in PI(A^\sigma)$, the local algebra $A^{-\sigma}_{a^\sigma}$ is isomorphic to the matrix algebra $M_t(\Delta)$ for $t = rank(a^\sigma)$ (see below the proof of Theorem 6.3).

Theorem 6.2: *Let A be a primitive associative pair. Then, for* $\sigma = \pm$:

- (i) *If* $a^\sigma \in A^\sigma$ *has* $rank(a^\sigma) = 1$, *then* $a^\sigma A^{-\sigma} A^\sigma$ *is a minimal left ideal of A.*
- (ii) *If* $a^\sigma \in A^\sigma$ *has* $rank(a^\sigma) = t \geq 1$, *then there exist rank one elements* $a_1^\sigma, \ldots, a_t^\sigma \in A^\sigma$ *such that* $a^\sigma = \sum_{i=1}^t a_i^\sigma$.
- (iii) $Soc(A^\sigma)$ *is the set of elements of finite rank.*

Proof: As mentioned in 6.2 above it suffices to review [11, Lemma 7.1.11, Lemma 7.1.12, Theorem 7.1.13] introducing the obvious changes to obtain the corresponding results for associative pairs. ■

Now we can state the announced analogue of Amitsur's theorem for Associative pairs.

Theorem 6.3: *Let A be an associative pair. Then the following are equivalent*:

- (i) *A is strongly primitive.*
- (ii) *A is prime and* $Soc(A) = PI(A) \neq 0$.
- (iii) *A is prime and the local algebra at some nonzero element is a simple unital PI-algebra.*

Proof: $(i) \Rightarrow (ii)$ Let A be a strongly primitive associative pair. Then A is prime (see 2.12) and by Theorem 6.1 its standard imbedding $\mathcal{E}$ is a strongly primitive algebra. Thus, by [11, Proposition 7.5.17], $Soc(\mathcal{E}) = PI(\mathcal{E})$ is a nonzero ideal of $\mathcal{E}$ and the equality $Soc(A) = PI(A)$ follows from 2.10 and Proposition 2.3. Moreover $Soc(A) = PI(A) \neq 0$ by Lemma 3.2.

$(ii) \Rightarrow (iii)$ Suppose now that A is a prime associative pair with nonzero socle equal to its PI-ideal, and take a nonzero element $0 \neq a \in A^{-\sigma} = Soc(A^{-\sigma}) = PI(A^{-\sigma})$. By Theorem 6.2(iii) we can assume a has finite rank $rank(a) = t$. Suppose also that $a = a^- \in A^-$. Then A is primitive by [17, 2.8], hence a dense subpair of $\mathcal{H} = (Hom_\Delta(M^-, M^+), Hom_\Delta(M^+, M^-))$ for a suitable pair of right vector spaces M^+ and M^- over a division ring Δ. Write $M = M^- \oplus M^+$ and take the local algebra A_a^+. Then $\widetilde{M} = M/lann_M(a)$ where $lann_M(a) = \{m \in M = M^- \oplus M^+ \mid ma = 0\} = M^- \oplus lann_{M^+}(a)$ is a faithful irreducible right A_a^+-module, and $[\widetilde{M} : \Delta] = rank(a) = t$. Thus, the local algebra A_a^+ is a primitive associative PI-algebra satisfying the polynomial standard identity S_{2t}. Hence, by Kaplansky's Theorem [11, Theorem 1.5.16], A_a^+ is a simple unital (since the element a belongs to $Soc(A)$, hence it is von Neumann regular and the local algebra A_a^+ is unital) PI-algebra.

$(iii) \Rightarrow (i)$ Consider now a prime associative pair having a simple unital local PI-algebra A_a^σ for some $0 \neq a \in A^{-\sigma}$. Then, since as noted in 2.13, the local algebra $\mathcal{E}_a$ of the standard imbedding $\mathcal{E}$ of A at the same element satisfies $\mathcal{E}_a \cong A_a^\sigma$, we have that $\mathcal{E}_a$ is a simple unital associative PI-algebra. Moreover $\mathcal{E}$ is prime (see 2.6), hence the standard imbedding $\mathcal{E}$ of A is strongly primitive by [11, Proposition 7.5.17(ii)]. Now the strong primitivity of the associative pair A follows from Theorem 6.1. ■

We next address our Pair analogue of Kaplansky Theorem on PI algebras. Here, as in [3], the polynomial identities that we will consider will be homotope polynomial identities, so that we can make use of our results on PI-elements. We begin recalling some facts on homotope polynomial identities.

6.3. Associative HPI pairs

Homotope polynomials are the images of associative polynomials $f(x_1, \ldots, x_n)$ of the free associative algebra $FA[X \cup \{z\}]$ on a countable set of generators X and $z \notin X$ under homomorphims $FA[X] \to FA[X \cup \{z\}]^{(z)}$, extending the identity on X. Homotope polynomials are usually denoted by $f(z; x_1, \ldots, x_n) = f(x_1, \ldots, x_n)^{(z)}$. An associative pair A satisfies a *homotope PI* (HPI for short), equivalently, A is an *associative HPI-pair* if there exists $f(x_1, \ldots, x_n) \in FA[X]$ such that $f(y^{-\sigma}; x_1^\sigma, \ldots, x_n^\sigma)$ vanishes under all substitutions of elements $y^{-\sigma} \in A^{-\sigma}$, $x_i^\sigma \in A^\sigma$, $\sigma = \pm$. Note that any HPI is in particular a generalized polynomial identity (GPI). Indeed if an associative pair is homotope-PI, then all its local algebras satisfy the same PI.

Theorem 6.4: *Let A be a primitive associative pair.*

(i) *If the local algebra at any element of A is PI, then A is simple, equal to its socle.*
(ii) *If A is HPI, then A is simple, equal to its socle.*

Proof: Clearly (ii) is a straightforward consequence of (i) since all local algebras of an associative HPI pair satisfy the same polynomial identity. To prove (i) consider A to be a primitive associative pair such that all its local algebras are PI-algebras. Then, by Theorem 6.3, we have $A = PI(A) = Soc(A)$, hence A is simple by [19, Theorem 1], since it is prime by 2.12, so 2.9 applies. ■

6.4. Simple associative pairs with finite capacity

Simple associative pairs coinciding with their socle are of the form $\mathcal{F}(\mathcal{P}_1, \mathcal{P}_2)$ for two pairs of dual vector spaces $\mathcal{P}_i = (X_i, Y_i)$, $i = 1, 2$, over the same associative division algebra

Δ [19, Theorem 2]. Recall, see [19, Remark p.483], that such an associative pair has finite capacity if and only if at least one of the vector spaces is finite-dimensional over Δ. The classification of simple associative pairs having finite capacity is given in [12, Theorem 11.16].

The existence of a homotope-PI on a primitive associative pair provides a bound for the dimension of (at least) one of the pairs of dual vector spaces, ensuring then the finite capacity of the associative pair.

Corollary 6.5: *Let A be a primitive associative pair. If A satisfies a homotope-PI of degree d, then A has finite capacity at most $[d/2]$.*

Proof: Assume that A is a primitive associative pair satisfying a homotope-PI of degree d. Then, by Theorem 6.4, A is simple and equal to its socle. Indeed, by [17, Theorem 2], $A = Soc(A) = (\mathcal{F}_\Delta(M^-, M^+), \mathcal{F}_\Delta(M^+, M^-))$, where (M^+, M^-) is a faithful irreducible right A-module, and $\Delta = End_{\mathcal{E}}(M^+ \oplus M^-)$ is a division algebra where $\mathcal{E}$ is the standard imbedding of A.

Take now an element $a \in A^{-\sigma} = Soc(A^{-\sigma})$. By Theorem 6.2(iii) we can assume that a has finite rank $rank(a) = r$. Then the local algebra A_a^σ is simple (2.13), and according to [18] (see also [27]) we can assume that A_a^σ is contained into a matrix algebra $M_r(F)$, where F denotes a maximal subfield of the division ring Δ. As a result, by [11, 1.4.1] A_a^σ satisfies the standard identity S_{2r}, hence $2r \le d$. Thus A has finite capacity at most $[d/2]$. ■

Next, the closeness between the central closures of semiprime associative pairs and that of their standard imbeddings makes it possible to obtain the following associative pair version of Martindale Theorem for prime associative algebras satisfying a generalized polynomial identity.

Theorem 6.6: *Let A be a prime associative pair. If $PI(A) \neq 0$, then the central closure $\mathcal{C}(A)A$ of A is a primitive associative pair with nonzero socle equal to $PI(\mathcal{C}(A)A)$. Moreover $PI(A) = A \cap Soc(\mathcal{C}(A)A)$.*

Proof: Let A be a prime associative pair having nonzero PI-elements. Then, by 2.6, its standard imbedding $\mathcal{E}$ is a prime associative algebra with nonzero PI-ideal $PI(\mathcal{E})$ by Proposition 2.3. Hence the central closure $\mathcal{C}(\mathcal{E})\mathcal{E}$ of $\mathcal{E}$ is a strongly primitive associative algebra with nonzero socle by [11, Theorem 7.6.15]. Moreover, since by [11, Proposition 7.5.17], $Soc(\mathcal{C}(\mathcal{E})\mathcal{E}) = PI(\mathcal{C}(\mathcal{E})\mathcal{E})$, it holds that $PI(\mathcal{E}) = \mathcal{E} \cap Soc(\mathcal{C}(\mathcal{E})\mathcal{E})$.

On the other hand, by Theorem 4.2, we have that $\mathcal{C}(\mathcal{E})\mathcal{E}$ is isomorphic to the standard imbedding of the central closure $\mathcal{C}(A)A$ of the associative pair A. Thus, by Theorem 6.1, $\mathcal{C}(A)A$ is a strongly primitive associative pair with $Soc(\mathcal{C}(A)A) = PI(\mathcal{C}(A)A) \neq 0$ as a result of Theorem 6.3.

Finally we claim that $PI(A) = A \cap PI(\mathcal{C}(A)A) = A \cap Soc(\mathcal{C}(A)A)$. Note that it suffices to prove $PI(A) \subseteq A \cap PI(\mathcal{C}(A)A)$. Take $a = a^{-\sigma} \in PI(A^{-\sigma})$. Then, by Proposition 2.3, $a \in A^{-\sigma} \cap PI(\mathcal{E}) = A^{-\sigma} \cap Soc(\mathcal{C}(\mathcal{E})\mathcal{E})$, which implies that a is von Neumann regular in $\mathcal{C}(\mathcal{E})\mathcal{E}$. Then, as a result of 2.13, Theorem 4.2 and Remark 4.1, we obtain the following containments of local algebras $A_a^\sigma \subseteq \mathcal{C}(A)A_a^\sigma \cong \mathcal{C}(\mathcal{E})\mathcal{E}_a \subseteq Q_s(\mathcal{E})_a$, where $Q_s(\mathcal{E})$ denotes the Martindale symmetric ring of quotients of the standard imbedding $\mathcal{E}$ of A. Moreover, from the (von Neumann) regularity of a in $\mathcal{C}(\mathcal{E})\mathcal{E}$, hence in $Q_s(\mathcal{E})$, we have $Q_s(\mathcal{E})_a \cong Q_s(\mathcal{E}_a)$ by [28, Theorem 3]. Therefore $A_a^\sigma \subseteq \mathcal{C}(A)A_a^\sigma \cong \mathcal{C}(\mathcal{E})\mathcal{E}_a \subseteq Q_s(\mathcal{E})_a \cong Q_s(\mathcal{E}_a)$. Thus, as A_a^σ is prime by 2.13, $Q_s(\mathcal{E}_a)$ is a PI-algebra by [7, Corollary 6.1.7], and that implies that $\mathcal{C}(A)A_a^\sigma$ is a PI-algebra, and therefore we finally obtain $a \in A \cap PI(\mathcal{C}(A)A)$. ■

Theorem 6.7: *Let A be a prime associative pair.*

(*i*) *If the local algebra $A_a^{-\sigma}$ at each element $a \in A^{-\sigma}$ of A is PI, then the central closure $\mathcal{C}(A)A$ of A is simple equal to its socle.*
(*ii*) *If A is HPI, then $\mathcal{C}(A)A$ is simple, equal to its socle.*

Proof: Under any of the above assumptions, the central closure $\mathcal{C}(A)A$ of A is a strongly primitive associative pair by Theorem 6.6. Besides, $A = PI(A)$ is contained in $Soc(\mathcal{C}(A)A)$. Thus $\mathcal{C}(A)A = Soc(\mathcal{C}(A)A)$ and the simplicity of $\mathcal{C}(A)A$ follows from [19, Theorem 1]. ■

As a consequence of the previous results, we obtain the following Associative Pair version of Posner's Theorem:

Corollary 6.8: *Let A be a prime associative pair. If A satisfies a homotope-PI of degree d, then its central closure $\mathcal{C}(A)A$ has finite capacity at most $[d/2]$.*

Proof: It follows from Theorem 6.7 as a result of Corollary 6.5. ■

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