Propiedades de forma y positividad total en números de Stirling generalizados

Shape properties and total positivity in generalized Stirling numbers

Emilio Casanova Biscarri

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Directora: Carmen Sangüesa Lafuente

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Preface

The notions of Stirling numbers of the first and second kinds were introduced by James Stirling around mid-18th century. From a combinatoric point of view, Stirling numbers of the first kind $s(n,k)$ (when considered in absolute value) represent the number of permutations of n elements with k disjoint cycles, and Stirling numbers of the second kind $S(n,k)$ represent all the possible ways to partition a set consisting of n elements into k nonempty subsets. Both are double sequences. Such numbers have emerged in various domains, in fields such as combinatorics and mathematical analysis.

As of today, and as might be expected, such numbers have been widely studied but still hide promising results. Numerous contributions have been made in an effort to generalize Stirling numbers. This document will focus on two particular generalizations of Stirling numbers of the second kind: one that comes from a generalization by the means of the generating function denoted by $S_g(n,m)$, and a second one from a probabilistic point of view introduced by Adell and Lekuona (2019) [1] denoted by $S_Y(n,m)$.

The aim of the present master's thesis is to study the log-concavity and total positivity of order 2 that such generalizations hold. In order to do so, several mathematical ingredients will be introduced in the first two chapters. The first chapter contains the necessary notions on Stirling numbers and a first approach to the probabilistic generalization. The second chapter focuses on familiarizing the shape properties (log-concavity) and total positivity for double sequences, and also includes two particular stochastic orders. The third and final chapter introduces the two generalizations previously mentioned. It also uses the previous chapters to demonstrate that such generalizations reveal interesting properties that allow to guarantee that, under certains hypothesis:

(i) $S_g(n,m)$ can be expected to be log-concave and totally positive of order 2,

(ii) $S_Y(n,m)$ are likely to be expected to be totally positive of order 2.

Let me end this preface by wishing you good reading, because in a year as complicated as this one, at least we still have mathematics.

Keywords: *generalized Stirling numbers, probabilistic Stirling numbers, Stirling numbers, Stirling numbers of the first kind, Stirling numbers of the second kind, log-concavity, total positivity*

Resumen

En el presente Trabajo de Fin de Máster se realiza un estudio de propiedades de log-concavidad y positividad total de los números de Stirling generalizados. El objetivo de dicha generalización es tratar números conocidos en el ámbito de la combinatoria de manera unificada, de forma y manera que las propiedades que satisfagan los números generalizados sean heredadas inmediatamente por todos aquellos números que engloban.

El trabajo se encuentra dividido en tres capítulos, cuyo contenido a grandes rasgos es el siguiente:

Capítulo 1: Basándose en Comtet (1974) [5], previa definición de las funciones generatrices, se definen los números de Stirling de primera $s(n,m)$ y segunda $S(n,m)$ especies con diversas definiciones equivalentes. También se presentan las relaciones de recurrencia que dichos números satisfacen, y se emplean para ejemplificar su utilidad. Finalmente, se presenta una generalización probabilística de los números de Stirling introducida por Adell y Lekuona (2019) [1], los llamados polinomios de Stirling de segunda especie asociados a una variable aleatoria. Dichos polinomios serán empleados en el tercer capítulo.

Capítulo 2: En este segundo capítulo, se introducen tres propiedades que serán de gran utilidad en el tercer y último capítulo: positividad total, log-concavidad y órdenes estocásticos.

Respecto a la positividad total, siguiendo la línea de Marshall y Olkin (2007) [11], se introduce la definición de funciones totalmente positivas de orden k, y posteriormente se enfatiza la definición de sucesiones dobles totalmente positivas de orden 2 (definición que es un caso particular de la primera).

Más adelante, tal y como se desarrolla en Nihn y Prékopa (2013) [13], se introduce la noción de log-concavidad en términos de positividad total, para acto seguido plantear una definición equivalente que será de mayor utilidad. Diversos ejemplos de distribuciones log-cóncavas y log-convexas se encuentran incluidos.

Finalmente, se presenta la noción de orden estocástico retomando la ruta de Marshall y Olkin [11]. La idea fundamental que motiva los órdenes estocásticos es poder comparar distintas distribuciones. Para ello, habrá que tener cierta cautela a la hora de escoger el criterio de comparación, como se podrá constatar en dicha sección. Asimismo, se introducen dos órdenes estocásticos que serán de gran utilidad posteriormente: el orden estocástico usual y el orden de razón de verosimilitud.

Capítulo 3: en el último capítulo, se introduce una primera generalización de los números de Stirling de segunda especie $S_g(n,m)$, y se demuestra que dicha generalización posee buenas propiedades de log-concavidad y de positividad total de orden 2 bajo ciertas hipótesis. Posteriormente, se estudia también la generalización probabilística de los números de Stirling introducida por Adell y Lekuona (2019) [1] y se inicia asimismo la comprobación de una posible propiedad de positividad total de orden 2, basada en el orden de razón de verosimilitud, y de nuevo bajo ciertas suposiciones.

El trabajo cierra con una conclusión que enfatiza en la importancia del trabajo de manera unificada sobre los números de Stirling (entre otros), y cómo este enfoque permite solventar cálculos y comprobaciones individuales de una forma asequible y exportable.

Palabras clave: *números de Stirling generalizados, números de Stirling probabilísticos, números de Stirling, números de Stirling de primera especie, números de Stirling de segunda especie, log-concavidad, positividad total*.

Contents

Chapter 1

Introduction to Stirling numbers

This first chapter serves as an introduction to Stirling numbers of the first, but mainly of the second kind. The following definitions and properties can be found in *Advanced Combinatorics*, by Comtet [5].

1.1 Generating functions

Let us introduce three generating functions, henceforth abbreviated GF.

Definition 1.1. *Let be given a real or complex sequence (in our particular case, it will consist of positive integers with a combinatorial meaning). The ordinary GF, exponential GF and, more generally, GF according to Ω_n of the sequence a_n, are by definition the following three formal series Φ, Ψ and Φ_Ω respectively, where Ω_n is a fixed given sequence:*

$$\Phi(t) := \sum_{n\geq 0} a_n t^n, \quad \Psi(t) := \sum_{n\geq 0} a_n \frac{t^n}{n!}, \quad \Phi_\Omega := \sum_{n\geq 0} \Omega_n a_n t^n.$$

In case that, at least, one of the previous series has a positive nonzero radious of convergence R, and converges for $|t| < R$ to a composition of elementary known functions, the properties of the three GF functions allow to obtain new information about the sequence a_n. Let us introduce some examples to illustrate the previous concepts, but first, a definition to simplify the notation.

Definition 1.2. $(x)_k := x(x-1)\cdots(x-k+1)$ *is called the* ***falling factorial*** *of x down to k. Besides,* $\langle x\rangle_k := x(x+1)\cdots(x+k-1)$ *is called the* ***rising factorial*** *of x up to k. Finally, the double sequence $\binom{n}{k}$ (namely binomial coefficients if $n,k \in \mathbb{N}^2$) is defined from now on also for $(x,y) \in \mathbb{C}^2$, such as*

$$\binom{x}{y} := \begin{cases} \frac{(x)_y}{y!} & \text{if } x \in \mathbb{C}, y \in \mathbb{N}, \\ 0 & \text{if } x \in \mathbb{C}, y \notin \mathbb{N}. \end{cases}$$

Example 1.1. *Consider $a_n := \binom{x}{n}$, where $x \in \mathbb{R}$ or $\mathbb{C}$. Hence $\Phi(t) = \sum_{n\geq 0}\binom{x}{n}t^n = \sum_{n\geq 0}(x)_n\frac{t^n}{n!} = (1+t)^x$, which converges for $|t| < 1$ (if $t \in \mathbb{C}$, one chooses the value of $\Phi(t)$ that equals 1 for $t = 0$). Comparing the coefficients of $\frac{t^n}{n!}$ in the first and the last member in definition 1.1.*

$$\sum_{n\geq 0}(x+y)_n\,\frac{t^n}{n!}=(1+t)^{x+y}=(1+t)^x(1+t)^y=$$
$$=\Big(\sum_{k\geq 0}(x)_k\,\frac{t^k}{k!}\Big)\Big(\sum_{l\geq 0}(y)_l\,\frac{t^l}{l!}\Big),$$

and the so called Vandermonde convolution is obtained in two forms, using the previous formula,

(i) $(x+y)_n=\sum_{0\leq k\leq n}\binom{n}{k}\cdot(x)_k\,(y)_{n-k}$,

(ii) $\binom{x+y}{n}=\sum_{0\leq k\leq n}\binom{x}{k}\binom{y}{n-k}$.

Similarly, using $\sum_{n\geq 0}\langle x\rangle_n(\frac{t^n}{n!})=(1-t)^{-x}$*, it is shown that*

(i) $\langle x+y\rangle_n=\sum_{0\leq k\leq n}\binom{n}{k}\cdot\langle x\rangle_k\,\langle y\rangle_{n-k}$,

(ii) $\left\langle {x+y \atop n}\right\rangle=\sum_{0\leq k\leq n}\left\langle {x \atop k}\right\rangle\left\langle {y \atop n-k}\right\rangle$,

where $\left\langle {n \atop p}\right\rangle:=\frac{\langle n\rangle_p}{p!}$.

Example 1.2. ***Fibonacci numbers.*** *Such Fibonacci numbers are integers* F_n *defined by*

$$F_n=F_{n-1}+F_{n-2},\ n\geq 2;\quad F_0=F_1=1.$$

To find the ordinary GF, $\Phi=\sum_{n\geq 0}F_nt^n$,

$$\Phi=1+t+\sum_{n\geq 2}(F_{n-1}+F_{n-2})t^n=1+t\Phi+t^2\Phi.$$

Comparing the first and the last terms of the equalities,

$$\Phi=\sum_{n\geq 0}F_nt^n=\frac{1}{1-t-t^2}.$$

Decomposing this rational function into partial fractions, naming the roots of $1-t-t^2=0$ *as* $-\alpha$ *and* $-\beta$*, where* $\alpha:=\frac{1-\sqrt{5}}{2}$ *and* $\beta:=\frac{1+\sqrt{5}}{2}$,

$$\Phi=\frac{1}{\beta-\alpha}\Big(\frac{\beta}{1-\beta t}-\frac{\alpha}{1-\alpha t}\Big)=\frac{1}{\sqrt{5}}\Big(\sum_{n\geq 0}\beta^{n+1}t^n-\sum_{n\geq 0}\alpha^{n+1}t^n\Big).$$

This allows to identify the coefficients of t^n,

$$F_n=\frac{\beta^{n+1}-\alpha^{n+1}}{\sqrt{5}},$$

1.2 Stirling numbers

Let us present a first definition of Stirling numbers of the first and the second kinds via double GF's. Meaningful definitions will be given in Definitions 4 , 1.5 and so on.

Definition 1.3. *The* ***Stirling numbers of the first kind****, denoted by $s(n,k)$, and the* ***Stirling numbers of the second kind****, denoted by $S(n,k)$, can be defined via the following double GF*

(i) $(1+t)^u := 1+\sum_{n\geq 1}\sum_{1\leq k\leq n} s(n,k)\frac{t^n}{n!}u^k$,

(ii) $\exp\{u(e^t-1)\} := 1+\sum_{n\geq 1}\sum_{1\leq k\leq n} S(n,k)\frac{t^n}{n!}u^k$.

Indeed, such double GF in their definition can be avoided, since

$$(1+t)^u = \exp\{u\log(1+t)\} = \sum_{k\geq 0} u^k \frac{log^k(1+t)}{k!} \implies \frac{\log^k(1+t)}{k!} := \sum_{n\geq k} s(n,k)\frac{t^n}{n!}, \qquad (1.1)$$

and

$$\exp\{u(e^t-1)\} = \sum_{k\geq 0} u^k \frac{(e^t-1)^k}{k!} \implies \frac{(e^t-1)^k}{k!} := \sum_{n\geq k} S(n,k)\frac{t^n}{n!}. \qquad (1.2)$$

These are the 'vertical' GF's of Stirling numbers of the first and the second kinds respectively.

1.2.1 Stirling numbers of the second kind

In order to introduce the formal definition of the *Stirling numbers of the second kind*, the following definition is needed.

Definition 1.4. *Let N be a subset of $\mathbb{N}$. Denoting $B(N)$ as the set of all subsets of N, and $B'(N)$ as the set of all nonempty subsets, a non-ordered (finite) set P of p blocks of N such that $P \subset B'(N)$ is called a* ***partition*** *of N, or p-partition if one wants to specify the number of its blocks, if the union of all blocks of P equals N, and these blocks are mutually disjoint.*

Example 1.3. *For instance, let us consider $N = \{1,2,3\}$. A 2-partition of N is $\{\{1\},\{2,3\}\}$.*

The formal definition of the *Stirling numbers of the second kind* is the following.

Definition 1.5. *Let $|N| = n$. The number $S(n,k)$ of k-partitions of N is called* ***Stirling number of the second kind****. Thus, $S(n,k) > 0$ for $1 \leq k \leq n$, and $S(n,k) = 0$ if $1 \leq n \leq k$. It is considered that $S(0,0) = 1$, and $S(0,k) = 0$ for $k \geq 1$.*

Example 1.4. *Going back to Example 1.3, with $N = \{1,2,3\}$, the number of 2-partitions of N is $S(3,2) = 3$, which are*

$$\{\{1\},\{2,3\}\}, \qquad \{\{1,2\},\{3\}\}, \qquad \textit{and} \qquad \{\{1,3\},\{2\}\}.$$

In other words, $S(n,k)$ is the number of equivalence relations with k classes on N. It can be conceived as the number of distributions of n distinct balls into k indistinguishable boxes, where the order of the boxes is not considered, and such that no box is empty.

The proof of the equivalence of the definition given in the definitions 1.3 *(ii)* and 1.5 can be found in [5] pp. 204-205.

Theorem 1.1. *The following are equivalent definitions for Stirling numbers of the second kind:*

(1) Definition 1.5.

(2) Via the 'double' GF, Definition 1.3 (ii).

(3) Via the 'vertical' GF, in eq. (1.2).

(4) Via the 'horizontal' GF, which is often taken as the usual definition, that is

$$x^n = \sum_{0 \le k \le n} S(n,k)(x)_k.$$

The proof of such equivalent definitions can be found in [5] pp. 204-207.

Let us introduce the recurrence relations that Stirling numbers of the second kind hold, which are useful to compute the first values of $S(n,k)$.

Theorem 1.2. *The Stirling numbers of the second kind S(n,k) satisfy the 'triangular' recurrence relation*

$$\begin{aligned} &S(n,k) = S(n-1,k-1) + kS(n-1,k), \qquad n,k \ge 1, \\ &S(n,0) = S(0,k) = 0, \qquad \textit{except } S(0,0) = 1. \end{aligned}$$

Let us illustrate the first Stirling numbers of the second kind $S(n,k)$, which can be easily computed using the previous triangular recurrence.

n \ k	0	1	2	3	4	5	6
0	1						
1	0	1					
2	0	1	1				
3	0	1	3	1			
4	0	1	7	6	1		
5	0	1	15	25	10	1	
6	0	1	31	90	65	15	1

Theorem 1.3. *The Stirling numbers of the second kind S(n,k) satisfy the 'vertical' recurrence relations*

$$\begin{aligned} S(n,k) &= \sum_{k-1 \le l \le n-1} \binom{n-1}{l} S(l,k-1), \\ S(n,k) &= \sum_{k \le l \le n} S(l-1,k-1)k^{n-1}. \end{aligned}$$

Theorem 1.4. *The Stirling numbers of the second kind S(n,k) satisfy the 'horizontal' recurrence relations*

$$S(n,k) = \sum_{0 \le j \le n-k} (-1)^j \langle k+1 \rangle_j \, S(n+1,k+j+1),$$

$$k! \, S(n,k) = k^n - \sum_{j=1}^{k-1} (k)_j \, S(n,j).$$

1.2.2 Stirling numbers of the first kind

At this point, two different definitions of the Stirling numbers of the first kind have been introduced.

Definition 1.6. *The Stirling numbers of the first kind are usually defined via the 'horizontal' GF, that is*

$$(x)_n = \sum_{0\le k\le n} s(n,k)x^k.$$

Theorem 1.5. *The following are equivalent definitions for Stirling numbers of the first kind:*

(1) Via the 'double' GF, Definition 1.3 (i).

(2) Via the 'vertical' GF, in eq. (1.1).

(3) Via the 'horizontal' GF, Definition 1.6

(4) Via another 'horizontal' GF, that is

$$\Psi_n(u) = \sum_{1\le k\le n} s(n,k)u^{n-k} = (1-u)(1-2u)\cdots(1-(n-1)u).$$

As mentioned before, the proof of this theorem can be found in [5] pp. 212-213.

Stirling numbers of the first kind hold, as well, recurrence relations.

Theorem 1.6. *The Stirling numbers of the first kind s(n,k) satisfy the 'triangular' recurrence relation*

$$s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k),\ n,k\ge 1,$$
$$s(n,0) = s(0,k) = 0,\ \textit{except}\ s(0,0) = 1.$$

As in the previous subsection, let us illustrate the first Stirling numbers of the first kind $s(n,k)$, which can be easily computed using the previous triangular recurrence.

n						
0	1					
1	-1	1				
2	2	-3	1			
3	-6	11	-6	1		
4	24	-50	35	-10	1	
5	-120	274	-225	85	-15	1
	0	1	2	3	4	5
			k			

Theorem 1.7. *The Stirling numbers of the first kind s(n,k) satisfy the 'vertical' recurrence relations*

$$ks(n,k) = \sum_{k-1\le l\le n-1} (-1)^{n-l-1}\binom{n}{l}s(l,k-1),$$
$$s(n+1,k+1) = \sum_{k\le l\le n} (-1)^{n-1}(l+1)(l+2)\cdots(n)s(l,k).$$

Theorem 1.8. *The Stirling numbers of the first kind s(n,k) satisfy the 'horizontal' recurrence relations*

$$(n-k)s(n,k) = \sum_{k+1\leq l\leq n} (-1)^{l-k}\binom{l}{k-1}s(n,l),$$
$$s(n,k) = \sum_{k\leq l\leq n} s(n+1,l+1)n^{l-k}.$$

1.3 Probabilistic generalization of Stirling numbers of the second kind

This section presents a probabilistic generalization of Stirling numbers of the second kind proposed by Adell and Lekuona [1]. In order to present such generalization, the following ingredients are needed.

Let us consider $\mathbb{N}$ to be the set of positive integers and $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. Consider as well that $f : \mathbb{R} \to \mathbb{R}$ is an arbitrary function, m, $n \in \mathbb{N}_0$, $x \in \mathbb{R}$, set $z = it$ (where i is the imaginary unit) and $t \in \mathbb{R}$ with $|t| < r$ (where $r > 0$). Finally consider the usual mth forward difference of f, which is defined as

$$\Delta^m f(x) := \sum_{k=0}^{m} \binom{m}{k}(-1)^{m-k} f(x+k). \tag{1.3}$$

Under this notation, the **Stirling polynomials of the second kind** are denoted by

$$S(n,m,x) := \frac{\Delta^m I_n(x)}{m!}, \quad m \leq n,$$

so that $S(n,m) := S(n,m,0)$, for $m \leq n$.

The Stirling polynomials of the second kind present a formula with increments of order one. The main idea behind the probabilistic generalization that is going to be presented is to substitute such increments by random variables. In order to do so, it is required that such random variable Y satisfies the moment conditions

$$E|Y|^n < \infty, \quad n \in \mathbb{N}_0, \quad \lim_{n\to\infty} \frac{|t|^n\, E|Y|^n}{n!} = 0, \quad |t| < r,$$

where E stands for mathematical expectation.

Letting $(Y_j)_{j\geq 1}$ be a sequence of independent copies of Y, and using the notation

$$S_k = \sum_{i=1}^{k} Y_i, \quad k \in \mathbb{N} \quad (S_0 = 0),$$

the following definition can be introduced.

Definition 1.7. *The **Stirling polynomials of the second kind associated to a random variable** Y satisfying the moment conditions 1.3 are defined as*

$$S_Y(n,m,x) := \frac{1}{m!}\sum_{k=0}^{m} \binom{m}{k}(-1)^{m-k} E[x+S_k]^n, \quad m \leq n.$$

As a remark, notice that such generalization of Stirling polynomials of the second kind is coherent, since if $Y = 1$ is considered,

$$S_1(n,m,x) = S(n,m,x), \quad \text{and } S_1(n,m) = S(n,m).$$

Let us introduce an interesting result that was proven in Adell and Lekuona (2019) [1].

Theorem 1.9. *Consider $(U_j)_{j\geq 1}$ to be a sequence of independent identically distributed random variables following the uniform distribution in $[0,1]$. Assuming that $(U_j)_{j\geq 1}$ and $(Y_j)_{j\geq 1}$ are mutually independent. Hence, for any $m,n \in \mathbb{N}_0$ with $m \leq n$, we have*

$$\begin{aligned} S_Y(n,m,x) &= \frac{1}{m!} E\Delta^m_{Y_1,\ldots,Y_m} I_n(x) \\ &= \binom{n}{m} EY_1 \cdots Y_m I_{n-m}(x + Y_1U_1 + \cdots + Y_mU_m) \\ &= \frac{1}{m!}\sum_{j=0}^{n} S(n,j) \sum_{k=0}^{m} (-1)^{m-k} E[(x+S_k)_j], \end{aligned}$$

where, we recall, $(\cdot)_j$ is the falling factorial. Equivalently, the polynomials $S_Y(n,m;x)$ are defined via their generating function as

$$\frac{e^{zx}}{m!}(Ee^{zY})^m = \sum_{n=m}^{\infty} \frac{S_Y(n,m,x)}{n!} z^n. \tag{1.4}$$

Let us introduce some examples.

Example 1.5. *Let $\alpha \in \mathbb{R}$. The* ***r-Whitney numbers of the second kind*** *$W_\alpha(n,m;x)$ are defined via the generating function*

$$\sum_{n=m}^{\infty} W_\alpha(n,m,x)\frac{z^n}{n!} = \frac{e^{zx}}{m!}\Big(\frac{e^{\alpha z}-1}{\alpha}\Big)^m. \tag{1.5}$$

Simply considering $Y = \alpha$ and comparing 1.4 with 1.5, it is immediate that

$$S_Y(n,m;x) = \alpha^m W_\alpha(n,m,x), \quad m \leq n.$$

Example 1.6. *Let $0 < p \leq 1$. Suppose that Y follows the Bernoulli law, that is*

$$P(Y=1) = p = 1 - P(Y=0).$$

Considering $Y = 1$ in 1.4,

$$\frac{e^{zx}}{m!}(e^z-1)^m = \sum_{n=m}^{\infty} \frac{S(n,m,x)}{n!} z^n, \tag{1.6}$$

i.e., it recovers one of the equivalent definitions of the Stirling polynomials of the second kind. Since when Y is a Bernoulli variable it can be written that

$$Ee^{zY} - 1 = p(e^z - 1),$$

from 1.6 it can be obtained that

$$\frac{e^{zx}}{m!}(Ee^{zY}-1)^m = p^m \frac{e^{zx}}{m!}(e^z-1)^m = p^m \sum_{n=m}^{\infty} \frac{S(n,m,x)}{n!} z^n.$$

This previous result combined with 1.4 shows that

$$S_Y(n,m,x) = p^m S(n,m,x), \quad m \le n.$$

As it can be observed, several well-known classical numbers in Number Theory and Combinatorics can be represented in a joint notation thanks to such Stirling polynomials of the second kind associated to a random variable.

Chapter 2

Total positivity, log-concavity and stochastic orders

This chapter contains three mathematical tools that will be useful in Chapter 3.

2.1 Total positivity

This present section summarizes some total positivity results that can be found in *Life Distributions: Structure of Nonparametric, Semiparametric, and Parametric Families*, by Marshall and Olkin [11], which is a brief survey based in the book *Inequalities: Theory of Majorization and Its Applications*, 1979 [14], from the same authors. Total positivity is an indispensable tool in the study of shape parameters. Even though the theory of total positivity was already well advanced in the early 1950s, it is not as well known as it deserves to be. A very complete discussion of the field can be found in Karlin (1968) [8]), and more recent collection of papers on the subject has been edited by Gasca and Micchelli (1996)[7]. A survey of the subject is presented by Barlow and Proschan (1965) [2]. Further publications have continued this path. As an example, Fomin and Zelevinsky (2000)[6] and, later on, Peña (2013)[15], have developed tests to identify such totally positive matrices (functions), since they can be difficult to detect due to high computational costs.

Let us present the definition of totally positive functions.

Definition 2.1. *Let A and B be subsets of the real line. A function K defined on $A \times B$ is said to be* ***totally positive of order k****, denoted TP_k if for all $x_1 < \cdots < x_m$, $y_1 < \cdots < y_m$ (where $x_i \in A$, $y_i \in B$), and for all m, $1 \leq m \leq k$,*

$$K\begin{pmatrix} x_1, & \ldots & ,x_n \\ y_1, & \ldots & ,y_n \end{pmatrix} := \det \begin{bmatrix} K(x_1,y_1) & \ldots & K(x_1,y_m) \\ \vdots & & \vdots \\ K(x_m,y_1) & \ldots & K(x_m,y_m) \end{bmatrix} \geq 0. \tag{2.1}$$

When the inequalities are all strict, K is said to be ***strictly totally positive of order k*** *(STP_k). If K is TP_k (STP_k) for all $k = 1,2\ldots$, then it is said to be* ***totally positive*** *(strictly totally positive)* ***of order*** *∞, written TP_∞ (STP_∞).*

Remark that being STP is a stronger condition than being TP. In other words, being STP_k implies being TP_k. The following consequences follow from the previous definition.

Corollary 2.1. *(i) If g and h are nonnegative functions defined on A and B respectively, and if K is TP_k on $A \times B$, then $g(x)h(y)K(x,y)$ is TP_k on $A \times B$.*

(ii) If g and h are increasing functions defined on A and B respectively, and if K is TP_k on $g(A)\times h(B)$, then $K(g(x),h(y))$ is TP_k on $A\times B$.

(iii) If K is TP_k on $A\times B$ and $A_0\subset A$, $B_0\subset B$, then K is TP_k on $A_0\times B_0$.

In this document, the previous properties will be applied to (finite or infinite) matrices, since it can be observed that Stirling numbers can be presented in the shape of a matrix with infinite rows and columns. In such case, $A=1,2,...$ will represent the number of rows, $B=1,2,...$ the number of columns, and K will be the function assigning to each (i,j) the corresponding element from the matrix. Let us introduce an example to illustrate this remark.

Example 2.1. *Let us study the totally positiveness of a general square Vandermonde matrix (which can be seen as the colocation matrix of the monomial basis $\{1,t,...,t^n\}$). Considering $0<t_0<\cdots<t_n$, with $t_i\in\mathbb{R}\;\forall\;0\leq i\leq n$, and*

$$M=\begin{pmatrix}1 & t_0 & \dots & t_0^n\\ 1 & t_1 & \dots & t_1^n\\ \vdots & \vdots & & \vdots\\ 1 & t_n & \dots & t_n^n\end{pmatrix},$$

since M is a Vandermonde matrix, $\det(M)=\prod_{i>j}(t_i-t_j)>0$. Indeed, all of its minors are nonnegative. In conclusion, M is a STP_∞ matrix (in other words, K is STP_∞). Since it is STP_∞, it is TP_∞ as well.

2.2 Total positivity of Stirling numbers

Let us introduce a definition of total positivity of order 2 for sequences of numbers. This definition is just a specific instance of Definition 2.1 restricted to $k=2$, where $A=B=\mathbb{N}$ and $K(\cdot,\cdot)=s(\cdot,\cdot)$, a general double sequence.

Definition 2.2. *A double sequence $s(n,m)$, where n and m vary on subsets of the integer numbers, is* ***totally positive of order 2*** *(TP_2) if for every $n_1\leq n_2$ and $m_1<m_2$ the determinant*

$$\begin{vmatrix}s(n_1,m_1) & s(n_1,m_2)\\ s(n_2,m_1) & s(n_2,m_2)\end{vmatrix}=s(n_1,m_1)s(n_2,m_2)-s(n_1,m_2)s(n_2,m_1)\geq 0. \tag{2.2}$$

Indeed, the following result can be found in Karlin (1968) [8], as well as in Sibuya (1998) [16], pg. 701.

Theorem 2.1. *As double sequences, Stirling numbers of the first and second kinds are strictly totally positive of order 2 in the sense that for any $n_1<n_2$ and $m_1<m_2$*

$$\begin{vmatrix}S(n_1,m_1) & S(n_1,m_2)\\ S(n_2,m_1) & S(n_2,m_2)\end{vmatrix}=\begin{cases}>0 & \text{if } S(n_1,m_1)S(n_2,m_2)\neq 0,\\ =0 & \text{if } S(n_1,m_1)S(n_2,m_2)=0.\end{cases} \tag{2.3}$$

This means that the sequences

$$\frac{S(n_2,m)}{S(n_1,m)}\;,\;m=1,2,...,n_1\;,\;(n_1<n_2),$$

and

$$\frac{S(n,m_2)}{S(n,m_1)}\;,\;n=m_2,m_2+1,....,\;,\;(m_1<m_2),$$

are strictly increasing. The previous result is analogously held for $s(n,m)$ as well.

2.3 Log-concavity

This section will serve to introduce a shape property of double sequences which will be used later on. Let us introduce a definition that will be frequently used from now on.

Definition 2.3. *Let us consider $(a_n)_{n=0,1,...}$ and $(b_n)_{n=0,1,...}$, two sequences of non-negative real numbers. The* ***convolution*** *and the* ***binomial convolution*** *of $(a_n)_{n=0,1,...}$ and $(b_n)_{n=0,1,...}$ are defined as*

$$c_n := \sum_{i=0}^{n} a_i b_{n-i} = \sum_{i=0}^{n} b_i a_{n-i} \qquad \textit{convolution}$$
$$d_n := \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i} = \sum_{i=0}^{n} \binom{n}{i} b_i a_{n-i} \quad \textit{binomial convolution}$$

As indicated in Nihn, Prékopa (2013) [13], pg. 3018, the notion of a log-concave sequence was first introduced under the name of 2-times or twice positive sequence as a special case of a r-times positive sequence, when $r = 2$. The sequence of nonnegative elements $\dots a_{-2}, a_{-1}, a_0, \dots$ (remark that this is a notion of log-concavity extended to the set of all the integers by defining $a_n = 0$, $n = -1, -2, \dots$) is said to be r-times positive if the matrix

$$A = \begin{pmatrix} \ddots & \ddots & \ddots & & \\ \ddots & a_0 & a_1 & a_2 & \\ \ddots & a_{-1} & a_0 & a_1 & \ddots \\ & a_{-2} & a_{-1} & a_0 & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix} \tag{2.4}$$

has no negative minor of order smaller than or equal to r (remember that a minor is the determinant of a finite square part of the matrix traced out by the same number of rows as columns).

The twice-positive sequences are those for which

$$\begin{vmatrix} a_i & a_j \\ a_{i-t} & a_{j-t} \end{vmatrix} = a_i a_{j-t} - a_j a_{i-t} \geq 0, \tag{2.5}$$

for every $i \leq j$ and $t \geq 1$. This motivates the following definition.

Definition 2.4. *A sequence $\dots a_{-2}, a_{-1}, a_0, \dots$ is said to be* ***log-concave*** *if its corresponding matrix (2.4) is TP_2.*

Let us introduce the following important theorem.

Theorem 2.2. *The convolution of two r-times positive sequences is at least r-times positive.*

Thus, a very useful property for log-concave sequences is that, if $(a_n)_{n=0,1,...}$ and $(b_n)_{n=0,1,...}$ are log-concave, so it is their convolution. Many authors may define log-concavity of a sequence $a_0, a_1 \dots$ by requiring only

$$a_n^2 \geq a_{n-1} a_{n+1} \ , n = 1, 2, \dots \tag{2.6}$$

However, this property alone does not imply that the convolution of two log-concave sequences are also log-concave. For instance, consider the two sequences to be defined as follows:

$$a_0 = 0, \quad a_1 = \frac{1}{2}, \quad a_2 = a_3 = 0, \quad a_4 = \frac{1}{2}, \quad a_5 = a_6 = \cdots = 0;$$
$$b_0 = b_1 = b_2 = b_3 = \frac{1}{4}, \quad b_4 = b_5 = \cdots = 0.$$

Hence, their convolution $(c_n)_{n=0,1,\ldots}$ is not log-concave since $c_3^2 < c_2 c_4$. In view of this, log-concavity should be defined in the same way as the notion of the twice positive sequence. Otherwise, another requirement has to be present in addition to (2.6).

Proposition 2.1. *A sequence of non-negative real numbers $(a_n)_{n=0,1,\ldots}$ is log-concave if and only if (2.6) is satisfied and $(a_n)_{n=0,1,\ldots}$ does not have any internal zero, i.e., there are no indices $0 \le i < j < k \le n$ such that $a_i \neq 0$, $a_j = 0$, $a_k \neq 0$.*

If the sequence $i!a_i$ is log-concave, then a_i is said to be ultra-log-concave. A univariate discrete probability distribution, defined on the integers, is said to be log-concave if the sequence of the corresponding probabilities is log-concave. To cite a few examples, the Poisson and the binomial distributions are log-concave.

Let us remark that the dual definition of **log-convexity** of a sequence just requires the reverse inequality in (2.6), with no further conditions needed. For instance, the negative binomial distribution, which posseses as probability density function

$$f(k;\lambda,p) = \binom{k+\lambda-1}{\lambda-1}(1-p)^k p^\lambda, \quad 0 \le p \le 1, \quad \lambda > 0, \quad k = 1,2,\ldots$$

is log-concave if the shape parameter $\lambda \ge 1$, but it is log-convex if $\lambda \le 1$. In particular, the geometric distribution with parameter $\lambda = 1$ is both log-concave and log-convex.

In addition, let us foresee that, as it can be found in Sibuya (1988) [16], Stirling numbers of the first and second kinds are log-concave with a fixed n when m varies. A general result will be introduced in Chapter 3.

2.4 Stochastic orders

The present section summarizes the content in Marshall and Olkin (2007) [11] pp. 47-56. In order to give the reader a highlight of this section, the idea of an stochastic order (or stochastic ordering) is to compare the whole distribution of random variables. This will allow to determine if such distributions are non-comparable, or if they are, which is greater than the other according to the comparison criteria that has been chosen.

Characteristics of distributions such as location, dispersion, skewness, and kurtosis have long been used for descriptive purposes. It is common to compare distributions by means of the previous characteristics. For example, standard deviation is used as a measure of dispersion. However, stochastic orders are tools which allow to compare distributions in a more general way.

Let F and $\overline{F}$ be the distribution and survival functions respectively of a random variable X, that is,

$$F(x) := P(X \le x), \quad \overline{F}(x) := 1 - F(x).$$

Now let F and G be distribution functions associated to the random variables X and Y respectively. Given a certain characteristic, an ordering $F \leq_* G$ is introduced to make precise the idea that "F has less of the characteristic than does G". Remark that the notation $F \leq_* G$ has been chosen to represent that such comparison is in some sense not yet determined. This approach was used by Mann and Whitney (1947)[10], who introduced what is now called "stochastic order".

Introducing an ordering is not so easy, since it must present the idea that one distribution presents more of some characteristic than another, and this will strongly depend on the choice of the characteristic's nature.

Once the ordering based on a certain has been chosen, it could be verified if measures of the characteristic are *order-preserving*. In other words, if $F \leq G$ in the ordering, then a measure m (which could be, for instance, the mean) of that characteristic should satisfy $m(F) \leq m(G)$.

The orderings $\leq_*$ of distributions considered in this section have two properties; they are both reflexive and transitive. That is,

$$F \leq_* F, \qquad \text{for all distribution functions } F \text{ (reflexivity)},$$

and

$$F \leq_* G \quad \text{and} \quad G \leq_* H \implies F \leq_* H \text{ (transitivity)}.$$

Orderings with these two properties are called *preorders*; if in addition they satisfy the condition that

$$F \leq_* G \quad \text{and} \quad G \leq_* F \implies F = G,$$

then the orders are called *partial orders*. Preorders and partial orders, unlike numerical measures of a characteristic as it could be the mean, can identify if two distributions are too disparate to be compared: $F \leq_* G$ and $G \leq_* F$ may both be false.

In what follows, it is sometimes convenient to write $X \leq_* Y$ to mean that the distribution F of X and the distribution G of Y satisfy $F \leq_* G$.

The usual stochastic order, defined below, has a number of properties that make it what might be called a "magnitude order". It will not be the only stochastic order to appear in this document, since the likelihood ratio order will be introduced as well.

Suppose that X' and Y' are random variables such that always $X' \leq Y'$ (for instance, $X' = X$ and $Y' = X + 1$). Although X' and X have the same distribution, Y' and Y have the same distribution, $X \leq Y$ could not be true. Nevertheless, it still seems reasonable to think that, in some sense, X is less than Y. This idea leads to the "usual" concept of stochastic order, an order in which X is less than Y if and only if the survival function of X is everywhere less than the survival function of Y.

The previous fundamental idea is not very old; apparently, it was first introduced by Mann and Whitney (1947)[10].

Definition 2.5. *If X and Y are random variables such that $P(X > x) \leq P(Y > x)$ for all $x \in \mathbb{R}$, then X is said to be* ***stochastically smaller*** *than Y. This relationship is notated by $X \leq_{st} Y$, or $F \leq_{st} G$, where X and Y have distribution F and G respectively.*

Intuitively, this definition means that if $X \leq_{st} Y$, then Y tends to take larger values than X with a higher probability. To illustrate a little more, $F \geq_{st} G$ means that $\overline{F}(x) \leq \overline{G}(x)$ for all x, and hence, $F(x) \geq G(x)$ for all x. The condition $\overline{F}(x) \geq \overline{G}(x)$, for all x, that one survival function dominates another is generally easy to check, suggestive both of examples and exponential applications, and it arises in a number of contexts. It is useful, for instance, while comparing medical treatments in a experiment in which X could be either the convalescent or survival time associated with one medical treatment and Y could be the corresponding time for another medical treatment. Or X and Y could be interpreted as the earnings resulting from various business strategies.

Let us introduce another interesting order that will be present in Chapter 3.

Definition 2.6. *The random variable X is said to be* ***smaller in the likelihood ratio ordering*** *than the random variable Y if for all u,*

$$P(X > u \mid a < X \leq b) \leq P(Y > u \mid a < Y \leq b),$$

whenever $a < b$ and the conditional probabilities are defined. This relationship is denoted by $X \leq_{lr} Y$ or by $F \leq_{lr} G$, where X and Y have distribution F and G respectively.

As a remark, the reader can verify that the likelihood ratio ordering implies the usual stochastic ordering, i.e., $X \leq_{lr} Y \implies X \leq_{st} Y$. It suffices to consider a sequence a_n tending to $-\infty$, another sequence b_n tending to ∞, and take limits when n tends to ∞.

The following proposition provides a useful way of verifying a likelihood ratio order when densities exist.

Proposition 2.2. *If X and Y are absolutely continuous random variables, then $X \leq_{lr} Y$ if and only if there are versions f and g of the corresponding densities such that*

$$f(u)g(v) \geq f(v)g(u), \text{ for all } u \leq v. \tag{2.7}$$

The proof can be found in Marshall and Olkin [11] pp. 56-57. The terminology of the previous definition comes from the fact that when denominators are positive, (2.7) can be written as the comparison of likelihood ratios

$$\frac{f(u)}{g(u)} \geq \frac{f(v)}{g(v)}, \text{ for all } u \leq v. \tag{2.8}$$

The previous proposition is still valid in the case of discrete variables (taking values in $\mathbb{Z}$) just by replacing the density by the probability function of the variables.

Example 2.2. *Let us consider the gamma distribution with shape parameter λ and scale parameter ν, whose density is given by*

$$f(x \mid \lambda, \nu) = \frac{\nu^{\lambda} x^{\lambda-1} e^{-\nu x}}{\Gamma(\lambda)}, \quad x > 0, \quad \lambda > 0, \quad \nu > 0,$$

where $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$, for all $\Re(z) > 0$ is the gamma function . Let us check the previous ratio of densities, taking into account that $f(x|\alpha,\rho)/f(x|\alpha,\nu) = Cx^{\rho-\nu}$, where C is a positive constant. Hence, this ratio is decreasing in x whenever $\rho \leq \nu$, so that in the likelihood ratio ordering, the distribution is increasing in the shape parameter. As the remark below the definition of likelihood ratio ordering, this implies that the distribution is also increasing in the *shape parameter in the usual stochastic ordering. This result is easily inherited, but it would be difficult to be obtained otherwise, since the gamma distributions with a non-integer shape parameter present no explicit distribution.*

Let us introduce a result contained in [12], pg. 52, which will be useful in Chapter 3.

Theorem 2.3. $X \leq_{lr} Y$ *if and only if*

$$E[g(X^*,Y^*)] \leq E[g(Y^*,X^*)], \quad \text{for all } g \in G_{ir}, \tag{2.9}$$

where

$$G_{ir} = \{g : g(x,y) - g(y,x) \geq 0 \text{ for all } x \geq y\}$$

and X^* *and* Y^* *are independent with* $X^* =_{st} X$ *and* $Y^* =_{st} Y$.

This result reveals that the stochastic relation $<_{lr}$ admits a bivariate characterization. Further details can be found in [12] pg.51.

Chapter 3

Generalization of Stirling numbers of the second kind

This final chapter contains a new generalization of Stirling numbers of the second kind (section 3.1) and a throwback to the generalized Stirling polynomials introduced in Chapter 1, section 3. The main objective is to verify if such generalizations present a nice behaviour in terms of log-concavity and total positivity of order 2, so that all the numbers included in such generalizations could inherit the properties as well under appropiate conditions.

3.1 A generalization of Stirling numbers of the second kind by means of the 'vertical' generating function

Let us recall that Stirling numbers of the second kind are defined by means of the 'vertical' exponential generating function

$$\sum_{k=m}^{\infty} S(n,m)\frac{z^n}{n!} = \frac{1}{m!}(e^z - 1)^m.$$

When studying generalizations of Stirling numbers of second kind $S_g(n,m)$, it is usual to find the following pattern in the 'vertical' exponential generating function

$$\sum_{k=m}^{\infty} S_g(n,m)\frac{z^n}{n!} = \frac{f(z)}{m!}(g(z))^m, \tag{3.1}$$

in which

$$g(z) = \sum_{n=1}^{\infty} \frac{b_n z^n}{n!},$$

being $(b_n)_{n=1,2,\dots}$ a sequence of real numbers and $f(z)$ admitting a generating function of the form

$$f(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!},$$

In which $(a_n)_{n=0,1,\dots}$ is a sequence of real numbers. Usually both sequences consist of non-negative terms. Due to the properties that are going to be studied (log-concavity, TP_2) this assumption is usually taken in the following results. Stirling numbers of the second kind are thus a particular case of this setting by taking $f(z) = 1$ and $g(z) = e^z - 1$, that is

$$a_0 = 1 \ \ (0 \text{ otherwise}) \quad \textit{and} \quad b_n = 1,\ n = 1,2,\dots \tag{3.2}$$

This class of numbers presents very nice properties in relation to Chapter 2: they are log-concave in m for each n, log-concave in n for each m and totally positive of order 2 in (n,m). The log-concavity in m is usually hard to prove, because the usual methods are either recursions or the study of the zeroes in the 'horizontal' generating function

$$\sum_{m=1}^{n} S_g(n,m) z^m,$$

which in many occasions cannot be expressed in a closed form. Many generalizations of the Stirling numbers are shown to exhibit the log-concavity in m for n fixed. For instance, the r-Whitney numbers, whose generating function is, as it has already been seen, given by

$$\sum_{k=m}^{\infty} W_{r,a}(n,m)\frac{z^n}{n!} = \frac{e^{rz}}{m!}\left(\frac{e^{az}-1}{a}\right)^m,$$

for some $a > 0$, and therefore, in this case:

$$a_n = r^n\ ,\ n = 1,2,\dots \quad \textit{and} \quad b_n = a^{n-1}, n = 1,2,\dots \tag{3.3}$$

Note that $W_{r,1}$ are the generalized Stirling numbers $S_r(n,m)$ and $W_{0,1}(n,m)$ are the usual Stirling numbers $S(n,m)$ (in particular, if r is an integer, $W_{r,1} = S(n+r, m+r)$.

Moreover, some authors studied the numbers $W_{1,m}$ (see Benoumhani (1996) [3]). Many other examples (Lah numbers, whose generating function is as in (3.1), with $(b_n)_{n=0,1,\dots}$ being the moments of a random variable) will be dealt with in the next section. Many other sequences can be defined as in (4), starting with a $(b_n)_{n=1,2,\dots}$ representing the moments of a non-negative random variable (Catalan numbers, Bell numbers, Fubini numbers...).

It seems natural thus to study conditions on the initial sequences $(a_n)_{n=0,1,\dots}$ and $(b_n)_{n=0,1,\dots}$ in order to guarantee that the previous properties for Stirling numbers are preserved. Several resuls in this direction for Whitney numbers, for instance, are well-known. Our aim is to give a unified treatment to numbers responding to the above-mentioned generating function (3.1). Let us first prove a recursive expression which will be useful for this purpose.

Lemma 3.1. *The generalized Stirling numbers $S_g(n,m)$ satisfy the following recursions*

$$S_g(n,m) = \frac{1}{m}\sum_{l=m-1}^{m-1}\binom{n}{l} S_g(l,m-1) b_{n-l}. \tag{3.4}$$

Proof. To show the expression, take into account that

$$f(z)\frac{g(z)^{m-1}}{(m-1)!} = \sum_{n=m-1}^{\infty} S_g(n,n-1,x)\frac{z^n}{n!}.$$

On the other hand,

$$g(z) = \sum_{n=1}^{\infty} b_n \frac{z^n}{n!}.$$

Multiplying both expressions and dividing by m, the generating function of $S_g(n,m)$ is obtained. Since the product of generating functions is the generating function of the convolution,

$$\frac{S_g(n,m)}{n!} = \frac{1}{m}\sum_{l=m-1}^{n-1} \frac{S_g(l,m-1)}{l!}\frac{b_{n-l}}{(n-l)!},$$

thus having (3.4).

□

Let us remark that the 'vertical' recurrence relation (1.3) from chapter 1 is a particular instance of (3.4).

Remark 3.1. *The previous recursion expresses the generalized Stirling numbers as a binomial convolution, that is the sequence* $(s_n)_{n=0,1,\ldots}$ *defined by* $s_k = S_g(k,m)$ *is the binomial convolution of the sequence* $(s'_n)_{n=0,1,\ldots}$ *defined as* $s'_k = S_g(k,m-1)$ *and the sequence* $(b_n)_{n=0,1,\ldots}$ $(b_0 = 0)$. *For the particular case of r-Whitney numbers, this recursion appears, for instance in Benoumhani (1996) [3].*

The previous recursive formulas allow us to give conditions for the log-concavity in n of $S_g(n,m)$ when m is fixed.

Proposition 3.1. *Let* $S_g(n,m)$ *be the generalized Stirling numbers, with sequences* $(a_n)_{n=0,1,\ldots}$ *and* $(b_n)_{n=0,1,\ldots}$ *consisting on non-negative terms. If* $(b_n)_{n=0,1,\ldots}$ *is a log-concave sequence and the binomial convolution*

$$d_n := \sum_{l=0}^{n-1} \binom{n}{l} a_l b_{n-l}, \quad n = 1,2,\ldots \tag{3.5}$$

is log-concave, then $S_g(n,m)$ *is log-concave in n for m fixed.*

Proof. The log-concavity of $S_g(n,1)$ follows as $S_g(n,1) = d_n$, as it can be easily verified considering $m = 1$ in (3.1). The general case for m follows by induction on m, by taking into account that

$$S_g(n,m) = \frac{1}{m}\sum_{l=m-1}^{n-1} \binom{n}{l} S_g(l,m-1) b_{n-l},$$

and the preservation of log-concavity by binomial convolutions (recall the previous remark).

□

Remark that, due to the preservation of log-concavity by binomial convolutions, the log-concavity of $(a_n)_{n=0,1,\ldots}$ and $(b_n)_{n=0,1,\ldots}$ is a sufficient condition. On the other hand, (3.5) is a necessary condition, as $S_g(n,1) = d_n$.

The recursions obtained in Proposition 3.1 allows verifying the TP_2 property for $S_g(n,m)$ (see Chapter 2).

Proposition 3.2. *Let* $S_g(n,m)$ *be the generalized Stirling numbers, with sequences* $(a_n)_{n=0,1,\ldots}$ *and* $(b_n)_{n=0,1,\ldots}$ *consisting on non-negative terms. If* $(b_n)_{n=0,1,\ldots}$ *is a log-concave sequence and the binomial convolution (3.5) is log-concave, then* $S_g(n,m)$ *is* TP_2 *in* (n,m).

Proof. It is enough to show the property for two consecutive n and m, that is

$$S_g(n,m)S_g(n+1,m+1) - S_g(n,m+1)S_g(n+1,m) \geq 0. \tag{3.6}$$

Let us apply (3.4) to expand terms containing the $m+1$-th second coefficient. Thus, the term on the left can be written as

$$\frac{S_g(n,m)}{m+1}\left(\sum_{l=m}^{n}\binom{n+1}{l}S_g(l,m)b_{n+1-l}-\frac{S_g(n+1,m)}{m+1}\sum_{l=m}^{n-1}\binom{n}{l}S_g(l,m)b_{n-l}\right)\geq$$

$$\frac{1}{m+1}\sum_{l=m}^{n-1}\left(\binom{n+1}{l+1}S_g(n,m)S_g(l+1,m)-\binom{n}{l}S_g(n+1,m)S_g(l,m)\right)b_{n-l}, \tag{3.7}$$

in which, in the last inequality, the first term in the first addend has been disregarded, and a term-by-term comparison has been efectuated directly after. On the other hand, it is easily verified by calculus that

$$\binom{n+1}{l+1}-\binom{n}{l}=\binom{n}{l}\left(\frac{n+1}{l+1}-1\right)\geq 0, \quad l\leq n.$$

Thus, it can be written as

$$\binom{n+1}{l+1}(S_g(n,m)S_g(l+1,m)-\binom{n}{l}S_g(n+1,m)S_g(l,m))\geq$$

$$\binom{n+1}{l+1}(S_g(n,m)S_g(l+1,m)-S_g(n+1,m)S_g(l,m)) \tag{3.8}$$

Now notice that by Proposition 3.1, $S_g(n,m)$ is a log-concave sequence in n, and therefore (recall Proposition 2.1).

$$S_g(n,m)S_g(l+1,m)-S_g(n+1,m)S_g(l,m)\geq 0, \quad l\leq n. \tag{3.9}$$

Thus, (3.7) - (3.9) show (3.6).

□

A similar argument to the one used in the proof of Proposition 3.3 allows to show the log-concavity in m of $S_g(n,m)$.

Proposition 3.3. *Let $S_g(n,m)$ be the generalized Stirling numbers, with sequences $(a_n)_{n=0,1,\ldots}$ and $(b_n)_{n=0,1,\ldots}$ consisting of non-negative terms. If $S_g(n,m)$ is TP_2, then $S_g(n,m)$ is log-concave in m for each $n=1,2,\ldots$.*

Proof. The following expression needs to be verified to conclude the proof.

$$S_g(n,m+1)^2-S_g(n,m+2)S_g(n,m)\geq 0$$

Using (3.4), the term on the left can be written as

$$\frac{S_g(n,m+1)}{m+1}\sum_{l=m}^{n-1}\binom{n}{l}S_g(l,m)b_{n-l}-\frac{S_g(n,m)}{m+2}\sum_{l=m+1}^{n-1}\binom{n}{l}S_g(l,m+1)b_{n-l}\geq$$

$$\frac{1}{m+1}\sum_{l=m+1}^{n-1}\binom{n}{l}(S_g(n,m+1)S_g(l,m)-S_g(n,m)S_g(l,m+1))b_{n-l}\geq 0.$$

From the last inequality, recalling Proposition 3.2, $S_g(n,m)$ is TP_2. □

The following result appears as an immediate consequence of Propositions 3.1 and 3.3.

Corollary 3.1. *Let $S_g(n,m)$ be the generalized Stirling numbers, with sequences $(a_n)_{n=0,1,\ldots}$ and $(b_n)_{n=0,1,\ldots}$ consisting on non-negative terms. If $(b_n)_{n=0,1,\ldots}$ is a log-concave sequence and the binomial convolution is log-concave, then $S_g(n,m)$ is both log-concave in n for m fixed and log-concave in m for n fixed.*

3.2 Probabilistic generalization of Stirling numbers of the second kind (reprise)

Recall that the probabilistic polynomial generalization of Stirling numbers of the second kind appearing in Adell and Lekuona (2019) [1] has been introduced in Chapter 1, section 3. They can be defined in terms of the 'vertical' exponential generating function as

$$S_Y(n,m,x) := \frac{1}{m!}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}E[x+S_k]^n, \quad m \leq n. \tag{3.10}$$

where Y is a random variable. Let us remark again that considering $x=0$ in the previous expression, $S_Y(n,m) := S_Y(n,m,0)$ is obtained. In addition, if $Y=1$, the Stirling numbers of the second kind $S(n,m)$ are obtained. In this section the notation $S_X(n,m,x) := S_X(n,m)$ will be used for the Stirling polynomials, as in Adell and Lekuona (2019) [1].

Note therefore that, in this general setting, $b_n = E[Y^n]$. As b_n is a moment sequence, it is necessarily log-convex, as by Cauchy-Schwartz's inequality

$$E[Y^{n+1}] = E[(Y^{n+1}Y^{n-1})^{1/2}] \leq (E[Y^{n+1}]E[Y^{n-1}])^{1/2}.$$

Thus, the only moment sequences which can verify Propositions 3.1 and 3.3 are those in which b_n is log-linear. This includes $Y=c$, with $c>0$ (deterministic case). The deterministic case thus includes Stirling numbers and Whitney numbers. In general, the only moment sequences being log-linear are those in which

$$b_1^2 \leq b_2, \quad b_n = b_2\left(\frac{b_2}{b_1}\right)^{n-2}.$$

In such case, denoting $b := b_2 b_1^{-1}$, it can be written that

$$Ee^{zY} - 1 = \frac{b_2}{b^2}\left(e^{zb}-1\right),$$

and, in fact, $S_Y(n,m) = (\frac{b_2}{b^2})W(n,m,b)$, so that these numbers are trivial extensions of the Whitney numbers.

As a conclusion, excluding the previous particular case, $b_n = E[Y^n]$ is a strictly log-convex sequence. Remark that, considering $x=0$ in (3.10), $S_Y(n,1) = E[Y^n]$ is a strictly log-convex sequence. It is well known that binomial convolutions preserve log-convexity. Hence, it is a natural question to ask if $S_Y(n,m)$ is a log-convex sequence in m. Observe that this is not straightforward as, recalling Remark 3.1 we have a binomial convolution starting with $b_0=0$, which breaks log-convexity for the sequence b_k.

Moreover, the preservation of log-convexity cannot be expected, even in the case that $S(n,1)$ is a strictly log-convex sequence. An example is provided with the classical Lah numbers.

Example 3.1. *Let Y be an exponential random variable with mean equal to 1. Note that*

$$Ee^{xY} - 1 = \frac{x}{1-x},$$

*and thus $S_Y(n,m)$ are the classical **Lah numbers**, which are defined by means of the previous generating function (see Adell and Lekuona (2019) [1] for instance). These numbers have an explicit expression, namely*

$$L(n,m) = \binom{n-1}{m-1}\frac{n!}{m!}.$$

In this case, $L(n,1) = n!$ is a strictly log-convex function. However, it is easy to check that $L(n,3) = n!(n-1)(n-2)/12$ is not log-convex. In fact, it can be observed that the sequence $l_n = n!(n-1)(n-2)$ is neither log-concave nor log-convex. In fact

$$\frac{l_{n+1}^2 - l_n l_{n+2}}{(n+1)!\; n!\; (n+1)\; n\; (n-1)} = n(n-1) - (n+2)(n-2) = 4-n,$$

which is obviously positive for $n = 3$ and negative for $n > 4$. Thus, $S(n,3)$ is neither log-concave nor log-convex. However, Lah numbers are shown to be totally positive in (n,m) (stronger property than TP_2). This fact is shown in Brenti (1995) [4].

The natural question is whether the TP_2 property holds true for general probabilistic Stirling numbers or not. It will be shown that, for a wide class of random variables Y, $S_Y(m,n,x)$ are in general TP_2 in (m,n), and therefore log-concave in m for n fixed. First of all, let us introduce a preliminar result which will be useful for this purpose.

Lemma 3.2. *Let Y be a random variable. The generalized Stirling polynomials $S_Y(n,m,x)$ satisfy the following recursion*

$$S_Y(n,m,y) = \sum_{l=m}^{n}\binom{n}{l} S_Y(l,m,x)(y-x)^{n-l}, \quad x,\, y \in \mathbb{R}.$$

Proof. The proof is similar to the deterministic formula (case $Y = 1$) for the classical Stirling Polynomials (see Kim and Kim (2017) [9]). We only have to take into account that the generating function for $S_Y(n,m,y)$ obeys to the following expression:

$$\frac{e^{zy}}{m!}(Ee^{zY} - 1)^m = \sum_{n=m}^{\infty} S_Y(n,m,y)\frac{z^n}{n!}. \tag{3.11}$$

Observe that

$$Ee^{zY} - 1 = \sum_{n=1}^{\infty} E[Y^n]\frac{z^n}{n!}.$$

On the other hand,

$$e^{z(y-x)} = \sum_{n=0}^{\infty} \frac{(y-x)^n}{n!} z^n.$$

Now let us take into account that

$$\frac{e^{zy}}{m!}(Ee^{zY} - 1)^m = e^{z(y-x)}\frac{e^{zx}}{m!}(Ee^{zY} - 1)^m,$$

and the formula follows as the product of generating functions of two sequences is the generating function of its convolution, that is

$$\frac{S_Y(n,m,y)}{n!} = \sum_{l=m}^{n} \frac{S_Y(l,m,x)}{l!} \frac{(y-x)^{n-1}}{(n-l)!}.$$

□

The following lemma provides a simple recursion for the generalized Stirling polynomials, which will be very useful in successive results.

Lemma 3.3. *Let Y be a random variable. The generalized Stirling polynomials $S_Y(n,m,x)$ satisfy the following recursion:*

$$mS_Y(n,m,x) = E[S_Y(n,m-1,x+Y_1)] - S_Y(n,m-1,x) = \frac{n}{m} E\left[\int_0^{Y_1} S_Y(n-1,m-1,u)du\right]$$

Proof. The proof of the first equality applies the simple fact that

$$\Delta^m_{Y_1,\dots,Y_m} f(x) = E[\Delta^{m-1}_{Y_2,\dots,Y_m}(f(x+Y_1)-f(x)) \mid Y_2,\dots,Y_m]$$
$$= \Delta^{m-1}_{Y_2,\dots,Y_m}(E[f(x+Y_1)]-f(x)) = \Delta^{m-1}_{Y_2,\dots,Y_m} E[f(x+Y_1)] - \Delta^{m-1}_{Y_2,\dots,Y_m} f(x),$$

which, applied to $f(x) = x^n$, shows the result (see (1.3) if necessary to recall the definition of the mth forward difference). □

As a remark, if $Y = 1$ and $x = 0$, the previous Lemma leads us to the well known two-term recursion for Stirling numbers of the second kind, namely

$$S(n+1,m) = mS(n,m) + S(n,m-1). \tag{3.12}$$

The reason is that, if Lemma 3.3 is applied in this case, it leads to

$$mS(n,m) = S(n,m-1,1) - S(n,m-1))$$

As mentioned before, $S(n,m-1,1)$ coincides with the Whitney number $W_1(n,m-1) = S(n+1,m)$. Combining this result together with the previous expression, shows (3.12).

Finally, the following possible TP_2 property for the probabilistic Stirling numbers, under appropriate likelihood ordering properties of Y, can be started to be verified.

Theorem 3.1. *Let Y be a random variable, and let Y_1, Y_2 be two independent copies of Y. Let $S_Y(n,m,x)$ be the generalized Stirling polynomials. If $Y_1 \leq_{lr} Y_1 + Y_2$, then $S_Y(n,m)$ is TP_2 in (m,n) for x fixed.*

Proof. Let us proceed by induction on m. Firstly, let us check the case $m = 1$, that is

$$p_{n,1} := S_Y(n+1,2)S_Y(n,1) - S_Y(n,2)S_Y(n+1,1) \geq 0. \tag{3.13}$$

Observe that

$$S_Y(n,1) = EY_1^n.$$

On the other hand,

$$S_Y(n,2) = \frac{1}{2}(ES_2^n - 2EY_1^n).$$

A simple calculus shows the following

$$\begin{aligned} p_{n,1} &= (ES_2^{n+1} - 2EY_1^{n+1})EY_1^n - (ES_2^n - 2EY_1^n)EY_1^{n+1} \\ &= ES_2^{n+1}EY_1^n - ES_2^nEY_1^{n+1} \end{aligned} \tag{3.14}$$

Let us define the function

$$g(x,y) := x^n y^{n+1}, \quad x,y \geq 0 \quad \text{(0 otherwise)},$$

and observe that

$$g(x,y) - g(y,x) \geq 0 \text{ for all } y \geq x.$$

Hence, using the bivariate characterization of the likelihood ratio order (recall Theorem 2.3), $Y_1 \leq_{lr} Y_1 + Y_2$ implies that

$$ES_2^{n+1}EY_1^n \geq ES_2^nEY_1^{n+1}, \tag{3.15}$$

thus, (3.14) and (3.15) show (3.13). Now, let us assume that the property is true for all n and for the set $1,2,\ldots,m$. The property will be checked in the set $1,2,\ldots,m+1$, that is

$$p_{n,m} := S_Y(n+1,m+1)S_Y(n,m) - S_Y(n,m+1)S_Y(n+1,m) \geq 0. \tag{3.16}$$

Using Lemma 3.3, it can be written that

$$S_Y(n,m+1) = \frac{E[S_Y(n,m,Y_1)] - S_Y(n,m)}{m+1}. \tag{3.17}$$

Using Lemma 3.2 we have

$$E[S_Y(n,m-1,Y_1)] = \sum_{l=m-1}^{n} \binom{n}{l} S_Y(l,m-1)(EY_1^{n-l}). \tag{3.18}$$

Thus, combining (3.16) - (3.18),

$$\begin{aligned} \frac{p_{n,m}}{(m+1)} = S_Y(n,m) &\sum_{l=m-1}^{n+1} \binom{n+1}{l} S_Y(l,m-1)EY_1^{n+1-l} \\ - S_Y(n+1,m) &\sum_{l=m-1}^{n} \binom{n}{l} S_Y(l,m-1)EY_1^{n-l}. \end{aligned}$$

Disregarding the first term in the first summation and making a comparison term by term leads us, finally, to

$$\frac{p_{n,m}}{2m(m+1)} \geq \sum_{l=m-1}^{n} \left(\binom{n+1}{l+1} S_Y(n,m)S_Y(l+1,m-1) - \binom{n}{l} S_Y(n+1,m)S_Y(l,m-1) \right) \geq 0,$$

as it follows by induction, since $\binom{n+1}{l+1} \geq \binom{n}{l}$. The case $x = 0$ being proved, this work stops here and future work on the subject could be very fruitful. $\square$

Conclusion

As observed in the third chapter, the generalizations of Stirling numbers of the second kind introduced in this document present a very nice behaviour in terms of log-concavity and total positivity of order 2. This leads to conclude that all the numbers that can be derived from the generalizations present such behaviour as well.

However, quite a few lines of future work remain open. The same applies to the search for generalizations for Stirling numbers of the first kind. Other generalizations for Stirling numbers of the second kind could be also considered. For instance, a deep study on the total positivity properties of the probabilistic generalization could be fruitful.

The main moral of this work is the importance of working in a unified way on Stirling numbers (among others), and how this approach allows solving calculations and individual checks in an affordable and exportable way.

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