IEJME
OPEN ACCESS
INTERNATIONAL ELECTRONIC JOURNAL OF MATHEMATICS EDUCATION
e-ISSN: 1306-3030. 2020, Vol. 15, No. 3, em0593
https://doi.org/10.29333/iejme/8279

Infinite Limit of Sequences and Its Phenomenology

Mónica Arnal-Palacián [1*], Javier Claros-Mellado [2], María Teresa Sánchez-Compaña [3]

[1] *University of Zaragoza, SPAIN*
[2] *Complutense University of Madrid, SPAIN*
[3] *University of Malaga, SPAIN*

* CORRESPONDENCE: marnalp@unizar.es

ABSTRACT

In this document, we search for and define an infinite limit of sequences that is correct and accepted by the mathematical experts, the final purpose of which is to analyze its phenomenology, in Freudenthal's sense. To make the choice, experts were consulted on two issues. The first one was not decisive because of the effect that the divergence term causes, and for this reason, we did a second expert consultation where this term was removed and we selected the definition we have analyzed in this document. Once the definition was chosen, two approaches were considered for analysis: the intuitive approach and the formal approach. Based on these two approaches, we specify certain phenomena organized by the definition: unlimited intuitive growth and unlimited intuitive decrease (intuitive approach) and one way and return infinite limit of sequences (formal approach), and show examples of such phenomena by graphical, verbal and tabular representation systems. All this aim to be a help to overcome the difficulties that pre-university students have with the concept of limit.

Keywords: infinite limit of sequences, phenomenology, representation systems, intuitive approach, formal approach

INTRODUCTION

A great number of researches (Belmonte & Sierra, 2011; Blázquez & Ortega, 2001; Marín, 2014; Morales et al., 2013; Salat, 2011; Sierra, González & López, 1999; Tall, 1991; Valls, Pons & Llinares, 2011) indicate how the teachers have revealed the difficulties that pre-university students present when they have to approach the concepts of limit and infinite. In these researches the finite limit of a sequence or the different limits of a function have been studied but not the infinite limit in a sequence. This research analyses this last one, through the search and definition of the organized phenomena by its definition, in Freudenthal's sense (1983) due we consider that difficulties are not only limited to finite limit. We also emphasize the use of phenomenology can help to overcome the difficulties mentioned above. For that reason, we will define the phenomena that are organized by the definition of infinite limit of a sequence. The definition has a different role to the technical use of a construct in contrast with its intuitive and colloquial uses. (Tall & Vinner, 1981; Vinner, 1991). An initial differentiating feature between both notions is that the definition must be formal, unequivocal, concise, and exact (Fernández-Plaza, 2015). Definition can be considered as the best representation of certain concepts, in the sense of being synthetic and complete, as all the properties of the concept are deduced logically.

The first step, in order to carry out this research, would be to choose a definition of an infinite limit of sequence and then perform an analysis of its phenomenology. When we refer to phenomenology, we do in Freudenthal's sense (1983). We should clarify what we understand as divergent sequence, a term that

Article History: Received 25 February 2020 ◆ Revised 9 May 2020 ◆ Accepted 13 May 2020

interfered initially in the choice of the definition, due to a disagreement within the mathematical community on its interpretation. The consideration by Pestana (2000) of a divergent sequence is used: a sequence with a plus infinite limit or a minus infinite limit, which are considered in this document. An example of divergent sequence is the next one: $s(n) = n, s(n) = \ln(n), s(n) = -n^2, s(n) = 2^n$.

The second step involves the characterization and definition of the phenomena organized by the definition of infinite limit chosen in step one.

We then describe the structure of the document, which is comprised of five sections. The first theoretical framework section shows a selection of authors that have carried out research on the phenomenology, representation systems and advanced mathematical thinking, the foundations of our research. In the second section, we show the process we followed to the selection of the definition of infinite limit of sequence which we work with. In the third, we define the phenomena of unlimited intuitive growth and unlimited intuitive decrease. We devote the fourth section 2 defining the phenomenon of the so-called one way and return infinite limit of sequences. In the last section, we present our conclusions and future prospects.

The first step constitutes the first objective, whereas the second objective is determined by the characterization of the phenomena established in the third and fourth sections.

LITERATURE REVIEW

This research begins with a study of the classroom difficulties relating to the notion of an infinite limit of sequences, both in our experience as teachers and in the researches reviewed. According to Claros (2010) and Sánchez (2012), we situate our research into the infinite limit of sequence as part of the theoretical framework constituted by the foundations of: phenomenology, representation systems and advanced mathematical thinking.

Phenomenology

When we refer to phenomenology, we do so as a component of its didactic analysis in Freudenthal's sense (1983), who calls "phenomenology" his method of analysis of mathematical content, based on a contrast between "phenomenon" and "noumenon". This philosophical analysis between objects constructed as concepts are "noumena" and the situations that these mathematical objects organize are "phenomena". As pointed out by Gravemeijer and Terwel (2000, p.784), "the objective of phenomenology research is to encounter problem situations in which generalized situations on how to approach it and encounter situations that can evoke paradigmatic solution procedures as a basis for vertical mathemetisation. Finding phenomena that can be mathematized enables us to understand how they were invented".

Puig (1997), following Freudenthal, points out that for the concepts of limit, continuity and the infinite:

"The phenomena for which mathematicians have developed these concepts for organization belong to the world of mathematics in which there are objects that produce them or they are produced in highly mathematized contexts. A teaching approach to these concepts must also take into account that only good mental objects can be constructed with them on the condition that we are able to experiment with the phenomena they organize".

Representation Systems

We present each phenomenon defined with examples in different representation systems. For Janvier (1987) and Rico (2009), all mathematical concepts require a variety of representation for their recording, understanding and structuring, as well as to establish relations between different representation systems. These representation systems emphasize certain properties of the concept and prevent others. In addition, an individual has a better understanding of a mathematical concept as bigger his knowledge of representations and properties of the same is. (Molina 2014) In particular, in the infinite limit of sequences we consider the following representation systems: verbal, graphical, symbolic and tabular (called numeric by Blázquez (2000)), although in this document we only use the verbal, graphic and tabular systems. The reason is because they are the most commonly used systems in textbooks for Secondary Education and Pre-university Studies (Claros, Sánchez, & Coriat, 2016).

Advanced Mathematical Thinking

Infinite limit belongs to Advanced Mathematical Thinking (AMT), as it requires a high level of logical-abstract ability to learn and take part in the process of abstraction, formalization, representation and definition. These processes begin to develop in educational stage corresponding to the 15-20 years (Garbin, 2015), which this work is in. All these elements are the features of cognitive development used in this type of thinking. In Tall's sense (1991), we research further into the cognitive development present in teaching and learning processes in concepts related to infinitesimal calculus. In addition, Tall highlights several concepts that should have been part of AMT due to their difficulty, including limits and infinite. In spite of the fact that we have just included the infinite limit of sequences as part of AMT, according to Edwards, Dubinsky and McDonald (2005), the concept of infinite limit forms part of elemental or advanced mathematical thinking, depending on the work related to it. Also, most students lack conceptual knowledge, even those, who are labeled as average in their performance, their knowledge is dominated by procedure manipulation (Sebsibe & Feza, 2019).

After describing the theoretical framework of our research, we deal with the choice of the definition of the infinite limit of sequence for its subsequent phenomenological analysis.

METHODOLOGY

For the definition of the phenomena organized by a definition of an infinite limit of sequences, we decided to previously consult experts to choose a correct definition that is accepted by the mathematical community.

The consultation gave rise to a questionnaire, which we call Questionnaire 1, formed by the definitions given by the next authors in university books: Bradley and Smith (1998), Díaz (1998), Baenas and Martínez de Santiago (2007), Brinton (2005) and Pestana (2000). The sample taken was of 5 secondary education teachers and 4 university professors in Didactics of Mathematics. All of them had teaching experience ranging from 5 to 35 years. These people were chosen by the High Schools and Faculties of Education we have access. In that study we aim to note the phenomena organized by the concept of infinite limit of a sequence in the student answers to the questions about that concept.

Each professor and teacher had to assign an order of suitability to the 6 proposed definitions: "1" being the most suitable from a mathematical perspective, "2" the next suitable, etc., up to "6", the least suitable. They were also required to assign a "0" to the definitions that could not be considered as such or were unsuitable. Each one was provided with space to explain why the definition was not considered valid or unsuitable for pre-university students (See **Annex I**).

Each teacher has the name of "expert" and was accompanied by the expression of "x.1", where "x" is a number between 1 and 9, and it refers to each one of the 9 teachers of the sample and "1" because we refer to the result of the Questionnaire 1. We use this nomenclature in order to ensure anonymity of the expert during all data analysis. Quantitative results are in **Annex III**.

The comments provided in the questionnaire gave rise to a new problem relating to the mathematical experts' different perceptions of the term divergent. This made the score obtained by each definition dependent on what they understood as divergent sequence and not actually on the suitability or not of the definition proposed in the questionnaire.

It was therefore decided to perform another expert consultation, which gave rise to Questionnaire 2. A new search for definitions of an infinite limit of sequence took place and those without a reference to the term divergent were selected. They were taken from the next university books and textbooks: Bradley and Smith (1998), Linés (1983), Vizmanos et al. (2011), Colera, Olivera and Fernández (1997) and Pestana (2000).

The sample used in Questionnaire 2 was comprised of 5 secondary education teachers and 2 university professors, chosen in the same way as Questionnaire 1. The format of the consultation and its subsequent evaluation were the same as in the previous questionnaire (See **Annex II**).

From Questionnaire 2 the following definition, included in a manual that the pre-university students and university students can use, obtained the highest score. (See **Annex III**). This would be the definition used in all our research:

> "Let K be an ordered field and $\{a_n\}$ a sequence of elements of K. The sequence $\{a_n\}$has a "plus infinite limit", if for each H element of K, there exists a natural number v, so that $a_n > H$ for all $n \geq v$" (Linés, 1983).

Despite the fact that the definition of "minus infinite" did not appear in either of the two questionnaires, according to the results obtained in Questionnaire 2, the following definition by the same author and in the same manual is used:

> "Let K be an ordered field and $\{a_n\}$ a sequence of elements of K. The sequence $\{a_n\}$ has a "minus infinite limit", if for each H element of K, there exists a natural number v, so that $a_n < H$, every time $n \geq v$" (Linés, 1983).

RESULTS

The following is a definition of the phenomena organized by the chosen definition, from an intuitive and a formal approach. Examples of each phenomenon are shown in graphic, verbal and tabular form and deal with growing and decreasing sequences.

Unlimited Intuitive Growth

We observed that a growing sequence fulfills the idea that the values of the sequence become increasingly greater. If $n > m$, we expect $s(n) > s(m)$. After checking this with a series of values, we intuitively deduced that the sequence is growing.

Growing sequences may have an upper limit or not. In the former case, the limit would be finite and not the subject of this research, as it has already been studied by Claros (2010). In the latter case, for unlimited growing sequences, we do not have a real number greater than all the values of the sequence and would therefore have unlimited growth. We call this phenomenon unlimited intuitive growth (u.i.g.) and it is observed in the sequences with a plus infinite limit.

In conclusion, we can characterize unlimited intuitive growth, u.i.g., as the phenomenon observed when the values of a sequence become increasingly greater as the sequence continues. As a result of this phenomenon, it can be deduced that the sequence is growing and unlimited, in other words, it grows without limit. We could therefore say that this phenomenon features unlimited growing sequences with an infinite limit.

The following are some examples of the phenomenon using graphical, verbal and tabular representation systems.

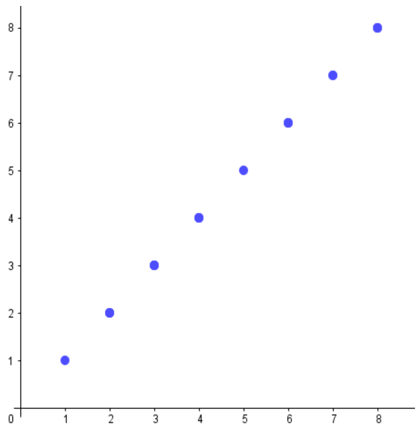

Figure 1. Example graphical representation system. Sequence $s(n) = n$. Own elaboration

Based on the graph, **Figure 1**, we can deduce that the terms of the sequence are increasingly greater as it advances. This leads us to believe that we will have unlimited growth, in which there is no real number greater than all the other values of the sequence. Growth without limit leads us to the conclusion that its limit is $+\infty$.

In the following example we can observe a limit of sequence in the verbal representation system.

Ex. The sequence with natural numbers 1, 4, 9, 16, … can grow as much as we want, then we say it has infinite limit.

We observe that the values of the sequence $1 < 4 < 9 < 16 < 25 < 36 < ..$ are increasingly higher as we continue, without finding a number higher than all of them and therefore producing unlimited growth. The sequence appears to be growing and not subject to an upper limit and we can therefore observe the u.i.g. phenomenon.

n	1	10	100	1 000	…	→	+ ∞
a_n	2	101	10 001	1 000 001	…	→	+ ∞

Figure 2. Example tabular representation system. Sequence $s(n) = n^2 + 1$. Vizmanos and Anzola (1996, p.160).

The u.i.g. phenomenon is observed, as the terms of row $s(n)$, **Figure 2**, corresponding to the terms of the sequence are increasingly greater. The sequence is not subject to an upper limit and appears to be growing. This allows us to deduce that the sequence has a limit $+\infty$.

Unlimited Intuitive Decrease

As in unlimited intuitive growth, we observe that a decreasing sequence fulfills the idea that the values of the sequence become increasingly smaller, small being understood as the negative numbers whose absolute value is increasingly higher. If $n > m$, we expect $s(n) < s(m)$. By checking this with two or three values, we intuitively deduced that the sequence is decreasing.

Decreasing sequences may have a lower limit or not. In the former case, the limit would be finite and not the subject of this research (Claros, 2010). In the latter case, for unlimited decreasing sequences, we do not have a real number lower than all the values of the sequence and would therefore have unlimited decrease. Given that the sequence has infinite terms and is not subject to a lower limit, it decreases in an unlimited way. We call this phenomenon unlimited intuitive decrease (u.i.d.). It is observed in the sequences with minus infinite limit.

We characterize unlimited intuitive decrease, u.i.d. as the phenomenon observed when the values of a sequence become increasingly smaller, small being understood as the negative numbers whose absolute value is increasingly higher, meaning that for any value we establish, we will always find a value of the sequence that is lower. As a result of this phenomenon, it can be deduced that the sequence is decreasing and unlimited, in other words decreases without limit. We could therefore say that this phenomenon features unlimited decreasing sequences with an infinite limit.

The following are some examples using the graphic, verbal and tabular representation systems.

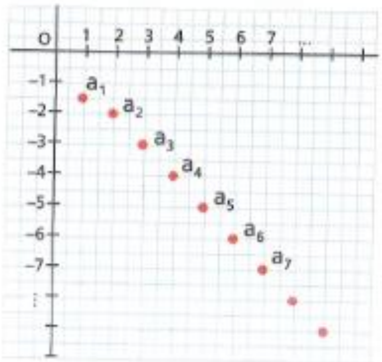

Figure 3. Example graphical representation system. Sequence $s(n) = -n$. Vizmanos and Anzola (1996, p.160)

Observing the graph, **Figure 3**, we can deduce that the terms of the sequence are increasingly lower as it advances and we do not have a real number greater than all the other values of the sequence. This leads us to observe the u.i.d. phenomenon. The sequence appears to be decreasing and we can therefore deduce that its limit will be $-\infty$.

In the following example we can observe a limit of sequence in the verbal representation system.

Ex. The sequence with natural numbers -1, -4, -9, -16, … can decrease as much as we want, then we say it has minus infinite limit.

We observe that the values of the sequence $-1 > -4 > -9 > -16 > -25 > -36 > \ldots$ are increasingly lower as we continue, without finding a number lower than all of them and therefore producing unlimited decrease. The sequence appears to be decreasing and not subject to a lower limit and we can therefore observe the u.i.d. phenomenon.

n	1	10	100	1 000	…	→	+ ∞
a_n	0	−99	−9 999	−999 999	…	→	− ∞

Figure 4. Example tabular representation system. Sequence $s(n) = -n^2 + 1$. Vizmanos and Anzola (1996, p.160)

The u.i.d. phenomenon is observed as the terms of the row $s(n)$, **Figure 4**, corresponding to the terms of the sequence, are increasingly smaller. The sequence is not subject to a lower limit and appears to be decreasing. This allows us to deduce that the sequence has a limit $-\infty$.

We call the two phenomena unlimited intuitive growth and unlimited intuitive decrease, represented as u.i.g. and u.i.d. respectively. The two phenomena are both intuitive and unlimited and therefore referred to hereinafter as u.i.-g. and u.i.-d.

One Way and Returned Phenomena in Sequences with an Infinite Limit (from Feedback)

In a formal approach to the definition chosen, Linés (1983), we can observe two processes that determine the phenomena of one way and return infinite limit of sequences:

- The first process called "one way" corresponds to the fragment: "if for every element H of K, there exists a natural number v".
- The second process called "return" corresponds to the fragment "so that $a_n > H$, every time$n \geq v$".

The feedback comes by observing the two processes jointly, specifically when interpreting and applying the processes included in the definition of the infinite limit of a sequence. This requires the construction of a function $H \to n(H)$.

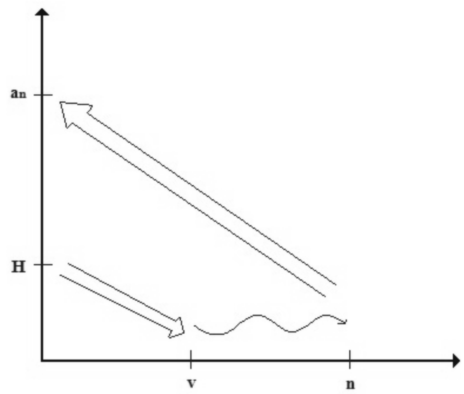

Figure 5. One way and return phenomenon

Establishing an H (**Figure 5**), we go "one way" to a v belonging to (not unique) natural numbers and "return" considering $n \geq v$ for which we will have $a_n > H$. In this way, a real function is built that takes on natural values and that we simply call $(H, n(H))$. This constructed function is unmistakably linked to the sequence we are working with. The specific feature of this function is that it starts at Axis Y and goes to Axis X.

We will hereinafter call this phenomenon o.w.r.i.s., one way and return infinite limit of sequences.

The following are some examples using graphic, verbal and tabular representation systems.

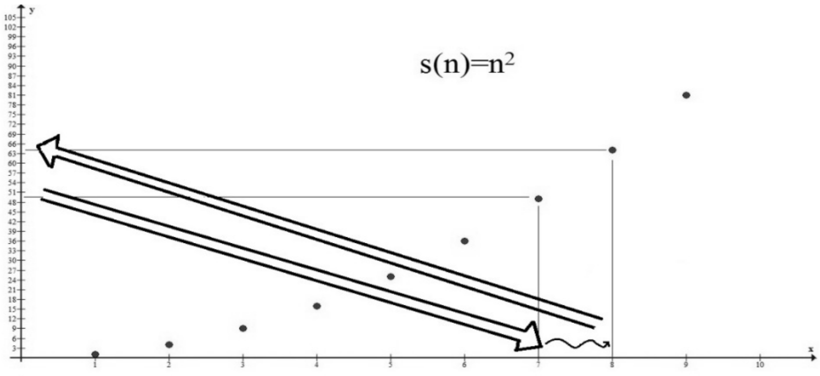

Figure 6. Example graphical representation system. One way and return phenomenon $s(n) = n^2$. Own elaboration

We start at a real number $H = 49$ (**Figure 6**), situated at the Axis Y and go “one way” to a natural number v, in our example $v = 7$, situated at the Axis X, and “return” from $n = 8$ ($n \geq v$) to a real number of the sequence, $a_8 = 64$ ($a_n > H$).

- One way: Given $H = 49$, there is a v natural number, for example $v = 7$.
- Return: with $n \geq v$, for example$n = 8$, we have $a_8 = 64 > 49 = H$.

The following are examples of some of the values taken on by the function $n(H)$ for different values of H, in which n is the lowest number that gives the inequality $a_n > H$:

$$H = 10, n = 4 \rightarrow s(4) = 16 > 10$$

$$H = 16, n = 5 \rightarrow s(5) = 25 > 16$$

$$H = 27, n = 6 \rightarrow s(6) = 36 > 27$$

$$H = 39, n = 7 \rightarrow s(7) = 49 > 39$$

$$H = 50, n = 8 \rightarrow s(8) = 64 > 50$$

$$\dots$$

$$H = 100, n = 11 \rightarrow s(11) = 121 > 100$$

$$\dots$$

$$H = 10\,000, n = 101 \rightarrow s(101) = 10\,201 > 10\,000$$

In light of all the above, it can be deduced that $n(H) = \lceil \sqrt{H} \rceil + 1$ if $H \geq 1$. This $n(H)$ is obtained by resolving the equation $n^2 > H$. We considered this natural number, n, and added 1 for it to be greater than the v established.

$n(H)$ is the real function that takes on natural values and has been constructed unmistakably for the sequence we are working with.

Below we can see an example in the verbal representation system, with the symbolic representation system support. For each real number H situated at the ordinate access, we have a natural number v at the Axis X. If we take a natural number that is higher or the same as such number, $n \geq v$, we can find a sequence value that is determined by the value of n, taking (a_n) and that will confirm that a_n is higher than the real number H established. We studied the specific case of $a_n = n^2$, in which taking $H = 5$ there is a natural number, for example $v = 3$, for which we can find a same higher number, $n = 4 \geq 3$ and obtain $a_n = 3^2 = 9 > 5$.

Table 1. Example tabular representation system. One way and return phenomenon.$s(n) = n^2$

H	5	10	17	26	39	…	8.286	…	99.820.086
v	3	4	5	6	7	…	92	…	9.992
n	4	5	6	7	8	…	93	…	9.993
s(n)	16	25	36	49	64	…	8.649	…	99.860.049

One way: Given $H = 5$, there exists a v natural number, $v = 3$.

Return $n \geq v, n = 4$, we have $a_n = 16 > 5 = H$.

Where $n(H) = \lfloor (\sqrt{H}) \rfloor + 2$, if $H \geq 0$.

In light of all the above, it can be deduced that $n(H) = \lfloor(\sqrt{H})\rfloor + 2$, if $H \geq 0$. This $n(H)$ is obtained by resolving the equation $n^2 > H$. We considered the whole part of $\sqrt{H}$ for n to be a natural number and added 2 for it to be greater than the v established. We did not want to take the same function as in the graphic representation system, to be able to observe that the real function is not unique.

Phenomenology in Sequences with and Infinite Limit

As shown in the following image, **Figure 7**, it is necessary to use both approaches to characterize all the phenomena organized by the definition of the infinite limit of a sequence and calculate the limit of the sequence presented. The first of them to guess the limit candidate and the second to prove that the selected candidate is the true limit of the succession presented.

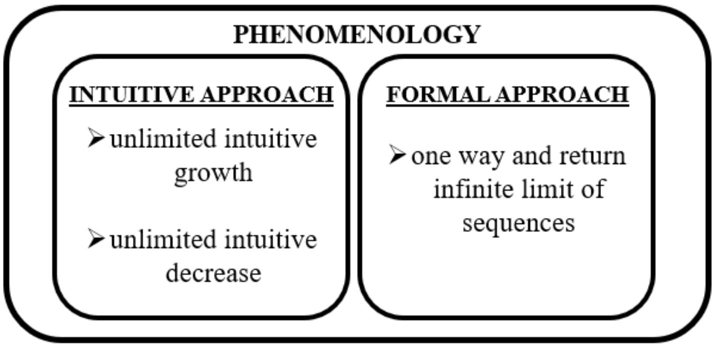

Figure 7. Phenomenology: intuitive and formal approach

The phenomena u.i.-g. and u.i.-d. provide "moral trust" depending on whether the limit is $+\infty$ or $-\infty$ and from this result, performing an abstraction of the notion, we will provide confirmation using the phenomena o.w.r.i.s.

CONCLUSIONS AND FUTURE PROSPECTS

Following the completion of Questionnaire 1 and Questionnaire 2, a definition of the infinite limit of a sequence was chosen. The chosen definition was that of Linés (1983), which was obtained after Questionnaire 2. This means the fulfillment of objective 1 which we established in step 1.

Once the definition was chosen, we performed a phenomenology study. This phenomenology study involved the meticulous characterization of the phenomena organized by said definition using an intuitive as well as formal approach, thus fulfilling objective 2. Specifically, we defined the phenomena of unlimited intuitive growth and unlimited intuitive decrease (u.i.-g. and u.i.-d.) from an intuitive approach and the phenomena of feedback or one way-return sequences with an infinite limit using a formal approach (o.w.r.i.s.). Furthermore, these phenomena organized can be found in the systems of verbal, tabular and graphic representations, as we have shown.

As the result of our research, we can conclude that phenomena u.i.-g. and u.i.-d. show a firs candidate to limit, $+\infty$ or $-\infty$, as the case may be, and it will be confirmed, through of the phenomena o.w.r.i.s., when we are capable of build a function n(H) that satisfies both processes. All this aim to be a help to overcome the difficulties that pre-university students have when they have to approach and understand task related with the concept of limit.

Although we have advanced in the phenomenology study of infinite limit, we should point out that there is still work pending. The pending research work includes a comparison and listing of the phenomena characterized and the mathematical terms used in this study, with the phenomena organized by a finite limit of sequences (Claros, 2010) and the finite limit of a function at a specific point (Sánchez, 2012), from both an intuitive and formal approach.

Disclosure statement

No potential conflict of interest was reported by the authors.

Notes on contributors

Mónica Arnal-Palacián – University of Zaragoza, Spain.

Javier Claros-Mellado – Complutense University of Madrid, Spain.

María Teresa Sánchez-Compaña – University of Malaga, Spain.

REFERENCES

Baenas, T., & Martínez de Santiago, C. (2007). Cálculo de variable natural. Editorial Club Universitario.

Belmonte, J. L., & Sierra, M. (2011). Modelos intuitivos del infinito y patrones de evolución nivelar. *Revista Latinoamericana de Investigación en Matemática Educativa*, *14*(2), 139-171. Retrieved from http://www.relime.org/

Blázquez, S. (2000). *Noción de límite en matemáticas aplicadas a las ciencias* (Doctoral dissertation). Universidad de Valladolid, Valladolid, España.

Blázquez, S., & Ortega, T. (2001). Los sistemas de representación en la enseñanza del límite. R*evista Latinoamericana de Investigación en Matemática Educativa*, *4*(3), 219-236. Retrieved from http://www.relime.org/

Brinton, G. (2005). *Cálculo: una variable*. Ed. Pearson Education.

Claros, J. (2010). *Límite finito de una sucesión: fenómenos que organiza* (Doctoral dissertation). University of Granada, Didactic Department of Mathematics, Granada, Spain.

Claros, J., Sánchez, M. T., & Coriat, M. (2016). Estructura cognitiva y fenomenología: el caso de la sucesión convergente. *Enseñanza de las Ciencias*, *34*(2), 87-105. https://doi.org/10.5565/rev/ensciencias.1859

Díaz, J. M. (1998). *Introducción a la topología de los espacios métricos*. Ed. Universidad de Cádiz.

Edwards, B. S., Dubinsky, E., & McDonald, M. (2005). Advanced mathematical thinking. *Mathematical Thinking and Learning, 7*(1), 15-25. https://doi.org/10.1207/s15327833mtl0701_2

Fernández-Plaza, J. A., Ruiz-Hidalgo J. F., & Rico, L. (2015). Razonamientos basados en el concepto de límite finito de una función en un punto. *Enseñanza de las Ciencias, 33*(2), 211-229. https://doi.org/10.5565/rev/ensciencias.1575

Freudenthal, H. (1983). *Didactical Phenomenology of Mathematics Structures*. Dordrech: Reidel Publishing Company.

Garbin, S. (2015). Investigar en pensamiento matemático avanzado. In J. Ortiz & M. Iglesias (Eds.), *Investigaciones en educación matemática. Aportes desde una unidad de investigación* (pp. 137-153). Maracay, Venezuela: Universidad de Carabobo. Retrieved from http://funes.uniandes.edu.co/8361/

Gravemeijer, K., & Terwel, J. (2000). Hans Freudenthal: a mathematician on didactics and curriculum theory. *Journal of Curriculum Studies, 32*(6), 777-796. https://doi.org/10.1080/00220270050167170

Janvier, C. (1987). *Problems of representations in the teaching and learning of mathematics*. Hillsdale, NJ: Lawrence Erlbaum Associated. https://doi.org/10.1007/BF00311018

Linés E. (1983). *Principios de Análisis Matemático*. Ed. Reverté.

Marín, I. D. (2014). Sobre el infinito y sus dificultades antes de Georg Cantor y sus obras. *Revista Ingeniería, Matemáticas y Ciencias de la Información, 1*(2). https://doi.org/10.21017/rimci

Molina, M. (2014). Traducción del simbolismo algebraico al lenguaje verbal: indagando en la comprensión de estudiantes de diferentes niveles educativos. *Gaceta de la RSME, 17*(3), 559-579. Retrieved from http://gaceta.rsme.es/

Morales, A., González, J., & Sigarreta, J. M. (2013). Concepciones sobre el infinito: Un estudio a nivel universitario. *Revista Digital: Matemática, Educación e Internet, 13*(2). https://doi.org/10.18845/rdmei.v13i2.1061

Pestana, D. (2000). *Curso práctico de Cálculo y Precálculo*. Ed. Ariel Ciencia.

Puig, L. (1997). *Análisis Fenomenológico. La Educación Matemática en la Enseñanza Secundaria*. Barcelona: Horsori.

Rico, L. (2009). Sobre las nociones de representación y comprensión en la investigación en educación matemática. *PNA, 4*(1), 1-14. Retrieved from https://revistaseug.ugr.es

Salat, R. S. (2011). El infinito en matemáticas. Números. *Revista de Didáctica de las Matemáticas, 77*, 75-83. Retrieved from http://www.sinewton.org/numeros/

Sánchez, T. (2012). *Límite finito de una función en un punto: fenómenos que organiza* (Doctoral dissertation). University of Granada, Didactic Department of Mathematics, Granada, Spain.

Sebsibe, A. S., & Feza, N. N. (2019). Assessment of Students' Conceptual Knowledge in Limit of Functions. *International Electronic Journal of Mathematics Education*, *15*(2). https://doi.org/10.29333/iejme/6294

Sierra, M., González, M. T., & López, C. (1999). Evolución histórica del concepto de límite funcional en los libros de texto de bachillerato y curso de orientación universitaria. *Enseñanza de las Ciencias, 17*(3), 463-476. Retrieved from https://www.raco.cat/index.php/Ensenanza/article/view/21606

Tall, D. (1991). The Psychology of Advanced Mathematical Thinking. *Advanced mathematical thinking*. Dordrech: Kluwer. https://doi.org/10.1007/0-306-47203-1_1

Tall, D., & Vinner, S. (1981). Concept Image and Concept Definition in Mathematics with particular reference to Limits and Continuity. *Educational Studies in Mathematics, 12*(2), 151-169. https://doi.org/10.1007/BF00305619

Valls, J., Pons, J., & Llinares, S. (2011). Coordinación de los procesos de aproximación en la comprensión del límite de una función. *Enseñanza de las Ciencias, 29*(3), 0325-338. https://doi.org/10.5565/rev/ec/v29n3.637

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In *Advanced mathematical thinking*. Dordrech: Kluwer. https://doi.org/10.1007/0-306-47203-1_5

Vizmanos, J. R., & Anzola, M. (1996). *Ciencias Naturales y de la salud-tecnología, matemáticas. 1º Bachillerato*. Ed. SM.

Vizmanos J. R., Hernández J., Alcalde F., Moreno M., & Serrano E. (2011). *2º Bachillerato*. Ed. SM.

ANNEX I

DEFINITIONS	SCORE
Definition 1 $\lim_{n\to+\infty} a_n = +\infty$ means that, for every real number A, $a_n > A$ for every sufficiently large n.	
Definition 2 A real sequence (x_n) is divergent and we write $\lim_{n\to+\infty} x_n = +\infty$, if for every $k > 0$ there is $m \in \mathbb{N}$ so that $\|x_n\| > k$ for every $n \geq m$.	
Definition 3 We say that $\{a_n\}$ has a limit $+\infty$, or that $\{a_n\}$ diverges to $+\infty$ and we represent it as $\lim_{n\to+\infty} a_n = +\infty$ if $\forall M \in \mathbb{R}, \exists n_0 \in \mathbb{N} \| a_n > M \forall n \geq n_0$. In such case, it is also said that sequence $\{a_n\}$ is divergent.	
Definition 4 The definition $\{a_n\}$ diverges to infinite if for every number M there is a whole N, so that every n is greater than N, $a_n > M$. If this condition is fulfilled, we will write $\lim_{n\to+\infty} a_n = +\infty$ o $a_n \to +\infty$	
Definition 5 We write $x_n \to +\infty$ and say that the sequence $\{x_n\}$ tends to $+\infty$ if for every c$\in \mathbb{R}$ there is $N \in \mathbb{N}$ N$\in\mathbb{N}$ so that $d(x_n, c) > 0$ every time $n > N$.	
Definition 6 We say that $\lim_{n\to+\infty} a_n = +\infty$, if for every M$\in \mathbb{R}$ there is a $N \in \mathbb{R}$ so that $a_n > M$ if $n > N$	

ANNEX II

DEFINITIONS	SCORE
Definition 1 $\lim_{n\to+\infty} a_n = +\infty$ means that, for every real number A, it is proven that $a_n > A$ for very sufficiently large n.	
Definition 2 Let K be an ordered field and {an} a sequence of elements of K. The sequence {an} has a "plus infinite limit", if for each H element of K, there exists a natural number v, so that $a_n > H$ for all $n \geq \nu$.	
Definition 3 $\lim_{n\to+\infty} a_n = +\infty \Leftrightarrow \forall M > 0$ can find a $n_0 \in \mathbb{N}$ so that $n > n_0 \Longrightarrow a_n > M$	
Definition 4 The limit of a sequence is ∞, $\lim_{n\to+\infty} s_n = +\infty \Leftrightarrow$ We can make s_n as large as we wish by giving n sufficiently high values.	
Definition 5 We say that $\lim_{n\to+\infty} a_n = +\infty$ if every time $M \in \mathbb{R}$ there is a N$\in \mathbb{R}$ so that $a_n > M$ if $n > N$.	

ANNEX III

The following scores were obtained in Questionnaire 1 and Questionnaire 2.

Questionnaire 1

	Def. 1	Def. 2	Def. 3	Def. 4	Def. 5	Def. 6
Expert 1.1	1	0	1/2	1/3	0	1/4
Expert 2.1	1/2	0	1	1/5	0	1/6
Expert 3.1	1	0	0	1/4	1/2	1/3
Expert 4.1	1/6	0	0	0	0	1/6
Expert 5.1	1	0	1/2	1/3	0	1/4
Expert 6.1	1/4	1/3	1	1/2	1/6	1/3
Expert 7.1	1/5	1/4	1	1/2	1/6	1/3
Expert 8.1	1/5	0	1/2	0	1/4	1/3
Expert 9.1	1/3	0	1	0	1/4	1/2

Questionnaire 2

	Def. 1	Def. 2	Def. 3	Def. 4	Def. 5
Expert 1.2	1/4	1/2	1	0	1/3
Expert 2.2	1/2	1	0	0	1/3
Expert 3.2	1/3	1	1/2	1/4	1/5
Expert 4.2	1/3	1/2	1	1/4	1/5
Expert 5.2	0	1	1/3	0	1/2
Expert 6.2	1/3	0	1	0	1/2
Expert 7.2	1/2	1	1/4	0	1/3
Total	2 +1/4	**5**	4 +1/12	1/2	2 + 2/5

http://www.iejme.com