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# A NEW PRODUCT INTEGRATION APPROACH FOR A WEAKLY SINGULAR HAMMERSTEIN EQUATION

**Abstract.** A new product integration scheme is proposed to solve Hammerstein equations which are weakly singular. The standard way of implementing the product integration method to a nonlinear equation is to transform the functional equation to an nonlinear finite dimensional algebraic system by the product integration scheme and then linearize the system to solve it. In this paper, we propose to treat the nonlinearity first. We construct a Newton sequence in the infinite dimensional functional space. Then we approximate the Newton iterates by the product integration method. We prove that the iterates, issued from our method, tend to the exact solution of the nonlinear Hammerstein equation when the number of Newton iterations tends to infinity, whatever the mesh size can be. This is not the case when the discretization is done first: in this case, the accuracy of the approximation is limited by the mesh size.

## 1. Introduction

The general framework of this paper is the following. Let $\mathcal{X}$ be a complex Banach space and $K : \mathcal{O} \subseteq \mathcal{X} \to \mathcal{X}$ a nonlinear Fréchet differentiable integral operator of the Hammerstein type defined on a nonempty open set $\mathcal{O}$ of $\mathcal{X}$:

$$K(x)(s) := \int_a^b H(s,t)L(s,t)F(t,x(t))dt, \quad \text{for all } x \in \mathcal{O}, \tag{1.1}$$

where $H$ is the singular part of the kernel, $L$ is the regular part of the kernel and $F$, the nonlinear part of the operator, is a real-valued function of two variables : $(t,u) \in [a,b]\times\mathbb{R} \mapsto F(t,u) \in \mathbb{R}$, with enough regularity so that $K$ is twice Fréchet-differentiable on $\mathcal{O}$.

The problem is a nonlinear Fredholm integral equation of the second kind: for a given function $y \in \mathcal{X}$,

$$\text{find } \varphi \in \mathcal{O} : \quad \varphi - K(\varphi) = y, \tag{1.2}$$

Let $T := K'$ denote the Fréchet derivative of $K$, i.e., for all $x \in \mathcal{O}$,

$$T(x)h(s) = \int_a^b H(s,t)L(s,t)\frac{\partial F}{\partial u}(t,x(t))h(t)dt, \quad h \in \mathcal{X}, s \in [a,b]. \tag{1.3}$$

In the following, $\mathcal{X}$ will be the space of the real valued continuous functions over a real interval $[a,b]$, $C^0([a,b],\mathbb{R})$, equipped with the supremum norm $\|.\|$.

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If we consider a singular kernel such as $H(s,t) := \log(|s-t|)$ or $|s-t|^{\alpha}$, $0<\alpha<1$, an approximation based on standard numerical integration is a poor idea. The product integration method consists in performing a piecewise polynomial interpolation of the smooth part of the kernel times the function involving the unknown. This method is called product trapezoidal rule when the interpolation is linear. The solution of a second kind Fredholm integral equation with weakly singular kernel is typically nonsmooth near the boundary of the domain of integration. In order to obtain a high order of convergence, taking into account the singular behavior of the exact solution, polynomial spline on graded mesh can be developed (e.g., Brunner [4], Pedas and Vainikko [12], Schneider [14]). In [9], Kaneko, Noren and Xu discuss a standard product integration method with a general piecewise polynomial interpolation for weakly singular Hammerstein equation and indicate its superconvergence properties.

In [1], Chapter 6, Anselone studies the Newton method to approximate the solution of nonlinear equations $P(x) = 0$, where $P$ is a nonlinear differentiable operator from a Banach space into itself. When dealing with the convergence of approximate solutions these are defined as the solution of $P_n(x) = 0$, where $P_n$ is an approximation of $P$. The Newton method is then applied to the functional equation $P_n(x) = 0$. The philosophy of most of the papers dealing with the numerical approximation of nonlinear integral operator equation consist in defining an approximate operator $P_n$ to $P$ and then to apply the Newton method (e.g. [3], [4], [6], [9], [10] and [12]).

We propose to apply the functional version of Newton's method directly to the operator equation $P(x) = 0$ and to discretize the linear operator equations, issued from the Newton's iterations, after, using a product integration method. We will prove that the approximate iterates tend to the exact solution of the operator equation as the number of iterations tends to infinity. The important fact is that the convergence holds whatever the discretization parameter, defining the size of the linear system to be solved, can be.

The paper is organized as follows. Section 2 is devoted to the description of our method (linearization via Newton's method followed by discretization by the product integration method). In Section 3 the convergence result is proved. In the last section, the classical – discretization followed by linearization – method is recalled and we compare it with our method through a numerical example.

## 2. Description of the new method

For a given function $y$ in $C^0([a,b],\mathbb{R})$, the application of the Newton method to the equation $\varphi - K(\varphi) = y$ leads to the sequence $(\varphi^{(k)})_{k\geqslant 0} \in O$:

$$\varphi^{(0)} \in O, \quad (I - T(\varphi^{(k)}))(\varphi^{(k+1)} - \varphi^{(k)}) = -\varphi^{(k)} + K(\varphi^{(k)}) + y,\ k \geqslant 0. \tag{2.4}$$

Then we discretize this equation with the product integration method associated with the piecewise linear interpolation. Let $\Delta_n$, defined by

$$a =: t_{n,0} < t_{n,1} < \cdots < t_{n,n} := b, \tag{2.5}$$

be the uniform grid of $[a,b]$ with mesh $h_n := \dfrac{b-a}{n}$.
Let $\pi_n$ denote the piecewise linear interpolation. If

$$f_x(t) := \frac{\partial F}{\partial u}(t, x(t)),$$

$\forall s \in [a,b], \forall i = 1, \ldots, n$:

$$\begin{aligned}\pi_n\left(L(s,.)f_x(.)h(.)\right)(t) \quad &:= \quad \frac{1}{h_n}(t_{n,i}-t)L(s,t_{n,i-1})f_x(t_{n,i-1})h(t_{n,i-1}) \\ &\quad + \frac{1}{h_n}(t-t_{n,i-1})L(s,t_{n,i})f_x(t_{n,i}))h(t_{n,i})\end{aligned}$$

for $t \in [t_{n,i-1}, t_{n,i}]$.

We define the approximate operator $T_n$ by

(2.6) $$T_n(x)(h)(s) := \int_a^b H(s,t)\pi_n\left(L(s,.)f_x(.)h(.)\right)(t)dt, \quad h \in X, s \in [a,b].$$

Then the approximate problem is:

(2.7) Find $\varphi_n^{(k+1)} \in X : (I - T_n(\varphi_n^{(k)}))(\varphi_n^{(k+1)} - \varphi_n^{(k)}) = -\varphi_n^{(k)} + K(\varphi_n^{(k)}) + y.$

We have

$$\begin{aligned} T_n(x)(h)(s) \quad &:= \quad \sum_{j=0}^{n} w_{n,j}(s)L(s,t_{n,j})f_x(t_{n,j})h(t_{n,j}), \\ w_{n,0}(s) \quad &:= \quad \frac{1}{h_n}\int_{t_{n,0}}^{t_{n,1}} H(s,t)(t_{n,1}-t)dt, \\ w_{n,n}(s) \quad &:= \quad \frac{1}{h_n}\int_{t_{n,n-1}}^{t_{n,n}} H(s,t)(t-t_{n,n-1})dt, \end{aligned}$$

and for $1 \leqslant j \leqslant n-1$,

$$w_{n,j}(s) \quad := \quad \frac{1}{h_n}\int_{t_{n,j-1}}^{t_{n,j}} H(s,t)(t-t_{n,j-1})dt + \frac{1}{h_n}\int_{t_{n,j}}^{t_{n,j+1}} H(s,t)(t_{n,j+1}-t)dt.$$

Hence (2.7) can be rewritten as

$$\begin{aligned} &\varphi_n^{(k+1)}(s) - \sum_{j=0}^{n} w_{n,j}(s)L(s,t_{n,j})f_k(t_{n,j})\varphi_n^{(k+1)}(t_{n,j}) \\ = \quad &K(\varphi_n^{(k)})(s) + y(s) - \sum_{j=0}^{n} w_{n,j}(s)L(s,t_{n,j})f_k(t_{n,j})\varphi_n^{(k)}(t_{n,j}), \end{aligned}$$

where

$$f_k(t_{n,j}) := \frac{\partial F}{\partial u}(t_{n,j}, \varphi_n^{(k)}(t_{n,j})).$$

Setting

$$x_n^{(k+1)}(j) := \varphi_n^{(k+1)}(t_{n,j}),$$

from the evaluations of equation (2.8) at the nodes of the grid, it is straightforward that the vector $x_n^{(k+1)}$ is the solution of the linear system

$$(I - A_n^{(k)})x_n^{(k+1)} = b_n^{(k)}, \tag{2.8}$$

where

$$\begin{aligned} A_n^{(k)}(i,j) &:= w_{n,j}(t_{n,i})L(t_{n,i}, t_{n,j})f_k(t_{n,j}), \\ b_n^{(k)}(i) &:= K(\varphi_n^{(k)})(t_{n,i}) + y(t_{n,i}) - A_n^{(k)}x_n^{(k)}. \end{aligned}$$

Finally, $\varphi_n^{(k+1)}$ is recovered with the following formula :

$$\begin{aligned} \varphi_n^{(k+1)}(s) &= \sum_{j=1}^{n} w_{n,j}(s)L(s,t_{n,j})f_k(t_{n,j})\left(x_n^{(k+1)}(t_{n,j}) - x_n^{(k)}(t_{n,j})\right) \\ &+ K(\varphi_n^{(k)})(s) + y(s). \end{aligned}$$

## 3. Convergence property of the new method

Existence, uniqueness and regularity properties of the solution of equation (1.2) have been already considered (for example by Kaneko, Noren and Xu [8] or Pedas and Vainikko [13]). In this section, we focus on the proof of the convergence of $\varphi_n^{(k)}$ towards $\varphi$ when $k \to \infty$ and whatever $n$ can be.

**Hypotheses:**

**(H0)** $F$, defined in (1.1), is twice continuously differentiable on $[a,b] \times \mathbb{R}$.

**(H1)** $L \in C^0([a,b] \times [a,b], \mathbb{R})$.

**(H2)** $H$ verifies:

**(H2.1)** $c_H := \sup_{s\in[a,b]} \int_a^b |H(s,t)|dt < +\infty$.

**(H2.2)** $\lim_{h\to 0} \omega_H(h) = 0$, where

$$\omega_H(h) := \sup_{|s-\tau|\leqslant h,\, s,\tau\in[a,b]} \int_a^b |H(s,t) - H(\tau,t)|dt.$$

**(H3)** $\varphi \in O$ is an isolated solution of $\varphi - K(\varphi) = y$.

**(H4)** $I - T(\varphi)$ is invertible.

These assumptions ensure that equation (1.2) is uniquely solvable (see [2]).

The existence of the Newton iterates defined by (2.7) depends on the domain of inversibility of the operator

$$x \mapsto (I - T_n(x)).$$

Let us treat this question first.

Let $a > 0$ such that $B(\varphi, a) \subset O$, where $B(\varphi, a)$ denotes the open ball in $C^0([a,b],\mathbb{R})$ centered in $\varphi$ and of radius $a$. Let us define the following constants:

$$M_2(a) \quad := \sup_{t\in[a,b],|u|\leqslant a+\|\varphi\|} |\frac{\partial^2 F}{\partial u^2}(t,u)|. \tag{3.9}$$

$$c_L := \max_{s,t\in[a,b]} |L(s,t)|. \tag{3.10}$$

From the mean value theorem applied to $\dfrac{\partial F}{\partial u}$ we get, for all $x \in B(\varphi, a)$,

$$\begin{aligned} \|T(x) - T(\varphi)\| &\leqslant \|x - \varphi\| \sup_{s\in[a,b]} \int_a^b |H(s,t)||L(s,t)| \sup_{t\in[a,b],|u|\leqslant a+\|\varphi\|} |\frac{\partial^2 F}{\partial u^2}(t,u)|dt \\ &\leqslant c_H c_L M_2(a)\|x - \varphi\|. \end{aligned}$$

We have

$$I - T(x) = (I - T(\varphi))[I + (I - T(\varphi))^{-1}(T(\varphi) - T(x))].$$

Let $0 < r < a$ be such that $r \leqslant \dfrac{1}{2\mu c_H c_L M_2(a)}$. Then, for all $x \in B(\varphi, r)$,

$$\|(I - T(\varphi))^{-1}(T(\varphi) - T(x))\| \leqslant \mu\|T(\varphi) - T(x)\| \leqslant \mu c_H c_L M_2(a) r \leqslant \frac{1}{2},$$

where

$$\mu := \|(I - T(\varphi))^{-1}\|.$$

Then $I - T(x)$ is invertible and its inverse is uniformly bounded on $B(\varphi, r)$. In fact

$$(I - T(x))^{-1} = \left[I + (I - T(\varphi))^{-1}(T(\varphi) - T(x))\right]^{-1}(I - T(\varphi))^{-1},$$

so that

$$\|(I - T(x))^{-1}\| \leqslant \mu \sum_{k=0}^{\infty} \|(I - T(\varphi))^{-1}(T(\varphi) - T(x))\|^k \leqslant 2\mu.$$

The function $(s,t) \mapsto L(s,t)\dfrac{\partial F}{\partial u}(t,x(t))$ is in $C^0([a,b]\times[a,b],\mathbb{R})$. Hence, according to [2] or [1], $T_n(x)$ tends to $T(x)$ pointwise and the sequence $(T_n(x))_{n\geqslant 0}$ is collectively compact. For all $x\in B(\varphi,r)$, $T_n(x)$ is a collectively compact approximation of $T(x)$ (see [1]). Hence for $n$ large enough, $I-T_n(x)$ is invertible and $(I-T_n(x))^{-1}$ is uniformly bounded in $n$. This means that there is a constant $c_x$ such that for $n$ large enough,

$$\|(I-T_n(x))^{-1}\| \leqslant c_x.$$

We just proved that, there exists $r>0$ such that, for all $x\in B(\varphi,r)$, $I-T_n(x)$ is invertible. So that the following operator $A_n$ is defined on $B(\varphi,r)$:

$$A_n(x) := x + (I-T_n(x))^{-1}\,(K(x)+y-x)\,. \tag{3.11}$$

For simplicity, $S_n(x)$ will denote $(I-T_n(x))^{-1}$. Notice that (2.7) is equivalent to

$$\varphi_n^{(k+1)} = A_n(\varphi_n^{(k)})) \tag{3.12}$$

and that the solution $\varphi$ of $\varphi - K(\varphi) = y$ solves

$$\varphi = A_n(\varphi). \tag{3.13}$$

The problem is now a fixed point problem that we will treat with a successive approximations convergence result (see [10] Theorem 2.3. pp 21).

THEOREM 1. *Under assumptions* **(H0) to (H4)**, *there exists $r>0$ such that, for a fixed n large enough to have*

$$\rho_n := \rho\left(A_n'(\varphi)\right) < 1,$$

*and for any $\varepsilon>0$ with $\rho_n+\varepsilon<1$, there exists $B(\varphi,r'_{n,\varepsilon})\subset B(\varphi,r)$ and $B(\varphi,r_{n,\varepsilon})\subset B(\varphi,r)$ such that the sequence $(\varphi_n^{(k)})_{k\geqslant 0}$, with $\varphi_n^{(0)}\in B(\varphi,r'_{n,\varepsilon})$, solution of*

$$\varphi_n^{(k+1)} = A_n(\varphi_n^{(k)})),$$

*is well defined, belongs to $B(\varphi,r_{n,\varepsilon})$ and*

$$\varphi_n^{(k)} \to \varphi \text{ as } k\to\infty.$$

*Moreover, the following estimation holds:*

$$\|\varphi_n^{(k)}-\varphi\| \leqslant r_{n,\varepsilon}\,(\rho_n+\varepsilon)^k\,. \tag{3.14}$$

*Proof.* The operator $A_n$ is Fréchet differentiable at $\varphi$ and

$$A_n'(\varphi) = I - S_n(\varphi)(I-T(\varphi)). \tag{3.15}$$

We have

$$\begin{aligned}\rho\,(I-S_n(\varphi)(I-T(\varphi))) &= \inf_n \|(I-S_n(\varphi)(I-T(\varphi)))^n\|^{\frac{1}{n}}\\ &\leqslant \|(I-S_n(\varphi)(I-T(\varphi)))^2\|^{\frac{1}{2}}.\end{aligned}$$

Since $(I-S_n(\varphi)(I-T(\varphi))) = S_n(\varphi)\,(T(\varphi)-T_n(\varphi))$,

$$\begin{aligned}\|\,(I-S_n(\varphi)(I-T(\varphi)))^2\,\| &= \|S_n(\varphi)\,(T(\varphi)-T_n(\varphi))\,S_n(\varphi)\,(T(\varphi)-T_n(\varphi))\,\|\\ &\leqslant \|S_n(\varphi)\|\|\,(T(\varphi)-T_n(\varphi))\,S_n(\varphi)\,(T(\varphi)-T_n(\varphi))\,\|\\ &\leqslant c_\varphi\|\,(T(\varphi)-T_n(\varphi))\,S_n(\varphi)\,(T(\varphi)-T_n(\varphi))\,\|.\end{aligned}$$

As $S_n(\varphi)$ is uniformly bounded and $(T(\varphi)-T_n(\varphi))$ is collectively compact, the closure of the set $S := \cup_n\{S_n(\varphi)(T(\varphi)-T_n(\varphi))x, \|x\| \leqslant 1\}$ is compact so that

$$\|\,(I-S_n(\varphi)(I-T(\varphi))^2\,\| \leqslant c_\varphi \sup_{x\in S}\|\,(T(\varphi)-T_n(\varphi))\,x\| \to 0 \text{ as } n\to\infty.$$

Then

$$\rho(A'_n(\varphi)) \to 0 \text{ as } n\to\infty$$

Therefore, for $n$ large enough,

$$\rho(A'_n(\varphi)) < 1.$$

The conditions needed to apply the successive approximation result of [10] (Theorem 2.3. pp 21) to the operator $A_n$ are satisfied. We thus get the convergence result. □

## 4. Numerical Evidence

### 4.1. The standard product integration method

The standard product integration approximation $\psi_n$ solves the nonlinear equation

$$\psi_n(s) - \int_a^b H(s,t)\pi_n[L(s,.)F(.,\psi_n(.))](t)dt = y(s),$$

which leads, by collocation at the grid points $t_{n,i},\ i = 0,\dots,n$, to the algebraic nonlinear system

$$\mathsf{X}_n - \mathsf{A}_n\mathsf{F}(\mathsf{X}_n) = \mathsf{Y}_n, \tag{4.16}$$

where

$$\begin{aligned}
\mathsf{Y}_n(i) &:= y(t_{n,i}), \quad \mathsf{A}_n(i,j) := w_{n,j}(t_{n,i})L(t_{n,i},t_{n,j}),\\
\mathsf{X}_n &:= \begin{bmatrix} \psi_n(t_{n,0}) \\ \vdots \\ \psi_n(t_{n,n}) \end{bmatrix},\\
\mathsf{F}(\mathsf{X}_n) &:= \begin{bmatrix} F(t_{n,0},\psi_n(t_{n,0})) \\ \vdots \\ F(t_{n,n},\psi_n(t_{n,n})) \end{bmatrix}.
\end{aligned}$$

This system is solved by the classical finite dimensional Newton method, leading to a sequence $\psi_n^{(k)}$ satisfying

$$\psi_n^{(k)} \to \psi_n \text{ as } k \to \infty.$$

Notice that these Newton iterates tend to $\psi_n$ and not to the exact solution $\varphi$. It means that the accuracy of the standard solution is limited by $n$ which is not the case for our method.

### 4.2. Numerical Illustration

Numerical experiments are now carried out to illustrate the accuracy of our method. Let us consider in $\mathcal{X} := \mathcal{C}^0([0,1],\mathbb{R})$ the operator

$$K(\varphi)(s) := \int_0^1 \kappa(s,t,\varphi(t))dt, \quad \varphi \in \mathcal{O} \subseteq \mathcal{X},\ s \in [0,1],$$

with the real valued kernel function $\kappa$ :

$$(s,t,u) \in [0,1]\times[0,1]\times\mathbb{R} \mapsto \kappa(s,t,u) := \log(|s-t|)\sin(\pi u)$$

and the right hand side is $y := 1$.

The standard product integration method can provide accurate solutions whenever the grid size is fine enough. Indeed, for large values of $n$, $\psi_n$ is sufficiently close to the exact solution $\varphi$. In order to illustrate the benefit of our new approach, a coarse grid must be considered. To address this point, a small grid of size $n = 10$ is chosen.

Table 1 shows the convergence history in terms of relative errors, for the standard and the new methods. With such a small grid the new proposed approach can reach almost machine precision within a small number of iterations.

The disadvantage of our method is its computational cost. It is worth mentioning that the computation of $K(\varphi_n^{(k)})$ is not easy due to its singularity and it has to be approximate at each iteration. Implementation details are however hidden from the presentation but include a careful treatment of $K(\varphi_n)$. The handling of this term deserves however further research, which may include different evaluation strategies according to the accuracy of the iterate.

Table 1: Convergence history for the standard method and the new approach

| | relative error | |
|---|---|---|
| iteration | standard | new |
| 1 | $3.89 \times 10^{-01}$ | $4.24 \times 10^{-01}$ |
| 2 | $3.89 \times 10^{-01}$ | $1.72 \times 10^{-01}$ |
| 3 | $3.87 \times 10^{-01}$ | $5.43 \times 10^{-02}$ |
| 4 | $3.87 \times 10^{-01}$ | $1.16 \times 10^{-02}$ |
| 5 | $3.85 \times 10^{-01}$ | $4.06 \times 10^{-03}$ |
| 6 | $3.74 \times 10^{-01}$ | $1.83 \times 10^{-07}$ |
| 7 | $3.19 \times 10^{-01}$ | $2.82 \times 10^{-15}$ |
| 8 | $1.48 \times 10^{-01}$ | – |
| 9 | $9.77 \times 10^{-03}$ | – |

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