U.PORTO
FACULDADE DE CIÊNCIAS
UNIVERSIDADE DO PORTO

Summation and Transformation formulas related with Special Functions

Pedro Manuel Macedo Ribeiro
Mestrado em Matemática
Departamento de Matemática
2019-2020

Orientador
Semyon Borisovich Yakubovich, Professor Associado,
Faculdade de Ciências da Universidade do Porto

U.PORTO

FACULDADE DE CIÊNCIAS
UNIVERSIDADE DO PORTO

Todas as correções determinadas pelo júri, e só essas, foram efetuadas.

O Presidente do Júri,

Porto, ______/______/_________

MSc

Agradecimentos

Aos meus pais, irmãos e avós Ritinha e Macedo, que me acompanharam e apoiaram desde o primeiro momento em que comecei a rabiscar uns "estranhos símbolos" nos cadernos da escola. Espero que esta tese reflita, de certa forma, os valores de trabalho e dedicação que sempre tentaram incutir em mim.

Aos meus amigos que, na jornada da Física, tornaram o meu percurso infinitamente mais fácil. Ao meu amigo Slav, por ter cunhado na minha memória um turbilhão de aventuras e conversas desde os primeiros tempos da Física. Aos meus amigos Duarte e João por todas as discussões de Mecânica Quântica e Física Estatística e por todos os fados e guitarradas. Ao meu amigo Francisco (Cho), por ter sido sempre a minha grande companhia desde os imemoráveis "tempos da Laje". À Ana e ao Olavo, pela amizade e paciência para com as minhas excentricidades combinadas com as do Francisco. Pela sanidade e companhia numa casa muitas vezes tomada pela "loucura".

Tendo sido um nómada de cursos e locais, gostaria de agradecer a todas as pessoas que me fizeram sentir bem-vindo em todos estes: ao Leandro, pelo companheirismo durante a Licenciatura em Matemática. Aos meus colegas de mestrado Jorge e Cláudio, pela bondade, companhia e constante prontidão em trocar informações e conhecimentos. Ao Nuno, ao Monsenhor e ao Dr. Cimini por terem criado em Roma uma colónia portuguesa chamada casa.

Ao meu orientador, o Professor Semyon Yakubovich, por toda a disponibilidade e ajudas indispensáveis em muitos pontos desta tese. Por todas as histórias que me contou e pelo encorajamento a elas sempre inerente.

Por último, à Minha Sofi, por estar sempre presente em todos os momentos, mesmo nos mais difíceis. Por ser a minha melhor amiga e por, em todos os segundos, fazer sempre jus ao seu carinhoso nome "Pirilampo".

Abstract/Resumo

Abstract (English)

In this thesis we study summation formulas of Poisson, Voronoï, and Hardy-Landau type associated with Dirichlet characters.

The developed methods are extensively employed to study further details of the analytic continuation of certain Dirichlet series, such as Dirichlet's $L-$functions and Epstein's $\zeta-$function.

Key-words: Summation formulas; Dirichlet series; Arithmetic functions; Dirichlet Characters; Integral Transforms; Bessel functions.

Resumo (Português)

Nesta tese estudamos fórmulas de soma do tipo Poisson, Voronoï e Hardy-Landau associadas a caráteres de Dirichlet.

Os métodos desenvolvidos são amplamente usados no estudo de propriedades do prolongamento analítico de certas séries de Dirichlet, tais como as funções L de Dirichlet e a função zeta de Epstein.

Palavras-chave: Fórmulas de Soma; Séries de Dirichlet; Funções Aritméticas; Caráter de Dirichlet; Transformada Integrais; Funções de Bessel.

Contents

Notation and Conventions

Because of the amount of different definitions and notations used throughout the text, we thought it was better to include a small glossary containing the most important definitions used during this thesis, from the arithmetical functions used to the important special functions.

Sets and Functions

$\mathbb{H}$	complex upper half-plane $= \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$
ℓ	usually used to denote the modulus of a Dirichlet character χ
τ	generally denoting an element of $\mathbb{H}$
Q	binary and positive definite quadratic form, $Q(x, y) := ax^2 + bxy + cy^2$
Q_0	binary quadratic form associated with the identity matrix $Q_0(m, n) := x^2 + y^2$
Q_1	binary quadratic form associated with $\chi_3(n)$, $Q_1(m, n) := x^2 + xy + y^2$
Q_{ℓ_1}	binary and positive-definite quadratic form defined by $Q_{\ell_1}(x, y) := Q\left(\frac{x}{\ell_1}, y\right)$
d	usually used to denote the discriminant of a binary quadratic form Q, $d := b^2 - 4ac$
$\sum_{n \leq x} f(n)$	A sum taken over all natural numbers not exceeding x
$\sum_{n \leq x}' f(n)$	if x is an integer, the last term is counted as $\frac{f(x)}{2}$

Functional Aspects

$f(x) = O(g(x))$	there exists a positive constant C such that $\lvert f(x)\rvert \leq C\, g(x)$
$f(x) \sim g(x)$	$\lim_{x \to \infty} f(x)/g(x) = C$
l.i.m.	limit taken with respect to the norm in $L_p(\mathbb{R}_+)$, $1 \leq p < \infty$. We will take $p = 2$
$f(x) \overset{\mu}{=} g(x)$	$f(x) = g(x) \quad \mu -$ a.e., where μ is the Lebesgue measure in $\mathbb{R}_+$

$C^n(\Omega)$	set of functions $f : \Omega \longmapsto \mathbb{R}$ whose first n derivatives exist and are continuous
$L_p(\sigma)$	Space of complex functions which belong to $L_p(\sigma - i\infty,\, \sigma + i\infty)$
$\mathcal{M}_{\alpha,n}$	Class of functions of Müntz - type

Integral Transforms

$f^*(s)$	Mellin transform of f, both in L_1 and L_2
$\hat{f}(x)$	Complex Fourier transform $\hat{f}(x) = \int_{-\infty}^{\infty} f(y)\, e^{-ixy} dy$
$g(x)$	Integral transform associated to the even character modulo ℓ
$h(x)$	Integral transform associated with the odd character modulo ℓ
$\mathcal{L}f(x)$	Laplace transform of f, $\mathcal{L}f(x) = \int_0^{\infty} f(y)\, e^{-xy} dy$
$K_{i\tau}[f]$	Kontorovich - Lebedev transform, $K_{i\tau}[f] = \int_0^{\infty} K_{i\tau}(x)\, f(x)\, dx$

Bessel Functions

$J_\nu(z)$	Bessel function of the first kind
$Y_\nu(z)$	Bessel function of the second kind (Neumann's function)
$I_\nu(z)$	Modified Bessel function of the first kind
$K_\nu(z)$	Modified Bessel function of the second kind
$H_\nu(z)$	Struve function of the first kind
$L_\nu(z)$	Modified Struve function (of the first kind)
$M_\nu(z)$	Modified Struve function of the second kind

Arithmetic functions

$\chi(n)$	Dirichlet character (denoted always with modulus ℓ)
$G(z,\chi)$	Gauss sum $\sum_{r=1}^{\ell-1} \chi(r)\, e^{2\pi i r z/\ell}$
$G(\chi)$	$G(1,\chi)$
χ_0	The trivial character modulo 1, i.e., $\chi_0(n) = 1$ for all $n \in \mathbb{Z}$
χ_1	principal Dirichlet character

χ_3	the primitive character modulo 3
χ_4	the primitive character modulo 4
$\Lambda_\chi(x)$	character counting function $\sum'_{n\leq x}\chi(n)$
$d \mid n$, $d \nmid n$	d divides or does not divide n
(m, n)	the greatest common divisor of m and n
$[x]$	denotes the integer part of x (floor function)
$\lceil x \rceil$	ceiling function of x
$\varphi(n)$	Euler's totient function $\varphi(n) = \#\{m \leq n : (m,n) = 1\}$
$p(n)$	Partition function
$d(n)$	the divisor function $d(n) = \sum_{d\mid n} 1$
$d_\chi(n)$	the character analogue of the divisor function $d_\chi(n) = \sum_{d\mid n}\chi(d)$
$\prod_p$	denotes the infinite product over the set of prime numbers
$\sigma_a(n)$	generalized divisor function $\sigma_a(n) = \sum_{d\mid n} d^a$
$\sigma_{a,\chi}(n)$	generalized weighted divisor function $\sigma_{a,\chi}(n) = \sum_{d\mid n}\chi(d)\, d^a$
$\sigma_a(n,\chi_1,\chi_2)$	generalized double-weighted divisor function $\sigma_a(n,\chi_1,\chi_2) = \sum_{d\mid n}\chi_1(d)\,\chi_2\,(n/d)\,d^a$
$r_k(n)$	sum of k squares function $\#\left\{(m_1,...,m_k) \in \mathbb{Z}^k : n = m_1^2 + ... + m_k^2\right\}$
$r_Q(n)$	$\#\left\{(m_1,m_2) \in \mathbb{Z}^2 : n = Q(m_1,m_2)\right\}$, with Q being a quadratic form.
$(a \star b)\,(n)$	Dirichlet convolution of arithmetic functions $\sum_{d\mid n} a(d)\, b\,(n/d)$
$\Delta(x)$	error term in the Dirichlet divisor problem (3^{rd} chapter)
$\Delta_\chi(x)$	error term in the character version of the divisor problem (3^{rd} chapter)
$\Delta_Q(x)$	error term in the generalized circle problem (4^{th} chapter)
$h_\chi(x)$	$\Lambda_\chi(x)/x$ (2^{nd} chapter) and $\Delta_\chi(x)/x$ (3^{rd} chapter)
$h_Q(x)$	$\Delta_Q(x)/x$ (4^{th} chapter)

Classical constants and Higher Transcendental functions

$\Gamma(s)$	Euler's $\Gamma-$function $\int_0^\infty x^{s-1}e^{-x}dx$ for $\mathrm{Re}(s)>0$
γ	Euler - Mascheroni constant $\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log(n)\right) = 0.5772157...$
$\gamma_n(a)$	Stieltjes constant
B_n	Bernoulli number of order n
$B_{n,\chi}$	character analogue of B_n
$\psi(s)$	Euler's digamma function, $\psi(s)=\Gamma^{'}(s)/\Gamma(s)$ when the argument is s. Denotes Jacobi's $\psi-$function in Example 2 of the third chapter with argument z
$\zeta(s,a)$	Hurwitz $\zeta-$function
$\zeta(s)$	Riemann's $\zeta-$function
$L(s,\chi)$	Dirichlet $L-$function
$L(s,\chi,a)$	Character Analogue of $\zeta(s,a)$
$Z_2(s,\,Q)$	Epstein's $\zeta-$function associated with the positive-definite binary quadratic form Q
$Z_2(s,\,Q,\,\chi)$	Character analogue of Epstein's $\zeta-$function
$\Phi(s)$	Riemann's symmetric function $\Phi(s)=\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\,\zeta(s)$
$\Phi_2(s)$	Analogue of Riemann's symmetric function for $\zeta_2(s)$, $\Phi_2(s) = \pi^{-s}\Gamma(s)\,\zeta_2(s)$
$A(s)$	usually denoting the Dirichlet series associated with the arithmetic function $a(n)$, $A(s):=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}$.
$\eta(\tau)$	Dedekind $\eta-$function.
$\eta_\chi(\tau)$	Character analogue of Dedekind $\eta-$function
$G_k(\tau)$	Eisenstein series of order k
$G_k(\tau,\,\chi)$	Character analogue of $G_k(\tau)$
$\mathbf{Ei}(x)$	Exponential integral function $:= \int_{-\infty}^{x}\frac{e^y}{y}\,dy$

Introduction

The present thesis deals with classical summation formulas and with the existence of character analogues for these. Informally, a summation formula is a connection between an infinite series of the type

$$\sum_{n=0}^{\infty} a(n)\, f(n), \tag{1}$$

where $a(n)$ is a suitable sequence/arithmetic function allowing the convergence of the infinite series (1), with a corresponding infinite series of the type

$$\sum_{n=0}^{\infty} b(n) \int_0^{\infty} f(y)\, K(ny)\, dy, \tag{2}$$

where $K(x)$ is a suitable kernel, which will be specified later.

Of course, the most famous summation formula in the form

$$\sum_{n=0}^{\infty} a(n)\, f(n) = \sum_{n=0}^{\infty} b(n) \int_0^{\infty} f(y)\, K(ny)\, dy \tag{3}$$

is due to Poisson and achieved if $a(0) = \frac{1}{2}$ and $a(n) = 1$, $n \geq 1$, $b(0) = 1$, $b(n) = 2$, $n \geq 1$. With these substitutions, one obtains the recognizable summation formula, for the kernel $K(x) = \cos(2\pi nx)$,

$$\frac{1}{2} f(0) + \sum_{n=1}^{\infty} f(n) = \int_0^{\infty} f(y)\, dy + 2 \sum_{n=1}^{\infty} \int_0^{\infty} f(y)\, \cos(2\pi ny)\, dy. \tag{4}$$

Since Dirichlet's proof of Poisson's formula via Fourier series [36], it is also known that, if f is a continuous function and of bounded variation on $[a, b]$, a finite version of Poisson's summation formula (4)

$$\sum_{n=a}^{b}{}' f(n) = \int_a^b f(y)\, dy + 2 \sum_{n=1}^{\infty} \int_a^b f(y)\, \cos(2\pi n\, y)\, dy \tag{5}$$

holds. In (5), the prime on the summation sign at the left indicates that, if a or b is an integer, then only $\frac{1}{2}\, f(a)$ or $\frac{1}{2}\, f(b)$ is counted in the sum.

Imposing conditions for the validity of (4) is a relatively studied subject as there are several ways of approaching Poisson's formula through different methods. The first one was given by Dirichlet in the form (5) [36] and it consists in seeing (4) as an expansion of a periodic function into Fourier series.

Another approach due to Cauchy [53] uses Complex Analysis and consists in assuming that $f(x)$ extends to $\mathbb{C}$ as a complex-analytic functions. This assumption means that the meromorphic function $g(x) = \pi \cot(\pi\, z)\, f(z)$ has simple poles located at the integers and an immediate application of the Residue Theorem [94] gives (4).

These are, however, very special cases and that can be treated under very strict conditions. One may ask under which conditions a summation formula of the type (3) holds and how the kernel $K(x)$, present in the integral transform, depends on the coefficients $a(n)$ and $b(n)$. This is a long standing problem, which we shall study into the light of Dirichlet series and for particular arithmetic functions.

In 1904, Voronoï [20] made the following conjecture, similar to the one informally exposed above: if $a(n)$ is an arithmetical function and if f is continuous on $(a,\, b)$ with only a finite number of maxima and minima there, then there exist analytic functions $\delta(x)$ and $K(x)$, depending only on $a(n)$, such that a generalized form of (5) holds, i.e.,

$$\sum_{n=a}^{b}{}' a(n)\, f(n) = \int_a^b f(y)\, \delta(y)\, dy + \sum_{n=1}^{\infty} a(n) \int_a^b f(y)\, K(ny)\, dy. \tag{6}$$

After considerable efforts, Voronoï was able to prove that his conjecture was true when the arithmetic function $a(n)$ is the classic divisor function $d(n)$. In this case, the associated functions $\delta(x)$ and $K(x)$ are given by

$$\delta(x) = \log(x) + 2\gamma,$$

where γ denotes Euler-Mascheroni's constant [45], and $K(x)$ denotes the kernel

$$K(x) = 4\, K_0\left(4\pi\sqrt{x}\right) - 2\pi\, Y_0\left(4\pi\sqrt{x}\right),$$

composed by Bessel functions of the second kind [106]. This special case gives the interesting formula

$$\sum_{n=a}^{b}{}' d(n)\, f(n) = \int_a^b f(y)\, \left(\log(y) + 2\gamma\right)\, dy + \sum_{n=1}^{\infty} d(n) \int_a^b f(y)\, \left[4\, K_0(4\pi\sqrt{ny}) - 2\pi\, Y_0(4\pi\sqrt{ny})\right]\, dy, \tag{7}$$

usually known as Voronoï's summation formula.

Formula (7) was later proved by Koshliakov (1928) [20] with the assumption of f being analytic. After Koshliakov's proof, several other arguments began to appear, all of them assuming different conditions over f. A. L. Dixon and W. L. Ferrar (1931) [42] gave a proof of (7) under the condition that $f \in C^2[a, b]$ and Wilton (1932) [110] extended it to the case $b = \infty$.

After proving his conjecture for the arithmetic function $d(n)$, Voronoï also announced a corresponding result for another arithmetic function, $r_2(n)$ [20]. This function counts the number of ways in which a given positive integer n can be expressed as a sum of two squared integers [60]. Voronoï presented the following formula, analogous to (7),

$$\sum_{n=a}^{b}{}' r_2(n)\, f(n) = \pi \int_a^b f(y)\, dy + \pi \sum_{n=1}^{\infty} r_2(n) \int_a^b f(y)\, J_0\left(2\pi\sqrt{ny}\right)\, dy, \tag{8}$$

where J_ν denotes the Bessel function of the first kind [106]. Another version of (8) was also established by Sierpiński and Laudau [20] for functions of bounded variation. An extension to $b = \infty$, invoking additional conditions on the decay of f, was made by Dixon and Ferrar [40]. Due to the generality presented in the works of Hardy and Landau concerning the circle problem [28], formula (8) is usually known as "Hardy-Landau Summation formula" although some authors (see [82] for instance) use the designation "Sierpiński's formula".

As itemized by Berndt [20], all the hypothesis under the proofs of Voronoï's formula generally fall in three classes. The first is the class of smooth functions, generally $f \in C^1[a, b]$ or $f \in C^2[a, b]$. Secondly, the finite versions of (7) are usually taken into consideration for functions of bounded variation on $[a, b]$.

Lastly, a third approach uses the theory of functions in $L_2(\mathbb{R}_+)$ and the theory of Mellin and Fourier transforms for this class. This approach was proposed for the first time by Ferrar [51] and Guinand [54]. Following this theory, other papers by Nasim [80] and Pearson [83] were given and employed these methods to reprove Voronoï's and Hardy-Landau summation formulas.

No matter what the conditions imposed over the functions are, these are only required in order to assure the convergence of the integral transform with kernel K given in (6), as well as the infinite series involving it.

The main purpose of Voronoï's conjecture was to study the interdependence between the arithmetic function $a(n)$ and the kernels $\delta(x)$ and $K(x)$.

The first step taken into a greater generality is due to Ferrar [50] (1935-1937), who proved that the kernel $K(x)$ owes its behavior to the functional equation for the analytic continuation of the Dirichlet series

$$A(s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s},\quad \mathrm{Re}(s)>\sigma_A.$$

Ferrar also pointed out one intriguing fact concerning the summation formulas. He observed, from the point of view of integral transforms [82], that the kernels lying in each of the summation formulas proved by Voronoï satisfy

$$g(x)=\int_0^{\infty}f(y)\,K(x\,y)\,dy, \tag{9}$$

$$f(x)=c\int_0^{\infty}g(y)\,K(x\,y)\,dy \tag{10}$$

where c is some normalization constant. The kernel $K(x)$ is usual called a Fourier kernel or Fourier-Watson kernel [102].

This striking connection between the summation formulas involving the arithmetic functions $a(n)$ and $b(n)$ and their respective Dirichlet series is the main motivation for the study developed in this thesis.

This Thesis:

In light of the theory of Dirichlet series and the analytic continuation of these, as well as the reciprocity lying in the integral transforms above, this thesis is devoted to study some extensions of the classical formulas (4), (7) and (8) mainly based on the classical theory of Fourier and Mellin transforms, as well as the theory of Dirichlet series.

This study involves the extension of the usual arithmetic functions, $a(n)=1$, $d(n)$, $r_2(n)$ to a somewhat generalized version of these involving Dirichlet characters. We shall call to these modified arithmetic functions by the name "character analogues", although no general definition is given for an "analogue" throughout our work.

Character Analogues:

The study of Dirichlet series and characters began with Dirichlet's work on the distribution of primes in arithmetic progressions. In his work, Dirichlet introduced the $L-$function as a necessary tool to approach the problem. His methods were essential to give the first description of a Dirichlet character as the coefficient of an important Dirichlet series

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}.$$

Informally speaking, given a positive integer ℓ, a Dirichlet character χ (modulo ℓ) is a function $\chi:\,\mathbb{Z}\longmapsto\mathbb{C}$ which is completely multiplicative, periodic (with period ℓ) and satisfying the property $\chi(n)=0$ iff $(n,\,\ell)>1$.

Associated with the study of Dirichlet characters are the periodic sums of the form

$$\sum_{r=1}^{\ell-1}\chi(r)\,e^{2\pi irn/\ell}:=G(n,\chi),$$

known as Gauss sums [5]. The arithmetic and analytic properties of sums of this type play an essential role in the most part of the computations given in the main text.

During our developments, we shall furnish several character analogues: some of them are

- $$\eta(\tau)\longmapsto\eta_{\chi}(\tau),\;\;2^{\text{nd}}\text{ chapter}$$

- $$\sum_{d|n}1=d(n)\longmapsto d_{\chi}(n)=\sum_{d|n}\chi(d),\;\;3^{\text{rd}}\text{ chapter}$$

- $$\sum_{d|n}d^a=\sigma_a(n)\;\;\longmapsto\;\;\sigma_{a,\chi}(n)=\sum_{d|n}\chi(d)\,d^a,\;\;4^{\text{th}}\text{ chapter.}$$

Furthermore, the proofs presented depend on the functional equations for the Dirichlet series associated to each of the extended arithmetic functions above-mentioned, as well as the asymptotic behavior of their analytic continuation near the critical line (chapters 2 and 3) or in some suitable regions of the complex plane (chapter 4).

The Classes of Functions:

In this thesis we consider two classes of functions which will be studied in order to assure the summation formulas.

The first class (which we call 'L_2 class'), firstly considered by Guinand [53], Nasim [80] and Yakubovich [118, 119], is composed by functions which are absolutely continuous on $\mathbb{R}_+$ and whose Mellin transform, $f^*(s)$, satisfies

$$\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}|s\,f^*(s)|^2\,|ds|<\infty. \tag{11}$$

In the second and third chapters of this thesis, all functions will belong to this class.

The second type of functions was introduced for the first time by Yakubovich [116] and was baptized with the name "Müntz class". Essentially [73], for $n \geq 2$, a function of Müntz-type is a function belonging to $C^n\left(\mathbb{R}_0^+\right)$ which decays, as well as its first n derivatives, in the form $f^{(k)}(x) = O\left(x^{-\alpha-k}\right)$, for $\alpha > 1$ and $x \to \infty$.

This class will be fundamental in the fourth chapter of this thesis.

The Structure of the Text:

- The first chapter gives a brief description of the concept of Dirichlet characters. Since we are mainly focused on their analytic properties and with their relation with summation formulas, this characterization will be made enlightening the importance of their primitivity. Although there are several ways to describe primitive Dirichlet characters and to study these under different viewpoints, we are mainly interested in their property of "splitting" the Gauss sum, i.e., in the equality

$$\sum_{r=1}^{\ell-1} \chi(r)\, e^{2\pi i r n/\ell} = \overline{\chi}(n) \sum_{r=1}^{\ell-1} \chi(r)\, e^{2\pi i r/\ell}, \quad \forall\, n \in \mathbb{N},$$

  which they satisfy. There, we also set up the main tools and ideas used throughout this thesis. The ultimate goal in this chapter is to prove, via a relatively unknown formula due to Plana [86], a functional equation for the analytic continuation of the Hurwitz $\zeta-$function

$$\zeta(s,a) = \sum_{n=1}^{\infty} \frac{1}{(n+a)^s}, \quad 0 < a \leq 1, \quad \text{Re}(s) > 1.$$

- The second chapter is devoted to the use of Plancherel's theory for the Mellin and Fourier transforms and aims to prove Poisson's summation formula (3) for the class of functions with the property (11). Moreover, our goal is to extend (3) for the coefficients $a(n) = \chi(n)$ and $b(n) = \overline{\chi}(n)$. We arrive to a formula proved by Guinand for the first time [53], but the methods employed by us are more general and can be adapted to prove other summation formulas under the same conditions. We also establish several examples, which allow to extend some well-known formulas to versions involving Dirichlet characters.

- In the third chapter we repeat some of the arguments given at the second, pursuing a proof of Voronoï's summation formula (with $a = 0$, $b = \infty$) and a character extension of it, when $a(n) = d_{\chi}(n)$ and $b(n) = d_{\overline{\chi}}(n)$. Our argument adapts Yakubovich's [118]. However, the conditions imposed over f will be, at first, different from those required by the condition (11). We do this on purpose, intentionally proving a weaker version of the desired summation formula in order to emphasize the importance of the behavior of the divisor functions $d(n)$ and $d_{\chi}(n)$. At the end of this chapter, we claim an estimate, also proved by Voronoï for the first time [31], which allows to prove Voronoï's summation formula for our first class of functions.

- In the fourth chapter we continue our study using as reference another function described by a Dirichlet series,
$$Z_2(s,Q)=\sum_{n=1}^{\infty}\frac{r_Q(n)}{n^s}, \tag{12}$$
where Q is a positive definite quadratic form and $r_Q(n)$ counts the number of ways in which we can express n ad $Q(a,b)$ for a, $b \in \mathbb{Z}$. The series (12), absolutely convergent for $\text{Re}(s)>1$, is usually known as Epstein's $\zeta-$function [21]. However, the type of summation formulas proved there are only valid for the class of functions of Müntz-type. We also rederive some examples given in the second and third chapters via a more general method.

- The fifth chapter is composed of two interesting applications of the work developed throughout the thesis. The first one is a proof that all Dirichlet $L-$functions attached to a non-principal and real character χ do not vanish at $s=1$. The second application consists in a modification of an argument due to Deuring [38] to prove the infinitude of zeros of $\zeta(s)$ at the critical line (Hardy's Theorem). Our proof invokes different methods which are based on results proved at the fourth chapter. We also deduce some results related with theorems of Hardy-type for Epstein $\zeta-$functions of higher orders.

We remark also that we wrote two supplementary notes [89] to this thesis. These contain some secondary results derived by us and additional details regarding specific examples of the second and third chapters.

In the first set of supplementary notes, we provide a proof we furnish some details regarding Example 2.6 and 2.7 of the second chapter.

The second set of supplementary notes is a direct proof, although with all necessary details exposed, of an identity due to Voronoï (see eq. (3.125), third chapter), which is essential for the understanding a strong estimate for Dirichlet's divisor problem [31].

Chapter 1

Chapter I: Preliminary study and the analytic continuations of $\zeta(s)$ and $L(s,\chi)$

In this chapter we introduce the main results that will guide this thesis. We start with the definition of Dirichlet characters and remark some of their useful features, enlightening also the importance of their primitivity in order to establish forthcoming Theorems.

Next, we introduce briefly the usual series representations for $\zeta(s)$, $L(s,\chi)$ and $\zeta(s,a)$ and study their analytic continuation via the Abel-Plana (summation) formula, providing an interesting integral representation dating back to Hermite's work [107].

As a corollary of this representation, we derive a new proof of the functional equation for $\zeta(s,a)$, the Hurwitz $\zeta-$function, from which both functional equations for $\zeta(s)$ and $L(s,\chi)$ are obtained.

We finish this chapter with some important identities for the derivatives of $L(s,\chi)$ which will be essential in future computations.

1.1 Preliminary results - Part I (Dirichlet characters and Gauss sums)

In this first section we briefly define the concept of Dirichlet character and explore some of its remarkable properties. We explore the importance of primitivity of these mathematical entities to allow further considerations in this thesis.

Definition 1.1. (Dirichlet character): Let ℓ be any positive integer. We say that χ is a Dirichlet character modulo ℓ if it is a function $\chi:\ \mathbb{Z} \longmapsto \mathbb{C}$ having the following properties

1. $\chi(1) = 1$.

2. $\chi(m \cdot n) = \chi(m) \cdot \chi(n)$ for all $m,\, n \in \mathbb{Z}$.

3. $\chi(n + \ell) = \chi(n)$ for all $n \in \mathbb{Z}$.

4. $\chi(n) = 0$ iff $(n,\, \ell) > 1$.

Properties 1. and 2. establish that every Dirichlet character is a completely multiplicative arithmetic function, a property which will be very important in the upcoming developments.

Property 3. says that χ is a periodic function having ℓ as period. Thus, if $a \equiv b \mod \ell$ then $\chi(a) = \chi(b)$, which proves that, if $k \equiv 1 \mod \ell$ then $\chi(k) = \chi(1) = 1$ by property 1.

Also, Fermat-Euler's theorem [5] states that, if $(a,\, \ell) = 1$, then

$$a^{\varphi(\ell)} \equiv 1 \mod \ell, \tag{1.1}$$

where $\varphi(\ell)$ denotes Euler's φ - function (see definition at the Glossary),

$$\varphi(n) := \#\left\{k \in \mathbb{N}:\ k \leq n \text{ and } (k, \ell) = 1\right\}. \tag{1.2}$$

From (1.1), and the properties 1. and 2. and 3. of χ, we can see that

$$1 = \chi(1) = \chi\left(a^{\varphi(\ell)}\right) = \chi(a)^{\varphi(\ell)}.$$

Thus, if $(a, \ell) = 1$, we see that $\chi(a)$ is a $\varphi(\ell)-$th root of unity, i.e., for some $0 \leq k < \varphi(\ell)$, $\chi(a) = e^{2\pi i k/\varphi(\ell)}$.

In what follows, we provide some useful definitions and elementary examples of Dirichlet characters following Apostol's exposition [5].

Definition 1.2.: Trivial and principal character A character χ is called trivial if it has period 1. Evidently, this means that $\chi(n) = 1$ for all $n \in \mathbb{Z}$. Usually, we denote this character by χ_0 and it will be sometimes referred in Chapters 2 and 3 as the case in which our summation formula will depend on Riemann's $\zeta-$function.

A character χ is called principal if it is given by

$$\chi_1(n) = \begin{cases} 1 & (n, \ell) = 1 \\ 0 & (n, \ell) > 1. \end{cases} \tag{1.3}$$

Given its definition (1.3), it is immediate to see that, for each $\ell \in \mathbb{N}$, there is exactly one principal character having ℓ as its modulus.

Whenever a character χ does not have the property (1.3), we call it "nonprincipal character".

From properties 1. and 2., it is easily seen that $\chi(-1) = \pm 1$. During this thesis we will study how the value taken by χ at -1 afftects the summation formulas involving χ.

Such a vital quality of χ can be described by the following definition.

Definition 1.3.: the sign of a character Let χ be a Dirichlet character. If:

1. $\chi(-1) = 1$, then we say that χ is an even character.
2. $\chi(-1) = -1$, then we say that χ is an odd character.

It is immediate to see that, for any $\ell \in \mathbb{N}$, the principal character modulo ℓ, χ_1, is even: since, for any ℓ, $(\ell - 1, \ell) = 1$, then, from (1.3) and property 3., $\chi_1(-1) = \chi_1(\ell - 1) = 1$.

In fact, this simple property that principal characters have will be fundamental in, for example, deriving a limit formula of Kronecker-type for a modified Epstein $\zeta-$function (to be defined later, on the fourth chapter).

Of course, there is a correspondence between Dirichlet characters modulo ℓ and the group characters on $(\mathbb{Z}/\ell\mathbb{Z})^*$ (see Apostol's book [5] for details). A result that follows immediately from this correspondence is the following one

Proposition 1.1.: There are exactly $\varphi(\ell)$ Dirichlet characters modulo ℓ, where $\varphi(\ell)$ denotes Euler's totient function (1.2).

Example 1.A:

Let $\ell = 3$ and $\ell = 4$. Then there are $\varphi(3) = \varphi(4) = 2$ Dirichlet characters modulo 3 and 4 respectively.

Therefore, for each $0 \leq n \leq 3$, $\chi(n)^{\varphi(3)} = \chi(n)^2 = 1$, so that every Dirichlet character modulo 3 (resp. 4) only takes the values 1 and -1.

Since we have a principal character in each of these cases, it is not hard to find that the unique nonprincipal characters modulo 3 and 4 are the following

$$\chi_3(n) = \begin{cases} 1 & n \equiv 1 \bmod 3 \\ -1 & n \equiv 2 \bmod 3 \end{cases}, \quad \chi_4(n) = \begin{cases} 1 & n \equiv 1 \bmod 4 \\ -1 & n \equiv 3 \bmod 4. \end{cases} \tag{1.4}$$

From the correspondence between Dirichlet characters and the group characters on $(\mathbb{Z}/\ell\mathbb{Z})^*$ above-mentioned, we can easily see that the orthogonality relations for an Abelian group can be translated immediately in the following relations for Dirichlet characters,

Proposition 1.2.: (Orthogonality relation for Dirichlet characters) Let $\ell \in \mathbb{N}$ and χ_a and χ_b be two Dirichlet characters modulo ℓ. Then the orthogonality relation holds

$$\sum_{r=1}^{\ell-1} \chi_a(r)\,\overline{\chi}_b(r) = \begin{cases} 0 & \text{if } \chi_a \neq \chi_b \\ \varphi(\ell) & \text{if } \chi_a = \chi_b. \end{cases} \tag{1.5}$$

An immediate consequence of (1.5) is obtained once we take $\chi_b = \chi_1$, where χ_1 is defined by (1.3). This immediately gives

$$\sum_{r=1}^{\ell-1} \chi(r) = \begin{cases} 0 & \text{if } \chi \neq \chi_1 \\ \varphi(\ell) & \text{if } \chi = \chi_1. \end{cases} \tag{1.6}$$

For a detailed proof of the orthogonality relation (1.5), the reader is invited to consult [5].

As we shall see in this chapter, (1.6) is fundamental to study the analytic continuation of the Dirichlet series (from now on called Dirichlet $L-$function),

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \qquad \mathrm{Re}(s) > 1 \tag{1.7}$$

as either an entire complex function or a meromorphic one. To do so, we still need to introduce the concepts of Gauss sum and primitive characters, which will allow us to describe a suitable functional equation for (1.7).

Definition 1.4. (Gauss sum): For any Dirichlet character χ modulo $\ell \geq 2$, the sum

$$G(n,\chi) = \sum_{r=1}^{\ell-1} \chi(r)\, e^{2\pi i r n/\ell} \tag{1.8}$$

is called the Gauss sum associated with χ. If $\ell = 1$, we simply put $G(n,\chi_0) = 1$, so this case won't be considered during our work. When $n = 1$, we denote $G(n,\chi)$ by $G(\chi)$, i.e.,

$$G(\chi) := \sum_{r=1}^{\ell-1} \chi(r)\, e^{2\pi i r/\ell}. \tag{1.9}$$

We say that a Gauss sum $G(n,\chi)$ is separable if $G(n,\chi) = \overline{\chi}(n)\, G(\chi)$. This property has significant consequences, as we shall see throughout this thesis. The following proposition, whose proof may be found on page 165 of [5] gives a sufficient condition for a given Gauss sum $G(n,\chi)$ to be separable.

Proposition 1.3.: If χ is any Dirichlet character modulo ℓ and if $(n,\ell)=1$, then

$$G(n,\chi)=\overline{\chi}(n)\,G(\chi). \tag{1.10}$$

The reader may ask what happens to $G(n,\chi)$ when n is an integer such that $(n,\ell)>1$: to handle this case, we have the following theorem

Proposition 1.4.: If χ is a character modulo ℓ, the Gauss sum $G(n,\chi)$ is separable for every n if, and only if, $G(n,\chi)=0$ whenever $(n,\ell)>1$.

Another important consequence of the separability of Gauss sums, invoked several times during this dissertation, is the following proposition.

Proposition 1.5.: Let χ be a Dirichlet character modulo ℓ. If $G(n,\chi)$ is separable for every n, then

$$|G(\chi)|^2=\ell. \tag{1.11}$$

Proof: For $\ell=1$, $\chi=\chi_0$ and so (1.11) is clear. For $\ell\geq 2$, we know that $\chi(\ell)=0$ (property 4.) and so, for the sake of the computations done below, we may write $G(n,\chi)$ as $\sum_{r=1}^{\ell}\chi(r)\,e^{2\pi irn/\ell}$. This gives,

$$\begin{aligned}
|G(\chi)|^2 &= G(\chi)\,\overline{G(\chi)} = G(\chi)\sum_{r=1}^{\ell}\overline{\chi}(r)\,e^{-2\pi ir/\ell}\\
&=\sum_{r=1}^{\ell}G(r,\chi)\,e^{-2\pi ir/\ell}=\sum_{r=1}^{\ell}\sum_{k=1}^{\ell}\chi(k)\,e^{2\pi ikr/\ell}e^{-2\pi ir/\ell}\\
&=\sum_{k=1}^{\ell}\chi(k)\sum_{r=1}^{\ell}e^{2\pi ir(k-1)/\ell}=\ell\chi(1)=\ell,
\end{aligned} \tag{1.12}$$

since the last sum over the index r is a geometric sum which vanishes when $k\neq 1$. ■

Since $G(n,\chi)$ is separable if $(n,\ell)=1$ and this separability is equivalent to the vanishing of $G(n,\chi)$ for $(n,\ell)>1$, it seems reasonable to study the Dirichlet characters such that $G(n,\chi)=0$ whenever $(n,\ell)>1$.

Following Apostol's textbook [5], the following theorem gives a necessary condition for $G(n,\chi)$ to be nonzero for $(n,\ell)>1$.

Proposition 1.6.: Let χ be a Dirichlet character modulo ℓ and assume that $G(n,\chi) \neq 0$ for some n satisfying $(n,\,\ell) > 1$. Then there exists $d < \ell$ such that $d \mid \ell$ and

$$\chi(a) = 1 \quad \text{whenever } (a,\ell) = 1 \text{ and } a \equiv 1 \mod d. \tag{1.13}$$

This theorem is important for us to introduce the following definition, which shall be essential for all the considerations on this thesis.

Definition 1.5. (Primitive character): A Dirichlet character χ modulo ℓ is said to be primitive if, for every divisor d of ℓ, $0 < d < \ell$, there exists an integer $a \equiv 1 \bmod d$ satisfying $(a,\,\ell) = 1$ and such that $\chi(a) \neq 1$.

For example, if $\ell > 1$, the principal character modulo ℓ, χ_1, is not primitive. To see this, take $d = 1$: then d clearly divides ℓ and every a coprime with ℓ satisfies $a \equiv 1 \bmod d$ and $\chi_1(a) = 1$ (because χ_1 is principal).

For example, it is also simple to see that, if χ is nonprincipal and has modulo ℓ being a prime number, then χ is also primitive.

Finally, we have the following theorem, which relates the primitivity of a given character with the properties of the Gauss sum associated with it.

Theorem 1.1. (Characterization of primitive characters) Let χ be a primitive character modulo ℓ. Then χ has the following properties:

1. $G(n,\chi) = 0$ for every n with $(n,\,\ell) > 1$.
2. $G(n,\chi) = \overline{\chi}(n)\, G(\chi)$ for every $n \in \mathbb{N}$.
3. $|G(\chi)|^2 = \ell$.

Proof: Let n be an integer such that $(n,\,\ell) > 1$ and satisfying $G(n,\chi) \neq 0$. Then by Proposition 1.6. and definition 1.5., χ cannot be primitive. This proves 1.

Clearly, 1. implies 2. by Proposition 1.4.

Finally, since $G(n,\chi)$ is separable, we immediately have 3. by the previous Proposition 1.5.. This concludes the proof. ∎

Example 1.B.: Primitive characters and Legendre symbols

For any odd prime p and $n \in \mathbb{Z}$, one may define the Legendre symbol [36] as

$$\left(\frac{n}{p}\right) = \begin{cases} 1 & \text{if } n \equiv x^2 \bmod p \\ -1 & \text{if } n \not\equiv x^2 \bmod p \\ 0 & \text{if } (n,\, p) \neq 1. \end{cases} \tag{1.14}$$

Then $\chi(n) = \left(\frac{n}{p}\right)$ is a primitive Dirichlet character. One way of proving this consists in using the characterization given by Theorem 1.1., together with the aid of Gauss's identity for the Gauss sum [29]

$$\sum_{n=0}^{M-1} e^{2\pi i n^2/M} = \begin{cases} \sqrt{M} & \text{if } M \equiv 1 \bmod 4 \\ 0 & \text{if } M \equiv 2 \bmod 4 \\ i\sqrt{M} & \text{if } M \equiv 3 \bmod 4 \\ (1+i)\sqrt{M} & \text{if } M \equiv 0 \bmod 4. \end{cases}$$

It should be also noted that, after combining 2. and 3. in Theorem 1.1., every primitive Dirichlet character modulo ℓ satisfies the relation

$$G(\chi)\, G(\overline{\chi}) = \chi(-1)\, \ell. \tag{1.15}$$

From now on, the characterization of primitive Dirichlet characters provided by Theorem 1.1. will be the one used by us in this thesis. One could actually prove that the separability of the Gaussian sum $G(n, \chi)$ is a sufficient condition for a character to be primitive.

This shows indeed interesting theorems, such as the one proved by Apostol [4] relating the uniqueness of the functional equation for $L(s, \chi)$ with the primitivity of the character χ.

However, in order to avoid more considerations, we remark that points 1., 2., 3. in Theorem 1.1. and relation (1.15) are enough to understand the conditions imposed over the primitive characters along our work.

We finish this preliminary section by stating a result which provides a relation between a given character $\chi(n)$ and a primitive one $\chi'(n)$, which will be useful at some points during this thesis (see, for instance, eq. (5.7) on the fifth chapter, an essential relation to prove the Main Theorem 1 stated in there).

Theorem 1.2. (Primitivity): Let χ be a Dirichlet character modulo ℓ. Then there exists a divisor $1 < \ell' < \ell$ of ℓ and a primitive character χ' modulo ℓ' such that, for all $n \in \mathbb{Z}$, χ can be expressed as a product

$$\chi(n) = \chi'(n)\,\chi_1(n), \tag{1.16}$$

where χ_1 is the principal character modulo ℓ.

By other words, we can express $\chi(n)$ as

$$\chi(n) = \begin{cases} \chi'(n) & \text{if } \ (n,\,\ell) = 1 \\ 0 & \text{if } \ (n,\ell) > 1. \end{cases} \tag{1.17}$$

It is generally said that, if χ' satisfies (1.16), then it induces χ.

After this short introduction, we are ready to study not only the Dirichlet $L-$functions (which informally have appeared in (1.7)) but also to relate these with the Hurwitz and Riemann's $\zeta-$functions, $\zeta(s,a)$ and $\zeta(s)$.

1.2 Preliminary Results - Part II: The functions $\zeta(s,a)$, $\zeta(s)$ and $L(s,\chi)$ and our methods of study

In this brief section, we present some elementary facts concerning the functions $\zeta(s)$, $\zeta(s,a)$ and $L(s,\chi)$ as direct consequences of their representation by a Dirichlet series.

Generally, a Dirichlet series is a series of the form

$$A(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

where $a(n)$ is an arithmetic function. The study of the analytic continuation of these series constitutes one of the most fundamental tools in Analytic number theory.

Here we introduce some important Dirichlet series present in our work, starting with the most famous of them. When $\text{Re}(s) > 1$, the Riemann $\zeta-$function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \text{Re}(s) > 1, \tag{1.18}$$

with the series converging absolutely for $\text{Re}(s) > 1$ by the integral test.

Moreover, since $f_n(s) = n^{-s}$ is bounded by $n^{-1-\epsilon}$ for all s in the half-plane $\text{Re}(s) \geq 1+\epsilon$, it follows from Weierstrass test that the series (1.18) also converges uniformly in every half-plane $\text{Re}(s) \geq 1+\epsilon$, $\forall \epsilon > 0$.

Since the series (1.18) is uniformly convergent in each region $\text{Re}(s) \geq 1+\epsilon$, it follows that [94, 107] $\zeta(s)$ is analytic in the half-plane $\text{Re}(s) > 1$ and its derivatives there may be obtained by termwise differentiation of the series (1.18).

The $\zeta-$function was firstly considered by Euler during his study of the particular values attained by the series at the right-hand side of (1.18) when s is a positive integer [11].

Later, Euler proved that, for real $s > 1$, the product formula holds [11, 48]

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \tag{1.19}$$

where the product in the right-hand side of (1.19) is taken over all prime numbers.

Formula (1.19) was the first step to establish the connection between $\zeta(s)$ and Number Theory. Indeed, as can be seen in [103], (1.19) still holds for $\text{Re}(s) > 1$.

In analogy with $\zeta(s)$, if $\ell \in \mathbb{N}$ and χ is a Dirichlet character modulo ℓ, then the Dirichlet $L-$function associated to χ is defined to be the Dirichlet series

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \text{Re}(s) > 1. \tag{1.20}$$

From the above discussion, it is clear that $L(s,\chi)$ is absolutely convergent for $\text{Re}(s) > 1$ and analytic in this region. Moreover, since χ is a completely multiplicative function (see [103] and [63] for details), $L(s,\chi)$ is also given by Euler's infinite product

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1. \tag{1.21}$$

In particular, if χ is the principal character modulo ℓ, χ_1, we have that, for $\text{Re}(s) > 1$,

$$L(s,\chi) = \prod_{p\nmid\ell} \left(1-p^{-s}\right)^{-1} = \prod_p \left(1-p^{-s}\right)^{-1} \prod_{p|\ell} \left(1-p^{-s}\right) = \zeta(s) \prod_{p|\ell} \left(1-p^{-s}\right), \tag{1.22}$$

which proves that $L(s,\chi)$ has a similar behavior to $\zeta(s)$ when we consider the character χ as principal.

It is the purpose of this chapter to study the analytic continuation of the Dirichlet series (1.18) and (1.20) to the complex plane. To proceed in a unified way, we need to treat the two Dirichlet series (1.18) and (1.20) as particular cases of the following series

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \quad \text{Re}(s) > 1, \tag{1.23}$$

where a is a fixed real number such that $0 < a \leq 1$. The $\zeta-$function presented in (1.23) is called Hurwitz $\zeta-$function, which is well-defined and analytic in the region $\mathrm{Re}(s) > 1$, as well as the previous series (1.18) and (1.20).

One sees immediately that Riemann's $\zeta-$function is a particular case of $\zeta(s, a)$ obtained when $a = 1$.

Moreover, if we fix $\ell \in \mathbb{N}$, any natural number n can be written as $n = q\ell + r$, where $q \in \mathbb{N}_0$ and $1 \leq r \leq \ell - 1$. Appealing to the periodic properties of the Dirichlet characters (see property 3. in Definition 1.1.), we have that

$$L(s,\chi) = \sum_{r=1}^{\ell-1} \sum_{q=0}^{\infty} \frac{\chi(q\ell + r)}{(q\ell + r)^s} = \ell^{-s} \sum_{r=1}^{\ell-1} \chi(r) \sum_{q=0}^{\infty} \frac{1}{(q + r/\ell)^s}$$
$$= \ell^{-s} \sum_{r=1}^{\ell-1} \chi(r)\, \zeta\left(s, \frac{r}{\ell}\right), \quad \mathrm{Re}(s) > 1. \tag{1.24}$$

Thus, if we study the analytic continuation of the Hurwitz series (1.23), we end up studying the very same continuation for $\zeta(s)$ and $L(s, \chi)$, which is one of the motivations to study summation formulas with Dirichlet characters.

1.3 Methods in this chapter

This chapter inaugurates the methods used throughout our thesis, as we state and prove some of the most important identities or functional relations which will be of utter significance in forthcoming sections.

The goal hereby proposed is a new proof, based on a relatively unknown representation of $\zeta(s, a)$, of the functional relation for the Hurwitz $\zeta-$function,

Main Theorem:

For $0 < a \leq 1$, the analytic continuation of Hurwitz's $\zeta-$function, $\zeta(s, a)$, defined in (1.23) satisfies the functional equation

$$\zeta(1-s, a) = \frac{2\Gamma(s)}{(2\pi)^s} \left[\cos\left(\frac{\pi s}{2}\right) \sum_{n=1}^{\infty} \frac{\cos(2\pi n a)}{n^s} + \sin\left(\frac{\pi s}{2}\right) \sum_{n=1}^{\infty} \frac{\sin(2\pi n a)}{n^s}\right], \quad \mathrm{Re}(s) > 1. \tag{1.25}$$

There are several proofs of this elegant identity in the literature: for instance, in the classical text of Titchmarsh [103] (or in Whittaker and Watson's "Modern Analysis" [107]), it is invoked the classical integral representation

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{\Gamma(s)}\int\limits_{0}^{\infty}x^{s-1}e^{-(n+a)\,x}dx=\frac{1}{\Gamma(s)}\int\limits_{0}^{\infty}\frac{x^{s-1}e^{-ax}}{1-e^{-x}}\,dx, \tag{1.26}$$

with the right-hand side of (1.26) being modified to be a contour of Hankel-type. This is also the approach given in Apostol's text [5]. It is evident that this proof is a modification of Riemann's first proof of the functional equation for $\zeta(s)$, which can be consulted in [103] p. 18.

As remarked by Fine [52], one can even adapt Riemann's second proof of the functional equation for $\zeta(s)$ (see [103] pp.21-22), using the reflection formula Jacobi's $\psi-$function, to derive (1.25), presenting it in a somewhat more symmetric form.

Apostol [7,8] also approaches the functional relation (1.25) in two papers: in the first one, he proves particular cases (over the integers) of the functional relation for Lerch's $\zeta-$function [44] and extends these to Hurwitz's. In a subsequent paper, he actually proves that the functional equation for Lerch's $\zeta-$function implies the functional relation (1.25).

Knopp, Robins [68] and Oberhettinger [81] appeal to a generalized version of Poisson's summation formula: in the first of these papers, the methods employed allow to derive the Lipschitz summation formula, while in the second they are more straightforward , although not elementary, since they invoke the inversion formula for the Laplace transform.

The proof presented in this chapter is different and somewhat more elementary than the above-cited, although there are natural similarities with N. J. Fine's proof, as well as with Knopp and Robbin's.

Finally, we note that, in a broader sense, the proof presented here can be extended to other known Dirichlet series, although this study is not covered by our thesis.

It should be remarked that, in the second chapter (example 2.6), we also present a formula similar to (1.25) for the series

$$L(s,\chi,a)=\sum_{n=1}^{\infty}\frac{\chi(n)}{(n+a)^{s}},\quad 0\leq a<1.$$

However, we shall not use the methods developed here to prove it, although these could work. Instead, our proof is based on a modification of Fine's argument [52] together with a character version of Poisson's summation formula, which will be the matter of study in the next chapter (details of these computations are shown in the Supplementary document to the second chapter [89]).

We should also remark the existence of secondary goals for this chapter: using the analytic continuation for $L(s,\chi)$ and $\zeta(s)$, we also aim to evaluate $L^{'}(1,\chi)$ as well as $L^{'}(0,\chi)$ for some suitable character χ (we will see some closed formulas when χ is primitive). It appears that our methods are new, as we appeal to Hermite's integral formula (see (1.55)) and prove Lerch's identity (see (1.62)) by invoking functional properties of the $\Gamma-$function.

We start our approach with a short revision of important identities for $\Gamma(s)$, always focused on its functional properties.

1.4 Preliminary Results - Part III: Euler's Γ and ψ functions

One of the novelties introduced in this chapter is our proof of Lerch's identity [24, 37]

$$\log\Gamma(a) = \zeta'(0,a) - \zeta'(0), \tag{1.27}$$

which relates the well-known $\Gamma-$function with the derivatives at zero of the analytic continuations of $\zeta(s,a)$ and $\zeta(s)$. Our method of proving (1.27) is inspired by the techniques presented in E. Artin's treatise on the $\Gamma-$function [9]: we invoke the uniqueness that some of the functional aspects of $\Gamma(s)$ offer, namely, Bohr-Mollerup's theorem (see Theorem 1.3. below).

Thus, it seems appropriate to develop in this section a brief revision of some known identities for $\Gamma(s)$.

Although not historically precise, the Gamma function is usually defined by the integral

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx, \quad \text{Re}(s) > 0. \tag{1.28}$$

It is well-known that the function so defined by (1.28) can be analytically continued beyond the line $\text{Re}(s) = \sigma = 0$.

Using (1.28), it is simple to check one of the most basic properties of $\Gamma(s)$,

$$\Gamma(s+1) = s\Gamma(s), \tag{1.29}$$

which is also the main feature of the factorial function.

Using property (1.29), one can establish the analytic continuation of $\Gamma(s)$ [94]. By an inductive reasoning, it is simple to see that (1.29) yields the equality, for $\text{Re}(s) > -n-1$,

$$\Gamma(s) = \frac{\Gamma(s+n+1)}{s(s+1)\ldots(s+n)}, \quad s \neq 0, -1, ..., -n \tag{1.30}$$

which allows to carry out the analytic continuation of $\Gamma(s)$ into the half-plane $\text{Re}(s) > -n-1$, for any $n \in \mathbb{N}$. Representation (1.30) tells that $\Gamma(s)$ can be extended to $\mathbb{C}$ as a meromorphic function with simple poles located at the nonpositive integers $s = 0, -1, -2, \ldots$. Furthermore, it also shows that the residue of $\Gamma(s)$ at $s = -n$ is equal to $(-1)^n/n!$.

The unique feature of $\Gamma(s)$ used to prove the analytic continuation (1.30) was the reduction formula (1.29) and it would very interesting if this property characterized $\Gamma(s)$.

Nevertheless, this property does not determine $\Gamma(s)$ [87]: a well-known additional condition, which is sufficient for characterizing $\Gamma(s)$ in a "functional way", is the convexity of the function $F(x) = \log \Gamma(x)$ for $x \in \mathbb{R}_+$ [9].

The most well-known theorem that sets the conditions to provide this characterization was proved by H. Bohr and J. Mollerup in 1922 and its statement is given as as follows [9]

Theorem 1.3. (Bohr-Mollerup): Let $F : \mathbb{R}_+ \longmapsto \mathbb{R}_+$ be a real-valued function having the properties:

1. $F(1) = 1$,
2. $F(x+1) = x\, F(x)$ for all $x > 0$ and
3. $\log\left(F(x)\right)$ is a convex real function.

Then for all $x > 0$, $F(x) = \Gamma(x)$.

By analytic continuation [94], $\Gamma(s)$ is the unique meromorphic function in $\mathbb{C}$ satisfying properties 1. and 2. and whose logarithm is a convex function when restricted to $\mathbb{R}_+$, so that the above properties 1., 2. and 3. furnish the desired functional characterization of $\Gamma(s)$.

In Artin's book [9], it is shown that the functional property provided by Theorem 1.3. can be used to deduce several identities, some of them dating back to Euler's time and others related with the evaluation of elliptic integrals.

As pointed out by R. Remmert in [87], another elegant functional characterization for $\Gamma(s)$ was discovered by Helmut Wielandt in 1939, which seems to require less than the convexity imposed by Bohr-Mollerup's theorem.

In the elegant paper [87], as well as in Artin's book, it is possible to find interesting proofs of the following identities, whose arguments appeal to Bohr-Mollerup and Wielandt's Theorems.

1. Weierstrass Product formula: $\Gamma(s)$ obeys to the product formula

$$\Gamma(s) = \frac{e^{-\gamma s}}{s} \prod_{n=1}^{\infty} \frac{n}{s+n}\, e^{s/n} = \lim_{n\to\infty} \frac{n!\, n^s}{s\,(s+1) \ldots (s+n)}, \tag{1.31}$$

when assuming that the reciprocal of the product in the second term of (1.31) converges uniformly on every compact subset of $\mathbb{C}$ to an entire function $\Delta(s) := 1/\Gamma(s)$ (see [107] p.235 for a very elegant justification of this fact).

2. Stirling's formula: let $s = \sigma + it$ be a complex number. Then $\Gamma(s)$ has the following asymptotic behavior
$$|\Gamma(s)| \sim (2\pi)^{1/2}\, |t|^{\sigma-\frac{1}{2}} \exp\left(-\frac{1}{2}\pi|t|\right), \tag{1.32}$$
for fixed σ and $|t| \to \infty$. Moreover, one has the exact expansion of $\log(\Gamma(s))$ as follows
$$\log\Gamma(s) = \left(s - \frac{1}{2}\right)\log(s) - s + \frac{1}{2}\log(2\pi) + \mu(s), \tag{1.33}$$
where the "error function" $\mu(s)$ satisfies the inequality $|\mu(s)| \leq \frac{1}{6|s|}$ for complex s with $|\arg(s)| \leq \frac{\pi}{2}$.

3. Euler's reflection formula: For all $s \in \mathbb{C} \setminus \mathbb{Z}$, the identity holds
$$\Gamma(s)\,\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}, \tag{1.34}$$
from which one can deduce the remarkable product formula
$$\frac{\sin(x)}{x} = \prod_{k=1}^{\infty}\left(1 - \frac{x^2}{\pi^2 k^2}\right), \qquad x \in \mathbb{R}, \tag{1.35}$$
obtained by Euler (1734 - 1735) [11]. Using (1.35), Euler was able to solve not only the Basel problem[1], but also provided a closed-form evaluation for $\zeta(2n)$[2] and its relation with the Bernoulli numbers, which we shall reprove by other methods in this chapter.

4. Gauss Multiplication Theorem: Let n be an integer greater than 1. Then, for any $s \in \mathbb{C} \setminus \{0, -1, -2, ...\}$, $\Gamma(s)$ satisfies the product formula
$$\prod_{r=0}^{n-1}\Gamma\left(s + \frac{r}{n}\right) = (2\pi)^{(n-1)/2}\, n^{1/2-ns}\,\Gamma(ns), \tag{1.36}$$
from which one obtains, upon taking $n = 2$, the identity
$$\Gamma(2s) = \frac{2^{2s-1}}{\sqrt{\pi}}\,\Gamma(s)\,\Gamma\left(s + \frac{1}{2}\right), \tag{1.37}$$
known as Legendre's duplication formula [44].

[1] According to Ayoub's description [11], this was "Euler's first triumph".

[2] Although the tradition in the History of Mathematics often emphasizes Euler's proof of the identity for $\zeta(2n)$ as a natural extension of his proof for $\zeta(2)$, the extension in the full generality was not immediate. Although Euler claims in *De summis serierum reciprocarum* (1734) [47] to have derived a general identity for $\zeta(2n)$, in this paper he only had computed the values for $\zeta(2n)$ when $n = 1, ..., 6$. However, in a paper written 6 years later, *De Seribus Quibusdam Considerationes* [46] Euler wrote the evaluation of $\zeta(2n)$ in the form $A_{2n}\,\pi^{2n}$ and found a recursive relation for the constants A_{2n}, which can also work as a characterization of the Bernoulli numbers.

Naturally, identities (1.31 - 1.37) given above can be translated to the digamma function $\psi(s)$, which is defined as

$$\psi(s) := \frac{d}{ds}\log(\Gamma(s)) = \frac{\Gamma'(s)}{\Gamma(s)}. \tag{1.38}$$

It is easily seen from the identities (1.34) and (1.31) that $\Gamma(s)$ has no zeros: this means that $\psi(s)$ can have no singularities other than the poles of $\Gamma(s)$ located at $\mathbb{Z}_0^-$.

Furthermore, around the point $s = -n$, $\psi(s)$ admits the representation

$$\psi(s) = -\frac{1}{s+n} + \Phi(s+n),$$

where $\Phi(s)$ is an entire function.

From Weierstrass product formula (1.31), it is clear that

$$\log\Gamma(s) = -\log(s) - \gamma s + \sum_{n=1}^{\infty}\left[\frac{s}{n} - \log\left(1+\frac{s}{n}\right)\right] \tag{1.39}$$

and

$$\log\Gamma(s) = \lim_{n\to\infty}\log\left(\frac{n!\,n^s}{s\,(s+1)\dots(s+n)}\right). \tag{1.40}$$

Hence, if we take $|s| < 1$ in (1.39), we see from the power series expansion for $\log(1+z)$, $|z| < 1$, that the series in (1.39) converges uniformly with respect to s.

Therefore, a termwise differentiation gives

$$\psi(s) = -\frac{1}{s} - \gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{s+n}\right) = -\gamma - \frac{1}{s} + \sum_{n=1}^{\infty}\frac{s}{n\,(s+n)}, \quad |s| < 1, \tag{1.41}$$

which clearly can be extended for all $s \in \mathbb{C}\setminus\mathbb{Z}_0^-$ via the reduction formula (1.29).

From a straightforward differentiation of (1.40) we can also check that

$$\psi(s) = -\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{s+n} - \log(n)\right), \tag{1.42}$$

from which we find the particular value $\psi(1) = -\gamma$, via the classic definition of γ (see our glossary),

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log(n)\right).$$

Of couse, from (1.29) and the relations given in (1.34) and (1.36) , it is immediate to see that ψ satisfies the identities

$$\psi\,(s+1) = \frac{1}{s} + \psi(s), \tag{1.43}$$

$$\psi(1-s)-\psi(s)=\pi\,\cot(\pi s),\quad s\in\mathbb{C}\setminus\mathbb{Z} \tag{1.44}$$

$$\sum_{r=0}^{n-1}\psi\left(s+r/n\right)+n\,\log(n)=n\,\psi(ns), \tag{1.45}$$

with the last one implying Legendre's duplication formula $\psi(s)+\psi\left(s+\frac{1}{2}\right)+2\,\log(2)=2\,\psi(2s)$.

It is interesting to observe that (1.44) also allows to deduce the meromorphic expansion of $\cot(\pi z),\ \ z\in\mathbb{C}\setminus\mathbb{Z}$: taking z instead of s there and applying (1.41), we arrive to

$$\begin{aligned}\pi\cot(\pi z)&=\frac{1}{z-1}+\frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{1-z+n}\right)\\&=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2-n^2},\qquad z\in\mathbb{C}\setminus\mathbb{Z},\end{aligned} \tag{1.46}$$

which was also deduced by Euler for the first as a corollary of the identity (1.35) and as an auxiliary tool to deduce an identity for $\zeta(2n)$ [3, 11].

The reader can also see in [2] a beautiful argument (due to Herglotz) for a derivation of (1.46). A similar argument was used by Artin in his classical book on the Gamma Function [9] to obtain (1.34) with the use of Bohr Mollerup's theorem.

Finally, from the particular values $\psi(1)=-\gamma$ and $\psi(1/2)=-\gamma-2\log(2)$, we may ask if there exists a closed-form evaluation of $\psi(p/q)$, when p and q are integers and $0<p<q$. In fact, Gauss's digamma theorem [44] states that an evaluation of this kind exists and can be written as

$$\psi\left(\frac{p}{q}\right)=-\gamma-\log(2q)-\frac{\pi}{2}\,\cot\left(\frac{p}{q}\pi\right)+2\sum_{k=1}^{\lceil q/2\rceil-1}\cos\left(\frac{2\pi p\,k}{q}\right)\,\log\sin\left(\frac{\pi k}{q}\right). \tag{1.47}$$

The usual proofs given to (1.47) use the natural representation for the digamma function,

$$\psi(s)=-\gamma+\int_0^1\frac{1-x^{s-1}}{1-x}\,dx,\qquad \mathrm{Re}(s)>0, \tag{1.48}$$

also deduced by Euler for the first time [11]. In Lemma 1.3. below, we also describe a new way of deducing (1.47), based on the computation of an integral coming from Hermite's representation (see (1.79) below).

After this introduction to some of the most remarkable features of $\Gamma(s)$ and $\psi(s)$, we move on to describe the analytic continuation of the functions $\zeta(s,a)$, $\zeta(s)$ and $L(s,\chi)$.

Our method is motivated by Abel-Plana's summation formula: although some of our conclusions overlap the ones outlined in Whittaker and Watson's Modern Analysis [107], our use of an immediate corollary of Abel-Plana's formula will be essential in deriving the functional equation for $\zeta(s,a)$. As far as we know, this constitutes a new proof of this result.

Furthermore, we use extensively this formula to give new proofs of some classical results regarding the values attained by $\zeta(s,a)$, $\zeta(s)$ and $L(s,\chi)$ when s is an integer.

1.5 Abel-Plana formula and the Analytic continuation of $\zeta(s,a)$

This section is devoted to the proof of the analytic continuation of $\zeta(s,a)$, as well as $\zeta(s)$ and $L(s,\chi)$. The methods employed here will be mimicked in the fourth chapter of this thesis, so it is very important to work these details comprehensively.

1.5.1 Abel-Plana formula

Although this work motivates some aspects lying in the fourth chapter, we should notice that, in this forthcoming work, Poisson's summation formula will be used to obtain an extension of Epstein's $\zeta-$function to $\mathbb{C}$. Here, however, we restrict ourselves to an easier formula, due to Plana [86], whose content is relatively unknown. We state it as the following theorem.

Theorem 1.4.: Let $f(z)$ be a function of one complex variable such that:

1. $f(z)$ is analytic in the region $\mathrm{Re}(z) \geq 0$;
2. $\lim_{|t|\to\infty} e^{-2\pi|t|}\, f(\sigma+it) = 0$ uniformly for $\sigma \geq 0$;
3. $\lim_{\sigma\to\infty} \int_{-\infty}^{\infty} e^{-2\pi|t|}\, |f(\sigma+it)|\, dt = 0$.

Then the following formula holds

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2}\, f(0) + \int_0^{\infty} f(x)\, dx + i\int_0^{\infty} \frac{f(ix)-f(-ix)}{e^{2\pi x}-1} dx. \tag{1.49}$$

Plana gave (1.49) (which he described as "remarquable" [86]) in 1820 and Abel arrived at this formula three years later. The usual proof of Theorem 1.4., under the conditions given, follows from straightforward considerations of the Theory of residues and it is attributed to Cauchy (1826) [86].

The reader can find in [[61], p. 274] a standard proof of (1.49), as well as some interesting examples which are particular cases of the ones developed in this chapter.

After its first appearance, (1.49) was also used in Genocchi's (1852) and Weber's (1903) proofs of the reciprocity law for quadratic Gauss sums [29, 74]. In Lindelöf's book it is possible to consult a simplification of Weber's proof. Formula (1.49) is even prior to a more famous one, known by the name of Poisson's summation formula.

By its turn, it seems that the first rigorous proof of Poisson's formula came only with Dirichlet's work (1829) [36], who also applied it later to derive the reciprocity law for quadratic Gauss sums (1834) [29].

A century later, and based upon the functional equation for $\zeta(s)$, Koshliakov [70] gave another proof of (1.49) under different conditions. Furthermore, he even extended (1.49) to a series of the form

$$\sum a(n)\, f(n),$$

where the coefficients $a(n)$ come from a suitable arithmetic function. Since Koshliakov's paper is given in Russian (although a brief explanation of his methods is given in english at the end of the paper [69]), the reader can find in a very recent article [72], a simplification of his proof.

Finally, it should be remarked that we took the liberty of taking the prefix "summation" out of the name "Abel-Plana summation formula", which is a more common designation to (1.49).

The reason why we did this is that (1.49) does not give a relation between a lattice sum of the form $\sum f(n)$ with an analogous one involving some integral transform $\sum K[f](n)$, as in the way exposed at the introduction.

Before using (1.49) to prove the analytic continuation of the ζ and $L-$functions, we introduce a very simple example, which will be useful later.

Example 1.1.: Sine identity

Taking $f(y) = e^{-xy}\cos(2\pi a\, y)$, for $x > 0$ and $0 \leq a < 1$, it is not hard to see that $f(y)$ extends to an analytic function $f(z)$ in $\text{Re}(z) \geq 0$ and satisfies the conditions of the above Theorem. A straightforward application of (1.49) yields

$$\begin{aligned}\sum_{n=1}^{\infty} e^{-nx}\cos(2\pi n a) &= -\frac{1}{2} + \int_0^{\infty} e^{-xy}\cos(2\pi a y)\, dy + 2\int_0^{\infty} \frac{\sin(xy)\,\cosh(2\pi a y)}{e^{2\pi y}-1} dy \\ &= -\frac{1}{2} + \frac{x}{x^2+4\pi^2 a^2} + 2\int_0^{\infty} \frac{\sin(xy)\,\cosh(2\pi a y)}{e^{2\pi y}-1} dy. \qquad (1.50)\end{aligned}$$

From elementary calculations regarding the computation of the series in the left-hand side of (1.50), we derive the formula

$$\int_0^{\infty} \frac{\sin(xy)\,\cosh(2\pi ay)}{e^{2\pi y}-1}dy = \frac{1}{2}\left(\frac{e^x\cos(2\pi a)-1}{e^{2x}-2e^x\cos(2\pi a)+1}+\frac{1}{2}-\frac{x}{x^2+4\pi^2a^2}\right). \tag{1.51}$$

Moreover, from well-known properties of the Laplace transform (regarding the computation of the Laplace transform of $\sin(xy)$) and the absolute convergence of the power series defining the function $1/(e^x-1)$, we arrive to

$$\int_0^{\infty} \frac{\sin(xy)\,\cosh(2\pi ay)}{e^{2\pi y}-1}dy = \frac{x}{2}\sum_{k=1}^{\infty}\left[\frac{1}{4\pi^2(k-a)^2+x^2}+\frac{1}{4\pi^2(k+a)^2+x^2}\right]. \tag{1.52}$$

Analogously, if we apply Abel-Plana's formula to $f(y)=e^{-xy}\sin(2\pi a\,y)$, $x>0$ and $0<a<1$, we are also able to deduce the identities

$$\begin{aligned}\sum_{n=1}^{\infty} e^{-nx}\sin(2\pi a\,n) &= \int_0^{\infty} e^{-xy}\sin(2\pi ay)\,dy - 2\int_0^{\infty}\frac{\cos(xy)\,\sinh(2\pi ay)}{e^{2\pi y}-1}dx\\ &= \frac{2\pi a}{x^2+4\pi^2a^2}-2\int_0^{\infty}\frac{\cos(xy)\,\sinh(2\pi ay)}{e^{2\pi y}-1}dy,\end{aligned} \tag{1.53}$$

as well as

$$\int_0^{\infty} \frac{\cos(xy)\sinh(2\pi ay)}{e^{2\pi y}-1}dx = \pi\sum_{k=1}^{\infty}\left[\frac{(k-a)}{4\pi^2(k-a)^2+x^2}-\frac{(k+a)}{4\pi^2(k+a)^2+x^2}\right]. \tag{1.54}$$

1.5.2 The Analytic continuation of $\zeta(s,a)$ and its corollaries

Now, we explore the consequences of (1.49) in order to extend the functions $\zeta(s)$, $\zeta(s,a)$ and $L(s,\chi)$ to the complex plane and to prove several of their properties.

These considerations seem to be new, although there are some aspects in our derivation inspired by Watson and Whittaker's text [107].

Nevertheless, Watson and Whittaker do not give a proof of the functional equations for $\zeta(s)$ and $\zeta(s,a)$ based upon (1.49) and it is precisely our use of the above-example 1.1. which constitutes one of the novelties introduced in this chapter.

As remarked in the second section, we focus our study on $\zeta(s,a)$. Let us take s real satisfying $s > 1$, $0 < a \leq 1$ and the complex function $f(z) = \frac{1}{(z+a)^s}$.

The considered function is clearly analytic in the region $\mathrm{Re}(z) \geq 0$ and satisfies the conditions imposed by Theorem 1.4.. Using Abel-Plana formula (1.49), we immediately find that, for real $s > 1$, and $0 < a \leq 1$,

$$\begin{aligned}\zeta(s,a) = \sum_{n=0}^{\infty} f(n) &= \frac{a^{-s}}{2} + \int_0^{\infty} \frac{dx}{(x+a)^s} + i \int_0^{\infty} \frac{1/(a+ix)^s - 1/(a-ix)^s}{e^{2\pi x}-1} dx \\ &= \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} + 2\int_0^{\infty} \frac{\sin\left(s \arctan(x/a)\right)}{(a^2+x^2)^{s/2}\left(e^{2\pi x}-1\right)}\, dx, \end{aligned} \tag{1.55}$$

where the last step comes from the fact that the argument of the complex number $a + ix$ is precisely $\arctan(x/a)$ and s is real.

Representation (1.55) is usually called Hermite's integral representation [44], and it was published in 1901, although a similar formula had been given before by Jensen [107].

Now, we study the properties of the integral in (1.55) as a complex function of s, whose behavior is crucial to understand the extension of $\zeta(s,a)$.

To do so, let us briefly recall the following theorem, whose proof and motivations can be found in [107], section 5.31, p. 92.

Proposition 1.7.: Let $s \in \Omega \subset \mathbb{C}$ be a domain (open and connected subset of $\mathbb{C}$) and $\mathcal{F}(s)$ a complex function defined by the infinite integral $\int_0^{\infty} F(x,s)\, dx$. Then $\mathcal{F}(s)$ is analytic on Ω if:

1. the integral defining $\mathcal{F}(s)$ converges for all $s \in \Omega$.
2. For a fixed $x \in \mathbb{R}_+$, $f(s) := F(x,s)$ defines an analytic function on Ω.
3. $\frac{\partial F(x,s)}{\partial s}$ is a continuous function in $\mathbb{R}_+ \times \Omega$.
4. $\int_0^{\infty} \frac{\partial F(x,s)}{\partial s}\, dx$ converges uniformly for any $s \in \Omega$.

Moreover, $\mathcal{F}^{'}(s)$ is obtained by differentiating under the integral sign.

Using the previous proposition, we prove the following Lemma.

Lemma 1.1.: Let $\mathcal{I}(s)$ be the complex-function defined by the improper integral in (1.55),

$$\mathcal{I}(s) = \int_0^{\infty} \frac{\sin\left(s \arctan(x/a)\right)}{(a^2+x^2)^{s/2}\left(e^{2\pi x}-1\right)}\, dx. \tag{1.56}$$

Then $\mathcal{I}(s)$ is entire.

Proof: It suffices to use Proposition 1.7. by replacing Ω by $\mathbb{C}$, $\mathcal{F}(s)$ by $\mathcal{I}(s)$ and letting

$$F(x,s)=\frac{\sin\left(s\ \arctan(x/a)\right)}{\left(a^2+x^2\right)^{s/2}\ \left(e^{2\pi x}-1\right)}.$$

Clearly, conditions 2. and 3. in Proposition 1.7. hold, so we just need to work the details for 1. and 4..

For instance, to see that 1. holds, we follow Whittaker and Watson's argument. Let $s=\sigma+it$: using the elementary inequalities

$$\arctan\left(\frac{x}{a}\right)<\frac{x}{a},\quad x\leq\frac{\pi}{2}a,$$

$$\arctan\left(\frac{x}{a}\right)<\frac{\pi}{2},\quad x>\frac{\pi}{2}a,$$

we find that the integral in (1.56) obeys to the bound

$$\begin{aligned}\left|\int\limits_0^\infty\frac{\sin\left(s\ \arctan(x/a)\right)}{(1+x^2)^{s/2}(e^{2\pi x}-1)}\,dx\right| &\leq\int\limits_0^\infty\frac{\sinh\left(|t|\ \arctan(x/a)\right)+|\sin(\sigma\ \arctan(x/a))|}{(a^2+x^2)^{\sigma/2}(e^{2\pi x}-1)}\,dx\\ &\leq\int\limits_0^{\pi a/2}\frac{\sinh(|t|\,x/a)}{\left(a^2+x^2\right)^{\sigma/2}\left(e^{2\pi x}-1\right)}\,dx+\frac{\sigma}{a}\int\limits_0^{\pi a/2}\frac{x}{\left(a^2+x^2\right)^{\sigma/2}\left(e^{2\pi x}-1\right)}\,dx\\ &+C_s\int\limits_{\pi a/2}^\infty\frac{dx}{(a^2+x^2)^{\sigma/2}(e^{2\pi x}-1)},\end{aligned}\qquad(1.57)$$

where $C_s\geq 1+\sinh(\pi|t|/2)$ is a constant only depending on s. Clearly, every term in (1.57) is a convergent integral and so we see that $\mathcal{I}(s)$ satisfies 1.

In order to verify 4., notice that

$$\frac{\partial F(x,s)}{\partial s}=\frac{\cos\left(s\ \arctan(x/a)\right)\ \arctan(x/a)}{\left(a^2+x^2\right)^{s/2}\ \left(e^{2\pi x}-1\right)}-\frac{1}{2}\log\left(a^2+x^2\right)\ \frac{\sin(s\ \arctan(x/a))}{\left(a^2+x^2\right)^{s/2}\left(e^{2\pi x}-1\right)}\qquad(1.58)$$

and so we just need to check property 4. for the integrals with respect to each term of (1.58), which we shall denote by $\mathcal{J}_1(s)$ and $\mathcal{J}_2(s)$.

For the first one, the required computations resemble the ones done previously: taking $s=\sigma+it$, it is easy to verify that

$$\left|\frac{\cos\left(s\ \arctan(x/a)\right)\ \arctan(x/a)}{\left(a^2+x^2\right)^{s/2}\ \left(e^{2\pi x}-1\right)}\right|\leq\frac{x\ \cosh\left(\pi t/2\right)}{a\left(a^2+x^2\right)^{\sigma/2}\ \left(e^{2\pi x}-1\right)}$$

from which we obtain

$$|\mathcal{J}_1(s)| \leq \frac{\cosh(\pi t/2)}{a} \int_0^\infty \frac{x}{(a^2+x^2)^{\sigma/2}\,(e^{2\pi x}-1)}\,dx, \tag{1.59}$$

with the last term being a constant depending only on s. In order to find an estimate for the second, take the partition of the interval into $\left(0,\, \frac{\pi a}{2}\right)$ and $\left(\frac{\pi a}{2},\, \infty\right)$ and apply the bounds used in (1.57). This gives

$$\begin{aligned}|\mathcal{J}_2(s)| \leq \frac{1}{2} &\int_0^{\pi a/2} \frac{\sinh(|t|\,x/a)\,|\log(a^2+x^2)|}{(a^2+x^2)^{\sigma/2}\,(e^{2\pi x}-1)}\,dx + \frac{\sigma}{2a}\int_0^{\pi a/2} \frac{x\,|\log(a^2+x^2)|}{(a^2+x^2)^{\sigma/2}\,(e^{2\pi x}-1)}\,dx \\ &+ D_s \int_{\pi a/2}^{\infty} \frac{dx}{(a^2+x^2)^{\sigma/2}(e^{2\pi x}-1)},\end{aligned}$$

where D_s is again depending solely on s. This concludes the proof. ■

With this Lemma proved, we are now ready to study the continuation that Hermite's integral furnishes to $\zeta(s,a)$. Before doing this, let us recall the following classic theorem, which can be found in [94]

Proposition 1.8. (Principle of Analytic continuation/Identity Theorem): Suppose that f and g are analytic in a domain $\Omega \subset \mathbb{C}$ and $f(s) = g(s)$ for all s in some non-empty open subset Ω' of Ω. Then $f(s) = g(s)$ for all $s \in \Omega$.

More generally, it suffices to impose that $f(s) = g(s)$ for s in some sequence of distinct points with limit point in Ω or assume that Ω' has an accumulation point in Ω.

It should be noticed that, if f and F are analytic in the domains Ω' and Ω respectively, with $\Omega' \subset \Omega$ and if the two functions agree on the smaller set Ω', then F is the analytic continuation of f into Ω, whose uniqueness is provided by the previous proposition.

We are now ready to prove the following theorem:

Theorem 1.5.: Hermite's formula provides the analytic continuations of $\zeta(s)$ and $\zeta(s,a)$ as complex meromorphic functions having a simple pole at $s=1$ with residue 1.

Moreover, it provides also the analytic continuation of $L(s,\chi)$ as:

1. an entire function in $\mathbb{C}$, if χ is a nonprincipal character;

2. a meromorphic function with a simple pole at $s=1$ with residue $\varphi(\ell)/\ell$, if χ is the principal character χ_1.

Proof: We know, from (1.55), that $\zeta(s,a)$ and the function of s given at the right-hand side of (1.55) coincide in the real interval $(1,\infty)$. Since $\zeta(s,a)$ is analytic in this interval (as it is a subset of the half-plane $\mathrm{Re}(s)>1$) and the right-hand side of (1.55) is analytic for all $s\in\mathbb{C}\setminus\{1\}$ (by Lemma 1.1.), it is clear from Proposition 1.8. that this extends $\zeta(s,a)$ to the whole complex plane as a meromorphic function having only a simple pole located at $s=1$ and having residue 1, since

$$\lim_{s\to 1}(s-1)\left[\frac{a^{-s}}{2}+\frac{a^{1-s}}{s-1}+2\int_0^\infty \frac{\sin\left(s\ \arctan(x/a)\right)}{(a^2+x^2)^{s/2}\left(e^{2\pi x}-1\right)}\,dx\right]=1.$$

If we take $a=1$, we are also able to represent the continuation of $\zeta(s)$ to $\mathbb{C}$ as

$$\zeta(s)=\zeta(s,\,1)=\frac{1}{2}+\frac{1}{s-1}+2\int_0^\infty \frac{\sin\left(s\ \arctan(x)\right)}{(x^2+1)^{s/2}\left(e^{2\pi x}-1\right)}\,dx, \tag{1.60}$$

from which we can also get that $\zeta(s)$ has a continuation to $\mathbb{C}$ as a meromorphic function with a simple pole at $s=1$ with residue 1.

Using the relation (1.24) between $\zeta(s,a)$ and $L(s,\chi)$, from (1.55) we also arrive at

$$\begin{aligned}L(s,\chi)&=\frac{1}{2}\sum_{r=1}^{\ell-1}\chi(r)\,r^{-s}+\frac{\ell^{-1}}{s-1}\sum_{r=1}^{\ell-1}\chi(r)\,r^{1-s}\\&\quad+2\sum_{r=1}^{\ell-1}\chi(r)\int_0^\infty\frac{\sin\left(s\ \arctan(x\ell/r)\right)}{(r^2+\ell^2x^2)^{s/2}}\,\frac{dx}{e^{2\pi x}-1}.\end{aligned} \tag{1.61}$$

Since each summand lying at the finite sum on the right-hand side of (1.61) defines an entire function of s, the only singular part of (1.61) may be located at the second term, where we have a singularity at $s=1$.

If χ is a nonprincipal character, it is simple to check that, since $\sum_{n=1}^{\ell-1}\chi(n)=0$ (by the orthogonality relation (1.6)), the second term in (1.61) can be written as

$$\begin{aligned}\frac{\ell^{-1}}{s-1}\sum_{r=1}^{\ell-1}\chi(r)\,r^{1-s}&=\frac{\ell^{-1}}{s-1}\sum_{r=1}^{\ell-1}\chi(r)\,\left(1-\log(r)\,(s-1)+O(s-1)^2\right)\\&=-\ell^{-1}\sum_{r=1}^{\ell-1}\chi(r)\,\log(r)+O(s-1),\end{aligned}$$

and so $s=1$ is removable and $L(s,\chi)$ is continued to an entire function.

If, otherwise, χ is the principal character χ_1, then by the above computations,

$$\frac{\ell^{-1}}{s-1}\sum_{r=1}^{\ell-1}\chi_1(r)\,r^{1-s} = \frac{\varphi(\ell)}{\ell}\,\frac{1}{s-1} - \ell^{-1}\sum_{r=1}^{\ell-1}\chi_1(r)\,\log(r) + O(s-1),$$

and so $s=1$ is a simple pole with residue $\varphi(\ell)/\ell$ of the function $L(s,\chi_1)$. ■

Now, we shall study the properties of the analytic continuation provided by Theorem 1.5.. In particular, by invoking the properties of Hermite's representation, we will derive the meromorphic expansions of $\zeta(s,a)$, $\zeta(s)$ and $L(s,\chi_1)$ around the simple pole that all these have at $s=1$.

But first, we shall derive two interesting consequences of Hermite's representation (1.55), together with Bohr-Mollerup theorem (theorem 1.3.) and Stirling's formula for $\Gamma(s)$ (1.32).

The first consequence is a formula due to M. Lerch who, in 1894 [24], proved

Lemma 1.2. (Lerch's formula) Let $0 < a \leq 1$: then the following identity holds

$$\log\Gamma(a) = \zeta'(0,a) - \zeta'(0), \tag{1.62}$$

where $'$ denotes the derivative of $\zeta(s,a)$ with respect to the complex variable s.

The reader can consult some of the most famous proofs of (1.62) in [107] and [24]. The first authors invoke Binet's second formula for $\log\Gamma(a)$, as well as Hermite's integral representation (1.55) to arrive at (1.62). The second author invokes a more familiar representation of $\zeta(s,a)$ and deduces that, following this representation and making the calculations required at the right-hand side of (1.62), one arrives at $\log\Gamma(a)$ by invoking the identity (1.31).

Before starting our own proof, based on Bohr-Mollerup's theorem, it should be noted that, although we took $0 < a \leq 1$ in the definition of Hurwitz's $\zeta-$function (1.23), it is easily seen that this can be extended for all $a \in \mathbb{R}_+$, since both the infinite series

$$\zeta(s,a) = \sum_{n=0}^{\infty}\frac{1}{(n+a)^s}, \tag{1.63}$$

and Hermite's representation of it (1.55) are still well-defined if $a > 1$.

In the forthcoming proof, we take $a \in \mathbb{R}_+$ in the definition of $\zeta(s,a)$ since there is no loss of generality in doing so. Hence, it is clear that (1.62) holds for all $a \in \mathbb{R}_+$ as well.

Proof of Lemma 1.2.: We shall prove the extension of formula (1.62) to all $a \in \mathbb{R}_+$. First, from (1.63), it is easy to check that

$$\zeta(s,a+1) = \zeta(s,a) - a^{-s}, \quad \mathrm{Re}(s) > 1 \tag{1.64}$$

and a differentiation of (1.64) with respect to s obeys to the recursive relation

$$\zeta'(s, a+1) = \zeta'(s,\, a) + a^{-s}\, \log(a), \quad \mathrm{Re}(s) > 1, \tag{1.65}$$

which can be analytically continued to all $s \in \mathbb{C} \setminus \{1\}$. Taking $s = 0$ in (1.65) gives

$$\zeta'(0,\, a+1) = \zeta'(0, a) + \log(a). \tag{1.66}$$

Now, let $F:\ \mathbb{R}_+ \to \mathbb{R}_+$ be the real-valued function defined by $F(a) = e^{\zeta'(0,a)}$ (note that we are using the fact that $\zeta'(0, a)$ is well-defined for $a \in \mathbb{R}_+$). From (1.66), F satisfies

$$F(a+1) = e^{\zeta'(0,a+1)} = a\, e^{\zeta'(0,a)} = a\, F(a), \tag{1.67}$$

i.e., the same functional relation as $\Gamma(a)$.

Now, from Hermite's representation (1.55) and the uniform convergence of the integral $\mathcal{I}(s)$ (1.56), we can find a formula for $\zeta^{'}(0, a)$ if we differentiate under the integral sign (which is justified by Proposition 1.7.) and take the limit $s \to 0$. This gives

$$\zeta'(0, a) = \left(a - \frac{1}{2}\right) \log(a) - a + 2 \int_0^\infty \frac{\arctan\,(x/a)}{e^{2\pi x} - 1} dx. \tag{1.68}$$

Using (1.68) we can see that, as a real function, $\log F(a)$ is convex: indeed, from (1.68),

$$\frac{d^2}{da^2} \log(F(a)) = \frac{d^2}{da^2} \zeta'(0, a) = \frac{1}{2a^2} + \frac{1}{a} + 4a \int_0^\infty \frac{x}{(x^2 + a^2)^2\, (e^{2\pi x} - 1)} dx > 0, \tag{1.69}$$

since, for every $a \in \mathbb{R}_+$, every term in the last expression is strictly positive.

Actually, it is simple to verify that, once we put $s = 2$ in (1.55), we get the last expression on the right-hand side of (1.69), so we can actually write (1.69) as

$$\frac{d^2}{da^2} \log(F(a)) = \zeta(2, a) > 0, \tag{1.70}$$

which is quite unexpected since it relates the derivatives of $\zeta(s, a)$ with a particular value attained at $s = 2$.

Since $F(1) = e^{\zeta'(0,1)} = e^{\zeta'(0)}$, we see that the function $G:\ \mathbb{R}_+ \longmapsto \mathbb{R}_+$ defined as

$$G(a) := F(a)\, e^{-\zeta'(0)}$$

satisfies $G(1) = 1$, $G(a+1) = a\, G(a)$ (by (1.67)) and $\log(G(a))$ is convex by (1.69). Finally, an application of Bohr-Mollerup Theorem (Theorem 1.3.) yields

$$F(a)\, e^{-\zeta'(0)} = \Gamma(a),$$

which is equivalent to Lerch's formula (1.62). ■

There are several corollaries of Lerch's formula: for instance, in [24], it is shown that Gauss's multiplication and Euler's reflection formulas, (1.36) and (1.34), follow immediately from (1.62). In the same paper, it is shown that, as a consequence of both (1.62) and Hurwitz's functional equation (equation (1.25) above, still unproved by us), one could also obtain the beautiful formula discovered by Kummer,

$$\log \Gamma(a) = \frac{1}{2} \log\left(\pi \csc(\pi a)\right) + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(\gamma + \log(2\pi n))\, \sin(2\pi n a)}{n}, \quad 0 < a < 1,$$

which was historically one of the main motivations in Lerch's approach to (1.62) [37].

In the next lemma we prove a result which is not commonly introduced as a consequence of (1.62), but rather proved by other means, completely equivalent to ours (see [107]).

This formula is attributed to J. P. M. Binet (1839) and it is of great use, as we shall see, to prove more accurate estimates for $\log \Gamma(s)$ when $|s| \to \infty$ and to establish several other properties of the meromorphic expansion of the functions $\zeta(s)$, $L(s, \chi)$ and $\zeta(s, a)$.

This formula can be established as follows:

Lemma 1.3.: (Binet's (second) formula) For $a \in \mathbb{R}_+$, $\log \Gamma(a)$ has the integral representation

$$\log \Gamma(a) = \left(a - \frac{1}{2}\right) \log(a) - a + \frac{1}{2} \log(2\pi) + 2 \int_0^{\infty} \frac{\arctan(x/a)}{e^{2\pi x} - 1}\, dx. \tag{1.71}$$

Proof: Using Lerch's identity and (1.68), we have that $\log \Gamma(a)$ is expressible as

$$\log \Gamma(a) = \left(a - \frac{1}{2}\right) \log(a) - a + 2 \int_0^{\infty} \frac{\arctan\,(x/a)}{e^{2\pi x} - 1} dx - \zeta'(0). \tag{1.72}$$

Thus, to determine an exact expression for $\log \Gamma(a)$ we need to find the unknown constant $\zeta'(0)$. In order to determine it, note the inequality

$$\left| \int_0^{\infty} \frac{\arctan(x/a)}{e^{2\pi x} - 1} dx \right| \leq \frac{1}{a} \left| \int_0^{\infty} \frac{x}{e^{2\pi x} - 1} dx \right| = \frac{\zeta(2)}{4\pi^2 a} = \frac{1}{24\, a}, \tag{1.73}$$

which comes from the elementary inequality $0 \leq \arctan(x) \leq x$, $x \in \mathbb{R}_+$, and from the integral representation

$$\zeta(s)\,\Gamma(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1}\,dx, \quad \mathrm{Re}(s) > 1. \tag{1.74}$$

From (1.73) and (1.72) it is simple to obtain

$$\left|\log\Gamma(a) - \left(a - \frac{1}{2}\right)\log(a) + a + \zeta'(0)\right| \leq \frac{1}{12a}, \tag{1.75}$$

which shows that the left-hand side of (1.75) vanishes once we take the limit $a \to \infty$. Comparing with Stirling's formula (see eq. (1.33) above),

$$\lim_{a\to\infty}\left[\log\Gamma(a) - \left(a - \frac{1}{2}\right)\log(a) + a - \frac{1}{2}\log(2\pi)\right] = 0,$$

we immediately arrive to $\zeta'(0) = -\frac{1}{2}\log(2\pi)$ and the identity (1.71) follows. ■

We have made all the efforts to derive the results regarding $\Gamma(s)$ in closed form, i.e., by appealing only to its functional properties, following a similar exposition to Artin's book and Remmert's paper [87].

This is the reason why we have derived (1.71) by invoking Lerch's identity (1.62). Of course, it is probably possible to establish (1.71) directly via functional properties of $\Gamma(s)$. However, since both formulas (1.62) and (1.71) are greatly needed for future developments, we have decided to present them by this order.

Notice that, by taking $a = 1$ in (1.71), we derive the interesting identity

$$\int_0^\infty \frac{\arctan(x)}{e^{2\pi x} - 1}\,dx = \frac{1}{2} - \frac{1}{4}\log(2\pi). \tag{1.76}$$

Furthermore, differentiating (1.71) and appealing to the definition of the digamma function (1.38), we can also obtain the representations

$$\psi(a) = \log(a) - \frac{1}{2a} - 2\int_0^\infty \frac{x}{(x^2 + a^2)\,(e^{2\pi x} - 1)}\,dx, \tag{1.77}$$

$$\gamma = \psi(1) = \frac{1}{2} + 2\int_0^\infty \frac{x}{(x^2 + a^2)\,(e^{2\pi x} - 1)}\,dx. \tag{1.78}$$

Note also that Binet's formula (1.71) improves the estimate given by Stirling's formula in the form (1.33): in (1.75), we have seen that the remainder function $\mu(s)$ satisfies the estimate

$$|\mu(s)| \leq \frac{1}{12\,|s|}, \quad \mathrm{Re}(s) > 0,$$

which is more precise than the one given in (1.33).

Finally, one can also use Binet's formula to derive a new proof of Gauss's digamma theorem (1.47). To do this, take $a = p/q$ and add equations (1.77) and (1.78): this gives the identity

$$\psi\left(\frac{p}{q}\right) + \gamma = \frac{1}{2} - \frac{q}{2p} + \log\left(\frac{p}{q}\right) + 2\left(1 - \frac{p^2}{q^2}\right)\int_0^\infty \frac{x}{(x^2+1)\,(x^2+p^2/q^2)}\,\frac{dx}{e^{2\pi x}-1} \tag{1.79}$$

with the last integral being given by straightforward computation from the residue theorem which, when evaluated, gives (1.47).

We are now ready to find the constant terms in the meromorphic expansions of $\zeta(s,a)$, $\zeta(s)$ and $L(s,\chi)$ when $\chi = \chi_1$. The meromorphic expansions of these functions are stated now as follows

Theorem 1.6.: (Meromorphic expansions) The constant term in the meromorphic expansion for $\zeta(s,a)$ is given by

$$A_0 := \lim_{s\to 1}\left[\zeta(s,a) - \frac{1}{s-1}\right] = -\psi(a). \tag{1.80}$$

Moreover, (1.80) implies

$$A_0 := \lim_{s\to 1}\left[\zeta(s) - \frac{1}{s-1}\right] = \gamma, \tag{1.81}$$

for $\zeta(s)$ and

$$\begin{aligned} A_0 :&= \lim_{s\to 1}\left(L(s,\chi_1) - \frac{\varphi(\ell)}{\ell}\,\frac{1}{s-1}\right) \\ &= \frac{\varphi(\ell)}{\ell}\gamma + \frac{\varphi(\ell)}{\ell}\,\log(2) - \frac{2}{\ell}\sum_{k=1}^{\lceil \ell/2\rceil - 1} G(k,\chi_1)\,\log\sin\left(\frac{\pi k}{\ell}\right), \end{aligned} \tag{1.82}$$

for the Dirichlet $L-$function associated to the principal character modulo ℓ.

Proof: From Hermite's representation of $\zeta(s,a)$ (1.55) it is immediate to see that, after using (1.77),

$$R(a) := \lim_{s\to 1}\left[\zeta(s,a) - \frac{1}{s-1}\right] = \frac{1}{2a} - \log(a) + 2\int_0^\infty \frac{x}{x^2+a^2}\,\frac{dx}{e^{2\pi x}-1} = -\psi(a). \tag{1.83}$$

Clearly, since $\zeta(s,1)=\zeta(s)$ and

$$R(1) := \lim_{s\to 1}\left(\zeta(s)-\frac{1}{s-1}\right)=\frac{1}{2}+2\int\limits_0^\infty \frac{x}{x^2+1}\,\frac{dx}{e^{2\pi x}-1} = -\psi(1)=\gamma, \tag{1.84}$$

we arrive immediately to (1.80) and (1.81)

Now, if $\chi=\chi_1$, $L(s,\chi)$ has a simple pole located at $s=1$ with residue $\varphi(\ell)/\ell$ (by Theorem 1.5.). Furthermore, from (1.80),

$$\begin{aligned} L(s,\chi_1) &= \ell^{-s}\sum_{r=1}^{\ell-1}\chi(r)\,\zeta\left(s,\frac{r}{\ell}\right) = \ell^{-s}\sum_{r=1}^{\ell-1}\chi_1(r)\,\left(\frac{1}{s-1}-\psi\left(\frac{r}{\ell}\right)+O(s-1)\right) \\ &= \ell^{-s}\,\frac{\varphi(\ell)}{s-1} - \ell^{-s}\sum_{r=1}^{\ell-1}\chi_1(r)\,\psi\left(\frac{r}{\ell}\right)+O(s-1) = \\ &= \frac{\varphi(\ell)}{s-1}\left[\frac{1}{\ell}-\frac{\log(\ell)}{\ell}(s-1)+O(s-1)^2\right]-\ell^{-s}\sum_{r=1}^{\ell-1}\chi_1(r)\,\psi\left(\frac{r}{\ell}\right)+O(s-1), \end{aligned} \tag{1.85}$$

and so, from Gauss's digamma theorem (1.47),

$$\begin{aligned} &\lim_{s\to 1}\left[L(s,\chi_1)-\frac{\varphi(\ell)}{\ell}\,\frac{1}{s-1}\right] = -\frac{\varphi(\ell)}{\ell}\log(\ell)-\ell^{-1}\sum_{r=1}^{\ell-1}\chi_1(r)\,\psi\left(\frac{r}{\ell}\right) \\ &= \frac{\varphi(\ell)}{\ell}\gamma+\frac{\varphi(\ell)}{\ell}\log(2)+\frac{\pi}{2\ell}\sum_{r=1}^{\ell-1}\chi_1(r)\,\cot\left(\frac{\pi r}{\ell}\right)-\frac{2}{\ell}\sum_{k=1}^{\lceil \ell/2\rceil-1}\log\sin\left(\frac{\pi k}{\ell}\right)\sum_{r=1}^{\ell-1}\chi_1(r)\,\cos\left(\frac{2\pi r}{\ell}\,k\right). \end{aligned} \tag{1.86}$$

From the fact that χ_1 is even, we can check that the third term in (1.86) vanishes since

$$\frac{\pi}{2\ell}\sum_{r=1}^{\ell-1}\chi_1(r)\,\cot\left(\frac{\pi r}{\ell}\right)=\frac{\pi}{2\ell}\sum_{r=1}^{\ell-1}\chi_1(\ell-r)\,\cot\left(\frac{\pi(\ell-r)}{\ell}\right)=-\frac{\pi}{2\ell}\sum_{r=1}^{\ell-1}\chi_1(r)\,\cot\left(\frac{\pi r}{\ell}\right)=0 \tag{1.87}$$

Furthermore, the last term in (1.86) can be rewritten in the form

$$\begin{aligned} &\sum_{k=1}^{\lceil \ell/2\rceil-1}\log\sin\left(\frac{\pi k}{\ell}\right)\sum_{r=1}^{\ell-1}\chi_1(r)\,\cos\left(\frac{2\pi r}{\ell}\,k\right)=\frac{1}{2}\sum_{k=1}^{\lceil \ell/2\rceil-1}\log\sin\left(\frac{\pi k}{\ell}\right)\sum_{r=1}^{\ell-1}\chi_1(r)\,\left[e^{i\frac{2\pi r}{\ell}k}+e^{-i\frac{2\pi r}{\ell}k}\right] \\ &=\frac{1}{2}\sum_{k=1}^{\lceil \ell/2\rceil-1}\log\sin\left(\frac{\pi k}{\ell}\right)\,(G(k,\chi_1)+G(-k,\chi_1))=\sum_{k=1}^{\lceil \ell/2\rceil-1}G(k,\chi_1)\,\log\sin\left(\frac{\pi k}{\ell}\right), \end{aligned} \tag{1.88}$$

where the last step comes from the fact that, for an even character χ, $G(-n,\chi)=G(n,\chi)$. A substitution of (1.88) into (1.86) and the use (1.87) yields (1.82). ■

Remark 1.1.: We can still extend the meromorphic representation of $\zeta(s,a)$, $\zeta(s)$ and $L(s,\chi_1)$ to powers of higher order. We define the Stieltjes constants as multiples of the coefficients occurring in the Laurent expansion for $\zeta(s,a)$, i.e.,

$$\zeta(s,a) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n(a)\,(s-1)^n. \tag{1.89}$$

If we use (1.55) and the power expansions of the integrand around $s=1$, we are able to find that these constants are equal to

$$\gamma_n(a) = \left[\frac{1}{2a} - \frac{\log(a)}{n+1}\right] \log^n(a) - i \int_0^{\infty} \left[\frac{\log^n(a-ix)}{a-ix} - \frac{\log^n(a+ix)}{a+ix}\right] \frac{dx}{e^{2\pi x}-1}. \tag{1.90}$$

However, there are other more natural representations of the constants $\gamma_n(a)$. For example, Wilton [111] and Berndt [23] have presented an evaluation which depends on the limit

$$\gamma_n(a) = \lim_{m\to\infty} \left(\sum_{k=0}^{m} \frac{\log^n(k+a)}{k+a} - \frac{\log^{n+1}(m+a)}{n+1}\right). \tag{1.91}$$

Of course, comparing (1.91) with (1.42), we immediately see that $\gamma_0(a) = -\psi(a)$ as expected. Formula (1.91) also extends the classical definition of the constants $\gamma_n := \gamma_n(1)$ appearing on the meromorphic expansion of $\zeta(s)$,

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n\,(s-1)^n. \tag{1.92}$$

Using the computations given in the previous theorem, we are also able to establish

Theorem 1.7.: Let χ be a nonprincipal Dirichlet character modulo ℓ. In general, we have that

$$L(1,\chi) = -\ell^{-1} \sum_{r=1}^{\ell-1} \chi(r)\,\psi\left(\frac{r}{\ell}\right), \tag{1.93}$$

which implies, depending on the sign of χ, the identities

$$L(1,\chi) = -\frac{2}{\ell} \sum_{k=1}^{\lceil \ell/2\rceil -1} G(k,\chi)\,\log\sin\left(\frac{\pi k}{\ell}\right), \quad \chi \text{ even}, \tag{1.94}$$

$$L(1,\chi) = \frac{\pi}{2\ell} \sum_{r=1}^{\ell-1} \chi(r)\,\cot\left(\frac{\pi r}{\ell}\right), \quad \chi \text{ odd}. \tag{1.95}$$

Proof: From (1.24) and the definition of $\gamma_n(a)$ (1.89)

$$L(s,\chi) = \ell^{-s} \sum_{r=1}^{\ell-1} \chi(r) \left(\frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n\left(\frac{r}{\ell}\right)(s-1)^n \right)$$
$$= -\ell^{-s} \sum_{r=1}^{\ell-1} \chi(r)\, \psi\left(\frac{r}{\ell}\right) + O(s-1), \tag{1.96}$$

since χ is nonprincipal. Putting $s=1$ in (1.96), we arrive at (1.93) as expected.

From Gauss's digamma theorem (1.47), we can rewrite (1.93) in the form

$$L(1,\chi) = \frac{\pi}{2\ell} \sum_{r=1}^{\ell-1} \chi(r) \cot\left(\frac{\pi r}{\ell}\right) - \frac{2}{\ell} \sum_{k=1}^{\lceil \ell/2 \rceil - 1} \log\sin\left(\frac{\pi k}{\ell}\right) \sum_{r=1}^{\ell-1} \chi(r) \cos\left(\frac{2\pi r}{\ell}\, k\right). \tag{1.97}$$

If χ is even, recall from (1.87) and (1.88) that the first term vanishes and that the second term of (1.97) can be written in terms of the Gaussian sum $G(k,\chi)$. This shows (1.94).

If χ is odd, we can proceed as in (1.87) to see that the second term in (1.97) vanishes since

$$\sum_{r=1}^{\ell-1} \chi(r) \cos\left(\frac{2\pi r}{\ell}\, k\right) = \sum_{r=1}^{\ell-1} \chi(\ell-r) \cos\left(\frac{2\pi(\ell-r)}{\ell}\, k\right) = -\sum_{r=1}^{\ell-1} \chi(r) \cos\left(\frac{2\pi r}{\ell} k\right) = 0,$$

which proves (1.95). ■

To finish this section, we compute the values of $L'(0,\chi)$ for any Dirichlet character based upon Lerch's formula and the previous computations. The main details are given in the following theorem.

Theorem 1.8. (Derivatives of $L-$functions at $s=0$) Let χ be a nonprincipal Dirichlet character modulo ℓ. Then its Dirichlet $L-$function satisfies the identities

$$L'(0,\chi) = \sum_{r=1}^{\ell-1} \chi(r) \log\Gamma\left(\frac{r}{\ell}\right), \quad \chi \text{ even}, \tag{1.98}$$

$$L'(0,\chi) = \frac{\log(\ell)}{\ell} \sum_{r=1}^{\ell-1} \chi(r)\, r + \sum_{r=1}^{\ell-1} \chi(r) \log\Gamma\left(\frac{r}{\ell}\right), \quad \chi \text{ odd}. \tag{1.99}$$

Additionally, if $\chi = \chi_1$ is the principal character modulo ℓ, $L'(0,\chi)$ satisfies the identity

$$L'(0,\chi_1) = -\frac{\varphi(\ell)}{2} \log(2\pi) + \sum_{r=1}^{\ell-1} \chi_1(r) \log\Gamma\left(\frac{r}{\ell}\right). \tag{1.100}$$

Proof: Since $\zeta(0,a) = \frac{1}{2} - a$ (take $s = 0$ in (1.55)), the use of (1.24) together with Lerch's identity (1.62) gives

$$\begin{aligned}
L^{'}(0,\chi) &= -\log(\ell) \sum_{r=1}^{\ell-1} \chi(r)\, \zeta\left(0,\, \frac{r}{\ell}\right) + \sum_{r=1}^{\ell-1} \chi(r)\, \zeta^{'}\left(0,\, \frac{r}{\ell}\right) \\
&= -\log(\ell) \sum_{r=1}^{\ell-1} \chi(r)\, \zeta\left(0,\, \frac{r}{\ell}\right) + \sum_{r=1}^{\ell-1} \chi(r) \left(\log\Gamma\left(\frac{r}{\ell}\right) - \frac{1}{2}\log(2\pi)\right) \qquad (1.101) \\
&= -\log(\ell) \sum_{r=1}^{\ell-1} \chi(r) \left(\frac{1}{2} - \frac{r}{\ell}\right) + \sum_{r=1}^{\ell-1} \chi(r)\, \log\Gamma\left(\frac{r}{\ell}\right) \\
&= \frac{\log(\ell)}{\ell} \sum_{r=1}^{\ell-1} \chi(r)\, r + \sum_{r=1}^{\ell-1} \chi(r)\, \log\Gamma\left(\frac{r}{\ell}\right), \qquad (1.102)
\end{aligned}$$

proving immediately (1.99). For the case where χ is even and nonprincipal, the first sum in (1.102) reduces to

$$\sum_{r=1}^{\ell-1} \chi(r)\, r = \sum_{r=1}^{\ell-1} \chi(\ell - r)\, (\ell - r) = \ell \sum_{r=1}^{\ell-1} \chi(\ell - r) - \sum_{r=1}^{\ell-1} \chi(\ell - r)\, r = -\sum_{r=1}^{\ell-1} \chi(r)\, r,$$

and so it vanishes, proving (1.98).

Finally, when $\chi = \chi_1$, notice that formula (1.22) holds for all $s \in \mathbb{C}$ by analytic continuation and so, after taking $s = 0$ there, we obtain $L(0, \chi_1) = 0$.

From the computations held for even χ (1.102), we are able to obtain

$$\begin{aligned}
L^{'}(0,\chi) &= -\log(\ell) \sum_{r=1}^{\ell-1} \chi(r)\, \zeta\left(0,\, \frac{r}{\ell}\right) + \sum_{r=1}^{\ell-1} \chi(r) \left(\log\Gamma\left(\frac{r}{\ell}\right) - \frac{1}{2}\log(2\pi)\right) \\
&= -\log(\ell)\, L(0,\chi) + \sum_{r=1}^{\ell-1} \chi(r)\, \log\Gamma\left(\frac{r}{\ell}\right) - \frac{\varphi(\ell)}{2}\log(2\pi) \\
&= \sum_{r=1}^{\ell-1} \chi(r)\, \log\Gamma\left(\frac{r}{\ell}\right) - \frac{\varphi(\ell)}{2}\log(2\pi).\ \blacksquare \qquad (1.103)
\end{aligned}$$

Now, we can still study other properties of $\zeta(s,a)$, $\zeta(s)$ and $L(s,\chi)$ that the representation (1.55) offers.

In the following section, we prove three classical properties of these functions, which can now be deduced by appealing to the integral representation (1.55).

As far as we are concerned, the proofs presented next are new. We start with the classical Euler's identity, which connects the values of $\zeta(2n)$ with the sequence $(B_{2n})_{n\in\mathbb{N}}$ of Bernoulli numbers

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n)!} B_{2n}, \tag{1.104}$$

with B_n (Bernoulli numbers) being a sequence defined by the generating function

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad |z| < 2\pi. \tag{1.105}$$

According to Edwards [43], it is possible that Euler's formula (1.104) may have had a significant impact on Riemann's work.

As it is known, at the beginning of his paper (the reader can consult an english translation by David Wilkins at [88]), Riemann proposes to find a formula for $\zeta(s)$ which "remains always valid" for all $s \in \mathbb{C}$ by considering the integral representation

$$\zeta(s)\,\Gamma(s) = \int_0^{\infty} \frac{x^{s-1}}{e^x - 1}\, dx, \quad \operatorname{Re}(s) > 1. \tag{1.106}$$

As remarked by Edwards, a particular case of the formula (1.106) appears in a paper of N. H. Abel [1] which was published in 1839 and it is very likely that Riemann was aware of this. It is impressive that in the afore-mentioned paper, Abel also presented another version of the now called Abel-Plana formula (1.49).

So, if Riemann was aware of Abel's paper and contributions, it is interesting to wonder why he did not use Abel's formula to achieve the analytic continuation for $\zeta(s)$ as the way we did above.

Edwards makes another intriguing assertion regarding the discovery of the analytic continuation of $\zeta(s)$ and the proof of its functional equation. In his famous book on the Riemann hypothesis [43] p.12, it is also stated that *"There is no easy way to deduce this famous formula of Euler's* [*i.e., eq. (1.104)*] *from Riemann's integral formula for* $\zeta(s)$ [*i.e., eq. (1.106)*] *and it may well have been this problem of deriving anew* [*eq.(1.104)*] *which led Riemann to the discovery of the functional equation of the zeta function".* It is not unfair to conjecture about other possible attempts that Riemann himself could try in order to evaluate $\zeta(2n)$, as well as to prove analytic properties of $\zeta(s)$.

The purpose of this little historical remark is to emphasize that there is, indeed, a very immediate way to prove a formula for $\zeta(2n)$ via Abel-Plana's formula, which Riemann probably knew, and the integral formula (*1.106*).

It is not also unfair to conjecture that, by using this approach, Riemann could also have derived Hermite's formula for $\zeta(s)$ (1.60), which, as we shall see, conduces to another proof of the functional equation for $\zeta(s)$.

With this historical motivation, we now move on to the next section, where we prove Euler's famous formula for $\zeta(2n)$ by using (1.106) and Abel-Plana's summation formula.

1.5.3 Consequences: Particular values for $\zeta(s,a)$ and $\zeta(s)$

Corollary 1.1: Euler's identity for $\zeta(2n)$ When $s = 2n$, $\zeta(s)$ satisfies (1.104).

Proof: In the Example 1.1 given for Abel-Plana's formula take $a = 0$ in (1.51) and let $0 < x < 2\pi$. To evaluate the integral at the left-hand side of (1.51),

$$\int_0^\infty \frac{\sin(xy)}{e^{2\pi y}-1}\,dy,$$

we may express $\sin(xy)$ by its power series and interchange the order of integration with the summation. This procedure is well-justified since

$$\begin{aligned}\int_0^\infty \left|\sum_{n=1}^\infty \frac{(-1)^{n-1}}{(2n-1)!}\,\frac{(xy)^{2n-1}}{e^{2\pi x}-1}\right| dy &\leq \int_0^\infty \sum_{n=1}^\infty \frac{(xy)^{2n-1}}{(2n-1)!\,(e^{2\pi y}-1)}\,dy \\ &= \int_0^\infty \frac{\sinh(xy)}{e^{2\pi y}-1}\,dy < \infty,\end{aligned}$$

by the condition imposed over x and the absolute convergence of the last integral. By (1.51) this yields

$$\begin{aligned}\frac{1}{2}\left(\frac{1}{e^x-1}+\frac{1}{2}-\frac{1}{x}\right) &= \int_0^\infty \frac{\sin(xy)}{e^{2\pi y}-1}dy = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{(2n-1)!}x^{2n-1}\int_0^\infty \frac{y^{2n-1}}{e^{2\pi y}-1}\,dy \\ &= \sum_{n=1}^\infty \frac{(-1)^{n-1}}{(2\pi)^{2n}}\,\zeta(2n)\,x^{2n-1}, \quad 0 < x < 2\pi \qquad (1.107)\end{aligned}$$

where the last equality came from Riemann's integral formula (1.106) for $s = 2n$.

Finally, recall the definition of Bernoulli numbers given by their generating function (1.105): note that the left-hand side of (1.107) can be expressed as

$$\frac{1}{2}\left(\sum_{n=0}^\infty \frac{1}{n!}\,B_n x^{n-1}+\frac{1}{2}-\frac{1}{x}\right) = \frac{1}{2}\sum_{n=1}^\infty \frac{B_{2n}}{(2n)!}x^{2n-1}, \quad 0 < x < 2\pi \qquad (1.108)$$

where the last step came from the fact that $B_{2n+1} = 0$ for $n \geq 1$, $B_0 = 1$ and $B_1 = -\frac{1}{2}$.

A direct comparison of both power series in (1.107) and (1.108) yields immediately (1.104). ■

Using the previous corollary, as well as Hermite's representation for $\zeta(s, a)$, we are now able to prove a celebrated identity for $\zeta(-n, a)$, involving the Bernoulli polynomials.

Furthermore, using this, we can also prove similar identities for $\zeta(-n)$ and $L(-n, \chi)$.

Before proving the next result, we need two important definitions, which we now give:

Definition 1.6.: For real $x \in \mathbb{R}$, the Bernoulli polynomials $B_n(x)$ are defined as coefficients of the series expansion,

$$\frac{z\, e^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x)\, \frac{z^n}{n!}, \quad |z| < 2\pi. \tag{1.109}$$

Related with the previous definition, we introduce a character analogue of the Bernoulli numbers, which we define as follows:

Definition 1.7.: Let χ be a Dirichlet character modulo ℓ. We define the sequence of the Bernoulli numbers with character weight, $B_{n,\chi}$, as the coefficients of the series expansion

$$\sum_{r=1}^{\ell-1} \chi(r)\, \frac{r\, e^{rz}}{e^{\ell z} - 1} = \sum_{n=0}^{\infty} B_{n,\chi}\, \frac{z^n}{n!}, \quad |z| < 2\pi. \tag{1.110}$$

It follows from elementary computations that we can express $B_n(x)$ and $B_{n,\chi}$ as the polynomials

$$B_n(x) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) B_k\, x^{n-k} \tag{1.111}$$

and

$$B_{n,\chi} = \ell^{n-1} \sum_{r=1}^{\ell-1} \chi(r)\, B_n\left(\frac{r}{\ell}\right). \tag{1.112}$$

It is also immediate to check that $B_n = B_n(0) = B_n(1)$ and that $B_n(x)$ obeys to the reflection formula

$$B_n(1 - x) = (-1)^n B_n(x), \quad n \geq 0. \tag{1.113}$$

With these important definitions, we are now ready to prove the following identity:

Corollary 1.2.: (Particular values for $\zeta(-n,\, a)$) The Hurwitz $\zeta-$function obeys to the Euler-type formula,

$$\zeta(-n, a) = -\frac{B_{n+1}(a)}{n+1}, \quad 0 < a \leq 1, \tag{1.114}$$

where $B_n(a)$ denotes the Bernoulli polynomial of order n, defined by (1.109).

Moreover, $\zeta(s)$ and $L(s,\chi)$ have, respectively, the particular values at the negative integers,

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}, \tag{1.115}$$

$$L(-n, \chi) = -\frac{B_{n+1,\chi}}{n+1}, \tag{1.116}$$

where $B_{n,\chi}$ denotes the character analogue of the sequence of Bernoulli numbers (1.112).

Proof: From Hermite's representation of $\zeta(s,a)$ (1.55) we can write

$$\zeta(-n,\, a) = \frac{a^n}{2} - \frac{a^{n+1}}{n+1} - 2\int_0^\infty \frac{\sin\left(n\, \arctan(x/a)\right)}{\left(a^2+x^2\right)^{-n/2}\left(e^{2\pi x}-1\right)}\, dx, \tag{1.117}$$

and a closed-form evaluation of (1.117) now depends on the last integral, which we denote by $I_n(a)$.

To evaluate it, we appeal to the multi-angle formula for $\sin(n\theta)$ and to the elementary trigonometric relations $\sin(\arctan(x)) = x/\sqrt{1+x^2}$, $\cos(\arctan(x)) = 1/\sqrt{1+x^2}$, which give

$$\begin{aligned} I_n(a) &= \sum_{k=0}^{n-1} \binom{n}{k} \sin\left[\frac{1}{2}(n-k)\,\pi\right]\, a^k \int_0^\infty \frac{x^{n-k}}{e^{2\pi x}-1}\, dx \\ &= \sum_{k=0}^{n-1} \frac{n!}{k!} \sin\left[\frac{1}{2}(n-k)\,\pi\right]\, a^k\, (2\pi)^{k-n-1}\zeta(n-k+1) && (1.118) \\ &= \frac{1}{2}\sum_{k=0}^{n-1} \frac{n!}{k!\,(n-k+1)!}\, B_{n-k+1}\, a^k && (1.119) \end{aligned}$$

where the last equality comes from the fact that the terms of the sum in the right-hand side of (1.118) are only nonzero if $n-k+1$ is even together with (1.104).

From (1.111),

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}\, x^k = 1 - \frac{n}{2}\, x + \sum_{k=2}^{n} \binom{n}{k} B_{n-k}\, x^k,$$

and due to the elementary properties of the Bernoulli numbers, the last sum runs over the integers k such that $n-k$ is even.

Since the index in (1.119) is such that that $n-k+1$ must be even, we obtain

$$I_n(a) = \frac{1}{2}\sum_{k=0}^{n-1} \frac{n!}{k!\,(n-k+1)!}\, B_{n-k+1}\, a^k = \frac{1}{2n+2}\left(B_{n+1}(a) + \frac{1}{2}(n+1)\, a^n - a^{n+1}\right), \quad (1.120)$$

which, when combined with (1.117), gives (1.114).

Now, (1.115) and (1.116) follow immediately: taking $a = 1$ and using the fact that $B_n(1) = B_n$ we arrive at (1.115).

In order to obtain (1.116), it is sufficient to use the result (1.114) together with (1.112). ■

Finally, an immediate consequence of the previous identities is the information regarding the localization of the trivial zeros of $L(s,\chi)$ and $\zeta(s)$.

Corollary 1.3.: Zeros of $L(s,\chi)$ and $\zeta(s)$ The Riemann $\zeta-$function $\zeta(s)$ has real zeros located at $s = -2n$, $n \in \mathbb{N}$.

Analogously,

1. if χ is nonprincipal and even, then $L(s,\chi)$ has trivial zeros located at $s = -2n$, $n \in \mathbb{N}_0$.
2. if χ is nonprincipal and odd, then $L(s,\chi)$ has trivial zeros located at $s = -2n-1$, $n \in \mathbb{N}_0$.

Proof: The first assertion immediately follows from (1.115), since $B_{2n+1} = 0$ for $n \in \mathbb{N}$.

From the previous corollary, assume that χ is even and use the reflection formula for $B_n(x)$ (1.113). We can derive

$$\begin{aligned} L(-2n,\,\chi) &= -\frac{\ell^{2n}}{2n+1}\sum_{r=1}^{\ell-1}\chi(r)\, B_{2n+1}\left(\frac{r}{\ell}\right) = -\frac{\ell^{2n}}{2n+1}\sum_{r=1}^{\ell-1}\chi(\ell-r)\, B_{2n+1}\left(1-\frac{r}{\ell}\right) \\ &= \frac{\ell^{2n}}{2n+1}\sum_{r=1}^{\ell-1}\chi(r)\, B_{2n+1}\left(\frac{r}{\ell}\right), \end{aligned}$$

which proves that $L(-2n,\chi) = 0$. Of course, a similar argument holds for the case where χ is odd and $s = -2n-1$, $n \in \mathbb{N}_0$. ■

1.6 Main section: Functional equation for $\zeta(s)$, $\zeta(s,a)$ and $L(s,\chi)$

After proving Hermite's integral representation for $\zeta(s,a)$ and deduce anew several immediate properties of $\zeta(s)$ and $L(s,\chi)$, we are ready to prove the functional relation that the continuation of $\zeta(s,a)$ to $\mathbb{C}$ satisfies, given by the main theorem. It is now time to prove it:

Proof of the Main Theorem: We first prove (1.25) assuming that $0 < a < 1$. To do so, let us denote by $f(a,\,x)$ the following function, expressable by the series

$$f(a,\,x) := \frac{e^{2x}-1}{2\left(e^{2x}-2e^{x}\cos(2\pi a)+1\right)} = \frac{1}{2} + \sum_{n=1}^{\infty} e^{-nx}\cos(2\pi n a). \tag{1.121}$$

We have seen in the Example 1.1. of Abel-Plana formula that $f(a,\,x)$ obeys to the relation (see equations (1.50) and (1.52))

$$\begin{aligned} f(a,\,x) &= \frac{x}{x^2+4\pi^2a^2} + 2\int_0^{\infty} \frac{\sin(xy)\,\cosh(2\pi a y)}{e^{2\pi y}-1} dx \\ &= \frac{x}{x^2+4\pi^2a^2} + \sum_{k=1}^{\infty}\left[\frac{x}{4\pi^2(k-a)^2+x^2} + \frac{x}{4\pi^2(k+a)^2+x^2}\right]. \end{aligned} \tag{1.122}$$

From (1.121), it is simple to check that, as $x \to \infty$, $f(a,\,x)$ tends to the value $1/2$ exponentially fast. Moreover, from identity (1.122), it is simple to check that $f(a,\,x)$ tends to 0 as $x \to 0$ linearly. Hence, the function defined by the Mellin integral

$$F_1(a,\,s) = \int_0^1 x^{s-1} f(a,x)\,dx \tag{1.123}$$

is analytic in the region $\mathrm{Re}(s) > -1$. Furthermore, the function defined by

$$F_2(a,\,s) = \int_1^{\infty} x^{s-1}\left(f(a,\,x) - \frac{1}{2}\right)\,dx \tag{1.124}$$

is entire.

Let us now define

$$H_1(a,\,s) = F_1(a,\,s) + F_2(a,\,s) - \frac{1}{2s}. \tag{1.125}$$

Then $H_1(a,s)$, as a function of s, is analytic at every point of the half-plane $\mathrm{Re}(s) > -1$ except at the origin, where it has a simple pole with residue $-\frac{1}{2}$.

It is also easy to see that, if $\mathrm{Re}(s) > 1$,

$$\begin{aligned} H_1(a,s) &= \int_0^{\infty} x^{s-1}\left(f(a,\,x) - \frac{1}{2}\right)\,dx = \sum_{n=1}^{\infty}\cos(2\pi n a)\int_0^{\infty} x^{s-1}e^{-nx}dx \\ &= \Gamma(s)\sum_{n=1}^{\infty}\frac{\cos(2\pi n a)}{n^s}, \end{aligned} \tag{1.126}$$

while, for $\mathrm{Re}(s) < 0$,

$$H_1(a,s) = \int_0^1 x^{s-1}\, f(a,x)\, dx + \int_1^\infty x^{s-1} \left(f(a,x) - \frac{1}{2} \right) dx - \frac{1}{2s}$$
$$= \int_0^\infty x^{s-1}\, f(a,x)\, dx. \tag{1.127}$$

Using (1.122) we obtain, for $-1 < \mathrm{Re}(s) < 0$,

$$\begin{aligned} H_1(a,\, s) &= \int_0^\infty \left\{ \frac{x^s}{x^2 + 4\pi^2 a^2} + \sum_{k=1}^\infty \left[\frac{x^s}{4\pi^2 (k-a)^2 + x^2} + \frac{x^s}{4\pi^2 (k+a)^2 + x^2} \right] \right\} dx \\ &= \frac{\pi\, (2\pi a)^{s-1}}{2 \cos\left(\frac{\pi s}{2}\right)} + \frac{\pi\, (2\pi)^{s-1}}{2 \cos\left(\frac{\pi s}{2}\right)} \sum_{k=1}^\infty \left[\frac{1}{(k-a)^{1-s}} + \frac{1}{(k+a)^{1-s}} \right] \\ &= \frac{\pi\, (2\pi)^{s-1}}{2 \cos\left(\frac{\pi s}{2}\right)} \left[\zeta(1-s,\, a) + \zeta(1-s,\, 1-a) \right] := \phi_1(a,s), \end{aligned} \tag{1.128}$$

where we could change the orders of the integration and summation due to the fact that $-1 < \mathrm{Re}(s) < 0$ and via the elementary integral of Mellin type,

$$\int_0^\infty \frac{x^s}{1+x^2} dx = \frac{\pi}{2\, \cos\left(\frac{\pi s}{2}\right)}, \quad -1 < \mathrm{Re}(s) < 1.$$

Since $\zeta(s,a)$ has a simple pole at $s = 1$ with residue 1, we see that $\phi_1(a,s)$ has a simple pole at this point as well, with residue equal to $-1/2$.

Moreover, since the (simple) zeros of $\cos\left(\frac{\pi s}{2}\right)$ in the region $\mathrm{Re}(s) > -1$ are located at $s = 2n-1$, $n = 1, 2, ...,$ the singularities of $\phi_1(a,s)$ are located at these points as well.

However, these are removable since, by the identity (1.114) and the reflection property for $B_n(x)$ (1.113),

$$\zeta\left(-2n,\, a\right) + \zeta\left(-2n,\, 1-a\right) = -\frac{B_{2n+1}(a) + B_{2n+1}(1-a)}{2n+1} = 0.$$

This proves that $\phi_1(a,s)$ and $H_1(a,s)$ have the same properties in the half-plane $\mathrm{Re}(s) > -1$ and they are both analytic in all points except at the origin. Since both coincide in the strip $-1 < \mathrm{Re}(s) < 0$, we conclude from the Principle of Analytic continuation (Prop. 1.8.) that (1.128) is valid in the whole half-plane $\mathrm{Re}(s) > -1$. In particular, it is valid in the region $\mathrm{Re}(s) > 1$, which shows that

$$\Gamma(s) \sum_{n=1}^\infty \frac{\cos(2\pi n a)}{n^s} = \frac{\pi\, (2\pi)^{s-1}}{2 \cos\left(\frac{\pi s}{2}\right)} \left[\zeta(1-s,\, a) + \zeta(1-s,\, 1-a) \right], \quad \mathrm{Re}(s) > 1. \tag{1.129}$$

We may also consider the function

$$g(a,x) := \frac{e^x \sin(2\pi a)}{e^{2x} - 2e^x \cos(2\pi a) + 1} = \sum_{n=1}^{\infty} e^{-nx} \sin(2\pi an), \tag{1.130}$$

which, as $x \to \infty$, tends to 0 exponentially fast, and, as $x \to 0$, approaches the constant value $\frac{1}{2}\cot(\pi a)$ linearly. Therefore, the function

$$G_1(a,\, s) = \int_0^1 x^{s-1} \left[g(a,x) - \frac{\cot(\pi a)}{2} \right] dx \tag{1.131}$$

is analytic in the whole region $\mathrm{Re}(s) > -1$ and

$$G_2(a,\, s) = \int_1^{\infty} x^{s-1} g(a,x)\, dx \tag{1.132}$$

is entire for all $s \in \mathbb{C}$. If we define, in analogy with (1.125),

$$H_2(a,s) = G_1(a,s) + G_2(a,s) + \frac{\cot(\pi a)}{2s}, \tag{1.133}$$

we see that (1.133) is analytic in the half-plane $\mathrm{Re}(s) > -1$ except at the point $s = 0$ and, for $\mathrm{Re}(s) > 1$, it satisfies

$$H_2(a,s) = \Gamma(s) \sum_{n=1}^{\infty} \frac{\sin(2\pi na)}{n^s}, \qquad \mathrm{Re}(s) > 1. \tag{1.134}$$

We can now mimic the computations above, after invoking relation (1.54) given at example 1 to show that

$$\Gamma(s) \sum_{n=1}^{\infty} \frac{\sin(2\pi na)}{n^s} = \frac{\pi\, (2\pi)^{s-1}}{2\, \sin\left(\frac{\pi s}{2}\right)} \left[\zeta(1-s,a) - \zeta(1-s,1-a)\right], \quad \mathrm{Re}(s) > 1. \tag{1.135}$$

Combining both (1.129) and (1.135), we immediately prove the desired functional equation,

$$\zeta(1-s,a) = \frac{2\Gamma(s)}{(2\pi)^s} \left[\cos\left(\frac{\pi s}{2}\right) \sum_{n=1}^{\infty} \frac{\cos(2\pi na)}{n^s} + \sin\left(\frac{\pi s}{2}\right) \sum_{n=1}^{\infty} \frac{\sin(2\pi na)}{n^s} \right], \quad \mathrm{Re}(s) > 1. \blacksquare \tag{1.136}$$

Remark 1.2.: In the previous proof, we were under the hypothesis that $0 < a < 1$. It should be noted that we still need to extend (1.136) to $a = 1$.

In this remark we show that this case is already covered by (1.136): if one takes $a = \frac{1}{2}$ in this formula, it gives

$$\zeta\left(1 - s,\, \frac{1}{2}\right) = 2^{1-s}\pi^{-s}\Gamma(s)\,\cos\left(\frac{\pi s}{2}\right)\sum_{n=1}^{\infty}\frac{(-1)^n}{n^s}$$
$$= 2^{1-s}\pi^{-s}\Gamma(s)\,\cos\left(\frac{\pi s}{2}\right)\left(2^{1-s} - 1\right)\zeta(s,\, 1). \tag{1.137}$$

In the meantime, if $\mathrm{Re}(s) > 1$,

$$\zeta\left(s, \frac{1}{2}\right) = \sum_{n=0}^{\infty}\frac{2^s}{(2n+1)^s} = 2^s\left\{\sum_{n=1}^{\infty}\frac{1}{n^s} - 2^{-s}\sum_{n=1}^{\infty}\frac{1}{n^s}\right\} = (2^s - 1)\,\zeta(s) = (2^s - 1)\,\zeta(s, 1), \tag{1.138}$$

which is valid for all $s \in \mathbb{C}$ by analytic continuation. A substitution of (1.138) in (1.137) immediately implies (1.136) for the case in which $a = 1$.

Remark 1.3.: The formula proved in the main theorem can be rewritten in a more symmetric form,

$$\zeta(1 - s,\, a) = \frac{\Gamma(s)}{(2\pi)^s}\left[e^{-\pi i s/2}F(a, s) + e^{\pi i s/2}F(-a, s)\right], \quad \mathrm{Re}(s) > 1, \tag{1.139}$$

where $F(a, s)$ denotes the periodic zeta function,

$$F(a, s) = \sum_{n=1}^{\infty}\frac{e^{2\pi i n a}}{n^s}, \quad \mathrm{Re}(s) > 1. \tag{1.140}$$

From elementary properties of the Dirichlet series (1.140), it is also clear that we can write the functional equation for $\zeta(s,\, a)$, for $0 < a = p/q \leq 1$, as

$$\zeta\left(1 - s,\, \frac{p}{q}\right) = \frac{2\Gamma(s)}{(2\pi q)^s}\sum_{r=1}^{q}\cos\left(\frac{\pi s}{2} - \frac{2\pi r p}{q}\right)\zeta\left(s, \frac{r}{q}\right). \tag{1.141}$$

We finish this chapter with three important corollaries, used extensively throughout this thesis.

Corollary 1.4.: (Functional equations for $\zeta(s)$ and $L(s,\chi)$) The Riemann $\zeta-$function can be analytically continued to a meromorphic function that obeys to the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\, \zeta(1-s). \tag{1.142}$$

Moreover, if χ is a nonprincipal and primitive character modulo ℓ, then $L(s,\chi)$ can be analytically continued to an entire function satisfying

$$L(s,\chi) = (\ell/2\pi)^{1-s}\, \ell^{-1} G(\chi)\Gamma(1-s)\, L(1-s,\overline{\chi})\, \left\{e^{-\pi i(1-s)/2} + \chi(-1)\, e^{\pi i(1-s)/2}\right\}. \tag{1.143}$$

Proof: To prove (1.142), take $a = 1$ in (1.136) and use elementary properties of $\Gamma(s)$.

To prove (1.143), we need to express $L(s,\chi)$ in terms of $\zeta(s,a)$ in order to apply the functional equation for $\zeta(s,a)$ to $L(s,\chi)$.

First, use $a = r/\ell$, $1 \le r \le \ell - 1$ in Hurwitz's formula (1.141) and then multiply each term by $\chi(r)$ and sum over the index r. From (1.139) this gives, for $\mathrm{Re}(s) > 1$,

$$L(1-s,\chi) = \ell^{s-1} \sum_{r=1}^{\ell-1} \chi(r)\, \zeta\left(1-s,\, \frac{r}{\ell}\right) = \frac{\Gamma(s)\ell^{s-1}}{(2\pi)^s} \left[e^{-\pi i s/2} \sum_{r=1}^{\ell-1} \chi(r)\, F\left(\frac{r}{\ell}, s\right) \right.$$
$$\left. + e^{\pi i s/2} \sum_{r=1}^{\ell-1} \chi(r)\, F\left(-\frac{r}{\ell}, s\right)\right]. \tag{1.144}$$

Since $F(a,s)$ is periodic with respect to a and has period 1, $F\left(\frac{\ell-r}{\ell},\, s\right) = F\left(-\frac{r}{\ell}, s\right)$, and we can rewrite the second sum in (1.144) as

$$\sum_{r=1}^{\ell-1} \chi(r)\, F\left(-\frac{r}{\ell}, s\right) = \sum_{r=1}^{\ell-1} \chi(\ell - r)\, F\left(\frac{r-\ell}{\ell},\, s\right) = \chi(-1) \sum_{r=1}^{\ell-1} \chi(r)\, F\left(\frac{r}{\ell},\, s\right). \tag{1.145}$$

After recombining (1.145) with (1.144), we obtain

$$L(1-s,\chi) = \frac{\Gamma(s)\ell^{s-1}}{(2\pi)^s} \sum_{r=1}^{\ell-1} \chi(r)\, F\left(\frac{r}{\ell},\, s\right) \left\{e^{-\pi i s/2} + \chi(-1)\, e^{\pi i s/2}\right\}. \tag{1.146}$$

Finally, since $\mathrm{Re}(s) > 1$, we can express $F(a,s)$ by the Dirichlet series given in (1.140), which proves

$$\sum_{r=1}^{\ell-1} \chi(r)\, F\left(\frac{r}{\ell},\, s\right) = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{r=1}^{\ell-1} \chi(r)\, e^{2\pi i n r/\ell} = \sum_{n=1}^{\infty} \frac{G(n,\chi)}{n^s}$$
$$= G(\chi) \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{n^s} = G(\chi)\, L(s,\overline{\chi}), \tag{1.147}$$

where the equality $G(n,\chi) = \overline{\chi}(n)\, G(\chi)$ comes from the fact that χ is primitive (see Theorem 1.1.).

Replacing the expression (1.147) into (1.146) and exchanging s with $1-s$ we arrive to (1.143). ■

Remark 1.4.: The primitivity of the character χ was essential to present a functional equation for $L(s,\chi)$ in the symmetric form (1.147). We should also remark that one can also write functional equations for $L(s,\chi)$ with χ not being primitive, but not in a symmetric form, i.e., with both sides of the equation depending explicitly on the Dirichlet $L-$function.

One way of seeing this is by following the result stated at the introduction of this chapter (see the Theorem 1.2. (1.16)). We know that any nonprincipal Dirichlet character can be written as

$$\chi(n) = \chi'(n)\, \chi_1(n),$$

where χ_1 is the principal character modulo ℓ and χ' is primitive. Following Apostol [5], this gives

$$L(s,\chi) = L(s,\chi') \prod_{p|\ell} \left(1 - \frac{\chi'(p)}{p^s}\right), \qquad \mathrm{Re}(s) > 1, \tag{1.148}$$

and, of course, this equality can be extended to all $s \in \mathbb{C}$ by analytic continuation. Using the functional equation for $L(s,\chi')$, one may find an expression for $L(s,\chi)$ in terms of $L(1-s,\overline{\chi})$ but this equation will in fact depend on the character χ' and, of course, will involve additional factors such as products of the form displayed in (1.148). Although we shall not consider functional equations of this type, the fact that we can write any $L-$function attached to a nonprincipal Dirichlet character χ in the form (1.148) will play an important role in the fifth chapter (see the Main Theorem 1 there).

Corollary 1.5.: Particular values for $L(n,\chi)$ when χ is primitive Let χ be a nonprincipal and primitive Dirichlet character modulo ℓ. Then we have

1. If χ is even, then $L(2n,\chi)$ satisfies the Euler-type identity (one should compare this with (1.104))
$$L(2n,\chi) = \frac{(-1)^{n-1}\,(2\pi)^{2n}\, G(\chi)}{2\ell^{2n}\,(2n)!}\, B_{2n,\overline{\chi}}. \tag{1.149}$$

2. If χ is odd then $L(2n-1,\chi)$ satisfies the Euler-type identity

$$L(2n-1,\chi) = i\,\frac{(-1)^{n-1}\,(2\pi)^{2n-1}\,G(\chi)}{2\ell^{2n-1}(2n-1)!}\,B_{2n-1,\overline{\chi}}. \tag{1.150}$$

Proof: For χ even or odd take, respectively, $s=2n$ and $s=2n-1$ and use equation (1.116). ■

Analogously to the case of $\zeta(2n-1)$, whose arithmetic nature is widely unknown, nothing arithmetically is known about the values $L(2n,\chi)$ when χ is odd and $L(2n-1,\chi)$ when χ is even, although there are some formulas which are analogous of well-known identities for $\zeta(2n-1)$ [56, 65].

We finish this chapter with the computation of the derivatives of $L(s,\chi)$ at $s=0$ and $s=1$ when χ is a primitive character.

For example, when χ is odd and primitive, the computation of $L^{'}(1,\chi)$ will be extremely useful in the third chapter, allowing to evaluate in closed form an extension of Koshliakov and Soni's formulas to odd characters [93].

Corollary 1.6.: Derivative of $L(s,\chi)$ at 0 and 1 Let χ be a nonprincipal and primitive character modulo ℓ. Then $L(s,\chi)$ satisfies the identities

$$L^{'}(0,\chi) = \frac{G(\chi)}{2}\,L(1,\overline{\chi}), \quad \chi \text{ even}, \tag{1.151}$$

$$L^{'}(0,\chi) = \frac{i}{\pi}G(\chi)\,\log(\ell)\,L(1,\overline{\chi}) + \sum_{r=1}^{\ell-1}\chi(r)\,\log\Gamma\left(\frac{r}{\ell}\right), \quad \chi \text{ odd}, \tag{1.152}$$

$$L^{'}(1,\chi) = \left(\gamma - \log\left(\frac{\ell}{2\pi}\right)\right)\,L(1,\chi) - \frac{G(\chi)}{\ell}L^{''}(0,\overline{\chi}), \quad \chi \text{ even}, \tag{1.153}$$

$$L^{'}(1,\chi) = i\pi\frac{G(\chi)}{\ell}\left[(\gamma+\log(2\pi))\,\frac{i}{\pi}G(\overline{\chi})L(1,\chi) + \sum_{r=1}^{\ell-1}\chi(r)\,\log\Gamma\left(\frac{r}{\ell}\right)\right], \quad \chi \text{ odd}. \tag{1.154}$$

Proof: If χ is even, we see from the functional equation for $L(s,\chi)$ (1.143) that

$$L(1,\chi) = \frac{2G(\chi)}{\ell}\,L^{'}(0,\overline{\chi}), \tag{1.155}$$

and so, after substituting χ by $\overline{\chi}$ and using the relation $G(\chi)\,G(\overline{\chi}) = \chi(-1)\,\ell$, (1.151) follows.

From the functional equation for $L(s,\chi)$ when χ is odd and using (1.99), we deduce

$$
\begin{aligned}
L^{'}(0,\chi) &= -\log(\ell)\, L(0,\chi) + \sum_{r=1}^{\ell-1} \chi(r)\, \log\Gamma\left(\frac{r}{\ell}\right) \\
&= \frac{i}{\pi} G(\chi)\, \log(\ell)\, L(1,\overline{\chi}) + \sum_{r=1}^{\ell-1} \chi(r)\, \log\Gamma\left(\frac{r}{\ell}\right), \quad \chi \text{ odd.} \qquad (1.156)
\end{aligned}
$$

(Formula (1.156) will be of extreme importance in the third chapter in order to obtain a generalization of a summation formula attributed to Dixon and Ferrar [40]). A straightforward application of the functional equation for $L(s,\chi)$ also gives the formulas

$$
L^{'}(1,\chi) = \left(\gamma - \log\left(\frac{\ell}{2\pi}\right)\right) L(1,\chi) - \frac{G(\chi)}{\ell} L^{''}(0,\overline{\chi}), \quad \chi \text{ even,} \qquad (1.157)
$$

$$
L^{'}(1,\chi) = -i\pi \frac{G(\chi)}{\ell} \left(\left(\gamma - \log\left(\frac{\ell}{2\pi}\right)\right) L(0,\overline{\chi}) - L^{'}(0,\overline{\chi})\right), \quad \chi \text{ odd.} \qquad (1.158)
$$

If, in the last one, we use (1.156), (1.154) immediately follows. ■

We end our considerations by remarking that the formulas derived here are classical and belong to the nineteenth century. The new points in the derivations presented in this chapter are the use of Abel-Plana summation formula to evaluate particular values for $\zeta(s,a)$ and to prove its functional equation.

We've also introduced a proof of Lerch's identity (1.62) based on Bohr-Mollerup's theorem and Hermite's representation for $\zeta(s,a)$ which seems to bring novelty to the literature regarding this topic.

Formula (1.158) was obtained also by M. Lerch and A. Berger [37]. It was also proved by A. Selberg and S. Chowla in [90], with the use of the theory of Epstein $\zeta-$function. This canonical work of A. Selberg and S. Chowla will be central in our fourth chapter.

Regarding formula (1.157), it is still possible to adapt classic arguments to evaluate $L^{''}(0,\chi)$. In particular, one needs to introduce the theory of the double-gamma function [37] to proceed with this computation. Since the values of $L^{'}(1,\chi)$ (for even and primitive χ) won't be necessary in the forthcoming work, we omit their evaluations.

Chapter 2

The Poisson summation formula and its character analogues

In this section, we prove a strong version of Poisson's summation formula and its character analogues by using the functional equation for the Riemann $\zeta-$function and for the Dirichlet $L-$function associated with a primitive character.

Our approach is similar to the ones given by Guinand, Ferrar, Berndt and Yakubovich [51, 53, 54, 112], but the conditions we deal with are imposed in a different framework, as well as our proof of the result.

We also present seven examples, which extend some identities already existing to a character version.

Examples 2.1 and 2.5 are of special importance, as they will be invoked several times during the fourth chapter to also prove an analogue of Selberg-Chowla's formula.

Nevertheless, we still have some standard theory to expose before the main results, so the first section of this chapter is devoted to a revision of classical results regarding Mellin's transform for functions belonging to $L_1(\mathbb{R}_+)$ and $L_2(\mathbb{R}_+)$, following closely the theory given in Titchmarsh's textbook [102].

Since our class of functions will generally be $L_2(\mathbb{R}_+)$, we will fix the following notation.

Notation remarks:

- Given a sequence of measurable functions $(f_n(x))_{n\in\mathbb{N}} \in L_2(\mathbb{R}_+)$, we say that

$$\underset{n\to\infty}{\text{l.i.m.}} f_n(x) = f(x), \tag{2.1}$$

  if the limit (2.1) exists in the L_2-norm, i.e.,

$$\lim_{n\to\infty} \int_0^\infty |f_n(x) - f(x)|^2 \, dx = 0. \tag{2.2}$$

- Also, if $f(x) = g(x)$, $\mu -$ a.e., where μ is the Lebesgue measure in $\mathbb{R}$, then we denote this equality by $f(x) \stackrel{\mu}{=} g(x)$.

- For simplicity, we will use at some points the following convention: $f^*(s) \in L_p(\sigma)$ if $f^*(s) \in L_p\,(\sigma - i\infty,\, \sigma + i\infty)$.

2.1 Preliminary results: Part I - The Fourier and Mellin integrals for the L_1 and L_2-class. Plancherel Theory

We start with the classical definition of Fourier transform for the class of functions belonging to $L_1(\mathbb{R})$.

Definition 2.A-1 (The L_1 - Fourier Transform) Let $f(x) \in L_1(\mathbb{R})$. We define the Fourier transforms of $f(x)$ as

$$g(x) = \int_0^\infty f(y)\, \cos(xy)\, dy, \text{ cosine transform,} \tag{2.3}$$

$$h(x) = \int_0^\infty f(y)\, \sin(xy)\, dy, \text{ sine transform,} \tag{2.4}$$

$$\hat{f}(x) = \int_{-\infty}^\infty f(y)\, e^{-ixy} dy, \text{ complex Fourier transform.} \tag{2.5}$$

Clearly, the condition $f(x) \in L_1(\mathbb{R})$ assures that the integrals defining the Fourier transforms (2.3), (2.4) and (2.5) are absolutely convergent and bounded since, for all $x \in \mathbb{R}$,

$$|g(x)|\,,\, |h(x)|,\, |\hat{f}(x)| \leq \| f \|_{L_1(\mathbb{R})}\,.$$

Now we present the inversion formula for the Fourier transform in the L_1- class, whose proof may be found in Titchmarsh's book [102].

Theorem 2.A-1 (Fourier inversion formula in L_1): Let $f(x)$ be a continuous function such that $f(x)$, $g(x) \in L_1(\mathbb{R})$ (resp. $h(x)$ and $\hat{f}(x)$). Then, for all $x \in \mathbb{R}$,

$$f(x) = \frac{2}{\pi} \int_0^\infty g(y)\, \cos(xy)\, dy. \tag{2.6}$$

Analogously, we may write the inversion formulas for the sine and the complex Fourier transforms as

$$f(x) = \frac{2}{\pi} \int_0^{\infty} h(y) \sin(xy)\, dy, \tag{2.7}$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(y)\, e^{ixy}\, dy. \tag{2.8}$$

Analogously to the case where $f \in L_1(\mathbb{R}_+) \cap L_2(\mathbb{R}_+)$, we can still define a theory of Fourier integrals for $f(x) \in L_2(\mathbb{R}_+)$. Following Titchmarsh [102], we also give the following definition.

Definition 2.A-2 (The L_2−Fourier transform): Let $f(x) \in L_2(\mathbb{R})$. We define the Fourier transforms of $f(x)$ as the following limits in the L_2 mean

$$g(x) = \underset{N\to\infty}{\text{l.i.m.}} \int_{1/N}^{N} f(y) \cos(xy)\, dy, \text{ cosine transform,} \tag{2.9}$$

$$h(x) = \underset{N\to\infty}{\text{l.i.m.}} \int_{1/N}^{N} f(y) \sin(xy)\, dy, \text{ sine transform,} \tag{2.10}$$

$$\hat{f}(x) = \underset{N\to\infty}{\text{l.i.m.}} \int_{-N}^{N} f(y)\, e^{-ixy}\, dy, \text{ complex Fourier transform.} \tag{2.11}$$

Analogously to the Fourier theory for the class of functions in the space $L_1(\mathbb{R})$, the following inversion formulas take place.

Theorem 2.A-2 (Fourier inversion formula in L_2): Let $f(x) \in L_2(\mathbb{R})$ and let $g(x)$ be its Fourier cosine transform ($h(x)$ and $\hat{f}(x)$ denote, respectively, its sine and complex Fourier transforms),

$$g(x) = \underset{N\to\infty}{\text{l.i.m.}} \int_{1/N}^{N} f(y) \cos(xy)\, dy.$$

Thus, we have reciprocally that

$$f(x) = \frac{2}{\pi} \underset{N\to\infty}{\text{l.i.m.}} \int_{1/N}^{N} g(y) \cos(xy)\, dy. \tag{2.12}$$

Furthermore, we have, almost everywhere, the following pair of transforms

$$g(x) \stackrel{\mu}{=} \frac{d}{dx} \int_0^\infty f(y) \, \frac{\sin(xy)}{y} \, dy, \quad f(x) \stackrel{\mu}{=} \frac{2}{\pi} \frac{d}{dx} \int_0^\infty g(y) \, \frac{\sin(xy)}{y} \, dy. \tag{2.13}$$

For the cases where we have the sine and complex transforms, we obtain the following analogous cases of (2.12) and (2.13)

$$f(x) = \frac{2}{\pi} \operatorname*{l.i.m.}_{N \to \infty} \int_{1/N}^{N} h(y) \, \sin(xy) \, dy, \tag{2.14}$$

$$h(x) \stackrel{\mu}{=} \frac{d}{dx} \int_0^\infty f(y) \, \frac{1 - \cos(xy)}{y} \, dy, \quad f(x) \stackrel{\mu}{=} \frac{2}{\pi} \frac{d}{dx} \int_0^\infty h(y) \, \frac{1 - \cos(xy)}{y} \, dy, \tag{2.15}$$

and

$$f(x) = \frac{1}{2\pi} \operatorname*{l.i.m.}_{n \to \infty} \int_{-N}^{N} \hat{f}(y) \, e^{ixy} \, dy, \tag{2.16}$$

$$\hat{f}(x) \stackrel{\mu}{=} i \, \frac{d}{dx} \int_{-\infty}^\infty f(y) \, \frac{e^{-ixy} - 1}{y} \, dy, \quad f(x) \stackrel{\mu}{=} -\frac{i}{2\pi} \frac{d}{dx} \int_{-\infty}^\infty \hat{f}(y) \, \frac{e^{ixy} - 1}{y} \, dy. \tag{2.17}$$

Furthermore, we can invoke Parseval's theorem for the Fourier transform in the L_2- class.

Theorem 2.A-3 (Parseval theorem for the Fourier transform): Let $f_1(x)$, $f_2(x) \in L_2(\mathbb{R}_+)$ and $g_1(x)$, $g_2(x) \in L_2(\mathbb{R}_+)$ be their Fourier cosine transforms.

Then the following equality holds

$$\frac{2}{\pi} \int_0^\infty f_1(x) \, f_2(x) \, dx = \int_0^\infty g_1(x) \, g_2(x) \, dx. \tag{2.18}$$

Analogously for the sine and the complex Fourier transforms, we obtain the formulas

$$\frac{2}{\pi} \int_0^\infty f_1(x) \, f_2(x) \, dx = \int_0^\infty h_1(x) \, h_2(x) \, dx, \tag{2.19}$$

$$2\pi \int_{-\infty}^\infty f_1(x) \, \overline{f}_2(x) \, dx = \int_{-\infty}^\infty \hat{f}_1(x) \, \overline{\hat{f}_2}(x) \, dx. \tag{2.20}$$

To prove the main theorem, we will make a large use of Mellin's transform theory for the L_2 class of functions, as well as we shall use the L_1-theory for the class of Müntz-type functions. Now, we furnish some useful definitions and results regarding this transform.

Definition 2.B-1 (Mellin transform in the L_1 class): Let $f(x)$ be a function such that $x^{\sigma-1} f(x) \in L_1(\mathbb{R}_+)$. Then, for a complex s satisfying $\mathrm{Re}(s) = \sigma$, we define the L_1-Mellin transform as the function

$$f^*(s) = \int_0^\infty x^{s-1} f(x)\, dx. \tag{2.21}$$

The condition $f(x)\, x^{\sigma-1} \in L_1(\mathbb{R}_+)$ assures that the integral (2.21) is absolutely convergent and that $f^*(s)$ is a bounded complex function in the line $\mathrm{Re}(s) = \sigma$, since

$$|f^*(s)| \leq \int_0^\infty x^{\sigma-1}|f(x)|\, dx = \left\|x^{\sigma-1} f(x)\right\|_{L_1(\mathbb{R}_+)}.$$

It is also known that the Mellin transform (2.21) can be written in terms of the complex Fourier transform in the following form

$$f^*\left(\sigma + it\right) = \int_0^\infty x^{\sigma+i\tau-1} f(x)\, dx = \int_{-\infty}^\infty f\left(e^{-u}\right) e^{-\sigma u}\, e^{-iut} du = \widehat{F}(t),$$

where $F(u) = f\left(e^{-u}\right) e^{-\sigma u}$.

Hence, by using Fourier's inversion formula for the L_1-class, we can analogously obtain the following inversion formula for the Mellin transform.

Theorem 2.B-1 (Mellin's inversion formula in L_1): Let $f(x)$ be a continuous function such that $x^{\sigma-1} f(x) \in L_1(\mathbb{R}_+)$ and its Mellin transform (2.21) satisfies $f^*(s) \in L_1\left(\sigma - i\infty, \sigma + i\infty\right)$. Then, for every $x \in \mathbb{R}_+$, the following inversion formula takes place

$$f(x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} f^*(s)\, x^{-s} ds. \tag{2.22}$$

In analogy with Fourier's, we can now introduce the Theory of the Mellin transforms for the L_2 class of functions. We start with the definition regarding an extension of (2.21) to functions belonging to $L_2(\mathbb{R}_+)$.

Definition 2.B-2 (Mellin transform in the L_2 class): Let $x^\sigma f(x) \in L_2\left(\mathbb{R}_+, \frac{dx}{x}\right)$. For $\mathrm{Re}(s) = \sigma$, we define the L_2-Mellin transform by the limit in the L_2- norm

$$f^*(s) = \underset{N\to\infty}{\mathrm{l.i.m.}} \int_{1/N}^{N} x^{s-1} f(x)\, dx. \tag{2.23}$$

For the transform (2.23) we also have an inversion formula similar to the one in L_1, whose expression will depend on the convergence in the L_2-mean of the integral. We state it in the same way as it is stated in Titchmarsh's textbook [102].

Theorem 2.B-2 (Mellin's inversion formula in L_2) If $x^\sigma f(x) \in L_2\left(\mathbb{R}_+, \frac{dx}{x}\right)$, consider the Mellin transform defined in (2.23),

$$f^*(s) = \underset{N\to\infty}{\text{l.i.m.}} \int_{1/N}^{N} x^{s-1} f(x)dx, \quad \text{Re}(s) = \sigma.$$

Conversely, we have that $f(x)$ is given by the limit in the L_2-norm,

$$f(x) = \underset{N\to\infty}{\text{l.i.m.}} \int_{\sigma-iN}^{\sigma+iN} f^*(s)\, x^{-s} ds. \tag{2.24}$$

Moreover, from Plancherel's theorem for the L_2-Fourier transform it also follows that

$$\int_0^\infty |f(x)|^2 x^{2\sigma-1} dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f^*(\sigma+it)|^2\, dt. \tag{2.25}$$

Finally, we state Parseval equality for the Mellin transform in these conditions, which will be useful to derive the necessary Fourier kernels in the main theorem.

Theorem 2.B-3 (Parseval Theorem): Assume that $f(x)$ and $g(x)$ are such that $x^\sigma f(x)$, $x^{1-\sigma} g(x) \in L_2(\mathbb{R}_+, \frac{dx}{x})$. Then the following formula holds

$$\int_0^\infty f(xt)\, g(t)\, dt = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} f^*(s)\, g^*(1-s)\, x^{-s} ds, \tag{2.26}$$

where the right-hand side of (2.26) can be seen as a Mellin inverse transform in L_1, since $f^*(s)\, g^*(1-s) \in L_1\,(\sigma - i\infty,\, \sigma + i\infty)$ by the Plancherel theorem for the Mellin transform (2.25) and Cauchy-Schwarz inequality [102].

In the next section we use the theory of Mellin and Fourier transforms for the class of L_2 functions to understand why the conditions imposed over the first class of functions defined at the introduction are necessary.

2.2 Preliminary Results: Part II - Our conditions

It is our aim for this section to explain why the conditions imposed over the functions $f(x)$ belonging to the 'L_2 class' are relevant to prove the version of Poisson's summation formula in this chapter as Main Theorem.

However, before doing this comprehensively, let us introduce the following definition:

Definition 2.1.: A function $f\,:\,[a,b]\longmapsto\mathbb{R}$ is said to be absolutely continuous on $[a,b]$ if, given $\epsilon>0$, there exists some $\delta>0$ such that

$$\sum_{i=1}^{n}|f(y_i)-f(x_i)|<\epsilon,$$

whenever $\{[x_i,\,y_i]\,:\,i=1,...,n\}$ is a finite collection of mutually disjoint intervals of $[a,b]$ with $\sum_{i=1}^{n}|y_i-x_i|<\delta$.

Clearly, any absolutely continuous function is uniformly continuous. Moreover, any absolutely continuous function f on $[a,b]$ has a derivative $f'(x)$ which exists for almost every $x\in[a,b]$ and is Lebesgue-integrable on the interval $[a,b]$ satisfying, for all $x\in[a,b]$, the condition [13]

$$f(x)=f(a)+\int_{a}^{x}f^{'}(y)\,dy.$$

In what follows, we say that f is absolutely continuous on $\mathbb{R}_+$ if it is absolutely continuous in every interval $[a,b]\subset\mathbb{R}_+$.

Our main purpose for this chapter is to prove the following Theorem, which can be seen as an extended version of Poisson's summation formula.

Main Theorem:

Let χ be a nonprincipal and primitive Dirichlet character modulo ℓ and $f(x)$ an absolutely continuous function on $\mathbb{R}_+$ such that $s\,f^{*}(s)\in L_2\left(\frac{1}{2}-i\infty,\,\frac{1}{2}+i\infty\right)$, where $f^{*}(s)$ is the Mellin transform (in the L_2-sense) of $f(x)$.

Then, depending only on the parity of the character χ, the following summation formulas hold

$$\lim_{N\to\infty}\sum_{n=1}^{N}\chi(n)\,f(nx)=\frac{2G(\chi)}{\ell x}\,\lim_{N\to\infty}\sum_{n=1}^{N}\overline{\chi}(n)\,g\left(\frac{2\pi n}{\ell x}\right),\quad\chi\text{ even},\tag{2.27}$$

$$\lim_{N\to\infty}\sum_{n=1}^{N}\chi(n)\,f(nx) = -\frac{2iG(\chi)}{\ell x}\lim_{N\to\infty}\sum_{n=1}^{N}\overline{\chi}(n)\,h\left(\frac{2\pi n}{\ell x}\right),\quad \chi \text{ odd}, \tag{2.28}$$

$$\lim_{N\to\infty}\left[\sum_{n=1}^{N}f(nx) - \frac{1}{x}\int_{1/N}^{N} f(y)\,dy\right] = \frac{2}{x}\lim_{N\to\infty}\left[\sum_{n=1}^{N}g\left(\frac{2\pi n}{x}\right) - \int_{1/N}^{N} g\left(\frac{2\pi y}{x}\right)\,dy\right],\quad \chi \text{ trivial}, \tag{2.29}$$

where $g(x)$ and $h(x)$ are the integral transforms defined, respectively, by

$$g(x) = -\frac{1}{x}\int_{0}^{\infty} f'(y)\,\sin(xy)\,dy, \tag{2.30}$$

$$h(x) = -\frac{2}{x}\int_{0}^{\infty} f'(y)\,\sin^2\left(\frac{xy}{2}\right)dy \tag{2.31}$$

and $G(\chi)$ denotes the Gauss sum, defined in the previous chapter (see relation (1.9) there).

By naming χ trivial, we simply mean that χ is the unique character modulo 1, i.e., $\chi(n) = \chi_0(n) \equiv 1$ (see the first chapter). Of course, the Dirichlet $L-$function associated to this character, which is 'principal' in the sense defined, is obviously Riemann's $\zeta-$function.

Let us also remark that the integral transforms (2.30) and (2.31) correspond, almost everywhere, to the cosine and sine Fourier transforms of $f(x) \in L_2(\mathbb{R}_+)$, since their primitives satisfy (see [102] and the previous Theorem 2.A-2)

$$\int_{0}^{z} g(x)\,dx = \int_{0}^{\infty} f(y)\,\frac{\sin(zy)}{y}\,dy, \tag{2.32}$$

$$\int_{0}^{z} h(x)\,dx = \int_{0}^{\infty} f(y)\,\frac{1-\cos(zy)}{y}\,dy. \tag{2.33}$$

As a corollary of the main theorem, we derive several interesting examples, some of them related with the Kontorovich-Lebedev transform (see, for instance, the summation formula (2.167) in Example 2.7.).

Serving as motivation, we cite some of them: for example, if $a > 0$ and $\mathrm{Re}(\nu) > 0$, the following formulas hold

$$\sum_{n\le a}\chi(n)\,(a^2-n^2)^{\nu-\frac{1}{2}} = G(\chi)\ell^{\nu-1}\pi^{\frac{1}{2}-\nu}a^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)}{n^{\nu}}\,J_{\nu}\left(\frac{2\pi n a}{\ell}\right),\quad \chi \text{ even},$$

$$\sum_{n\leq a}\chi(n)\,(a^2-n^2)^{\nu-\frac{1}{2}}=-iG(\chi)\ell^{\nu-1}\pi^{\frac{1}{2}-\nu}a^{\nu}\,\Gamma\left(\nu+\frac{1}{2}\right)\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)}{n^{\nu}}H_{\nu}\left(\frac{2\pi na}{\ell}\right),\quad \chi \text{ odd},$$

where $J_\nu(x)$ and $H_\nu(x)$ denote, respectively, Bessel's and Struve's functions of the first kind [44, 102, 106].

Using the properties of the modified Bessel function [106], we are also able to prove, for $0<b<2a$ and $k:=\sqrt{4ac-b^2}/2a$,

$$4\sum_{n=1}^{\infty}\cos\left(\pi\frac{bn}{a}\right)K_0\left(2\pi k\,n\right)=\sum_{n\in\mathbb{Z}}\left\{\frac{1}{\sqrt{(n+b/2a)^2+k^2}}-\frac{1}{|n+b/2a|}\right\}$$
$$+\log\left(\frac{k^2}{4}\right)-2\psi\left(\frac{b}{2a}\right)-\pi\cot\left(\frac{\pi b}{2a}\right),$$

extending, to any quadratic form, a formula deduced by Watson for the first time [105].

We are now ready to study the consequences of the main preliminary hypothesis for our function $f(x)$. In the next Proposition, we study some of the properties of the functions belonging to the so called 'L_2 class'.

Proposition 2.1. : Main preliminary hypothesis Let $f(x)$ be an absolutely continuous function on $\mathbb{R}_+$ such that $s\,f^*(s)\in L_2\left(\frac{1}{2}-i\infty,\,\frac{1}{2}+i\infty\right)$. Then we have the following

1. $f(x)\in L_2(\mathbb{R}^+)$ and $f^*(s)\in L_2\left(\frac{1}{2}-i\infty,\,\frac{1}{2}+i\infty\right)\cap L_1\left(\frac{1}{2}-i\infty,\,\frac{1}{2}+i\infty\right)$.
2. For $\sigma=\frac{1}{2}$, the integral (2.24) can be written as an absolutely convergent integral.
3. $\lim_{x\to 0}x^{1/2}\,f(x)=\lim_{x\to\infty}x^{1/2}\,f(x)=0$.

Proof: Let us prove these facts by their order in the statement: first, let us see that $f^*(s)\in L_2\left(\frac{1}{2}-i\infty,\,\frac{1}{2}+i\infty\right)$. Note that for all s in the line $\text{Re}(s)=1/2$, one has $|s|^2\geq\frac{1}{4}$, and so, by hypothesis,

$$\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}|f^*(s)|^2|ds|\leq 4\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}|s\,f^*(s)|^2\,|ds|<\infty.$$

The proof that $f^*(s)\in L_1\left(\frac{1}{2}-i\infty,\,\frac{1}{2}+i\infty\right)$ comes also easily from the Cauchy-Schwarz inequality,

$$\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}|f^*(s)||ds|\leq\left(\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}|s\,f^*(s)|^2\,|ds|\right)^{1/2}\left(\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{|ds|}{|s|^2}\right)^{1/2}<\infty.$$

Finally, from the fact that $f^*(s) \in L_2\left(\frac{1}{2} - i\infty,\, \frac{1}{2} + i\infty\right)$, it follows from Plancherel theorem for the Mellin transform (eq. (2.25) with $\sigma = 1/2$) that $f(x) \in L_2(\mathbb{R}_+)$, which concludes the proof of the first part of the statement.

To prove the second topic on the statement, recall that, since $f^*(s) \in L_1\left(\frac{1}{2} - i\infty,\, \frac{1}{2} + i\infty\right) \cap L_2\left(\frac{1}{2} - i\infty,\, \frac{1}{2} + i\infty\right)$, it follows from the Mellin inversion Theorem in L_1 that we can write (2.24) as

$$\underset{n\to\infty}{\text{l.i.m.}}\, \frac{1}{2\pi i} \int_{\frac{1}{2}-iN}^{\frac{1}{2}+iN} f^*(s)\, x^{-s} ds = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} f^*(s)\, x^{-s} ds. \tag{2.34}$$

To prove the final statement, consider the L_1 representation of $f(x)$ as a Mellin inverse transform (right-hand side of (2.34)). Using this, we can write it in terms of the complex Fourier transform in the same way described above. In fact, taking $F(t) = f^*\left(\frac{1}{2} + it\right) \in L_1(\mathbb{R})$ (by point 1.), we can write (2.34) as follows

$$\begin{aligned} x^{1/2} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f^*\left(\frac{1}{2} + it\right) e^{-i\log(x)\,t} dt = \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(t)\, e^{-i\log(x)\,t} dt = \frac{1}{2\pi} \hat{F}(\log(x)) \,. \end{aligned}$$

Since $F \in L_1(\mathbb{R})$ by the first statement and $x^{1/2} f(x)$ can be seen as a Fourier transform of $F(x)$, it follows from the Riemann-Lebesgue theorem [94] that

$$\lim_{x\to\infty} \hat{F}(\log(x)) = 0, \tag{2.35}$$

which clearly implies that $\lim_{x\to\infty} x^{1/2} f(x) = 0$. To conclude the same when $x \to 0$, take the change of variables $x \leftrightarrow 1/x$ and use the previous argument. This concludes the proof of the Preliminary Hypothesis. ■

Having established conditions over the class of functions obeying the desired summation formulas (2.27 - 2.29), as well as the necessary theory and motivation to develop these, we are now able to proceed with another set of preliminary results.

2.3 Preliminary results: Part III

For a nonprincipal and primitive Dirichlet character χ modulo ℓ, let us introduce the function

$$\Lambda_\chi(x) = {\sum_{n\le x}}' \chi(n), \tag{2.36}$$

where, as observed on the Glossary, the prime on the summation implies that if x is an integer then only $\frac{1}{2}\chi(x)$ is counted as the last term.

We assume that χ is nonprincipal and has modulus ℓ, so that $\sum_{n=1}^{\ell-1}\chi(n) = 0$ and $|\Lambda_\chi(x)| \le \varphi(\ell) \le \ell - 1$ for all $x > 0$.

From the definition (2.36), $\Lambda_\chi(x) = 0$ for $0 < x < 1$ and $\Lambda_\chi(x) = O(1)$ as $x \to \infty$. Thus, if we introduce the auxiliary function

$$h_\chi(x) = \frac{\Lambda_\chi(x)}{x}, \tag{2.37}$$

we have that $h_\chi(x) = O(x^{-1})$ as $x \to \infty$, as well as $h_\chi(x) = 0$ for $0 < x < 1$. Hence, if we write its Mellin transform in $L_1(\mathbb{R}_+)$ (2.21),

$$h^*_\chi(s) = \int_0^\infty h_\chi(x)\, x^{s-1} dx,$$

we see that this transform is well-defined for $\mathrm{Re}(s) < 1$ as an analytic function. Firstly, if we assume that $\mathrm{Re}(s) < 0$, by the definition of $h_\chi(x)$, $\Lambda_\chi(x)$ and $L(s, \chi)$ as a Dirichlet series (see relation (1.20) on the previous chapter), we obtain

$$\begin{aligned}
h^*_\chi(s) &= \int_1^\infty \Lambda_\chi(x)\, x^{s-2} dx = \int_1^\infty \Lambda_\chi(x)\, x^{s-2} dx \\
&= \sum_{k=1}^\infty \int_k^{k+1} \Lambda_\chi(x)\, x^{s-2} dx = \sum_{k=1}^\infty \Lambda_\chi(k + 1/2) \int_k^{k+1} x^{s-2} dx \\
&= \frac{1}{s-1} \sum_{k=1}^\infty \Lambda_\chi(k+1/2) \left((k+1)^{s-1} - k^{s-1}\right) \\
&= -\frac{\Lambda_\chi\left(\frac{3}{2}\right)}{s-1} + \frac{1}{s-1} \sum_{k=2}^\infty k^{s-1} \left(\Lambda_\chi(k - 1/2) - \Lambda_\chi(k+1/2)\right) \\
&= -\frac{\chi(1)}{s-1} - \frac{1}{s-1} \sum_{k=2}^\infty k^{s-1} \chi(k) = \frac{L(1-s,\, \chi)}{1-s}, \qquad \mathrm{Re}(s) < 0.
\end{aligned} \tag{2.38}$$

Recall that, by imposing χ as nonprincipal, $L(s, \chi)$ is an entire function (see the first chapter of this thesis). This means that $(1-s)\, h^*_\chi(s)$ and $L(1-s, \chi)$ are analytic functions which coincide in the half-plane $\mathrm{Re}(s) < 0$: by the principle of analytic continuation (see Proposition 1.8. at the first chapter), we can extend (2.38) to the strip $0 < \mathrm{Re}(s) < 1$, from which we obtain the representation

$$h_{\chi}^{*}(s) = \frac{L(1-s,\,\chi)}{1-s}, \quad 0 < \mathrm{Re}(s) < 1. \tag{2.39}$$

Now, for all $0 < \sigma < 1$, we have that $x^{\sigma} h_{\chi}(x) \in L_2\left(\mathbb{R}_+,\, \frac{dx}{x}\right)$ and so we can represent $h_{\chi}(x)$ via Mellin's inversion theorem for the L_2-class (2.24),

$$h_{\chi}(x) = \frac{1}{2\pi i} \underset{N\to\infty}{\mathrm{l.i.m.}} \int_{\sigma-iN}^{\sigma+iN} \frac{L(1-s,\,\chi)}{1-s} x^{-s} ds. \tag{2.40}$$

Since (2.40) is valid for all $0 < \mathrm{Re}(s) < 1$, it holds in particular when $\mathrm{Re}(s) = \frac{1}{2}$. Hence, a simple change of variables allows us to write

$$\Lambda_{\chi}(x) = \frac{1}{2\pi i} \underset{N\to\infty}{\mathrm{l.i.m.}} \int_{\frac{1}{2}-iN}^{\frac{1}{2}+iN} \frac{L(s,\,\chi)}{s} x^{s}\, ds. \tag{2.41}$$

In what follows, we prove the main Lemma in this chapter, which will allow us to connect an integral of the type (2.41) with an infinite series involving Dirichlet characters.

Before establishing this main Lemma, we need a preliminary result related with the theory of functions, which was established for the first time by Lindelöf [43], commonly known nowadays by "Phragmén-Lindelöf" principle. We state Lindelöf's theorem in the way Edwards states it in the section 9.1. of his book on the Riemann zeta function [43].

Lemma 2-A: Lindelöf's Theorem (or a version of Phragmén-Lindelöf Principle): Let $f(s)$ be defined and analytic in a half-strip $\mathcal{S} = \{s \in \mathbb{C} : \sigma_1 \leq \mathrm{Re}(s) \leq \sigma_2,\, \mathrm{Im}(s) \geq t_0 > 0\}$.

If p and q are such that $|f(\sigma_1 + it)| \leq M_1\, |t|^p$ and $|f(\sigma_2 + it)| \leq M_2\, |t|^q$, then there exists a constant M satisfying

$$|f(\sigma + it)| \leq M\, t^{\mu(\sigma)}, \quad \sigma + it \in \mathcal{S},$$

where

$$\mu(\sigma) = p + \frac{q-p}{\sigma_2 - \sigma_1}\,(\sigma - \sigma_1),$$

is the affine function connecting the points (σ_1, p) and (σ_2, q) on the plane.

An immediate Corollary of Lindelöf's Theorem is the following Proposition, which will be useful in this chapter and in the next one.

Proposition 2.2. (Lindelöf's Estimate): Let χ be a nonprincipal and primitive Dirichlet character modulo ℓ. Then, given $t_0 > 0$, there exists a $M > 0$ such that, for all $t \in \mathbb{R}$ satisfying $|t| \geq t_0$ and any $\epsilon > 0$,

$$|\zeta(\sigma \pm it)|,\, |L(\sigma \pm it,\, \chi)| \leq \begin{cases} M & \sigma > 1 \\ M\,|t|^{\frac{1-\sigma}{2}+\epsilon} & 0 \leq \sigma \leq 1 \\ M\,|t|^{\frac{1}{2}-\sigma} & \sigma < 0. \end{cases} \tag{2.42}$$

Proof: Since $\zeta(s)$ possesses only a simple pole located at $s = 1$, it is analytic in every half-strip of the form described by the above Lemma.

We know that, for every $\epsilon > 0$, $\zeta(1 + \epsilon + it)$ is bounded, since it is given by an absolutely convergent series. Moreover, note that we can write the functional equation for $\zeta(s)$ in the following symmetric form:

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{2.43}$$

Hence, using (2.43), we see that, for a sufficiently large t,

$$|\zeta(-\epsilon - it)| < C\left|\frac{\Gamma\left(\frac{1+\epsilon+it}{2}\right)}{\Gamma\left(\frac{-\epsilon-it}{2}\right)}\zeta(1+\epsilon+it)\right| < M_2|t|^{\frac{1}{2}+\epsilon},$$

by virtue of Stirling's formula for $\Gamma(s)$.

Thus, applying Lindelöf's theorem (or Phragmén-Lindelöf principle) for $q = 0$ and $p = \frac{1}{2} + \epsilon$, $\sigma_1 = -\epsilon$ and $\sigma_2 = 1 + \epsilon$, we obtain immediately (2.42) for the case of Riemann's $\zeta-$function.

When χ is nonprincipal and primitive, we know that $L(s, \chi)$ has a functional equation which is similar to (2.43). Thus, applying the same argument yields the estimate (2.42) for $L(s, \chi)$ as well. ■

Another Lemma which will be important in the proof of our Main result is the following one, which commonly appears as an exercise in Measure Theory textbooks [13].

Lemma 2.1.: Let μ denote the Lebesgue measure on $\mathbb{R}_+$ and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions converging to f in the L_p mean (with $p \geq 1$), i.e.,

$$\lim_{n\to\infty} \int_{\mathbb{R}_+} |f_n - f|^p \, d\mu = 0.$$

Then there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that $f_{n_k} \longmapsto f \;\; \mu -$ a.e.

With the necessary results stated, we are now ready to prove:

Main Lemma:

Let χ be a nonprincipal and primitive Dirichlet character modulo ℓ and $f(x)$ an absolutely continuous function on $\mathbb{R}_+$ such that its Mellin transform satisfies the condition $s\, f^*(s) \in L_2\left(\frac{1}{2} - i\infty,\, \frac{1}{2} + i\infty\right)$.

Then for any $x > 0$, the following representation is valid

$$\frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} L(s,\chi)\, f^*(s)\, x^{-s}\, ds = \lim_{N\to\infty} \sum_{n=1}^{N} \chi(n)\, f(nx). \tag{2.44}$$

Proof of the Main Lemma: By hypothesis, we have $s\, f^*(s) \in L_2\left(\frac{1}{2} - i\infty,\, \frac{1}{2} + i\infty\right)$ and, since $L\left(\frac{1}{2} + it,\, \chi\right) = O\left(|t|^{\frac{1}{4}+\epsilon}\right)$ when $|t| \to \infty$ (see (2.42)), $L(s,\chi)/s \in L_2\left(\frac{1}{2} - i\infty,\, \frac{1}{2} + i\infty\right)$.

By Cauchy-Schwarz inequality, we see that the left-hand side of (2.44) satisfies

$$\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} |L(s,\chi)\, f^*(s)|\; |ds| \leq \left\| \frac{L(s,\chi)}{s} \right\|_{L_2(\sigma)} \|s\, f^*(s)\|_{L_2(\sigma)} < \infty,$$

and so the integral on (2.44) exists as an absolutely convergent one.

Recalling (2.41) and using the simple fact that $\Lambda_\chi\left(\frac{1}{x}\right) \in L_2\left(\mathbb{R}_+\right)$, the L_2-inversion formula for the Mellin transform (2.24) allows to write

$$\frac{L(s,\chi)}{s} = \underset{N\to\infty}{\text{l.i.m.}} \int_{1/N}^{N} \Lambda_\chi\left(\frac{1}{x}\right) x^{s-1} dx = \underset{N\to\infty}{\text{l.i.m.}}\, \Lambda^*_{N,\chi}(s). \tag{2.45}$$

After the elementary verification that $d\Lambda_\chi(x)$ satisfies the properties of a Lebesgue-Stieltjes measure[1] [13], the previous integral (2.45) can be written by appealing to an integration by parts as

$$\Lambda^*_{N,\chi}(s) = \frac{1}{s} \int_{1/N}^{N} x^{-s} d\Lambda_\chi(x) - \Lambda_\chi(N)\, \frac{N^{-s}}{s} + \Lambda_\chi\left(\frac{1}{N}\right) \frac{N^s}{s}. \tag{2.46}$$

[1]This is done after decomposing χ into real and imaginary parts and then check for each one the properties that characterize Lebesgue-Stieltjes measures.

From (2.45) we see that $\left(s\,\Lambda^*_{N,\chi}(s)\right)_{N\in\mathbb{N}}$ is a sequence that converges to $L(s,\chi)$ in the L_2 mean, i.e.,

$$\lim_{N\to\infty}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\left|s\,\Lambda^*_{N,\chi}(s)-L(s,\chi)\right|^2\,d\nu=0 \tag{2.47}$$

where $\nu\,(:=|ds|)$ denotes the Lebesgue measure in the real line $\mathrm{Re}(s)=\frac{1}{2}$.

From Lemma 2.1. above, we know that (2.47) implies that exists a subsequence $\left(s\,\Lambda^*_{N_k,\chi}(s)\right)_{k\in\mathbb{N}}$ such that $s\,\Lambda^*_{N_k,\chi}(s)\to L(s,\chi)$ $\nu-$a.e..

Now, the conclusion of the proof follows from the Claims given below:

Claim 1: Let s be fixed complex number on the critical strip $0<\mathrm{Re}(s)=\sigma<1$. Then $\left(s\,\Lambda^*_{N,\chi}(s)\right)_{N\in\mathbb{N}}$ is a Cauchy sequence.

Proof of the Claim 1: The proof is immediate: for all $\epsilon>0$ and $\mathrm{Re}(s)=\sigma>0$ and, of course, taking N and M sufficiently large,

$$\left|\Lambda^*_{N,\chi}(s)-\Lambda^*_{M,\chi}(s)\right|\leq K\int_M^N\left|\Lambda_\chi(x)\right|x^{-\sigma-1}dx\leq K\,\frac{\ell-1}{\sigma}\,\left|N^{-\sigma}-M^{-\sigma}\right|<\epsilon.\ \blacksquare$$

Claim 2: The sequence $\left(\Lambda^*_{N,\chi}(s)\right)_{N\in\mathbb{N}}$ converges $\nu-$a.e. to $L(s,\chi)/s$.

Proof of the Claim 2: Let $\mathcal{A}=\left\{s\in\mathbb{C}:\ \mathrm{Re}(s)=\frac{1}{2}\text{ and }\lim_{k\to\infty}s\,\Lambda^*_{N_k,\chi}(s)=L(s,\chi)\right\}$. By Lemma 2.1. above, we know that $\nu\left(\mathcal{A}^c\right)=0$.

From the previous Claim, we know that, for any $s\in\mathcal{A}$, $\left(s\,\Lambda^*_{N,\chi}(s)\right)_{N\in\mathbb{N}}$ is a Cauchy-sequence which has a convergent subsequence.

Therefore, by an elementary property of Cauchy sequences, we see that, for all $s\in\mathcal{A}$, $\lim_{N\to\infty}s\,\Lambda_{N,\chi}(s)=L(s,\chi)$ and so $s\,\Lambda^*_{N,\chi}(s)$ converges $\nu-$ a.e. to $L(s,\chi)$. $\blacksquare$

Claim 3: We have the following equality

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}L(s,\chi)\,f^*(s)\,ds=\frac{1}{2\pi i}\lim_{N\to\infty}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}s\,\Lambda^*_{N,\chi}(s)\,f^*(s)\,ds. \tag{2.48}$$

Proof of the Claim 3: Consider the sequence of functions $\varphi_N(s) = s\,\Lambda^*_{N,\chi}(s)\,f^*(s)$. Then, by (2.46), it is not hard to see that for all $N \in \mathbb{N}$, $\varphi_N(s) \in L_1\left(\frac{1}{2} - i\infty,\, \frac{1}{2} + i\infty\right)$ since $f^*(s) \in L_1\left(\frac{1}{2} - i\infty,\, \frac{1}{2} + i\infty\right)$. Moreover[2], for $\mathrm{Re}(s) = \frac{1}{2}$, we clearly have that exists a constant $K > 0$ such that

$$
\begin{aligned}
|\varphi_N(s)| &\leq K\,|f^*(s)|\left(\left|\sum_{n=1}^{N}\frac{\chi(n)}{n^s}\right| + \frac{\ell-1}{\sqrt{N}}\right) \\
&= K\,|f^*(s)|\left(\left|L(s,\chi) + O\left(N^{-s}\right)\right| + \frac{\ell-1}{\sqrt{N}}\right) \\
&= K\,|f^*(s)|\left[|L(s,\chi)| + O\left(N^{-1/2}\right)\right] \in L_1\left(\frac{1}{2} - i\infty,\, \frac{1}{2} + i\infty\right) \qquad (2.49)
\end{aligned}
$$

Moreover, from the previous claim we know that $\varphi_N(s)$ converges $\nu -$ a.e. to $L(s,\chi)\,f^*(s) \in L_1\left(\frac{1}{2} - i\infty,\, \frac{1}{2} + i\infty\right)$.

From an application of Lebesgue's dominated convergence Theorem [13] to the sequence $\varphi_N(s)$, we immediately obtain (2.48). ■

Now, on the right-hand side of (2.48), let us use (2.46): this immediately gives

$$
\lim_{N\to\infty}\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\int_{1/N}^{N} x^{-s} f^*(s)\,d\Lambda_\chi(x)\,ds - \Lambda_\chi(N)\,f(N) + \Lambda_\chi\left(\frac{1}{N}\right)\,f\left(\frac{1}{N}\right) \qquad (2.50)
$$

$$
= \lim_{N\to\infty}\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\int_{1/N}^{N} x^{-s} f^*(s)\,d\Lambda_\chi(x)\,ds \qquad (2.51)
$$

where the latter terms were obtained via Mellin's inversion formula applied to $f^*(s) \in L_1\left(\frac{1}{2} - i\infty,\, \frac{1}{2} + i\infty\right)$. These clearly vanish when $N \to \infty$ because $\Lambda_\chi(x) = 0$ for $0 < x < 1$ and $f(x)\,x^{1/2} \to 0$ as $x \to \infty$ (see point 3. in the Main Preliminary Hypothesis, Proposition 2.1.).

Finally, after combining (2.48) with (2.50), we are left with the equality

$$
\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} L(s,\chi)\,f^*(s)\,ds = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\int_{1/N}^{N} x^{-s} f^*(s)\,d\Lambda_\chi(x)\,ds, \qquad (2.52)
$$

which we explore by evaluating its right-hand side. Note that

$$
\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\int_{1/N}^{N} \left|x^{-s} f^*(s)\,d\Lambda_\chi(x)\,ds\right| \leq \|f^*(s)\|_{L_1\left(\frac{1}{2}\right)}\int_{1/N}^{N} x^{-\frac{1}{2}}\,d\Lambda_\chi(x) < \infty,
$$

[2] To see this it suffices to adapt the computations in (2.55).

and so, by Fubini's theorem, we can interchange the order of integration on the right-hand side of (2.52) and this gives

$$\lim_{N\to\infty} \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \int_{1/N}^{N} x^{-s} f^{*}(s)\, d\Lambda_{\chi}(x)\, ds = \lim_{N\to\infty} \int_{1/N}^{N} f(x)\, d\Lambda_{\chi}(x), \tag{2.53}$$

by Mellin's inversion formula for the L_1-class.

Since $f(x)$ is absolutely continuous, it possesses a derivative for almost every $x \in [1/N,\, N]$ which is Lebesgue-integrable (see definition 2.1. above). Hence, we can integrate once more (2.53) by parts and we obtain

$$\int_{1/N}^{N} f(x)\, d\Lambda_{\chi}(x) = f\left(N\right) \Lambda_{\chi}\left(N\right) - f\left(\frac{1}{N}\right) \Lambda_{\chi}\left(\frac{1}{N}\right) - \int_{1/N}^{N} f'(x)\, \Lambda_{\chi}(x)\, dx. \tag{2.54}$$

To evaluate the last integral in (2.54), we simply proceed with the same trick as in (2.38) and we finally get

$$\begin{aligned} \int_{1/N}^{N} f'(x)\, \Lambda_{\chi}(x)\, dx = \int_{1}^{N} f'(x)\, \Lambda_{\chi}(x)\, dx &= \sum_{n=1}^{N-1} \int_{n}^{n+1} f'(x)\, \Lambda_{\chi}\left(n+\frac{1}{2}\right) dx \\ &= -\sum_{n=1}^{N-1} \chi\left(n\right)\, f(n) + \Lambda_{\chi}\left(N-\frac{1}{2}\right) f(N). \end{aligned} \tag{2.55}$$

Finally, using evaluation (2.55) together with (2.54), we obtain (2.44) after a simple substitution $f(y) \leftrightarrow f(xy)$, $x > 0$. ■

We now move to the main section of this chapter, where we prove the Main Theorem stated above.

2.4 Main Results: A character version of Poisson's summation formula

The main strategy in our proof, which will be used several times during this thesis, relies on using the symmetries of $\zeta(s)$ and $L(s,\chi)$ with respect to the critical line $\mathrm{Re}(s) = \sigma = 1/2$.

In what follows, we prove formulas (2.27) and (2.28) given in the main theorem. Since they invoke a different transform, we separate their proofs and we emphasize the importance that the parity of the character χ plays in their statements.

Proof of the Main Theorem for even χ In the main lemma of this chapter we have proved that

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} L(s,\chi)\, f^{*}(s)\, x^{-s}\, ds = \lim_{N\to\infty}\sum_{n=1}^{N}\chi(n)\, f(nx), \tag{2.56}$$

and the right-hand side of the previous equality is precisely the left-hand side of eq. (2.27).

Hence, to relate the limit of the sum involving f with its homologous g, we need to study the left-hand side of (2.56), i.e., the absolutely convergent integral involving $L(s,\chi)$.

The idea now is to invoke the functional equation for $L(s,\chi)$: recall that, if χ is nonprincipal, primitive and even, the functional equation for its $L-$function is given by (see eq. (1.143) in the first chapter)

$$L(1-s,\chi) = 2\left(\frac{\ell}{2\pi}\right)^{s}\ell^{-1}G(\chi)\,\Gamma(s)\,L(s,\overline{\chi})\,\cos\left(\frac{\pi s}{2}\right), \tag{2.57}$$

which allows to write the left-hand side of (2.56) as

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} L(s,\chi)\, f^{*}(s)\, x^{-s}\, ds = \frac{2G(\chi)}{\ell x}\,\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} L(s,\overline{\chi})\,\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\, f^{*}(1-s)\,\left(\frac{2\pi}{\ell x}\right)^{-s} ds. \tag{2.58}$$

To obtain a summation formula, we need to study the Mellin inverse of the function $g^{*}(s) = \Gamma(s)\,\cos\left(\frac{\pi s}{2}\right)\, f^{*}(1-s)$, which will be found by appealing to Parseval's equality (2.26).

Taking $s=\sigma+it$, we can easily see from Stirling's formula for the $\Gamma-$function (1.32) that, when $|t|\to\infty$, $\Gamma(s)\cos\left(\frac{\pi s}{2}\right)$ behaves as

$$\Gamma(s)\,\cos\left(\frac{\pi s}{2}\right) = O\left(|t|^{\sigma-\frac{1}{2}}e^{-\frac{\pi}{2}|t|}\cosh\left(\frac{\pi}{2}t\right)\right) = |t|^{\sigma-\frac{1}{2}},\quad |t|\to\infty, \tag{2.59}$$

which is $O(1)$ if $\sigma=1/2$. The estimate (2.59) clearly implies that

$$\frac{\Gamma(s)}{1-s}\,\cos\left(\frac{\pi s}{2}\right)\in L_{2}\left(\frac{1}{2}-i\infty,\,\frac{1}{2}+i\infty\right).$$

Thus, $g^{*}(s)\in L_{1}\left(\frac{1}{2}-i\infty,\,\frac{1}{2}+i\infty\right)$, since $f^{*}(s)\in L_{1}\left(\frac{1}{2}-i\infty,\,\frac{1}{2}+i\infty\right)$ by point 1. given at the preliminary hypothesis. We can now write the integral

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma(s)\,\cos\left(\frac{\pi s}{2}\right)\, f^{*}(1-s)\,\left(\frac{2\pi}{\ell x}\right)^{-s} ds$$

using the considerations of the Mellin transform on L_1. Now we want to find $g(x)$ given by the Mellin integral,

$$g(x) = \frac{1}{2\pi i} \int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma(s)\, \cos\left(\frac{\pi s}{2}\right)\, f^*(1-s)\, x^{-s} ds$$
$$= \frac{1}{2\pi i} \int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Gamma(s)}{1-s}\, \cos\left(\frac{\pi s}{2}\right) \times (1-s)\, f^*(1-s)\, x^{-s} ds. \tag{2.60}$$

It is easy to observe that $s\, f^*(s)$ is the L_2-Mellin transform of $-x\, f'(x)$, which exists almost everywhere by the absolute continuity of $f(x)$.

Since both factors in the integrand of (2.60) belong to $L_2\left(\frac{1}{2} - i\infty,\, \frac{1}{2} + i\infty\right)$, the conditions to apply Parseval's theorem are met. Applying it to $g^*(s) = \Gamma(s)\, \cos\left(\frac{\pi s}{2}\right)\, f^*(1-s)$ we obtain, with the use of the elementary integral [102],

$$\int\limits_{0}^{\infty} \frac{\sin(x)}{x}\, x^{s-1} dx = \cos\left(\frac{\pi s}{2}\right) \frac{\Gamma(s)}{1-s}, \quad 0 < \sigma < 1, \tag{2.61}$$

that $g(x)$ can be expressed as the transform

$$g(x) = \frac{1}{2\pi i} \int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Gamma(s)}{1-s} \cos\left(\frac{\pi s}{2}\right)\, (1-s) f^*(1-s)\, x^{-s} ds$$
$$= -\frac{1}{x} \int\limits_{0}^{\infty} f'(y)\, \sin(xy) dy. \tag{2.62}$$

Finally, to arrive at the right-hand side of (2.27), we just need to prove that $s\, g^*(s) \in L_2\left(\frac{1}{2} - i\infty,\, \frac{1}{2} + i\infty\right)$: to check this, notice that

$$s\, g^*(s) = \Gamma(s)\, \cos\left(\frac{\pi s}{2}\right)\, s\, f^*(1-s)$$
$$= -\Gamma(s)\, \cos\left(\frac{\pi s}{2}\right) (1-s)\, f^*(1-s) + \Gamma(s)\, \cos\left(\frac{\pi s}{2}\right) f^*(1-s).$$

Therefore, from the fact that $\Gamma(s)\, \cos\left(\frac{\pi s}{2}\right)$ is bounded in the critical line (eq. (2.59) above) and both $(1-s)\, f^*(1-s)$ and $f^*(1-s)$ belong to $L_2\left(\frac{1}{2} - i\infty,\, \frac{1}{2} + i\infty\right)$, an application of Minkowski's inequality proves that $s\, g^*(s) \in L_2\left(\frac{1}{2} - i\infty,\, \frac{1}{2} + i\infty\right)$ as well.

From the Main Lemma, the left-hand side of (2.56) can now be written as

$$\frac{2G(\chi)}{\ell x}\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} L(s,\overline{\chi})\Gamma(s)\cos\left(\frac{\pi s}{2}\right) f^{*}(1-s)\left(\frac{2\pi}{\ell x}\right)^{-s} ds$$

$$= \frac{2G(\chi)}{\ell x}\lim_{N\to\infty}\sum_{n=1}^{N}\overline{\chi}(n)\, g\left(\frac{2\pi n}{\ell x}\right),$$

which, together with (2.62), concludes the proof of the first part of the Main Theorem. ■

Remark 2.1.: Note that the integral transform $g(x)$ has a primitive satisfying

$$\int_0^z g(x)\,dx = \int_0^z -\frac{1}{x}\int_0^\infty f'(y)\,\sin(xy)\,dy\,dx$$

$$= -\int_0^\infty f'(y)\int_0^z \frac{\sin(xy)}{x}dx\,dy$$

$$= \int_0^\infty f(y)\,\frac{\sin(zy)}{y}\,dy,$$

which means that, almost everywhere, $g(x)$ coincides with the L_2 - Fourier cosine transform of f (see (2.13)), i.e.,

$$g(x) \overset{\mu}{=} \underset{N\to\infty}{\text{l.i.m.}} \int_{1/N}^{N} f(y)\,\cos(xy)\,dy,$$

where μ denotes the Lebesgue measure on $\mathbb{R}_+$.

Using similar techniques, we can prove the analogous formula for primitive odd Dirichlet characters. We give the main details below.

Proof of the Main theorem for odd χ Assuming that χ is odd and primitive, recall that the functional equation for its Dirichlet $L-$function can be expressed as

$$L(1-s,\,\chi) = -2i\left(\frac{\ell}{2\pi}\right)^{s}\ell^{-1}G(\chi)\Gamma(s)\,L\left(s,\,\overline{\chi}\right)\sin\left(\frac{\pi s}{2}\right). \tag{2.63}$$

The substitution of the functional equation in the left hand side integral of (2.56) and an elementary change of variables provides the relation

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} L(s,\chi)\, f^{*}(s)\, x^{-s}ds = \frac{-2iG(\chi)}{\ell x}\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} L(s,\overline{\chi})\,\Gamma(s)\,\sin\left(\frac{\pi s}{2}\right) f^{*}(1-s)\left(\frac{2\pi}{\ell x}\right)^{-s} ds.$$

Using the previous argument we obtain similarly

$$\frac{1}{2\pi i}\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} L(s,\chi)\, f^*(s)\, ds = \frac{-2iG(\chi)}{\ell x}\,\frac{1}{2\pi i}\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} L\left(s,\overline{\chi}\right)\, h^*(s)\, \left(\frac{2\pi}{\ell x}\right)^{-s} ds,$$

where $h(x)$ is given by the integral transform

$$h(x) = \int\limits_0^\infty -y\, f'(y)\, \varphi(xy)\, dy, \tag{2.64}$$

whith the kernel $\varphi(x)$ being the L_2-Mellin inverse of $\frac{\Gamma(s)}{1-s}\,\sin\left(\frac{\pi s}{2}\right)$.

Also, from the elementary integral (see [102])

$$\int\limits_0^\infty \frac{1-\cos(x)}{x}\, x^{s-1} dx = \frac{\Gamma(s)}{1-s}\,\sin\left(\frac{\pi s}{2}\right), \qquad 0<\sigma<1, \tag{2.65}$$

we have that $\varphi(x) = \frac{1-\cos(x)}{x}$ and an elementary substitution in (2.64) shows that

$$h(x) = -\frac{2}{x}\int\limits_0^\infty f'(y)\, \sin^2\left(\frac{xy}{2}\right) dy, \tag{2.66}$$

which provides (2.28). ■

Remark 2.2.: One can also check that

$$\int\limits_0^z h(x)\, dx = \int\limits_0^\infty f(y)\, \frac{1-\cos(zy)}{y}\, dy,$$

and so $g(x)$ equals almost everywhere to the Fourier sine transform of f, i.e.,

$$h(x) \overset{\mu}{=} \underset{N\to\infty}{\text{l.i.m.}} \int\limits_{1/N}^{N} f(y)\, \sin(xy)\, dy.$$

The approach used in this chapter can be also adapted to prove the classical Poisson summation formula for the L_2-class given by equation (2.29).

Notice that the transform $g(x)$ used in (2.29) is the very same as the one in (2.27), which is not a surprise, since the functional equation for the Riemann $\zeta-$function is very similar to the one for the Dirichlet $L-$function with even characters.

We briefly describe the main steps leading to (2.29).

Proof of the Classical Poisson summation formula: The proof itself is basically the same as the one for an even character, so we will skip the details. However, the aspects derived at the section "Preliminary results - III" will be slightly different because $\zeta(s)$ has a simple pole at $s=1$, while $L(s,\chi)$ is entire (recall Theorem 1.6. on the previous chapter).

We itemize the main differences in this proof, omitting the details in the computations, as they can be handled out in an analogous way.

1. Instead of $\Lambda_\chi(x)$ we introduce the remainder $\Lambda(x)=[x]-x$ if $x\notin\mathbb{N}$ and $\Lambda(x)=\frac{x}{2}-1$ if $x\in\mathbb{N}$. If, by analogy, we take $h(x)=\frac{\Lambda(x)}{x}$, it is simple to see that $h(x)=O(1/x)$ as $x\to\infty$ and $h(x)=O(1)$ as $x\to 0$, so that the L_1-Mellin transform of $h(x)$ is well-defined at the critical strip. Furthermore, using the L_2-theory for Mellin transforms, we can easily show that
$$h^{*}(s)=\frac{\zeta(1-s)}{1-s},\quad 0<\text{Re}(s)<1. \tag{2.67}$$
2. From (2.67) we can deduce an analogous version of the Main Lemma:
$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\zeta(s)\,f^{*}(s)\,ds=\lim_{N\to\infty}\sum_{n=1}^{N}f(n)-\int_{1/N}^{N}f(x)\,dx. \tag{2.68}$$
3. Finally, using the functional equation for the Riemann $\zeta-$function in the left-hand side of (2.68) and appealing to the transform (2.62) we deduce immediately the Poisson summation formula for the L_2 - class (2.29). ■

2.5 Poisson summation formula for the Müntz Class

As said at the introduction, one of the classes of functions studied in this thesis is the class of functions of Müntz-type, a concept introduced for the first time by Yakubovich in [116].

Lima [73] generalized the definition given in Yakubovich's paper and we will work under Lima's definition, which we now state formally.

Definition 2.2.: A function $f(x)$, defined for $x\in\mathbb{R}_0^{+}$, belongs to the class of functions of Müntz type, $\mathcal{M}_{\alpha,n}$, where $\alpha>1$ and $n\geq 2$, if $f\in C^{n}(\mathbb{R}_0^{+})$ and $f^{(k)}(x)=O\left(x^{-\alpha-k}\right)$, $x\to\infty$ for all $k=0,\,1,\,...,\,n$.

In the following corollary, we show that (2.27), (2.28) and (2.29) also hold for the Müntz class of functions, with the series and integrals involving these being absolutely convergent (recall the definition given at the introduction).

However, in order to establish this, we need to state a lemma proved by Yakubovich in [116] and extended by Lima in [73]. We state the theorem as it is given in [73]:

Lemma 2.2.: Let $f \in \mathcal{M}_{\alpha,n}$, for $n \geq 2$. Then the Mellin transform of $f(x)$, $f^*(s)$, can be analytically continued to the strip $-n < \operatorname{Re}(s) < \alpha$, where it is analytic except at the points $s = -k$, $k = 0, 1, ..., n-1$. At these, $f^*(s)$ may have a simple pole with residue $\frac{f^{(k)}(0)}{k!}$, if $f^{(k)}(0) \neq 0$, or a removable singularity, if $f^{(k)}(0) = 0$.

Moreover, for any $-n < \sigma < \alpha$ there exists $C(\sigma) \in \mathbb{R}_+$ such that $|f^*(\sigma + it)| \leq C(\sigma)\,|t|^{-n}$ for all $t \in \mathbb{R} \setminus \{0\}$.

The previous Lemma 2.2., whose proof may be found in [73] is now essential for the next Corollary, which extends a result proved in [116] to Dirichlet characters.

Corollary 2.1.: Poisson Summation formula for the class of Müntz functions Let $f \in \mathcal{M}_{\alpha,2}$, $\alpha > 1$ and χ be a nonprincipal and primitive Dirichlet character modulo ℓ.

Then, for all $x > 0$, the summation formulas hold

$$\sum_{n=1}^{\infty} \chi(n)\, f(nx) = \frac{2G(\chi)}{\ell x} \sum_{n=1}^{\infty} \overline{\chi}(n)\, g\left(\frac{2\pi n}{\ell x}\right), \ \chi \text{ even}, \tag{2.69}$$

$$\sum_{n=1}^{\infty} \chi(n)\, f(nx) = -\frac{2iG(\chi)}{\ell x} \sum_{n=1}^{\infty} \overline{\chi}(n)\, h\left(\frac{2\pi n}{\ell x}\right), \ \chi \text{ odd}, \tag{2.70}$$

$$\frac{f(0)}{2} + \sum_{n=1}^{\infty} f(nx) = \frac{1}{x}\, g(0) + \frac{2}{x} \sum_{n=1}^{\infty} g\left(\frac{2\pi n}{x}\right) \quad \text{classical Poisson sum. formula}, \tag{2.71}$$

where $g(x)$ and $h(x)$ are the integral transforms given by

$$g(x) = \int_0^{\infty} f(y)\, \cos(xy)\, dy$$

and

$$h(x) = \int_0^{\infty} f(y)\, \sin(xy)\, dy.$$

The transforms given above are the Fourier transforms for the L_1-class.

Proof: Since $f \in \mathcal{M}_{\alpha,2}$, we have by Lemma 2.2. that, for $-2 < \sigma < \alpha$, $f^*(\sigma + it) = O\left(|t|^{-2}\right), \ \ |t| \to \infty$.

Therefore, f satisfies the conditions of the main theorem, as $s\, f^*(s) \in L_2\left(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty\right)$. Moreover, $f \in C^2(\mathbb{R}_0^+)$ so that its derivative exists for all $x \in \mathbb{R}_+$ and we can apply the previous considerations.

Since $f(x) = O(x^{-\alpha})$, $\alpha > 1$, by hypothesis, the series $\sum_{n=1}^{\infty} \chi(n)\, f(n)$ converges absolutely by the integral test, and so

$$\lim_{N\to\infty} \sum_{n=1}^{N} \chi(n)\, f(n) = \sum_{n=1}^{\infty} \chi(n)\, f(n).$$

Moreover, the integral defining $g(x)$ (2.30) converges absolutely and uniformly on $\mathbb{R}_+$. To see this, note that an integration by parts yields

$$g(x) = -\frac{1}{x}\int_0^{\infty} f'(y)\,\sin(xy)\,dy = \frac{1}{x^2}\left[f'(0) - \int_0^{\infty} f''(y)\,\cos(xy)\,dy\right] = O\left(\frac{1}{x^2}\right),$$

as $x \to \infty$, which proves that the series $\sum_{n=1}^{\infty} \chi(n)\, g(n)$ also converges absolutely.

Since f and g clearly belong to $L_1(\mathbb{R}_+) \cap L_2(\mathbb{R}_+)$, we can view $g(x)$ as the L_1-Fourier cosine transform of f,

$$g(x) = \int_0^{\infty} f(y)\,\cos(xy)\,dy, \tag{2.72}$$

for which we have the classical inversion formula (see Theorem 2.A-1)

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \cos(xy)\,g(y)dy. \tag{2.73}$$

Analogous considerations hold for the sine transform and the case where χ is odd.

Using (2.73), can write the latter integral in (2.29) as

$$\int_0^{\infty} g\left(\frac{2\pi y}{x}\right)\,dy = \frac{x}{4}\,f(0),$$

and an application of the Main Theorem allows to prove (2.69 - 2.71). ■

2.6 Examples:

In this section, we introduce some examples of the proved summation formulas and which will be used throughout this thesis.

We start with a generalization of the reflection formula for the Dedekind $\eta-$function, which will be used in the fourth chapter.

Example 2.1.: The reflection formula for Dedekind $\eta-$function and a character version of it

Consider the function $f(x)=x^{\nu-1}e^{-\alpha x}$, for $\mathrm{Re}(\nu)>0$ and $\alpha>0$. It is simple to check that $f^{*}(s)=\alpha^{-(s+\nu)}\Gamma(s+\nu-1)$ satisfies the conditions of the main theorem and $f\in L_1(\mathbb{R}_+)\cap L_2(\mathbb{R}_+)$. Using the integrals [44]

$$\int_0^{\infty} y^{\nu-1}e^{-\alpha y}\cos(xy)\,dy=\frac{\Gamma(\nu)}{(\alpha^2+x^2)^{\nu/2}}\,\cos\left(\nu\,\arctan\left(x/\alpha\right)\right), \tag{2.74}$$

$$\int_0^{\infty} y^{\nu-1}e^{-\alpha y}\sin(xy)\,dy=\frac{\Gamma(\nu)}{(\alpha^2+x^2)^{\nu/2}}\,\sin\left(\nu\,\arctan\left(x/\alpha\right)\right), \tag{2.75}$$

we see that (2.69), (2.70) and (2.71) yield the formulas

$$\sum_{n=1}^{\infty}\chi(n)\,n^{\nu-1}e^{-\alpha n}=\frac{2\,G(\chi)\Gamma(\nu)}{\ell}\left(\frac{2\pi}{\ell}\right)^{-\nu}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)\,\cos\left(\nu\,\arctan(2\pi n/\ell\alpha)\right)}{\left(n^2+\left(\frac{\alpha\ell}{2\pi}\right)^2\right)^{\nu/2}},\quad \chi\,\text{even},$$

$$\sum_{n=1}^{\infty}\chi(n)\,n^{\nu-1}e^{-\alpha n}=-\frac{2iG(\chi)\Gamma(\nu)}{\ell}\left(\frac{2\pi}{\ell}\right)^{-\nu}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)\,\sin\left(\nu\,\arctan\left(2\pi n/\ell\alpha\right)\right)}{\left(n^2+\left(\frac{\alpha\ell}{2\pi}\right)^2\right)^{\nu/2}},\quad \chi\,\text{odd},$$

$$\sum_{n=1}^{\infty}n^{\nu-1}e^{-\alpha n}=\Gamma(\nu)\,\alpha^{-\nu}+2\Gamma(\nu)(2\pi)^{-\nu}\sum_{n=1}^{\infty}\frac{\cos\left(\nu\,\arctan\left(2\pi n/\alpha\right)\right)}{\left(n^2+\left(\frac{\alpha}{2\pi}\right)^2\right)^{\nu/2}}.$$

Let us consider the simpler case $\nu=1$ above. Using elementary trigonometric identities, we obtain the particular cases

$$\sum_{n=1}^{\infty}\chi(n)\,e^{-\alpha n}=\frac{\alpha}{2\pi^2}G(\chi)\ell\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)}{n^2+\left(\frac{\alpha\ell}{2\pi}\right)^2},\quad \chi\,\text{even}, \tag{2.76}$$

$$\sum_{n=1}^{\infty}\chi(n)\,e^{-\alpha n}=-\frac{iG(\chi)}{\pi}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)\,n}{n^2+\left(\frac{\alpha\ell}{2\pi}\right)^2},\quad \chi\,\text{odd}, \tag{2.77}$$

$$\frac{1}{2}+\sum_{n=1}^{\infty}e^{-\alpha n}=\frac{1}{\alpha}+\frac{\alpha}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2+\left(\frac{\alpha}{2\pi}\right)^2}. \tag{2.78}$$

Using these identities, we construct this example providing a new proof of the functional equation for the Dedekind $\eta-$function and studying a character version of it, for an even Dirichlet character which is primitive and real. These studies will play an important role in the fourth Chapter, where we derive a character version of Kronecker limit formula for Epstein's $\zeta-$function.

Related with the theory of partitions, the Dedekind $\eta-$function is defined by the infinite product [94],

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{m=1}^{\infty} \left(1 - e^{2\pi i m\tau}\right), \quad \mathrm{Im}(\tau) > 0. \tag{2.79}$$

One can show that $\eta(\tau)$ can be expressed in terms of the generating function for the partition function $p(n)$, which counts all ways to express n as a sum of positive integers. Indeed, an alternative way of writing (2.79) is

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} f(\tau)^{-1}, \tag{2.80}$$

where $f(\tau)$ is the generating function for $p(n)$, i.e.,

$$f(\tau) = \sum_{n=0}^{\infty} p(n)\, e^{2\pi i n\tau}. \tag{2.81}$$

It is also well-known that (2.79) obeys to the reflection formula [5, 94]

$$\eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{1/2}\eta(\tau). \tag{2.82}$$

Using Poisson summation formula in the L_2-form stated at the beginning of the chapter, we shall derive the reflection formula (2.82) and extend it to a character analogue involving even characters.

From (2.78) it is simple to deduce that, if $f(x) = -\log(1-e^{-\alpha x})$, then its Fourier cosine transform is given by

$$\begin{aligned} g(x) &= -\int_0^{\infty} \log\left(1-e^{-\alpha y}\right)\, \cos(xy)dy = \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} e^{-\alpha n y} \cos(xy)\, dy \\ &= \frac{1}{\alpha}\sum_{n=1}^{\infty} \frac{1}{n^2+\left(\frac{x}{\alpha}\right)^2} = \frac{\pi}{x}\left(\frac{1}{2} - \frac{\alpha}{2\pi x} + \sum_{n=1}^{\infty} e^{-\frac{2\pi x n}{\alpha}}\right) \\ &= \frac{\pi}{x}\left(\frac{1}{2} - \frac{\alpha}{2\pi x} + \frac{1}{e^{2\pi x/\alpha}-1}\right), \end{aligned} \tag{2.83}$$

and, by the Basel identity, $g(0) = \frac{1}{\alpha}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6\alpha}$. Using Poisson summation formula (2.29) for $f(x) = -\log\left(1-e^{-\alpha x}\right)$, one obtains

$$-\sum_{n=1}^{\infty} \log\left(1-e^{-\alpha n}\right) - \frac{\pi^2}{6\alpha} = 2\lim_{N\to\infty}\left[\sum_{n=1}^{N} g\left(2\pi n\right) - \int_{1/N}^{N} g\left(2\pi x\right)\, dx\right] \tag{2.84}$$

where $g(x)$ is given by (2.83). To evaluate the right-hand side of (2.84), we need to compute the limit

$$\begin{aligned}
\lim_{N\to\infty}\sum_{n=1}^{N} g(2\pi n) &= \lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{2n}\left(\frac{1}{2}-\frac{\alpha}{4\pi^2 n}+\frac{1}{e^{4\pi^2 n/\alpha}-1}\right)\\
&= \lim_{N\to\infty}\left[\frac{1}{4}H_N+\frac{1}{2}\sum_{n=1}^{N}\frac{1}{n\left(e^{4\pi^2 n/\alpha}-1\right)}-\frac{\alpha}{8\pi^2}\sum_{n=1}^{N}\frac{1}{n^2}\right]\\
&= \frac{\gamma}{4}+\frac{1}{4}\lim_{N\to\infty}\log(N)-\frac{1}{2}\sum_{m=1}^{\infty}\log\left(1-e^{-\frac{4\pi^2 m}{\alpha}}\right)-\frac{\alpha}{48},
\end{aligned}\tag{2.85}$$

where we have used the definition of the Euler-Mascheroni constant γ. We join the logarithmic limit with the integral in the right-hand side of (2.84) to obtain

$$\begin{aligned}
&\lim_{N\to\infty}\left(\int_{1/N}^{N} g(2\pi x)\,dx-\frac{1}{4}\log(N)\right)\\
&=\lim_{N\to\infty}\int_{1/N}^{N}\frac{1}{4x}-\frac{\alpha}{4\pi x^2}+\frac{1}{2x\left(e^{4\pi^2 x/\alpha}-1\right)}dx-\frac{1}{4}\log(N)\\
&=\lim_{N\to\infty}\int_{1/N\alpha}^{1}\frac{1}{4x}-\frac{1}{4\pi\, x^2}+\frac{1}{2x\left(e^{4\pi^2 x}-1\right)}\,dx\\
&\quad+\int_{1}^{N/\alpha}\frac{1}{2x\left(e^{4\pi^2 x}-1\right)}-\frac{1}{4\pi x^2}\,dx-\frac{1}{4}\log(\alpha)=I-\frac{1}{4}\log(\alpha),
\end{aligned}\tag{2.86}$$

where I is a constant expressed by the sum of integrals

$$\int_0^1\frac{1}{4x}-\frac{1}{4\pi\, x^2}+\frac{1}{2x\left(e^{4\pi^2 x}-1\right)}\,dx+\int_1^{\infty}\frac{1}{2x\left(e^{4\pi^2 x}-1\right)}-\frac{1}{4\pi x^2}\,dx.$$

Combining (2.84), (2.85) and (2.86) yields

$$\begin{aligned}
&-\sum_{n=1}^{\infty}\log\left(1-e^{-\alpha n}\right)-\frac{\pi^2}{6\alpha}\\
&=\frac{\gamma}{2}-\sum_{m=1}^{\infty}\log\left(1-e^{-\frac{4\pi^2 m}{\alpha}}\right)-\frac{\alpha}{24}-2I+\frac{1}{2}\log(\alpha).
\end{aligned}\tag{2.87}$$

Now, to determine the value of the constant I, take $\alpha=2\pi$ and solve (2.87) to obtain

$$I=\frac{\gamma}{4}+\frac{1}{4}\log(2\pi),$$

so that, if we take $\alpha = 2\pi x$ in (2.87) we derive the identity

$$-\sum_{n=1}^{\infty} \log\left(1-e^{-2\pi n x}\right) - \frac{\pi}{12x} = -\sum_{n=1}^{\infty} \log\left(1-e^{-\frac{2\pi n}{x}}\right) - \frac{\pi x}{12} + \frac{1}{2}\log(x). \qquad (2.88)$$

From the definition of $\eta(\tau)$, if $x > 0$,

$$\log \eta(ix) = -\frac{\pi x}{12} + \sum_{n=1}^{\infty} \log\left(1-e^{-2\pi n x}\right),$$

which means that (2.88) is equivalent to

$$\log(\eta(ix)) = \log(\eta(i/x)) + \frac{1}{2}\log(x). \qquad (2.89)$$

Taking the exponential function at both sides of (2.89), we obtain (2.82) for $\tau = ix$, $x > 0$. However, the whole result holds by analytic continuation, since $\eta(\tau)$ defines an analytic function on the upper half-complex plane $\mathbb{H}$ [94]. This proves the elegant reflection formula

$$\eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{1/2}\,\eta(\tau), \quad \text{Im}(\tau) > 0. \qquad (2.90)$$

We can still deduce a character analogue of (2.90) and prove a new relation of this type. Invoking (2.76), we obtain

$$\begin{aligned} -\sum_{n=1}^{\infty} \chi(n)\,\log\left(1-e^{-\alpha n}\right) &= \frac{2G(\chi)}{\ell}\sum_{n=1}^{\infty} \overline{\chi}(n)\left[\frac{\ell}{4n} - \frac{\alpha\ell^2}{8\pi^2 n^2} + \frac{\ell}{2n\left(e^{4\pi^2 n/\ell\alpha}-1\right)}\right] \\ &= \frac{G(\chi)}{2}L(1,\overline{\chi}) - \frac{\alpha\ell G(\chi)}{4\pi^2}\,L(2,\overline{\chi}) + G(\chi)\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)}{n\left(e^{4\pi^2 n/\ell\alpha}-1\right)} \\ &= \frac{G(\chi)}{2}L(1,\overline{\chi}) - \frac{\alpha\ell G(\chi)}{4\pi^2}\,L(2,\overline{\chi}) + G(\chi)\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)}{n}\sum_{m=1}^{\infty} e^{-4\pi^2 nm/\ell\alpha} \\ &= \frac{G(\chi)}{2}L(1,\overline{\chi}) - \frac{\alpha\ell G(\chi)}{4\pi^2}\,L(2,\overline{\chi}) + G(\chi)\sum_{n=1}^{\infty}\sigma_{-1,\overline{\chi}}(n)\,e^{-4\pi^2 n/\ell\alpha}, \end{aligned} \qquad (2.91)$$

where $\sigma_{a,\chi}(n) = \sum_{d|n}\chi(d)\,d^a$ denotes the character analogue of the generalized divisor function $\sigma_a(n)$ (see the glossary). As we shall see, (2.91) has a connection with a character analogue of Guinand's summation formula, which we shall deduce in the fourth chapter using the properties of the series representation for the Epstein $\zeta-$function (see the details on Corollary 4.2.3.).

However, the last series in (2.91) can be simplified if we introduce the character analogue of the logarithmic function, which we now define.

If χ is a Dirichlet character and $|z| < 1$, we define the character analogue of the logarithmic function as the power series of Mercator type

$$\log_\chi(1-z) = -\sum_{k=1}^{\infty} \frac{\chi(k)}{k}\, z^k, \quad |z| < 1, \tag{2.92}$$

which can be employed in the series in (2.91) with a reversion of the order of summation, giving

$$\sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{n} \sum_{m=1}^{\infty} e^{-4\pi^2 nm/\ell\alpha} = -\sum_{m=1}^{\infty} \log_{\overline{\chi}}\left(1 - e^{-4\pi^2 m/\ell\alpha}\right). \tag{2.93}$$

Finally, if we take $\alpha = 2\pi x$ in (2.91), we arrive to the identity

$$-\sum_{n=1}^{\infty} \chi(n) \log\left(1 - e^{-2\pi n x}\right) = \frac{G(\chi)}{2} L(1,\overline{\chi}) - \frac{\ell G(\chi)\, x}{2\pi}\, L(2,\overline{\chi}) - G(\chi) \sum_{m=1}^{\infty} \log_{\overline{\chi}}\left(1 - e^{-\frac{2\pi m}{\ell x}}\right), \tag{2.94}$$

which can be seen as a character analogue of formula (2.82) for $\eta(\tau)$.

Now, if we define for a real, even and primitive character modulo ℓ, the function

$$\eta_\chi(\tau) = e^{i\frac{\ell G(\chi)\, L(2,\chi)}{2\pi}\tau} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n\tau}\right)^{\chi(n)}, \tag{2.95}$$

we arrive to the functional equation

$$\eta_\chi(\tau) = e^{-\frac{G(\chi)}{2}\, L(1,\chi)} \prod_{n=1}^{\infty} \exp\left(G(\chi)\, \log_\chi\left(1 - e^{-\frac{2\pi i m}{\ell\tau}}\right)\right), \quad \mathrm{Im}(\tau) > 0, \tag{2.96}$$

which seems an extension of the reflection formula (2.90), although it only makes sense for real, even and primitive characters.

However, if $\chi(n)$ is the trivial character $\chi(n) \equiv 1$, $L(2,\overline{\chi}) = \zeta(2) = \frac{\pi^2}{6}$ and, in this case, $\eta_\chi(\tau)$ reduces to the usual Dedekind $\eta-$function (2.79) and $\log_\chi$ to the usual logarithmic function.

Since χ is even, we know that $L(2,\chi)$ obeys to the identity (see relation (1.149) on the first chapter)

$$L(2,\chi) = \frac{\pi^2}{\ell^2}\, G(\chi)\, B_{2,\chi}, \tag{2.97}$$

where $B_{2,\chi}$ is the character analogue of B_2 which can be written in terms of Bernoulli polynomials (recall (1.112))

$$B_{2,\chi} = \ell \sum_{r=1}^{\ell-1} \chi(r)\, B_2\left(\frac{r}{\ell}\right) = \ell \sum_{r=1}^{\ell-1} \chi(r) \left[\frac{r^2}{\ell^2} - \frac{r}{\ell} + \frac{1}{6}\right] = \ell^{-1} \sum_{r=1}^{\ell-1} \chi(r)\, r^2, \tag{2.98}$$

since the other terms are zero due to the sign of χ and the fact that it is nonprincipal.

Alternatively, we can write the extended Dedekind $\eta-$function in the compact form

$$\eta_{\chi}(\tau)=e^{i\frac{\pi}{2}B_{2,\chi}\tau}\prod_{n=1}^{\infty}\left(1-e^{2\pi in\tau}\right)^{\chi(n)}. \tag{2.99}$$

When χ is the principal character modulo $\ell>1$, we shall see on the fourth chapter that (2.99) plays an important role in studying a character analogue for Epstein's $\zeta-$function.

Example 2.2.: Eisenstein series and formulas of Nasim-type

Before considering the following example, it is important to mention other famous representations of the summation formulas proved above: note that we can write (2.27), (2.28) and (2.29) by summing also over the negative integers and appealing to the complex Fourier transform (2.11). Doing this yields the complex versions of Poisson's summation formula

$$\lim_{N\to\infty}\sum_{n=-N}^{N}\chi(n)\,f(n)=\frac{G(\chi)}{\ell}\lim_{N\to\infty}\sum_{n=-N}^{N}\overline{\chi}(n)\,\hat{f}\left(\frac{2\pi n}{\ell}\right),\quad\chi\text{ even}, \tag{2.100}$$

$$\lim_{N\to\infty}\sum_{n=-N}^{N}\chi(n)\,f(n)=-\frac{G(\chi)}{\ell}\lim_{N\to\infty}\sum_{n=-N}^{N}\overline{\chi}(n)\,\hat{f}\left(\frac{2\pi n}{\ell}\right),\quad\chi\text{ odd}, \tag{2.101}$$

$$\lim_{N\to\infty}\left[\sum_{1\leq|n|\leq N}f(n)-\int_{-N}^{N}f(y)\,dy\right]=\lim_{N\to\infty}\left[\sum_{1\leq|n|\leq N}\hat{f}\left(2\pi n\right)-\int_{-N}^{N}\hat{f}\left(2\pi y\right)\,dy\right]. \tag{2.102}$$

If k is an integer such that $k\geq 3$, the Eisenstein series of order k is defined by the double series [94],

$$G_{k}(\tau)=\sum_{(m,n)\neq(0,0)}\frac{1}{(m+n\tau)^{k}}, \tag{2.103}$$

where τ is a complex number belonging to the upper half-complex plane. For $k\geq 3$, the double series in (2.103) converges absolutely and uniformly in every half-plane $0<\delta\leq\text{Im}(\tau)$, which implies that $G_{k}(\tau)$ is analytic in the upper half-plane $\text{Im}(\tau)>0$.

It is also easy to check that, if k is an odd integer, then $G_{k}(\tau)=0$. Furthermore, $G_{k}(\tau)$ is invariant under modular transformations, i.e.,

$$G_{k}\left(\tau+1\right)=G_{k}\left(\tau\right),\qquad G_{k}(\tau)=\tau^{-k}\,G_{k}(-1/\tau), \tag{2.104}$$

so that $G_{k}(\tau)$, together with $\eta(\tau)$, is the standard example of a modular form [94]. In what follows, by analogy with $\eta(\tau)$, we shall discuss character analogues of (2.103) and the summation formulas that we can derive by considering their symmetries.

If χ is a nonprincipal and primitive character modulo ℓ, the associated Eisenstein series can be expressed by the double series

$$G_k(\tau,\,\chi) = \sum_{(m,n)\neq(0,0)} \frac{\chi(m)}{(m+n\tau)^k}. \tag{2.105}$$

Note that, if χ and k have different "parity" (i.e., if χ is even and k is odd and vice versa), then $G_k(\tau,\chi)=0$. Therefore, for $k\geq 2$, we shall consider, for the trivial character and for the case where χ is even, the even indexed Eisenstein series,

$$G_{2k}(\tau) = \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^{2k}}, \tag{2.106}$$

$$G_{2k}(\tau,\chi) = \sum_{(m,n)\neq(0,0)} \frac{\chi(m)}{(m+n\tau)^{2k}}. \tag{2.107}$$

By other hand, when χ is odd and $k\geq 2$, we also consider the odd indexed Eisenstein series,

$$G_{2k-1}(\tau,\chi) = \sum_{(m,n)\neq(0,0)} \frac{\chi(m)}{(m+n\tau)^{2k-1}}. \tag{2.108}$$

To study other representations of (2.105), (2.106) and (2.107), we apply Poisson's summation formulas (2.100), (2.101) and (2.102). If $\mathrm{Im}(\tau)>0$ and $k\geq 2$, it is not hard to show (see [94], Exercise 7 Chapter 4), via the theory of Residues that the complex Fourier transform of the family of functions

$$f_k(x) = \frac{1}{(x+\tau)^k},$$

is given by

$$\hat{f}_k\,(\xi) = \begin{cases} \frac{(-i)^k\,2\pi}{(k-1)!}\,\xi^{k-1}e^{i\tau\xi} & \xi>0 \\ 0 & \xi\leq 0. \end{cases} \tag{2.109}$$

Hence, an immediate application of (2.100 - 2.102), yields

$$\sum_{m\in\mathbb{Z}} \frac{\chi(m)}{(m+n\tau)^{2k}} = \frac{(-1)^k(2\pi)^{2k}G(\chi)}{\ell^{2k}\,(2k-1)!} \sum_{m=1}^{\infty} \overline{\chi}(m)\,m^{2k-1}\,e^{2\pi i\,\frac{m\,n}{\ell}\tau}, \quad \chi \text{ even}, \tag{2.110}$$

$$\sum_{m\in\mathbb{Z}} \frac{\chi(m)}{(m+n\tau)^{2k-1}} = i\,\frac{(-1)^{k-1}\,(2\pi)^{2k-1}\,G(\chi)}{\ell^{2k-1}\,(2k-2)!} \sum_{m=1}^{\infty} \overline{\chi}(m)\,m^{2k-2}e^{2\pi i\,\frac{m\,n}{\ell}\tau}, \quad \chi \text{ odd}, \tag{2.111}$$

$$\sum_{m\in\mathbb{Z}} \frac{1}{(m+n\tau)^{2k}} = \frac{(-1)^k\,(2\pi)^{2k}}{(2k-1)!} \sum_{m=1}^{\infty} m^{2k-1} e^{2\pi i\, mn\tau}. \tag{2.112}$$

We can now use the previous formulas to describe (2.106) and (2.107). Summing (2.110) and (2.112) over the index n, we obtain immediately

$$\begin{aligned} G_{2k}(\tau,\chi) &= 2\,L\,(2k,\chi) + \frac{(-1)^k 2\,(2\pi)^{2k}\,G(\chi)}{(2k-1)!\,\ell^{2k}} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \overline{\chi}(m)\, m^{2k-1}\, e^{2\pi i \frac{mn}{\ell}\tau} \\ &= 2\,L\,(2k,\chi) + \frac{(-1)^k 2\,(2\pi)^{2k}\,G(\chi)}{(2k-1)!\,\ell^{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1,\overline{\chi}}(n)\, e^{2\pi i \frac{n}{\ell}\tau}, \quad \chi \text{ even}, \end{aligned} \tag{2.113}$$

$$G_{2k}(\tau) = 2\,\zeta(2k) + \frac{(-1)^k 2(2\pi)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)\, e^{2\pi i n\tau}, \tag{2.114}$$

where $\sigma_{a,\chi}(n)$ is the character analogue of the generalized divisor function, which appeared in (2.91).

Furthermore, if χ is odd and primitive, we also derive the representation

$$G_{2k-1}(\tau,\chi) = 2\,L\,(2k-1,\chi) + i\,\frac{(-1)^{k-1} 2\;(2\pi)^{2k-1}\,G(\chi)}{(2k-2)!\,\ell^{2k-1}} \sum_{n=1}^{\infty} \sigma_{2k-2,\overline{\chi}}(n)\, e^{2\pi i \frac{n}{\ell}\tau}. \tag{2.115}$$

We now use the identities (2.113 - 2.115) to obtain a new proof of an identity appearing in [78]. Using (2.114) and appealing to the second reflection formula in (2.104) we get

$$\begin{aligned} &2\,\zeta(2k) + \frac{(-1)^k 2(2\pi)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)\, e^{2\pi i n\tau} \\ &= 2\tau^{-2k}\,\zeta(2k) + \frac{(-1)^k 2(2\pi)^{2k}\tau^{-2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)\, e^{-2\pi i n/\tau} \end{aligned}$$

from which we obtain, after using Euler's identity for $\zeta(2k)$ (1.104)

$$4\sum_{n=1}^{\infty} \sigma_{2k-1}(n)\, e^{2\pi i n\tau} - \tau^{-2k} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)\, e^{-2\pi i n/\tau} = \frac{B_{2k}}{4k} - \frac{B_{2k}}{4k}\,\tau^{-2k}. \tag{2.116}$$

Finally, we end this example by remarking that we can still derive character analogues of (2.116). The main difference with the characters is that we cannot apply directly a reflection formula of the form (2.104): however, we can argue by interchanging the order of summation in (2.107) and (2.108), which is always possible due to the fact that $k \geq 2$.

For instance, assume that χ is even (the odd case is completely analogous) and sum firstly over the index n in (2.107). Applying (2.102), we obtain

$$\sum_{(m,n)\neq(0,0)} \frac{\chi(m)}{(m+n\tau)^{2k}} = 2\tau^{-2k} \sum_{m=1}^{\infty} \chi(m) \sum_{n\in\mathbb{Z}} \frac{1}{(n+m\tau^{-1})^{2k}}$$
$$= \frac{(-1)^k 2\,(2\pi)^{2k}}{(2k-1)!} \tau^{-2k} \sum_{m=1}^{\infty} \chi(m) \sum_{n=1}^{\infty} n^{2k-1} e^{-2\pi i\,\frac{n\,m}{\tau}}$$
$$= \frac{(-1)^k 2\,(2\pi)^{2k}}{(2k-1)!} \tau^{-2k} \sum_{n=1}^{\infty} \sigma_{1-2k,\,\chi}(n)\, n^{2k-1} e^{-2\pi i \frac{n}{\tau}}. \qquad (2.117)$$

Comparing (2.117) with (2.113) and using the generalization of Euler's identity (1.149),

$$L(2n,\chi) = \frac{(-1)^{n-1}\,(2\pi)^{2n}\,G(\chi)}{2\ell^{2n}\,(2n)!}\,B_{2n,\overline{\chi}},$$

we derive the extension of (2.116)

$$-\frac{G(\chi)}{4k\,\ell^{2k}}\,B_{2k,\overline{\chi}} + \frac{G(\chi)}{\ell^{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1,\overline{\chi}}(n)\, e^{2\pi i \frac{n}{\ell}\tau}$$
$$= \tau^{-2k} \sum_{n=1}^{\infty} \sigma_{1-2k,\,\chi}(n)\, n^{2k-1} e^{-2\pi i n/\tau},\ \chi \text{ even}. \qquad (2.118)$$

Using the same ideas for the case in which χ is odd, we can also derive

$$\frac{G(\chi)}{\ell^{2k-1}\,(4k-2)}\,B_{2k-1,\overline{\chi}} + \frac{G(\chi)}{\ell^{2k-1}} \sum_{n=1}^{\infty} \sigma_{2k-2,\overline{\chi}}(n)\, e^{2\pi i \frac{n}{\ell}\tau}$$
$$= \tau^{1-2k} \sum_{n=1}^{\infty} \sigma_{2-2k,\,\chi}(n)\, n^{2k-1} e^{-2\pi i n/\tau},\ \chi \text{ odd}. \qquad (2.119)$$

In the fourth chapter, we will extend (2.116) and (2.118) to the case where $k=1$ by using a character version of a formula obtained by A. Selberg and S. Chowla in [90].

Example 2.3.: A Character extension of a formula attributed to Phillips

In this example, we extend to characters a well-known formula firstly proved by Phillips [105]. For $\text{Re}(\nu) > -\frac{1}{2}$, let $f(x)$ be defined by

$$f(x) = \begin{cases} \left(a^2 - x^2\right)^{\nu-1/2} & 0 < x < a \\ 0 & x \geq a. \end{cases} \qquad (2.120)$$

Recall that, for $\text{Re}(\nu) > -1/2$, the Bessel function of the first kind is given by the power series [44]

$$J_{\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{-\nu-2n}}{n!\,\Gamma(n+\nu+1)}\, x^{\nu+2n}, \qquad (2.121)$$

and the Struve function of the first kind can be written as well as [44]

$$H_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \, 2^{-2n-\nu-1}}{\Gamma\left(n+\frac{3}{2}\right) \, \Gamma\left(n+\nu+\frac{3}{2}\right)} \, x^{2n+\nu+1}. \tag{2.122}$$

To find Fourier cosine transform of $f(x)$ we can use the power series expansion for the cosine function and obtain

$$\begin{aligned}
\int_0^a (a^2-y^2)^{\nu-\frac{1}{2}} \cos(xy)\, dy &= \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n}}{(2n)!} \int_0^a (a^2-y^2)^{\nu-\frac{1}{2}} \, y^{2n} \, dy \\
&= \frac{a^{2\nu}}{2} \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{(2n)!} \, \frac{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\nu+n+1\right)} x^{2n} \\
&= \frac{a^\nu}{2}\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right) \sum_{n=0}^{\infty} \frac{(-1)^n (ax)^{\nu+2n}}{2^{2n} n! \, \Gamma\left(\nu+n+1\right)} \\
&= 2^{\nu-1}\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right) a^\nu \, x^{-\nu} J_\nu(ax),
\end{aligned} \tag{2.123}$$

where in the second and fourth equalities it was invoked, respectively, the power series of the Bessel function of the first kind (2.121) and the definition of Euler's Beta function.

For the sine transform, we can invoke a similar proceeding and get

$$\begin{aligned}
\int_0^a (a^2-y^2)^{\nu-\frac{1}{2}} \sin(xy)\, dy &= \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+1}}{(2n+1)!} \int_0^a (a^2-y^2)^{\nu-\frac{1}{2}} y^{2n+1} dy \\
&= \frac{a^{2\nu}}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \, \frac{\Gamma\left(\nu+\frac{1}{2}\right) n!}{\Gamma\left(\nu+n+\frac{3}{2}\right)} \, (xa)^{2n+1} \\
&= \frac{a^\nu}{2}\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right) \sum_{n=0}^{\infty} \frac{(-1)^n \, n!}{(2n+1)! \, \Gamma\left(\nu+n+\frac{3}{2}\right)} \, (ax)^{2n+1+\nu} \\
&= 2^{\nu-1}\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right) a^\nu \, x^{-\nu} \, H_\nu\left(ax\right).
\end{aligned} \tag{2.124}$$

From (2.123) and (2.124) we obtain the interesting identities

$$\sum_{n\le a} \chi(n)\,(a^2-n^2)^{\nu-\frac{1}{2}} = G(\chi)\ell^{\nu-1}\pi^{\frac{1}{2}-\nu} a^\nu \Gamma\left(\nu+\frac{1}{2}\right) \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{n^\nu} \, J_\nu\left(\frac{2\pi n a}{\ell}\right), \quad \chi \, \text{even}, \tag{2.125}$$

$$\sum_{n\le a} \chi(n)\,(a^2-n^2)^{\nu-\frac{1}{2}} = -iG(\chi)\ell^{\nu-1}\pi^{\frac{1}{2}-\nu} a^\nu \, \Gamma\left(\nu+\frac{1}{2}\right) \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{n^\nu} H_\nu\left(\frac{2\pi n a}{\ell}\right), \quad \chi \, \text{odd}, \tag{2.126}$$

$$\frac{a^{2\nu-1}}{2} + \sum_{n\leq a} \left(a^2 - n^2\right)^{\nu-\frac{1}{2}} = \frac{\pi}{2} + \pi^{\frac{1}{2}-\nu}\Gamma\left(\nu + \frac{1}{2}\right) a^{\nu} \sum_{n=1}^{\infty} \frac{J_{\nu}(2\pi n a)}{n^{\nu}}, \quad \text{Re}(\nu) > -\frac{1}{2}, \quad (2.127)$$

where the last one of these was proved by E.G. Phillips for the first time and appeared in print in a paper written by Watson on self-reciprocal functions [105].

Example 2.4.:

Consider $f(x) = J_0(\alpha x)/(x^2+\beta^2)$: its cosine and sine transforms satisfy, respectively (see relations 1.12.14 and 2.12.12 in [45] Vol. 1)

$$g(x) = \begin{cases} 0 & x \leq \alpha \\ \frac{\pi}{2\beta}\, e^{-\beta x} I_0(\alpha\beta) & x > \alpha, \end{cases}$$

$$h(x) = \begin{cases} \frac{\sinh(\beta x)}{\beta}\, K_0(\alpha\beta) & x < \alpha \\ 0 & x \geq \alpha, \end{cases}$$

where I_0 and K_0 denote the modified Bessel functions of order zero [106]. An application of these transforms gives the following identities

$$\sum_{n=1}^{\infty} \frac{\chi(n)\, J_0(\alpha n)}{n^2+\beta^2} = \frac{\pi G(\chi)}{\ell\beta} I_0(\alpha\beta) \sum_{n\geq 1+[\alpha]}^{\infty} \overline{\chi}(n)\, e^{-\frac{2\pi\beta}{\ell}n}, \quad \chi \text{ even}, \quad (2.128)$$

$$\sum_{n=1}^{\infty} \frac{\chi(n)\, J_0(\alpha n)}{n^2+\beta^2} = -\frac{2iG(\chi)}{\ell\beta} K_0(\alpha\beta) \sum_{n\leq[\alpha]} \overline{\chi}(n)\, \sinh\left(\frac{2\pi\beta n}{\ell}\right), \quad \chi \text{ odd}, \quad (2.129)$$

$$\frac{1}{2\beta^2} + \sum_{n=1}^{\infty} \frac{J_0(\alpha n)}{n^2+\beta^2} = \frac{\pi}{\beta} I_0(\alpha\beta)\, \frac{e^{-2\pi\beta([\alpha]+1)}}{1-e^{-2\pi\beta}}, \quad (2.130)$$

where $[\alpha]$ denotes the integer part of α (see our glossary).

Example 2.5: An example of Watson-type

In this example we extend a formula proved by Watson [105] to a general polynomial with degree 2.

Let ν be a complex number satisfying $\text{Re}(\nu) > \frac{1}{2}$. Consider also the function

$$f_{\nu}(x) = \frac{1}{Q(x)^{\nu}} = \frac{1}{(ax^2+bx+c)^{\nu}},$$

where the denominator is a second degree polynomial which is always positive, i.e., with its coefficients satisfying $-d = 4ab - c^2 > 0$, $a > 0$. Instead of finding the Fourier cosine transform, we consider easier to compute directly the complex exponential and apply Poisson summation formula in the form given in the second example (eq. (2.100) to (2.102)).

In the following calculations, we will find the respective Fourier transform, i.e., for $\xi \in \mathbb{R}$

$$\hat{f}_{\nu}(\xi) = \int_{-\infty}^{\infty} f_{\nu}(x)\, e^{-i\xi x}\, dx = \frac{1}{\Gamma(\nu)} \int_{-\infty}^{\infty} e^{-i\xi x} \int_{0}^{\infty} y^{\nu-1} e^{-Q(x)\, y} dy\, dx, \tag{2.131}$$

since $Q(x)$ is positive-definite by hypothesis. Note that

$$\int_{-\infty}^{\infty} \int_{0}^{\infty} \left| y^{\nu-1} e^{-i\xi x} e^{-Q(x)\, y} \right| dy\, dx = \Gamma(\text{Re}(\nu)) \int_{-\infty}^{\infty} \frac{dx}{Q(x)^{\sigma}} < \infty,$$

since we are under the hypothesis $\text{Re}(\nu) > \frac{1}{2}$. Therefore, the orders of integration in (2.131) may be reversed, giving

$$\begin{aligned} \hat{f}_{\nu}(\xi) &= \frac{1}{\Gamma(\nu)} \int_{0}^{\infty} y^{\nu-1} \int_{-\infty}^{\infty} e^{-i\xi x} e^{-Q(x)\, y} dx\, dy \\ &= \frac{1}{\Gamma(\nu)} \sqrt{\frac{\pi}{a}}\, e^{i\, \frac{b}{2a} \xi} \int_{0}^{\infty} y^{\nu-\frac{3}{2}}\, e^{-\frac{\xi^2}{4ay}}\, e^{-\frac{|d|}{4a}\, y}\, dy. \end{aligned} \tag{2.132}$$

Now (2.132) can be evaluated if we take two separate cases: $\xi = 0$ or $\xi \neq 0$. For the first, it is simple to see that

$$\hat{f}_{\nu}(0) = \frac{1}{\Gamma(\nu)} \sqrt{\frac{\pi}{a}} \int_{0}^{\infty} y^{\nu-\frac{3}{2}} e^{-\frac{|d|}{4a}\, y} dy = \sqrt{\pi}\, \frac{\Gamma\left(\nu - 1/2\right)}{\Gamma(\nu)} a^{-\nu} k^{1-2\nu}, \tag{2.133}$$

where $k = \sqrt{|d|}/2a$ with d being the discriminant of $Q(x)$. We now compute (2.133) for $\xi \neq 0$: there are several ways to do this (see, for example, 186/187 of Watson's treatise [106]), but our method to prove it is by invoking the famous Cahen-Mellin integral representation for the exponential

$$e^{-\frac{\xi^2}{4ay}} = \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} \Gamma(z) \left(\frac{\xi^2}{4ay} \right)^{-z} dz, \tag{2.134}$$

where $\mu > 0$. Let us substitute (2.134) inside the integral (2.132) and change the order integration (this is once more justified due to absolute convergence): we obtain

$$\hat{f}_\nu(\xi) = \frac{1}{\Gamma(\nu)}\sqrt{\frac{\pi}{a}}\, e^{i\,\frac{b}{2a}\xi}\, \frac{1}{2\pi i}\int\limits_{\mu-i\infty}^{\mu+i\infty} \Gamma(z)\, \left(\frac{\xi^2}{4a}\right)^{-z} \int\limits_{0}^{\infty} y^{\nu+z-\frac{3}{2}}\, e^{-\frac{|d|}{4a}\,y}\, dy\, dz$$

$$= \frac{1}{\Gamma(\nu)}\sqrt{\frac{\pi}{a}}\, e^{i\,\frac{b}{2a}\xi}\, \left(k^2 a\right)^{\frac{1}{2}-\nu}\, \frac{1}{2\pi i}\int\limits_{\mu-i\infty}^{\mu+i\infty} \Gamma(z)\,\Gamma(\nu+z-\frac{1}{2})\, \left(\frac{\xi^2 k^2}{4}\right)^{-z} dz.$$

Taking the change of variable $z = \frac{w+1/2-\nu}{2}$ and using the Mellin representation for the Modified Bessel function [106],

$$K_\nu(x) = \frac{1}{2\pi i}\int\limits_{\mu-i\infty}^{\mu+i\infty} 2^{s-2}\,\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right)\, x^{-s} ds, \quad x>0, \quad \mu > |\mathrm{Re}(\nu)|,$$

we immediately arrive to

$$\hat{f}_\nu(\xi) = \frac{2}{\Gamma(\nu)}\sqrt{\frac{\pi}{a}}\, e^{i\,\frac{b}{2a}\xi}\, \left(\frac{2ka}{|\xi|}\right)^{\frac{1}{2}-\nu} K_{\nu-\frac{1}{2}}\left(k\,|\xi|\right), \quad \xi \neq 0. \tag{2.135}$$

Applying the summation formulas (2.100-2.102) for the transform (2.135), we obtain

$$\sum_{n\in\mathbb{Z}} \frac{\chi(n)}{\left(an^2+bn+c\right)^\nu} = \frac{4G(\chi)}{\ell\Gamma(\nu)}\sqrt{\frac{\pi}{a}}\left(\frac{ka\ell}{\pi}\right)^{\frac{1}{2}-\nu}\sum_{n=1}^{\infty}\overline{\chi}(n)\,\cos\left(\pi\frac{bn}{a\ell}\right)\, n^{\nu-\frac{1}{2}}\, K_{\nu-\frac{1}{2}}\left(\frac{2\pi k\, n}{\ell}\right), \tag{2.136}$$

$$\sum_{n\in\mathbb{Z}} \frac{\chi(n)}{\left(an^2+bn+c\right)^\nu} = -\frac{4iG(\chi)}{\ell\Gamma(\nu)}\sqrt{\frac{\pi}{a}}\left(\frac{ka\ell}{\pi}\right)^{\frac{1}{2}-\nu}\sum_{n=1}^{\infty}\overline{\chi}(n)\,\sin\left(\pi\frac{bn}{a\ell}\right)\, n^{\nu-\frac{1}{2}}\, K_{\nu-\frac{1}{2}}\left(\frac{2\pi k\, n}{\ell}\right), \tag{2.137}$$

$$\sum_{n\in\mathbb{Z}} \frac{1}{\left(an^2+bn+c\right)^\nu} = \sqrt{\pi}\,\frac{\Gamma\left(\nu-1/2\right)}{\Gamma(\nu)} a^{-\nu} k^{1-2\nu}$$
$$+\frac{4}{\Gamma(\nu)}\sqrt{\frac{\pi}{a}}\left(\frac{ka}{\pi}\right)^{\frac{1}{2}-\nu}\sum_{n=1}^{\infty}\cos\left(\pi\frac{bn}{a}\right)\, n^{\nu-\frac{1}{2}}\, K_{\nu-\frac{1}{2}}\left(2\pi k\, n\right). \tag{2.138}$$

Formulas (2.136) and (2.138) will play an important role in proving the meromorphic extension of the Epstein $\zeta-$function, which shall be deduced on the fourth chapter.

Related with the meromorphic continuation of Epstein's $\zeta-$function, we can still extend (2.136-2.138) to the case where $\nu = \frac{1}{2}$. We can achieve this extension if we use Poisson's summation in the L_2-form (as in Example 2.1.), but we prefer to adapt Watson's method [105] and invoke some principles of analytic continuation.

Since the left-hand sides of (2.136) and (2.138) converge conditionally, this extension is simple to obtain in these cases, as the right-hand side is also well-defined and the associated series converges absolutely. This gives

$$\sum_{n\in\mathbb{Z}} \frac{\chi(n)}{\sqrt{an^2+bn+c}} = \frac{4G(\chi)}{\ell\sqrt{a}} \sum_{n=1}^{\infty} \overline{\chi}(n) \cos\left(\pi \frac{bn}{a\ell}\right) K_0\left(\frac{2\pi k\, n}{\ell}\right), \quad \chi \text{ even}, \tag{2.139}$$

$$\sum_{n\in\mathbb{Z}} \frac{\chi(n)}{\sqrt{an^2+bn+c}} = -\frac{4iG(\chi)}{\ell\sqrt{a}} \sum_{n=1}^{\infty} \overline{\chi}(n) \sin\left(\pi \frac{bn}{a\ell}\right) K_0\left(\frac{2\pi k\, n}{\ell}\right), \quad \chi \text{ odd}. \tag{2.140}$$

Dealing with (2.138) is more delicate, since the series at the left-hand side of (2.138) does not converge for $\nu = \frac{1}{2}$ and the right-hand side has a simple pole at $\nu = \frac{1}{2}$ coming from the $\Gamma-$factor $\Gamma\left(\nu - \frac{1}{2}\right)$.

However, we can study the meromorphic continuation of the left-hand side of (2.138) in the following way: let us assume, for simplicity, that $0 < b < 2a$ and consider $N \in \mathbb{N}$.

Since $k^2 = |d|/4a^2$, it is clear, from the definition of the Hurwitz $\zeta-$function, that we can extend the left-hand side of (2.138) in the following way

$$\begin{aligned} & a^{-\nu} \sum_{n\in\mathbb{Z}} \left\{ \frac{1}{\left((n+b/2a)^2+k^2\right)^{\nu}} - \sum_{m=0}^{N-1} \binom{-\nu}{m} \frac{k^{2m}}{(n+b/2a)^{2\nu+2m}} \right\} \\ & + a^{-\nu} \sum_{m=0}^{N-1} \binom{-\nu}{m} k^{2m} \left\{ \zeta\left(2\nu+2m,\, \frac{b}{2a}\right) + \zeta\left(2\nu+2m,\, 1-\frac{b}{2a}\right) \right\}, \end{aligned} \tag{2.141}$$

i.e., for $\text{Re}(\nu) > \frac{1}{2}$ $(N = 0)$ the left-hand side of (2.138) and (2.141) coincide.

Let M be a positive integer such that, for $|n| > M$, the inequality

$$\frac{k}{|n+b/2a|} < 1$$

is satisfied. From the generalized Binomial theorem, we can write, for $|n| > M$, the first summand in (2.141) as the power series

$$\frac{1}{\left((n+b/2a)^2+k^2\right)^{\nu}} = \sum_{m=0}^{\infty} \binom{-\nu}{m} \frac{k^{2m}}{(n+b/2a)^{2\nu+2m}}, \tag{2.142}$$

and we can easily see that (2.141) may be expressed as

$$a^{-\nu}\sum_{|n|\le M}\frac{1}{\left((n+b/2a)^2+k^2\right)^{\nu}}+a^{-\nu}\sum_{|n|>M}\left\{\sum_{m=N}^{\infty}\left(\begin{array}{c}-\nu\\ m\end{array}\right)\frac{k^{2m}}{(n+b/2a)^{2\nu+2m}}\right\}$$
$$+a^{-\nu}\sum_{m=0}^{N-1}\left(\begin{array}{c}-\nu\\ m\end{array}\right)k^{2m}\left\{\zeta\left(2\nu+2m,\,\frac{b}{2a}\right)+\zeta\left(2\nu+2m,\,1-\frac{b}{2a}\right)\right\}. \tag{2.143}$$

Note that the general term of the infinite series $\sum_{|n|>M}$ is of the form $O\left(\frac{1}{n^{2\nu+2N}}\right)$ for large n.

This means that this series converges absolutely if $\mathrm{Re}(\nu)>-N+\frac{1}{2}$ and (2.141) defines a complex function of ν extended to the half-plane $\mathrm{Re}(\nu)>-N+\frac{1}{2}$ and which is analytic everywhere except at the poles of Hurwitz $\zeta-$function present in the third term of (2.143).

By analytic continuation, for $\mathrm{Re}(\nu)>-N+\frac{1}{2}$, the following equality holds

$$a^{-\nu}\sum_{n\in\mathbb{Z}}\left\{\frac{1}{\left((n+b/2a)^2+k^2\right)^{\nu}}-\sum_{m=0}^{N-1}\left(\begin{array}{c}-\nu\\ m\end{array}\right)\frac{k^{2m}}{(n+b/2a)^{2\nu+2m}}\right\}$$
$$+a^{-\nu}\sum_{m=0}^{N-1}\left(\begin{array}{c}-\nu\\ m\end{array}\right)k^{2m}\left\{\zeta\left(2\nu+2m,\,\frac{b}{2a}\right)+\zeta\left(2\nu+2m,\,1-\frac{b}{2a}\right)\right\}$$
$$=\sqrt{\pi}\,\frac{\Gamma(\nu-1/2)}{\Gamma(\nu)}a^{-\nu}k^{1-2\nu}+\frac{4}{\Gamma(\nu)}\sqrt{\frac{\pi}{a}}\left(\frac{ka}{\pi}\right)^{\frac{1}{2}-\nu}\sum_{n=1}^{\infty}\cos\left(\pi\frac{bn}{a}\right)n^{\nu-\frac{1}{2}}K_{\nu-\frac{1}{2}}\left(2\pi k\,n\right). \tag{2.144}$$

From (2.144), we are finally ready to see what happens at the limiting case $\nu\to\frac{1}{2}$: take $N=1$ above and let $\nu\to\frac{1}{2}$. We obtain

$$4a^{-\frac{1}{2}}\sum_{n=1}^{\infty}\cos\left(\pi\frac{bn}{a}\right)K_0\left(2\pi k\,n\right)=a^{-\frac{1}{2}}\sum_{n\in\mathbb{Z}}\left\{\frac{1}{\sqrt{(n+b/2a)^2+k^2}}-\frac{1}{|n+b/2a|}\right\}$$
$$+\lim_{\nu\to\frac{1}{2}}\left\{a^{-\nu}\zeta\left(2\nu,\,\frac{b}{2a}\right)+a^{-\nu}\zeta\left(2\nu,\,1-\frac{b}{2a}\right)-\sqrt{\pi}a^{-\nu}k^{1-2\nu}\,\frac{\Gamma(\nu-1/2)}{\Gamma(\nu)}\right\}. \tag{2.145}$$

From Hermite's representation of the Hurwitz $\zeta-$function (see Theorem 1.6. given at the first chapter), we have deduced that its meromorphic expansion around the simple $s=1$ is given by

$$\zeta(s,\,a)=\frac{1}{s-1}-\psi(a)+O\left(s-1\right). \tag{2.146}$$

Hence, applying this formula to (2.146) and invoking the Laurent series

$$\sqrt{\pi}a^{-\nu}k^{1-2\nu}\,\frac{\Gamma\left(\nu-1/2\right)}{\Gamma(\nu)}=a^{-1/2}\left[\frac{1}{\nu-\frac{1}{2}}+\log\left(\frac{4}{ak^2}\right)+O\left(\nu-\frac{1}{2}\right)\right],$$

we obtain

$$\lim_{\nu\to\frac{1}{2}}\left\{a^{-\nu}\zeta\left(2\nu,\,\frac{b}{2a}\right)+a^{-\nu}\zeta\left(2\nu,\,1-\frac{b}{2a}\right)-\sqrt{\pi}a^{-\nu}k^{1-2\nu}\,\frac{\Gamma\left(\nu-1/2\right)}{\Gamma(\nu)}\right\}$$
$$=a^{-1/2}\,\log\left(\frac{k^2}{4}\right)-a^{-1/2}\left[\psi\left(\frac{b}{2a}\right)+\psi\left(1-\frac{b}{2a}\right)\right]=a^{-1/2}\,\log\left(\frac{k^2}{4}\right)-a^{-1/2}\left[2\psi\left(\frac{b}{2a}\right)+\pi\,\cot\left(\frac{\pi b}{2a}\right)\right], \tag{2.147}$$

where the last equality came from the reflection formula for the digamma function (1.44)

$$\psi\left(1-x\right)-\psi\left(x\right)=\pi\cot(\pi x).$$

Joining (2.147) with (2.145), we are able to obtain a generalization of Watson's formula

$$4\sum_{n=1}^{\infty}\cos\left(\pi\frac{bn}{a}\right)\,K_0\left(2\pi k\,n\right)=\sum_{n\in\mathbb{Z}}\left\{\frac{1}{\sqrt{\left(n+b/2a\right)^2+k^2}}-\frac{1}{|n+b/2a|}\right\}$$
$$+\log\left(\frac{k^2}{4}\right)-2\psi\left(\frac{b}{2a}\right)-\pi\cot\left(\frac{\pi b}{2a}\right). \tag{2.148}$$

Watson's formula can be obtained if we put $b=0$ in (2.138). Under this assumption, note that the extension given in (2.141) is slightly different and can be written in the form

$$(ak^2)^{-\nu}+2a^{-\nu}\sum_{n=1}^{\infty}\left\{\frac{1}{\left(n^2+k^2\right)^{\nu}}-\sum_{m=0}^{N-1}\begin{pmatrix}-\nu\\ m\end{pmatrix}\frac{k^{2m}}{n^{2\nu+2m}}\right\}+2a^{-\nu}\sum_{m=0}^{N-1}\begin{pmatrix}-\nu\\ m\end{pmatrix}k^{2m}\zeta\left(2\nu+2m\right) \tag{2.149}$$

where, instead of the Hurwitz $\zeta-$function we have the particular case of Riemann's $\zeta-$function. Proceeding as before, we take the limit $\nu\to\frac{1}{2}$ in (2.149) for $b=0$ and from the meromorphic expansions of $\zeta(2\nu)$ and $\sqrt{\pi}a^{-\nu}k^{1-2\nu}\Gamma(\nu-1/2)/\Gamma(\nu)$ around $\nu=\frac{1}{2}$, we derive

$$4\sum_{n=1}^{\infty}K_0\left(2\pi k\,n\right)=\frac{1}{k}+2\sum_{n=1}^{\infty}\left\{\frac{1}{\sqrt{n^2+k^2}}-\frac{1}{n}\right\}+2\gamma+2\log\left(k/2\right). \tag{2.150}$$

It is also interesting to note that, for the case where $a,\,b\in\mathbb{N}$, we can write (2.148) in an elegant way, by invoking Gauss's digamma theorem (see relation (1.47) in the first chapter). Since $b<2a$ by hypothesis, after a straightforward application of this theorem we obtain

$$\psi\left(\frac{b}{2a}\right)=-\gamma-\log(4a)-\frac{1}{2}\pi\cot\left(\frac{b}{2a}\pi\right)+2\sum_{m=1}^{a-1}\cos\left(\frac{\pi b}{a}\,m\right)\log\sin\left(\frac{\pi m}{2a}\right) \tag{2.151}$$

so that (2.148) can be written in the compact form

$$4\sum_{n=1}^{\infty}\cos\left(\pi\frac{bn}{a}\right)\,K_0\left(2\pi k\,n\right)=\sum_{n\in\mathbb{Z}}\left\{\frac{1}{\sqrt{(n+b/2a)^2+k^2}}-\frac{1}{|n+b/2a|}\right\}$$
$$+\log\left(|d|\right)+2\gamma-4\sum_{m=1}^{a-1}\cos\left(\frac{\pi b}{a}\,m\right)\log\sin\left(\frac{\pi m}{2a}\right). \qquad (2.152)$$

Finally, although we were assuming that $0<b<2a$, we can find another expression for (2.152) without imposing the condition $b<2a$. If we further assume that a, $b\in\mathbb{N}$, then exists there exist $n\in\mathbb{N}_0$ and $0\leq r\leq 2a-1$ such that

$$\psi\left(\frac{b}{2a}\right)=\psi\left(\frac{2an+r}{2a}\right)=\psi\left(\frac{r}{2a}+n\right)=-\gamma-\log(4a)-\frac{\pi}{2}\cot\left(\frac{\pi b}{2a}\right)$$
$$+\,2a\sum_{j=1}^{n}\frac{1}{b-2aj}+\,2\sum_{m=1}^{a-1}\cos\left(\frac{\pi b}{a}m\right)\,\log\sin\left(\frac{\pi m}{2a}\right) \qquad (2.153)$$

where $n=\left[\frac{b}{2a}\right]$. This extends (2.152) in the following form

$$4\sum_{n=1}^{\infty}\cos\left(\pi\frac{bn}{a}\right)\,K_0\left(2\pi k\,n\right)=\sum_{n\in\mathbb{Z}}\left\{\frac{1}{\sqrt{(n+b/2a)^2+k^2}}-\frac{1}{|n+b/2a|}\right\}$$
$$+\log\left(|d|\right)+2\gamma-4a\sum_{j=1}^{\left[\frac{b}{2a}\right]}\frac{1}{b-2aj}\;-4\sum_{m=1}^{a-1}\cos\left(\frac{\pi b}{a}m\right)\,\log\sin\left(\frac{\pi m}{2a}\right). \qquad (2.154)$$

Example 2.6.: A Character analogue Hurwitz $\zeta-$function

In this example, we briefly argue a proof of the functional equation for an analogue of Hurwitz's $\zeta-$function

$$L(s,\chi,a)=\sum_{n=1}^{\infty}\frac{\chi(n)}{(n+a)^s},\quad \mathrm{Re}(s)>1,\;-1<a<1. \qquad (2.155)$$

To do so, let us consider $f(x)=e^{-\alpha x^2}\cos(\beta x)$ for even χ and $f(x)=e^{-\alpha x^2}\sin(\beta x)$ for odd χ. Then their cosine and sine transforms are given respectively by

$$g(x)=\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\,e^{-\frac{x^2+\beta^2}{4\alpha}}\,\cosh\left(\frac{\beta x}{2\alpha}\right),$$

$$h(x)=\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\,e^{-\frac{x^2+\beta^2}{4\alpha}}\,\sinh\left(\frac{\beta x}{2\alpha}\right).$$

Therefore, using the main summation formulas of this chapter (2.27) and (2.28),

$$\sum_{n=1}^{\infty} \chi(n)\, e^{-\alpha n^2} \cos(\beta n) = \frac{G(\chi)}{\ell} \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{\beta^2}{4\alpha}} \sum_{n=1}^{\infty} \overline{\chi}(n)\; e^{-\frac{\pi^2 n^2}{\alpha \ell^2}} \cosh\left(\frac{\pi n \beta}{\alpha \ell}\right)$$
$$= \frac{G(\chi)}{2\ell} \sqrt{\frac{\pi}{\alpha}} \sum_{n=-\infty}^{\infty} \overline{\chi}(n)\, e^{-\frac{1}{\alpha}\left(\frac{\pi n}{\ell}+\frac{\beta}{2}\right)^2}, \quad \chi \text{ even}, \tag{2.156}$$

$$\sum_{n=1}^{\infty} \chi(n)\, e^{-\alpha n^2} \sin(\beta n) = -\frac{iG(\chi)}{\ell} \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{\beta^2}{4\alpha}} \sum_{n=1}^{\infty} \overline{\chi}(n)\, e^{-\frac{\pi^2 n^2}{\alpha \ell^2}} \sinh\left(\frac{\pi n \beta}{\alpha \ell}\right)$$
$$= -\frac{iG(\chi)}{2\ell} \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{\beta^2}{4\alpha}} \sum_{n=1}^{\infty} \overline{\chi}(n)\, e^{-\frac{1}{\alpha}\left(\frac{\pi n}{\ell}+\frac{\beta}{2}\right)^2}, \quad \chi \text{ odd}. \tag{2.157}$$

By appealing to these identities, we can adapt the proof of the functional equation for $\zeta(s,a)$ given at the first chapter. Invoking this reasoning (the reader can consult the complete argument in the supplementary document [89]), we derive the functional equations for $L(s,\chi,a)$,

$$L\left(1-s,\, \overline{\chi},\, \frac{\beta \ell}{2\pi}\right) = \frac{2\Gamma(s)}{G(\chi)} \left(\frac{2\pi}{\ell}\right)^{-s} \left[\cos\left(\frac{\pi s}{2}\right) \sum_{n=1}^{\infty} \frac{\chi(n)\, \cos(\beta n)}{n^s} + \sin\left(\frac{\pi s}{2}\right) \sum_{n=1}^{\infty} \frac{\chi(n)\, \sin(\beta n)}{n^s}\right], \tag{2.158}$$

for even χ and

$$L\left(1-s, \overline{\chi}, \frac{\beta \ell}{2\pi}\right) = -\frac{2i\,\Gamma(s)}{G(\chi)} \left(\frac{2\pi}{\ell}\right)^{-s} \left[\cos\left(\frac{\pi s}{2}\right) \sum_{n=1}^{\infty} \frac{\chi(n)\, \sin(\beta n)}{n^s} - \sin\left(\frac{\pi s}{2}\right) \sum_{n=1}^{\infty} \frac{\chi(n)\, \cos(\beta n)}{n^s}\right], \tag{2.159}$$

for odd χ.

Equations (2.158) and (2.159) extend the functional equation for $\zeta(s,a)$ for Dirichlet characters. Furthermore taking $\beta = 0$ in both and using the simple property $G(\chi)\, G(\overline{\chi}) = \chi(-1)\, \ell$, we recover, respectively, the functional equation for the Dirichlet $L-$function for even and odd characters.

Example 2.7.: The Kontorovich-Lebedev Transform

In this final example, we consider one of the standard representations of the modified Bessel function of the second kind [106]

$$K_{\nu}(x) = \int_0^{\infty} e^{-x \cosh(u)} \cosh(\nu u)\, du, \qquad x > 0. \tag{2.160}$$

The usual way to get (2.160) is by using Basset's integral representation,

$$K_\nu(x) = \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}} \left(\frac{2}{x}\right)^\nu \int_0^\infty \frac{\cos(xy)}{(1+y^2)^{\nu+\frac{1}{2}}} dy, \quad \mathrm{Re}(\nu) > -\frac{1}{2}, \ x > 0,$$

which, in its turn, is easier to prove if one invokes Mellin's representation of $K_\nu(x)$ [106]

$$K_\nu(x) = \frac{1}{\pi i} \int_{\mu-i\infty}^{\mu+i\infty} 2^{s-3} \Gamma\left(\frac{s+\nu}{2}\right) \Gamma\left(\frac{s-\nu}{2}\right) x^{-s} ds, \ x > 0, \ \mu > |\mathrm{Re}(\nu)|$$

and then uses Parseval's Theorem for the Mellin transform.

If we put $\nu = i\tau$ in (2.160), we arrive to the Fourier transform of the Kontorovich-Lebedev kernel

$$K_{i\tau}(x) = \int_0^\infty e^{-x\cosh(u)} \cos(\tau u)\, du \qquad x > 0. \tag{2.161}$$

For a fixed and positive x, let $f(u) = e^{-x\cosh(u)}$. Then clearly $f \in L_1(\mathbb{R}_+) \cap L_2(\mathbb{R}_+)$ and obeys to the conditions of the main theorem. Also, its Fourier cosine transform $g(\tau)$ follows the same behavior due to the inequality [115]

$$|g(\tau)| = |K_{i\tau}(x)| \leq e^{-\pi|\tau|/3} K_0\left(\frac{x}{2}\right). \tag{2.162}$$

From (2.162) it is clear that the series appearing on both sides of (2.69) and (2.71) converge and an application of these formulas allows the proof of the interesting identities

$$\sum_{n=1}^\infty \chi(n)\, e^{-x\cosh(n\alpha)} = \frac{\beta G(\chi)}{\pi \ell} \sum_{n=1}^\infty \overline{\chi}(n)\, K_{\frac{i\beta n}{\ell}}(x), \quad \chi\,\text{even}, \tag{2.163}$$

$$\frac{\alpha e^{-x}}{2} + \alpha \sum_{n=1}^\infty e^{-x\cosh(n\alpha)} = K_0(x) + 2\sum_{n=1}^\infty K_{i\beta n}(x), \tag{2.164}$$

where $\alpha > 0$ and $\beta = \frac{2\pi}{\alpha}$. We can still derive an odd version of (2.163) and (2.164). Integrating by parts in (2.161), we see that

$$K_{i\tau}(x) = \frac{x}{\tau} \int_0^\infty e^{-x\cosh(u)} \sinh(u)\, \sin(\tau u)\, du, \tag{2.165}$$

from which we deduce, after invoking (2.70)

$$\sum_{n=1}^\infty \chi(n)\, e^{-x\cosh(n\alpha)} \sinh(n\alpha) = -\frac{iG(\chi)\beta^2}{\pi \ell^2 x} \sum_{n=1}^\infty \overline{\chi}(n)\, n\, K_{\frac{i\beta n}{\ell}}(x), \quad \chi\,\text{odd}. \tag{2.166}$$

Using (2.163), (2.164) and (2.166), we can state and prove the following theorem, which extends to Dirichlet characters some results already proved in [113]. The proof of this theorem can be found in the supplementary document to this chapter [89], where we also expose other computations and applications related with the Kontorovich-Lebedev transform.

Theorem 2.7.1.: Assume that χ is a nonprincipal and primitive Dirichlet character modulo ℓ and $f(x) \in L_1(\mathbb{R}_+,\, K_\mu(\xi x))$, for some $\mu > \frac{1}{2}$ and $0 < \xi < \frac{1}{2}$. Then the following summation formulas hold

$$\sum_{n=1}^{\infty} \chi(n)\, \mathcal{L}f\,\left(\cosh(n\alpha)\right) = \frac{\beta G(\chi)}{\pi \ell} \sum_{n=1}^{\infty} \overline{\chi}(n)\, K_{\frac{i\beta n}{\ell}}[f], \quad \chi\,\text{even}, \tag{2.167}$$

$$\frac{\alpha}{2}\mathcal{L}f\,(1) + \alpha \sum_{n=1}^{\infty} \mathcal{L}f\,\left(\cosh(n\alpha)\right) = K_0[f] + 2\sum_{n=1}^{\infty} K_{i\beta n}[f], \tag{2.168}$$

where $\mathcal{L}f(x)$ is the Laplace transform

$$\mathcal{L}f(x) = \int_0^\infty f(y)\, e^{-xy} dy \tag{2.169}$$

and $K_{i\tau}[f]$ denotes the Kontorovich-Lebedev transform of f,

$$K_{i\tau}[f] = \int_0^\infty f(x)\, K_{i\tau}(x)\, dx. \tag{2.170}$$

If $f(x) \in L_1(\mathbb{R}_+,\, K_\mu(\xi x))$, for some $\mu > \frac{3}{2}$ and $0 < \xi < \frac{1}{2}$, we have the summation formula for the odd case

$$\sum_{n=1}^{\infty} \chi(n)\, \sinh(n\alpha)\, \mathcal{L}f\,\left(\cosh(n\alpha)\right) = -\frac{iG(\chi)\beta^2}{\pi \ell^2 x} \sum_{n=1}^{\infty} \overline{\chi}(n)\, n\, K_{\frac{i\beta n}{\ell}}[f], \quad \chi\,\text{odd}. \tag{2.171}$$

Example 2.7.1.:

Consider the function $f(x) = x^{\eta-1}e^{-x}$, for $\eta > \frac{1}{2}$. To find its Kontorovich-Lebedev transform (2.170) we use relations 10.2.32 in [45], vol.2 and 2.16.6.4. in [85], vol.2, giving

$$K_{i\tau}[f] = 2^{-\eta}\sqrt{\pi}\, \frac{|\Gamma\left(\eta + i\tau\right)|^2}{\Gamma\left(\frac{1}{2} + \eta\right)}. \tag{2.172}$$

Furthermore

$$\mathcal{L}f\ (\cosh(n\alpha)) = \int_0^\infty x^{\eta-1}e^{-(1+\cosh(n\alpha))x}dx = \frac{\Gamma(\eta)}{(1+\cosh(n\alpha))^\eta}. \tag{2.173}$$

Applying (2.167) and (2.168) to (2.172) and (2.173) we obtain

$$\Gamma(\eta)\sum_{n=1}^\infty \frac{\chi(n)}{(1+\cosh(n\alpha))^\eta} = \frac{\beta G(\chi)}{2^\eta\sqrt{\pi}\ell\,\Gamma\left(\frac{1}{2}+\eta\right)}\sum_{n=1}^\infty \overline{\chi}(n)\left|\Gamma\left(\eta+\frac{i\beta n}{\ell}\right)\right|^2, \quad \chi\,\text{even}, \tag{2.174}$$

$$\alpha\Gamma(\eta)\left[2^{-\eta-1}+\sum_{n=1}^\infty\frac{1}{(1+\cosh(n\alpha))^\eta}\right] = \frac{2^{-\eta}\sqrt{\pi}}{\Gamma\left(\eta+\frac{1}{2}\right)}\left[\Gamma^2(\eta)+2\sum_{n=1}^\infty\left|\Gamma\left(\eta+i\beta n\right)\right|^2\right]. \tag{2.175}$$

For the case where χ is an odd character, in order to meet the conditions of the Theorem above, we need to impose $\eta > \frac{3}{2}$. This gives

$$\Gamma(\eta)\sum_{n=1}^\infty\frac{\chi(n)\,\sinh(n\alpha)}{(1+\cosh(n\alpha))^\eta} = -\frac{i\,2^{-\eta}\beta^2 G(\chi)}{\sqrt{\pi}\ell^2\,\Gamma\left(\frac{1}{2}+\eta\right)\,x}\sum_{n=1}^\infty\overline{\chi}(n)\,n\left|\Gamma\left(\eta+\frac{i\beta n}{\ell}\right)\right|^2. \tag{2.176}$$

If we take $\eta = 1$ and use Euler's reflection formula for the $\Gamma-$function, we obtain the interesting particular cases

$$\sum_{n=1}^\infty\frac{\chi(n)}{1+\cosh(n\alpha)} = \frac{4\pi^2 G(\chi)}{\alpha^2\ell^2}\sum_{n=1}^\infty\frac{\overline{\chi}(n)\,n}{\sinh\left(\frac{2\pi^2 n}{\alpha\ell}\right)}, \quad \chi\,\text{even}, \tag{2.177}$$

$$\frac{\alpha}{4}+\alpha\sum_{n=1}^\infty\frac{1}{1+\cosh(n\alpha)} = 1+\frac{4\pi^2}{\alpha}\sum_{n=1}^\infty\frac{n}{\sinh\left(\frac{2\pi^2 n}{\alpha}\right)}. \tag{2.178}$$

If we now take χ odd and let $\eta = 2$, we deduce from (2.176),

$$\sum_{n=1}^\infty\frac{\chi(n)\,\sinh(n\alpha)}{(1+\cosh(n\alpha))^2} = -\frac{8\pi^3 i\,G(\chi)}{3\ell^3}\sum_{n=1}^\infty\frac{\overline{\chi}(n)\,n^2}{\sinh\left(\frac{2\pi^2 n}{\alpha\ell}\right)}\left[1+\frac{4\pi^2 n^2}{\alpha^2\ell^2}\right]. \tag{2.179}$$

Example 2.7.2.:

Consider the function $f(x) = \sinh(r\,x)$, for $0 < r < \frac{1}{2}$. Then $f(x) \in L_1\left(\mathbb{R}_+,\, K_\mu(\xi x)\right)$ for some $\mu > \frac{1}{2}$ $0 < \xi < \frac{1}{2}$ (we need to take $\xi > r$). Its Kontorovich-Lebedev transform takes the form (see relation 10.2.32 on [45], vol.2)

$$K_{i\tau}[f] = \frac{\pi}{2\sqrt{1-r^2}}\text{csch}\left(\frac{\pi\tau}{2}\right)\,\sinh\left(\tau\,\sin^{-1}(r)\right).$$

And a direct application of (2.167) and (2.168) allows to obtain the identities

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{\cosh^2(n\alpha) - r^2} = \frac{\beta G(\chi)}{2\ell r\sqrt{1-r^2}} \sum_{n=1}^{\infty} \overline{\chi}(n) \operatorname{csch}\left(\frac{\pi\beta n}{2\ell}\right) \sinh\left(\frac{\beta n}{\ell} \sin^{-1}(r)\right), \quad \chi \text{ even}, \tag{2.180}$$

$$\frac{1}{1-r^2} + 2\sum_{n=1}^{\infty} \frac{1}{\cosh^2(n\alpha) - r^2} = \frac{\beta \sin^{-1}(r)}{\pi\, r\sqrt{1-r^2}} + \frac{\beta}{r\sqrt{1-r^2}} \sum_{n=1}^{\infty} \operatorname{csch}\left(\frac{\pi\beta n}{2}\right) \sinh\left(\beta n \sin^{-1}(r)\right). \tag{2.181}$$

Chapter 3

The Voronoï Summation formula and its Character Analogues

In this chapter, following the same lines and principles of the previous one, we use the L_2-theory of integral transforms to prove a character analogue of Voronoï's summation formula [80]. As we shall see, the connection with Dirichlet's divisor problem will motivate a stronger version, which will be only proved at the end. Besides, we will set some new identities analogous to Voronoï's involving character weighted series. These can be used to establish a character version of Voronoï's estimate for the divisor problem and prove other interesting results, such as the fact that $L(s,\chi)$ does not vanish at $s=1$ when χ is a nonprincipal and real Dirichlet character (see the Main Theorem 1 on the fifth chapter).

The main theorem of this section, which we shall prove only at the very end, is stated as follows:

Main Theorem:

Let χ be a nonprincipal and primitive character modulo ℓ and $f(x)$ an absolutely continuous function on $\mathbb{R}_+$ such that $s\,f^*(s) \in L_2\left(\frac{1}{2}-i\infty,\,\frac{1}{2}+i\infty\right)$. Then the following summation formulas hold

$$\lim_{N\to\infty}\left[\sum_{n=1}^{N} d_{\chi}(n)\,f(n) - L(1,\chi)\int_{1/N}^{N} f(x)\,dx\right]$$

$$=\frac{4G(\chi)}{\ell}\lim_{N\to\infty}\left[\sum_{n=1}^{N} d_{\overline{\chi}}(n)\,g\left(\frac{4\pi^2}{\ell}n\right) - L(1,\overline{\chi})\int_{1/N}^{N} g\left(\frac{4\pi^2}{\ell}x\right)\,dx\right],\quad \chi \text{ even}, \tag{3.1}$$

$$\lim_{N\to\infty}\left[\sum_{n=1}^{N} d_{\chi}(n)\, f(n) - L(1,\chi)\int_{1/N}^{N} f(x)\, dx\right]$$
$$= -\frac{4iG(\chi)}{\ell}\lim_{N\to\infty}\left[\sum_{n=1}^{N} d_{\overline{\chi}}(n)\, h\left(\frac{4\pi^2}{\ell}n\right) - L(1,\overline{\chi})\int_{1/N}^{N} h\left(\frac{4\pi^2}{\ell}x\right) dx\right], \quad \chi \text{ odd}, \tag{3.2}$$

$$\lim_{N\to\infty}\left[\sum_{n=1}^{N} d(n)\, f(n) - \int_{1/N}^{N} (\log(x)+2\gamma)\ f(x)\, dx\right]$$
$$= 4\lim_{N\to\infty}\left[\sum_{n=1}^{N} d(n)\, g(4\pi^2 n) - \int_{1/N}^{N} (\log(x)+2\gamma)\ g(4\pi^2 x)\, dx\right], \quad \chi \text{ trivial}, \tag{3.3}$$

where $d_{\chi}(n)$ is a character analogue of the classical divisor function $d(n) = \sum_{d|n} 1$ (see our glossary), being defined as

$$d_{\chi}(n) = \sum_{d|n} \chi(d),$$

and $g(x)$ and $h(x)$ are the integral transforms given by

$$g(x) = -\frac{\pi}{2\sqrt{x}}\int_{0}^{\infty}\left(\frac{2}{\pi}K_1(2\sqrt{xy}) + Y_1(2\sqrt{xy})\right)\sqrt{y}\, f'(y)\, dy, \tag{3.4}$$

$$h(x) = -\frac{\pi}{2\sqrt{x}}\int_{0}^{\infty}\sqrt{y}\, J_1(2\sqrt{xy})\, f'(y)\, dy. \tag{3.5}$$

Here K_ν and Y_ν denote the Bessel functions of the second kind and J_ν is the classical Bessel function of the first kind. In the sequel, as in the previous chapter, we shall invoke Watson Theory for the L_2 transforms [102] to write the integral transforms (3.4) and (3.5) as integral transforms for the L_2 class of functions (see Remarks 2.1 and 2.2 in the second chapter).

In contrast with the previous chapter, here we do not attack directly the proof of the Main theorem stated as above. Instead, we develop the theory and preliminary results aiming at a weaker version of it and only at the last pages we argue that the Main theorem is also true, but its foundations are based upon non-elementary results from the Analytic Theory of Numbers.

This has to do with the fact that, in a strong contrast with the summation formula at the second chapter, the conditions in the Main theorem are very narrow to lie in rather elementary considerations on the asymptotic properties of the remainder terms

$$\Delta_\chi(x) = \sum_{n\leq x}{}' d_\chi(n) - L(1,\chi)\, x, \tag{3.6}$$

$$\Delta(x) = \sum_{n\leq x}{}' d(n) - x\left(\log(x) + 2\gamma - 1\right), \tag{3.7}$$

considered in Dirichlet's divisor problem and related with the Dirichlet series whose coefficients are the divisor functions $d_\chi(n)$ and $d(n)$. Since getting a suitable bound for (3.6) and (3.7) is very difficult [63, 103], we will focus at a weaker version of the Main Theorem above, which we state as follows,

Theorem 3.1. Let $f(x)$ be an absolutely continuous function on $\mathbb{R}_+$ so that its Mellin transform satisfies $s\, f^*(s) \in L_2\left(\frac{1}{2} - i\infty,\, \frac{1}{2} + i\infty\right)$. Furthermore, assume that exists a $\delta > 0$ such that $f^*(s)$ is analytic in the strip $\frac{1}{2} - \delta \leq \sigma = \mathrm{Re}(s) \leq \frac{1}{2} + \delta$ and $f^*(\sigma + it) = O\left(|t|^{-\eta}\right)$, for $\eta > \frac{3}{2}$.

Then the formulas (3.1), (3.2) and (3.3) are valid under these conditions.

As in the previous chapter, we will prove first the character formulas (3.1) and (3.2), arguing that the third one (3.3) can be proved analogously, as in the Poisson case, with the extra factors coming from the higher order of the poles of Riemann's $\zeta-$function.

We start with a few preliminary results, related with an extension of the classical Dirichlet estimate for the divisor problem, invoking the analytic behavior of $L(s,\chi)$ around $s = 1$.

3.1 Preliminary results - Part I:

First, we need to establish some results related with the behavior of the Riemann $\zeta-$function near the critical line. In the previous chapter, we have used the fact that the asymptotic order of $\zeta(s)$ in the line $\mathrm{Re}(s) = \frac{1}{2}$ is given by

$$\zeta\left(\frac{1}{2} + it\right) = O\left(|t|^{\frac{1}{4}+\epsilon}\right), \quad |t| \to \infty, \tag{3.8}$$

which was established via well-known methods, such as Phragmén-Lindelöf principle [43] (see Proposition 2.2. at the previous chapter). However, to study Voronoï's summation formula in the L_2-sense, we shall need a sharper estimate, which we state at the next lemma. The proof of this estimate can be found in Theorem 5.5., page 99 of Titchmarsh's textbook on the $\zeta-$function [103].

Lemma 3.A.: An extension of the Phragmén-Lindelöf Principle The order of the Riemann $\zeta-$function in the line $\mathrm{Re}(s) = 1/2$ satisfies the asymptotic order

$$\zeta\left(\frac{1}{2} + it\right) = O\left(|t|^{\frac{1}{6}+\epsilon}\right), \quad |t| \to \infty. \tag{3.9}$$

Now we formally define the weighted divisor function used in the statement of the main theorem.

Definition 3.1. Let χ be any character modulo ℓ. We define the arithmetic function $d_{\chi}(n)$ by the following sum

$$d_{\chi}(n) = \sum_{d|n} \chi(d), \tag{3.10}$$

which, in some sense, extends the notion of the classical divisor function $d(n)$.

Example 3.A.:

Consider, for instance, the odd character modulo 4 (see the elementary example 1.A given at the first chapter)

$$\chi_4(n) = \begin{cases} 0 & n = 0,\ 2 \\ 1 & n = 1 \\ -1 & n = 3. \end{cases}$$

Then we have that

$$\begin{aligned} d_{\chi_4}(n) &= \sum_{d|n} \chi(d) = \sum_{d \equiv 1 \mod 4} 1 - \sum_{d \equiv 3 \mod 4} 1 \\ &= d_{1,4}(n) - d_{3,4}(n), \end{aligned}$$

where the notation $d_{j,\ell}(n)$ denotes the number of divisors of n which are congruent with j modulo ℓ.

Since, as $n \to \infty$, $d(n) = O\left(n^{\epsilon}\right)$ for every positive ϵ [60], $|d_{\chi}(n)| \leq d(n) = O\left(n^{\epsilon}\right)$ as well, so that the Dirichlet series

$$\mathcal{A}(s) = \sum_{n=1}^{\infty} \frac{d_{\chi}(n)}{n^{s}}$$

is absolutely convergent for all s in the half-plane $\mathrm{Re}(s) > 1$. It is also easy to check that, for $\mathrm{Re}(s) > 1$ (with $\star$ denoting the Dirichlet convolution [5])

$$\zeta(s)\, L(s,\chi) = \sum_{n=1}^{\infty} \frac{1 \star \chi(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\sum_{d|n} \chi(d)}{n^s} = \sum_{n=1}^{\infty} \frac{d_\chi(n)}{n^s},$$

and so the Dirichlet series having the arithmetic function $d_\chi(n)$ as coefficient equals to the product $\zeta(s)\, L(s,\chi)$.

Before proving the weaker theorem stated in this section, let us recall, from the first chapter (1.24), one of the representations of the Dirichlet $L-$function via the Hurwitz $\zeta-$function

$$L(s,\chi) = \ell^{-s} \sum_{r=1}^{\ell-1} \chi(r)\, \zeta\left(s,\, \frac{r}{\ell}\right), \tag{3.11}$$

which is usually invoked in some arguments aiming to prove the functional equation for $L(s,\chi)$. To estimate $\Delta_\chi(x)$ given in (3.6), we are interested in a truncated version of (3.11). Note that

$$\sum_{n\le x} \frac{\chi(n)}{n^s} = \sum_{r=1}^{\ell-1} \sum_{n=0}^{\left[\frac{x}{\ell}\right]} \frac{\chi\,(n\ell + r)}{(n\ell + r)^s} = \ell^{-s} \sum_{r=1}^{\ell-1} \chi(r) \sum_{n=0}^{\left[\frac{x}{\ell}\right]} \frac{1}{\left(n + \frac{r}{\ell}\right)^s}, \tag{3.12}$$

and the second sum is a truncated version of the Hurwitz $\zeta-$function. One of the natural conclusions of the Hermite representation of $\zeta(s,a)$, seen in the first chapter (1.89), was that, around $s = 1$, this function admits the Laurent expansion

$$\zeta(s,\, a) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\, \gamma_n(a)\, (s-1)^n, \quad 0 < a \le 1 \tag{3.13}$$

where $\gamma_n(a)$ are the so called Stieltjes constants (for $a = 1$, $\gamma_n(a) = \gamma_n$ and (3.13) describes the Laurent expansion for Riemann's $\zeta-$function). For example (see eq. (1.90) at the first chapter), one can show that $\gamma_n(a)$ obeys to the integral representation [44]

$$\gamma_n(a) = \left[\frac{1}{2a} - \frac{\log(a)}{n+1}\right] \log^n(a) - i \int_0^{\infty} \left[\frac{\log^n(a - ix)}{a - ix} - \frac{\log^n(a + ix)}{a + ix}\right] \frac{dx}{e^{2\pi x} - 1}. \tag{3.14}$$

However, another relation due to Wilton [23, 111] is more natural, and generalizes the usual definition of the Euler constant $\gamma = \gamma_0(1)$. This relation gives the formula

$$\gamma_n(a) = \lim_{m\to\infty} \left[\sum_{k=0}^{m} \frac{\log^n(k+a)}{k+a} - \frac{\log^{n+1}(m+a)}{n+1}\right]. \tag{3.15}$$

From (3.13) it is natural to expect the truncated version

$$\sum_{k=0}^{m} \frac{1}{k+a} = -\psi(a) + \log(m+a) + O\left(\frac{1}{m}\right). \tag{3.16}$$

Recall also that, using (3.13) as well as the fact that $\gamma_0(a) = -\psi(a)$, one can obtain the particular value (see Theorem 1.7. on the first chapter)

$$L(1,\chi) = -\ell^{-1} \sum_{r=1}^{\ell-1} \chi(r)\, \psi\left(\frac{r}{\ell}\right). \tag{3.17}$$

We are now ready to prove the following result, which can be seen as a character analogue of Dirichlet's theorem for the divisor function.

Lemma 3.1. Let χ be a nonprincipal Dirichlet character modulo ℓ and $\Delta_\chi(x)$ be the remainder of the weighted divisor function (3.10),

$$\Delta_\chi(x) = \sum_{n\leq x}{}' d_\chi(n) - L(1,\chi)\, x. \tag{3.18}$$

Then the following estimate holds

$$|\Delta_\chi(x)| = O\left(x^{1/2}\right). \tag{3.19}$$

Proof: Our proof uses the key ideas of Dirichlet's own proof for $\Delta(x)$ presented in Titchmarsh's textbook. Recall from the previous chapter the function $\Lambda_\ell(x) = \sum_{n\leq x}' \chi(n)$, which satisfies $|\Lambda_\ell(x)| \leq \ell - 1$. From elementary considerations, we find the estimates

$$\begin{aligned}
\sum_{n\leq x}{}' d_\chi(n) &= \sum_{m} \sum_{n\cdot m\leq x} \chi(n) \\
&= \sum_{m\leq\sqrt{x}} \sum_{n\leq\sqrt{x}} \chi(n) + \sum_{m\leq\sqrt{x}} \sum_{\sqrt{x}<n\leq\frac{x}{m}} \chi(n) \\
&\quad + \sum_{n\leq\sqrt{x}} \sum_{\sqrt{x}<m\leq\frac{x}{n}} \chi(n) \\
&= \left[\sqrt{x}\right] \Lambda_\ell\left(\sqrt{x}\right) + \sum_{m\leq\sqrt{x}} \left(\Lambda_\ell\left(\frac{x}{m}\right) - \Lambda_\ell\left(\sqrt{x}\right)\right) \\
&\quad + \sum_{n\leq\sqrt{x}} \chi(n) \left(\left[\frac{x}{n}\right] - \left[\sqrt{x}\right]\right) \\
&= \sum_{m\leq\sqrt{x}} \Lambda_\ell\left(\frac{x}{m}\right) + \sum_{n\leq\sqrt{x}} \chi(n) \left(\frac{x}{n} + O(1)\right) - \left[\sqrt{x}\right] \Lambda_\ell\left(\sqrt{x}\right) \\
&= x \sum_{n\leq\sqrt{x}} \frac{\chi(n)}{n} + \sum_{m\leq\sqrt{x}} \Lambda_\ell\left(\frac{x}{m}\right) - \left[\sqrt{x}\right] \Lambda_\ell\left(\sqrt{x}\right) + O(1).
\end{aligned} \tag{3.20}$$

Using (3.12), (3.16) and (3.17), we deduce easily

$$\begin{aligned}
\sum_{n\le\sqrt{x}} \frac{\chi(n)}{n} &= \ell^{-1}\sum_{r=1}^{\ell-1}\chi(r)\sum_{n=0}^{\left[\frac{\sqrt{x}}{\ell}\right]}\frac{1}{n+\frac{r}{\ell}} \\
&= \ell^{-1}\sum_{r=1}^{\ell-1}\chi(r)\left(-\psi\left(\frac{r}{\ell}\right)+\log\left(\left[\frac{\sqrt{x}}{\ell}\right]+\frac{r}{\ell}\right)+O\left(\frac{1}{\sqrt{x}}\right)\right) \\
&= L(1,\chi)+\ell^{-1}\sum_{r=1}^{\ell-1}\chi(r)\,\log\left(\left[\frac{\sqrt{x}}{\ell}\right]+\frac{r}{\ell}\right)+O\left(\frac{1}{\sqrt{x}}\right). \qquad (3.21)
\end{aligned}$$

Consider the second factor in (3.21): since we will take x arbitrarily large, take $x>\ell$. From the power series for the logarithmic function we obtain

$$\begin{aligned}
\sum_{r=1}^{\ell-1}\chi(r)\,\log\left(\left[\frac{\sqrt{x}}{\ell}\right]+\frac{r}{\ell}\right) &= \log\left(\left[\frac{\sqrt{x}}{\ell}\right]\right)\sum_{r=1}^{\ell-1}\chi(r)+\sum_{r=1}^{\ell-1}\chi(r)\,\log\left(1+\frac{r}{\ell\,[\sqrt{x}/\ell]}\right) \\
&= \sum_{r=1}^{\ell-1}\chi(r)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\left(\frac{r}{\ell}\right)^k\frac{1}{[\sqrt{x}/\ell]^k}=O\left(\frac{1}{\sqrt{x}}\right)
\end{aligned}$$

where the second equality came from the fact that χ is nonprincipal. Hence, we have that

$$\sum_{n\le\sqrt{x}}\frac{\chi(n)}{n}=L(1,\chi)+O\left(\frac{1}{\sqrt{x}}\right),$$

which, together with (3.20) yields the desired estimate (3.19). ■

We now introduce another class of preliminary results which is similar to the one presented in the Main Lemma of the second chapter.

3.2 Preliminary Results - Part II:

As before, let us assume that χ is a nonprincipal and primitive Dirichlet character modulo ℓ: consider the function (the average order of the arithmetic function $d_\chi(n)$)

$$h_\chi(x)=\frac{1}{x}\sum_{n\le x}{}'\, d_\chi(n)-L\left(1,\chi\right).$$

From lemma 3.1., $h_\chi(x)$ satisfies the asymptotic estimates

$$\begin{aligned}
h_\chi(x) &= O\left(x^{-1/2}\right), \quad x\to\infty \\
&= O(1), \qquad x\to 0.
\end{aligned}$$

As in the previous chapter, we are now allowed to write the L_1-Mellin transform

$$h^*_\chi(s) = \int_0^\infty x^{s-1} h_\chi(x)\, dx, \tag{3.22}$$

which is well-defined for $0 < \sigma < \frac{1}{2}$. It is not surprising that (3.22) can be analytically continued into the region $\sigma = \mathrm{Re}(s) < 0$ by a similar process to the one given in the previous chapter (see eq. (2.38)) , i.e., we are allowed to write $h^*_\chi(s)$ as follows

$$\begin{aligned} h^*_\chi(s) &= \int_0^1 x^{s-1} h_\chi(x)\, dx + \int_1^\infty x^{s-1} h_\chi(x)\, dx \\ &= -\int_0^1 x^{s-1}\, L(1,\chi)\, dx + \int_1^\infty x^{s-1} h_\chi(x)\, dx \\ &= -\frac{L(1,\chi)}{s} + \int_1^\infty x^{s-1} h_\chi(x)\, dx, \end{aligned} \tag{3.23}$$

and these equalities are valid for $\sigma < \frac{1}{2}$, since the latter integral exists and it is analytic in this region.

In (3.23), let us take $\mathrm{Re}(s) = \sigma < 0$: performing the same type of computations as in the previous chapter, the latter integral can be written as

$$\int_1^\infty x^{s-1} h_\chi(x)\, dx = \int_1^\infty {\sum_{n\leq x}}' d_\chi(n)\, x^{s-2} dx - \int_1^\infty L(1,\chi)\, x^{s-1} dx$$

$$= \int_1^\infty {\sum_{n\leq x}}' d_\chi(n)\, x^{s-2} dx + \frac{L(1,\chi)}{s} = \frac{\zeta(1-s)\, L(1-s,\chi)}{1-s} + \frac{L(1,\chi)}{s} \tag{3.24}$$

Since the function in (3.24) is analytic in $\mathbb{C}\setminus\{1\}$, having $s=0$ as a removable singularity, it follows from analytic continuation that, for all complex s belonging to the region $\mathrm{Re}(s) = \sigma < \frac{1}{2}$

$$\int_1^\infty x^{s-1} h_\chi(x)\, dx = \frac{\zeta(1-s)\, L(1-s,\chi)}{1-s} + \frac{L(1,\chi)}{s}, \tag{3.25}$$

and a direct comparison with (3.23) yields

$$h^*_\chi(s) = \frac{\zeta\,(1-s)\;L\,(1-s,\chi)}{1-s}, \quad 0 < \mathrm{Re}(s) < \frac{1}{2}.$$

Now, for all $0 < \sigma < \frac{1}{2}$, we know that $x^{\sigma} h_{\chi}(x) \in L_2\left(\mathbb{R}_+,\, \frac{dx}{x}\right)$ and so we can apply Mellin's inversion formula in L_2 to obtain (see the details in the second chapter),

$$h_{\chi}(x) = \underset{N\to\infty}{\text{l.i.m.}} \int_{\sigma-iN}^{\sigma+iN} \frac{\zeta(1-s)\, L(1-s,\chi)}{1-s}\, x^{-s} ds, \quad 0 < \sigma < \frac{1}{2}. \tag{3.26}$$

Our main obstacle now is the extension of (3.26) to the critical line $\sigma = \frac{1}{2}$, where $\zeta(s)$ and $L(s,\chi)$ possess a suitable symmetry to provide a summation formula.

To do this, we cannot use, as in the previous chapter, the classical estimate given by Phragmén-Lindelöf (3.8). Doing so would imply that the integrand in (3.26), for $\sigma = \frac{1}{2}$, would be $O\left(|t|^{-\frac{1}{2}+\epsilon}\right)$, therefore not belonging to $L_2\left(\frac{1}{2}\right)$.

To surpass this first obstacle, we use Lemma 3.A., which implies that, at the critical line, for any $\epsilon > 0$, $h_{\chi}^{*}(s)$ obeys to the estimate

$$h_{\chi}^{*}(s) = \frac{\zeta(1-s)\, L(1-s,\chi)}{1-s} = O\left(|t|^{-\frac{7}{12}+\epsilon}\right), \quad |t| \to \infty$$

and therefore $h_{\chi}^{*}(s) \in L_2\left(\frac{1}{2} - i\infty,\, \frac{1}{2} + i\infty\right)$. This means that the L_2 integral of Mellin type

$$\frac{1}{2\pi i} \underset{N\to\infty}{\text{l.i.m.}} \int_{\frac{1}{2}-iN}^{\frac{1}{2}+iN} \frac{\zeta(1-s)\, L(1-s,\chi)}{1-s}\, x^{-s} ds, \tag{3.27}$$

is well-defined in the sense remarked by eq. (2.2) of the second chapter.

Now, let $g(x)$ be defined as the limit (in the L_2 mean) given by (3.27): then $g(x) \in L_2(\mathbb{R}_+)$ by Plancherel theorem for the Mellin transform in L_2 [102] (see eq. (2.25) in the second chapter).

For any $0 < \sigma < \frac{1}{2}$, consider the rectangular contour $\Gamma_{\sigma,N}$ having $\sigma \pm iN$ and $\frac{1}{2} \pm iN$ as vertices. Since $\zeta(1-s)\, L(1-s,\chi)/(1-s)\, x^{-s}$ is analytic on the interior of $\Gamma_{\sigma,N}$, by Cauchy's theorem we can write, for each $0 < \sigma < \frac{1}{2},\;\; N > 0$,

$$\int_{\Gamma_{\sigma,N}} \frac{\zeta(1-s)\, L(1-s,\chi)}{1-s}\, x^{-s} ds = 0. \tag{3.28}$$

Since $\zeta(1-s)\, L(1-s,\chi)/(1-s) = O\left(|t|^{\sigma+2\epsilon-1}\right)$ (recall (2.42)), it follows that the integrals along the lines $\left[\sigma \pm iN,\, \frac{1}{2} \pm iN\right]$ obey to the inequality

$$\int_{\sigma\pm iN}^{\frac{1}{2}\pm iN} \left| \frac{\zeta\,(1-s)\, L\,(1-s)}{1-s}\, x^{-s} \right| |ds| \leq \frac{K\, |N|^{-\sigma+2\epsilon} x^{-\sigma}}{\log\left(|N|\, x\right)} \left[1 - \left(|N|\, x\right)^{\sigma-\frac{1}{2}}\right],$$

so that

$$\underset{N\to\infty}{\text{l.i.m.}} \int_{\sigma\pm iN}^{\frac{1}{2}\pm iN} \left|\frac{\zeta^2(1-s)}{1-s}\, x^{-s}\right| |ds| = 0.$$

From (3.28) we conclude that $g(x) = h_\chi(x)$ a.e., which means that $h_\chi(x) \in L_2(\mathbb{R}_+)$ as well. Hence, for a complex number s belonging to the line $\text{Re}(s) = \frac{1}{2}$, we can write $\frac{\zeta(1-s)\,L(1-s,\chi)}{1-s}$ as the L_2-Mellin transform,

$$\frac{\zeta(1-s)\,L(1-s,\chi)}{1-s} = \underset{N\to\infty}{\text{l.i.m.}} \int_{1/N}^{N} x^{s-1} h_\chi(x)\, dx, \quad 0 < \sigma \le \frac{1}{2}, \tag{3.29}$$

and after an elementary change of variables, we can rewrite (3.29) as

$$\frac{\zeta(s)\,L(s,\chi)}{s} = \underset{N\to\infty}{\text{l.i.m.}} \int_{1/N}^{N} x^{s-1}\, \Delta_\chi\left(\frac{1}{x}\right) dx = \underset{N\to\infty}{\text{l.i.m.}}\ \Delta^*_{\chi,N}(s), \quad \frac{1}{2} \le \text{Re}(s) < 1. \tag{3.30}$$

Equation (3.30) plays an analogous role to equation (2.41) in the second chapter, which was essential for the establishment of the main lemma. Now we can prove the following:

Main Lemma: Summation representation in the region $\sigma > 1/2$

Let $f(x)$ be an absolutely continuous function on $\mathbb{R}_+$ such that $s\,f^*(s) \in L_2\,(\sigma - i\infty,\, \sigma + i\infty)$, for $\frac{1}{2} < \sigma < 1$. Then the following representation holds

$$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \zeta(s)\,L(s,\chi)\,f^*(s)\,ds = \lim_{N\to\infty} \left[\sum_{n=1}^{N} d_\chi(n)\,f(n) - L(1,\chi) \int_{1/N}^{N} f(x)\,dx\right], \quad \frac{1}{2} < \sigma < 1. \tag{3.31}$$

Proof: After the proof of (3.30), which is the most important part of the preliminary results, it is quite immediate to deduce (3.31) from an adaptation of the work done in the previous chapter.

Since $\zeta(s)\,L(s,\chi)/s$ and $s\,f^*(s)$ are both $L_2(\sigma)$, $\frac{1}{2} < \sigma < 1$, Cauchy-Schwarz inequality establishes that the integrand in the left-hand side of (3.31) is $L_1(\sigma)$ and we can use the theory of Mellin transforms for the L_1-class of functions.

First, using an integration by parts in the second term of (3.30) (we can still justify the integration in the sense of Lebesgue-Stieltjes [1]), we adapt the proof given in the previous chapter

[1] When χ is a real character, one can check that $\Delta_\chi(x)$ is monotone by looking at Lemma 5.1., Chapter V, of this thesis. If χ is not real, one may decompose $d_\chi(n)$ into real and imaginary parts and then verify the properties of a Lebesgue-Stieltjes measure for each one of these.

to conclude

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\zeta(s)\,L(s,\chi)\,f^{*}(s)\,ds=\frac{1}{2\pi i}\lim_{N\to\infty}\int_{\sigma-i\infty}^{\sigma+i\infty}s\,\Delta_{\chi,N}^{*}(s)\,f^{*}(s)\,ds$$

$$=\lim_{N\to\infty}\left[\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\int_{1/N}^{N}f^{*}(s)\,x^{-s}d\Delta_{\chi}(x)\,ds\;-\Delta_{\chi}\left(N\right)f(N)+\Delta_{\chi}\left(\frac{1}{N}\right)\,f\left(\frac{1}{N}\right)\right]\tag{3.32}$$

where the latter terms come from a direct application of Mellin's inversion formula in $L_1\left(\sigma-i\infty,\,\sigma+i\infty\right)$ (recall that the hypothesis $s\,f^{*}(s)\in L_2\left(\sigma\right)$ implies $f^{*}(s)\in L_1\left(\sigma\right)$).

Furthermore, in (3.32) we have an ordinary limit, while $s\,\Delta_{\chi,N}^{*}(s)$ converges in the L_2-mean to $\zeta(s)\,L(s,\chi)$. This apparent change in the nature of limits can be also justified in the same way done previously, i.e., by invoking the fact that convergence in L_2 implies convergence $\mu-$ a.e. for some subsequence (see Lemma 2.1.) and that $\left(s\,\Delta_{\chi,N}^{*}\left(s\right)\right)_{N\in\mathbb{N}}$ is Cauchy.

Moreover, the exchange of the limit with the integral in the first equality is possible due to Lebesgue's dominated convergence theorem for the class of L_1 functions (see the previous chapter).

Since $s\,f^{*}(s)\in L_2\left(\sigma-i\infty,\,\sigma+i\infty\right)$, it follows from the preliminary results given in the previous chapter that $\lim_{x\to0,\,\infty}x^{\sigma}f(x)=0$. Hence, using the fact that $\sigma>\frac{1}{2}$ and the estimates for $\Delta_{\chi}(x)$,

$$\begin{aligned}|\Delta_{\chi}(x)|&=O(x^{1/2})\quad x\to\infty\\&=O\left(x\right)\quad x\to0,\end{aligned}$$

we immediately see that the latter terms on (3.32) vanish and $x^{-\sigma-1}|\Delta_{\chi}(x)|\in L_1(\mathbb{R}_{+})$. This gives the equality

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\zeta(s)\,L(s,\chi)\,f^{*}(s)\,ds=\lim_{N\to\infty}\int_{\sigma-i\infty}^{\sigma+i\infty}\int_{1/N}^{N}f^{*}(s)\,x^{-s}d\Delta_{\chi}(x)\,ds,\tag{3.33}$$

which can be treated by appealing to Fubini's theorem since

$$\lim_{N\to\infty}\int_{\sigma-i\infty}^{\sigma+i\infty}\int_{1/N}^{N}\left|f^{*}(s)\,x^{-s}d\Delta_{\chi}(x)\,ds\right|\leq\|f^{*}(s)\|_{L_1(\sigma)}\lim_{N\to\infty}\int_{1/N}^{N}x^{-\sigma}\,d\,|\Delta_{\chi}(x)|$$

$$=\sigma\,\|f^{*}(s)\|_{L_1(\sigma)}\lim_{N\to\infty}\int_{1/N}^{N}x^{-\sigma-1}\,|\Delta_{\chi}(x)|\,dx<\infty.\tag{3.34}$$

Interchanging the order of integration in (3.33) (which is allowed by (3.34)), we obtain

$$\lim_{N\to\infty}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\int_{1/N}^{N} f^{*}(s)\,x^{-s}d\Delta_{\chi}(x)\,ds=\lim_{N\to\infty}\int_{1/N}^{N} f(x)\,d\Delta_{\chi}(x), \tag{3.35}$$

which finally gives, after an integration by parts and appealing to the absolute continuity of f (it suffices to mimic the computations given in (2.55))

$$\lim_{N\to\infty}\int_{1/N}^{N} f'(x)\,\Delta_{\chi}(x)\,dx=\lim_{N\to\infty}\left[\sum_{n=1}^{N} d_{\chi}(n)\,f(n)-L(1,\chi)\int_{1/N}^{N} f(x)\,dx\right].\ \blacksquare \tag{3.36}$$

The lemma just proved ensures a representation of an absolutely convergent integral of Mellin-type as a series involving the weighted divisor function $d_{\chi}(n)$. But this representation is only valid in the region $\sigma>\frac{1}{2}$. As we've seen in the previous chapter, to study a summation formula over the coefficients that arise from the Dirichlet series $\zeta(s)$ or $L(s,\chi)$, we need to use the symmetries of the product $\zeta(s)\,L(s,\chi)$ via a functional equation. The symmetry we now need is only provided if, in the left-hand side of (3.31), we make an integration over the critical line $\sigma=\frac{1}{2}$.

To allow this, we need to impose additional conditions over $f^{*}(s)$ and this is precisely what is done in the next lemma.

Lemma 3.2. (extension to the line $\sigma=\frac{1}{2}$) Let $f(x)$ be an absolutely continuous function on $\mathbb{R}_{+}$ such that $s\,f^{*}(s)\in L_{2}\left(\frac{1}{2}-i\infty,\,\frac{1}{2}+i\infty\right)$.

Moreover, assume that exists $\delta>0$ such that $f^{*}(s)$ is analytic in the strip $\frac{1}{2}-\delta\leq\sigma=\mathrm{Re}(s)\leq\frac{1}{2}+\delta$ and $f^{*}(\sigma+it)=O\left(|t|^{-\eta}\right)$, for $\eta>\frac{3}{2}$.

Then the following representation holds

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\zeta(s)\,L(s,\chi)\,f^{*}(s)\,ds=\lim_{N\to\infty}\left[\sum_{n=1}^{N} d_{\chi}(n)\,f(n)-L(1,\chi)\int_{1/N}^{N} f(x)\,dx\right]. \tag{3.37}$$

Proof: Since, by hypothesis, $f^{*}(s)$ is analytic in the region $\frac{1}{2}\leq\mathrm{Re}(s)\leq\frac{1}{2}+\delta$, Cauchy's theorem allows to write, for the rectangle $\Gamma_{\delta,N}=\left[\frac{1}{2}\pm iN,\,\frac{1}{2}+\delta\pm iN\right]$,

$$\int_{\Gamma_{\delta,N}}\zeta(s)\,L(s,\chi)\,f^{*}(s)\,ds=0.$$

By hypothesis, $f^*(\sigma + it) = O\left(|t|^{-\eta}\right)$, for $\frac{1}{2} \le \sigma \le \frac{1}{2} + \delta$ and $\eta > \frac{3}{2}$, so that, for any $\epsilon > 0$,

$$\zeta(s)\, L(s,\chi)\, f^*(s) = O\left(|t|^{-1+2\epsilon}\right),$$

which implies

$$\lim_{N\to\infty} \int\limits_{\frac{1}{2}\pm iN}^{\sigma\pm iN} \zeta(s)\, L(s,\chi)\, f^*(s)\, ds = 0.$$

This proves the equality

$$\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \zeta(s)\, L(s,\chi)\, f^*(s)\, ds = \int\limits_{\sigma-i\infty}^{\sigma+i\infty} \zeta(s)\, L(s,\chi)\, f^*(s)\, ds,$$

and the fact that $s\, f^*(s) = O\left(|t|^{1-\eta}\right),\ \eta > \frac{3}{2}$ implies $s\, f^*(s) \in L_2(\sigma - i\infty, \sigma + i\infty)$, and the result now follows from the main lemma above. ■

Now, we establish the first approach to deduce the form of the integral transforms (3.4) and (3.5) associated to the summation formulas (3.1), (3.2) and (3.3). This approach uses a chain transform of Fourier type and it will be deduced later that this chain transform is related with the Bessel functions presented in (3.4) and (3.5).

Theorem 3.2. (Reciprocity Fourier-Watson Transforms for characters) Let $f(x)$ be an absolutely continuous function in $\mathbb{R}_+$ such that $f^*(s)$ satisfies the condition $s\, f^*(s) \in L_2\left(\frac{1}{2} - i\infty,\, \frac{1}{2} + i\infty\right)$.

Then there exists a tetrad of functions $\varphi(x),\ \psi(x),\ g(x),\ h(x) \in L_2(\mathbb{R}_+)$ such that

$$f(x) \overset{\mu}{=} \frac{2}{\pi} \underset{N\to\infty}{\text{l.i.m.}} \int\limits_{1/N}^{N} \varphi(y)\, \cos(xy)\, dy, \tag{3.38}$$

$$g(x) \overset{\mu}{=} \underset{N\to\infty}{\text{l.i.m.}} \int\limits_{1/N}^{N} \frac{1}{y}\, \varphi\left(\frac{1}{y}\right) \cos(xy)\, dy, \tag{3.39}$$

$$f(x) \overset{\mu}{=} \frac{2}{\pi} \underset{N\to\infty}{\text{l.i.m.}} \int\limits_{1/N}^{N} \psi(y)\, \sin(xy)\, dy \tag{3.40}$$

$$h(x) \overset{\mu}{=} \underset{N\to\infty}{\text{l.i.m.}} \int\limits_{1/N}^{N} \frac{1}{y}\, \psi\left(\frac{1}{y}\right) \cos(xy)\, dy. \tag{3.41}$$

Moreover, both $s\, g^*(s)$ and $s\, h^*(s)$ belong to $L_2\left(\frac{1}{2} - i\infty,\, \frac{1}{2} + i\infty\right)$ and the following identities hold

$$\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \zeta(s)\, L(s,\chi)\, f^*(s)\, ds = \frac{4G(\chi)}{\ell} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \zeta(s)\, L(s,\overline{\chi})\, g^*(s) \left(\frac{4\pi^2}{\ell}\right)^{-s} ds, \quad \chi\, \text{even}, \tag{3.42}$$

$$\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \zeta(s)\, L(s,\chi)\, f^*(s)\, ds = -\frac{4iG(\chi)}{\ell} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \zeta(s)\, L(s,\overline{\chi})\, h^*(s) \left(\frac{4\pi^2}{\ell}\right)^{-s} ds, \quad \chi\, \text{odd}, \tag{3.43}$$

where $g^*(s)$ and $h^*(s)$ denote, repectively, the Mellin transforms of $g(x)$ and $h(x)$ in L_2.

Proof: We will divide the proof into two parts as the different nature of the transforms (3.39) and (3.41) is due to the sign of the character χ.

Assume first that χ is even and consider the absolutely convergent integral

$$\frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \zeta(s)\, L(s,\chi)\, f^*(s)\, ds. \tag{3.44}$$

Let us use the functional equations for $L(s,\chi)$ and $\zeta(s)$ and take the change of variable $s \leftrightarrow 1-s$. The integral in (3.44) can also be expressed as

$$\frac{4G(\chi)}{\ell} \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \cos^2\left(\frac{\pi s}{2}\right) \Gamma^2(s)\, \zeta(s)\, L(s,\, \overline{\chi})\, f^*(1-s) \left(\frac{4\pi^2}{\ell}\right)^{-s} ds. \tag{3.45}$$

Let us denote the factor $\cos\left(\frac{\pi s}{2}\right) \Gamma(s)\, f^*(1-s)$ by $\varphi^*(s)$: clearly, from considerations on the second chapter (recall that $\varphi^*(s)$ is the function $g^*(s)$ appearing there), $s\, \varphi^*(s) \in L_2\left(\frac{1}{2} - i\infty,\, \frac{1}{2} + i\infty\right)$. Moreover, following eq. (2.62) and Remark 2.1 given at the previous chapter, we deduce that the Mellin inverse of $\varphi^*(s)$, $\varphi(x)$, can be expressed by the transform

$$\varphi(x) = -\frac{1}{x} \int_0^{\infty} f'(y)\, \sin(xy)\, dy \overset{\mu}{=} \underset{N\to\infty}{\text{l.i.m.}} \int_{1/N}^{N} f(y)\, \cos(xy)\, dy \tag{3.46}$$

Therefore, by using the Fourier inversion formula in the L_2-class (recall relation (2.12)), we deduce the converse

$$f(x) \overset{\mu}{=} \frac{2}{\pi} \underset{N\to\infty}{\text{l.i.m.}} \int_{1/N}^{N} \varphi(y)\, \cos(xy)\, dy.$$

Now, take (3.46) and define $g^*(s)$ as

$$g^*(s) = \cos\left(\frac{\pi s}{2}\right)\Gamma(s)\,\varphi^*(s).$$

Clearly $s\,g^*(s) \in L_2\left(\frac{1}{2} - i\infty,\, \frac{1}{2} + i\infty\right)$, as $\cos\left(\frac{\pi s}{2}\right)\Gamma(s)$ is bounded at the critical line and $s\,\varphi^*(s) \in L_2\left(\frac{1}{2} - i\infty,\, \frac{1}{2} + i\infty\right)$.

To find $g(x)$ we appeal to Parseval's equality and we obtain (with the notation $\Phi^*(s) = \varphi^*(1-s)$),

$$\begin{aligned} g(x) &= \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Gamma(s)}{1-s}\cos\left(\frac{\pi s}{2}\right)(1-s)\Phi^*(1-s)\,x^{-s}ds \\ &= -\frac{1}{x}\int_0^\infty \Phi'(y)\,\sin(xy)\,dy = -\frac{1}{x}\int_0^\infty \left(\frac{1}{y}\varphi\left(\frac{1}{y}\right)\right)'\sin(xy)\,dy \\ &\overset{\mu}{=} \underset{N\to\infty}{\text{l.i.m.}}\int_{1/N}^{N}\frac{1}{y}\varphi\left(\frac{1}{y}\right)\cos(xy)\,dy, \end{aligned}$$

where we have used the fact that $\Phi(x) = \frac{1}{x}\varphi\left(\frac{1}{x}\right)$ and

$$g(x) = \frac{d}{dx}\int_0^\infty \frac{1}{y}\varphi\left(\frac{1}{y}\right)\frac{\sin(xy)}{y}\,dy,$$

together with Remark 2.1 at the second chapter.

This concludes the first part of the theorem. For the second one assume that χ is odd and use in (3.44) the functional equations for $L(s,\chi)$ and $\zeta(s)$: this gives

$$\left(-\frac{4iG(\chi)}{\ell}\right)\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\cos\left(\frac{\pi s}{2}\right)\sin\left(\frac{\pi s}{2}\right)\Gamma^2(s)\,\zeta(s)\,L(s,\overline{\chi})\,f^*(1-s)\left(\frac{4\pi^2}{\ell}\right)^{-s}ds. \tag{3.47}$$

Taking $\psi^*(s) = \sin\left(\frac{\pi s}{2}\right)\Gamma(s)\,f^*(1-s)$, it is easy to check that $\psi^*(s)$ has similar properties to the Mellin transform $h^*(s)$ given at the second chapter and its inverse Mellin transform satisfies (compare this with (2.66))

$$\psi(x) = -\frac{2}{x}\int_0^\infty f'(y)\,\sin^2\left(\frac{xy}{2}\right)dy \overset{\mu}{=} \underset{N\to\infty}{\text{l.i.m.}}\int_{1/N}^{N} f(y)\,\sin(xy)\,dy, \tag{3.48}$$

from which the converse holds

$$f(x) \stackrel{\mu}{=} \frac{2}{\pi} \underset{N\to\infty}{\text{l.i.m.}} \int\limits_{1/N}^{N} \psi(y)\, \sin(xy)\, dy.$$

Finally, if we define $h^*(s)$ as

$$h^*(s) = \cos\left(\frac{\pi s}{2}\right) \Gamma(s)\, \psi^*(s),$$

it is easily seen that $s\, h^*(s) \in L_2\left(\frac{1}{2} - i\infty,\, \frac{1}{2} + i\infty\right)$ and an application of Parseval equality yields once more (with $\Psi^*(s) = \psi^*(1-s)$)

$$\begin{aligned} h(x) &= \frac{1}{2\pi i} \int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Gamma(s)}{1-s} \cos\left(\frac{\pi s}{2}\right) (1-s)\Psi^*(1-s)\, x^{-s} ds \\ &= -\frac{1}{x} \int\limits_{0}^{\infty} \Psi^{'}(y)\, \sin(xy)\, dy \stackrel{\mu}{=} \underset{N\to\infty}{\text{l.i.m.}} \int\limits_{1/N}^{N} \frac{1}{y}\, \psi\left(\frac{1}{y}\right) \cos(xy)\, dy, \end{aligned}$$

which proves (3.41). Equalities (3.42) and (3.43) come immediately from equations (3.45) and (3.47) and the definitions of $g^*(s)$ and $h^*(s)$. ■

Finally we are ready to prove a first version of Theorem 3.1, which invokes the chain transforms invoked above. We state it as follows

Theorem 3.3. (A character version of Voronoï's summation formula): Assume that $f(x)$ is an absolutely continuous function on $\mathbb{R}_+$ whose Mellin transform satisfies the conditions of Lemma 3.2.. Moreover, let χ be a nonprincipal and primitive Dirichlet character modulo ℓ.

Then the identities (3.1) and (3.2) hold, with $f(x)$, $g(x)$ and $h(x)$ being related by (3.38), (3.39) and (3.41) and $\varphi(x),\ \psi(x) \in L_2(\mathbb{R}_+)$.

Proof: From (3.42) and (3.43), the proof is almost immediate. To conclude it, we just need to check that $g^*(s)$ and $h^*(s)$ satisfy the conditions of Lemma 3.2. in order to transform the right-hand sides of (3.42) and (3.43) into a series involving the arithmetic function $d_\chi(n)$.

We have seen that $s\, g^*(s),\, s\, h^*(s) \in L_2\left(\frac{1}{2} - i\infty,\, \frac{1}{2} + i\infty\right)$, so we just need to check the analyticity of $g^*(s)$ and $h^*(s)$ in the critical line and that they decay as $O(|t|^{-\eta}),\ \eta > \frac{3}{2}$, in some strip containing it.

We check these properties for $g^*(s)$ (as similar computations can be held for $h^*(s)$). Since

$$g^*(s) = \cos^2\left(\frac{\pi s}{2}\right) \Gamma^2(s)\, f^*(1-s), \tag{3.49}$$

$g^*(s)$ is clearly analytic in the region $\frac{1}{2} - \delta \leq \sigma \leq \frac{1}{2} + \delta$ since, by hypothesis, $f^*(s)$ is.

Since $f^*(s) = O\left(|t|^{-\eta}\right)$, $\eta > \frac{3}{2}$, by Stirling's formula $g^*(s)$ obeys to the asymptotic order

$$\cos^2\left(\frac{\pi s}{2}\right)\Gamma^2(s)\, f^*(1-s) = O\left(|t|^{2\sigma-1-\eta}\right) = O\left(|t|^{2\delta-\eta}\right). \tag{3.50}$$

The result now follows from the choice of δ, which may be taken as smaller than $\frac{1}{2}\eta - \frac{3}{4}$. Since $f^*(s)$, $g^*(s)$ and $h^*(s)$ obey to the conditions of Lemma 3.2., we have immediately (3.1) and (3.2). ■

The previous theorem furnished a description of the transforms $g(x)$ and $h(x)$ in means of a chain transform of Fourier type [53, 54].

However, we can write explicitly these integral transforms. In the next corollaries, we deduce (3.4) and (3.5) and we describe the kernel in both of these for the L_2-class of functions.

Corollary 3.1. (Explicit representation of $g(x)$ and $h(x)$) The integral transforms $g(x)$ and $h(x)$ given in (3.39) and (3.41) can be written explicitly as the following transforms

$$g(x) = -\frac{\pi}{2\sqrt{x}} \int_0^\infty \left(\frac{2}{\pi}\, K_1(2\sqrt{xy}) + Y_1(2\sqrt{xy})\right) \sqrt{y}\, f'(y)\, dy,$$

$$h(x) = -\frac{\pi}{2\sqrt{x}} \int_0^\infty \sqrt{y}\, J_1(2\sqrt{xy})\, f'(y)\, dy.$$

Proof: Let s be a complex number such that$-\nu < \mathrm{Re}(s) < \frac{3}{2}$. Consider the following Mellin integral (see [102], p. 196, relation 7.9.2.)

$$\int_0^\infty x^{s-1} J_\nu(x)\, dx = \frac{2^{s-1}\Gamma\left(\frac{s+\nu}{2}\right)}{\Gamma\left(1+\frac{\nu-s}{2}\right)}. \tag{3.51}$$

From (3.51), and using one of the definitions of the Bessel function of the second kind [106],

$$Y_\nu(x) = \frac{J_\nu(x)\, \cos(\pi\nu) - J_{-\nu}(x)}{\sin(\pi\nu)}, \tag{3.52}$$

we obtain, for $\nu + |\nu| < \mathrm{Re}(s) < \nu + \frac{3}{2}$, also the Mellin integral for Y_ν,

$$\int_0^\infty x^{s-1-\nu}\, Y_\nu(x)\, dx = -2^{s-\nu-1}\pi^{-1}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}-\nu\right)\cos\left(\pi\left(\frac{s}{2}-\nu\right)\right), \tag{3.53}$$

from which we deduce immediately

$$\int_0^\infty x^{s-1}\, \frac{1}{\sqrt{x}} Y_1\left(4\pi\sqrt{x}\right) dx = 2^{1-2s}\pi^{-2s}\Gamma(s)\,\Gamma(s-1)\, \cos(\pi s), \quad 1 < \mathrm{Re}(s) < \frac{5}{4}. \tag{3.54}$$

Now, for $0 < \sigma < \frac{1}{2}$, the right-hand side of (3.54) belongs to $L_1(\sigma - i\infty, \sigma + i\infty)$. Therefore, the evaluation of the integral

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} 2^{1-2s}\pi^{-2s}\Gamma(s)\,\Gamma(s-1)\,\cos(\pi s)\,x^{-s}ds, \quad 0 < \sigma < \frac{1}{2}$$

is straightforward by means of the Residue theorem. Using the double poles of the $\Gamma-$function at the negative integers and using the series expansion for $Y_\nu(z)$ [106], we easily deduce

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} 2^{1-2s}\pi^{-2s}\Gamma(s)\,\Gamma(s-1)\,\cos(\pi s)\,x^{-s}ds$$
$$= \frac{1}{\sqrt{x}}\,Y_1\left(4\pi\sqrt{x}\right) + \frac{1}{2\pi^2 x}, \quad 0 < \sigma < \frac{1}{2}. \tag{3.55}$$

Since the integrand in (3.54) is bounded by $O\left(|t|^{2\sigma-2}\right)$ and it is analytic in the region $0 < \mathrm{Re}(s) < 1$, we can change the line of integration to $\sigma = \frac{1}{2}$ and from Cauchy's Theorem we obtain

$$\frac{1}{2\pi i}\,\underset{N\to\infty}{\mathrm{l.i.m.}}\int_{\frac{1}{2}-iN}^{\frac{1}{2}+iN} 2^{1-2s}\pi^{-2s}\Gamma(s)\,\Gamma(s-1)\,\cos(\pi s)\,x^{-s}ds$$
$$= \frac{1}{\sqrt{x}}\,Y_1\left(4\pi\sqrt{x}\right) + \frac{1}{2\pi^2 x}. \tag{3.56}$$

It is also well-known that, for $\mu > 1$, the representation holds (see Example 2.5 at the second chapter)

$$\frac{2}{\pi\sqrt{x}}\,K_1(4\pi\sqrt{x}) = \frac{2}{\pi\sqrt{x}}\,\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} 2^{s-2}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s-1}{2}\right)\left(4\pi\sqrt{x}\right)^{-s}ds$$
$$= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} 2^{1-2s}\pi^{-2s}\,\Gamma(s)\,\Gamma(s-1)\,x^{-s}ds, \tag{3.57}$$

where $\sigma = \frac{\mu+1}{2} > 1$. Although we do not give the details in any point of our work, one should note that (3.57) can be also proved via the elementary relation [106]

$$K_\nu(x) = \frac{\pi i}{2}\,e^{\pi i\nu/2}\left(J_\nu(x) + i\,Y_\nu(x)\right), \tag{3.58}$$

together with (3.51). In (3.57) we may change the line of integration to $\sigma = \frac{1}{2}$ and an application of the Residue Theorem gives (once we count the pole of $\Gamma(s-1)$ at $s=1$),

$$\frac{2}{\pi\sqrt{x}}\, K_1(4\pi\sqrt{x}) = \frac{1}{2\pi i}\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} 2^{1-2s}\pi^{-2s}\,\Gamma\left(s\right)\Gamma\left(s-1\right)\, x^{-s} ds - \frac{1}{2\pi^2 x}. \tag{3.59}$$

By virtue of Stirling's formula, the integrand in (3.59) belongs also to $L_2\left(\frac{1}{2}-i\infty,\,\frac{1}{2}+i\infty\right)$: thus, if we sum both sides (3.59) with (3.56) we get

$$\frac{1}{2\pi i}\operatorname*{l.i.m.}_{N\to\infty}\int\limits_{\frac{1}{2}-iN}^{\frac{1}{2}+iN} (2\pi)^{1-2s}\,\Gamma(s)\,\Gamma(s-1)\,\cos^2\left(\frac{\pi s}{2}\right)\, x^{-s} ds$$
$$= \frac{\pi}{2\sqrt{x}}\left(Y_1\left(4\pi\sqrt{x}\right) + \frac{2}{\pi}\, K_1\left(4\pi\sqrt{x}\right)\right). \tag{3.60}$$

Now we are ready to prove the representation (3.4). Using the definition of $g^{*}(s)$ given above and Parseval equality for the Mellin transform, we obtain

$$g(x) = \frac{1}{2\pi i}\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Gamma^2(s)\,\cos^2\left(\frac{\pi s}{2}\right)}{1-s}\,(1-s)\,f^{*}(1-s)\,x^{-s} ds$$
$$= -\int\limits_0^{\infty} \mathfrak{F}_1(xy)\,y\,f'(y)\,dy, \tag{3.61}$$

where

$$\mathfrak{F}_1(x) = \frac{1}{2\pi i}\operatorname*{l.i.m.}_{N\to\infty}\int\limits_{\frac{1}{2}-iN}^{\frac{1}{2}+iN} \frac{\Gamma^2(s)\,\cos^2\left(\frac{\pi s}{2}\right)}{1-s}\, x^{-s} ds$$
$$= \frac{1}{2\pi i}\operatorname*{l.i.m.}_{N\to\infty}\int\limits_{\frac{1}{2}-iN}^{\frac{1}{2}+iN} \Gamma(s)\Gamma\left(s-1\right)\,\cos^2\left(\frac{\pi s}{2}\right)\, x^{-s} ds$$
$$= \frac{\pi}{2\sqrt{x}}\left(Y_1\left(2\sqrt{x}\right) + \frac{2}{\pi}\, K_1\left(2\sqrt{x}\right)\right), \tag{3.62}$$

which implies (3.4).

Now, for the case where χ is odd, we need to find the Mellin inverse of $h^{*}(s) = \frac{1}{2}\,\sin(\pi s)\,\Gamma^2(s)\,f^{*}(1-s)$, i.e., to compute

$$h(x) = \frac{1}{2\pi i} \int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{1}{2}\,\sin(\pi s)\,\Gamma^2(s)\,f^*(1-s)\,x^{-s}ds$$
$$= \frac{1}{2\pi i} \int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\pi\,\Gamma(s)}{2\,\Gamma(1-s)\,(1-s)}\,(1-s)\,f^*(1-s)\,x^{-s}ds$$
$$= -\int\limits_0^\infty \mathfrak{F}_2(xy)\,y\,f'(y)\,dy, \tag{3.63}$$

where

$$\mathfrak{F}_2(x) = \frac{1}{2\pi i}\,\underset{N\to\infty}{\text{l.i.m.}} \int\limits_{\frac{1}{2}-iN}^{\frac{1}{2}+iN} \frac{\pi\Gamma(s)}{2\,\Gamma(1-s)}\,x^{-s}ds. \tag{3.64}$$

Taking $s=\sigma+it$, we can easily see from Stirling's formula that the ratio of $\Gamma-$functions behaves as $O\left(|t|^{2\sigma-1}\right)$ and so $\frac{\Gamma(s)}{\Gamma(1-s)\,(1-s)} \in L_1(\sigma)$ for $\sigma<\frac{1}{2}$. Thus, for $\sigma<\frac{1}{2}$ and $x>0$, by computing the residues of $\Gamma(s)$ at $s=-k$, $k\in\mathbb{N}_0$, we are able to obtain

$$\mathfrak{F}_2(x) = \frac{1}{2\pi i} \int\limits_{\sigma-i\infty}^{\sigma+i\infty} \frac{\pi\,\Gamma(s)}{2\,\Gamma(1-s)\,(1-s)}\,x^{-s}ds$$
$$= \frac{\pi}{2}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!^2(k+1)}\,x^k = \frac{\pi}{2\sqrt{x}}J_1(2\sqrt{x}), \tag{3.65}$$

where in the last equality we have used the well-known series representation for the Bessel function of the first kind (you can consult equation (2.121) on the second chapter).

Note that we could also see that the integrand in (3.65) is the Mellin transform of $\frac{\pi}{2\sqrt{x}}J_1(2\sqrt{x})$ by invoking (3.53).

Changing the line of integration to $\sigma=\frac{1}{2}$ and using (3.64) and (3.65), we arrive at (3.5). ■

Remark 3.1.: As in the previous chapter, we stress that $g(x)$ and $h(x)$ can be expressed as integral transforms for the L_2-class of functions. We will invoke Watson's theory, exposed in [102].

We start with the transform $g(x)$: from elementary properties of the derivatives of the Bessel functions

$$\frac{d}{dx}\left(x^\nu K_\nu(x)\right) = -x^\nu K_{\nu-1}(x), \quad \frac{d}{dx}\left(x^\nu Y_\nu(x)\right) = x^\nu Y_{\nu-1}(x), \tag{3.66}$$

one can immediately deduce that

$$g(x) = \frac{d}{dx} \int_0^\infty \frac{\chi_1(xy)}{y}\, f(y)\, dy, \tag{3.67}$$

where

$$\chi_1(x) = \int_0^x K_0(2\sqrt{u}) - \frac{\pi}{2}\, Y_0(2\sqrt{u})\, du.$$

It follows from the Watson theory for L_2-integral transforms [102] that we can express $g(x)$, almost-everywhere, as the L_2-integral transform

$$g(x) \overset{\mu}{=} \underset{N\to\infty}{\text{l.i.m.}} \int_{1/N}^N \left(K_0\left(2\sqrt{xy}\right) - \frac{\pi}{2}\, Y_0\left(2\sqrt{xy}\right)\right) f(y)\, dy. \tag{3.68}$$

Moreover, the integral transform (3.67) is given by a Fourier-Watson kernel, i.e., it has a reciprocal relation with its transform. Using (3.49), note that

$$G(x) = \frac{d}{dx} \int_0^\infty \frac{\chi_1(xy)}{y}\, g(y)\, dy$$

admits a Mellin transform given by

$$G^*(s) = \cos^2\left(\frac{\pi s}{2}\right)\, \Gamma^2(s)\, g^*(1-s) = \frac{\pi^2}{4}\, f^*(s).$$

And so we have conversely

$$f(x) = \frac{4}{\pi^2} \frac{d}{dx} \int_0^\infty \frac{\chi_1(xy)}{y}\, g(y)\, dy. \tag{3.69}$$

Furthermore, from the also elementary relation,

$$\frac{d}{dx}\left(x^\nu J_\nu(x)\right) = x^\nu J_{\nu-1}(x), \tag{3.70}$$

we deduce

$$h(x) = \frac{d}{dx} \int_0^\infty \frac{\chi_2(xy)}{y}\, f(y)\, dy,$$

where

$$\chi_2(x) = \frac{\pi}{2} \int_0^x J_0\left(2\sqrt{u}\right)\, du,$$

which allows to write $h(x)$ as the Hankel-type transform

$$h(x) \stackrel{\mu}{=} \underset{N\to\infty}{\text{l.i.m.}} \int_{1/N}^{N} \frac{\pi}{2}\, J_0\left(2\sqrt{xy}\right)\, f(y)\, dy. \tag{3.71}$$

It is also clear that if

$$H(x) = \frac{d}{dx} \int_0^{\infty} \frac{\chi_2(xy)}{y}\, h(y)\, dy,$$

then the Mellin transform of $H(x)$ can be given by

$$H^*(s) = \frac{\pi^2}{4}\, f^*(s),$$

and so we have reciprocally

$$f(x) = \frac{4}{\pi^2} \int_0^{\infty} \frac{\chi_2(xy)}{y}\, h(y)\, dy. \tag{3.72}$$

This concludes the main theoretical aspects of this chapter. In what follows, we shall describe how the previous computations can be adapted for the case where, instead of the product $\zeta(s)\, L(s,\chi)$, we have $\zeta^2(s)$ as main Dirichlet series.

The Classical Voronoï's Formula: the $\zeta-$function case

If we deal with the Dirichlet series for $\zeta^2(s)$ instead of the product $\zeta(s)\, L(s,\chi)$ we face a different type of computation as $\zeta^2(s)$ has a double pole located at $s = 1$ while, in the case of the product $\zeta(s)\, L(s,\chi)$, this pole is a simple one.

We describe the main differences in the proof of the classical Voronoï's summation formula, skipping the main details as they are completely analogous.

1. First, instead of $\Delta_{\chi}(x)$, we need to have the remainder term $\Delta(x) = \sum_{n\leq x}{}' d(n) - x\left(\log(x) + 2\gamma - 1\right)$, where the additive term comes from straightforward computation of the residues of $\zeta^2(s)\, x^s/s$ at $s = 1$. Furthermore, from Dirichlet's estimate for the divisor problem ([103] p. 312), we also have $\Delta(x) = O(x^{1/2})$ as $x \to \infty$.

2. An analogue of Lemma 3.2. holds in the same circumstances, i.e., if $f(x)$ satisfies the conditions imposed by its statement, then the following representation is valid

$$\frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \zeta^2(s)\, f^*(s)\, ds = \lim_{N\to\infty} \left[\sum_{n=1}^{N} d(n)\, f(n) - \int_{1/N}^{N} \left(\log(x) + 2\gamma\right)\, f(x)\, dx \right]. \tag{3.73}$$

3. The Voronoï summation formula holds, i.e.,

$$\lim_{N\to\infty}\left[\sum_{n=1}^{N} d(n)\, f(n) - \int\limits_{1/N}^{N} (\log(x)+2\gamma)\; f(x)\, dx\right]$$
$$= 4\lim_{N\to\infty}\left[\sum_{n=1}^{N} d(n)\, g(4\pi^2 n) - \int\limits_{1/N}^{N} (\log(x)+2\gamma)\; g(4\pi^2 x)\, dx\right], \tag{3.74}$$

where $g(x)$ is the transform given by (3.4) or, equivalently, by (3.68).

As in the previous chapter, we introduce the summation formulas (3.1), (3.2) and (3.3) for the class of functions of Müntz-type. We extend to Dirichlet characters a result already proved by Yakubovich in [116], Theorem 2.

Corollary 3.2.: Voronoï's summation formula for a Class of Müntz functions Suppose that both $f(x)$ and $x^{-1}\varphi(x^{-1})$ belong to the class $\mathcal{M}_{\alpha,2}$ for $\alpha > 2$, (where $\varphi(x)$ is the L_1 cosine transform of f given by (3.38)).

The the formulas hold

$$\sum_{n=1}^{\infty} d_\chi(n)\, f(n) - L(1,\chi)\int\limits_0^{\infty} f(x)\, dx = \frac{4G(\chi)}{\ell}\sum_{n=1}^{\infty} d_{\overline{\chi}}(n)\, g\left(\frac{4\pi^2}{\ell} n\right), \quad \chi \text{ even}, \tag{3.75}$$

$$\sum_{n=1}^{\infty} d(n)\, f(n) - \int\limits_0^{\infty} f(x)\; (\log(x)+2\gamma)\; dx = \frac{f(0)}{4} + 4\sum_{n=1}^{\infty} d(n)\, g\left(4\pi^2 n\right), \tag{3.76}$$

where $g(x)$ is the L_1-transform given by

$$g(x) = \int\limits_0^{\infty}\left(K_0\left(2\sqrt{xy}\right) - \frac{\pi}{2}\, Y_0\left(2\sqrt{xy}\right)\right)\; f(y)\, dy. \tag{3.77}$$

Moreover, if $f(x)$ and $x^{-1}\,\psi(x^{-1})$ also belong to $\mathcal{M}_{\alpha,2}$, $\alpha > 2$ (ψ is the L_1-sine transform of f given in (3.40)), the following summation formula holds

$$\sum_{n=1}^{\infty} d_\chi(n)\, f(n) - L(1,\chi)\int\limits_0^{\infty} f(x)\, dx$$
$$= \frac{iG(\chi)}{2\pi}\, L(1,\overline{\chi})\, f(0) - \frac{4iG(\chi)}{\ell}\sum_{n=1}^{\infty} d_{\overline{\chi}}(n)\, h\left(\frac{4\pi^2}{\ell} n\right), \quad \chi \text{ odd}, \tag{3.78}$$

where $h(x)$ is the Hankel-type transform for the L_1-class given by

$$h(x) = \frac{\pi}{2}\int\limits_0^{\infty} J_0\left(2\sqrt{xy}\right)\; f(y)\, dy. \tag{3.79}$$

Proof: Let $f(x) \in \mathcal{M}_{\alpha,2}$. By Lemma 2.2. given in the second chapter, for $-2 < \sigma < \alpha$, $f^*(\sigma + it) = O\left(|t|^{-2}\right)$ as $|t| \to \infty$. Thus, $s\, f^*(s) \in L_2\left(\frac{1}{2} - i\infty,\, \frac{1}{2} + i\infty\right)$ and $f^*(\sigma + it) = O\left(|t|^{-2}\right)$, for every σ in the interval $\left[\frac{1}{2} - \delta,\, \frac{1}{2} + \delta\right]$, $0 < \delta < \frac{1}{2}$. This shows that f satisfies the conditions of Lemma 3.2..

Moreover, since $f(x) \in \mathcal{M}_{\alpha,2}$ and $\alpha > 2$, we have $f(x) = O(x^{-\alpha})$ as $x \to \infty$, so that the series and integral at the left-hand side of (3.75), (3.76) and (3.78) converge.

From the fact that $f \in L_1(\mathbb{R}_+) \cap L_2(\mathbb{R}_+)$, we can write (3.4) as a transform for the class of absolutely integrable functions in $\mathbb{R}_+$, which is achieved after employing an integration by parts and using the relation (3.66),

$$g(x) = \int_0^\infty \left(K_0\left(2\sqrt{xy}\right) - \frac{\pi}{2}\, Y_0\left(2\sqrt{xy}\right)\right) f(y)\, dy.$$

Since $x^{-1}\,\varphi(x^{-1}) \in \mathcal{M}_{\alpha,2} \subset L_1(\mathbb{R}_+) \cap L_2(\mathbb{R}_+)$, we obtain from (3.39)

$$\begin{aligned} g(x) &= -\frac{1}{x} \int_0^\infty \left(y^{-1}\varphi\left(y^{-1}\right)\right)' \sin(xy)\, dy \\ &= -\frac{1}{x^2}\left[\left(y^{-1}\,\varphi\left(y^{-1}\right)\right)'(0) + \int_0^\infty \left(y^{-1}\,\varphi\left(y^{-1}\right)\right)'' \cos(xy)\, dy\right] \\ &= O\left(\frac{1}{x^2}\right), \quad x \to \infty, \end{aligned}$$

which proves that the series and integrals in the right-hand side of (3.75) and (3.76) converge and so we can write Voronoï's summation formula as

$$\begin{aligned} &\sum_{n=1}^\infty d_\chi(n)\, f(n) - L(1,\chi) \int_0^\infty f(x)\, dx \\ &= \frac{4G(\chi)}{\ell}\left[\sum_{n=1}^\infty d_{\overline{\chi}}(n)\, g\left(\frac{4\pi^2}{\ell} n\right) - \frac{\ell}{4\pi^2}\, L(1,\overline{\chi})\, g^*(1)\right]. \end{aligned} \tag{3.80}$$

Since

$$g^*(s) = \cos^2\left(\frac{\pi s}{2}\right) \Gamma^2(s)\, f^*(1-s),$$

we see that $s = 1$ is a simple zero of $g^*(s)$ (recall that the pole of $f^*(1-s)$ at $s = 1$ is simple), which proves (3.75). To prove (3.76), note that equation (3.74) implies that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} d(n)\, f(n) - \int_0^{\infty} (\log(x) + 2\gamma)\; f(x)\, dx \\
&= 4 \sum_{n=1}^{\infty} d(n)\, g(4\pi^2 n) - \frac{1}{\pi^2} \left[\frac{d}{ds}\, g^*(s)\right]_{s=1},
\end{aligned} \tag{3.81}
$$

and the derivative given above is simple to be evaluated, giving the value $-\frac{\pi^2}{4} f(0)$.

Finally, to deal with the case where χ is odd, recall that

$$
h^*(s) = \frac{\pi\, \Gamma(s)}{2\, \Gamma(1-s)}\, f^*(1-s),
$$

which proves $h^*(1) = \frac{\pi}{2}\, f(0)$ and making a similar reasoning as above we prove (3.78). ■

Actually, Voronoï's summation formula for the odd character case (see (3.2) and (3.78)) allows to prove another famous formula, sometimes called 'Hardy-Landau formula' or 'Sierpiński's formula' ([83, 93]). In the next chapter we will study a generalized version of it, related with the behavior of Epstein's $\zeta-$function. But for now we prove this elegant particular case.

Corollary 3.3. (Hardy-Landau summation formula): Let $f(x)$ be a function satisfying the conditions of Theorem 3.1.

If $r_2(n)$ represents the number of ways in which n can be expressed in the form $n = a^2 + b^2$, $a,\, b \in \mathbb{Z}$ (see our glossary of arithmetic functions or, alternatively, [60]), the following summation formula holds

$$
\lim_{N\to\infty} \left(\sum_{n=1}^{N} r_2(n)\, f(n) - \pi \int_{1/N}^{N} f(x)\, dx\right) = 2 \lim_{N\to\infty} \left(\sum_{n=1}^{N} r_2(n)\, h\left(\pi^2 n\right) - \pi \int_{1/N}^{N} h\left(\pi^2 x\right)\, dx\right), \tag{3.82}
$$

where $h(x)$ denotes the Hankel-type transform in the class of L_2 functions given in (3.5).

Moreover, if we assume that $f(x)$ and $x^{-1}\,\psi\left(x^{-1}\right)$ belong to $\mathcal{M}_{\alpha,2}$, $\alpha > 2$, we can also prove

$$
f(0) + \sum_{n=1}^{\infty} r_2(n)\, f(n) = \pi \int_0^{\infty} f(x)\, dx + 2 \sum_{n=1}^{\infty} r_2(n)\, h\left(\pi^2 n\right). \tag{3.83}
$$

Proof: The proof follows from Jacobi's two-square theorem. This theorem [60] states that the number of ways of expressing a given integer n in the form $a^2 + b^2$, $(a,\, b) \in \mathbb{Z}^2$, is equal to

$$
r_2(n) = 4\left(d_{1,4}(n) - d_{3,4}(n)\right), \tag{3.84}
$$

where $d_{1,4}(n) - d_{3,4}(n)$ corresponds to the weighted divisor function $d_{\chi_4}(n)$ (see Example 3.A. given at the beginning of this chapter).

Applying Voronoï's summation formula for odd characters with the particular values $\ell = 4$ and $G(\chi_4) = 2i$, we just need to find $L(1, \chi_4)$ in order to complete the proof of (3.82). But this is quite elementary, since

$$L(1, \chi_4) = \sum_{n=1}^{\infty} \frac{\chi_4(n)}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)} = \frac{\pi}{4},$$

by the well-known Gregory-Leibniz formula for π. Substituting these values in the summation formulas (3.2) and (3.78), we obtain the desired results and the proofs are complete. ■

3.3 Examples:

In what follows, we shall introduce a set of examples yielding new identities which involve special functions. The most important of these are identities of Koshliakov and Soni type, which will be extremely important in the fifth chapter to derive the positivity of $L(1, \chi)$, when χ is nonprincipal and primitive.

Example 3.1 (Koshliakov formula and its character analogues):

In 1928, Koshliakov proved the summation formula [71]

$$\begin{aligned} &\sum_{n=1}^{\infty} d(n)\, K_0\left(2\pi n z\right) - \frac{1}{z} \sum_{n=1}^{\infty} d(n)\, K_0\left(\frac{2\pi n}{z}\right) \\ &= \frac{1}{4z}\left(\gamma - \log\left(4\pi z\right)\right) - \frac{1}{4}\left(\gamma - \log\left(\frac{4\pi}{z}\right)\right), \quad z > 0 \end{aligned} \tag{3.85}$$

using Voronoï's summation formula for a different class of functions [32].

A. L. Dixon and W. L. Ferrar also proved (3.85) using a similar approach and the joint work of F. Oberhettinger and K. L. Soni established a generalization of (3.85).

Later, Soni [93] derived identities equivalent to (3.85) and therefore equivalent to the functional equation for $\zeta(s)$. Nasim also proved (3.85) under similar assumptions as ours.

Berndt, Dixit and Sohn proved that (3.85) could also be generalized to a version having characters [30]. But their proof is very particular and invokes computations regarding the derivatives of the Dirichlet $L-$function, which we have computed at the first chapter. Avoiding their computations, we prove directly (3.85) and its character extensions based on equalities (3.42) and (3.43) and the inverse Mellin transform, following a reasoning similar to Yakubovich's [112].

Consider, for some parameter $z > 0$, the function $f(x) = K_0\left(\frac{2\pi z}{\sqrt{\ell}}\, x\right)$: applying the well-known integral representation

$$\int_0^\infty x^{s-1}\, K_\nu(cx)\, dx = 2^{s-2}c^{-s}\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right), \quad \mathrm{Re}(c) > 0,\, \mathrm{Re}(s) > |\mathrm{Re}(\nu)|,$$

we deduce immediately that, for $\mathrm{Re}(s) > 0$,

$$f^{*}(s) = \frac{1}{4}\left(\frac{\pi z}{\sqrt{\ell}}\right)^{-s}\Gamma^2\left(\frac{s}{2}\right). \tag{3.86}$$

This function clearly satisfies the conditions of Lemma 3.2., due to Stirling's formula and the fact that it is analytic in the half-plane $\mathrm{Re}(s) > 0$. Therefore, we are ready to apply Theorem 3.1: from straightforward calculations

$$g^{*}(s) = \cos^2\left(\frac{\pi s}{2}\right)\Gamma^2(s)\, f^{*}(1-s) = \frac{\sqrt{\ell}}{4\,z}\,\frac{1}{4}\left(\frac{4\pi z}{\sqrt{\ell}}\right)^{s}\Gamma^2\left(\frac{s}{2}\right)$$

we obtain immediately

$$g(x) = \frac{\sqrt{\ell}}{4z}\, K_0\left(\frac{\sqrt{\ell}}{2\pi z}\, x\right). \tag{3.87}$$

The asymptotic estimate $K_0(x) \sim \left(\frac{\pi}{2x}\right)^{1/2} e^{-x}$ [106] tells us that the series and the integrals on both sides of the summation formula (3.1) converge absolutely, proving the formula

$$\begin{aligned}&\frac{G(\chi)}{4}\, L(1,\overline{\chi}) + \sum_{n=1}^{\infty} d_{\chi}(n)\, K_0\left(\frac{2\pi n z}{\sqrt{\ell}}\right)\\ &= \frac{\sqrt{\ell}}{4z} L(1,\chi) + \frac{G(\chi)}{z\sqrt{\ell}}\sum_{n=1}^{\infty} d_{\overline{\chi}}(n)\, K_0\left(\frac{2\pi n}{z\sqrt{\ell}}\right).\end{aligned} \tag{3.88}$$

Formula (3.88) appeared for the first time in [30]. However, the authors did not observe that (3.88) can be used to establish a character analogue of Soni's formula. Of course, we may obtain (3.85) by similar considerations and computations so we shall skip this derivation.

To prove a character analogue of Soni's formula, take $f(z) = z\, K_0\left(\frac{2\pi a z}{\sqrt{\ell}}\right)$, $a \in \mathbb{R}_+ \setminus \mathbb{N}$, multiply both sides of (3.88) by $f(z)$ and integrate over $\mathbb{R}_+$ in the variable z.

Of course, due to the exponential decay of $K_0(x)$ when $x \to \infty$ we can interchange the orders of integration and summation during this process.

For the first summand in (3.88), we obtain an integral of the form [44], vol.2.

$$\int_0^\infty z\, K_0\left(\frac{2\pi a z}{\sqrt{\ell}}\right) K_0\left(\frac{2\pi n z}{\sqrt{\ell}}\right) dz = \frac{\ell}{4\pi^2} \frac{\log(a/n)}{a^2-n^2}, \tag{3.89}$$

and for the second

$$\int_0^\infty K_0\left(\frac{2\pi a z}{\sqrt{\ell}}\right) K_0\left(\frac{2\pi n}{z\sqrt{\ell}}\right) dz = \frac{\sqrt{\ell}}{2a} K_0\left(4\pi\sqrt{\frac{a}{\ell}\,n}\right). \tag{3.90}$$

Using (3.89) and (3.90) in (3.88) and interchanging the orders of summation and integration, we arrive to

$$\begin{aligned} &\frac{G(\chi)\ell}{16\pi^2 a^2} L(1,\overline{\chi}) + \frac{\ell}{4\pi^2}\sum_{n=1}^{\infty} \frac{d_\chi(n)\,\log(a/n)}{a^2-n^2} \\ &= \frac{\ell}{16a} L(1,\chi) + \frac{G(\chi)}{2a}\sum_{n=1}^{\infty} d_{\overline{\chi}}(n)\, K_0\left(4\pi\sqrt{\frac{a}{\ell}\,n}\right), \quad \chi \text{ even}, \end{aligned} \tag{3.91}$$

which is a character analogue of Soni's formula [93]. Of course, using the same method for (3.85), we can derive Soni's classical formula

$$\begin{aligned} &2\sum_{n=1}^{\infty} d(n)\, K_0\left(4\pi\sqrt{an}\right) + \frac{\gamma}{2} + \frac{\log(a)}{4}\left(1+\frac{1}{\pi^2 a}\right) \\ &= \frac{a}{\pi^2}\sum_{n=1}^{\infty}\frac{d(n)\,\log(a/n)}{a^2-n^2} - \frac{\log(2\pi)}{2\pi^2 a}. \end{aligned} \tag{3.92}$$

Up to now, we have assumed that χ was even. In what follows, we prove a version of (3.88) for odd characters: note that $h^*(s)$ is given by

$$h^*(s) = \frac{\pi\,\Gamma(s)}{2\,\Gamma(1-s)} f^*(1-s) = \frac{\pi^2}{4}\left(\frac{4\pi z}{\sqrt{\ell}}\right)^{s-1} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)\cos\left(\frac{\pi s}{2}\right)},$$

where we have used the classical relations for the $\Gamma-$function (see the first chapter of this thesis). To find the Hankel-type transform $h(x)$, we find the integral

$$h(x) = \frac{\pi\sqrt{\ell}}{16z}\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)\cos\left(\frac{\pi s}{2}\right)}\left(\frac{\sqrt{\ell}x}{4\pi z}\right)^{-s} ds, \tag{3.93}$$

by appealing to Slater's theorem [85] and by computing the residues located at $s=-2n$ and $s=-2n-1$, $n\in\mathbb{N}_0$. After elementary manipulations and writing the integral as a power series, we obtain

$$h(x) = \frac{\pi\sqrt{\ell}}{8z}\left[\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\left(\frac{\sqrt{\ell}x}{4\pi z}\right)^{2n} - \frac{\sqrt{\ell}x}{4\pi z}\sum_{n=0}^{\infty}\frac{1}{\Gamma^2\left(\frac{3}{2}+n\right)}\left(\frac{\sqrt{\ell}x}{4\pi z}\right)^{2n}\right], \tag{3.94}$$

which can be simplified by invoking well-known special functions. Recalling the definitions of the modified Bessel function $I_\nu(z)$ and Struve's function $L_\nu(z)$ ([44] Vol. 2),

$$I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2m}, \tag{3.95}$$

$$L_\nu(x) = \left(\frac{x}{2}\right)^{\nu+1} \sum_{n=0}^{\infty} \frac{1}{\Gamma\left(\frac{3}{2}+n\right)\Gamma\left(\frac{3}{2}+n+\nu\right)} \left(\frac{x}{2}\right)^{2m}, \tag{3.96}$$

we immediately deduce that $h(x)$ is given by

$$h(x) = \frac{\pi\sqrt{\ell}}{8z}\left[I_0\left(\frac{\sqrt{\ell}x}{2\pi z}\right) - L_0\left(\frac{\sqrt{\ell}x}{2\pi z}\right)\right] = -\frac{\pi\sqrt{\ell}}{8z} M_0\left(\frac{\sqrt{\ell}x}{2\pi z}\right), \tag{3.97}$$

where $M_\nu(x)$ denotes the modified Struve function of the second kind.

Also, from the integral representation for the modified Struve function [44, 106]

$$L_\nu(x) = I_{-\nu}(x) - \frac{2}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{x}{2}\right)^\nu \int_0^\infty \frac{\sin(xy)}{(1+y^2)^{\nu+\frac{1}{2}}}\,dy,$$

we can deduce that, from the behavior of the above integral,

$$I_{-\nu}(x) - L_\nu(x) = \frac{x^{\nu-1}}{2^{\nu-1}\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)} + O\left(x^{\nu-3}\right), \quad x \to \infty,$$

which gives the asymptotic estimate for $M_0(x)$

$$M_0(x) \sim -\frac{2}{\pi x} + O\left(\frac{1}{x^3}\right), \quad x \to \infty. \tag{3.98}$$

From (3.98), we see that $h(x) \in L_2(\mathbb{R}_+)$ but $h(x) \notin L_1(\mathbb{R}_+)$. Hence, Voronoï's summation formula in this case will be written in the L_2 form stated by (3.2). Precisely, we have the identity

$$\sum_{n=1}^{\infty} d_\chi(n)\,K_0\left(\frac{2\pi n z}{\sqrt{\ell}}\right) - \frac{\sqrt{\ell}}{4z}L(1,\chi)$$
$$= \frac{i\,\pi G(\chi)}{2z\sqrt{\ell}} \lim_{N\to\infty}\left[\sum_{n=1}^{N} d_{\overline{\chi}}(n)\,M_0\left(\frac{2\pi n}{z\sqrt{\ell}}\right) - L(1,\overline{\chi})\int_{1/N}^{N} M_0\left(\frac{2\pi}{z\sqrt{\ell}}x\right)dx\right], \quad \chi \text{ odd}, \tag{3.99}$$

which appears to be novel.

Example 3.2: An Extension of Dixon and Ferrar's formula

One of the interesting aspects of Koshliakov's formula (3.88) when χ is an even character is that it is a symmetric summation formula. With "symmetric" we mean that the same function appears (although with a different argument) in both sides of the infinite series representing the summation formula.

It is simple to construct examples of such functions for the classical Fourier cosine transform: the classical example is the Gaussian, $f(x) = e^{-\pi x^2}$ and the summation formula associated to it is the classical reflection formula for the Jacobi $\psi-$function[2]. However, there are still more examples, such as $f(x) = \text{sech}(x)$, for which the associated summation formula resembles Jacobi's reflection formula for ψ^2 [3].

In the same way, the Bessel function $K_0(x)$ is a "fixed point" of the transform given by (3.4). Using the terminology of Titchmarsh and Hardy [102], we say that $K_0(x)$ is a self-reciprocal function with respect to the transform (3.4).

Therefore, one may regard Koshliakov's formula (3.85) as an analogue of the transformation formula for the classical $\psi-$function, i.e.,

$$\frac{1}{2}+\sum_{n=1}^{\infty} e^{-\pi n^2 z} = \frac{1}{2\sqrt{z}}+\frac{1}{\sqrt{z}}\sum_{n=1}^{\infty} e^{-\frac{\pi n^2}{z}}, \quad z>0. \tag{3.100}$$

We shall present now this result in another direction. It is clear from relation (3.99) of the previous example that $f(x) = K_0(x)$ is not self-reciprocal with respect to the transform h.

However, a simple class of exponential functions is. Indeed, consider, for $z > 0$, $f(x) = e^{-\frac{2\pi z}{\sqrt{\ell}}x}$: then $f^{*}(s) = \left(\frac{2\pi z}{\sqrt{\ell}}\right)^{-s}\Gamma(s)$ and $h^{*}(s)$ can be expressed by

$$h^{*}(s) = \frac{\pi\,\Gamma(s)}{2\,\Gamma(1-s)}\,f^{*}(1-s) = \frac{\pi}{2}\Gamma(s)\left(\frac{2\pi z}{\sqrt{\ell}}\right)^{s-1},$$

providing the transform

$$h(x) = \frac{\sqrt{\ell}}{4z}\,e^{-\frac{\sqrt{\ell}}{2\pi z}x}. \tag{3.101}$$

Applying the summation formula (3.2) to $f(x) = e^{-\frac{2\pi z}{\sqrt{\ell}}x}$, we obtain the symmetric formula

$$\sum_{n=1}^{\infty} d_{\chi}(n)\,e^{-\frac{2\pi n z}{\sqrt{\ell}}}-\frac{iG(\chi)}{2\pi}L(1,\overline{\chi}) = -\frac{iG(\chi)}{\sqrt{\ell}z}\sum_{n=1}^{\infty} d_{\overline{\chi}}(n)\,e^{-\frac{2\pi n}{z\sqrt{\ell}}}+\frac{\sqrt{\ell}}{2\pi z}L(1,\chi), \quad \chi \text{ odd} \tag{3.102}$$

which, in some sense, also resembles Jacobi's formula (3.100). In fact, a particular case of (3.102) implies (3.100). From corollary 3.3., recall the arithmetic function $r_2(n)$ which satisfies

[2] Here, ψ is the "positive version" of the classical $\theta-$function. Although written with the same notation, this function has nothing to do with the sine transform $\psi(x) = \int_0^{\infty} f(y)\,\sin(xy)\,dy$ invoked above, nor with digamma's function invoked at previous chapters.

[3] In fact, one of Jacobi's proofs of the 2-square theorem follows this similarity. Jacobi realised that this method could be extended to prove the 4-square theorem. This was, maybe, the first time when a summation formula played an important role in proving a very important theorem. Perhaps the second time in history that a summation formula was crucial for establishing an important theorem was in Riemann's memoir, on which he proved the functional equation for $\zeta(s)$ via the reflection formula for $\psi(x)$.

$r_2(n) = 4\,d_{\chi_4}(n)$. Taking $\chi = \chi_4$ in (3.102) and replacing $\ell = 4$, $G(\chi_4) = 2i$ and $L(1, \chi_4) = \frac{\pi}{4}$, we obtain

$$1 + \sum_{n=1}^{\infty} r_2(n)\, e^{-\pi n z} = \frac{1}{z} + \frac{1}{z}\sum_{n=1}^{\infty} r_2(n)\, e^{-\frac{\pi n}{z}}. \tag{3.103}$$

However, the arithmetic function $r_2(n)$ only "counts" the integers which can be described as a sum of squares and so both series in (3.103) can be handled to provide

$$\sum_{n=1}^{\infty} r_2(n)\, e^{-\pi n z} = \sum_{(n,m)\in\mathbb{Z}^2\setminus(0,0)} e^{-\pi\left(n^2+m^2\right)z} = 4\,\left(\psi^2(z) + \psi(z)\right),$$

and so (3.103) implies

$$1 + 4\psi^2(z) + 4\psi(z) = \frac{1}{z} + \frac{4}{z}\left(\psi^2\left(\frac{1}{z}\right) + \psi\left(\frac{1}{z}\right)\right),$$

which is equivalent to (3.100), from the positivity of $\psi(z)$.

Returning to (3.102), the fact that $e^{-\frac{2\pi z}{\sqrt{\ell}}x}$ is self-reciprocal sets a question on the existence of a formula of Soni-type for odd characters, similar to (3.91) and (3.92). We shall see that such a formula exists for this case.

To do so, consider the function $f(z) = z^{\nu}\, e^{-\frac{2\pi a z}{\sqrt{\ell}}}$, $a > 0$, $\mathrm{Re}(\nu) > 0$ and multiply (3.102) by $f(z)$ and integrate over $\mathbb{R}_+$ in the variable z. Using the identity,

$$\int_0^{\infty} x^{\nu-1} e^{-\frac{\alpha}{x}} e^{-\beta x} dx = 2\left(\frac{\alpha}{\beta}\right)^{\nu/2} K_{\nu}\left(2\sqrt{\alpha\beta}\right), \quad \mathrm{Re}(\nu) > 0 \tag{3.104}$$

we arrive to a formula of Soni type given as follows

$$\begin{aligned}\Gamma(\nu+1)\ell^{\frac{\nu+1}{2}} a^{\nu+1}\sum_{n=1}^{\infty}\frac{d_{\chi}(n)}{(n+a)^{\nu+1}} - \frac{iG(\chi)}{2\pi}\,\Gamma(\nu+1)\,\ell^{\frac{\nu+1}{2}}\,L(1,\overline{\chi})\\ = -\frac{2iG(\chi)}{\sqrt{\ell}}\,(2\pi)^{\nu+1}a^{\nu/2+1}\sum_{n=1}^{\infty} d_{\overline{\chi}}(n)\, n^{\nu/2} K_{\nu}\left(4\pi\sqrt{\frac{an}{\ell}}\right) + a\ell^{\frac{\nu+1}{2}}\Gamma(\nu)\, L(1,\chi).\end{aligned} \tag{3.105}$$

For example, if we take $\chi = \chi_4$ in (3.105) and use the fact that $r_2(n) = 4\,d_{\chi_4}(n)$ (Jacobi's two-square theorem), we obtain the formula for $r_2(n)$

$$\begin{aligned}\frac{a^{\nu/2}\Gamma(\nu+1)}{2\pi^{\nu+1}}\sum_{n=1}^{\infty}\frac{r_2(n)}{(n+a)^{\nu+1}} + \frac{a^{\nu/2}\Gamma(\nu+1)}{2\pi^{\nu+1}}\\ = \sum_{n=1}^{\infty} r_2(n)\, n^{\frac{\nu}{2}} K_{\nu}\left(2\pi\sqrt{an}\right) + \frac{\Gamma(\nu)}{2\pi^{\nu}a^{\nu/2}},\end{aligned} \tag{3.106}$$

which was given for the first time by Dixon and Ferrar [40], eq. (3.12). Note that, from $\lim_{x\to0} x^{\nu/2} K_{\nu}(2\pi\sqrt{ax}) = \frac{\Gamma(\nu)}{2\pi^{\nu}a^{\nu/2}}$ and the convention $r_2(0)=1$, (3.106) can be written in the compact form

$$\frac{a^{\nu/2}\Gamma(\nu+1)}{2\pi^{\nu+1}}\sum_{n=0}^{\infty}\frac{r_2(n)}{(n+a)^{\nu+1}}=\sum_{n=0}^{\infty}r_2(n)\,n^{\frac{\nu}{2}}\,K_{\nu}\left(2\pi\sqrt{an}\right),\quad \mathrm{Re}(\nu)>0, \tag{3.107}$$

which may also be found in [28, 58]. In the next chapter, we shall generalize (3.107) by dealing with functional properties of Epstein's $\zeta-$function.

As in the fifth example of the second chapter, we can extend (3.105) to the case where $\nu=0$: in fact, taking $f(x)=K_0\left(2\pi\sqrt{ax}\right)$, it is immediate that $f^{*}(s)=\frac{1}{2}\left(\pi\sqrt{a}\right)^{-2s}\,\Gamma^2(s)$ and

$$h(x)=\frac{1}{4\pi a}\,\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{\pi}{\sin(\pi s)}\left(\frac{x}{\pi^2 a}\right)^{-s}ds=\frac{\pi}{4}\,\frac{1}{x+\pi^2 a}.$$

Therefore, an application of Voronoï's summation formula in the L_2 form (3.2) gives

$$\begin{aligned}&\sum_{n=1}^{\infty}d_{\chi}(n)\,K_0\left(2\pi\sqrt{an}\right)-\frac{L(1,\chi)}{2\pi^2 a}\\&=-\frac{iG(\chi)}{4\pi}\lim_{N\to\infty}\left[\sum_{n=1}^{N}\frac{d_{\overline{\chi}}(n)}{n+\frac{a\ell}{4}}-L(1,\overline{\chi})\log(N)\;+L(1,\overline{\chi})\,\log\left(\frac{a\ell}{4}\right)\right],\end{aligned} \tag{3.108}$$

which can be simplified by direct calculations.

Instead of explicitly make the calculations using the limit in (3.108), we use a method similar to the one given at Example 2.5 in the second chapter.

Notice that it is possible to simplify (3.105) by extending its left-hand side to the half-plane $\mathrm{Re}(\nu)>-1$ and that we can write the series at the left-hand side of (3.105) in the form

$$\sum_{n=1}^{\infty}d_{\chi}(n)\left[\frac{1}{(n+a)^{\nu+1}}-\frac{1}{n^{\nu+1}}\right]+\zeta\left(\nu+1\right)\,L\left(\nu+1,\,\chi\right),\quad \mathrm{Re}(\nu)>0, \tag{3.109}$$

which clearly extends it to the half-plane $\mathrm{Re}(\nu)>-1$ by an argument of analytic continuation.

Substituting (3.109) at (3.105) and taking the formal limit $\nu\to0$ as well as replacing χ by $\overline{\chi}$, we are able to derive

$$\begin{aligned}&\sum_{n=1}^{\infty}d_{\chi}(n)\,K_0\left(4\pi\sqrt{\frac{an}{\ell}}\right)+\frac{iG(\chi)}{4\pi}\left[L^{'}(1,\overline{\chi})+\gamma\,L(1,\overline{\chi})+\log(a)\,L(1,\overline{\chi})\right]\\&\qquad=-\frac{iG(\chi)}{4\pi}\sum_{n=1}^{\infty}d_{\overline{\chi}}(n)\left[\frac{1}{n+a}-\frac{1}{n}\right]+\frac{L(1,\chi)\ell}{8\pi^2 a}.\end{aligned} \tag{3.110}$$

Now, using formula (1.154) given at the first chapter for $L^{'}(1,\chi)$,

$$L^{'}(1,\chi)=i\pi\frac{G(\chi)}{\ell}\left[(\gamma+\log(2\pi))\ \frac{i}{\pi}G(\overline{\chi})L(1,\chi)+\sum_{r=1}^{\ell-1}\chi(r)\ \log\Gamma\left(\frac{r}{\ell}\right)\right],$$

we can simplify the computations in (3.110) and prove that

$$\sum_{n=1}^{\infty}d_{\chi}(n)\,K_0\left(4\pi\sqrt{\frac{an}{\ell}}\right)+\frac{1}{4}\sum_{r=1}^{\ell-1}\overline{\chi}(r)\ \log\Gamma\left(\frac{r}{\ell}\right)+\frac{iG(\chi)}{4\pi}\,L(1,\overline{\chi})\ \left[2\gamma+\log\left(2\pi a\right)\right]$$
$$=-\frac{iG(\chi)}{4\pi}\sum_{n=1}^{\infty}d_{\overline{\chi}}(n)\left[\frac{1}{n+a}-\frac{1}{n}\right]+\frac{L(1,\chi)\ell}{8\pi^2 a},\qquad (3.111)$$

which can be seen as an extension of a formula due to Dixon and Ferrar [40].

Note that, if we take $\chi=\chi_4$ and use Jacobi's two-square theorem, we derive the formula,

$$\sum_{n=1}^{\infty}r_2(n)\,K_0\left(2\pi\sqrt{\alpha n}\right)+2\log\Gamma\left(\frac{1}{4}\right)-\log\left(2\right)-\frac{3}{2}\log\left(\pi\right)-\frac{1}{2}\log(\alpha)$$
$$=\frac{1}{2\pi\alpha}+\frac{1}{2\pi}\sum_{n=1}^{\infty}r_2(n)\left[\frac{1}{n+\alpha}-\frac{1}{n}\right]+\gamma,\qquad (3.112)$$

obtained for the first time in [40].

We now move on to study Voronoï's formula for the function $f(x)=e^{-\frac{2\pi z}{\sqrt{\ell}}x}$ when χ is even which, as should be expected, does not possess a symmetric expression of the form given above. From Legendre's duplication formula,

$$g^{*}(s)=\pi\,2^{2s-2}\frac{\Gamma^2\left(\frac{s}{2}\right)}{\Gamma^2\left(\frac{1-s}{2}\right)}\,f^{*}(1-s)=\pi^{\frac{3}{2}}\,2^{s-2}\,\left(\frac{2\pi z}{\sqrt{\ell}}\right)^{s-1}\csc\left(\frac{\pi s}{2}\right)\,\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}.\qquad (3.113)$$

To find $g(x)$, we appeal to Parseval equality for the Mellin transform (although we could use the calculation of Residues, as in the previous example).

Note that

$$g\left(x\right)=\frac{\sqrt{\pi\ell}}{8z}\,\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\,\csc\left(\frac{\pi s}{2}\right)\,\left(\frac{\sqrt{\ell}x}{4\pi z}\right)^{-s}\,ds$$
$$=\frac{\sqrt{\ell}}{4z}\,\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma\left(s\right)\cos\left(\frac{\pi s}{2}\right)\,\sec\left(\frac{\pi}{2}\left(1-s\right)\right)\left(\frac{\sqrt{\ell}x}{2\pi z}\right)^{-s}\,ds,\qquad (3.114)$$

and from relation (2.61) in the previous chapter, (3.114) denotes the Mellin transform of a Fourier cosine transform. From the relation 7.2.(14) in [45] (Vol. 1),

$$\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \sec\left(\frac{\pi s}{2}\right) x^{-s} ds = \frac{2x}{\pi\,(1+x^2)}, \tag{3.115}$$

together with (2.5.9.12) in [85] (Vol. 1), we arrive to

$$\begin{aligned} g(x) &= \frac{\sqrt{\ell}}{2\pi z} \int_0^\infty \frac{y}{1+y^2} \cos\left(\frac{\sqrt{\ell} x}{2\pi z}\, y\right) dy \\ &= -\frac{\sqrt{\ell}}{4\pi z}\left[e^{-\frac{\sqrt{\ell}x}{2\pi z}} \operatorname{Ei}\left(\frac{\sqrt{\ell}x}{2\pi z}\right) + e^{\frac{\sqrt{\ell}x}{2\pi z}} \operatorname{Ei}\left(-\frac{\sqrt{\ell}x}{2\pi z}\right)\right], \end{aligned} \tag{3.116}$$

where $\operatorname{Ei}(z)$ denotes the exponential integral function [44].

Since $f(x), g(x) \in L_1(\mathbb{R}_+) \cap L_2(\mathbb{R}_+)$ and the series presented in (3.1) and (3.3) converge absolutely for such functions, we have also the summation formulas

$$\begin{aligned} &\sum_{n=1}^{\infty} d_{\chi}(n)\, e^{-\frac{2\pi z n}{\sqrt{\ell}}} - \frac{\sqrt{\ell}}{2\pi z} L(1,\chi) \\ &= -\frac{G(\chi)}{\pi z\sqrt{\ell}} \sum_{n=1}^{\infty} d_{\overline{\chi}}(n) \left[e^{-\frac{2\pi n}{z\sqrt{\ell}}} \operatorname{Ei}\left(\frac{2\pi n}{z\sqrt{\ell}}\right) + e^{\frac{2\pi n}{z\sqrt{\ell}}} \operatorname{Ei}\left(-\frac{2\pi n}{z\sqrt{\ell}}\right)\right], \quad \chi \text{ even}, \end{aligned} \tag{3.117}$$

$$\begin{aligned} &\sum_{n=1}^{\infty} d(n)\, e^{-2\pi z n} - \frac{1}{4} + \frac{\log(2\pi z) - \gamma}{2\pi z} \\ &= \frac{1}{\pi z} \sum_{n=1}^{\infty} d(n) \left[e^{-\frac{2\pi n}{z}} \operatorname{Ei}\left(\frac{2\pi n}{z}\right) + e^{\frac{2\pi n}{z}} \operatorname{Ei}\left(-\frac{2\pi n}{z}\right)\right]. \end{aligned} \tag{3.118}$$

Example 3.3:

Consider the function defined on $\mathbb{R}_+$

$$f(x) = \begin{cases} \log\left(\frac{1}{x}\right) & 0 < x \leq 1 \\ 0 & x > 1, \end{cases}$$

with Mellin transform $f^*(s) = \frac{1}{s^2}$ clearly satisfying all the conditions of Theorem 3.1. To find the transform $g(x)$ note that

$$g^*(s) = \frac{\Gamma(s)\,\Gamma(s-1)\,\cos^2\left(\frac{\pi s}{2}\right)}{s-1},$$

and so, using relation (3.57), the even transform $g(x)$ is given by

$$g(x) = -\frac{1}{x}\int_0^x \frac{1}{2\pi i}\ \underset{N\to\infty}{\text{l.i.m.}} \int_{\frac{1}{2}-iN}^{\frac{1}{2}+iN} \Gamma(s-1)\,\Gamma(s)\,\cos^2\left(\frac{\pi s}{2}\right) y^{-s} ds\, dy$$

$$= -\frac{1}{x}\int_0^x \frac{1}{\sqrt{y}}\left(K_1\left(2\sqrt{y}\right) + \frac{\pi}{2}\,Y_1\left(2\sqrt{y}\right)\right) dy = \frac{1}{x}\left(K_0\left(2\sqrt{x}\right) + \frac{\pi}{2} Y_0\left(2\sqrt{x}\right)\right).$$

After substituting $f(y)$ by $f(xy)$, $x > 0$, in (3.1) and (3.3),

$$\sum_{n \le x^{-1}} d_\chi(n)\,\log(nx) + \frac{1}{x}L(1,\chi) - \frac{G(\chi)}{4}L(1,\overline{\chi}) \tag{3.119}$$

$$= -\frac{G(\chi)}{\pi^2}\sum_{n=1}^{\infty}\frac{d_{\overline{\chi}}(n)}{n}\left[K_0\left(4\pi\sqrt{\frac{n}{\ell x}}\right) + \frac{\pi}{2}\,Y_0\left(4\pi\sqrt{\frac{n}{\ell x}}\right)\right], \quad \chi \text{ even},$$

$$\sum_{n \le x^{-1}} d(n)\,\log(nx) - \frac{1}{x}\left(2\left(1-\gamma\right) + \log(x)\right) + \frac{3}{2}\gamma$$

$$= -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{d(n)}{n}\left[K_0\left(4\pi\sqrt{\frac{n}{x}}\right) + \frac{\pi}{2}Y_0\left(4\pi\sqrt{\frac{n}{x}}\right)\right] - \frac{1}{4}\log\left(\frac{x}{16\pi^8}\right). \tag{3.120}$$

When χ is odd,

$$h^*(s) = \frac{\pi\Gamma(s)}{2\,\Gamma(2-s)\,(1-s)},$$

and so $h(x)$ can be expressed as

$$h(x) = \frac{\pi}{2x}\int_0^x \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{\Gamma(s)}{\Gamma(2-s)}\,y^{-s} ds\, dy$$

$$= \frac{\pi}{2x}\int_0^x \frac{J_1\left(2\sqrt{y}\right)}{\sqrt{y}}\,dy = \frac{\pi}{2x}\,\left(1 - J_0\left(2\sqrt{x}\right)\right).$$

From the asymptotic properties of the Bessel function $J_0(z)$ [106], we can see that h belongs to $L_2(\mathbb{R}_+)$ but not to $L_1(\mathbb{R}_+)$. Therefore, we will write Voronoï's summation formula in the form (3.2), providing the identity

$$\sum_{n \le x^{-1}} d_\chi(n)\,\log(nx) + \frac{1}{x}\,L(1,\chi)$$

$$= \frac{iG(\chi)}{2\pi}\lim_{N\to\infty}\left[\sum_{n=1}^{N} d_{\overline{\chi}}(n)\,\frac{1}{n}\left(1 - J_0\left(4\pi\sqrt{\frac{n}{\ell x}}\right)\right) - L(1,\overline{\chi})\int_{1/N}^{N}\frac{1-J_0(2\sqrt{y})}{y}\,dy\right]. \tag{3.121}$$

3.4 Final considerations and Main Theorem:

We end this chapter by discussing how to prove the main theorem stated above, which is related with the estimates used for the remainder functions $\Delta_{\chi}(x)$ and $\Delta(x)$.

The purpose of this final discussion is to comment that, using the Kontorovich-Lebedev transform,

$$K_{i\tau}[f] = \int_0^{\infty} K_{i\tau}(x)\, f(x)\, dx, \tag{3.122}$$

it is possible to prove the following identities (for $x \in \mathbb{R}_+ \setminus \mathbb{N}$)

$$\sum_{n\leq x}{}' d_{\chi}(n) - L(1,\chi)\, x = -\frac{2G(\chi)}{\pi\sqrt{\ell}} \sum_{n=1}^{\infty} d_{\overline{\chi}}(n) \left(\frac{x}{n}\right)^{1/2} \left[K_1\left(4\pi\sqrt{\frac{nx}{\ell}}\right) + \frac{\pi}{2} Y_1\left(4\pi\sqrt{\frac{nx}{\ell}}\right)\right], \tag{3.123}$$

for even χ,

$$\sum_{n\leq x}{}' d_{\chi}(n) - x\, L(1,\chi) + \frac{1}{2}L(0,\chi) = -\frac{iG(\chi)}{\sqrt{\ell}} \sum_{n=1}^{\infty} d_{\overline{\chi}}(n) \left(\frac{x}{n}\right)^{1/2} J_1\left(4\pi\sqrt{\frac{n\,x}{\ell}}\right), \tag{3.124}$$

for odd χ and

$$\sum_{n\leq x}{}' d(n) - x\left(\log(x) + 2\gamma - 1\right) - \frac{1}{4} = -\frac{2}{\pi}\sum_{n=1}^{\infty} d(n) \left(\frac{x}{n}\right)^{1/2} \left(K_1\left(4\pi\sqrt{xn}\right) + \frac{\pi}{2}Y_1\left(4\pi\sqrt{xn}\right)\right). \tag{3.125}$$

Equation (3.125) is the well-known Voronoï's identity for the remainder term in the classical divisor problem [63]. Still, employing our methods and invoking some properties of the kernel $K_{i\tau}(x)$, we can also establish (3.123) and (3.124), expanding the scope of this identity to Dirichlet characters.

In the supplementary document to this chapter [89], we prove (3.123-3.125) under the hypothesis that $x \in \mathbb{R}_+ \setminus \mathbb{N}_0$, which does not affect the asymptotic behavior of both sides of the equations.

It is also worthy of mention that identities (3.123-3.125) cannot be proved under the conditions imposed by Theorem 3.1, so the proof of these will be different from the ones developed above.

These identities are the departure point of the proof of Voronoï's upper bound for the Dirichlet divisor problem [31, 63, 103], which we now state:

Theorem 3.4. (Voronoï's estimate for the error term in the divisor function) Let $\Delta_\chi(x)$ and $\Delta(x)$ be the remainder terms defined by (3.6) and (3.7). Then, as $x \to \infty$, both $\Delta(x)$ and $\Delta_\chi(x)$ obey to the estimate

$$\Delta(x),\, |\Delta_\chi(x)| = O\left(x^{\frac{1}{3}+\epsilon}\right), \quad \epsilon > 0. \tag{3.126}$$

Using (3.126), which is far more precise than the one given in (3.19), we can now prove the stronger version of Voronoï's summation formula given by the Main Theorem.

Proof of the Main Theorem: In formula (3.34) of the Main Lemma, which is one of the main points of our argument, we have deduced that we could commute the integrals due to the convergence of the limit

$$\lim_{N\to\infty} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \int_{1/N}^{N} \left|x^{-s} f^*(s)\, d\Delta_\chi(x)\, ds\right| \leq \sigma\, \|f^*(s)\|_{L_1(\sigma)} \lim_{N\to\infty} \int_{1/N}^{N} x^{-\sigma-1} |\Delta_\chi(x)|\, dx < \infty, \tag{3.127}$$

which was obtained from the fact that $\frac{1}{2} < \sigma < 1$ and $|\Delta_\chi(x)| = O\left(x^{\frac{1}{2}}\right)$ for $x \to \infty$. However, with the new estimate $|\Delta_\chi(x)| = O\left(x^{\frac{1}{3}+\epsilon}\right)$ as $x \to \infty$, we can take $\sigma = \frac{1}{2}$ in (3.127) and still verify that the limit is finite, since $x^{-3/2}|\Delta_\chi(x)| \in L_1(\mathbb{R}_+)$.

Thus, following the same argument as in the Main Lemma, we obtain the representation

$$\frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \zeta(s)\, L(s,\chi)\, f^*(s)\, ds = \lim_{N\to\infty} \left[\sum_{n=1}^{N} d_\chi(n)\, f(n) - L(1,\chi) \int_{1/N}^{N} f(x)\, dx\right], \tag{3.128}$$

which, in this time, holds under the very simple assumption that $s\, f^*(s) \in L_2\left(\frac{1}{2} - i\infty,\, \frac{1}{2} + i\infty\right)$. From (3.128), Voronoï's summation formulas (3.1), (3.2) and (3.3) follow and our proof is complete. ■

Chapter 4

Summation formulas involving Quadratic forms and their character analogues

At the Corollary 3.3. of the previous chapter we have proved that, if $f(x)$ and $x^{-1}\,\psi\left(x^{-1}\right)$ belong to $\mathcal{M}_{\alpha,2}$, $\alpha > 2$, the summation formula takes place

$$f(0) + \sum_{n=1}^{\infty} r_2(n)\, f(n) = \pi \int_0^{\infty} f(x)\, dx + 2\sum_{n=1}^{\infty} r_2(n)\, h\left(\pi^2 n\right), \tag{4.1}$$

where $r_2(n)$ is the arithmetic function counting the number of ways in which one can express n as a sum of two squares (see our glossary), ψ denotes the L_1-sine transform of f and h is the transform given by equation (3.5) in the previous chapter.

The main purpose of this chapter is to extend (4.1) under slightly different conditions.

This extension is two-sided: the first side is, of course, concerned with Dirichlet characters. The other is related to a generalization of the arithmetic function $r_2(n)$ to a wider class.

To generalize $r_2(n)$, we consider binary quadratic forms which are positive definite. If $Q_0 : \mathbb{R}^2 \longmapsto \mathbb{R}$ is the quadratic form $Q_0(x,\,y) = x^2 + y^2$, the arithmetic function $r_2(n)$ counts the number of integer solutions of the equation $Q_0(x,\,y) = n$. So, we can see $r_2(n)$ as a very particular case of the arithmetic function that we now define:

Definition 4.1. Let a, b and c be real numbers such that $a > 0$ and $d = b^2 - 4ac < 0$, so that

$$Q(x,y) = ax^2 + bxy + cy^2 \tag{4.2}$$

is a positive definite binary quadratic form with discriminant d.

Then, for a given $t \in \mathbb{R}_+$, we define $r_Q(t)$ as the number of integer solutions of the equation $Q(x, y) = t$, i.e.,

$$r_Q(t) = \# \left\{ (m, n) \in \mathbb{Z}^2 : \, Q(m, n) = t \right\}. \tag{4.3}$$

The purpose of this chapter is to obtain a summation formula of the type (4.1) whose coefficients are of the form $r_Q(n)$, where Q denotes a positive definite quadratic form with integral coefficients.

To do this, we need to introduce the following consideration: if Q is a positive definite quadratic form over $\mathbb{R}^2$, it is clear that Q admits the representation

$$Q\left(\mathbf{x}\right) = \mathbf{x}^T \, Q \, \mathbf{x},$$

where $\mathbf{x} \in \mathbb{R}^2$ and Q denotes the square matrix

$$Q = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}, \tag{4.4}$$

which is invertible by hypothesis, since $d = -4\det(Q) < 0$. In the sequel, we define the inverse of the quadratic form $Q(\mathbf{x})$ and denote it by $Q^{-1}(\mathbf{x})$ as the following quadratic form

$$Q^{-1}\left(\mathbf{x}\right) = -\frac{d}{4} \, \mathbf{x}^T \, Q^{-1} \, \mathbf{x}, \tag{4.5}$$

where Q^{-1} denotes the inverse matrix of Q and d its discriminant. It is simple to check that, if Q is described by (4.2), then

$$Q^{-1}(x, y) = cx^2 - bxy + ay^2, \tag{4.6}$$

which defines Q^{-1} as positive definite as well.

After establishing the main obstacles concerning the notation at this section, we introduce the definition of Epstein's $\zeta-$function. [21]

Definition 4.2.: Let Q be a two-dimensional (said binary) real and positive definite quadratic form. We define the Epstein $\zeta-$function, when $\text{Re}(s) > 1$, by the following series over $\mathbb{Z}^2 \setminus \{(0,0)\}$,

$$Z_2(s, \, Q) = \sum_{(m,n) \neq (0,0)} \frac{1}{Q(m, n)^s}, \tag{4.7}$$

where Q is a quadratic form given by $Q(m, n) = am^2 + bmn + cn^2$ having discriminant $d = b^2 - 4ac < 0$.

In this notation for the $\zeta-$function (4.7), "2" denotes the dimension of the Quadratic form, s is the complex argument and Q stands for the Quadratic form itself. Note that we can write it as the Dirichlet series

$$Z_2\left(s,\, Q\right)=\sum_{\lambda_n>0}\frac{r_Q(\lambda_n)}{\lambda_n^s}, \tag{4.8}$$

where the sequence λ_n is the image of $\mathbb{Z}^2\setminus\{(0,0)\}$ by the application of Q. Naturally, if a, b, c are integers, $\lambda_n\in\mathbb{N}$ for every $n\in\mathbb{N}$ and we can write

$$Z_2(s,\, Q)=\sum_{n=1}^{\infty}\frac{r_Q(n)}{n^s}. \tag{4.9}$$

We remark also that the series above converges absolutely for $\mathrm{Re}(s)>1$, in a similar fashion to Riemann's $\zeta-$function (recall Chapter 1). It is simple to check that, for all $(x,y)\in\mathbb{R}^2$, $Q(x,y)\geq\lambda(x^2+y^2)$ with

$$\lambda=\frac{1}{2}\left(a+c-\sqrt{(a-c)^2+b^2}\right)>0,$$

and so, by comparison, we see that the series (4.7) converges absolutely for $\sigma>1$ and, by Weierstrass's test, uniformly in $\sigma\geq1+\epsilon$, for any $\epsilon>0$.

We will study the analytic continuation of the function defined by (4.7) and its functional equation. To understand how we obtain the analytic continuation of (4.7), it will be convenient to look at the following example.

Example 4.A.: Sum of squares function

The most trivial example of a quadratic form defined above is obtained when we take $a=1$, $b=0$ and $c=1$. In this case, $Q=Q_0(m,n):=m^2+n^2$ and the Epstein's $\zeta-$function associated to Q_0 is given by the series

$$Z_2(s,\, Q_0)=\sum_{(m,n)\neq(0,0)}\frac{1}{(m^2+n^2)^s}=\sum_{n=1}^{\infty}\frac{r_2(n)}{n^s},\quad \mathrm{Re}(s)>1. \tag{4.10}$$

At the corollary 3.3. on the previous chapter, we have invoked Jacobi's two-square theorem to deduce a summation formula similar to Hardy-Landau's (4.1). We can use this identity one more time: since $r_2(n)=4\,d_{\chi_4}(n)$, we see that, for $\mathrm{Re}(s)>1$,

$$Z_2(s,\, Q_0)=\sum_{n=1}^{\infty}\frac{r_2(n)}{n^s}=4\sum_{n=1}^{\infty}\frac{d_{\chi_4}(n)}{n^s}=4\,\zeta(s)\,L(s,\,\chi_4). \tag{4.11}$$

Therefore, the analytic continuation of $Z_2(s,\, Q_0)$ follows from the analytic continuation of the product $\zeta(s)\,L(s,\chi_4)$: for example, a functional equation may be derived for $Z_2(s,\, Q_0)$ by applying the functional equation for both $\zeta(s)$ and $L(s,\chi_4)$, which was previously done (see formula (3.47) in the third chapter). It is immediate to obtain the functional equation

$$\pi^{-s}\Gamma(s)\,Z_2(s,\,Q_0)=\pi^{s-1}\Gamma(1-s)\,Z_2(1-s,\,Q_0), \tag{4.12}$$

which yields an extension of $Z_2(s,\,Q_0)$ to the complex plane. Since $\zeta(s)\,L(s,\chi_4)$ has a simple pole located at $s=1$, we can also see that $Z_2(s,\,Q_0)$ has a simple pole at $s=1$ with residue given by $4\,L(1,\chi_4)=\pi$.

In the sequel, we shall denote $Z_2(s,\,Q_0)$ by $\zeta_2(s)$ when this notation is more convenient (the reader can consult our glossary).

Consider also the Epstein $\zeta-$function for the Quadratic form $Q_1(m,n)=m^2+mn+n^2$, which is clearly positive definite. From a formula attributed to Liouville [108] it is known that

$$r_{Q_1}(n)=6\,d_{\chi_3}(n)=6\left(d_{1,3}(n)-d_{2,3}(n)\right). \tag{4.13}$$

Therefore, we see that (4.13) implies

$$Z_2\left(s,\,Q_1\right)=6\sum_{n=1}^{\infty}\frac{d_{\chi_3}(n)}{n^s}=6\,\zeta(s)\,L\left(s,\chi_3\right) \tag{4.14}$$

from which we can analogously prove the functional equation

$$\left(\frac{2\pi}{\sqrt{3}}\right)^{-s}\Gamma(s)\,Z_2(s,\,Q_1)=\left(\frac{2\pi}{\sqrt{3}}\right)^{s-1}\Gamma(1-s)\,Z_2(1-s,\,Q_1). \tag{4.15}$$

Now, a summation formula of the type (4.1) can also be deduced, for f in the Müntz class[1] and for the quadratic form Q_1: since χ_3 is an odd and primitive character and $r_{Q_1}(n)=6\,d_{\chi_3}(n)$, we apply Voronoï's summation formula for odd characters (see equation (3.78) in the previous chapter) with the particular values $\ell=3$ and $G(\chi_3)=\sqrt{3}\,i$ and we arrive to

$$\sum_{n=1}^{\infty}r_{Q_1}(n)\,f(n)-\frac{2\pi}{\sqrt{3}}\int_0^{\infty}f(x)\,dx$$
$$=\frac{4\sqrt{3}}{3}\sum_{n=1}^{\infty}r_{Q_1}(n)\,h\left(\frac{4\pi^2}{3}n\right)-\frac{3\sqrt{3}}{\pi}\,L(1,\overline{\chi}_3)\,f(0). \tag{4.16}$$

Finally, to obtain a closed form of (4.16), we just need to find $L(1,\chi_3)$: we can compute this value by appealing to the elementary relations

[1]We could deduce the same type of result even for $f(x)$ is the L_2 class by adapting the argument given in the third chapter.

$$
\begin{aligned}
L(1,\chi_3) &= \sum_{n=0}^{\infty}\left(\frac{1}{3n+1}-\frac{1}{3n+2}\right) = \sum_{n=0}^{\infty}\int_0^1 \left(x^{3n}-x^{3n+1}\right)dx \\
&= \int_0^1 (1-x)\sum_{n=0}^{\infty} x^{3n}\,dx = \int_0^1 \frac{1-x}{1-x^3}\,dx = \int_0^1 \frac{dx}{1+x+x^2} \\
&= \frac{2}{\sqrt{3}}\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{dx}{1+x^2} = \frac{\pi}{3\sqrt{3}},
\end{aligned}
$$

from which we deduce the summation formula

$$
f(0)+\sum_{n=1}^{\infty} r_{Q_1}(n)\, f(n) = \frac{2\pi}{\sqrt{3}}\int_0^{\infty} f(x)\,dx + \frac{4\sqrt{3}}{3}\sum_{n=1}^{\infty} r_{Q_1}(n)\, h\left(\frac{4\pi^2}{3}n\right), \tag{4.17}
$$

where $h(x)$ is the transform obtained in the previous chapter,

$$
h(x) = \frac{\pi}{2}\int_0^{\infty} J_0\left(2\sqrt{xy}\right)\, f(y)\,dy. \tag{4.18}
$$

Hence, we can conclude that (4.1) is true for, at least, two quadratic forms which are positive definite: Q_0 and Q_1.

Indeed, following the same ideas, we can see that a formula of the type (4.1) holds whenever our quadratic form obeys to a theorem of Jacobi type, i.e., $r_Q(n) = c\, d_{\chi_Q}(n)$, for some primitive character χ_Q. In fact, there are several quadratic forms for which we have this relation: indeed, if the discriminant of Q is such that the number of classes of quadratic forms with the same discriminant is equal to 1, then a relation of this type holds [90].

However, we shall not discuss the details here, preferring to approach the extension of (4.7) by a more general way. To proceed with our considerations, we introduce the following definition, in which we introduce a character version of Epstein's $\zeta-$function (4.7).

Definition 4.3.: The "single-weighted Epstein $\zeta-$function" Let χ be an even character modulo $\ell > 1$ and Q a positive definite binary quadratic form. We define the "single-weighted Epstein $\zeta-$function", for $\text{Re}(s) > 1$, as the absolutely convergent series

$$
Z_2(s,\,Q,\,\chi) = \sum_{(m,n)\neq(0,0)} \frac{\chi(m)}{Q(m,n)^s}. \tag{4.19}
$$

Note that in this definition we allow χ to be principal or even nonprimitive. Although we do need the primitivity condition to derive a functional equation for (4.19), $Z_2\left(s,\,Q,\,\chi\right)$ is also well-defined even when χ is the principal character and we can actually study its analytic continuation to the complex plane as a meromorphic function.

The previous definition may look fairly unsymmetrical, as the values of the character χ only depend on the index m. Indeed, the first written version of this chapter dealt with a "double-weighted" Epstein $\zeta-$function (see eq.(5.42) in the next chapter). However, in the end it seemed to us more natural if we could replicate some remarkable identities obtained in previous chapters via the methods developed in the present chapter. That is the reason why we have chosen (4.19) as a relevant Dirichlet series. Because if we study it properly, then most of the examples derived previously will naturally follow.

Remark 4.1.: Both $Z_2\left(s,\,Q\right)$ and $Z_2(s,\,Q,\,\chi)$ can be respectively regarded as two-dimensional versions of Riemann's ζ and Dirichlet's $L-$functions. Indeed, if we consider a 1-dimensional quadratic form $Q(x)=ax^2$, with $a>0$, we see that the "low-dimensional analogues" of (4.7) and (4.19) are, for $\text{Re}(s)>1$,

$$Z_1(s,\,Q)=\sum_{m\neq 0}\frac{1}{(am^2)^s}=2a^{-s}\zeta(2s),$$

$$Z_1(s,\,Q,\,\chi)=\sum_{m\neq 0}\frac{\chi(m)}{(am^2)^s}=2a^{-s}L\left(2s,\,\chi\right).$$

Furthermore, we have seen that, for some particular quadratic forms, $Z_2(s,\,Q)$ can be expressed as a product of $\zeta(s)$ by $L(s,\chi)$.

Also, as we shall see below, $Z_2(s,\,Q)$ and $Z_2(s,\,Q,\,\chi)$ satisfy functional equations similar to those of $\zeta(2s)$ and $L(2s,\,\chi)$. Thus, one may conjecture whether results for $\zeta(s)$ and $L(s,\chi)$ may be "translated" for the continuation of the functions given by (4.7) and (4.19).

For instance, we may be bold enough to ask if the Riemann hypothesis, or even the generalized Riemann hypothesis are true for $Z_2(s,\,Q)$ and $Z_2(s,\,Q,\,\chi)$. This question was first raised by Titchmarsh and Potter [84] for the classical Epstein $\zeta-$function and they conjectured its falsity.

Later, Bateman and Grosswald [14], using a formula stated only by A. Selberg and S. Chowla [90], proved that Titchmarsh and Potter's conjecture was true. Here, we adapt their calculations to extend the falsity of the conjecture for $Z_2(s,\,Q,\,\chi)$, where χ is any Dirichlet character modulo $\ell>1$, by also establishing a character version of Selberg-Chowla's formula, to be stated below.

In the sequel, instead of using the notation $h(x)$ for the Voronoï-transform (4.18) valid for odd characters, we use $g(x)$ to denote the integral transform of Hankel-type,

$$g(x)=\int_0^{\infty}J_0\left(2\sqrt{xy}\right)\,f(y)dy, \tag{4.20}$$

which is proportional to (4.18).

The first part of this chapter resembles our first chapter since we prove the analytic continuation of a particular Dirichlet series via a well-established summation formula.

Although we do not invoke here Abel-Plana's summation formula, we use a formula due to A. Selberg and S. Chowla [90] and a character extension of it, which can be derived under the considerations given in Example 2.5 of the second chapter.

Using this tools, we develop the theory of the analytic continuation for (4.7) and (4.19), pursuing to prove the following theorem:

Main Theorem:

Let Q be the quadratic form given by $Q(m,n) = am^2 + bnm + cn^2$ with $(a,b,c) \in \mathbb{Z}^3$ such that $d = b^2 - 4ac < 0,\ \ a > 0$. Moreover, assume that χ is a nonprincipal, primitive and even character modulo ℓ and denote by $r_Q(a)$ the number of solutions of the diophantine equation $Q(m,n) = a$ (see definition 4.1. above).

If $f(x)$ belongs to the Müntz class $\mathcal{M}_{\alpha,3}$, then the following summation formulas hold

$$f(0) + \sum_{n=1}^{\infty} r_Q(n)\, f(nx) = \frac{2\pi}{\sqrt{|d|}} \int_0^{\infty} f(xy)\, dy + \frac{2\pi}{x\sqrt{|d|}} \sum_{n=1}^{\infty} r_Q(n)\, g\left(\frac{4\pi^2 n}{|d|\, x}\right), \tag{4.21}$$

$$\sum_{(m,n)\neq(0,0)} \chi(m)\, f\left(Q(m,n)\, x\right) = \frac{2\pi G(\chi)}{\ell\sqrt{|d|}\, x} \sum_{(m,n)\neq(0,0)} \overline{\chi}(m)\, g\left(\frac{4\pi^2}{|d|\, x}\, Q_{\ell}^{-1}(m,n)\right). \tag{4.22}$$

where d denotes the discriminant of Q and $g(x)$ denotes the L_1-transform of Hankel-type given in (4.20). Furthermore, $Q_{\ell}^{-1}(x,y)$ denotes the quadratic form

$$Q_{\ell}^{-1}(x,y) = Q^{-1}\left(\frac{x}{\ell},\, y\right), \tag{4.23}$$

with Q^{-1} being the inverse of Q (4.6).

Of course, using the convention $r_Q(0) = 1$ and the particular value $J_0(0) = 1$, we can rewrite (4.21) in the following compact form

$$\sum_{n=0}^{\infty} r_Q(n)\, f(nx) = \frac{2\pi}{x\sqrt{|d|}} \sum_{n=0}^{\infty} r_Q(n)\, g\left(\frac{4\pi^2 n}{|d|\, x}\right). \tag{4.24}$$

The reader may wonder why we do not cover this chapter under the conditions imposed to our first class of functions, i.e., for the 'L_2 class'.

This question is raised in accordance with the developments given in the previous chapter, as well as with the above example. One may expect that a summation formula of the type (4.24) may also hold for the class of functions satisfying $s\,f^{*}(s)\in L_{2}\left(\frac{1}{2}-i\infty,\,\frac{1}{2}+i\infty\right)$ since there are several particular cases for which one has $r_{Q}(n)=c\,d_{\chi}(n)$ for some odd and primitive Dirichlet character.

As we shall remark at the Main section of this chapter, this consideration/conjecture must be dealt very carefully since it depends on the behavior of Epstein's $\zeta-$function in the critical strip $0\leq\text{Re}(s)\leq 1$, whose study constitutes a very hard problem [101].

To prove (4.21) and (4.22) we introduce two theoretical sections dealing with the analytic continuations of (4.7) and (4.19): in the first one, we prove Selberg-Chowla formula and use it to rederive some interesting results obtained in the second and third chapters.

In the second, we prove the functional equations for (4.7) and (4.19) and, using these, we derive the asymptotic estimates for $Z_{2}(\sigma+it,\,Q)$ and $Z_{2}\left(\sigma+it,\,Q,\,\chi\right)$ as $|t|\rightarrow\infty$ via the Phragmén-Lindelöf principle. We also introduce for the first time a character version of Kronecker's limit formula.

4.1 Preliminary results I - Extension of Selberg-Chowla formula to characters and examples

In this section we derive Selberg-Chowla's formula, which firstly appeared in a paper of A. Selberg and S. Chowla (1949) but it was proved in its modern form by P. Bateman and E. Grosswald (1964) [14].

Although there are definitely more approaches to deduce the analytic continuation of the Epstein $\zeta-$function, even mimicking Riemann's own methods[2], this seems to be the simplest and more direct approach, and allows us to rederive, as we shall see, several summation formulas obtained in previous chapters.

Furthermore, as pointed out by A. Selberg, S. Chowla and other authors [37], this representation of the Epstein $\zeta-$function is also very convenient for the computation of particular values of certain Dirichlet $L-$function at critical points.

We proceed now with their proofs, which we state separately.

[2]This was the approach given by Paul Epstein. Audrey Terras's book [99] contains a modern version of his proof.

Theorem 4.1. (Selberg-Chowla formula for the Epstein $\zeta-$function): Let s be a complex number such that $\mathrm{Re}(s)>1$. Then the following identity for the Epstein $\zeta-$function (4.7) holds

$$a^{s}\Gamma(s)\,Z_2(s,\,Q)=2\Gamma(s)\zeta(2s)+2k^{1-2s}\pi^{1/2}\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1) \qquad (4.25)$$
$$+\,8k^{1/2-s}\pi^{s}\sum_{n=1}^{\infty}n^{s-1/2}\sigma_{1-2s}(n)\,\cos\left(n\pi b/a\right)K_{s-1/2}\left(2\pi k\,n\right),$$

where d is the discriminant of the quadratic form, $k^2:=|d|/4a^2$ and $\sigma_{\nu}(n)=\sum_{d|n}d^{\nu}$ is the generalized divisor function of index ν.

Equivalently, $Z_2(s,\,Q)$ can also be described by the analogous formula

$$c^{s}\Gamma(s)\,Z_2(s,\,Q)=2\Gamma(s)\zeta(2s)+2k'^{1-2s}\pi^{1/2}\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1) \qquad (4.26)$$
$$+\,8k'^{1/2-s}\pi^{s}\sum_{n=1}^{\infty}\sigma_{1-2s}(n)\,n^{s-1/2}\,\cos\left(n\pi b/c\right)K_{s-1/2}\left(2\pi k'\,n\right),$$

where $k'^2:=|d|/4c^2$.

Proof: The proof follows from an application of the Poisson summation formula to the series (4.7), being similar to Example 2.5. of the second chapter.

Note that, for $\mathrm{Re}(s)>1$,

$$Z_2(s,\,Q)=\sum_{m\neq 0}\frac{1}{Q(m,0)^s}+\sum_{n\neq 0}\sum_{m\in\mathbb{Z}}\frac{1}{Q(m,n)^s}$$
$$=2a^{-s}\zeta(2s)+2\sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}}\frac{1}{Q(m,n)^s}. \qquad (4.27)$$

In the last series in (4.27), fix $n\in\mathbb{N}$ and consider the sum

$$\mathcal{S}_n=\sum_{m\in\mathbb{Z}}\frac{1}{(am^2+bmn+cn^2)^s},$$

which can be easily computed by using Example 2.5. If, in formula (2.138) of the second chapter we replace $a\leftrightarrow a$, $b\leftrightarrow bn$, $c\leftrightarrow cn^2$, $k\leftrightarrow nk$ and $\nu\leftrightarrow s$ we obtain that $\mathcal{S}_n$ is given by

$$\mathcal{S}_n=\sqrt{\pi}\,\frac{\Gamma\left(s-1/2\right)}{\Gamma(s)}a^{-s}n^{1-2s}k^{1-2s}$$
$$+\,\frac{4}{\Gamma(s)}\,k^{\frac{1}{2}-s}\left(\frac{a}{\pi}\right)^{-s}\sum_{m=1}^{\infty}\cos\left(\pi\frac{b\,mn}{a}\right)\left(\frac{m}{n}\right)^{s-\frac{1}{2}}K_{s-\frac{1}{2}}\left(2\pi k\,mn\right). \qquad (4.28)$$

Therefore, if in (4.28) we sum over the index n (we take the change of index $mn \leftrightarrow n$ and we use the definition of the generalized divisor function $\sigma_\nu(n)$), we arrive to

$$2\sum_{n=1}^{\infty}\mathcal{S}_n = 2\sqrt{\pi}\,\frac{\Gamma\left(s-1/2\right)}{\Gamma(s)}\,a^{-s}k^{1-2s}\zeta(2s-1) + \frac{8\pi^s a^{-s}}{\Gamma(s)}\,k^{\frac{1}{2}-s}\sum_{n=1}^{\infty}\sigma_{1-2s}(n)\,n^{s-\frac{1}{2}}\,\cos\left(n\pi b/a\right)\,K_{s-\frac{1}{2}}\left(2\pi k\,n\right). \tag{4.29}$$

Using (4.29) and (4.27), we immediately get (4.25), which concludes the proof of the first identity stated in this theorem.

To obtain (4.26), we just need to reverse the order of summation and firstly sum over the index n: by doing so, we need to replace a by c and $k^2 = |d|/4a^2$ by $k'^2 = |d|/4c^2$. With these substitutions and analogous computations, formula (4.25) becomes (4.26), and the proof is established. ■

Now, still following Example 2.5, we can establish character analogues of (4.25) and (4.26), now stated and proved as follows.

Theorem 4.2. (Selberg-Chowla formula applied to Single-Weighted Epstein $\zeta-$function) Let χ be a nonprincipal, primitive and even character modulo $\ell > 1$. For $\mathrm{Re}(s) > 1$, the following formula holds

$$a^s\Gamma(s)\,Z_2(s,\,Q,\,\chi) = 2\Gamma(s)\,L(2s,\chi) + +8\pi^s\,G(\chi)\,k^{\frac{1}{2}-s}\ell^{-s-\frac{1}{2}}\sum_{n=1}^{\infty}\sigma_{2s-1,\overline{\chi}}(n)\,n^{\frac{1}{2}-s}\cos\left(\pi\frac{b}{a}\frac{n}{\ell}\right)\,K_{s-\frac{1}{2}}\left(\frac{2\pi kn}{\ell}\right). \tag{4.30}$$

Moreover, we also have that

$$c^s\Gamma(s)\,Z_2(s,\,Q,\,\chi) = 2\,k'^{1-2s}\pi^{1/2}\Gamma\left(s-\frac{1}{2}\right)\,L(2s-1,\,\chi) + +8\pi^s\,k'^{\frac{1}{2}-s}\sum_{n=1}^{\infty}\sigma_{1-2s,\chi}\left(n\right)\,n^{s-\frac{1}{2}}\cos\left(\frac{n\pi b}{c}\right)\,K_{s-\frac{1}{2}}\left(2\pi k' n\right), \tag{4.31}$$

where $\sigma_{a,\chi}(n)$ denotes the generalized weighted-divisor function (see our glossary or consult equation (2.91) of the second chapter),

$$\sigma_{a,\chi}(n) = \sum_{d|n}\chi(d)\,d^a. \tag{4.32}$$

Proof: Once more, the proof follows the same lines as the previous one. To prove (4.30), note the elementary calculation

$$Z_2\,(s,\,Q,\,\chi) = 2a^{-s}L(2s,\chi) + 2\sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}}\frac{\chi(m)}{Q(m,n)^s}. \tag{4.33}$$

For a fixed $n \in \mathbb{N}$, the second series $\mathcal{S}_{n,\chi}$ may be evaluated by using equation (2.136) in the second chapter. Making the same substitutions as the ones described above, we obtain

$$\mathcal{S}_{n,\chi} = \frac{4G(\chi)}{\Gamma(s)}\sqrt{\frac{k}{\ell}}\left(\frac{ka\ell}{\pi}\right)^{-s}\sum_{m=1}^{\infty}\overline{\chi}(m)\,\cos\left(\pi\frac{b}{a}\frac{mn}{\ell}\right)\left(\frac{m}{n}\right)^{s-\frac{1}{2}}\,K_{s-\frac{1}{2}}\left(\frac{2\pi k\,mn}{\ell}\right), \tag{4.34}$$

and an elementary sum over the index n, together with (4.33), yields

$$Z_2(s,\,Q,\,\chi) = 2a^{-s}L(2s,\chi)+\frac{8G(\chi)}{\Gamma(s)}\sqrt{\frac{k}{\ell}}\left(\frac{ka\ell}{\pi}\right)^{-s}\sum_{n=1}^{\infty}\sigma_{2s-1,\overline{\chi}}(n)\,n^{1/2-s}\,\cos\left(\pi\frac{b}{a}\frac{n}{\ell}\right)\,K_{s-\frac{1}{2}}\left(\frac{2\pi k\,n}{\ell}\right),$$

which is equivalent to (4.30).

To deduce (4.31), let us change once more the order of summation: considering the sum over n first,

$$Z_2(s,\,Q,\,\chi) = 2\sum_{m=1}^{\infty}\chi(m)\sum_{n\in\mathbb{Z}}\frac{1}{Q(m,n)^s} \tag{4.35}$$

and applying equation (2.138) at the second chapter with the substitutions $a \leftrightarrow c$, $b \leftrightarrow bm$, $c \leftrightarrow am^2$ and $k = m\,k'$, we immediately obtain

$$\sum_{n\in\mathbb{Z}}\frac{1}{Q(m,n)^s} = \sqrt{\pi}\,\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)\,m^{2s-1}}c^{-s}k'^{1-2s} + \frac{4\pi^s c^{-s}}{\Gamma(s)}\,k'^{\frac{1}{2}-s}$$
$$\times\sum_{n=1}^{\infty}\cos\left(\pi\frac{b}{c}\,nm\right)\left(\frac{n}{m}\right)^{s-\frac{1}{2}}K_{s-\frac{1}{2}}\left(2\pi k'\,nm\right), \tag{4.36}$$

and an application of (4.35) immediately implies (4.31). ■

Remark 4.2.: It should be pointed out that, even if χ is principal or a nonprimitive character, (4.31) is still valid, since the only tool required to prove it is the classical Poisson summation formula.

However, this is not true with (4.30) since its proof relies on the character version of Poisson's summation formula, whose validity strongly depends on the primitivity of χ.

Moreover, (4.30) and (4.31) can also be defined if one substitutes χ by an arbitrary even arithmetic function with suitable growing property.

Indeed, let $\mathfrak{q}(m)$ be an arithmetic function defined in $\mathbb{Z}$ such that $\mathfrak{q}(m) = \mathfrak{q}(-m)$ and $\mathfrak{q}(m) = O\left(m^{\alpha}\right), \ \alpha \geq 0$ as $m \to \infty$. For $\sigma > (\alpha+2)/2$, let $Z_2(s,\, Q,\, \mathfrak{q})$ denote the $\mathfrak{q}-$single weighted Epstein zeta function

$$Z_2(s,\, Q,\, \mathfrak{q}) = \sum_{(m,n)\neq(0,0)} \frac{\mathfrak{q}(m)}{Q(m,n)^s}, \quad \text{Re}(s) > 1+\frac{\alpha}{2}.$$

For $\sigma > 1+\alpha$, let $\mathfrak{Q}(s)$ denote the Dirichlet series associated with the arithmetic function $\mathfrak{q}(m)$,

$$\mathfrak{Q}(s) = \sum_{n=1}^{\infty} \frac{\mathfrak{q}(n)}{n^s}, \quad \text{Re}(s) > 1+\alpha.$$

Then, applying the same argument used to prove (4.31), we can easily deduce that

$$\begin{aligned} c^s\Gamma(s)\, Z_2(s,\, Q,\, \mathfrak{q}) &= 2\, k'^{1-2s}\pi^{1/2}\Gamma\left(s-\frac{1}{2}\right)\, \mathfrak{Q}\left(2s-1\right) + \\ &+ 8\pi^s\, k'^{\frac{1}{2}-s} \sum_{n=1}^{\infty} \mathfrak{s}_{1-2s}\left(n\right)\, n^{s-\frac{1}{2}} \cos\left(\frac{n\pi b}{c}\right)\, K_{s-\frac{1}{2}}\left(2\pi k' n\right), \end{aligned} \tag{4.37}$$

where $\mathfrak{s}_{\nu}(n) = \sum_{d|n} \mathfrak{q}(d)\, d^{\nu}$ is the generalized divisor function associated with $\mathfrak{q}(m)$.

As an example, consider the divisor function $d(n)$. Notice that we can extend its domain to $\mathbb{Z}$ in the following way: for $n \in \mathbb{Z}$, let $D(n) = \#\left\{m \in \mathbb{Z}:\ m|n\right\}$.

For example, $D(6) = \#\{-6,-3,-2,-1,\, 1,\, 2,\, 3,\, 6\} = 8$ since we can express 6 also as $-2\times(-3)$ or as $(-1)\times(-6)$. It is now simple to see that the function

$$d(n) = \frac{1}{2}\, D(n), \tag{4.38}$$

with the convention $d(0)=0$, extends $d(n)$ to all $\mathbb{Z}$ as an even arithmetic function.

Since $d(n) = O\left(n^{\epsilon}\right),\ \forall \epsilon > 0$ [60] $d(n)$ satisfies the above-mentioned conditions for any $\alpha > 0$.

Using (4.37) and the fact that, in this case, $\mathfrak{Q}(s) = \zeta^2(s)$, we deduce immediately, for $\text{Re}(s) > 1$,

$$\begin{aligned} c^s\Gamma(s) \sum_{(m,n)\neq(0,0)} \frac{d(m)}{Q(m,n)^s} &= 2\, k'^{1-2s}\pi^{1/2}\Gamma\left(s-\frac{1}{2}\right)\, \zeta^2\left(2s-1\right) + \\ + 8\pi^s\, k'^{\frac{1}{2}-s} \sum_{n=1}^{\infty} \mathfrak{s}_{1-2s}\left(n\right)\, n^{s-\frac{1}{2}} \cos\left(\frac{n\pi b}{c}\right)\, K_{s-\frac{1}{2}}\left(2\pi k' n\right), \end{aligned} \tag{4.39}$$

where $\mathfrak{s}_{\nu}(n) = \sum_{m|n} d(m)\, m^{\nu}$.

Still another example can be given for $\mathfrak{q}(m) = d(m^2)$, which clearly extends to $\mathbb{Z}$ as an even arithmetic function. Since [103] chpt. 2,

$$\mathfrak{Q}(s) = \sum_{n=1}^{\infty} \frac{d(n^2)}{n^s} = \frac{\zeta^3(s)}{\zeta(2s)}, \quad \mathrm{Re}(s) > 1, \tag{4.40}$$

an application of (4.31) yields the interesting identity

$$c^s \Gamma(s) \sum_{(m,n)\neq(0,0)} \frac{d(m^2)}{Q(m,n)^s} = 2\, k'^{1-2s} \pi^{1/2} \Gamma\left(s - \frac{1}{2}\right) \frac{\zeta^3(2s-1)}{\zeta(4s-2)} +$$

$$+8\pi^s\, k'^{\frac{1}{2}-s} \sum_{n=1}^{\infty} \mathfrak{s}_{1-2s}(n)\; n^{s-\frac{1}{2}} \cos\left(\frac{n\pi b}{c}\right) K_{s-\frac{1}{2}}\left(2\pi k' n\right), \tag{4.41}$$

where $\mathfrak{s}_{\nu}(n) = \sum_{m|n} d(m^2)\, m^{\nu}$.

Using summation formulas involving the coefficients $d(m)$ and $d(m^2)$ (Voronoï's formula and an analogue of it), we can still find other type of representations for (4.39) and (4.41), if one sums firstly over the index m. However, these considerations are beyond the scope of this thesis and the required computations seem very difficult to consider here.

The previous theorems established, respectively, two ways of expressing $Z_2(s,\, Q)$ and $Z_2(s,\, Q,\, \chi)$ as a series involving Bessel functions and terms depending on Riemann's $\zeta-$function.

Since the right-hand sides of equations (4.25) and (4.30) are well-defined and well-behaved for any $s \in \mathbb{C}$, the next step is to check that these equations provide, indeed, the analytic continuations of $Z_2(s,\, Q)$ and $Z_2(s,\, Q,\, \chi)$.

To do this, we need to study carefully the right-hand side given by the formulas of Selberg-Chowla type stated above.

Analogously to Abel-Plana's formula, which provided a representation (due to Hermite) in which the last term (i.e., the integral) was an entire function, we shall see that the function defined by the right-hand sides of (4.25) and (4.30) has similar properties.

In the result that follows, we state that the series involving the Modified Bessel functions $K_{s-\frac{1}{2}}(n)$ defines an entire function in both cases. Since a very short proof is presented in [14], we omit the details and we only focus on the symmetric properties of this series.

Lemma 4.1.: The analyticity of $H(s,\, Q)$ and $H_{\chi}(s,\, Q)$ and its symmetric properties Let $H(s,\, Q)$, $H_{\chi}(s,\, Q)$ denote, respectively, the functions lying at the second term of the left-hand side of (4.25) and (4.30),

$$H(s,\, Q) = \sum_{n=1}^{\infty} n^{s-1/2} \sigma_{1-2s}(n)\, \cos\left(n\pi b/a\right) K_{s-\frac{1}{2}}\left(2\pi k\, n\right), \tag{4.42}$$

$$H_{\chi}(s,\,Q)=\sum_{n=1}^{\infty}\sigma_{2s-1,\overline{\chi}}(n)\,n^{\frac{1}{2}-s}\cos\left(\pi\frac{b}{a}\frac{n}{\ell}\right)\,K_{s-\frac{1}{2}}\left(\frac{2\pi kn}{\ell}\right), \tag{4.43}$$

where χ is an even character modulo ℓ. Then we have that (4.42) and (4.43) are entire and obey, respectively, to the reflection formulas

$$H(s,\,Q)=H(1-s,\,Q), \tag{4.44}$$

$$H_{\overline{\chi}}(1-s,\,Q_{\ell}^{-1})=\sum_{n=1}^{\infty}\sigma_{1-2s,\chi}(n)\,n^{s-\frac{1}{2}}\cos\left(\frac{n\pi b}{c}\right)\,K_{\frac{1}{2}-s}\left(2\pi k'n\right), \tag{4.45}$$

where $Q_{\ell}^{-1}(x,y)$ denotes the quadratic form defined in (4.23).

Proof: Since proving the analyticity of $H(s,\,Q)$ and $H_{\chi}(s,\,Q)$ requires the same ideas and arguments, it suffices to prove the first part of the Lemma 4.1. for $H(s,\,Q)$. The proof of this fact can be found in Lemma 2, p. 368 of [14], so we just need to prove the reflection formulas (4.44) and (4.45).

To prove (4.44), let us use the elementary property of the generalized divisor function,

$$\sigma_{1-2s}(n)=\sum_{d|n}d^{1-2s}=\sum_{d|n}\left(\frac{n}{d}\right)^{1-2s}=n^{1-2s}\sigma_{2s-1}(n), \tag{4.46}$$

which implies

$$\begin{aligned}H(s,\,Q)&=\sum_{n=1}^{\infty}n^{s-1/2}\sigma_{1-2s}(n)\,\cos\left(n\pi b/a\right)K_{s-1/2}\left(2\pi k\,n\right)\\&=\sum_{n=1}^{\infty}n^{1/2-s}\sigma_{2s-1}(n)\,\cos\left(n\pi b/a\right)\,K_{1/2-s}\left(2\pi k\,n\right)\\&=H(1-s,\,Q),\end{aligned} \tag{4.47}$$

where in the second equality we have used the reflection property of the Bessel function $K_{\nu}(z)=K_{-\nu}(z)$ [106]. This proves (4.44).

To prove (4.45), note that $Q^{-1}(x,y)=cx^2-bxy+ay^2$ by (4.6). This gives

$$Q_{\ell}^{-1}(x,y)=Q^{-1}\left(x/\ell,\,y\right)=c\,\frac{x^2}{\ell^2}-b\,\frac{x}{\ell}y+a\,y^2,$$

and so if we replace Q by Q_{ℓ}^{-1} in the argument of $H_{\chi}(s,\,Q)$ we need to replace a by c/ℓ^2, b by $-b/\ell$, c by a and k by $\ell k'$. Furthermore, replacing χ by its conjugate and s by $1-s$, we obtain the right-hand side of (4.45), which concludes our proof. ■

Note that the previous theorem can be employed to study the analytic continuation of Epstein's $\zeta-$functions as a meromorphic complex function. Note that if, at the present point, we knew nothing about the analytic continuations of $\zeta(s)$ and $L(s,\chi)$, we could still deduce this from the previous formulas.

To see this, compare the right-hand sides of (4.25) with (4.26), which are equal when respectively multiplied by a^{-s} and c^{-s}.

It is immediate to arrive at the equality, for $\text{Re}(s) > 1$,

$$\frac{1}{4}\left(\frac{\sqrt{|d|}}{2}\right)^{s-\frac{1}{2}} \pi^{-s}\Gamma(s)\zeta(2s)\ \left(a^{-s}-c^{-s}\right)+\frac{1}{4}\left(\frac{\sqrt{|d|}}{2}\right)^{\frac{1}{2}-s} \pi^{\frac{1}{2}-s}\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)\ \left(a^{s-1}-c^{s-1}\right)$$
$$=\sum_{n=1}^{\infty} n^{s-1/2}\sigma_{1-2s}(n)\ \left[a^{-1/2}\cos\left(n\pi b/a\right)\ K_{s-1/2}\left(2\pi k n\right)-c^{-1/2}\cos\left(n\pi b/c\right)\ K_{s-1/2}\left(2\pi k' n\right)\right]. \tag{4.48}$$

Note that, from (4.47), the right-hand side of (4.48) is invariant under the reflection $s \leftrightarrow 1-s$. This means that the analytic continuation of the left-hand side of (4.48) is invariant under this operation as well. From this observation and after proceeding with elementary operations, we obtain

$$\frac{1}{4}\left(\frac{\sqrt{|d|}}{2}\right)^{s-\frac{1}{2}} \left(a^{-s}-c^{-s}\right)\left[\pi^{-s}\Gamma(s)\zeta(2s)-\pi^{s-\frac{1}{2}}\Gamma\left(\frac{1}{2}-s\right)\zeta(1-2s)\right]$$
$$=\frac{1}{4}\left(\frac{\sqrt{|d|}}{2}\right)^{\frac{1}{2}-s} \left(a^{s-1}-c^{s-1}\right)\left[\pi^{s-1}\Gamma(1-s)\zeta(2-2s)-\pi^{\frac{1}{2}-s}\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)\right]. \tag{4.49}$$

If we take, for example, $c = 2a$ and use the fact that a is an arbitrary positive real number, it follows from (4.49) that both sides must vanish for all $s \in \mathbb{C}$ (we are now assuming that $\zeta(s)$ represents the continuation of Riemann's $\zeta-$function). But if this happens then we obtain the relation

$$\pi^{-s}\Gamma(s)\zeta(2s)=\pi^{s-\frac{1}{2}}\Gamma\left(\frac{1}{2}-s\right)\zeta(1-2s), \tag{4.50}$$

which is equivalent to the functional equation for $\zeta(s)$.

Of course, this method can also be applied to deduce the functional equation for $L(s,\chi)$ when χ is even and primitive.

It is now clear that formulas (4.25) and (4.30) are fundamental to understand the properties of $Z_2(s,\, Q)$. The next corollary takes care of this characterization:

Corollary 4.1.: (Analytic continuations of $Z_2(s, Q)$ and $Z_2(s, Q, \chi)$): The Selberg-Chowla formulas (4.25) and (4.30) provide the analytic continuation of the Epstein $\zeta-$functions (4.7) and (4.19).

Moreover, $Z_2(s,\,Q)$ has a removable singularity at the points $s = 1/2 - k$, $k \in \mathbb{N}_0$ and a simple pole at $s = 1$ with residue $\frac{2\pi}{\sqrt{|d|}}$.

Furthermore,

1. If χ is a nonprincipal character, then $Z_2(s,\,Q,\,\chi)$ can be continued as a complex entire function.

2. If $\chi = \chi_1$ is the principal character modulo ℓ, then $Z_2(s,\,Q,\,\chi_1)$ can be continued as a meromorphic function having a simple pole at $s = 1$ with residue $\frac{2\pi\varphi(\ell)}{\ell\sqrt{|d|}}$.

Proof: We have seen at Lemma 4.1 that $H_Q(s)$ is an entire function, so the meromorphic part of the Epstein $\zeta-$function comes from the first two terms in (4.25), i.e.,

$$G(s,\,Q) = 2a^{-s}\zeta(2s) + 2k^{1-2s}a^{-s}\pi^{1/2}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\zeta(2s-1).$$

Clearly, from a standard verification, $Z_2(s,\,Q)$ has removable singularities at $s = \frac{1}{2} - k$, $k \in \mathbb{N}_0$. Since $\zeta(2s)$ and $\Gamma(s - 1/2)$ are analytic in a neighbourhood of $s = 1$, we can conclude that $Z_2(s,\,Q)$ must have a pole at $s = 1$ coming from the function $\zeta(2s-1)$ with residue $\pi/(ak) = 2\pi/\sqrt{|d|}$.

Since χ can be nonprimitive and principal, let us use (4.31) to study the properties of the continuation of $Z_2(s, Q, \chi)$ to $\mathbb{C}$: if χ is nonprincipal, it follows that $L(2s-1,\,\chi)$ is entire (see Theorem 1.5 at the first chapter). Moreover, from the previous lemma, the second term in the right-hand side of (4.31) defines an entire function, so that $Z_2(s, Q, \chi)$ is entire.

If $\chi = \chi_1$, we know as well from Theorem 1.5 that $L(2s-1, \chi_1)$ has a simple pole at $s = 1$ with residue $\varphi(\ell)/2\ell$, so we can easily see that $Z_2(s,\,Q,\,\chi_1)$ has a simple pole at $s = 1$ with residue $\frac{2\pi\varphi(\ell)}{\ell\sqrt{|d|}}$. ■

In the remaining part of this section, we revisit some corollaries of Poisson and Voronoï's summation formulas, arguing that these can be established as corollaries of the theorems 4.1 and 4.2 proved above. The first of these was derived for the first time by Guinand and reproved by Nasim 20 years later [32, 55, 79]. We establish, for the first time, character versions of this formula.

Corollary 4.2.: (Guinand's formula and its character analogues) Assume that $\alpha, \beta > 0$ are such that $\alpha\beta = \pi^2$ and χ is a nonprincipal and primitive even character modulo ℓ. For all $s \in \mathbb{C}$, the following identities hold

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-s}(n)\,n^{s/2}K_{\frac{s}{2}}(2n\alpha)-\sqrt{\beta}\sum_{n=1}^{\infty}\sigma_{-s}(n)\,n^{s/2}K_{\frac{s}{2}}(2n\beta)=$$
$$\frac{1}{4}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)\left\{\beta^{(1+s)/2}-\alpha^{(1+s)/2}\right\}+\frac{1}{4}\Gamma\left(\frac{s}{2}\right)\zeta(s)\left\{\beta^{(1-s)/2}-\alpha^{(1-s)/2}\right\}, \tag{4.51}$$

$$\alpha\pi^{s/2}\ell^{-s-1}G(\chi)\Gamma\left(-\frac{s}{2}\right)L(-s,\overline{\chi})-\Gamma\left(\frac{s}{2}\right)\pi^{1-\frac{s}{2}}L(s,\chi)$$
$$=4\pi^{1+\frac{s}{2}}\alpha^{-\frac{s}{2}}\sum_{n=1}^{\infty}\sigma_{-s,\chi}(n)\,n^{\frac{s}{2}}\,K_{\frac{s}{2}}(2\beta n)-4\alpha\beta^{\frac{s}{2}}\pi^{-\frac{s}{2}}\ell^{-1-\frac{s}{2}}G(\chi)\sum_{n=1}^{\infty}\sigma_{s,\overline{\chi}}(n)\,n^{-\frac{s}{2}}\,K_{\frac{s}{2}}\left(\frac{2\alpha n}{\ell}\right). \tag{4.52}$$

Proof: Assume that $\text{Re}(s) > 1$ and compare (4.25) and (4.26), using the quadratic form $Q_{\alpha}(m,n) = m^2 + \frac{\alpha^2}{\pi^2}n^2$ and replacing s by $\frac{1+s}{2}$. It is simple to see that we obtain the identity

$$\frac{1}{4}\left(\frac{\alpha}{\pi}\right)^{\frac{s}{2}}\pi^{-\frac{1+s}{2}}\Gamma\left(\frac{1+s}{2}\right)\zeta\left(1+s\right)\,\left(1-\alpha^{-1-s}\pi^{1+s}\right)+\frac{1}{4}\left(\frac{\alpha}{\pi}\right)^{-\frac{s}{2}}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)\,\left(1-\alpha^{s-1}\pi^{1-s}\right)$$
$$=\sum_{n=1}^{\infty}n^{\frac{s}{2}}\,\sigma_{-s}(n)\,\left[a^{-1/2}\,K_{\frac{s}{2}}\left(2\pi kn\right)-c^{-1/2}\,K_{\frac{s}{2}}\left(2\pi k'n\right)\right]. \tag{4.53}$$

Therefore, using the functional equation for $\zeta(1+s)$ and replacing $\frac{\pi^2}{\alpha^2}$ by β, we deduce (4.51) for all $s \in \mathbb{C}$ such that $\text{Re}(s) > 1$. To conclude that (4.51) is true for all $s \in \mathbb{C}$, note that, from lemma 4.1., both sides of (4.51) define complex-analytic functions which coincide in the half-plane $\text{Re}(s) > 1$: therefore, by the principle of analytic continuation, (4.51) holds for all $s \in \mathbb{C}$.

Finally, the proof of (4.52) follows the same lines and it is sufficient to invoke (4.30) and (4.31) with a quadratic form having the parameters $a = 1$, $b = 0$, $c = \frac{\alpha^2}{\pi^2}$ and to use the functional equation for the Dirichlet $L-$function. ■

Now, it is remarkable to see that Theorems 4.1 and 4.2 can also be employed to obtain Koshliakov's formula for even characters as well as the reflection formulas for the Dedekind $\eta-$function and the Eisenstein series studied on the second chapter. These results are obtained as the following corollaries.

Corollary 4.2.1. (Koshliakov's formula) Let $z > 0$ and χ be a nonprincipal, primitive and even character modulo ℓ. Then the following formulas hold

$$\begin{aligned}&\sum_{n=1}^{\infty} d(n)\, K_0\left(2\pi nz\right) - \frac{1}{z}\sum_{n=1}^{\infty} d(n)\, K_0\left(\frac{2\pi n}{z}\right) \\ &= \frac{1}{4z}\left(\gamma - \log\left(4\pi z\right)\right) - \frac{1}{4}\left(\gamma - \log\left(\frac{4\pi}{z}\right)\right),\end{aligned} \tag{4.54}$$

$$\frac{G(\chi)}{4}\, L(1,\overline{\chi}) + \sum_{n=1}^{\infty} d_{\chi}(n)\, K_0\left(\frac{2\pi nz}{\sqrt{\ell}}\right) = \frac{\sqrt{\ell}}{4z} L(1,\chi) + \frac{G(\chi)}{z\sqrt{\ell}}\sum_{n=1}^{\infty} d_{\overline{\chi}}(n)\, K_0\left(\frac{2\pi n}{z\sqrt{\ell}}\right). \tag{4.55}$$

Proof: The proof comes from Guinand's formula and the character analogues we've established for it. We clearly obtain (4.54) by letting $s \to 0$ in (4.51) and using the meromorphic expansions for $\Gamma(s)$ and $\zeta(s)$ around $s = 0$

$$\Gamma(s) = \frac{1}{s} - \gamma + O(s), \tag{4.56}$$

$$\zeta(s) = -\frac{1}{2} - \frac{1}{2}\log(2\pi)\, s + O\left(s^2\right) = -\frac{1}{2} - \frac{\log(4\alpha\beta)}{4}\, s + O\left(s^2\right). \tag{4.57}$$

Taking the limit $s \to 0$ yields for the left-hand side of (4.51)

$$\sqrt{\alpha}\sum_{n=1}^{\infty} d(n)\, K_0(2n\alpha) - \sqrt{\beta}\sum_{n=1}^{\infty} d(n)\, K_0(2n\beta). \tag{4.58}$$

By other hand, the right-hand side of (4.51) can be treated by using the series expansions for the entire functions $\beta^{(1-s)/2}$ and $\alpha^{(1-s)/2}$ around $s = 0$ which, when combined with (4.56) and (4.57) give

$$\begin{aligned}&\frac{1}{4}\lim_{s\to 0}\Gamma\left(\frac{s}{2}\right)\zeta(s)\left(\beta^{(1-s)/2} - \alpha^{(1-s)/2}\right) + \frac{1}{4}\lim_{s\to 0}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)\left(\beta^{(1+s)/2} - \alpha^{(1+s)/2}\right) \\ &= \frac{1}{4}\gamma\left(\sqrt{\beta} - \sqrt{\alpha}\right) - \frac{1}{4}\log(4\alpha\beta)\,(\sqrt{\beta} - \sqrt{\alpha}) + \frac{1}{4}\left(\sqrt{\beta}\log(\beta) - \sqrt{\alpha}\log(\alpha)\right).\end{aligned} \tag{4.59}$$

Joining (4.58) with (4.59) we obtain

$$\begin{aligned}&\sqrt{\alpha}\left(\frac{1}{4}\gamma - \frac{1}{4}\log(4\beta) + \sum_{n=1}^{\infty} d(n)\, K_0(2n\alpha)\right) \\ &= \sqrt{\beta}\left(\frac{1}{4}\gamma - \frac{1}{4}\log(4\alpha) + \sum_{n=1}^{\infty} d(n)\, K_0(2n\beta)\right),\end{aligned} \tag{4.60}$$

from which (4.54) follows after the substitution $\alpha = \pi z,\ z > 0$.

Analogously, formula (4.55) is obtained from (4.52) and the functional equation for the Dirichet $L-$function and taking $\alpha = \pi\sqrt{\ell}z$. ■

Corollary 4.2.2.: (Nasim summation formula and character analogues) The following formulas of Nasim-type take place

$$\sum_{n=1}^{\infty}\frac{\sigma(n)}{n}\,e^{-2\pi nz}-\sum_{n=1}^{\infty}\frac{\sigma(n)}{n}\,e^{-2\pi n/z}=\frac{1}{2}\,\log\left(z\right)+\frac{\pi}{12\,z}\left(1-z^{2}\right), \tag{4.61}$$

$$\frac{G(\overline{\chi})}{4\ell z^{2}}\,B_{2,\chi}=\frac{G(\overline{\chi})}{z^{2}\ell}\,\sum_{n=1}^{\infty}\sigma_{1,\chi}(n)\,e^{-\frac{2\pi n}{z\sqrt{\ell}}}+\sum_{n=1}^{\infty}\sigma_{-1,\overline{\chi}}(n)\,n\,e^{-\frac{2\pi nz}{\sqrt{\ell}}}, \tag{4.62}$$

extending the relations (2.116) and (2.118) preved on the second chapter to $k=1$.

Proof: To prove (4.61), take $s\to-1$ in (4.51) and use the well-known values $\Gamma(1/2)=\sqrt{\pi}$, $\Gamma(-1/2)=-2\sqrt{\pi}$ and $\zeta(-1)=-\frac{1}{12}$: this gives the identity

$$\begin{aligned}\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma(n)\,n^{-1/2}K_{1/2}(2n\alpha)-\sqrt{\beta}\sum_{n=1}^{\infty}\sigma(n)\,n^{-1/2}K_{1/2}(2n\beta)\\=\frac{1}{4}\Gamma\left(\frac{1}{2}\right)\lim_{s\to-1}\zeta(-s)\left\{\beta^{(1+s)/2}-\alpha^{(1+s)/2}\right\}+\frac{1}{4}\Gamma\left(-\frac{1}{2}\right)\zeta(-1)\left\{\beta-\alpha\right\}\\=\frac{\sqrt{\pi}}{8}\,\log\left(\frac{\alpha}{\beta}\right)+\frac{\sqrt{\pi}}{24}\left(\beta-\alpha\right).\end{aligned} \tag{4.63}$$

Using the particular value for $K_{\nu}(z)$,

$$K_{\frac{1}{2}}(z)=\sqrt{\frac{\pi}{2z}}\,e^{-z},\quad z>0, \tag{4.64}$$

we deduce, after taking $\alpha=\pi z$, the interesting formula

$$\sum_{n=1}^{\infty}\frac{\sigma(n)}{n}\,e^{-2\pi nz}-\sum_{n=1}^{\infty}\frac{\sigma(n)}{n}\,e^{-2\pi n/z}=\frac{1}{2}\,\log\left(z\right)+\frac{\pi}{12\,z}\left(1-z^{2}\right), \tag{4.65}$$

which, when differentiated with respect to z, yields

$$\sum_{n=1}^{\infty}\sigma(n)\,e^{-2\pi nz}+\frac{1}{z^{2}}\,\sum_{n=1}^{\infty}\sigma(n)\,e^{-2\pi n/z}=\frac{1}{24}\left(1+\frac{1}{z^{2}}\right)-\frac{1}{4\pi z}. \tag{4.66}$$

Analogously, using (4.52) and taking once more $s\to-1$ and $\alpha=\pi\sqrt{\ell}z$, we derive the following Nasim-type formula

$$\frac{1}{2}\,L(1,\overline{\chi})-\frac{\sqrt{\ell}}{2\pi z}\,L(2,\overline{\chi})=\frac{G(\overline{\chi})}{\ell}\,\sum_{n=1}^{\infty}\frac{\sigma_{1,\chi}(n)}{n}\,e^{-\frac{2\pi n}{z\sqrt{\ell}}}-\sum_{n=1}^{\infty}\sigma_{-1,\overline{\chi}}(n)\,e^{-\frac{2\pi nz}{\sqrt{\ell}}}. \tag{4.67}$$

Differentiating (4.67) with respect to z yields a more familiar representation,

$$\frac{\ell}{4\pi^{2}z^{2}}\,L(2,\overline{\chi})=\frac{G(\overline{\chi})}{z^{2}\ell}\,\sum_{n=1}^{\infty}\sigma_{1,\chi}(n)\,e^{-\frac{2\pi n}{z\sqrt{\ell}}}+\sum_{n=1}^{\infty}\sigma_{-1,\overline{\chi}}(n)\,n\,e^{-\frac{2\pi nz}{\sqrt{\ell}}}, \tag{4.68}$$

which can be rewritten by invoking the character version of Basel identity (see relation (1.149) of the first chapter)

$$L(2,\overline{\chi}) = \frac{\pi^2}{\ell^2} G(\overline{\chi})\, B_{2,\chi}, \tag{4.69}$$

giving the desired version (4.62), which is precisely what we obtain in equation (2.118) of the second chapter if we put $k=1$ and $\tau = i\sqrt{\ell}/z$ and replace χ by its conjugate $\overline{\chi}$.

It should be also noted that we can take $s = 2k-1$, $k \geq 2$ in (4.51) and appeal to the particular values of $K_{k-\frac{1}{2}}(z)$ to rederive formulas (2.116) and (2.118) of the second chapter as well. ■

Following the same ideas, we can also prove anew the reflection formula for the character analogue of Dedekind $\eta-$function, $\eta_{\chi}(\tau)$ (see Example 2.1 at the second chapter).

Corollary 4.2.3.: (Dedekind $\eta-$function and character analogues) If $\mathrm{Im}(\tau) > 0$, let $\eta(\tau)$ and $\eta_{\chi}(\tau)$ (with χ real, even and primitive) denote, respectively, the two versions of Dedekind's $\eta-$function,

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{m=1}^{\infty} \left(1 - e^{2\pi i m \tau}\right) \tag{4.70}$$

and

$$\eta_{\chi}(\tau) = e^{i\frac{\ell G(\chi) L(2,\chi)}{2\pi}\tau} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n \tau}\right)^{\chi(n)}. \tag{4.71}$$

Then $\eta(\tau)$ and $\eta_{\chi}(\tau)$ obey, respectively, to the reflection formulas

$$\eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{1/2}\, \eta(\tau), \tag{4.72}$$

$$\eta_{\chi}(\tau) = e^{-\frac{G(\chi)}{2}\, L(1,\chi)} \prod_{n=1}^{\infty} \exp\left(G(\chi)\, \log_{\chi}\left(1 - e^{-\frac{2\pi i m}{\ell\tau}}\right)\right), \tag{4.73}$$

where, for $|z|<1$, the character analogue of the logarithmic function, $\log_{\chi}(1-z)$, was defined by equation (2.92) in the second chapter.

Proof: Using (4.51) with $s \to 1$, we obtain,

$$\sqrt{\alpha}\sum_{n=1}^{\infty} \sigma_{-1}(n)\, \sqrt{n}\, K_{1/2}(2n\alpha) - \sqrt{\beta}\sum_{n=1}^{\infty} \sigma_{-1}(n)\, \sqrt{n}\, K_{1/2}(2n\beta)$$
$$= \frac{\sqrt{\pi}}{8} \log\left(\frac{\alpha}{\beta}\right) + \frac{\sqrt{\pi}}{24}\, (\beta - \alpha),$$

where the last equality came from the series expansion for $\zeta(s)$ and $\beta^{(1-s)/2}-\alpha^{(1-s)/2}$ around $s=1$.

Finally, using also relation (4.64), we deduce immediately

$$\sum_{n=1}^{\infty}\sigma_{-1}(n)\,e^{-2n\alpha}-\sum_{n=1}^{\infty}\sigma_{-1}(n)\,e^{-2n\beta}=\frac{\beta-\alpha}{12}+\frac{1}{4}\log\left(\frac{\alpha}{\beta}\right), \tag{4.74}$$

which is equivalent to formula (2.88) of the second chapter if we take $\alpha=\pi\ell x$.

If we also take the limit $s\to 1$ in (4.52) and use the relations invoked above, we arrive to

$$\frac{2G(\chi)}{\ell}\sum_{n=1}^{\infty}\overline{\chi}(n)\,\log\left(1-e^{-\frac{2\alpha n}{\ell}}\right)=\frac{\alpha}{\pi^{2}}L(2,\chi)-L(1,\chi)-2\sum_{n=1}^{\infty}\sigma_{-1,\chi}(n)\,e^{-2\beta n}, \tag{4.75}$$

which is equivalent to formula (2.94) of the second chapter when we take $\alpha=\pi\ell x$. Following now the argument given at Example 2.1, from the previous identity we are able to obtain (4.71). ■

After revisiting several particular examples which were worked extensively on previous chapters, we end this section with an important consequence of the representations (4.25) and (4.31). As remarked at the beginning of this chapter, $Z_2\left(s,Q\right)$ can be regarded as a two-dimensional version of Riemann's $\zeta-$function $\zeta(2s)$. Therefore, we may conjecture if the following versions of the Riemann hypothesis are true.

Conjecture 4.1.: An Extended Riemann Hypothesis All zeros of $Z_2\left(s,Q\right)$ in the strip $0<\text{Re}(s)<1$ are located at the critical line $\text{Re}(s)=\frac{1}{2}$.

Conjecture 4.2.: An Extended Generalized Riemann Hypothesis Let χ be an even Dirichlet character modulo $\ell>1$. Then all zeros of $Z_2(s,Q,\chi)$ in the strip $0<\text{Re}(s)<1$ are located at the critical line $\text{Re}(s)=\frac{1}{2}$.

We shall prove the falsity of both conjectures by straightforward applications of Selberg-Chowla formulas (4.25) and (4.31).

The proofs are given as the following corollaries.

Corollary 4.3.1.: Conjecture 4.1 is false.

Proof: To prove the falsity of the conjecture, it suffices to find a positive definite quadratic form Q for which $Z_2(s,\, Q)$ does not satisfy the extended Riemann hypothesis.

Using (4.25), we will study the value of $Z_2\left(\frac{1}{2},\, Q\right)$ and relate its positivity with the properties of the quadratic form Q. From (4.25), we can write $Z_2\left(\frac{1}{2},\, Q\right)$ as

$$Z_2\left(\frac{1}{2},\, Q\right) = \lim_{s\to\frac{1}{2}}\left[2a^{-s}\zeta(2s) + 2a^{-s}k^{1-2s}\pi^{1/2}\,\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\,\zeta(2s-1)\right] + 8a^{-1/2}\sum_{n=1}^{\infty} d(n)\,\cos\left(n\pi b/a\right)\,K_0\left(2\pi k\, n\right), \tag{4.76}$$

where the second series converges at exponential rate due to the asymptotic behavior of the modified Bessel function for large arguments [106].

To evaluate the first two terms in (4.76), we appeal to the following Laurent expansions around $s=\frac{1}{2}$,

$$\zeta\left(2s\right) = \frac{1}{2s-1} + \gamma + O\left(s-\frac{1}{2}\right), \tag{4.77}$$

$$\zeta(2s-1) = -\frac{1}{2} - \frac{1}{2}\log(2\pi)\,\left(2s-1\right) + O\left(s-1/2\right)^2, \tag{4.78}$$

$$\frac{1}{\Gamma(s)} = \frac{1}{\sqrt{\pi}}\left(1 + \left(\gamma + 2\log(2)\right)\left(s-1/2\right) + O\left(s-1/2\right)^2\right), \tag{4.79}$$

$$\Gamma\left(s-\frac{1}{2}\right) = \frac{1}{s-\frac{1}{2}} - \gamma + O\left(s-1/2\right). \tag{4.80}$$

Applying these to (4.76), we immediately find the value of $Z\left(\frac{1}{2},\, Q\right)$,

$$Z_2\left(\frac{1}{2},\, Q\right) = 2a^{-1/2}\gamma - 2a^{-1/2}\log(4\pi) + 2a^{-1/2}\log(k) + 8a^{-1/2}\sum_{n=1}^{\infty} d(n)\,\cos\left(n\pi b/a\right)\,K_0\left(2\pi k\, n\right). \tag{4.81}$$

Now, we study the series at the right-hand side of (4.81): from a well-known inequality for $K_0(z)$, [14, 106],

$$K_0(z) \leq \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\left[1 - \frac{1}{8z} + \frac{9}{128z^2}\right], \tag{4.82}$$

we can deduce that the series in (4.81) is bounded as follows

$$\left|\sum_{n=1}^{\infty} d(n)\, \cos\left(n\pi b/a\right)\, K_0\left(2\pi k\, n\right)\right| \leq \sum_{n=1}^{\infty} d(n)\, K_0(2\pi kn)$$

$$\leq \sum_{n=1}^{\infty} n \left(\frac{1}{4kn}\right)^{1/2} e^{-2\pi kn} \left[1 - \frac{1}{16\pi kn} + \frac{9}{512\pi^2 k^2 n^2}\right]$$

$$\leq \frac{1}{2\sqrt{k}} \left[\sum_{n=1}^{\infty} n\, e^{-2\pi kn} - \frac{1}{16\pi k} e^{-2\pi kn} + \frac{9}{512\pi^2 k^2}\, e^{-2\pi kn}\right]$$

$$\leq \frac{e^{-2\pi k}}{2\sqrt{k}} \left[\left(\frac{9}{512\pi^2 k^2} - \frac{1}{16\pi k}\right) \frac{e^{2\pi k}}{e^{2\pi k} - 1} + \frac{e^{4\pi k}}{\left(e^{2\pi k} - 1\right)^2}\right], \tag{4.83}$$

which tends to zero as $k \to \infty$.

Therefore, there exists $k_0 \in \mathbb{R}_+$ such that, for all $k \geq k_0$, the expression (4.83) is arbitrarily small and bounded by $\frac{e^{-2\pi k}}{2\sqrt{k}}$.

Hence, if we take $k > \max\{4\pi e^{-\gamma},\, k_0\}$, we see that the right-hand side of (4.81) is positive, since k_0 can be taken large enough to assure this.

It is now clear that if we take $|d|$ suitably large, then we have $Z\left(\frac{1}{2},\, Q\right) > 0$.

Now, the falsity of the conjecture follows immediately: if, for some quadratic form with large $|d|$, $Z_2\left(\frac{1}{2},\, Q\right) > 0$ and $\lim_{s\to 1^-}(s-1)\, Z_2(s,\, Q) = -\frac{2\pi}{\sqrt{|d|}} < 0$ (see Corollary 4.1 above), we see that $Z_2(s,\, Q)$ goes to $-\infty$ as $s \to 1^-$ through real values of s. Since $Z_2\left(\frac{1}{2},\, Q\right) > 0$, it follows from the intermediate value theorem that exists $s_0 \in]\frac{1}{2},\, 1[$ for which we have $Z_2\left(s_0,\, Q\right) = 0$, contradicting conjecture 4.1. ■

We can also prove the falsity of Conjecture 4.2 by using a similar approach.

Corollary 4.3.2.: Conjecture 4.2 is false.

Proof: It suffices to find a Dirichlet character χ and a positive definite real quadratic form Q for which this conjecture does not hold.

Assume that $\chi = \chi_{1,6}$, i.e., the principal character modulo 6 and Q is the trivial quadratic form $Q(m,n) := Q_0(m,n) = m^2 + n^2$. From the first chapter (see relation (1.22)), we know that

$$L\left(2s-1,\, \chi_{1,6}\right) = \zeta(2s-1) \prod_{p|6} \left(1 - \frac{1}{p^{2s-1}}\right)$$

$$= \zeta\left(2s-1\right) \left(1 - 2^{1-2s}\right) \left(1 - 3^{1-2s}\right), \tag{4.84}$$

which has a double zero at $s = \frac{1}{2}$. Moreover, $d_{\chi_{1,6}}(n) = \sum_{d|n} \chi_{1,6}(d) > 0$ since $\chi_{1,6}(n)$ is principal.

Note that, in this case, by Remark 4.2. we cannot apply formula (4.30) directly, since this depends on the validity of the character version of Poisson's summation formula.

However, we can apply (4.31) without any extra assumption, which yields

$$\begin{aligned} Z_2\left(\frac{1}{2},\, Q_0,\, \chi_{1,6}\right) &= 2\lim_{s\to\frac{1}{2}} \Gamma\left(s-\frac{1}{2}\right)\, L(2s-1,\, \chi_{1,6}) + 8\sum_{n=1}^{\infty} d_{\chi_{1,6}}(n)\; K_0\,(2\pi n) \\ &= 8\sum_{n=1}^{\infty} d_{\chi_{1,6}}(n)\; K_0\,(2\pi n) > 0, \end{aligned} \tag{4.85}$$

from the positivity of $d_{\chi_{1,6}}(n)$ and $K_0(z)$.

Now, from Corollary 4.1, we see that $Z_2\,(s,\, Q_0,\, \chi_{0,6})$ has a simple pole at $s=1$ with residue $\frac{\pi}{3}$, so that $\lim_{s\to 1^-} Z_2(s,\, Q_0,\, \chi_{0,6}) = -\infty$ and the theorem follows by the argument given before. ■

Furthermore, if χ is a nonprincipal, primitive, even and real Dirichlet character modulo ℓ, the positivity of $Z_2\left(\frac{1}{2},\, Q,\, \chi\right)$, for Q having large $|d|$, actually follows from the positivity of $L(1,\chi)$, a fact that plays an important role in Dirichlet theorem about prime numbers in arithmetic progressions [5, 10].

We state this Corollary below without proof, leaving this to the next chapter where we shall prove a result regarding the value $L(1,\chi)$ for a nonprincipal character χ.

Corollary 4.3.3.: Let χ be a nonprincipal, real and even character modulo ℓ. If Q is a positive definite real quadratic form with a sufficiently large $|d|$, then $Z_2\left(\frac{1}{2},\, Q,\, \chi\right) > 0$.

Finally, we present a last corollary of Selberg-Chowla's formula, related with a series involving the arithmetic function σ_{it}. Although it is possible to derive a summation formula involving this divisor function (the reader can consult Yakubovich's paper [119] where such a summation formula is proved), even with this at our disposal it is a very difficult problem to find a suitable asymptotic estimate for

$$H\left(\frac{1}{2}+it\right) = \sum_{n=1}^{\infty} n^{it}\sigma_{-2it}(n)\, K_{it}(2\pi n), \quad t\to\infty \tag{4.86}$$

directly from the asymptotic behavior of $K_{it}(2\pi k\, n)$ as $|t|\to\infty$ [115]. However, using Selberg-Chowla formula (4.25) together with the estimate (3.9) we can also deduce the following Corollary:

Corollary 4.4.: For every positive ϵ, $H\left(\frac{1}{2}+it\right)$ obeys to the asymptotic estimate

$$H\left(\frac{1}{2}+it\right)=O\left(|t|^{\frac{5}{12}+\epsilon}\,e^{-\frac{\pi}{2}|t|}\right),\quad |t|\to\infty. \tag{4.87}$$

Proof: Note that $H(s)$ corresponds to $H(s,\,Q)$ when $Q=Q_0$.

Using the notation $\zeta_2(s)=Z_2\,(s,\,Q_0)$ and invoking Selberg-Chowla formula (4.25), we derive

$$\begin{aligned} a^s k^{s-1/2}\pi^{-s}\Gamma(s)\,\zeta_2(s) &= 2k^{s-1/2}\Gamma\,(s)\,\zeta(2s)+ \\ &+2k^{1/2-s}\pi^{1/2-s}\Gamma\left(s-\frac{1}{2}\right)\zeta\,(2s-1)+8\,H(s). \end{aligned} \tag{4.88}$$

By Jacobi's two-square theorem, we know that $\zeta_2(s)=4\zeta(s)\,L(s,\chi_4)$ (see eq. (4.11) above), and so

$$\zeta_2\left(\frac{1}{2}+it\right)=4\zeta\left(\frac{1}{2}+it\right)\,L\left(\frac{1}{2}+it,\,\chi_4\right)=O\left(|t|^{\frac{5}{12}+\epsilon}\right), \tag{4.89}$$

from Lemma 3.A. given in the third chapter, which furnished the asymptotic estimate for $\zeta\left(\frac{1}{2}+it\right)$,

$$\zeta\left(\frac{1}{2}+it\right)=O\left(|t|^{1/6+\epsilon}\right).$$

From Stirling's formula and Lindelöf's estimate $\zeta(\sigma+it)=O\left(|t|^{\frac{1-\sigma}{2}+\epsilon}\right)$, we see that the first and the second terms in (4.88) have the asymptotic order $O\left(|t|^{\epsilon}e^{-\frac{\pi}{2}|t|}\right)$.

Thus, using (4.89) we easily conclude that $H(s)$ follows (4.87). ■

The previous corollaries showed that Selberg-Chowla formula furnishes a very important form of describing the meromorphic continuation of $Z_2\,(s,\,Q)$ and $Z_2\,(s,\,Q,\,\chi)$, providing also important corollaries derived in previous chapters.

However, to prove the Main theorem above, we need to assert functional equations for $Z_2\,(s,\,Q)$ and $Z_2\,(s,\,Q,\,\chi)$, as they assure, when stated in the form of those in (4.12) and (4.15), the existence of a transform of Hankel-type (4.20).

In the next section, we prove that these Epstein $\zeta-$functions satisfy suitable functional equations. Moreover, we rederive Kronecker's limit formula and provide, for the first time, an analogue of this formula for principal Dirichlet characters.

4.2 Preliminary Results II - The functional equation and Kronecker's limit formula

We start this section by proving the functional equations that $Z_2\left(s,\,Q\right)$ and $Z_2\left(s,\,Q,\,\chi\right)$ satisfy, which are given in the following theorem

Theorem 4.3.: The Epstein $\zeta-$functions (4.7) and (4.19) satisfy the functional equations

$$\left(\frac{2\pi}{\sqrt{|d|}}\right)^{-s}\Gamma(s)\,Z_2(s,\,Q)=\left(\frac{2\pi}{\sqrt{|d|}}\right)^{s-1}\Gamma(1-s)\,Z_2(1-s,\,Q), \tag{4.90}$$

$$G(\overline{\chi})\left(\frac{2\pi}{\sqrt{|d|}}\right)^{-s}\Gamma(s)\,Z_2(s,\,Q,\,\chi)=\left(\frac{2\pi}{\sqrt{|d|}}\right)^{s-1}\Gamma(1-s)\,Z_2\left(1-s,\,Q_{\ell}^{-1},\,\overline{\chi}\right), \tag{4.91}$$

where χ is a nonprincipal, primitive and even character modulo ℓ and Q_{ℓ}^{-1} denotes the normalized version of the inverse of the quadratic form Q given by (4.23).

Proof: The proofs of (4.90) and (4.91) come from the reflection property of the entire functions $H_Q(s)$ and $H_Q(s,\,\chi)$ (see relations (4.44) and (4.45) in Lemma 4.1), together with the functional equation for $\zeta(s)$ (1.142). We can rewrite (4.25) as

$$\begin{aligned}(ak/\pi)^{s}\,\Gamma(s)\,Z_2(s,\,Q)=2\,(k/\pi)^{s}\,\Gamma(s)\zeta(2s)+2k^{1-s}\pi^{\frac{1}{2}-s}\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)\\ +\,8k^{1/2}H_Q(s).\end{aligned} \tag{4.92}$$

If we replace s by $1-s$, the right-hand side of (4.92) equals to

$$\begin{aligned}&2\,(k/\pi)^{1-s}\,\Gamma(1-s)\zeta(2-2s)+2k^{s}\pi^{s-\frac{1}{2}}\Gamma\left(\frac{1}{2}-s\right)\zeta(1-2s)+8k^{1/2}H_Q(1-s)\\ &=2k^{1-s}\pi^{\frac{1}{2}-s}\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)+2\,(k/\pi)^{s}\,\Gamma(s)\zeta(2s)+8k^{1/2}H_Q(s),\end{aligned}$$

by the functional equation for $\zeta(s)$ and the reflection properties of $H_Q(s)$. Therefore, we conclude that the left-hand side of (4.92) is invariant under the reflection $s\leftrightarrow 1-s$, which proves (4.90), after using the definition $k=\sqrt{|d|}/2a$.

To prove (4.91), use both representations for $Z_2(s,\,Q,\,\chi)$ in (4.30) and (4.31),

$$\begin{aligned}Z_2(s,\,Q,\,\chi)=2a^{-s}L(2s,\chi)+\\ +8\,a^{-s}\pi^{s}\frac{G(\chi)}{\Gamma(s)}\,k^{\frac{1}{2}-s}\ell^{-s-\frac{1}{2}}\sum_{n=1}^{\infty}\sigma_{2s-1,\overline{\chi}}(n)\,n^{\frac{1}{2}-s}\cos\left(\pi\frac{b}{a}\frac{n}{\ell}\right)\,K_{s-\frac{1}{2}}\left(\frac{2\pi kn}{\ell}\right),\end{aligned} \tag{4.93}$$

$$Z_2(s,\,Q,\,\chi) = 2c^{-s}\,k'^{1-2s}\pi^{1/2}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\,L(2s-1,\,\chi)+$$
$$+\frac{8\pi^s c^{-s}}{\Gamma(s)}\,k'^{\frac{1}{2}-s}\sum_{n=1}^{\infty}\sigma_{1-2s,\chi}\,(n)\;n^{s-\frac{1}{2}}\cos\left(\frac{n\pi b}{c}\right)\,K_{s-\frac{1}{2}}\left(2\pi k' n\right). \tag{4.94}$$

The reflection formula for $H_\chi(s,\,Q)$ (4.45) tells that, if we replace s by $1-s$, χ by $\overline{\chi}$ and Q by Q_ℓ^{-1} at the infinite series in (4.93), we get precisely the very same infinite series as the one lying at the right-hand side of (4.94). Therefore, using this symmetry and appealing to the functional equation for the Dirichlet $L-$function at both sides, we are able to derive (4.91). ■

The functional equations established in the previous theorem allow to prove suitable estimates for $Z_2(s,\,Q)$ and $Z_2(s,\,Q,\,\chi)$ when $|t|\to\infty$ via Phragmén-Lindelöf principle [103]. These will be very useful in order to impose the conditions of the Main theorem. We state these at the following corollary.

Corollary 4.5.: On Estimates for $Z_2(s,\,Q)$ and $Z_2\,(s,\,Q,\,\chi)$: Let χ be a nonprincipal, primitive and even character modulo ℓ and Q a positive definite real quadratic form. We have the following estimates

$$|Z_2(s,\,Q)|\,,\,|Z_2(s,\,Q,\,\chi)| = \begin{cases} O(1) & \sigma>1 \\ O\left(|t|^{1-\sigma+\epsilon}\right) & 0\leq\sigma\leq 1 \\ O\left(|t|^{1-2\sigma}\right) & \sigma<0. \end{cases} \tag{4.95}$$

Proof: It suffices to adapt the proof of Proposition 2.2. We know that, for every $\epsilon>0$, $Z_2\,(1+\epsilon+it,\,Q)$ is bounded, since it is given by the absolutely convergent series (4.7). Moreover, by the functional equation (4.90) we see that, for a sufficiently large t,

$$|Z_2\,(-\epsilon-it,\,Q)\,| < C_1\left|\frac{\Gamma(1+\epsilon+it)}{\Gamma(-\epsilon-it)}\,Z_2(1+\epsilon+it,\,Q)\right| < C_2\,|t|^{1+2\epsilon}, \tag{4.96}$$

by Stirling's formula.

Since $Z_2(s,\,Q)$ has finite order in the strip $-\epsilon\leq\sigma\leq 1+\epsilon$, the usual Phragmén-Lindelöf principle implies (4.95) for $Z_2\,(s,\,Q)$. Since $Z_2(s,\,Q,\,\chi)$ has a similar functional equation to $Z_2(s,\,Q)$, it is simple to deduce (4.95) for $Z_2\,(s,\,Q,\,\chi)$ as well. ■

We introduce now an important result, due to Kronecker [37], which inspired several authors in generalizing the work of Paul Epstein, including Shintani [91], who extended some of its formulations to quadratic forms which are not necessarily positive definite.

We extend this important formula to a character version arriving to a new formula, but we remark once more that this extension is possible with other arithmetic functions [21]:

Theorem 4.4.1. (Classical Kronecker's limit formula): The Epstein $\zeta-$function admits the following meromorphic expansion

$$Z_2(s,\,Q) = \frac{2\pi}{\sqrt{|d|}}\,\frac{1}{s-1} + \frac{2\pi}{\sqrt{|d|}}\left(2\gamma - \log\left(\frac{|d|}{a}\right) - 4\log(|\eta(\tau)|)\right) + O(s-1), \tag{4.97}$$

where γ is the Euler-Mascheroni constant, $\eta(\tau)$ is Dedekind's $\eta-$function (4.70) and $\tau = \frac{b+i\sqrt{|d|}}{2a} \in \mathbb{H}$.

Proof: Around $s = 1$, the functions presented in Selberg-Chowla's formula (4.25) have the meromorphic expansions

$$\zeta(2s-1) = \frac{1}{2(s-1)} + \gamma + O\left(s-1\right), \tag{4.98}$$

$$\begin{aligned}\frac{\sqrt{\pi}\,\Gamma(s-\frac{1}{2})}{\Gamma(s)} &= \pi + \pi\left[\psi\left(\frac{1}{2}\right) - \psi(1)\right](s-1) + O\left(s-1\right)^2 \\ &= \pi - 2\pi\log(2)\,(s-1) + O\left(s-1\right)^2.\end{aligned} \tag{4.99}$$

Therefore, the second term in (4.25) can be written as

$$2k^{1-2s}a^{-s}\pi^{1/2}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\zeta(2s-1) = \frac{2\pi}{\sqrt{|d|}}\,\frac{1}{s-1} + \frac{2\pi}{\sqrt{|d|}}\left(2\gamma - \log\left(\frac{|d|}{a}\right)\right) + O\left(s-1\right), \tag{4.100}$$

which gives

$$\begin{aligned}Z_2(s,\,Q) &= \frac{2}{a}\zeta(2) + \frac{2\pi}{\sqrt{|d|}}\,\frac{1}{s-1} + \frac{2\pi}{\sqrt{|d|}}\left(2\gamma - \log\left(\frac{|d|}{a}\right)\right) \\ &+ \frac{8\pi}{a\sqrt{k}}\sum_{n=1}^{\infty}\sigma_{-1}(n)\,\sqrt{n}\,\cos\left(n\pi b/a\right)K_{1/2}\left(2\pi k\,n\right) + O(s-1).\end{aligned} \tag{4.101}$$

Using once more formula (4.64) and writing $\cos(n\pi b/a)$ as a combination of complex exponentials involving $\tau := \frac{b+i\sqrt{|d|}}{2a}$, we deduce that the last term in (4.101) equals to

$$\begin{aligned}&\frac{2\pi}{ak}\sum_{n=1}^{\infty}\sigma_{-1}(n)\,e^{-2\pi kn}\left(e^{in\pi b/a} + e^{-in\pi b/a}\right) = \frac{4\pi}{\sqrt{|d|}}\sum_{n,m=1}^{\infty}\frac{1}{m}\left(e^{2\pi imn\tau} + e^{-2\pi imn\overline{\tau}}\right) \\ &= -\frac{4\pi}{\sqrt{|d|}}\sum_{n=1}^{\infty}\left[\log\left(1-e^{2\pi in\tau}\right) + \log\left(1-e^{-2\pi in\overline{\tau}}\right)\right] = -\frac{4\pi}{\sqrt{|d|}}\sum_{n=1}^{\infty}\log\left|1-e^{2\pi in\tau}\right|^2.\end{aligned} \tag{4.102}$$

Finally, from the definition of the Dedekind $\eta-$function (4.70), it is simple to see that

$$\log\left(|\eta(\tau)|^2\right) = \log\left(e^{-\frac{\pi}{12}\frac{\sqrt{|d|}}{a}}\prod_{n=1}^{\infty}\left|1-e^{2\pi in\tau}\right|^2\right) = -\frac{\pi}{12}\frac{\sqrt{|d|}}{a} + \sum_{n=1}^{\infty}\log\left|1-e^{2\pi in\tau}\right|^2, \quad (4.103)$$

which proves

$$-\frac{4\pi}{\sqrt{|d|}}\sum_{n=1}^{\infty}\log\left|1-e^{2\pi in\tau}\right|^2 = -\frac{4\pi}{\sqrt{|d|}}\log\left(|\eta(\tau)|^2\right) - \frac{\pi^2}{3a}. \quad (4.104)$$

Combining (4.101) with (4.104) and using the Basel identity $\zeta(2) = \frac{\pi^2}{6}$ (see identity (1.104) on the first chapter), we prove immediately (4.97). ■

We can extend (4.97) to the meromorphic expansion of $Z_2(s,\, Q,\, \chi)$ when $\chi = \chi_1$. As seen in the Corollary 4.1., $Z_2(s,\, Q,\, \chi_1)$ can be continued as a meromorphic function with a simple pole at $s = 1$.

Before proving an extension of (4.97), we introduce the following lemma, which gives a closed-form evaluation of the character analogue of the logarithmic function (see (2.67) in the second chapter).

Lemma 4.2: Let $\chi = \chi_1$ be the principal Dirichlet character modulo ℓ and $\log_\chi$ be defined by the power series (see equation (2.92) at the second chapter)

$$\log_\chi(1-z) = -\sum_{k=1}^{\infty}\frac{\chi(k)}{k}z^k, \quad |z| < 1. \quad (4.105)$$

Then, for $|z| < 1$, $\log_{\chi_1}(1-z)$ can be expressed by

$$\log_{\chi_1}(1-z) = \log(1-z) - \sum_{p|\ell}\frac{1}{p}\log\left(1-z^p\right). \quad (4.106)$$

Proof: If χ denotes the trivial character, i.e., the principal character modulo $\ell = 1$, the second term in (4.106) vanishes and so the statement is clear from the classical Mercator expansion for the logarithm

Now, if $\ell > 1$, recall that $\chi_1(n) = 1$ iff $(n, \ell) = 1$ by definition of principal character. Therefore, (4.105) can be expressed as

$$\begin{aligned}\log_{\chi_1}(1-z) &= -\sum_{k=1}^{\infty}\frac{\chi_1(k)}{k}z^k = -\sum_{k=1}^{\infty}\frac{z^k}{k}+\sum_{p|\ell}\sum_{k=1}^{\infty}\frac{z^{kp}}{kp}\\ &= \log(1-z)+\sum_{p|\ell}\frac{1}{p}\sum_{k=1}^{\infty}\frac{(z^p)^k}{k}\\ &= \log(1-z)-\sum_{p|\ell}\frac{1}{p}\log\left(1-z^p\right),\end{aligned}$$

proving (4.106). ■

Using this lemma, we can now prove an extended version of Kronecker's limit formula, which seems to be unnoticed in the literature.

Theorem 4.4.2. (Extended version of Kronecker's Limit formula - Character analogues): Let $\chi=\chi_1$ be the principal character modulo $\ell>1$. Then the single-weighted Epstein $\zeta-$function admits the following meromorphic expansion around $s=1$

$$\begin{aligned}Z_2(s,\,Q,\,\chi_1) &= \frac{2\pi\varphi(\ell)}{\ell\sqrt{|d|}}\,\frac{1}{s-1}-\frac{2\pi}{\ell\sqrt{|d|}}\left[\varphi(\ell)\,\log\left(\frac{|d|}{c}\right)+2\sum_{r=1}^{\ell-1}\chi_1(r)\,\psi\left(\frac{r}{\ell}\right)+2\log(\ell)\varphi(\ell)\right]\\ &+\frac{4\pi}{\sqrt{|d|}}\,\log\left(\frac{\prod_{p|\ell}|\eta(p\tau')|^{2/p}}{|\eta(\tau')|^2}\right)+\frac{\pi^2}{3c}\left[\sum_{p|\ell}\frac{1}{p}-1\right]+O(s-1),\end{aligned}\tag{4.107}$$

where $\tau'=\frac{b+i\sqrt{|d|}}{2c}\in\mathbb{H}$, $\log_{\chi}(1-z)$ is given by (4.105) and φ and ψ denote, respectively, Euler's totient and Gauss's digamma functions.

Proof: Use the representation (4.31) of $Z_2(s,\,Q,\,\chi_1)$,

$$\begin{aligned}Z_2(s,\,Q,\,\chi_1) &= 2\,c^{-s}k'^{1-2s}\pi^{1/2}\frac{\Gamma\left(s-1/2\right)}{\Gamma(s)}\,L(2s-1,\,\chi_1)+\\ &+\frac{8}{\Gamma(s)}c^{-s}\pi^s\,k'^{\frac{1}{2}-s}\sum_{n=1}^{\infty}\sigma_{1-2s,\,\chi_1}\left(n\right)\,n^{s-\frac{1}{2}}\cos\left(\frac{n\pi b}{c}\right)\,K_{s-\frac{1}{2}}\left(2\pi k'n\right).\end{aligned}\tag{4.108}$$

To study the first term in the right-hand side of (4.108), recall the well-known relation (1.86) in the first chapter

$$L(s,\,\chi_1)=\frac{\varphi(\ell)}{\ell}\,\frac{1}{s-1}-\frac{1}{\ell}\sum_{r=1}^{\ell-1}\chi_1(r)\,\psi\left(\frac{r}{\ell}\right)-\frac{\varphi(\ell)}{\ell}\log(\ell)+O(s-1).\tag{4.109}$$

Now, replacing s by $2s-1$ in (4.109) and using the meromorphic expansions (4.99) and (4.109), we deduce that

$$Z_2(s,\,Q,\,\chi_1) = \frac{2\pi\varphi(\ell)}{\ell\sqrt{|d|}}\,\frac{1}{s-1} - \frac{2\pi}{\ell\sqrt{|d|}}\left[\varphi(\ell)\,\log\left(\frac{|d|}{c}\right) + 2\sum_{r=1}^{\ell-1}\chi_1(r)\,\psi\left(\frac{r}{\ell}\right) + 2\log(\ell)\varphi(\ell)\right]$$
$$+\,\frac{8\pi}{c}\,k'^{-\frac{1}{2}}\,\sum_{n=1}^{\infty}\sigma_{-1,\,\chi_1}(n)\;n^{1/2}\cos\left(\frac{n\pi b}{c}\right)\,K_{1/2}\left(2\pi k' n\right) + O(s-1).$$

Once more, appealing to the particular value of the modified Bessel function $K_{1/2}(z)$ (4.64) and using the same manipulation as in (4.102), we obtain

$$Z_2(s,\,Q,\,\chi_1) = \frac{2\pi\varphi(\ell)}{\ell\sqrt{|d|}}\,\frac{1}{s-1} - \frac{2\pi}{\ell\sqrt{|d|}}\left[\varphi(\ell)\,\log\left(\frac{|d|}{c}\right) + 2\sum_{r=1}^{\ell-1}\chi_1(r)\,\psi\left(\frac{r}{\ell}\right) + 2\log(\ell)\varphi(\ell)\right]$$
$$-\,\frac{4\pi}{\sqrt{|d|}}\,\sum_{n=1}^{\infty}\left[\log_{\chi_1}\left(1-e^{2\pi i n\tau'}\right) + \log_{\chi_1}\left(1-e^{-2\pi i n\overline{\tau'}}\right)\right] + O(s-1), \qquad (4.110)$$

which can be established as a character analogue of (4.97). Using lemma 4.2 and (4.103), it is straightforward to obtain

$$\sum_{n=1}^{\infty}\left[\log_{\chi_1}\left(1-e^{2\pi i n\tau'}\right) + \log_{\chi_1}\left(1-e^{-2\pi i n\overline{\tau'}}\right)\right]$$
$$= \sum_{n=1}^{\infty}\left[\log\left(\left|1-e^{2\pi i n\tau'}\right|^2\right) - \sum_{p|\ell}\frac{1}{p}\log\left(\left|1-e^{2\pi i n p\tau'}\right|^2\right)\right]$$
$$= \frac{\pi}{12}\,\frac{\sqrt{|d|}}{c} + \log\left(|\eta(\tau')|^2\right) - \sum_{p|\ell}\frac{1}{p}\left(\frac{\pi}{12}\,\frac{\sqrt{|d|}}{c} + \log\left(|\eta(p\tau')|^2\right)\right)$$
$$= \frac{\pi}{12}\,\frac{\sqrt{|d|}}{c}\left[1-\sum_{p|\ell}\frac{1}{p}\right] + \log\left(\frac{|\eta(\tau')|^2}{\prod_{p|\ell}|\eta(p\tau')|^{2/p}}\right), \qquad (4.111)$$

which concludes the proof. ■

Remark 4.3: Other analogues of Kronecker's limit formula (4.97) can be established if we use the Epstein $\zeta-$function in the form given by equation (4.37) at Remark 2. Indeed, it is not hard to show that Theorem 4.4.2. can be also extended if we replace $\chi(m)$ for any periodic arithmetic function $a(m)$.

We are now ready to prove the Main theorem of this section and to study two interesting examples related with it.

4.3 Main results: Summation formulas involving quadratic forms for the Müntz class

In this final section we prove the main theorem stated at the beginning of the chapter. However, as promissed at the introduction, we need to understand first why a summation formula involving any positive definite quadratic form does not hold under the same conditions as Poisson's and Voronoï's, i.e., that our attempts do not hold for the 'L_2 class'. In the next discussion, we start by trying to adapt the arguments given for this class of functions and replacing the previous arithmetic functions $\chi(n)$ and $d_\chi(n)$ by $r_Q(n)$ and then we arrive to the conclusion that the previous methods cannot be applied.

Why the Müntz Class?

Let us try to adapt the arguments given in the Preliminary results of the previous chapter.

For a matter of simplicity, we shall only explain why formula (4.21) (or a version of it for the class of L_2 functions) cannot be proved by the same methods (for the other formula (4.22) it suffices to adapt this forthcoming explanation).

In analogy with the remainder function $\Delta(x)$ given in the previous chapter, we can take the following function

$$\Delta_Q(x) = {\sum_{n\leq x}}' r_Q(n) - \frac{2\pi}{\sqrt{|d|}}\, x. \tag{4.112}$$

By a simple geometric reasoning (due to Gauss) [31], it is simple to see that the function

$$R_Q(x) = {\sum_{n\leq x}}' r_Q(n),$$

counts the number of lattice points in the area defined by the condition $Q(a,b)\leq x$. Therefore, as $x\to\infty$, $R_Q(x)$ satisfies

$$R_Q(x) = A\,x + O\left(x^{1/2}\right),$$

where A is the area of the "ellipse" defined by $Q(x,y)\leq 1$, which is precisely $2\pi/\sqrt{|d|}$.

Now, it is reasonable to define a function $h_Q(x)$ (similar to the functions $h_\chi(x)$ at Chapters 2 and 3) satisfying the same estimates as $h_\chi(x)$ at the previous chapter (see eq. (3.22) there),

$$h_Q(x) = \frac{\Delta_Q(x)}{x} = \begin{cases} O\left(x^{-1/2}\right), & x\to\infty \\ O\left(1\right), & x\to 0. \end{cases}$$

Now, we can adapt every single calculation given in equations (3.24), (3.25) and (3.26) at the third chapter regarding the L_1 Mellin transform of $h_Q(x)$, $h_Q^*(s)$, which exists in the region $0<\sigma<\frac{1}{2}$.

Invoking the series representation of $Z_2(s,Q)$, it is not hard to check that the computations give (once more, try to adapt the computations expressed in equation (3.24))

$$h_Q^*(s)=\frac{Z_2\left(1-s,\,Q\right)}{1-s},\quad 0<\text{Re}(s)<\frac{1}{2}. \tag{4.113}$$

Finally, since $x^{\sigma}h_Q(x)\in L_2\left(\mathbb{R}_+,\,\frac{dx}{x}\right)$ and $Z_2(1-s,Q)/(1-s)=O\left(|t|^{\sigma-1+\epsilon}\right),\ 0<\sigma<\frac{1}{2}$, we see that the right hand-side of (4.113) is $L_2(\sigma)$ and we can apply Mellin's inversion formula for the L_2 class (2.24)

$$h_Q(x)=\underset{N\to\infty}{\text{l.i.m.}}\int\limits_{\sigma-iN}^{\sigma+iN}\frac{Z_2(1-s,\,Q)}{1-s}\,x^{-s}ds,\quad 0<\sigma<\frac{1}{2}. \tag{4.114}$$

Recall now that, to use the computations given at the previous chapter in their full potential, we need to extend (4.114) to the critical line $\sigma=\text{Re}(s)=\frac{1}{2}$. However, it is precisely at this point where we face a very hard difficulty to surpass.

At the previous chapter we were able to invoke a better asymptotic estimate for $\zeta\left(\frac{1}{2}+it\right)$ in order to solve this problem (see eq. (3.9) there). At the present point, as far as we know, there is no better estimate than (4.95) for the Epstein $\zeta-$function equipped with a general positive definite quadratic form.

Thus, by using the estimate (4.95) we can only find that $Z_2\left(\frac{1}{2}+it,\,Q\right)=O\left(|t|^{\frac{1}{2}+\epsilon}\right)$, which means that, with an asymptotic behavior like this, it is not possible for us to conclude that $\frac{Z_2(1-s,\,Q)}{1-s}\in L_2\left(\frac{1}{2}\right)$ and, therefore, to extend (4.114) to the critical line and use the symmetries of $Z_2(s,\,Q)$ there.

With this impossibility, we cannot use the approach given at previous chapters. So, our strategy now is to change the class of functions and still prove a summation formula which is able to provide interesting examples. Although in a similar set of conditions, our hypothesis is more general than the one posed by Dixon and Ferrar [40] in two directions: firstly, we assume that Q is any binary and positive definite quadratic form, while they only took the simplest case $Q(m,n)=m^2+n^2$. Secondly, we impose the condition to f and their derivatives to be bounded in the form $O(x^{-\alpha-k})$, while they impose the weaker condition $O\left(\exp(-x^{u})\right)$, $u>0$.

Having briefly discussed why we need to change the class of functions to proceed with our considerations, we now deal with a direct proof of the Main Theorem. First, we establish a useful lemma, which allows to represent the left-hand side of (4.21) and (4.22) as an integral of Mellin type (one should compare the proof of this lemma, which works perfectly in the L_1-class with the proofs of the Main Lemmas given in the previous chapters).

Main Lemma: Representation for Müntz in $-\frac{1}{2} < \sigma < 0$

Let χ be a nonprincipal, primitive and even character modulo ℓ and Q a binary, positive definite and real quadratic form modulo ℓ with integer coefficients.

Assume also that $f \in \mathcal{M}_{\alpha,3}$ and $-\frac{1}{2} < \sigma < 0$. Then the following representations are valid

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} Z_2(s,\, Q)\, f^*(s)\, x^{-s} ds = \sum_{n=1}^{\infty} r_Q(n)\, f(nx) - \frac{2\pi}{\sqrt{|d|}}\int_0^{\infty} f(xy)\, dy + f(0) \qquad x > 0, \tag{4.115}$$

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} Z_2(s,\, Q,\, \chi)\, f^*(s)\, x^{-s} ds = \sum_{(m,n)\neq(0,0)} \chi(m)\, f\left(Q(m,n)\, x\right), \qquad x > 0. \tag{4.116}$$

Proof: We prove (4.115) first, as the proof of (4.116) can be similarly handled. Assume, without any loss of generality, that $1 < \alpha < 2$ and let $R_Q f(x)$ denote the Möbius-type operator,

$$R_Q f\,(x) = \sum_{n=1}^{\infty} r_Q(n)\, f(nx),\ \ x > 0. \tag{4.117}$$

One can take the Mellin transform on both sides of (4.117) for $1 < \sigma < \alpha$, i.e.,

$$\begin{aligned}(R_Q f)^*(s) &= \int_0^{\infty} x^{s-1}\sum_{n=1}^{\infty} r_Q(n)\, f(nx)\, dx = \sum_{n=1}^{\infty} r_Q(n)\int_0^{\infty} x^{s-1} f(nx)\, dx \\ &= \sum_{n=1}^{\infty}\frac{r_Q(n)}{n^s}\, f^*(s) = Z_2(s,\, Q)\, f^*(s),\end{aligned} \tag{4.118}$$

by the absolute convergence of the series (4.7) for $\mathrm{Re}(s) = \alpha > 1$.

Now, let us see that we can write the Möbius-type transformation $R_Q f(x)$ as a L_1-Mellin inverse transform: from the absolute convergence of the Dirichlet series for $Z_2(s,\, Q)$ when $\mathrm{Re}(s) > 1$,

$$\int_{\sigma-i\infty}^{\sigma+i\infty} \left|Z_2(s,\, Q)\, f^*(s)\, x^{-s}\right|\, |ds| \leq Z_2(\sigma,\, Q)\, x^{-\sigma}\int_{\sigma-i\infty}^{\sigma+i\infty} |f^*(s)|\, |ds| < \infty,$$

and so we have $Z_2(s, Q)\, f^*(s) \in L_1(\sigma - i\infty, \sigma + i\infty)$, so that the inversion theorem for the Mellin transform in the L_1-class (2.22) yields

$$R_Q f(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} Z_2(s, Q)\, f^*(s)\, x^{-s} ds, \quad 1 < \sigma < \alpha. \tag{4.119}$$

On the other hand, the integrand $Z_2(s, Q)\, f^*(s)\, x^{-s}$ is analytic in the strip $0 < \sigma < \alpha$ with the exception of a simple pole located at $s = 1$ with residue $\frac{2\pi}{\sqrt{|d|}}$ (see Corollary 4.1).

Recalling the asymptotic estimate for the Epstein $\zeta-$function (4.95) we get, for every positive ϵ,

$$Z_2(s, Q)\, f^*(s) = O\left(|t|^{-2-\sigma+\epsilon}\right), \ |t| \to \infty, \quad 0 \le \sigma \le 1, \tag{4.120}$$

since, by hypothesis, $f^*(\sigma + it) = O\left(|t|^{-3}\right)$, $-3 < \sigma < \alpha$ by Lemma 2.2.

Consequentely, $Z_2(s, Q)\, f^*(s) \in L_1(\sigma - i\infty, \sigma + i\infty)$ for all $0 \le \sigma \le 1$ and so, under the conditions of the Residue Theorem, we are allowed to shift the line of integration on the integral (4.119) into the region $0 < \sigma < 1$, with an additional counting of the residue located at the simple pole $s = 1$. Notice that, due to the estimate (4.120),

$$\lim_{|t| \to \infty} \int_{a+it}^{b+it} Z_2(s, Q)\, f^*(s)\, x^{-s} ds = 0, \quad 0 < a < b < \alpha \tag{4.121}$$

and so the only contribution in the change of the line of the integration is due to the residue coming from the simple pole $s = 1$, which can be written as (see Corollary 4.1),

$$\text{Res}_{s=1}\left[Z_2(s, Q)\, f^*(s)\, x^{-s}\right] = \frac{2\pi}{\sqrt{|d|}\, x}\, f^*(1) = \frac{2\pi}{\sqrt{|d|}} \int_0^\infty f(xy)\, dy. \tag{4.122}$$

Therefore, for $x > 0$, we arrive to the identity of Müntz type,

$$\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} Z_2(s, Q)\, f^*(s)\, x^{-s} ds = \sum_{n=1}^{\infty} r_Q(n)\, f(nx) - \frac{2\pi}{\sqrt{|d|}} \int_0^\infty f(xy)\, dy, \quad 0 < \sigma < 1. \tag{4.123}$$

Furthermore, we can now move the line of integration on the left-hand side of (4.123) to the strip $-\frac{1}{2} < \sigma < 0$, taking into account the pole of $f^*(s)$ at $s = 0$: this change in the line of integration is possible due to the fact that

$$Z_2(s, Q)\, f^*(s) = O\left(|t|^{-2-2\sigma}\right), \quad \sigma < 0, \tag{4.124}$$

which implies that, for $-\frac{1}{2} < \sigma \leq 0$, $Z_2(s,\, Q)\, f^*(s) \in L_1(\sigma - i\infty,\, \sigma + i\infty)$ and the integrals over the horizontal segments with height $|t| \to \infty$ vanish (as in (4.121)).

So we just need to find the residue located at $s = 0$: as remarked in Lemma 2.2. of the second chapter of this thesis (see also [73, 116]), $f^*(s)$ has a simple pole at $s = 0$ with residue $f(0)$, and so

$$\text{Res}_{s=0}\left[Z_2(s,\, Q)\, f^*(s)\, x^{-s}\right] = Z_2(0,\, Q)\, f(0) = -f(0). \tag{4.125}$$

Finally, after combining (4.123) with (4.125), we conclude the proof of (4.115).

To prove (4.116) we apply the same estimates. Since $Z_2\,(s,\, Q,\, \chi)$ is entire by Corollary 4.1, when we shift the line of integration to the region $0 < \sigma < 1$ we do not count the pole at $s = 1$. Furthermore, it is easily seen from the functional equation (4.91) that $Z_2(s,\, Q,\, \chi)$ has a simple zero at $s = 0$, which means that this point is a removable singularity of $Z_2(s,\, Q,\, \chi)\, f^*(s)\, x^{-s}$. This implies that $Z_2(s,\, Q,\, \chi)\, f^*(s)\, x^{-s}$ is analytic in the region $-\frac{1}{2} < \sigma < \alpha$, and so Cauchy's Theorem, together with the previous estimates (4.120), (4.124) proves (4.116). ■

After this lemma, we are ready to prove the Main theorem. As remarked at the beginning of this chapter, the argument to prove it follows similar computations as the one in the previous chapter for the odd case and so we shall omit some computations which these have in common.

Proof of the Main Theorem: We will only prove the first summation formula (4.21), since the proof of (4.22) follows the same argument.

From the Main Lemma we know that, for $\max\left(1-\alpha, -\frac{1}{2}\right) < \sigma < 0$, the integral representation is valid

$$\int\limits_{\sigma - i\infty}^{\sigma + i\infty} Z_2(s,\, Q)\, f^*(s)\, x^{-s} ds = \sum_{n=1}^{\infty} r_Q(n)\, f(nx) - \frac{2\pi}{\sqrt{|d|}} \int\limits_0^{\infty} f(xy)\, dy + f(0) \qquad x > 0.$$

Consider the Mellin integral at the left-hand side of (4.115): if we apply the functional equation for the Epstein ζ−function (4.90),

$$Z_2(s,\, Q) = \left(\frac{2\pi}{\sqrt{|d|}}\right)^{2s-1} \frac{\Gamma(1-s)}{\Gamma(s)}\, Z_2(1-s,\, Q),$$

and perform a change of variables $1 - s \leftrightarrow s$, we can write it as

$$\frac{2\pi}{\sqrt{|d|}x}\, \frac{1}{2\pi i} \int\limits_{\sigma' - i\infty}^{\sigma' + i\infty} \frac{\Gamma(s)}{\Gamma(1-s)}\, Z_2(s,\, Q)\, f^*(1-s) \left(\frac{4\pi^2}{|d|\, x}\right)^{-s} ds, \tag{4.126}$$

where $1 < \sigma' < \min\left(\alpha,\, \frac{3}{2}\right)$.

Since $Z_2(s,\, Q)$ can be expressed as the absolutely convergent series (4.9) in the strip $1 < \text{Re}(s) < \min\left(\alpha,\, \frac{3}{2}\right)$, we can write $Z_2(s, Q)$ in this form and then interchange the series with the integral, which is also possible since $\Gamma(s)/\Gamma(1-s)\, f^{*}(1-s) = O\left(|t|^{2\sigma'-4}\right) \in L_1(\sigma'-i\infty,\, \sigma'+i\infty)$.

Doing this and using (4.115), we obtain

$$\sum_{n=1}^{\infty} r_Q(n)\, f(nx) - \frac{2\pi}{\sqrt{|d|}} \int\limits_0^{\infty} f(xy)\, dy + f(0) = \frac{2\pi}{\sqrt{|d|}x} \sum_{n=1}^{\infty} r_Q(n)\, g\left(\frac{4\pi^2 n}{|d|\, x}\right), \tag{4.127}$$

where $g(x)$ can be described by the Mellin inversion formula in L_1

$$g(x) = \frac{1}{2\pi i} \int\limits_{\sigma'-i\infty}^{\sigma'+i\infty} \frac{\Gamma(s)}{\Gamma(1-s)}\, f^{*}(1-s)\, x^{-s} ds, \qquad 1 < \sigma' < \min\left(\alpha,\, \frac{3}{2}\right). \tag{4.128}$$

Note that we have seen this transform in the third chapter, which was related with the odd character version of Voronoï's summation formula (see equation (3.65) on the third chapter). On the right-hand side of (4.128), we can shift the line of integration into the region $0 < \mu = \text{Re}(s) < \frac{1}{2}$. Since the integrand of (4.128) is analytic on the strip $0 < \text{Re}(s) < \min\left(\alpha,\, \frac{3}{2}\right)$ and belongs to $L_1\left(\sigma - i\infty,\, \sigma + i\infty\right)$ for $0 < \sigma < \frac{3}{2}$, Cauchy's theorem allows to write

$$\int\limits_{\sigma'-i\infty}^{\sigma'+i\infty} \left(\frac{4\pi^2 n}{|d|\, x}\right)^{-s} \frac{\Gamma(s)}{\Gamma(1-s)}\, f^{*}(1-s)\, ds = \int\limits_{\mu-i\infty}^{\mu+i\infty} \left(\frac{4\pi^2 n}{|d|\, x}\right)^{-s} \frac{\Gamma(s)}{\Gamma(1-s)}\, f^{*}(1-s)\, ds, \qquad 0 < \mu < \frac{1}{2}. \tag{4.129}$$

Now, to compute the integral on the right-hand side of (4.129), we appeal to equations (3.63) and (3.65) at the third chapter, as well as Parseval's equality to derive

$$g(x) = -\frac{1}{\sqrt{x}} \int\limits_0^{\infty} \sqrt{y}\, J_1(2\sqrt{xy})\, f'(y)\, dy, \tag{4.130}$$

which can be further simplified by appealing to an integration by parts: recalling the properties of differentiation of the Bessel function $J_\nu(x)$ ([106] or see eq. (3.66) on the third chapter), we easily derive

$$g(x) = -\frac{1}{\sqrt{x}} \left[\sqrt{y}\, J_1\left(2\sqrt{xy}\right)\, f(y)\right]_0^{\infty} + \int\limits_0^{\infty} f(y)\, J_0(2\sqrt{xy})\, dy. \tag{4.131}$$

Finally, to get (4.21), we just need to justify that the boundary terms in (4.131) vanish. To do so, note that $J_1(0) = 0$ and

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) + O\left(x^{-3/2}\right), \quad x \to \infty,$$

which means that exists $M > 0$ such that, for every $y > M$,

$$\left|\sqrt{\frac{y}{x}} J_1\left(2\sqrt{xy}\right) f(y)\right| \leq \frac{M}{\sqrt{\pi} x^{3/4}} \left|y^{1/4} f(y)\right|. \tag{4.132}$$

From the fact that $f(y) = O\left(y^{-\alpha}\right), \ \alpha > 1$, we can easily conclude that the right-hand side of (4.132) vanishes for $y \to \infty$ and this gives the transform

$$g(x) = \int_0^\infty f(y) \, J_0\left(2\sqrt{xy}\right) \, dy, \tag{4.133}$$

which, together with (4.127) establishes (4.21).

To obtain the character analogue (4.22), we proceed analogously: invoke (4.116) for $\max\left(1 - \alpha, \, -\frac{1}{2}\right) < \sigma < 0$ and apply the functional equation for $Z_2\left(s, \, Q, \, \chi\right)$ (4.91) in the integral on its left-hand side. Using the same computations as above and invoking the elementary relation for even characters, $G(\chi) \, G(\overline{\chi}) = \ell$, we obtain (4.22). ■

4.4 Examples:

In the following lines, we give two important examples of the summation formulas proved above. They generalize the ones given in [28,40] for an arbitrary quadratic form, so it is very important for us to work these in general lines. It is also interesting to note that the second example will be used to give a brief explanation of the proof of Corollary 5.2.3 given in the next chapter.

Example 4.1.: A generalization of Dixon and Ferrar's example

Let us consider, for $\alpha > 0$ and $\text{Re}(\nu) > 0$, the following function

$$f_\nu(x) = x^{\frac{\nu}{2}} K_\nu\left(2\pi\sqrt{\alpha x}\right). \tag{4.134}$$

By the asymptotic estimates for $K_\nu(z)$, we immediately have that $f_\nu(0) = \frac{\Gamma(\nu)}{2\pi^\nu \alpha^{\nu/2}}$ and $f_\nu(x) = O\left(x^{\frac{\nu}{2}-\frac{1}{4}} e^{-2\pi\sqrt{\alpha x}}\right)$. Moreover, by the elementary relations for the modified Bessel function

$$\frac{d}{dx}\left(x^\nu K_\nu(x)\right) = -x^\nu K_{\nu-1}(x),$$

we can easily see that $f^{(k)}(x) = O\left(e^{-C\sqrt{x}}\right)$, for $k = 1,\, 2,\, 3$. Therefore, $f(x) \in \mathcal{M}_{\alpha,3}$, $\alpha > 1$, and satisfies the conditions of the Main Theorem in this chapter. The Mellin transform of $f_{\nu}(x)$ is given by

$$f_{\nu}^{*}(s) = \frac{1}{2}\left(\pi\sqrt{\alpha}\right)^{-2s-\nu}\,\Gamma(s)\,\Gamma(s+\nu), \tag{4.135}$$

which can be deduced by the relation given in equation (3.58) of the third chapter or by looking at Example 2.5 of the second chapter. To find the transform $g_{\nu}(x)$ we note that, from (4.129),

$$g_{\nu}^{*}(s) = \frac{1}{2}\left(\pi\sqrt{\alpha}\right)^{2s-2-\nu}\Gamma(s)\Gamma\left(1-s+\nu\right),$$

and so, from Slater's Theorem, we may compute the residues of the previous function and this gives

$$\begin{aligned} g_{\nu}(x) &= \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\frac{1}{2}\left(\pi\sqrt{\alpha}\right)^{2s-2-\nu}\Gamma(s)\Gamma\left(1-s+\nu\right)\,x^{-s}ds \\ &= \frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{k!}\left(\pi\sqrt{\alpha}\right)^{-2k-2-\nu}\Gamma\left(1+k+\nu\right)\,x^{k} \\ &= \frac{1}{2}\left(\pi\sqrt{\alpha}\right)^{-\nu-2}\Gamma(\nu+1)\;{}_2F_1\left(\nu+1,\,1;\,1;\,-\frac{x}{\pi^{2}\alpha}\right) \\ &= \frac{1}{2}\pi^{\nu}\alpha^{\nu/2}\,\Gamma(\nu+1)\,\frac{1}{\left(x+\pi^{2}\alpha\right)^{\nu+1}}, \end{aligned} \tag{4.136}$$

where the last equality came from the well-known special case for the hypergeometric function [45]

$${}_2F_1\left(a,b;\,b;\,z\right) = \frac{1}{(1-z)^{a}}.$$

Taking $x = \frac{4\pi^{2}n}{|d|}$, summing over the index n and using the convention $r_{Q}(0) = 1$ and $f_{\nu}(0) = \frac{\Gamma(\nu)}{2}(\pi\sqrt{\alpha})^{-\nu}$, we obtain the interesting identities

$$\sum_{n=0}^{\infty}r_{Q}(n)\,n^{\frac{\nu}{2}}K_{\nu}\left(2\pi\sqrt{\alpha n}\right) = \frac{\alpha^{\nu/2}\Gamma\left(\nu+1\right)}{2\,\pi^{\nu+1}}\left(\frac{|d|}{4}\right)^{\nu+\frac{1}{2}}\sum_{n=0}^{\infty}\frac{r_{Q}(n)}{\left(n+\frac{|d|}{4}\alpha\right)^{\nu+1}}, \tag{4.137}$$

$$\begin{aligned} &\sum_{(m,n)\neq(0,0)}\chi(m)\,Q(m,n)^{\nu/2}K_{\nu}\left(2\pi\sqrt{\alpha Q(m,n)}\right) = \\ &\quad = \frac{\alpha^{\nu/2}\Gamma\left(\nu+1\right)G(\chi)}{2\ell\,\pi^{\nu+1}}\left(\frac{|d|}{4}\right)^{\nu+\frac{1}{2}}\sum_{(m,n)\neq(0,0)}\frac{\overline{\chi}(m)}{\left(Q_{\ell}^{-1}(m,n)+\frac{|d|}{4}\alpha\right)^{\nu+1}} \end{aligned} \tag{4.138}$$

which generalize an identity proved for the first time in [40].

Now, let us see that we can easily recover some results proved in the previous chapter by extending (4.137) and (4.138) to $\nu = 0$. From the conditional convergence of both series in (4.138), we can simply take $\nu = 0$ and write the formula

$$\sum_{(m,n)\neq(0,0)} \chi(m)\, K_0\left(2\pi\sqrt{\alpha\, Q(m,n)}\right) = \frac{\sqrt{|d|}\, G(\chi)}{4\pi\ell} \sum_{(m,n)\neq(0,0)} \frac{\overline{\chi}(m)}{\left(Q_{\ell}^{-1}(m,n) + \frac{|d|}{4}\alpha\right)}. \quad (4.139)$$

However, since both sides of (4.137) are not absolutely convergent when we take $\nu = 0$ we need to argue in the same way we did at Example 2.5. To do so, we see that an extension of the series in the right-hand side of (4.137) to the half-plane $\text{Re}(\nu) > -1$ can be given by

$$\sum_{n=1}^{\infty} \frac{r_Q(n)}{\left(n + \frac{|d|}{4}\alpha\right)^{\nu+1}} = \sum_{n=1}^{\infty} r_Q(n) \left[\frac{1}{\left(n + \frac{|d|}{4}\alpha\right)^{\nu+1}} - \frac{1}{n^{\nu+1}}\right] + Z_2\left(\nu+1,\, Q\right), \quad (4.140)$$

since the general terms of the infinite series at the right-hand side of (4.140) are $O\left(n^{-\nu-2}\right)$ and $Z_2(\nu+1,\, Q)$ is defined in all the complex plane.

Using (4.140) in (4.137), taking the limit $\nu \to 0$ and using Kronecker's limit formula (4.97), we derive the extension of Ferrar's formula [40],

$$\begin{aligned} &\frac{1}{\pi\alpha\sqrt{|d|}} + \frac{\sqrt{|d|}}{4\pi} \sum_{n=1}^{\infty} r_Q(n) \left[\frac{1}{n + \frac{|d|}{4}\alpha} - \frac{1}{n}\right] \\ &= \sum_{n=1}^{\infty} r_Q(n)\, K_0\left(2\pi\sqrt{\alpha n}\right) + 2\log\left(|\eta(\tau)|\right) - \gamma - \log\left(\frac{\sqrt{\alpha a}}{2}\right). \end{aligned} \quad (4.141)$$

Note that if we take $Q = Q_0(m,n)$, then $a = 1$ and $\tau = i$ and from the particular value for the Dedekind $\eta-$function[3] [44],

$$\eta(i) = \frac{\Gamma\left(\frac{1}{4}\right)}{2\pi^{3/4}},$$

we arrive to

$$\begin{aligned} &\sum_{n=1}^{\infty} r_2(n)\, K_0\left(2\pi\sqrt{\alpha n}\right) + 2\log\Gamma\left(\frac{1}{4}\right) - \log(2) - \frac{3}{2}\log(\pi) - \frac{1}{2}\log(\alpha) \\ &= \frac{1}{2\pi\alpha} + \frac{1}{2\pi} \sum_{n=1}^{\infty} r_2(n) \left[\frac{1}{n+\alpha} - \frac{1}{n}\right] + \gamma \end{aligned} \quad (4.142)$$

which is equivalent with equation (3.112) in the third chapter, which was derived by other means.

[3] In fact, it is not hard to derive this result. Notice that we can use eq.(2.88) of the second chapter with $x = 1$ and appeal to the calculations of the first chapter involving $L^{'}(1,\chi)$ and then take $\chi = \chi_4$.

Example 4.2.: Diagonal quadratic forms and sum of squares function

Assume that Q is a real positive definite quadratic form in $\mathbb{R}^n$ and s is a complex number such that $\text{Re}(s) > \frac{n}{2}$. Usually, one defines the $n-$dimensional Epstein's $\zeta-$function [21, 97]

$$Z_n(s,\,Q) = \sum_{\mathbf{m}\in\mathbb{Z}^n\setminus 0} \frac{1}{Q\left(\mathbf{m}\right)^s}, \quad \text{Re}(s) > \frac{n}{2} \tag{4.143}$$

where, as in (4.7) the series (4.143) is evaluated over vectors with integral coordinates, not all of which are zero. Trivially, one can find that, for $n = 2$, we have the classical Epstein $\zeta-$function (4.7) and, for $n = 1$, we have $2a^{-s}\zeta(2s)$.

In this final example we prove a Selberg-Chowla formula for one particular case of (4.143) when $n = 4$ and Q is the diagonal quadratic form

$$Q(x_1, ..., x_4) = Q_{0,4}(x_1, ..., x_4) = x_1^2 + ... + x_4^2. \tag{4.144}$$

We denote the Epstein $\zeta-$function associated to (4.144) by $\zeta_4(s) := Z_4(s,\,Q_{0,4})$. From the definition of the arithmetic function $r_4(n)$, which describes the number of ways for which n can be expressed as a sum of 4 square numbers (see [60] or our glossary), we can write $\zeta_4(s)$ as the Dirichlet series

$$\zeta_4(s) = \sum_{m=1}^{\infty} \frac{r_4(m)}{m^s}, \quad \text{Re}(s) > 2. \tag{4.145}$$

Thus, taking $n = 4$ in (4.145) we obtain

$$\begin{aligned}\zeta_4(s) &= \sum_{m=1}^{\infty} \frac{r_4(m)}{m^s} = \sum_{(m_1,...,m_4)\neq\mathbf{0}} \frac{1}{\left(m_1^2 + ... + m_4^2\right)^s} \\ &= \sum_{(m,n)\neq(0,0)} \frac{r_2(m)\,r_2(n)}{\left(m+n\right)^s} = 2\zeta_2(s) + \sum_{m,n=1}^{\infty} \frac{r_2(m)\,r_2(n)}{\left(m+n\right)^s}\end{aligned} \tag{4.146}$$

where the first equality came from setting $m = m_1^2 + m_2^2$ and $n = m_3^2 + m_4^2$.

In the double series presented in (4.146), we sum firstly over the index m: to evaluate this sum, recall formula (4.137) derived in Example 4.1 above for $Q := Q_{0,2}$. This gives the equality

$$\frac{1}{\Gamma(s)}\left[\frac{\pi\Gamma(s-1)}{n^{s-1}} + \frac{2\pi^s}{n^{(s-1)/2}}\sum_{m=1}^{\infty} r_2(m)\,m^{\frac{s-1}{2}}K_{s-1}\left(2\pi\sqrt{mn}\right)\right] - \frac{1}{n^s} = \sum_{m=1}^{\infty}\frac{r_2(m)}{(m+n)^s}. \tag{4.147}$$

Now, summing (4.147) over the index n, we obtain, for $\text{Re}(s) > 2$, the following expression for its left-hand side

$$\frac{1}{\Gamma(s)}\left[\pi\Gamma(s-1)\,\zeta_2(s-1) + 2\pi^s\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} r_2(n)\,r_2(m)\,\left(\frac{m}{n}\right)^{\frac{s-1}{2}}\,K_{s-1}\left(2\pi\sqrt{mn}\right)\right] - \zeta_2(s). \tag{4.148}$$

Combining (4.146) with (4.148), we obtain a formula similar to Selberg-Chowla's for the higher-dimmensional Epstein $\zeta-$function, $\zeta_4(s)$,

$$\pi^{-s}\Gamma(s)\,\zeta_4(s)=\pi^{-s}\Gamma(s)\,\zeta_2(s)+\pi^{1-s}\Gamma(s-1)\,\zeta_2(s-1)$$
$$+\,2\sum_{m,n=1}^{\infty}r_2(n)\,r_2(m)\,\left(\frac{m}{n}\right)^{\frac{s-1}{2}}\,K_{s-1}\left(2\pi\sqrt{mn}\right). \tag{4.149}$$

which provides the analytic continuation of $\zeta_4(s)$.

Adapting Bateman and Grosswald's argument cited in Lemma 4.1 above, we can prove that the series on the right-hand side of (4.149) is entire and has a particularly simple reflection formula with respect to the reflection $s\leftrightarrow 2-s$. We can also see that $\zeta_4(s)$ has an analytic continuation to the complex plane as a meromorphic function with a simple pole at $s=2$ and obeys to the functional equation

$$\pi^{-s}\Gamma(s)\,\zeta_4(s)=\pi^{s-2}\Gamma\left(2-s\right)\,\zeta_4\left(2-s\right). \tag{4.150}$$

Furthermore, from Jacobi's 4-square theorem [60, 94], we know that

$$r_4(n)=8\sigma(n)-32\sigma(n/4), \tag{4.151}$$

where the second term is to be taken zero if $4\nmid n$. Using (4.151), we arrive at a simple expression for $\zeta_4(s)$,

$$\zeta_4(s)=\sum_{m=1}^{\infty}\frac{r_4(m)}{m^s}=8\sum_{m=1}^{\infty}\frac{\sigma(m)}{m^s}-32\sum_{4|m}\frac{\sigma(m/4)}{m^s}$$
$$=8\sum_{m=1}^{\infty}\frac{\sigma(m)}{m^s}-2^{-2s+5}\sum_{m=1}^{\infty}\frac{\sigma(m)}{m^s}$$
$$=8\left(1-4^{1-s}\right)\zeta(s)\,\zeta(s-1). \tag{4.152}$$

Combining (4.152) with the identity $\zeta_2(s)=4\zeta(s)\,L(s,\chi_4)$ (see (4.11)), we arrive at an interesting identity for $\zeta(s)$,

$$4\,\left(1-4^{1-s}\right)\pi^{-s}\Gamma(s)\,\zeta(s)\,\zeta(s-1)=\,2\pi^{-s}\Gamma(s)\,\zeta(s)\,L\left(s,\chi_4\right)+2\,\pi^{1-s}\,\Gamma(s-1)\,\zeta(s-1)\,L\left(s-1,\chi_4\right)$$
$$+\sum_{m,n=1}^{\infty}r_2(n)\,r_2(m)\,\left(\frac{m}{n}\right)^{\frac{s-1}{2}}\,K_{s-1}\left(2\pi\sqrt{mn}\right). \tag{4.153}$$

We can still derive further identities of the type (4.153): for example, the case for which $n=3$, Selberg-Chowla formula provides the interesting formula

$$\pi^{-s}\Gamma(s)\,\zeta_3(s) = \pi^{1-s}\Gamma(s-1)\,\zeta(2s-2) + \pi^{-s}\Gamma(s)\,\zeta_2(s) + 4\sum_{m,n=1}^{\infty} r_2(n)\left(\frac{n}{m^2}\right)^{\frac{1-s}{2}} K_{1-s}\left(2\pi m\sqrt{n}\right), \quad \mathrm{Re}(s) > \frac{3}{2} \tag{4.154}$$

which gives the analytic continuation for $\zeta_3(s)$ and also provides the functional equation

$$\pi^{-s}\Gamma(s)\,\zeta_3(s) = \pi^{s-\frac{3}{2}}\Gamma\left(\frac{3}{2}-s\right)\,\zeta_3\left(\frac{3}{2}-s\right). \tag{4.155}$$

Chapter 5

Consequences of the Work developed

In this final chapter we present some consequences of the work previously done, related with the behavior of $\zeta(s)$ and $L(s,\chi)$ at interesting regions and points of the complex plane.

We will cover two main results, whose statements can be read in the following items:

1. For a nonprincipal, primitive and real character modulo ℓ, $L(1,\chi) > 0$. Moreover, $L(1,\chi) \neq 0$ whenever χ is a nonprincipal and real character.

2. The Riemann $\zeta-$function has infinitely many zeros at the critical line $\text{Re}(s) = \frac{1}{2}$ if and only if the trivial Epstein's $\zeta-$function, $\zeta_2(s)$, has the same property. By other words, Hardy's theorem is true if and only if it also holds for $\zeta_2(s)$.

5.1 1^{st} Application: The non-vanishing of $L(s,\chi)$ at $s=1$

Dirichlet [5] proved in 1837 that an arithmetic progression of the form $A_n := a + bn$ with $(a,b) = 1$ and $1 \leq b \leq n-1$ contains infinitely many prime numbers. It is universally agreed that the most difficult step in Dirichlet's proof lies in showing that, for any nonprincipal and real character modulo ℓ, $L(1,\chi) \neq 0$ [5].

In the first part of this chapter, our goal is to establish a proof of this fact, based on the formulas of Koshliakov-type derived in the third chapter. We pursue the proof of the following theorem

Main Theorem 1:

Let χ be any nonprincipal real character modulo χ. Then $L(1,\chi) \neq 0$.

To prove Main Theorem 1, we first need to introduce an elementary lemma, usually given in some proofs of Dirichlet's theorem [62, 104]

Lemma 5.1. Let χ be a real character modulo ℓ. Then, for each $n \in \mathbb{N}$, $d_{\chi}(n) \geq 0$.

Proof: From the definition of $d_{\chi}(n)$ (see equation (3.10) at the third chapter)

$$d_{\chi}(n) = \sum_{d|n} \chi(d),$$

we have, for each prime number p dividing n,

$$d_{\chi}(n) = \prod_{p|n} \left(1 + \chi(p) + ... + \chi(p)^{\alpha_p}\right), \tag{5.1}$$

where α_p is the exponent of p in the prime decomposition of n. Since χ is real, we have $\chi(n) \in \{-1, 0, 1\}$ $\forall n \in \mathbb{N}$. Therefore, it is simple to conclude that

1. If α_p is even, then each factor of (5.1) is 1 or $\alpha_p + 1$.
2. If α_p is odd, then each factor of (5.1) is 0, 1 or $\alpha_p + 1$.

From 1. and 2. we have the desired. ■

Now it is time to use the character analogues of Koshliakov's formulas derived at the third chapter to prepare a proof of the Main Theorem 1.

Recall from the third chapter that the self-reciprocal functions for the transforms given by equations (3.4) and (3.5) were, respectively, functions in the families of $K_0(\alpha x)$ and $e^{-\alpha x}$. Associated to these, we've also deduced the following formulas, valid when $z > 0$,

$$\frac{G(\chi)}{4}\, L(1, \overline{\chi}) + \sum_{n=1}^{\infty} d_{\chi}(n)\, K_0\left(\frac{2\pi n z}{\sqrt{\ell}}\right) = \frac{\sqrt{\ell}}{4z} L(1, \chi) + \frac{G(\chi)}{z\sqrt{\ell}} \sum_{n=1}^{\infty} d_{\overline{\chi}}(n)\, K_0\left(\frac{2\pi n z}{\sqrt{\ell}}\right), \tag{5.2}$$

for nonprincipal, primitive and even χ and

$$\sum_{n=1}^{\infty} d_{\chi}(n)\, e^{-\frac{2\pi n z}{\sqrt{\ell}}} - \frac{iG(\chi)}{2\pi} L(1, \overline{\chi}) = -\frac{iG(\chi)}{\sqrt{\ell} z} \sum_{n=1}^{\infty} d_{\overline{\chi}}(n)\, e^{-\frac{2\pi n}{z\sqrt{\ell}}} + \frac{\sqrt{\ell}}{2\pi z} L(1, \chi), \tag{5.3}$$

for nonprincipal, primitive and odd χ.

In the following argument, our idea will be to use the previous formulas to study the limit case $z \to 0^{+}$. Our argument is an adaptation of the idea proposed by Berndt et al. in [30], but is aimed to extend the approach given in the paper [30] to Dirichlet odd characters.

It should be pointed out that, whatever is the argument invoking the above formulas, this approach does not close the question completely since the formulas are only valid for primitive characters.

Nevertheless, following the first chapter of this thesis, we shall see that proving the Main Theorem for primitive characters is all that is required to prove the general version. So, we are going prove the following "weak" version of the Main Theorem 1:

Theorem 5.1: Let χ be a nonprincipal, real and primitive character modulo ℓ. Then $L(1,\chi) > 0$.

Proof: Assume first that χ is an even character: since it is also real, we have that equation (5.2) can be written as

$$\frac{G(\chi)}{4} L(1,\chi) + \sum_{n=1}^{\infty} d_{\chi}(n)\, K_0\left(\frac{2\pi n z}{\sqrt{\ell}}\right) = \frac{\sqrt{\ell}}{4z} L(1,\chi) + \frac{G(\chi)}{z\sqrt{\ell}} \sum_{n=1}^{\infty} d_{\chi}(n)\, K_0\left(\frac{2\pi n}{z\sqrt{\ell}}\right). \tag{5.4}$$

Recall that, as $x \to \infty$, the modified Bessel function $K_0(x)$ satisfies the asymptotic estimate $K_0(x) = \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} \left[1 + O(1/x)\right]$ and, as $x \to 0$, $K_0(x) = -\log(x) + O(1)$ (see [106]).

If we let $z \to 0^+$ on (5.4), we obtain the following expression

$$\begin{aligned} &\frac{G(\chi)}{4} L(1,\chi) + \sum_{n=1}^{\infty} d_{\chi}(n)\, K_0\left(\frac{2\pi n z}{\sqrt{\ell}}\right) \\ &= \frac{\sqrt{\ell}}{4z} L(1,\chi) + \frac{G(\chi)}{2\ell^{1/4}} \sum_{n=1}^{\infty} \frac{d_{\chi}(n)}{\sqrt{n}} z^{-1/2}\, e^{-\frac{2\pi n}{z\sqrt{\ell}}} \left[1 + O\left(z/n\right)\right], \quad z \to 0^+ \end{aligned} \tag{5.5}$$

so that the series on the right-hand side of (5.5) vanishes when this limit is taken. From the facts that $d_{\chi}(n) \geq 0$ for all $n \in \mathbb{N}$ (by Lemma 5.1.) and $K_0(x)$ is a strictly positive function in $\mathbb{R}_+$, it follows that the series

$$\sum_{n=1}^{\infty} d_{\chi}(n)\, K_0\left(\frac{2\pi n z}{\sqrt{\ell}}\right)$$

is always positive for all $z > 0$ and so it goes to $+\infty$ when $z \to 0^+$. This shows that the right-hand side of (5.5) goes to $+\infty$. But this only means that

$$\lim_{z \to 0^+} \frac{L(1,\chi)}{z} = +\infty,$$

proving that $L(1,\chi)$ is a positive real number.

Now, consider the case where χ is odd: using (5.3) and the fact that χ is real, we obtain the formula

$$\sum_{n=1}^{\infty} d_{\chi}(n)\, e^{-\frac{2\pi n z}{\sqrt{\ell}}} - \frac{iG(\chi)}{2\pi} L(1,\overline{\chi}) = -\frac{iG(\chi)}{\sqrt{\ell} z} \sum_{n=1}^{\infty} d_{\chi}(n)\, e^{-\frac{2\pi n}{z\sqrt{\ell}}} + \frac{\sqrt{\ell}}{2\pi z} L(1,\chi). \tag{5.6}$$

Once more, let $z \to 0^{+}$ in (5.6). The nonnegativity of the arithmetic function $d_{\chi}(n)$ and the positivity of the exponential show that the left-hand side tends to $+\infty$, while the series in the right-hand side clearly goes to 0. In the same way, this shows the divergence of the second term in the right-hand side of (5.6), i.e.,

$$\lim_{z\to 0^{+}} \frac{L(1,\chi)}{z} = +\infty,$$

which closes the proof. ■

Following the lines briefly introduced in the first chapter, we can see that the main theorem holds. This is what we prove now:

Proof of the Main Theorem 1:

Let χ be any nonprincipal and real character modulo ℓ. If χ is primitive, the previous theorem assures that $L(1,\chi) > 0$, and so there is nothing to prove.

If χ is nonprimitive and has modulo ℓ, then, by Theorem 1.2 of the first chapter, there corresponds a divisor ℓ' of ℓ and a primitive character χ' modulo ℓ' such that

$$\chi(n) = \begin{cases} \chi'(n) & \text{if } \ (n,\,\ell) = 1 \\ 0 & \text{if } \ (n,\ell) > 1, \end{cases}$$

which, by its turn (see eq. (1.148) at the first chapter) proves that $L(s,\chi)$ can be written in the form

$$L(s,\chi) = L(s,\chi') \prod_{p|\ell} \left(1 - \chi'(p)\, p^{-s}\right), \quad \mathrm{Re}(s) > 1, \tag{5.7}$$

which holds by analytic continuation for every $s \in \mathbb{C}$.

Since, by Theorem 5.1, $L(1,\chi') > 0$ always, it follows immediately from (5.7) that $L(1,\chi) \neq 0$ for any nonprincipal and real character χ. ■

This closes the main part of this section. In the remaining part, we prove Corollary 4.3.3. stated in the fourth chapter. This proposition asserted that, for a sufficiently large $|d|$ and a nonprincipal and primitive character χ, the single-weighted Epstein $\zeta-$function satisfies

$$Z_2\left(\frac{1}{2},\, Q,\, \chi\right) > 0.$$

Now, by using our Main Theorem, we are ready to prove it.

Proof: Use formula (4.30) at the fourth chapter for $s = \frac{1}{2}$, which is valid only for nonprincipal and primitive characters. We easily obtain

$$Z_2\left(\frac{1}{2},\, Q,\, \chi\right) = 2\sqrt{\frac{\pi}{a}}\, L(1,\chi) + \frac{8G(\chi)}{\sqrt{a}\ell} \sum_{n=1}^{\infty} d_{\overline{\chi}}(n)\, \cos\left(\pi\frac{b}{a}\frac{n}{\ell}\right)\, K_0\left(\frac{2\pi k n}{\ell}\right), \tag{5.8}$$

and the estimates (4.83) given in Corollary 4.3.1 of the fourth chapter prove that the series in the right-hand side of (5.8) is $O\left(\frac{e^{-2\pi k}}{\sqrt{k}}\right)$. Therefore, if k is large enough, this second term becomes arbitrarily small. Since $L(1,\chi) > 0$ (by Theorem 5.1), we deduce immediately that $Z_2\left(\frac{1}{2},\, Q,\, \chi\right) > 0$. ■

In Corollaries 4.3.1. and 4.3.2. given in the fourth chapter, we have stressed that $Z_2\,(s,\, Q)$ and $Z_2\,(s,\, Q,\, \chi)$ do not satisfy multi-dimensional analogues of the Riemann hypothesis, and so we cannot "transport" to $Z_2(s,\, Q)$ and $Z_2\,(s,\, Q,\, \chi)$ some of the conjectured properties of $\zeta(s)$ and $L(s,\chi)$ respectively.

However, the positivity of $Z_2(s,\, Q,\, \chi)$ at $s = 1$ is a feature that $Z_2(s, Q, \chi)$ preserves, at least when Q has a large discriminant. We state this fact as an easy corollary, and it goes as follows

Corollary 5.1.1.: Let χ be a nonprincipal, real and even Dirichlet character modulo ℓ. Then for any quadratic form with large discriminant, $Z_2\,(1,\, Q,\, \chi) > 0$.

Proof: Let us use formula (4.31) in the fourth chapter,

$$\begin{aligned} c^s\Gamma(s)\, Z_2(s,\, Q,\, \chi) &= 2\, k'^{1-2s}\pi^{1/2}\Gamma\left(s - \frac{1}{2}\right)\, L(2s-1,\, \chi) + \\ &+ 8\pi^s\, k'^{\frac{1}{2}-s} \sum_{n=1}^{\infty} \sigma_{1-2s,\chi}\,(n)\; n^{s-\frac{1}{2}} \cos\left(\frac{n\pi b}{c}\right)\, K_{s-\frac{1}{2}}\left(2\pi k' n\right), \end{aligned} \tag{5.9}$$

which is valid even for nonprimitive characters (see the discussion there). Using once more estimate (4.83) given at the fourth chapter, it is again immediate to see that the second term of (5.9) is bounded as $O\left(\frac{e^{-2\pi k}}{\sqrt{k}}\right)$, so it is arbitrarily small when k is sufficiently large. Taking $s = 1$ in (5.9) and invoking the Main Theorem 1 above proves the desired claim. ■

5.2 2^{nd} Application: Hardy's theorem for $\zeta(s)$ and $\zeta_2(s)$

To finish this thesis, we explore another important application of the work done in the previous chapter. We have seen that, despite being regarded as a multi-dimensional analogue of $\zeta(s)$, Epstein's $\zeta-$function lacks some of some (still conjectured) features of $\zeta(s)$.

Moreover, only in some cases (for example, when $Z_2(s, Q) = \gamma\,\zeta(s)\,L(s,\chi_d)$ for some character χ_d and constant γ) it is described by an Euler product formula similar to the one of $\zeta(s)$ and $L(s,\chi)$ (see eq. (1.19) and (1.21) at the first chapter).

Despite this, it may seem surprising that, for the simplest version of the class of zeta functions given by quadratic forms, $Z_2(s, Q_0) = \zeta_2(s)$, the following theorem takes place:

Main Theorem 2:

Riemann's $\zeta-$ function has infinitely many zeros in the critical line $\mathrm{Re}(s) = \frac{1}{2}$ if and only if $\zeta_2(s)$ verifies this property as well.

By other words, what the previous theorem states is that Hardy's theorem is valid for Epstein's $\zeta-$function $\zeta_2(s)$ if it is also valid for $\zeta(s)$ and viceversa.

It is clear that one of the implications predicted by the Main Theorem 2 holds. In fact, since $\zeta_2(s) = 4\,\zeta(s)\,L\,(s,\chi_4)$ by Jacobi's two-square theorem (see chapters 3 and 4), it follows that if $\zeta(s)$ has infinitely many zeros of the form $s = \frac{1}{2} + it$, $\zeta_2\,(s)$ has the same complex numbers as its zeros.

However, the converse is not so obvious and, of course, one curious application of this non-trivial implication is the following corollary

Corollary 5.2.1: If $L(s,\chi_4)$ has infinitely many zeros with real part 1/2, then $\zeta(s)$ has the very same property.

In what follows, we shall prove the Main Theorem 2 by using Selberg-Chowla's formula. This proof results from an adaptation of an argument due to Max Deuring [38] and there are of course natural similarities with Hardy's proof itself (see [103], chpt. 12). However, the use of Selberg-Chowla formula to arrive to these consideration is a new approach and comes from a different motivation than Deuring's and so the forthcoming argument may be regarded as complementary to those of Hardy and Deuring.

First, we establish an important lemma, which allows to write the symmetric function associated with $\zeta_2(s)$,

$$\Phi_2(s) = \pi^{-s}\Gamma(s)\,\zeta_2(s),$$

with an integral similar to the one found in Hardy's proof [103].

Lemma 5.2.1.: Let $\zeta_2(s) = Z_2(s,\, Q_0)$ denote Epstein's $\zeta-$function for the simplest quadratic form $Q_0(m,\, n) = m^2 + n^2$ and

$$\Phi(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\,\zeta(s), \tag{5.10}$$

$$\Phi_2(s) = \pi^{-s}\Gamma(s)\,\zeta_2(s) \tag{5.11}$$

denote the symmetric functions associated with $\zeta(s)$ and $\zeta_2(s)$ respectively.

Then the identity holds

$$\Phi_2\left(\frac{1}{2}+it\right) = 4\Phi\left(1+2it\right) + 4\Phi\left(2it\right) + \frac{1}{\pi}\int_{-\infty}^{\infty}\Phi\left(\frac{1}{2}+i\left(y-t\right)\right)\,\Phi\left(\frac{1}{2}+i\left(y+t\right)\right)\,dy. \tag{5.12}$$

Proof: The proof of (5.12) is the result of an application of Selberg-Chowla formula when the Epstein $\zeta-$function is defined for the quadratic form $Q(m,n) = Q_0(m,n) := m^2 + n^2$.

Using formula (4.25) in the fourth chapter for $Q(m,n) = Q_0(m,n)$ we get

$$\begin{aligned}\pi^{-s}\Gamma(s)\,\zeta_2(s) = 2\pi^{-s}\Gamma(s)\zeta(2s) + 2\pi^{-s+1/2}\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1) \\ +\ 8\sum_{n=1}^{\infty}\sigma_{1-2s}(n)\,n^{s-1/2}\,K_{s-1/2}\left(2\pi n\right).\end{aligned} \tag{5.13}$$

Now recall that we can write the modified Bessel function lying in the series (5.13) as the inverse Mellin integral (note that $\sigma = \mathrm{Re}(s)$),

$$K_{s-\frac{1}{2}}\left(2\pi n\right) = \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} 2^{w-2}\,\Gamma\left(\frac{w-\frac{1}{2}+s}{2}\right)\,\Gamma\left(\frac{w+\frac{1}{2}-s}{2}\right)\left(2\pi n\right)^{-w}\,dw, \quad \mu > 1 + \left|\sigma - \frac{1}{2}\right|. \tag{5.14}$$

The absolute convergences of the integral (5.14) and the series (5.13) allow us to interchange the order of summation and integration in the third term of (5.13). This gives the formula

$$\sum_{n=1}^{\infty}\sigma_{1-2s}(n)\,n^{s-1/2}\,K_{s-1/2}\,(2\pi n)$$

$$=\frac{1}{8\pi i}\int\limits_{\mu-i\infty}^{\mu+i\infty}\pi^{-w}\,\Gamma\left(\frac{w-\frac{1}{2}+s}{2}\right)\,\Gamma\left(\frac{w+\frac{1}{2}-s}{2}\right)\sum_{n=1}^{\infty}\frac{\sigma_{1-2s}(n)}{n^{w-s+1/2}}\,dw$$

$$=\frac{1}{8\pi i}\int\limits_{\mu-i\infty}^{\mu+i\infty}\pi^{-w}\,\Gamma\left(\frac{w-\frac{1}{2}+s}{2}\right)\,\Gamma\left(\frac{w+\frac{1}{2}-s}{2}\right)\zeta\left(w+s-\frac{1}{2}\right)\,\zeta\left(w-s+\frac{1}{2}\right)\,dw, \quad (5.15)$$

where the last equality came from the fact that $\mathrm{Re}(w)=\mu>1+\left|\sigma-\frac{1}{2}\right|$ and the identity involving the Dirichlet series associated with the generalized divisor function (see [103] chpt. 2),

$$\sum_{n=1}^{\infty}\frac{\sigma_{1-2s}(n)}{n^{w-s+1/2}}=\zeta\left(w-s+1/2\right)\,\zeta\left(w+s-1/2\right),\quad \mathrm{Re}(w)>\max\left\{\sigma-\frac{1}{2},\,\frac{1}{2}-\sigma\right\}. \quad (5.16)$$

Now we can write the integral (5.15) as

$$\frac{1}{8\pi i}\int\limits_{\mu-i\infty}^{\mu+i\infty}\Phi\left(w-s+\frac{1}{2}\right)\Phi\left(w+s-\frac{1}{2}\right)\,dw, \quad (5.17)$$

where $\Phi(s)$ denotes the complex function (5.10), which satisfies the reflection formula $\Phi(s)=\Phi(1-s)$ due to the functional equation for $\zeta(s)$ (1.142).

Now let us assume that s is a complex number on the critical line, i.e., $s=\frac{1}{2}+it$, for some $t\in\mathbb{R}$. Then, from (5.13) and (5.17), together with definitions (5.10) and (5.11),

$$\Phi_2\left(\frac{1}{2}+it\right)=2\,\Phi\left(1+2it\right)+2\,\Phi\left(2it\right)+\frac{1}{\pi i}\int\limits_{\mu-i\infty}^{\mu+i\infty}\Phi\left(w-it\right)\,\Phi\left(w+it\right)\,dw,\quad \mu>1 \quad (5.18)$$

and we can move the line of integration to the critical line $\mu=\frac{1}{2}$. Note that, in the region $\frac{1}{2}\leq\mathrm{Re}(w)<\mu$, the product $\Phi\left(w-it\right)\Phi\left(w+it\right)$ has two simple poles located at $w=1\pm it$, and both come from the pole that $\zeta(s)$ has at $s=1$. Since $\mathrm{Res}_{s=1}\zeta(s)=1$, it is easy to see that

$$\begin{aligned}\mathrm{Res}_{w=1\pm it}\left[\Phi(w-it)\,\Phi(w+it)\right]&=\Phi\left(1+2it\right)+\Phi\left(1-2it\right)\\&=\Phi\left(1+2it\right)+\Phi(2it),\end{aligned}$$

where the last equality comes from the relation $\Phi(s)=\Phi(1-s)$. Therefore, by Cauchy's Residue Theorem, the integral term given in (5.18) can be written as

$$\frac{1}{\pi i}\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Phi\left(w-it\right)\,\Phi\left(w+it\right)\,dw+2\Phi\left(1+2it\right)+2\Phi(2it). \quad (5.19)$$

Thus, combining (5.19) with (5.18) yields the desired

$$\Phi_2\left(\frac{1}{2}+it\right) = 4\Phi\left(1+2it\right)+4\Phi\left(2it\right)+\frac{1}{\pi i}\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Phi\left(w-it\right)\,\Phi\left(w+it\right)\,dw$$

$$= 4\Phi\left(1+2it\right)+4\Phi\left(2it\right)+\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\Phi\left(\frac{1}{2}+i\left(y-t\right)\right)\,\Phi\left(\frac{1}{2}+i\left(y+t\right)\right)\,dy. \quad \blacksquare$$

Using representation (5.12) and well-known asymptotic estimates for $\Phi(s)$ and $\Phi_2(s)$ studied in previous chapters, we are ready to start the proof of the Main Theorem 2.

Proof of Main Theorem 2: In what follows, we prove that if $\zeta(s)$ has finitely many zeros on the line $\mathrm{Re}(s)=\frac{1}{2}$ then the same must happen for $\zeta_2(s)$, proving the implication which is left to prove.

By the functional equation for $\zeta(s)$, written in the symmetric form,

$$\Phi(s)=\Phi(1-s), \tag{5.20}$$

we see that the function $\varphi(y)=\Phi\left(\frac{1}{2}+i\left(y-t\right)\right)\,\Phi\left(\frac{1}{2}+i\left(y+t\right)\right)$ is real valued for $y\in\mathbb{R}$, since

$$\overline{\varphi(y)}=\Phi\left(\frac{1}{2}-i\left(y-t\right)\right)\Phi\left(\frac{1}{2}-i\left(y+t\right)\right)$$

$$=\Phi\left(\frac{1}{2}+i\left(y-t\right)\right)\,\Phi\left(\frac{1}{2}+i\left(y+t\right)\right)=\varphi(y).$$

Now, let us suppose, by contradiction, that $\zeta\left(\frac{1}{2}+it\right)$ vanishes for a finite number of real numbers t. From this hypothesis, we infer that exists $T\in\mathbb{R}^{+}$ such that, for all $t>T$, the real number $\Phi\left(\frac{1}{2}+it\right)$ has always the same sign.

For such T, let us take $t>T$ in (5.12). Due to the symmetry with respect to t in (5.20), the considerations are analogous for $t<0$, so, from now on, we assume that t is positive.

Since, by hypothesis, $\zeta(s)$ has a finite number of zeros on the critical line, $\varphi(y)$ has constant sign if $|y+t|,\ |y-t|>T$. Hence, if we take a partition of the integral (5.20) as

$$\int\limits_{-T+t}^{\infty}\Phi\left(\frac{1}{2}+i\left(y-t\right)\right)\,\Phi\left(\frac{1}{2}+i\left(y+t\right)\right)\,dy+\int\limits_{-\infty}^{-t+T}\Phi\left(\frac{1}{2}+i\left(y-t\right)\right)\,\Phi\left(\frac{1}{2}+i\left(y+t\right)\right)\,dy$$

$$+\int\limits_{-t+T}^{t-T}\Phi\left(\frac{1}{2}+i\left(y-t\right)\right)\,\Phi\left(\frac{1}{2}+i\left(y+t\right)\right)\,dy, \tag{5.21}$$

we see that, in the third of these, the integrand has a constant and positive sign, since $|y-t|>T$ and $|y+t|>T$.

Therefore, $\Phi\left(\frac{1}{2}+i\,(y-t)\right)\,\Phi\left(\frac{1}{2}+i\,(y+t)\right) = \left|\Phi\left(\frac{1}{2}+i\,(y-t)\right)\,\Phi\left(\frac{1}{2}+i\,(y+t)\right)\right|$ and so the third integral (5.21) can be expressed as

$$\int_{-T+t}^{\infty} \Phi\left(\frac{1}{2}+i\,(y-t)\right)\,\Phi\left(\frac{1}{2}+i\,(y+t)\right)\,dy + \int_{-\infty}^{-t+T} \Phi\left(\frac{1}{2}+i\,(y-t)\right)\,\Phi\left(\frac{1}{2}+i\,(y+t)\right)\,dy$$
$$+\int_{-t+T}^{t-T} \left|\Phi\left(\frac{1}{2}+i\,(y-t)\right)\,\Phi\left(\frac{1}{2}+i\,(y+t)\right)\right|\,dy.$$

The main hypothesis is now developed. To proceed with the proof, we will now state and prove two main claims, related with the asymptotic order of the integrals in (5.21) as $t\to\infty$.

Claim 1: Let $\mathcal{A}_1(t)$ and $\mathcal{A}_2(t)$ denote, respectively, the first two integrals in (5.21), i.e.,

$$\mathcal{A}_1(t) = \int_{-T+t}^{\infty} \Phi\left(\frac{1}{2}+i\,(y-t)\right)\,\Phi\left(\frac{1}{2}+i\,(y+t)\right)\,dy \tag{5.22}$$

and

$$\mathcal{A}_2(t) = \int_{-\infty}^{-t+T} \Phi\left(\frac{1}{2}+i\,(y-t)\right)\,\Phi\left(\frac{1}{2}+i\,(y+t)\right)\,dy. \tag{5.23}$$

Then both obey, as functions of t, to the following estimate

$$\mathcal{A}_1(t),\,\mathcal{A}_2(t) = O\left(|t|^{\epsilon}e^{-\frac{1}{2}\pi|t|}\right),\quad |t|\to\infty, \tag{5.24}$$

for any positive ϵ.

Proof of the Claim 1: First, let us note that, by the functional equation for $\zeta(s)$,

$$\begin{aligned}\overline{\mathcal{A}_1(t)} &= \int_{-T+t}^{\infty} \Phi\left(\frac{1}{2}-i\,(y-t)\right)\Phi\left(\frac{1}{2}-i\,(y+t)\right)\,dy = \\ &= \int_{-\infty}^{-t+T} \Phi\left(\frac{1}{2}+i\,(y+t)\right)\xi\left(\frac{1}{2}+i\,(y-t)\right)\,dy = \\ &= \mathcal{A}_2(t),\end{aligned}$$

and so it suffices to prove (5.24) for $\mathcal{A}_1(t)$ as an analogous estimate holds for $\mathcal{A}_2(t)$.

To estimate the integral $\mathcal{A}_1(t)$, recall that, from Lindelöf's estimate (2.42) given in Proposition 2.2 of the second chapter, $\zeta\left(\frac{1}{2}+it\right)=O\left(|t|^{\frac{1}{4}+\epsilon}\right)$ for every $\epsilon>0$. Moreover, from Stirling's formula,

$$\Gamma\left(\frac{1}{4}+i\,\frac{t}{2}\right)=O\left(|t|^{-\frac{1}{4}}e^{-\frac{\pi}{4}|t|}\right),$$

we have the estimate

$$\Phi\left(\frac{1}{2}+i\,(y-t)\right)=O\left(|y-t|^{\epsilon}\,e^{-\frac{\pi}{4}|y-t|}\right) \tag{5.25}$$

and

$$\Phi\left(\frac{1}{2}+i\,(y+t)\right)=O\left(|y+t|^{\epsilon}\,e^{-\frac{\pi}{4}|y+t|}\right). \tag{5.26}$$

Now, notice that we can rewrite $\mathcal{A}_1(t)$ by taking a partition of the integral in (5.22) and using the change of variable $u=y-t$, giving

$$\begin{aligned}\mathcal{A}_1(t)&=\int_{-T+t}^{\infty}\Phi\left(\frac{1}{2}+i\,(y-t)\right)\,\Phi\left(\frac{1}{2}+i\,(y+t)\right)\,dy\\&=\int_{-T+t}^{T+t}\Phi\left(\frac{1}{2}+i\,(y-t)\right)\,\Phi\left(\frac{1}{2}+i\,(y+t)\right)\,dy\\&+\int_{T+t}^{\infty}\Phi\left(\frac{1}{2}+i\,(y-t)\right)\,\Phi\left(\frac{1}{2}+i\,(y+t)\right)\,dy\\&=\int_{-T}^{T}\Phi\left(\frac{1}{2}+iu\right)\,\Phi\left(\frac{1}{2}+i\,(u+2t)\right)\,du\\&+\int_{T}^{\infty}\Phi\left(\frac{1}{2}+iu\right)\,\Phi\left(\frac{1}{2}+i\,(u+2t)\right)\,du.\end{aligned} \tag{5.27}$$

If we now let t be arbitrarily large in (5.27) and use the estimate (5.26), we obtain that the first integral in (5.27) can be bounded in the following way

$$\begin{aligned}\int_{-T}^{T}\Phi\left(\frac{1}{2}+iu\right)\,\Phi\left(\frac{1}{2}+i\,(u+2t)\right)\,du&\leq C\int_{-T}^{T}\Phi\left(\frac{1}{2}+iu\right)\,|u+2t|^{\epsilon}\,e^{-\frac{\pi}{4}|u+2t|}du\\&=O\left(|t|^{\epsilon}\,e^{-\frac{\pi}{2}|t|}\right),\end{aligned} \tag{5.28}$$

which follows from the compactness of the interval of integration and, certainly, from the obvious continuity of $\Phi\left(\frac{1}{2}+iu\right)$ as a function of u.

For the second integral given in (5.27) note that, since $u > T$ and T can be taken large enough in order to assure that $\zeta\left(\frac{1}{2}+it\right)$ has no zeros for $t > T$ (by the contradiction hypothesis), we can also apply Stirling's formula to $\Phi\left(\frac{1}{2}+iu\right)$ which gives in its turn

$$\int_T^\infty \Phi\left(\frac{1}{2}+iu\right)\,\Phi\left(\frac{1}{2}+i\left(u+2t\right)\right)\,du \le D\int_T^\infty u^\epsilon e^{-\frac{\pi}{4}u}\left(u+t\right)^\epsilon e^{-\frac{\pi}{4}|u+2t|}\,du =$$

$$= D\,e^{-\frac{\pi}{2}|t|}\int_T^\infty \left(u^2+t\,u\right)^\epsilon\,e^{-\frac{\pi}{2}u}du = O\left(|t|^\epsilon\,e^{-\frac{1}{2}\pi|t|}\right), \tag{5.29}$$

completing the proof of the first claim. ■

We now move on to a second claim, which provides a lower bound for the third integral appearing in the partition (5.21). This will be the final step towards our proof of the Main Theorem 2.

Claim 2: For a sufficiently large $|t|$, we have the inequality

$$\mathcal{A}_3(t) = \int_{-t+T}^{t-T}\left|\Phi\left(\frac{1}{2}+i\left(y-t\right)\right)\,\Phi\left(\frac{1}{2}+i\left(y+t\right)\right)\right|\,dy > C\,|t|^{\frac{1}{2}}e^{-\frac{\pi}{2}|t|}, \tag{5.30}$$

for some positive constant C.

Proof of the Claim 2: Throughout this proof we shall assume that $t > T > 0$, but similar considerations hold for $t < -T < 0$ as well.

In the interval of integration considered by (5.30) we have $|y-t| > T$ and $|y+t| > T$. So, if T is chosen sufficiently large and $t > T$, we conclude, by Stirling's formula, that exist two constant numbers $0 < C_1,\, C_2 < \sqrt{2\pi}$ such that the inequalities

$$\left|\Gamma\left(\frac{1}{4}+\frac{i}{2}\left(y-t\right)\right)\right| \ge C_1\,|y-t|^{-\frac{1}{4}}\,e^{-\frac{\pi}{4}|y-t|} \tag{5.31}$$

and

$$\left|\Gamma\left(\frac{1}{4}+\frac{i}{2}\left(y+t\right)\right)\right| \ge C_2\,|y+t|^{-\frac{1}{4}}\,e^{-\frac{\pi}{4}|y+t|} \tag{5.32}$$

are satisfied. Thus, if we multiply both terms (5.31) and (5.32), we get the inequality

$$\left|\Gamma\left(\frac{1}{4}+\frac{i}{2}\left(y-t\right)\right)\,\Gamma\left(\frac{1}{4}+\frac{i}{2}\left(y+t\right)\right)\right| \ge C\,\left|t^2-y^2\right|^{-\frac{1}{4}}\,e^{-\frac{\pi}{4}\left(|y+t|+|y-t|\right)}$$

$$\ge C\,|t|^{-\frac{1}{2}}\,e^{-\frac{\pi}{4}\left(|y+t|+|y-t|\right)} \tag{5.33}$$

since the function $h(y) := |t^2 - y^2|^{-\frac{1}{4}}$ attains its minimum for $y = 0$. Using (5.33), we find an estimate for $\mathcal{A}_3(t)$ as follows

$$\begin{aligned}
\mathcal{A}_3(t) &= \int_{-t+T}^{t-T} \left| \Phi\left(\frac{1}{2} + i\,(y-t)\right)\, \Phi\left(\frac{1}{2} + i\,(y+t)\right)\right| dy \\
&\geq C\,|t|^{-\frac{1}{2}} \int_{-t+T}^{t-T} \left|\zeta\left(\frac{1}{2} + i\,(y-t)\right)\, \zeta\left(\frac{1}{2} + i\,(y+t)\right)\right| e^{-\frac{\pi}{2}t}\, dy \\
&\geq C\,|t|^{-\frac{1}{2}}\, e^{-\frac{\pi}{2}|t|} \left| \int_{\gamma_1} \zeta\,(z - it)\, \zeta\,(z + it)\, dz \right|, \qquad (5.34)
\end{aligned}$$

where γ_1 is the line segment connecting $\frac{1}{2} - i(t-T)$ to $\frac{1}{2} + i(t-T)$.

Using the fact that the product $\zeta\,(z-it)\,\zeta\,(z+it)$ has no poles for $|\mathrm{Im}(z)| < t$, it defines an analytic function on the rectangular contour

$$\begin{aligned}
\Gamma = &\left[\frac{1}{2} - i(t-T),\, \frac{1}{2} + i(t-T)\right] \cup \left[\frac{1}{2} + i(t-T),\, 2 + i\,(t-T)\right] \\
&\cup [2 + i\,(t-T)\,,\, 2 - i\,(t-T)] \cup \left[2 + i\,(t-T)\,,\, \frac{1}{2} + i\,(t-T)\right]. \qquad (5.35)
\end{aligned}$$

The remaining part of this proof consists in applying Cauchy's theorem to the last integral in (5.34) and then evaluate the asymptotic behavior in each segment defined in (5.35). Note that we can write this path integral by using the equality

$$\left| \int_{\gamma_1} \zeta\,(z - it)\, \zeta\,(z + it)\, dz \right| = \left| \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4} (\zeta\,(z - it)\, \zeta\,(z + it))\, dz \right|,$$

where γ_2, γ_3 and γ_4 are the line segments appearing in (5.35) by its respective order.

To finish the proof, we will see that the integral along $\gamma_3 = [2 + i\,(t-T)\,,\, 2 - i\,(t-T)]$ satisfies (5.30), which will close the proof. Our task now is to check the following items:

1. Since $\gamma_2 = \left[\frac{1}{2} + i(t-T),\, 2 + i\,(t-T)\right]$, we have that, along this path, z has a fixed imaginary part. So, when t is taken arbitrarily large, $\zeta(z-it)\,\zeta(z+it) = O\left(|t|^{\frac{1}{2}}\right)$ for $\frac{1}{2} \leq \mathrm{Re}(z) \leq 2$ and since the domain of integration is a compact set, $\int_{\gamma_2} \zeta\,(z-it)\,\zeta\,(z+it)\,dz = O\left(|t|^{\frac{1}{2}}\right)$.

2. By the same reasons invoked in the previous item, the integral along γ_4 obeys to the estimate $\int_{\gamma_4} \zeta\,(z-it)\,\zeta\,(z+it)\,dz = O\left(|t|^{\frac{1}{2}}\right)$ as well, as γ_4 also defines an "horizontal segment".

3. Finally, consider the integral along γ_3, i.e., $\int_{\gamma_3} \zeta(z-it)\,\zeta(z+it)\,dz$. Since $\mathrm{Re}(z-it) = \mathrm{Re}(z+it) = 2$ along this path, we can represent the product of $\zeta-$functions by the Dirichlet series [103]

$$\zeta(z-it)\,\zeta(z+it) = \sum_{n=1}^{\infty} \frac{\sigma_{2it}(n)}{n^{z+it}}, \tag{5.36}$$

which is absolutely convergent along γ_3. So we can interchange the orders of integration and summation once we take the series (5.36) inside the integral $\int_{\gamma_3}$. This immediately gives

$$\begin{aligned}
-\int_{\gamma_3} \zeta(z-it)\,\zeta(z+it)\,dz &= \sum_{n=1}^{\infty} \sigma_{2it}(n) \int_{2-i(t-T)}^{2+i(t-T)} \frac{dz}{n^{z+it}} \\
&= i \sum_{n=1}^{\infty} \sigma_{2it}(n) \int_{-t+T}^{t-T} \frac{du}{n^{2+iu+it}} \\
&= 2it + i \sum_{n=2}^{\infty} \frac{\sigma_{2it}(n)}{n^{2+it}} \int_{-t+T}^{t-T} e^{-i\log(n)\,u} du \\
&= 2it + 2i \sum_{n=2}^{\infty} \frac{\sigma_{2it}(n)}{n^{2+it}} \frac{\sin((t-T)\,\log(n))}{\log(n)} \\
&= 2it + O(1),
\end{aligned} \tag{5.37}$$

where the bound for the second term comes from the trivial estimate

$$\left| \sum_{n=2}^{\infty} \frac{\sigma_{2it}(n)}{n^{2+it}} \frac{\sin((t-T)\,\log(n))}{\log(n)} \right| \leq 2 \sum_{n=1}^{\infty} \frac{d(n)}{n^2} = \frac{\pi^4}{18}.$$

Using (5.34) and the previous items, we finally prove the inequality

$$\begin{aligned}
\mathcal{A}_3(t) &\geq C\,|t|^{-\frac{1}{2}}\,e^{-\frac{\pi}{2}|t|} \left| \int_{\gamma_1} \zeta(z-it)\,\zeta(z+it)\,dz \right| \\
&\geq C\,|t|^{-\frac{1}{2}}\,e^{-\frac{\pi}{2}|t|} \left| 2it + O\left(|t|^{\frac{1}{2}}\right) \right| = C\,|t|^{\frac{1}{2}}\,e^{-\frac{\pi}{2}|t|} + O\left(e^{-\frac{\pi}{2}|t|}\right). \quad \blacksquare
\end{aligned}$$

Now, the proof of the Main Theorem 2 follows: we have seen in (5.21) that, once one assumes that $\zeta\left(\frac{1}{2}+it\right)$ has finitely many zeros, we can take the partition

$$\mathcal{I}(t) := \int_{-\infty}^{\infty} \Phi\left(\frac{1}{2} + i\,(y-t)\right) \Phi\left(\frac{1}{2} + i\,(y+t)\right)\, dy = \mathcal{A}_1(t) + \mathcal{A}_2(t) + \mathcal{A}_3(t),$$

where $\mathcal{A}_j(t)$ are the integrals defined in the statements of the previous claims.

By the conclusions driven out from Claim 1 and Claim 2, we know that exists $C > 0$ such that

$$\mathcal{I}(t) > C\,|t|^{1/2} e^{-\frac{\pi}{2}|t|}.$$

Moreover, since Φ has the estimate (2.42),

$$\Phi(1+2it) = \pi^{-\frac{1}{2}-it}\,\Gamma\left(\frac{1}{2}+it\right)\,\zeta\left(1+2it\right) = O\left(|t|^{\epsilon}\,e^{-\frac{\pi}{2}|t|}\right),$$

$$\Phi\left(2it\right) = \Phi(1-2it) = O\left(|t|^{\epsilon}\,e^{-\frac{\pi}{2}|t|}\right),$$

we have, for a sufficiently large $|t|$,

$$\left|\Phi_2\left(\frac{1}{2}+it\right)\right| = \left|4\Phi\left(1+2it\right) + 4\Phi\left(2it\right) + \frac{1}{\pi i}\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Phi\left(w-it\right)\,\Phi\left(w+it\right)\,dw\right|$$

$$> C\,|t|^{1/2} e^{-\frac{\pi}{2}|t|}, \tag{5.38}$$

which proves that $\Phi\left(\frac{1}{2}+it\right)$ can only vanish for a finite number of parameters $t \in \mathbb{R}$. ■

Note that the previous argument can be now used to derive Hardy's theorem, once we recall the relations (4.87) and (4.89) given in Corollary 4.4 of the fourth chapter.

Corollary 5.2.2 The Riemann $\zeta-$function has infinitely many zeros located at the critical line $\mathrm{Re}(s) = \frac{1}{2}$.

Proof The proof easily follows from (5.38): we have seen that, once we assume the finitude of zeros of $\zeta\left(\frac{1}{2}+it\right)$, we obtain, for some large T and $t > T$,

$$\left|\Phi_2\left(\frac{1}{2}+it\right)\right| = \left|\pi^{-\frac{1}{2}-it}\,\Gamma\left(\frac{1}{2}+it\right)\,\zeta_2\left(\frac{1}{2}+it\right)\right| > C\,|t|^{\frac{1}{2}} e^{-\frac{\pi}{2}|t|}. \tag{5.39}$$

However, for every $\epsilon > 0$ (see formula (4.89) given in the fourth chapter), Epstein's $\zeta-$function obeys to the estimate

$$\zeta_2\left(\frac{1}{2}+it\right) = 4\,\zeta\left(\frac{1}{2}+it\right)\,L\left(\frac{1}{2}+it,\,\chi_4\right) = O\left(|t|^{\frac{5}{12}+\epsilon}\right),$$

which proves that

$$\left|\Phi_2\left(\frac{1}{2}+it\right)\right| = \left|\pi^{-\frac{1}{2}-it}\Gamma\left(\frac{1}{2}+it\right)\,\zeta_2\left(\frac{1}{2}+it\right)\right| = O\left(|t|^{\frac{5}{12}+\epsilon}\,e^{-\frac{\pi}{2}|t|}\right),\quad |t|\to\infty. \tag{5.40}$$

It is now immediate to see that estimate (5.40) clearly contradicts (5.39) and from this contradiction we prove that there are infinitely many zeros with real part $\frac{1}{2}$. ■

We also remark that it is possible to prove extended versions of the Main theorem 2.

Indeed, following formula (4.155) given in the fourth chapter for $\zeta_3(s)$, $\text{Re}(s) > \frac{3}{2}$,

$$\pi^{-s}\Gamma(s)\,\zeta_3(s) = \pi^{1-s}\Gamma(s-1)\,\zeta(2s-2) + \pi^{-s}\Gamma(s)\,\zeta_2(s) + 4\sum_{m,n=1}^{\infty} r_2(n)\left(\frac{n}{m^2}\right)^{\frac{1-s}{2}} K_{1-s}\left(2\pi m\sqrt{n}\right), \tag{5.41}$$

and after mimicking the previous proof, it is also possible to prove that

Corollary 5.2.3: If $\zeta_3(s)$ has infinitely many zeros in the line $\text{Re}(s) = \frac{3}{4}$ then $\zeta_2(s)$ has infinitely many zeros in the line $\text{Re}(s) = \frac{1}{2}$ as well.

However, the study of the zeros of the "three-dimensional" Epstein $\zeta-$function $\zeta_3(s)$ is still open for explorations. For instance, Terras [98] proved that, for the Epstein $\zeta-$function associated to ternary quadratic forms

$$Z_3(s,\,Q) = \sum_{(m_1,m_2,m_3)\neq\mathbf{0}} \frac{1}{Q(m_1,m_2,m_3)^s}, \quad \text{Re}(s) > \frac{3}{2},$$

if the matrix representation of Q has a suitable determinant, one can show that $Z_3(s,\,Q)$ has a real zero lying on the open interval $\left(1,\,\frac{3}{2}\right)$.

This argument proves that a three dimensional analogue of Riemann hypothesis is not true and gives a three-dimensional analogue of the result given at Corollary 4.3.1 of the previous chapter.

Finally, we should also remark that we could prove Hardy's theorem by other means, i.e., by proving separately its version for $L(s,\chi_4)$ and appeal to the unexpected Corollary 5.2.1. However, if we wished to do so by considering a variation of the argument given above, it would be required to consider another version of Epstein's $\zeta-$function in the form

$$Z_2(s,\,Q,\,\chi_1,\,\chi_2) = \sum_{(m,n)\neq(0,0)} \frac{\chi_1(m)\,\chi_2(n)}{Q(m,n)^s}, \quad \text{Re}(s) > 1 \tag{5.42}$$

where χ_1 and χ_2 are nonprincipal and primitive characters modulo ℓ having the same parity.

Note that, if χ_1 and χ_2 possess a different parity, the series represented in (5.42) vanishes. In analogy with the "single-weighted Epstein $\zeta-$function" given in the previous chapter, we refer to (5.42) as "double-weighted Epstein $\zeta-$function".

It is also simple to show, via Example 2.5 of the second chapter, that $Z_2(s,\, Q,\, \chi_1,\, \chi_2)$ obeys to the formulas similar to Selberg-Chowla's,

1. If χ_1 and χ_2 are even, $Z_2\,(s,\, Q,\, \chi_1,\, \chi_2)$ has the representation

$$Z_2(s,\, Q,\, \chi_1,\, \chi_2) = \frac{8\pi^s a^{-s}}{\Gamma(s)}\, G(\chi_1)\, k^{1/2-s} \ell_1^{-(s+1/2)}$$
$$\sum_{n=1}^{\infty} \sigma_{1-2s}(n,\, \overline{\chi}_1,\, \chi_2)\, n^{s-1/2} \cos\left(\pi\, \frac{b}{a}\, \frac{n}{\ell_1}\right) K_{\frac{1}{2}-s}\left(\frac{2\pi k\, n}{\ell_1}\right), \quad \mathrm{Re}(s) > 1. \tag{5.43}$$

2. If χ_1 and χ_2 are odd, $Z_2\,(s,\, Q,\, \chi_1,\, \chi_2)$ has the representation

$$Z_2(s,\, Q,\, \chi_1,\, \chi_2) = -\frac{8i\, \pi^s a^{-s}}{\Gamma(s)}\, G(\chi_1)\, k^{1/2-s} \ell_1^{-(s+1/2)}$$
$$\sum_{n=1}^{\infty} \sigma_{1-2s}(n,\, \overline{\chi}_1,\, \chi_2)\, n^{s-1/2} \sin\left(\pi\, \frac{b}{a}\, \frac{n}{\ell_1}\right) K_{\frac{1}{2}-s}\left(\frac{2\pi k\, n}{\ell_1}\right), \quad \mathrm{Re}(s) > 1, \tag{5.44}$$

where $\sigma_z(n,\, \chi_a,\, \chi_b)$ is a character analogue of the generalized divisor function (called the "double-weighted divisor function") defined by

$$\sigma_z(n,\, \chi_a,\, \chi_b) = \sum_{d|n} \chi_a(d)\, \chi_b\left(\frac{n}{d}\right) d^z.$$

Adapting the methods studied at the previous chapter, one can prove that equations (5.43) and (5.44) provide the analytic continuation of (5.42) as an entire function in $\mathbb{C}$ satisfying the functional equation

$$G(\overline{\chi}_1)\, G(\overline{\chi}_2) \left(\frac{2\pi}{\sqrt{|d|}}\right)^{-s} \Gamma(s)\, Z_2\,(s,\, Q,\, \chi_1,\, \chi_2) = \left(\frac{2\pi}{\sqrt{|d|}}\right)^{s-1} \Gamma(1-s)\, Z_2\left(1-s,\, Q^{-1}_{\ell_1,\ell_2},\, \overline{\chi}_1,\, \overline{\chi}_2\right), \tag{5.45}$$

where $Q^{-1}_{\ell_1,\ell_2}$ denotes the Quadratic form

$$Q^{-1}_{\ell_1,\ell_2}\,(x,\, y) = Q^{-1}\left(\frac{x}{\ell_1},\, \frac{y}{\ell_2}\right).$$

Taking $\chi_1 = \chi_2$ and setting both equal to the character modulo 4, χ_4, we can show, via a similar argument given above and by using (5.45), that $L\,(s,\, \chi_4)$ has infinitely many zeros located in the critical line $\mathrm{Re}(s) = \frac{1}{2}$. However, despite the similarity with the proof of Corollary 5.2., dealing with the character version of the generalized divisor function, $\sigma_z(n,\, \chi_a,\, \chi_b)$, is very delicate, since estimates of series involving characters, even real ones, cannot be established with such straightforward methods.

Bibliography

[1] N. H. Abel, Solution de quelques problèmes à l'aide d'intégrales définies, in Oeuvres complètes de Niels Henrik Abel: Nouvelle édition *Cambridge Library Collection - Mathematics*, Cambridge University Press, 11–27, doi:10.1017/CBO9781139245807.003

[2] M. Aigner and G. M. Ziegler, Proofs from THE BOOK, Springer, Berlin, 2009.

[3] T. M. Apostol, Another Elementary proof of Euler's formula for $\zeta(2n)$, *Amer. Math. Monthly* **80** (1973) 425–431.

[4] ———, Dirichlet $L-$functions and Primitive Characters, *Proc. Amer. Math. Soc.* **31** (1972) 384–386.

[5] ———, Introduction to Analytic Number Theory, Undergraduate Texts in Mathematics, Springer-Verlag, New York-Heidelberg, 1976.

[6] ———, Modular Functions and Dirichlet Series in Number Theory, 2nd edition, Springer, New York (1990).

[7] ———, On the Lerch zeta function, *Pacific J. Math.*, Vol. 1., **2** (1951), 161–167.

[8] ———, Remark on the Hurwitz Zeta Function, Proc. Amer. Math. Soc. Vol. 2, **5** (1951), 690–693.

[9] E. Artin, The Gamma Function, Holt, Rinehart and Winston, 1964.

[10] R. Ayoub, An introduction to the analytic theory of numbers, Math. Surveys, no. **10**, Amer. Math. Soc., Providence, R.I., 1963.

[11] ———, Euler and the zeta function, *Amer. Math. Monthly*, **81** (1974), 1067–1086.

[12] ———, On L-functions, *Monatsh. Math.*, **71** (1967), 193–202.

[13] G. de Barra, Measure Theory and Integration, First New Age International Reprint, New Delhi, 2000.

[14] P. T. Bateman and E. Grosswald, On Epstein's zeta function, *Acta Arith.*, **9** (1964), 365-–373.

[15] B. C. Berndt, Generalized Dirichlet series and Hecke's functional equation, *Proc. Edinburgh Math. Soc.*, **15** (1967), 309–313.

[16] ———, Identities involving the coefficients of a class of Dirichlet series. I, *Trans. Amer. Math. Soc.*, **137** (1969), 345–359.

[17] B. C. Berndt, Identities involving the coefficients of a class of Dirichlet series. II, *Trans. Amer. Math. Soc.*, **137** (1969), 361–374.

[18] B. C. Berndt, Identities involving the coefficients of a class of Dirichlet series. III., *Trans. Amer. Math. Soc.*, **146** (1969), 323–348.

[19] B. C. Berndt, Identities involving the coefficients of a class of Dirichlet series. IV, *Trans. Amer. Math. Soc.*, **149** (1970), 179–185.

[20] B. C. Berndt, Identities involving the coefficients of a class of Dirichlet series. V., *Trans. Amer. Math. Soc.*, **160** (1971), 139–156.

[21] ———, Identities involving the coefficients of a class of Dirichlet series. VI, *Trans. Amer. Math. Soc.* **160** (1971), 157-167.

[22] ———, Modular transformations and generalizations of several formulae of Ramanujan, *Rocky Mountain J. Math.*, **7** (1977), 147–189.

[23] ———, On the Hurwitz zeta-function, *Rocky Mountain J. Math.*, **2** (1972) 151–157.

[24] ———, The Gamma function and the Hurwitz zeta-function, *Amer. Math. Monthly*, **92**, 126–130, 1985.

[25] ———, The functional equation of some Dirichlet series, II, *Proc. Amer. Math. Soc.*, **31** (1972), 24—26.

[26] ———, Two new proofs of Lerch's functional equation, *Proc. Amer. Math. Soc.*, **32** (1972), 403–408.

[27] B. C. Berndt, A. Dixit, J. Sohn, Character analogues of theorems of Ramanujan, Koshliakov, and Guinand, *Adv. Appl. Math.*, **46** (2011), 54—70.

[28] B. C. Berndt, A. Dixit, S. Kim, A. Zaharescu, Sums of squares and products of Bessel functions, *Advances in Mathematics*, **338** (2018), 305–338.

[29] B. C. Berndt, R. J. Evans, The determination of Gauss sums, *Bull. Amer. Math. Soc.* **5** (1981), 107—129.

[30] B. C. Berndt, S. Kim, A. Zaharescu, Analogues of Koshliakov's formula, *Ramanujan 125*, Contemp. Math. **627**, K. Alladi, F. Garvan, and A. J. Yee, editors, American Mathematical Society, Providence, RI, 2014, 41–48.

[31] ———, The Circle Problem of Gauss and the Divisor Problem of Dirichlet — Still Unsolved, *Amer. Math. Monthly*, **125** (2018), 99–114.

[32] B. C. Berndt, Y. Lee, and J. Sohn, Koshliakov's formula and Guinand's formula in Ramanujan's lost notebook, *Surveys in number theory, Dev. Math.*, **17** (2012), 21-42.

[33] S. Bochner and K. Chandrasekharan, On Riemann's Functional Equation, *Ann. of Math.*, **63** (1956), 336–360.

[34] K. Chandrasekharan and R. Narasimhan, Hecke's functional equation and arithmetical identities, *Ann. of Math.*, **74** (1961), 1-23.

[35] S. Chowla, L. J. Mordell, Note on the Non Vanishing of $L(1)$, *Proceedings of the American Mathematical Society*, **12** (1961), 283–284.

[36] H. Davenport, Multiplicative Number Theory, Springer-Verlag, Berlin Heidelberg, 2nd Edition, 1980.

[37] C. Deninger, On the analogue of the formula of Chowla and Selberg for real quadratic fields, *J. Reine Angew. Math.*, **351** (1984), 171—191.

[38] Max F. Deuring, On Epstein's zeta function, *Annals of Mathematics, Second Series*, **38** (1937), 585-593.

[39] A. L. Dixon, W. L. Ferrar, Lattice-point summation formulae, *Quart. J. Math. Oxford*, **2** (1931), 31-54.

[40] ———, Some summations over the lattice points of a circle I., *Quart. J. Math. Oxford*, **5** (1934), 48–63.

[41] ———, Some summations over the lattice points of a circle II., *Quart. J. Math. Oxford*, **5** (1934), 172–185.

[42] ———, On the summation formulae of Voronoï and Poisson, *Quart. J. Math. Oxford* **8** (1937), 66–74.

[43] H. M. Edwards, Riemann's Zeta Function, 1974, New York: Academic Press.

[44] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, Higher Transcendental Functions, Vols. **1**, **2** and **3**, McGraw-Hill, New York, London and Toronto, 1953.

[45] ———, Tables of integral transforms, Vol. **1** and **2**, McGraw-Hill, New York, 1954.

[46] L. Euler, De seriebus quibusdam considerationes, *Commentarii academiae scientiarum Petropolitanae*, **12**, 53–96. The reader can find the manuscript in latin in https://scholarlycommons.pacific.edu/euler-works/130/.

[47] ———, De summis serierum reciprocarum, *Commentarii academiae scientiarum Petropolitanae*, **7** (1734), 123–134. An english translation can be found in https://scholarlycommons.pacific.edu/euler-works/41/.

[48] ———, Variae observationes circa series infinitas, *Commentarii academiae scientiarum imperialis Petropolitanae*, **9** (1744), 160–188. An english translation can be found in https://scholarlycommons.pacific.edu/euler-works/72/.

[49] W. L. Ferrar, Some solutions of the equation $F(t) = F(t^{-1})$, *J. London Math. Soc.*, **11** (1936), 99–103.

[50] ———, Summation formulae and their relation to Dirichlet's series I., *Compositio Math.*, **1** (1935), 344-360.

[51] ———, Summation formulae and their relation to Dirichlet's series II., *Compositio Math.*, **4** (1937), 394-405.

[52] N. J. Fine, Note on the Hurwitz Zeta-Function, *Proceedings of the American Mathematical Society*, **2** (1951), 361–364.

[53] A. P. Guinand, On Poisson's summation formula, *Ann. Math.*, **42** (1941), 591–603.

[54] ———, Summation formulae and self-reciprocal functions. I, *Quart. J. Math.* (Oxford Ser. (2)), **9** (1938), 53–67.

[55] ———, Summation formulae and self-reciprocal functions (II), *Quart. J. Math.*, **1** (1939), 104—118.

[56] Emil Grosswald, Comments on some formulae of Ramanujan, *Acta Arith.*, **21** (1972), 25-34.

[57] G. H. Hardy, On Dirichlet's divisor problem, *Proc. London Math. Soc.* **15** (1916), 1—25.

[58] ———, On the expression of a number as the sum of two squares, *Quart. J. Pure Appl. Math.*, **46** (1915), 263—283.

[59] G. H. Hardy and E. C. Titchmarsh, Self reciprocal functions. *Quart. J. Math.*, **1** (1930), 196—231.

[60] G. H. Hardy, E. M. Wright, Introduction to the Theory of Numbers, Oxford University Press, Oxford, 6th edition, 2008.

[61] P. Henrici, Applied and computational complex analysis, vol. **1**, Wiley, New York, 1974.

[62] A. E. Ingham, Note on Riemann's zeta-Function and Dirichlet's $L-$ Functions, *J. London Math. Soc.* **5** (1930), 107–112.

[63] A. Ivić, The Riemann Zeta-function, John Wiley Sons, Inc., New York, 1985.

[64] Koji Katayama, On Ramanujans formula for values of Riemann zeta-function at positive odd integers, *Acta Arith.*, **22** (1973), 149-155.

[65] ———, Ramanujan's formulas for L-functions, *J. Math. Soc. Japan*, **26** (1974), 234-240.

[66] N. S. Koshliakov, On Voronoï's sum-formula, *Mess. Math.* **58** (1929), 30—32 (in Russian).

[67] M. I. Knopp, On Dirichlet series satisfying Riemann's functional equation. *Invent. Math.*, **117** (1994), 361–372.

[68] M. I. Knopp, S. Robins, Easy proofs of Riemann's functional equation for $\zeta(s)$ and of Lipschitz summation, *Proc. Amer. Math. Soc.*, **129** (2001), 1915–1922.

[69] N. S. Koshliakov, Application of Mellin's transformation to the deduction of some summation formulae, *Izv. Akad. Nauk SSSR Ser. Mat.*, **5** (1944), Issue 1, 43—56.

[70] ———, Investigation of some questions of analytic theory of the rational and quadratic fields II (Russian), *Izv. Akad. Nauk SSSR Ser. Mat.* **18** (1954), 213–260.

[71] ———, On Voronoï's sum-formula, *Mess. Math.* **58** (1928), 30-–32.

[72] H.L. Li, S. Kanemitsu, X. H. Wang, Plana's summation formula as a modular relation and applications, *Integral transforms and Special Functions*, **23** (2012), 183–190.

[73] H. Lima, On Müntz-type formulas related to the Riemann zeta function, *Journal of Mathematical Analysis and Applications*, **463** (2018), 398–411.

[74] E. Lindelöf, Le calcul des résidus, Chelsea, New York, 1947.

[75] L. J. Mordell, Some applications of Fourier series in the analytic theory of numbers, *Proc. Cambridge Philos. Soc.* (1928), 585–596.

[76] ———, Poisson's summation formula and the Riemann zeta-function, *J. London Math. Soc.* **4** (1929), 285-291.

[77] Y. Motohashi, A new proof of the limit formula of Kronecker, *Proc. Jap. Acad*, **44** (1968), 614–616.

[78] C. Nasim, A summation formula involving $\sigma_k(n)$, $k > 1$, *Canad. J. Math.* **21** (1969), 951-964.

[79] ———, A summation formula involving $\sigma(n)$, *Trans. Amer. Math. Soc.* **192** (1974), 307—317.

[80] ———, On the summation formula of Voronoï, *Trans. Amer. Math. Soc.*, **163** (1972), 35-45.

[81] F. Oberhettinger, Note on Lerch Zeta Function, *Pacific J. Math.*, **6** (1956), 117–120.

[82] F. Oberhettinger and K. L. Soni, On some relations which are equivalent to functional equations involving the Riemann zeta-function, *Math. Z.*, **127** (1972), 17–34.

[83] T. L. Pearson, Note on the Hardy-Landau summation formula, *Canad. Math. Bull.*, **8** (1965), 717–720.

[84] H. S. A. Potter, E. C. Titchmarsh, The zeros of Epstein's zeta-functions , *Proc. London Math. Soc. (2)*, **39** (1935), 372-384.

[85] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, Integrals and Series: Vol. **1**: Elementary Functions, Gordon and Breach, New York, 1986; Vol. **2**: Special Functions, Gordon and Breach, New York, 1986; Vol. **3**: More Special Functions, Gordon and Breach, New York, 1989.

[86] R. Remmert, Classical Topics in Complex Function Theory, Springer-Verlag, New York, 1998.

[87] ———, Wielandt's theorem about the Γ-function, *Amer. Math. Monthly*, **103** (1996), 214–220.

[88] B. Riemann, On the Number of Prime Numbers less than a Given Quantity (translation from the german Über die Anzahl der Primzahlen unter einer gegebenen Grösse),

[89] P. Ribeiro, Appendix to Master's Thesis, **A.1.**: Supplement to Chapter II: Kontorovich-Lebedev transform and Poisson-type formulas, https://drive.google.com/file/d/17YxiVUnR1s8Q-ppf4u5F2LxzKaYnlHmS/view?usp=sharing
A.2. Supplement to Chapter III: A proof of Voronoï's identity via Kontorovich-Lebedev Transform
https://drive.google.com/file/d/1E0pm0-etSaGY0Vq4CCA-BnNbkXmoe_OG/view?usp=sharing

[90] A. Selberg and S. Chowla, On Epstein' s zeta function, J. Reine Angew. Math., **227** (1967), 86-110.

[91] T. Shintani, A Proof of the Classical Kronecker Limit Formula, *Tokyo J. Math*, **3** (1980), 191–199.

[92] J. R. Smart, On the values of the Epstein zeta function, *Glasgow Math. J.*, **14** (1973), 1-12.

[93] K. Soni, Some relations associated with an extension of Koshliakov's formula, *Proc. Amer. Math. Soc.*, **17** (1966), 543–551.

[94] E. M. Stein, R. Shakarchi, Complex Analysis, Princeton University Press, Princeton, New Jersey, 2003.

[95] P. R. Taylor, The functional equation for Epstein's zeta-function, *The Quarterly Journal of Mathematics*, **11** (1940), 177–182.

[96] ———, On the Riemann zeta function, *The Quarterly Journal of Mathematics*, **16** (1945), 1–21

[97] A. Terras, Bessel series expansion of the Epstein zeta function and the functional equation, *Trans. Amer. Math. Soc.*, **183** (1973), 477–486.

[98] ———, Real zeroes of Epstein's zeta function for ternary positive definite quadratic forms, *Illinois J. Math.*, **23** (1979), 1–14.

[99] , ———, Harmonic Analysis on Symmetric Spaces and Applications, Vol. **1**, Springer, Verlag, 1985.

[100] ———, Some formulas for the Riemann zeta function at odd integer argument resulting from Fourier expansions of the Epstein zeta function, *Acta Arith.*, **29** (1976), 181-189.

[101] E. C. Titchmarsh, On Epstein's zeta-function, *Proc. London Math. Soc.* **36** (1934), 485—500.

[102] ———, The Theory of Fourier Integrals, 2nd ed., Oxford University Press, London, 1948.

[103] ———, The Theory of The Riemann Zeta-Function, 2nd ed., The Clarendon Press, Oxford, 1986.

[104] B. Veklych, A One-Formula Proof of the Nonvanishing of L -Functions of Real Characters at 1, *Amer. Math. Monthly*, **122** (2015), 484–485.

[105] G. N. Watson, Some self-reciprocal functions (1), *Quart. J. of Math.* (Oxford), **2** (1931), 298–309.

[106] ———, A Treatise on the Theory of Bessel Functions, second ed., Cambridge University Press, London, 1966.

[107] E. T. Whittaker, G. N. Watson, A Course of Modem Analysis, 4th ed., University Press, Cambridge, 1962.

[108] K. S. Williams, Number Theory in the Spirit of Liouville, London Mathematical Society Student Texts, Vol. **76**, Cambridge University Press, Cambridge, 2010.

[109] J. R. Wilton, A Proof of Poisson's Summation formula, *Journal Lond. Math. Soc.*, **1-5** (1930), 276–279

[110] ———, Voronoï's summation formula, *Quart. J. Math.* (Oxford Ser. 2), **3** (1932), 26-32.

[111] ———, A note on the coefficients in the expansion of $\zeta(s, x)$ in powers of $(s-1)$, *Quart. J. Pure Appl. Math.*, **50** (1927), 329–332.

[112] S. B. Yakubovich, A general class of Voronoï's and Koshliakov-Ramanujan's summation formulas involving $d_k(n)$, *Integral Transforms Spec. Funct.*, **22** (2011), 801–821.

[113] ———, Asymptotic and summation formulas related to the Lebedev integrals, *Integral Transforms and Special Functions*, **19** (2008), 293–304

[114] ———, Certain identities, connection and explicit formulas for the Bernoulli and Euler numbers and the Riemann zeta-values, *Analysis* (Berlin),**35** (2015), 59–71.

[115] ———, Index Transforms, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1996.

[116] ———, New summation and transformation formulas of the Poisson, Müntz, Möbius and Voronoï type, *Integral Transforms and Special Functions*, **26** (2015), 768–795.

[117] ———, On a progress in the Kontorovich-Lebedev transform theory and related integral operators, *Integral Transforms and Special Functions*, **19** (2008), 509–534.

[118] ———, Voronoï–Nasim summation formulas and index transforms, *Integral Transforms and Special Functions*, **23** (2012), 369-388.

[119] ———, Voronoï-type summation formula involving $\sigma_{i\tau}$ and index transforms, *Integral Transforms and Special Functions*, **24** (2013), 98-110.