INSTANTIATION OVERFLOW

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Abstract. The well-known embedding of full intuitionistic propositional calculus into the atomic polymorphic system $\mathbf{F_{at}}$ is possible due to the intriguing phenomenon of *instantiation overflow*. Instantiation overflow ensures that (in $\mathbf{F_{at}}$) we can instantiate certain universal formulas by any formula of the system, not necessarily atomic. Until now only three types in $\mathbf{F_{at}}$ were identified with such property: the types that result from the Prawitz translation of the propositional connectives ($\perp$, $\wedge$, $\vee$) into $\mathbf{F_{at}}$ (or Girard's system $\mathbf{F}$). Are there other types in $\mathbf{F_{at}}$ with instantiation overflow? In this paper we show that the answer is yes and we isolate a class of formulas with such property.

1. Introduction

Since 2006 [1], it is known that the restriction of Jean-Yves Girard's system $\mathbf{F}$ [6] to atomic universal instantiations embeds the full intuitionistic propositional calculus (**IPC**). Or, on recent terminology [3], the atomic polymorphic system $\mathbf{F_{at}}$ embeds **IPC**.

System $\mathbf{F_{at}}$ has exactly the same formulas as $\mathbf{F}$: the smallest class of expressions which includes the atomic formulas (propositional constants P, Q, R, ... and second-order variables X, Y, Z, ...) and is closed under implication and second-order universal quantification. The (natural deduction) rules of $\mathbf{F_{at}}$ only differ from the ones of $\mathbf{F}$ on the second-order universal elimination rule where a restriction to atomic instantiations is imposed. I.e., the introduction rules of $\mathbf{F_{at}}$ are as in $\mathbf{F}$:

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \to B}\ \to\mathrm{I} \qquad\qquad \dfrac{\begin{matrix}\vdots \\ A\end{matrix}}{\forall X.A}\ \forall\mathrm{I}$$

where, in the universal rule, X does not occur free in any undischarged hypothesis and the elimination rules of $\mathbf{F_{at}}$ are:

$$\dfrac{\begin{matrix}\vdots \\ A \to B\end{matrix} \qquad \begin{matrix}\vdots \\ A\end{matrix}}{B}\ \to\mathrm{E} \qquad\qquad \dfrac{\begin{matrix}\vdots \\ \forall X.A\end{matrix}}{A[C/X]}\ \forall\mathrm{E}$$

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where C is an *atomic* formula (free for X in A), and $A[C/X]$ is the result of replacing in A all the free occurrences of X by C. (Note that system $\mathbf{F}$ allows in the $\forall$E-rule the instantiation by any formula, not only by the atomic ones.)

For a formulation of $\mathbf{F_{at}}$ in the (operational) λ-calculus style see [3].

As opposed to Girard's $\mathbf{F}$, system $\mathbf{F_{at}}$ is predicative, has a good notion of subformula and enjoys the subformula property (see [1]).

Moreover, strong normalization for $\mathbf{F_{at}}$ can be proved by an easy adaptation of Tait's reducibility technique with no need for Girard's reducibility candidates, and an alternative proof of strong normalization for **IPC** can be derived [3].

The embedding of **IPC** into $\mathbf{F_{at}}$ [1, 2] is via the Prawitz translation of connectives [7]:

$$\begin{aligned}
&\perp := \forall X.X \\
&A \wedge B := \forall X((A \to (B \to X)) \to X) \\
&A \vee B := \forall X((A \to X) \to ((B \to X) \to X)),
\end{aligned}$$

where X is a second-order variable which does not occur in A nor in B;
and is made possible due to the property of *instantiation overflow*, which ensures that, from the universal formulas above, it is possible to deduce in $\mathbf{F_{at}}$ (respectively)

$$\begin{aligned}
&F \\
&(A \to (B \to F)) \to F \\
&(A \to F) \to ((B \to F) \to F),
\end{aligned}$$

for *any* (not necessarily atomic) formula F. In other words, although the $\forall$E-rule of $\mathbf{F_{at}}$ allows just atomic instantiations, for the three types above (i.e., for the translations of $\perp$, $A \wedge B$ or $A \vee B$), instantiation overflow ensures that we can (via a proper derivation in $\mathbf{F_{at}}$) do the instantiation with any formula. (Modulo derivations in $\mathbf{F_{at}}$) These three types are not affected by $\mathbf{F_{at}}$'s restriction. Instantiation overflow is crucial in the embedding of **IPC** into $\mathbf{F_{at}}$. For the proof of instantiation overflow[1] in the three cases above and the proof of the (sound) embedding of **IPC** into $\mathbf{F_{at}}$ see [1, 2, 3]. The faithfulness of the embedding can be seen in [4, 5].

In [3] we can read:

> *"The above three types correspond to the empty type, the product type and the sum type (respectively) in the terminology of Girard et al. [6]. We believe that it is an interesting question to characterize exactly which types enjoy the property of instantiation overflow."*

Note that if all formulas of $\mathbf{F_{at}}$ had instantiation overflow, the system would have the exact same expressive power as $\mathbf{F}$. This is of course very far from being the case[2].

Until now, the only formulas identified in $\mathbf{F_{at}}$ with the overflow property were the three types above. In general, from an universal formula, we do not have a derivation in $\mathbf{F_{at}}$ for its instantiation by an arbitrary formula of the system. See Appendix 3 for the illustration of that impossibility with a concrete example. A brief inspection over arbitrary universal formulas quickly made us wonder if there was any other formula in $\mathbf{F_{at}}$ with the overflow property.

[1]The proof is by induction on the complexity of F and provides an algorithmic method for obtaining the deductions for the three types above.

[2]It is not clear yet how $\mathbf{F_{at}}$ and **IPC** compare in terms of expressiveness.

This paper is a first contribution towards (what seems to be) the hard problem of characterizing the class of formulas of $\mathbf{F_{at}}$ with instantiation overflow. Inspired by the formula's structure imposed by the Prawitz translation of the **IPC** connectives $\bot$, $\wedge$, $\vee$, we construct a class of formulas stratified by levels and prove that the universal closure of all formulas in the first two levels (which properly include the translation of the three **IPC** connectives above) have the property of instantiation overflow. We also show that at each level we can find at least a formula whose universal closure has the overflow property.

2. Formulas with instantiation overflow

As mentioned in Section 1, we know that the three types

- $\forall X.X$,
- $\forall X((A \to (B \to X)) \to X)$,
- $\forall X((A \to X) \to ((B \to X) \to X))$,

with X a second-order variable which does not occur in A nor in B, have instantiation overflow. So far, no other formulas in $\mathbf{F_{at}}$ were known to have such property. When trying to answer the natural question: "Are there other formulas in $\mathbf{F_{at}}$ with instantiation overflow?" some easy candidates are the universal closure of the subformulas of the formulas above. Not surprisingly, as shown in the result below, they still have instantiation overflow. Proposition 1 follows as a particular case of more general results (see Corollary 11) presented later in this section. We opted for presenting its proof here to familiarize the reader with the algorithmic structure of a proof of instantiation overflow.

Proposition 1. *The following formulas*

(1) $\forall X\,(A \to X)$,
(2) $\forall X\,(A \to (B \to X))$,
(3) $\forall X\,((A \to X) \to X)$,

with X a second-order variable which does not occur in A nor in B, have instantiation overflow.

Proof. (1) From $\forall X\,(A \to X)$ we want to show that there is a derivation in $\mathbf{F_{at}}$ of $A \to F$, for any formula F. The proof is by induction on the complexity of the formula F. For F an atomic formula the result is immediate from the application of the $\forall$E-rule. For F of the form $D \to E$ we have

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall X\,(A \to X)}{A \to E}\,\text{(IH)} \qquad [A]}{E}}{D \to E}}{A \to (D \to E)}$$

For F of the form $\forall X.E$ we have

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall X\,(A \to X)}{A \to E}\,\text{(IH)} \qquad [A]}{E}}{\forall X.E}}{A \to \forall X.E}$$

Note that in the double lines above we are assuming (by induction hypothesis) that instantiation overflow is available for E.

Cases (2) and (3) are proved in a similar way. We present below the deduction trees for implication and universal quantification in the latter case.

(3) One has

$$
\begin{prooftree}
\AxiomC{$\forall X\,((A\to X)\to X)$}
\doubleLine
\RightLabel{(IH)}
\UnaryInfC{$(A\to E)\to E$}
\AxiomC{$[A\to (D\to E)]$}
\AxiomC{$[A]$}
\BinaryInfC{$D\to E$}
\AxiomC{$[D]$}
\BinaryInfC{E}
\UnaryInfC{$A\to E$}
\BinaryInfC{E}
\UnaryInfC{$D\to E$}
\UnaryInfC{$(A\to (D\to E))\to (D\to E)$}
\end{prooftree}
$$

and

$$
\begin{prooftree}
\AxiomC{$\forall X\,((A\to X)\to X)$}
\doubleLine
\RightLabel{(IH)}
\UnaryInfC{$(A\to E)\to E$}
\AxiomC{$[A\to \forall X.E]$}
\AxiomC{$[A]$}
\BinaryInfC{$\forall X.E$}
\UnaryInfC{E}
\UnaryInfC{$A\to E$}
\BinaryInfC{E}
\UnaryInfC{$\forall X.E$}
\UnaryInfC{$(A\to \forall X.E)\to \forall X.E$}
\end{prooftree}
$$

$\square$

In the following proposition we present a formula with instantiation overflow which is not a subformula of any of the three types in the beginning of this section.

Proposition 2. *The formula* $\forall X\,((A\to X)\to (B\to X))$*, with* X *not occurring in* A *nor in* B*, has instantiation overflow.*

Proof. Let us prove, by induction on the complexity of the formula F, that from $\forall X\,((A\to X)\to (B\to X))$ we can derive $(A\to F)\to (B\to F)$ for any formula F. For F an atomic formula, the result is immediate. We give below the deduction trees for $F :\equiv D\to E$ and for $F :\equiv \forall X.E$.

$$
\begin{prooftree}
\AxiomC{$\forall X\,((A\to X)\to (B\to X))$}
\doubleLine
\RightLabel{(IH)}
\UnaryInfC{$(A\to E)\to (B\to E)$}
\AxiomC{$[A\to (D\to E)]$}
\AxiomC{$[A]$}
\BinaryInfC{$D\to E$}
\AxiomC{$[D]$}
\BinaryInfC{E}
\UnaryInfC{$A\to E$}
\BinaryInfC{$B\to E$}
\AxiomC{$[B]$}
\BinaryInfC{E}
\UnaryInfC{$D\to E$}
\UnaryInfC{$B\to (D\to E)$}
\UnaryInfC{$(A\to (D\to E))\to (B\to (D\to E))$}
\end{prooftree}
$$

and

$$
\begin{prooftree}
\AxiomC{$\forall X\,((A \to X) \to (B \to X))$}
\doubleLine
\RightLabel{(IH)}
\UnaryInfC{$(A \to E) \to (B \to E)$}
\AxiomC{$[A \to \forall X.E]$}
\AxiomC{$[A]$}
\BinaryInfC{$\forall X.E$}
\UnaryInfC{E}
\UnaryInfC{$A \to E$}
\BinaryInfC{$B \to E$}
\AxiomC{$[B]$}
\BinaryInfC{E}
\UnaryInfC{$\forall X.E$}
\UnaryInfC{$B \to \forall X.E$}
\UnaryInfC{$(A \to \forall X.E) \to (B \to \forall X.E)$}
\end{prooftree}
$$

$\square$

Since the target system in the Prawitz translation of the full intuitionistic propositional calculus is a system with implication, the connectives translated are $\perp$, $\wedge$ and $\vee$, with no need to translate $\to$. Note however that $B \to A$ could be translated (following a similar strategy) by the formula of Proposition 2.

Could it be the case that the only formulas in $\mathbf{F_{at}}$ with instantiation overflow were the ones that came from the translation of the four logical connectives via the previous *extension* of Prawitz correspondence (and their subformulas)? The answer is "no".

Having in view to isolate a class of formulas with instantiation overflow we start with some definitions.

Definition 3. *Consider the formula $\forall X.A$. We say that A is a* Prawitz formula *if A can be obtained according to the following clauses:*

(i) $A \equiv X$.
(ii) $A \equiv B \to P$, *where X does not occur in B and P is a Prawitz formula.*
(iii) $A \equiv P \to Q$, *where P, Q are Prawitz formulas.*

Definition 4. *Let A be a Prawitz formula. We define $lv(A)$, the* level *of A, according to the following clauses:*

(i) $lv(A) := 0$, *if $A \equiv X$.*
(ii) $lv(A) := lv(P)$, *if $A \equiv B \to P$, where X does not occur in B and P is a Prawitz formula.*
(iii) $lv(A) := \max\left(lv\,(P) + 1, lv\,(Q)\right)$, *if $A \equiv P \to Q$, where P, Q are Prawitz formulas.*

Note that the formulas of level 0 are the ones obtained by restricting Definition 3 to the first two clauses, i.e. are the smallest class of formulas that includes X and is closed under $B \to$ with B any formula of $\mathbf{F_{at}}$ where X does not occur. It is easy to see that a formula A has level 0 if and only if A is of the form $B_n \to (B_{n-1} \to (... \to (B_1 \to B_0)...))$, for $n \in \mathbb{N}_0$, where X does not occur in B_i $(1 \leq i \leq n)$ and $B_0 \equiv X$.

Lemma 5. *Let A be a Prawitz formula such that $lv\,(A) = 0$. Let D, E be formulas in $\mathbf{F_{at}}$.*

(1) *If there is a proof of $A\,[E/X]$ in $\mathbf{F_{at}}$, possibly with undischarged hypothesis, then we can extend that proof to a proof of $A\,[D \to E/X]$ and discharge any hypothesis D.*
(2) *If there is a proof of $A\,[E/X]$ in $\mathbf{F_{at}}$, where X does not occur free in any undischarged hypothesis, then one can extend that proof to a proof of $A\,[\forall X.E/X]$.*

Proof. Take $A \equiv B_n \to (B_{n-1} \to (... \to (B_1 \to B_0) ...))$, where B_0 is X and X does not occur in B_i $(1 \leq i \leq n)$. In both cases the proof is by induction on $n \in \mathbb{N}_0$.
(1) For $n = 0$ the proof is trivial (an application of the $\to$I-rule). For the induction step assume that from a proof of $B_n \to (B_{n-1} \to (... \to (B_1 \to E) ...))$ we may derive $B_n \to (B_{n-1} \to (... \to (B_1 \to (D \to E)) ...))$ and discharge any hypothesis D. Then

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots \\ B_{n+1} \to (B_n \to (... \to (B_1 \to E) ...))\end{array} \qquad [B_{n+1}]}{B_n \to (B_{n-1} \to (... \to (B_1 \to E) ...))}}{B_n \to (B_{n-1} \to (... \to (B_1 \to (D \to E)) ...))}\,\text{(IH)}}{B_{n+1} \to (B_n \to (... \to (B_1 \to (D \to E)) ...))}$$

I.e., from a proof of $B_{n+1} \to (B_n \to (... \to (B_1 \to E) ...))$ we may derive $B_{n+1} \to (B_n \to (... \to (B_1 \to (D \to E)) ...))$ and discharge any hypothesis D.
(2) For $n = 0$ the proof is trivial (an application of the $\forall$I-rule). For the induction step assume that from a proof of $B_n \to (B_{n-1} \to (... \to (B_1 \to E) ...))$, where X does not occur free in any undischarged hypothesis we may derive a proof of $B_n \to (B_{n-1} \to (... \to (B_1 \to \forall X.E) ...))$. Then

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots \\ B_{n+1} \to (B_n \to (... \to (B_1 \to E) ...))\end{array} \qquad [B_{n+1}]}{B_n \to (B_{n-1} \to (... \to (B_1 \to E) ...))}}{B_n \to (B_{n-1} \to (... \to (B_1 \to \forall X.E) ...))}\,\text{(IH)}}{B_{n+1} \to (B_n \to (... \to (B_1 \to \forall X.E)) ...)}$$

I.e., from a proof of $B_{n+1} \to (B_n \to (... \to (B_1 \to E) ...))$, where X does not occur free in any undischarged hypothesis, we may derive a proof of $B_{n+1} \to (B_n \to (... \to (B_1 \to \forall X.E)) ...)$. $\square$

In what follows we will be interested not only on the existence of the derivations above but also in the concrete derivations displayed in the proof of Lemma 5.

Theorem 6. *If $lv(A) = 0$ then the formula $\forall X.A$ has instantiation overflow.*

Proof. Because $lv(A) = 0$ we have (i) $A \equiv X$ or (ii) $A \equiv B_n \to (B_{n-1} \to (... \to (B_1 \to X)...))$, for some $n \geq 1$. Case (i) was shown in [1]. To prove case (ii) we need to show that from $\forall X.A$ we can derive, in $\mathbf{F_{at}}$, $A\,[F/X]$, for any formula F. By induction on the complexity of F we study the cases $F :\equiv D \to E$ and $F :\equiv \forall X.E$. One has

$$\dfrac{\dfrac{\forall X\,(B_n \to (B_{n-1} \to (... \to (B_1 \to X) ...)))}{B_n \to (B_{n-1} \to (... \to (B_1 \to E) ...))}\,\text{(IH)}}{B_n \to (B_{n-1} \to (... \to (B_1 \to (D \to E)) ...))}\,\text{Lemma 5.1}$$

and

$$\dfrac{\dfrac{\forall X\,(B_n \to (B_{n-1} \to (... \to (B_1 \to X) ...)))}{B_n \to (B_{n-1} \to (... \to (B_1 \to E) ...))}\,\text{(IH)}}{B_n \to (B_{n-1} \to (... \to (B_1 \to (\forall X.E)) ...))}\,\text{Lemma 5.2}$$

□

We aim to show that any Prawitz formula A such that $lv\,(A) = 1$ also has instantiation overflow. In order to do so we need a kind of converse of Lemma 5 (for level 0 formulas) and a version of the same lemma for level 1 formulas.

Lemma 7. *Let A be a Prawitz formula such that $lv\,(A) = 0$. Let D, E be arbitrary formulas in $\mathbf{F_{at}}$. From $A\,[D \to E/X]$ and D we can derive, in $\mathbf{F_{at}}$, $A\,[E/X]$.*

Proof. Since $lv\,(A) = 0$, A has the form $B_n \to (B_{n-1} \to (... \to (B_1 \to B_0)\,...))$, where B_0 is X and X does not occur in B_i $(1 \leq i \leq n)$. The proof is by induction on $n \in \mathbb{N}_0$. For $n = 0$ the proof is trivial (an application of the →E-rule). For the induction step assume that from $B_n \to (B_{n-1} \to (... \to (B_1 \to (D \to E))...))$ and D we may derive $B_n \to (B_{n-1} \to (... \to (B_1 \to E)\,...))$. Then

$$\cfrac{\cfrac{\cfrac{B_{n+1} \to (B_n \to (... \to (B_1 \to (D \to E))\,...)) \qquad [B_{n+1}]}{B_n \to (B_{n-1} \to (... \to (B_1 \to (D \to E))\,...))} \qquad D}{\overline{B_n \to (B_{n-1} \to (... \to (B_1 \to E)\,...))}}\,\text{(IH)}}{B_{n+1} \to (B_n \to (... \to (B_1 \to E)\,...))}$$

I.e., from $B_{n+1} \to (B_n \to (... \to (B_1 \to (D \to E))\,...))$ and D we may derive $B_{n+1} \to (B_n \to (... \to (B_1 \to E)\,...))$.

□

Lemma 8. *Let A be a Prawitz formula such that $lv(A) = 0$. Let E be an arbitrary formula in $\mathbf{F_{at}}$. Then from $A\,[\forall X.E/X]$ we can derive, in $\mathbf{F_{at}}$, $A\,[E/X]$.*

Proof. We prove, by induction on n, that the formula $A \equiv B_n \to (B_{n-1} \to (... \to (B_1 \to B_0)...))$, where $B_0 \equiv X$ and X does not occur in B_i $(1 \leq i \leq n)$ has the desired property, for all $n \in \mathbb{N}_0$. For $n = 0$ the result is trivial (an application of the ∀E-rule). For the induction step assume that from $B_n \to (B_{n-1} \to (... \to (B_1 \to \forall X.E)...))$ we may derive $B_n \to (B_{n-1} \to (... \to (B_1 \to E)\,...))$. Then

$$\cfrac{\cfrac{\cfrac{B_{n+1} \to (B_n \to (... \to (B_1 \to \forall X.E)\,...)) \qquad [B_{n+1}]}{B_n \to (B_{n-1} \to (... \to (B_1 \to \forall X.E)\,...))}}{\overline{B_n \to (B_{n-1} \to (... \to (B_1 \to E)\,...))}}\,\text{(IH)}}{B_{n+1} \to (B_n \to (... \to (B_1 \to E)\,...))}$$

I.e., from $B_{n+1} \to (B_n \to (... \to (B_1 \to \forall X.E)\,...))$ we may derive $B_{n+1} \to (B_n \to (... \to (B_1 \to E)\,...))$. □

By Definition 4, we can see that the class of Prawitz formulas of level 1 is the smallest class of formulas which includes the formulas of the form $P_2 \to P_1$ with $lv\,(P_2) = lv\,(P_1) = 0$ and is closed under $B \to$ and $S \to$ where B is any formula in $\mathbf{F_{at}}$ where X does not occur and S is any level 0 formula.

Lemma 9. *Let A be a Prawitz formula such that $lv\,(A) = 1$. Let D, E be arbitrary formulas in $\mathbf{F_{at}}$. Then*

(1) *If there is a proof of $A\,[E/X]$ in $\mathbf{F_{at}}$, possibly with undischarged hypothesis, then we can extend that proof to a proof of $A\,[D \to E/X]$ and discharge any hypothesis D.*

(2) *If there is a proof of $A\,[E/X]$, in $\mathbf{F_{at}}$, where X does not occur free in any undischarged hypothesis, then one can extend that proof to a proof of $A\,[\forall X.E/X]$.*

Proof. The proof is, in both cases, by induction on A, noticing that, because $lv(A) = 1$, the Prawitz formula A has one of the following forms:

a) $A \equiv A_1 \to A_2$, with A_1, A_2 Prawitz formulas of level 0;
b) $A \equiv B \to A'$, where X does not occur in B and A' is a Prawitz formula of level 1;
c) $A \equiv A_1 \to A_2$, where A_1 is a Prawitz formula of level 0 and A_2 is a Prawitz formula of level 1.

(1) For cases a) and c) we have

$$\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$A_1\,[E/X] \to A_2\,[E/X]$}
\AxiomC{$[A_1\,[D \to E/X]]$}
\AxiomC{$[D]$}
\doubleLine
\RightLabel{Lemma 7}
\BinaryInfC{$A_1\,[E/X]$}
\BinaryInfC{$A_2\,[E/X]$}
\doubleLine
\RightLabel{Lemma 5.1 (for case a)) or IH (for case c)); D discharged}
\UnaryInfC{$A_2\,[D \to E/X]$}
\UnaryInfC{$A_1\,[D \to E/X] \to A_2\,[D \to E/X]$}
\end{prooftree}$$

For case b) we have

$$\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$B \to A'\,[E/X]$}
\AxiomC{$[B]$}
\BinaryInfC{$A'\,[E/X]$}
\doubleLine
\RightLabel{(IH); D discharged}
\UnaryInfC{$A'\,[D \to E/X]$}
\UnaryInfC{$B \to A'\,[D \to E/X]$}
\end{prooftree}$$

(2) For cases a) and c) we have

$$\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$A_1\,[E/X] \to A_2\,[E/X]$}
\AxiomC{$[A_1\,[\forall X.E/X]]$}
\doubleLine
\RightLabel{Lemma 8}
\UnaryInfC{$A_1\,[E/X]$}
\BinaryInfC{$A_2\,[E/X]$}
\doubleLine
\RightLabel{Lemma 5.2 (for case a)) or IH (for case c))}
\UnaryInfC{$A_2\,[\forall X.E/X]$}
\UnaryInfC{$A_1\,[\forall X.E/X] \to A_2\,[\forall X.E/X]$}
\end{prooftree}$$

For case b) we have

$$\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$B \to A'\,[E/X]$}
\AxiomC{$[B]$}
\BinaryInfC{$A'\,[E/X]$}
\doubleLine
\RightLabel{(IH)}
\UnaryInfC{$A'\,[\forall X.E/X]$}
\UnaryInfC{$B \to A'\,[\forall X.E/X]$}
\end{prooftree}$$

$\square$

Theorem 10. *If $lv(A) = 1$ then the formula $\forall X.A$ has instantiation overflow.*

Proof. The proof is by induction on the complexity of F. We need to show that from $\forall X.A$ we may derive $A\,[F/X]$ for any formula F. For F atomic the proof is trivial (an application of the $\forall$E-rule). If $F :\equiv D \to E$ then

$$\dfrac{\dfrac{\forall X.A}{A\,[E/X]}\,(\mathrm{IH})}{A\,[D \to E/X]}\,\text{Lemma 9.1}$$

If $F := \forall X.E$ then

$$\dfrac{\dfrac{\forall X.A}{A\,[E/X]}\,(\mathrm{IH})}{A\,[\forall X.E/X]}\,\text{Lemma 9.2}$$

We conclude that we may derive $A\,[F/X]$ for any formula F, hence the formula $\forall X.A$ has instantiation overflow. $\square$

Corollary 11. *Let A be the translation into $\mathbf{F_{at}}$ of a formula of the intuitionistic propositional calculus (through the embedding of* **IPC** *into $\mathbf{F_{at}}$ mentioned in the introductory section) then:*

- *every universal subformula of A (say $\forall X.B$) has instantiation overflow;*
- *the universal closure of the subformulas of B which have X as a free-variable have instantiation overflow.*

Proof. Immediately from Theorems 6 and 10. Note that $\forall X.B$ has to be the Prawitz's translation of $\perp$, conjunction or disjunction and so B (and its subformulas which have X as a free-variable) have level less than or equal to 1. $\square$

Remark 1. *Let A be an arbitrary closed universal formula, and let*

$$A^{io} := \forall X\,((A \to X) \to X)\,.$$

Trivially, $A \vdash_{\mathbf{F_{at}}} A^{io}$. On the other side, by definition, $(A \to X) \to X$ is a Prawitz formula of level 1, and so by Theorem 10, the formula A^{io} has instantiation overflow. Hence, $A^{io} \vdash_{\mathbf{F_{at}}} (A \to A) \to A$ and also $A^{io} \vdash_{\mathbf{F_{at}}} A$ does hold. The (not so surprising) conclusion is that any universal formula is equivalent, in $\mathbf{F_{at}}$, to a universal formula having instantiation overflow. And so, given that not all universal formulas have instantiation overflow (see Appendix 3), it turns out that the class of formulas having instantiation overflow is not closed under logical equivalence.

3. Prawitz formulas of level 2 and beyond

It remains an open question if levels greater or equal than 2 are also as well-behaved as levels 0 and 1 concerning instantiation. We are able to show that (even disregarding tautologies) each level n contains particular inhabitants whose universal closure has instantiation overflow, namely,

$$(\ldots((P \to \underbrace{X) \to X)\ldots) \to X}_{n+1\ \mathit{times}}$$

with P a propositional constant.

Definition 12. *Let P be a propositional constant. For all $n \in \mathbb{N}_0$, we define A_n recursively by*

$$\begin{cases} A_0 := P \\ A_{n+1} := A_n \to X. \end{cases}$$

Observe that for $n \geq 1$ the formula A_n is a Prawitz formula with level $n-1$. To show that for all $n \in \mathbb{N}$, the formula $\forall X.A_n$ has instantiation overflow we need the following lemma.

Lemma 13. *Let A_n be as defined above. For all $i, k \in \mathbb{N}_0$, we can extend, in $\mathbf{F_{at}}$, a proof of $A_i[F/X]$ to a proof of $A_{2k+i}[F/X]$, for any formula F in $\mathbf{F_{at}}$.*

Proof. Fix $i \in \mathbb{N}_0$. The proof is by induction on $k \in \mathbb{N}_0$. For $k = 0$ the result is obvious. Assume that it is possible to extend, in $\mathbf{F_{at}}$, a proof of $A_i\,[F/X]$ to a proof of $A_{2k+i}\,[F/X]$. Then

$$\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$A_i\,[F/X]$}
\doubleLine
\RightLabel{(IH)}
\UnaryInfC{$A_{2k+i}\,[F/X]$}
\AxiomC{$[A_{2k+i+1}\,[F/X]]$}
\BinaryInfC{F}
\UnaryInfC{$A_{2k+i+2}\,[F/X]$}
\end{prooftree}$$

$\square$

Theorem 14. *For all $n \in \mathbb{N}$, the formula $\forall X.A_n$ has instantiation overflow.*

Proof. We consider two cases: i) n is even (say $n \equiv 2m$ with $m \in \mathbb{N}$) and ii) n in odd (say $n \equiv 2m-1$ with $m \in \mathbb{N}$).

In the first case, let us prove, by induction on the complexity of the formula F, that from $\forall X.A_{2m}$ we can derive $A_{2m}[F/X]$ for any formula F. For F an atomic formula, the result is immediate. We give below the deduction trees for $F :\equiv D \to E$ and for $F :\equiv \forall X.E$.

$$\begin{prooftree}
\AxiomC{$\forall X.A_{2m}$}
\doubleLine
\RightLabel{(IH)}
\UnaryInfC{$A_{2m}\,[E/X]$}
\AxiomC{$[P \to (D \to E)]$}
\AxiomC{$[P]$}
\BinaryInfC{$D \to E$}
\AxiomC{$[D]$}
\BinaryInfC{E}
\UnaryInfC{$P \to E$}
\dashedLine
\UnaryInfC{$A_1[E/X]$}
\doubleLine
\RightLabel{Lemma 13}
\UnaryInfC{$A_{2m-1}\,[E/X]$}
\BinaryInfC{E}
\UnaryInfC{$D \to E$}
\UnaryInfC{$(P \to (D \to E)) \to (D \to E)$}
\dashedLine
\UnaryInfC{$A_2\,[D \to E/X]$}
\doubleLine
\RightLabel{Lemma 13}
\UnaryInfC{$A_{2m}\,[D \to E/X]$}
\end{prooftree}$$

where the dashed line means syntactic equality.

$$
\begin{prooftree}
\AxiomC{$\forall X.A_{2m}$}
\doubleLine
\RightLabel{(IH)}
\UnaryInfC{$A_{2m}\,[E/X]$}
\AxiomC{$[P \to \forall X.E]$}
\AxiomC{$[P]$}
\BinaryInfC{$\forall X.E$}
\UnaryInfC{E}
\UnaryInfC{$P \to E$}
\dashedLine
\UnaryInfC{$A_1[E/X]$}
\doubleLine
\RightLabel{Lemma 13}
\UnaryInfC{$A_{2m-1}\,[E/X]$}
\BinaryInfC{E}
\UnaryInfC{$\forall X.E$}
\UnaryInfC{$(P \to \forall X.E) \to \forall X.E$}
\dashedLine
\UnaryInfC{$A_2\,[\forall X.E/X]$}
\doubleLine
\RightLabel{Lemma 13}
\UnaryInfC{$A_{2m}\,[\forall X.E/X]$}
\end{prooftree}
$$

In the second case, let us prove, by induction on the complexity of the formula F, that from $\forall X.A_{2m-1}$ we can derive $A_{2m-1}[F/X]$ for any formula F. For F an atomic formula, the result is immediate. We give below the deduction trees for $F :\equiv D \to E$ and for $F :\equiv \forall X.E$.

$$
\begin{prooftree}
\AxiomC{$\forall X.A_{2m-1}$}
\doubleLine
\RightLabel{(IH)}
\UnaryInfC{$A_{2m-1}\,[E/X]$}
\AxiomC{$[P]$}
\dashedLine
\UnaryInfC{$A_0[E/X]$}
\doubleLine
\RightLabel{Lemma 13}
\UnaryInfC{$A_{2m-2}\,[E/X]$}
\BinaryInfC{E}
\UnaryInfC{$D \to E$}
\UnaryInfC{$P \to (D \to E)$}
\dashedLine
\UnaryInfC{$A_1[D \to E/X]$}
\doubleLine
\RightLabel{Lemma 13}
\UnaryInfC{$A_{2m-1}\,[D \to E/X]$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$\forall X.A_{2m-1}$}
\doubleLine
\RightLabel{(IH)}
\UnaryInfC{$A_{2m-1}\,[E/X]$}
\AxiomC{$[P]$}
\dashedLine
\UnaryInfC{$A_0[E/X]$}
\doubleLine
\RightLabel{Lemma 13}
\UnaryInfC{$A_{2m-2}\,[E/X]$}
\BinaryInfC{E}
\UnaryInfC{$\forall X.E$}
\UnaryInfC{$P \to \forall X.E$}
\dashedLine
\UnaryInfC{$A_1[\forall X.E/X]$}
\doubleLine
\RightLabel{Lemma 13}
\UnaryInfC{$A_{2m-1}\,[\forall X.E/X]$}
\end{prooftree}
$$

□

Corollary 15. *For all $n \in \mathbb{N}_0$ there exists a Prawitz formula of level n whose universal closure has instantiation overflow.*

Appendix A

In this appendix we prove that the formula $\forall X\,(X \to P)$, with P a propositional constant, does not have instantiation overflow. Observe that (according to Definition 3) $X \to P$ is not a Prawitz formula.

Theorem 16. *The formula $\forall X\,(X \to P)$, with P a propositional constant, does not have instantiation overflow.*

Proof. Suppose, in order to obtain a contradiction, that from $\forall X\,(X \to P)$ one could derive, in $\mathbf{F_{at}}$, $(P \to P) \to P$. Since $\mathbf{F_{at}}$ is strongly normalizable [3] take $\mathcal{D}$ a normal proof of $(P \to P) \to P$ from $\forall X\,(X \to P)$. We know that

(δ) every formula in $\mathcal{D}$ is either a subformula of $\forall X\,(X \to P)$ (the hypothesis) or a subformula of $(P \to P) \to P$ (the conclusion).[3]

Let us analyse $\mathcal{D}$. By (δ) the last rule in $\mathcal{D}$ must be an introduction rule, so $\mathcal{D}$ has the form

$$\dfrac{\begin{array}{ccc}\forall X\,(X \to P) & & [P \to P]\\ & \vdots & \\ & P & \end{array}}{(P \to P) \to P}$$

Since P is a propositional constant it has to be derived by an elimination rule. By (δ) it is the elimination of an implication. Thus, in the bottom of the proof we have

$$\dfrac{\dfrac{S \to P \qquad S}{P}}{(P \to P) \to P}$$

By (δ) three situations may occur:

($\star$) (i) S is the formula $P \to P$ or (ii) S is P or (iii) S is an atomic formula different from P.

Case (i) does not occur. Note that if it were the case we would have

$$\dfrac{\dfrac{(P \to P) \to P \qquad P \to P}{P}}{(P \to P) \to P}$$

which is impossible because $\mathcal{D}$ is a normal proof, so it cannot have an η-conversion. Case (iii) also does not occur because if it was the case we would have

$$\dfrac{\dfrac{S \to P \qquad S}{P}}{(P \to P) \to P}$$

and above S the rule could not be an introduction rule since S is an atomic formula and could not be an elimination rule by (δ). Thus, we would have case (ii) and the proof would be

$$\dfrac{\dfrac{P \to P \qquad P}{P}}{(P \to P) \to P}$$

Again, above P we would be in the previous ($\star$) situation. Case (iii) does not occur (see the argument above). Case (i) uses the conclusion, just postponing the problem, so we may assume it is not the case. By case (ii) we have

[3]Condition (δ) is known as the subformula property. See [1] for a proof of the subformula property in $\mathbf{F_{at}}$.

$$\dfrac{\dfrac{P \to P \qquad \dfrac{P \to P \qquad P}{P}}{P}}{(P \to P) \to P}$$

Another P and $(\star)$ situation was generated and it should be analysed exactly as before. We see that the process would go forever. This is a contradiction because $\mathcal{D}$ (being a natural deduction proof) has necessarily a finite number of steps.

$\square$

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References

[1] F. Ferreira. Comments on Predicative Logic. *Journal of Philosophical Logic*, 35:1–8, 2006.

[2] F. Ferreira and G. Ferreira. Commuting conversions vs. the standard conversions of the "good" connectives. *Studia Logica*, 92:63–84, 2009.

[3] F. Ferreira and G. Ferreira. Atomic Polymorphism. *Journal of Symbolic Logic*, 78(1):260–274, 2013.

[4] F. Ferreira and G. Ferreira. The faithfulness of atomic polymorphism. *Trends in Logic XIII*, A. Indrzejczak, J. Kaczmarek, M. Zawidzki (eds.), Łódź University Press, pp. 55–64, 2014.

[5] F. Ferreira and G. Ferreira. The faithfulness of $\mathbf{F_{at}}$: a proof-theoretic proof. To appear in Studia Logica, DOI: 10.1007/s11225-015-9620-5. Online version available at: http://link.springer.com/article/10.1007%2Fs11225-015-9620-5

[6] J.-Y. Girard, Y. Lafont and P. Taylor. *Proofs and Types.* Cambridge University Press, 1989.

[7] D. Prawitz. *Natural Deduction.* Almkvist & Wiskell, Stockholm, 1965. Reprinted in Dover Publications, 2006.

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