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On the Size of Subclasses of Quasi-Copulas and Their Dedekind–MacNeille Completion

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Abstract: We study some topological properties of the class of supermodular n -quasi-copulas and check that the topological size of the Dedekind–MacNeille completion of the set of n -copulas is small, in terms of the Baire category, in the Dedekind–MacNeille completion of the set of the supermodular n -quasi-copulas, and in turn, this set and the set of n -copulas are small in the set of n -quasi-copulas.

Keywords: copula; Dedekind–MacNeille completion; quasi-copula; supermodularity

MSC: 62H05; 26A21

1. Introduction

The classical procedure in multivariate stochastic models assumes the existence of a suitable (joint) probability distribution, expressed in terms of the univariate marginals and the copula, that provides an accurate description of the phenomenon under consideration, and all subsequent computations (i.e., risk calculations) are based on this model. However, in most practical cases, the joint distribution is only partially known, and this model uncertainty should be taken into consideration. In particular, various investigations have considered that we only know the marginal distributions, but do not know the dependence structure, i.e., we do not know the copula.

These latter studies typically provide dependence uncertainty bounds for the copula and/or for related functionals of the multivariate distributions. Their origin has a long history going back to earlier works by Hoeffding and Fréchet (see, for instance, [1] and the references therein), but they have been extended to many kinds of functionals of the joint distribution, especially related to quantile estimation and risk calculation in finance and insurance (see, e.g., [2,3] and the references therein). As a matter of fact, these bounds are in general not copulas, but quasi-copulas. Such objects were introduced in [4,5] and characterized in analytical terms in [6,7] (for a complete overview, we refer to [8,9]). Noticeably, the suprema and infima of copulas are actually quasi-copulas, although not all quasi-copulas can be expressed in terms of lattice operations on copulas in dimensions greater than three (see [10,11]).

Quasi-copulas have recently appeared in the study of the model-free procedure for pricing financial instruments (see [3,12–14]). In fact, as underlined in [3], since the bounds on the dependence structure of a given random vector \mathbf{X} may not be copulas, then the results from stochastic order theory

that translate the bounds on the copula of \mathbf{X} into the bounds on the expectation of $f(\mathbf{X})$ do not apply directly, and some concepts need to be extended into the quasi-copula framework.

Motivated by the study of bounds for special classes of (quasi-)copulas, here, we focus our attention on copulas, supermodular quasi-copulas (see, for instance, [15]), as well as their Dedekind–MacNeille completion. In fact, these sets may represent natural frameworks where lattice operations on copulas naturally lie. Our purpose is, hence, to study the relative size of these sets by using the concept of the Baire category as considered, for instance, in [16–19]. Specifically, we will determine the size of the Dedekind–MacNeille completion of the sets of copulas and supermodular quasi-copulas with respect to the topology induced by the distance (or metric) d_∞ , i.e., the uniform convergence in the set of quasi-copulas.

The paper is organized as follows. After recalling some preliminary definitions and notations (Section 2), in Section 3, we present some properties related to supermodular quasi-copulas and show that the set of supermodular quasi-copulas is compact under the metric d_∞ . In Section 4, we show that the Dedekind–MacNeille completion of the set of copulas is small, in terms of the Baire category, in the Dedekind–MacNeille completion of the set of the supermodular quasi-copulas, and in turn, this set and the set of copulas are small in the set of quasi-copulas. Finally, Section 5 concludes.

2. Preliminaries

First, we recall some basic aspects about copulas and quasi-copulas (see, e.g., [20,21]).

Let $n \geq 2$ be a natural number. We recall that an n -dimensional quasi-copula (briefly, n -quasi-copula) is a function Q from $[0, 1]^n$ to $[0, 1]$ satisfying:

- (Q1) Boundary conditions: For every $\mathbf{u} = (u_1, u_2, \dots, u_n) \in [0, 1]^n$, $Q(\mathbf{u}) = 0$ if at least one coordinate of \mathbf{u} is equal to zero; and $Q(\mathbf{u}) = u_k$ whenever all coordinates of \mathbf{u} are equal to one, except maybe u_k .
- (Q2) Monotonicity: Q is nondecreasing in each variable.
- (Q3) Lipschitz condition: For every $\mathbf{u}, \mathbf{v} \in [0, 1]^n$, it holds that $|Q(\mathbf{u}) - Q(\mathbf{v})| \leq \sum_{i=1}^n |u_i - v_i|$.

The set of n -quasi-copulas will be denoted by \mathcal{Q}_n .

Quasi-copulas are generalizations of the concept of a copula, which is recalled here. An n -copula is a function C from $[0, 1]^n$ to $[0, 1]$ that satisfies the condition (Q1) for n -quasi-copulas and, in place of (Q2) and (Q3), the stronger condition:

- (Q4) n -increasing property: $V_C(B) := \sum (-1)^{k(\mathbf{c})} C(\mathbf{c}) \geq 0$ for every n -box $B = \times_{i=1}^n [a_i, b_i]$ in $[0, 1]^n$, where the sum is taken over all the vertices $\mathbf{c} = (c_1, c_2, \dots, c_n)$ of B (i.e., each c_k is equal to either a_k or b_k), and $k(\mathbf{c})$ is the number of indices k 's such that $c_k = a_k$.

The set of n -copulas will be denoted by \mathcal{C}_n .

Every n -copula is an n -quasi-copula, and a proper n -quasi-copula is an n -quasi-copula, which is not an n -copula—the set of proper n -quasi-copulas will be denoted by $\mathcal{Q}_n \setminus \mathcal{C}_n$.

If we consider the standard partial order among real-valued functions in the space of quasi-copulas, then we can provide upper and lower bounds in \mathcal{Q}_n . In fact, every n -quasi-copula Q satisfies the following condition:

$$W_n(\mathbf{u}) := \max\left(\sum_{i=1}^n u_i - n + 1, 0\right) \leq Q(\mathbf{u}) \leq \min(u_1, u_2, \dots, u_n) =: M_n(\mathbf{u}) \quad \text{for all } \mathbf{u} \in [0, 1]^n$$

It is known that: (a) M_n is an n -copula for every $n \geq 2$, (b) W_2 is a two-copula, and (c) W_n is a proper n -quasi-copula for every $n \geq 3$. For several interesting similarities and differences between copulas and proper quasi-copulas, see, for example, [22–27].

In the following, we will also consider some notions from lattice theory (see, e.g., [28]), which are recalled here.

Given two elements x and y of a partially ordered set (i.e., poset) (P, \leq) , let $x \vee y$ denote the join (or the least upper bound) of x and y (when it exists); similarly for $\bigvee S$, where S is a subset of P ; $x \wedge y$ denotes the meet (or the greatest lower bound) of x and y (when it exists); and similarly for $\bigwedge S$. If the join or meet is found within a particular poset P , we subscript $\bigvee_P S$. Given two posets A and B , we say that A is join-dense (respectively, meet-dense) in B if, for every d in B , there exists a set $S \subseteq A$ such that $d = \bigvee_B S$ (respectively, $d = \bigwedge_B S$). A poset $P \neq \emptyset$ is a lattice if for every x, y in P , $x \vee y$ and $x \wedge y$ are in P ; and P is a complete lattice if, for every $S \subseteq P$, $\bigvee S$ and $\bigwedge S$ are in P .

If $\varphi: P \rightarrow L$ is an order-embedding (i.e., order-preserving injection) of a poset P into a complete lattice L , then we say that L is a completion of P . Moreover, if φ maps P onto L , then φ is referred to as an order-isomorphism (i.e., order-preserving bijection). We also have the following definition (see [28]).

Definition 1. A completion P of a lattice L is called a Dedekind–MacNeille completion of L if P is join-dense and meet-dense in L .

3. Quasi-Copulas and Related Subclasses

Now, we consider the class of quasi-copulas \mathcal{Q}_n equipped with the standard partial order among real-valued functions. For any pair Q_1 and Q_2 of quasi-copulas (or copulas), $Q_1 \vee Q_2 = \inf\{Q \in \mathcal{Q}_n \mid Q_1 \leq Q, Q_2 \leq Q\}$ and $Q_1 \wedge Q_2 = \sup\{Q \in \mathcal{Q}_n \mid Q \leq Q_1, Q \leq Q_2\}$. As is known (see [10,11]), \mathcal{Q}_n is a complete lattice; however, \mathcal{C}_n (respectively, $\mathcal{Q}_n \setminus \mathcal{C}_n$) is not even a lattice. We denote by $DM(\mathcal{C}_n)$ the Dedekind–MacNeille completion of the set of n -copulas in \mathcal{Q}_n .

It is known that:

- the set of two-quasi-copulas \mathcal{Q}_2 is order-isomorphic to the Dedekind–MacNeille completion of the set of two-copulas \mathcal{C}_2 (see [11]);
- for $n \geq 3$, \mathcal{Q}_n is not order-isomorphic to $DM(\mathcal{C}_n)$ (see [10]).

In the quest for a suitable subset of quasi-copulas that may be order-isomorphic to $DM(\mathcal{C}_n)$, supermodular quasi-copulas were considered in [15]. Here, we recall the definition of this concept (see, e.g., [29]).

Definition 2. A function $f: [0, 1]^n \rightarrow [0, 1]$ is called supermodular if, for all $\mathbf{x}, \mathbf{y} \in [0, 1]^n$, it holds that $f(\mathbf{x} \vee \mathbf{y}) + f(\mathbf{x} \wedge \mathbf{y}) \geq f(\mathbf{x}) + f(\mathbf{y})$.

The following result is a useful characterization of supermodular functions (see [30,31]). We recall that for any $\mathbf{u} \in [0, 1]^n$ and any set of indices $A \subset \{1, 2, \dots, n\}$ with $0 < \text{card}(A) = k < n$, the k -dimensional section of a function $f: [0, 1]^n \rightarrow [0, 1]$ with fixed values given by \mathbf{u} at the positions not in A is the function $f_{\mathbf{u},A}: [0, 1]^k \rightarrow [0, 1]$ defined by $f_{\mathbf{u},A}(\mathbf{x}) = f(\mathbf{y})$, where $y_j = x_j$ if $j \in A$ and $y_j = u_j$ if $j \notin A$.

Proposition 1. A function $f: [0, 1]^n \rightarrow [0, 1]$ is supermodular if, and only if, all of its two-dimensional sections are supermodular.

For $n = 2$, supermodularity and two-increasingness are equivalent. However, this is no longer true for $n \geq 3$ (see [22]). Furthermore, supermodularity together with the boundary conditions (Q1) implies increasingness and one-Lipschitz continuity, whence the following result is obtained (see [22]).

Proposition 2. If $S: [0, 1]^n \rightarrow [0, 1]$ is a supermodular function satisfying the condition (Q1) of an n -quasi-copula, then S is an n -quasi-copula.

Let \mathcal{SQ}_n denote the set of supermodular n -quasi-copulas. As can be easily seen, for $n \geq 3$, $\mathcal{C}_n \subset \mathcal{SQ}_n$ (see [22]). Moreover, $\mathcal{C}_2 = \mathcal{SQ}_2$. A relevant subset of \mathcal{SQ}_n is formed by all Archimedean

n -quasi-copulas (see [22]), as introduced in [32]. In particular, W_n is supermodular. For other examples of supermodular n -quasi-copulas, see [33].

Now, we consider the lattice structure of \mathcal{SQ}_n and its related Dedekind–MacNeille completion, denoted by $\text{DM}(\mathcal{SQ}_n)$. The following result follows from [15].

Proposition 3. For $n \geq 3$, the following results hold:

- (a) \mathcal{SQ}_n is join-dense in \mathcal{Q}_n ;
- (b) \mathcal{SQ}_n is not meet-dense in \mathcal{Q}_n , i.e., there exists an n -quasi-copula Q_L such that for any $A \subseteq \mathcal{SQ}_n$, $Q_L \neq \bigwedge_{\mathcal{Q}_n} A$.

Thus, \mathcal{SQ}_n is not a complete lattice.

As a consequence of Proposition 3, for $n \geq 3$, \mathcal{Q}_n is not order-isomorphic to $\text{DM}(\mathcal{SQ}_n)$. Furthermore, in the following example, we show that neither $\mathcal{SQ}_n \subset \text{DM}(\mathcal{C}_n)$ nor $\text{DM}(\mathcal{C}_n) \subset \mathcal{SQ}_n$ hold.

Example 1. Let $n \geq 3$. We know that $W_n \in \mathcal{SQ}_n$; however, $W_n \notin \text{DM}(\mathcal{C}_n)$ since there does not exist an n -copula C such that $C \leq W_n$.

On the other hand, consider the following two n -copulas: $C_i^*(\mathbf{u}) = C_i(u_1, u_2) \prod_{j=3}^n u_j$ for $i = 1, 2$ and for all $\mathbf{u} = (u_1, u_2, \dots, u_n) \in [0, 1]^n$, where C_1 and C_2 are the two-copulas given by $C_1(u_1, u_2) = \min(u_1, u_2, \max(0, u_1 - 2/3, u_2 - 1/3, u_1 + u_2 - 1))$ and $C_2(u_1, u_2) = C_1(u_2, u_1)$, respectively. Then, we have that $Q = C_1^* \vee C_2^*$ is a proper n -quasi-copula such that $Q \in \text{DM}(\mathcal{C}_n)$ (see [10]); furthermore, Q is not supermodular, since $C_1 \vee C_2$ is a proper two-quasi-copula [11], i.e., it is not supermodular.

We conclude this section with two additional properties about the structure of the set \mathcal{SQ}_n .

It is known that the sets \mathcal{C}_n and \mathcal{Q}_n are compact under the metric d_∞ (see [20,34]). This is also the case for the set \mathcal{SQ}_n , as the next result shows.

Proposition 4. The set \mathcal{SQ}_n is compact under the metric d_∞ .

Proof. Let $\{Q_r\}_{r \in \mathbb{N}}$ be a sequence in \mathcal{SQ}_n that converges pointwise to an n -quasi-copula Q_n . Since $Q_r(\mathbf{x} \vee \mathbf{y}) + Q_r(\mathbf{x} \wedge \mathbf{y}) \geq Q_r(\mathbf{x}) + Q_r(\mathbf{y})$ for every $r \in \mathbb{N}$ and for all $\mathbf{x}, \mathbf{y} \in [0, 1]^n$, taking the limits on both sides of the inequality, we have:

$$\begin{aligned} Q(\mathbf{x} \vee \mathbf{y}) + Q(\mathbf{x} \wedge \mathbf{y}) &= \lim_{r \rightarrow +\infty} Q_r(\mathbf{x} \vee \mathbf{y}) + \lim_{r \rightarrow +\infty} Q_r(\mathbf{x} \wedge \mathbf{y}) = \lim_{r \rightarrow +\infty} (Q_r(\mathbf{x} \vee \mathbf{y}) + Q_r(\mathbf{x} \wedge \mathbf{y})) \\ &\geq \lim_{r \rightarrow +\infty} (Q_r(\mathbf{x}) + Q_r(\mathbf{y})) = \lim_{r \rightarrow +\infty} Q_r(\mathbf{x}) + \lim_{r \rightarrow +\infty} Q_r(\mathbf{y}) = Q(\mathbf{x}) + Q(\mathbf{y}), \end{aligned}$$

i.e., \mathcal{SQ}_n is closed in \mathcal{Q}_n . Since \mathcal{Q}_n is compact under the metric d_∞ , the set \mathcal{SQ}_n is also compact, which completes the proof. \square

For the next result, we need to recall the concept of an ordinal sum for quasi-copulas. Let \mathcal{J} be a finite or countably infinite subset of \mathbb{N} , and let $F_k: [0, 1]^n \rightarrow [0, 1]$, $k \in \mathcal{J}$, be a collection of functions. The ordinal sum F of $(F_k)_{k \in \mathcal{J}}$ with respect to the family of pairwise disjoint intervals $(]a_k, b_k[)_{k \in \mathcal{J}}$ is defined, for all $\mathbf{u} \in [0, 1]^n$, by:

$$F(\mathbf{u}) := \begin{cases} a_k + (b_k - a_k)F_k \left(\frac{\min(u_1, b_k) - a_k}{b_k - a_k}, \dots, \frac{\min(u_n, b_k) - a_k}{b_k - a_k} \right), & \text{if } \min(u_1, \dots, u_n) \in]a_k, b_k[\text{ for some } k \in \mathcal{J}, \\ \min(u_1, \dots, u_n), & \text{otherwise.} \end{cases}$$

The sets \mathcal{C}_n and \mathcal{Q}_n are closed under ordinal sums (see [20,35,36]). In the next result, we study the ordinal sum of two supermodular n -quasi-copulas in intervals of the form $]0, \alpha[$ and $] \alpha, 1[$, with $\alpha \in]0, 1[$.

Proposition 5. *The ordinal sum of two n -quasi-copulas of \mathcal{SQ}_n with respect to the intervals $]0, \alpha[$ and $]\alpha, 1[$, with $\alpha \in]0, 1[$, is in \mathcal{SQ}_n .*

Proof. Let Q be the ordinal sum of $Q_1, Q_2 \in \mathcal{SQ}_n$ with respect to the intervals $]0, \alpha[$, $]\alpha, 1[$. Let $\mathbf{u}_{ij} := (u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_{j-1}, u_{j+1}, \dots, u_n) \in [0, 1]^{n-2}$ be a fixed point in $[0, 1]^{n-2}$, with $1 \leq i < j \leq n - 2$, and define the function $G(x, y) := Q(u_1, \dots, u_{i-1}, x, u_{i+1}, \dots, u_{j-1}, y, u_{j+1}, \dots, u_n)$ on $[0, 1]^2$. We check that G is two-increasing (note that, as a consequence of Proposition 1, we would have $G \in \mathcal{SQ}_n$). Observe that, for all $x_1, x_2, y_1, y_2 \in [0, 1]$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$, we have:

$$\begin{aligned} G(x_2, y_2) + G(x_1, y_1) - G(x_2, y_1) - G(x_1, y_2) &= Q(u_1, \dots, u_{i-1}, x_2, u_{i+1}, \dots, u_{j-1}, y_2, u_{j+1}, \dots, u_n) \\ &\quad + Q(u_1, \dots, u_{i-1}, x_1, u_{i+1}, \dots, u_{j-1}, y_1, u_{j+1}, \dots, u_n) \\ &\quad - Q(u_1, \dots, u_{i-1}, x_2, u_{i+1}, \dots, u_{j-1}, y_1, u_{j+1}, \dots, u_n) \\ &\quad - Q(u_1, \dots, u_{i-1}, x_1, u_{i+1}, \dots, u_{j-1}, y_2, u_{j+1}, \dots, u_n) \\ &= V_Q \left(\times_{h=1}^{i-1} [0, u_h] \times [x_1, y_1] \times_{h=i+1}^{j-1} [0, u_h] \times [x_2, y_2] \times_{h=j+1}^n [0, u_h] \right). \end{aligned}$$

Note also that the rectangle $[x_1, x_2] \times [y_1, y_2]$ can be decomposed into a union of rectangles, namely $R_k := [x_{k1}, x_{k2}] \times [y_{k1}, y_{k2}]$, of disjoint interiors so that, if $(]x_{k1}, x_{k2}[\times]y_{k1}, y_{k2}[) \cap]0, \alpha[^2 \neq \emptyset$ (respectively, $(]x_{k1}, x_{k2}[\times]y_{k1}, y_{k2}[) \cap]\alpha, 1[^2 \neq \emptyset$), then $R_k \subseteq [0, \alpha]^2$ (respectively, $R_k \subseteq [\alpha, 1]^2$). We consider three cases:

1. $R_k \cap (]0, \alpha[^2 \cup]\alpha, 1[^2) = \emptyset$. We consider two subcases:

1a. $R_k \subseteq [0, \alpha] \times [\alpha, 1]$. Then, we have $0 \leq x_{k1} < x_{k2} \leq \alpha \leq y_{k1} < y_{k2} \leq 1$; thus:

$$\begin{aligned} &Q(u_1, \dots, u_{i-1}, x_{ir}, u_{i+1}, \dots, u_{j-1}, y_{is}, u_{j+1}, \dots, u_n) \\ &= \alpha Q_1 \left(\frac{\min(u_1, \alpha)}{\alpha}, \dots, \frac{\min(u_{i-1}, \alpha)}{\alpha}, \frac{x_{ir}}{\alpha}, \frac{\min(u_{i+1}, \alpha)}{\alpha}, \dots, \frac{\min(u_{j-1}, \alpha)}{\alpha}, 1, \right. \\ &\quad \left. \frac{\min(u_{j+1}, \alpha)}{\alpha}, \dots, \frac{\min(u_n, \alpha)}{\alpha} \right), \end{aligned}$$

with $r, s \in \{1, 2\}$, whence $V_G(R_k) = 0$.

1b. $R_k \subseteq [\alpha, 1] \times [0, \alpha]$. In this case, we have $0 \leq y_{k1} < y_{k2} \leq \alpha \leq x_{k1} < x_{k2} \leq 1$; thus:

$$\begin{aligned} &Q(u_1, \dots, u_{i-1}, x_{ir}, u_{i+1}, \dots, u_{j-1}, y_{is}, u_{j+1}, \dots, u_n) \\ &= \alpha Q_1 \left(\frac{\min(u_1, \alpha)}{\alpha}, \dots, \frac{\min(u_{i-1}, \alpha)}{\alpha}, 1, \frac{\min(u_{i+1}, \alpha)}{\alpha}, \dots, \frac{\min(u_{j-1}, \alpha)}{\alpha}, \frac{y_{is}}{\alpha}, \right. \\ &\quad \left. \frac{\min(u_{j+1}, \alpha)}{\alpha}, \dots, \frac{\min(u_n, \alpha)}{\alpha} \right), \end{aligned}$$

with $r, s \in \{1, 2\}$, whence $V_G(R_k) = 0$.

2. $R_k \subseteq]0, \alpha[^2$. We need to consider two subcases:

2a. $\min(\mathbf{u}_{ij}) \leq \alpha$. Then, we have:

$$\begin{aligned} &Q(u_1, \dots, u_{i-1}, x_{ir}, u_{i+1}, \dots, u_{j-1}, y_{is}, u_{j+1}, \dots, u_n) \\ &= \alpha Q_1 \left(\frac{\min(u_1, \alpha)}{\alpha}, \dots, \frac{\min(u_{i-1}, \alpha)}{\alpha}, \frac{x_{ir}}{\alpha}, \frac{\min(u_{i+1}, \alpha)}{\alpha}, \dots, \frac{\min(u_{j-1}, \alpha)}{\alpha}, \frac{y_{is}}{\alpha}, \right. \\ &\quad \left. \frac{\min(u_{j+1}, \alpha)}{\alpha}, \dots, \frac{\min(u_n, \alpha)}{\alpha} \right), \end{aligned}$$

with $r, s \in \{1, 2\}$. Therefore, $V_G(R_k) = \alpha V_{Q_1}(R'_k) \geq 0$, where:

$$R'_k = \times_{h=1}^{i-1} \left[0, \frac{\min(u_h, \alpha)}{\alpha} \right] \times \left[\frac{x_{k1}}{\alpha}, \frac{y_{k1}}{\alpha} \right] \times_{h=i+1}^{j-1} \left[0, \frac{\min(u_h, \alpha)}{\alpha} \right] \times \left[\frac{x_{k2}}{\alpha}, \frac{y_{k2}}{\alpha} \right] \times_{h=j+1}^n \left[0, \frac{\min(u_h, \alpha)}{\alpha} \right].$$

2b. $\min(\mathbf{u}_{ij}) > \alpha$. We separately study the values:

$$V_Q \left(\times_{h=1}^{i-1} I_h \times [x_{k1}, x_{k2}] \times_{h=i+1}^{j-1} I_h \times [y_{k1}, y_{k2}] \times_{h=j+1}^n I_h \right),$$

where $I_j \in \{[0, \alpha], [\alpha, u_j]\}$. Unless $I_j = [\alpha, u_j]$ for all j , all the cases correspond to Case 2a; thus, their Q -volumes are non-negative, and:

$$V_Q \left(\times_{h=1}^{i-1} [\alpha, u_h] \times [x_{k1}, x_{k2}] \times_{h=i+1}^{j-1} [\alpha, u_h] \times [y_{k1}, y_{k2}] \times_{h=j+1}^n [\alpha, u_h] \right) = 0.$$

3. $R_k \subseteq]\alpha, 1]^2$. We have two subcases.

3a. $\min(\mathbf{u}_{ij}) \leq \alpha$. Then, we have:

$$Q(u_1, \dots, u_{i-1}, x_{ir}, u_{i+1}, \dots, u_{j-1}, y_{is}, u_{j+1}, \dots, u_n) = \alpha Q_1 \left(\frac{\min(u_1, \alpha)}{\alpha}, \dots, \frac{\min(u_{i-1}, \alpha)}{\alpha}, 1, \frac{\min(u_{i+1}, \alpha)}{\alpha}, \dots, \frac{\min(u_{j-1}, \alpha)}{\alpha}, 1, \frac{\min(u_{j+1}, \alpha)}{\alpha}, \dots, \frac{\min(u_n, \alpha)}{\alpha} \right),$$

with $r, s \in \{1, 2\}$. Therefore, $V_G(R_k) = 0$.

3b. $\min(\mathbf{u}_{ij}) > \alpha$. This case is similar to Case 2b.

Therefore, we have that G is two-increasing, and the result follows. \square

Notice that the previous result can be extended to the ordinal sum of a finite (or countable) set of copulas by using a similar procedure as in [36]. Specifically, the ordinal sum of k copulas, namely C_1, \dots, C_k , can be interpreted as the ordinal sum of two copulas, C_1 and C'_1 , where C'_1 is an ordinal sum of the copulas C_2, \dots, C_k (with respect to suitable intervals).

4. Baire Category Results for Subclasses of Quasi-Copulas

In this section, we check that the “size” of the set $DM(\mathcal{SQ}_n)$ is “small”—in terms of the Baire category—in \mathcal{Q}_n and, in turn, $DM(\mathcal{C}_n)$ is “small” in $DM(\mathcal{SQ}_n)$. We recall that a subset of a (complete) metric space is called nowhere dense if it is not dense in any open ball $B(\mathbf{x}, r)$ of centre \mathbf{x} and radius $r > 0$ (equivalently, if its closure has an empty interior). Thus, for example, the set \mathcal{C}_2 is nowhere dense in the class \mathcal{Q}_2 (see [18]). Using the same techniques as those used in [18], it can be proven that, for $n \geq 3$, \mathcal{C}_n is nowhere dense \mathcal{Q}_n .

To study the Baire category results for the set $DM(\mathcal{SQ}_n)$ in \mathcal{Q}_n , we need some preliminary results.

Lemma 1. *Let $Q \in \mathcal{Q}_n$. Then, $Q \in DM(\mathcal{SQ}_n)$ if, and only if, for every $\mathbf{x} \in [0, 1]^n$, there exist Q_x and Q^x in \mathcal{SQ}_n such that $Q_x(\mathbf{x}) = Q^x(\mathbf{x}) = Q(\mathbf{x})$ and $Q_x(\mathbf{y}) \leq Q(\mathbf{y}) \leq Q^x(\mathbf{y})$ for all $\mathbf{y} \in [0, 1]^n$.*

Proof. Suppose $Q \in DM(\mathcal{SQ}_n)$ in \mathcal{Q}_n . Then, there exists a set $A \subset \mathcal{SQ}_n$ such that, for all $\mathbf{x} \in [0, 1]^n$, $Q(\mathbf{x}) = \sup \{Q^*(\mathbf{x}) : Q^* \in A\}$. Fixing \mathbf{x} , there exists a sequence $\{Q_k\}_{k \in \mathbb{N}}$ in \mathcal{SQ}_n such that $Q_k(\mathbf{y}) \leq Q(\mathbf{y})$ for all $\mathbf{y} \in [0, 1]^n$ and for every $k \in \mathbb{N}$, and $Q_k(\mathbf{x})$ converges to $Q(\mathbf{x})$ as k goes to ∞ .

Since \mathcal{SQ}_n is compact with the topology induced by the metric d_∞ (recall Proposition 4), there exists a subsequence of $\{Q_k\}_{k \in \mathbb{N}}$ convergent to $Q_x \in \mathcal{SQ}_n$. The n -quasi-copula Q_x satisfies that

$Q_x(\mathbf{x}) = Q(\mathbf{x})$ and $Q_x(\mathbf{y}) \leq Q(\mathbf{y})$ for all $\mathbf{y} \in [0, 1]^n$. Analogously, we can obtain that $Q^x(\mathbf{x}) = Q(\mathbf{x})$ and $Q(\mathbf{y}) \leq Q^x(\mathbf{y})$ for all $\mathbf{y} \in [0, 1]^n$.

Conversely, note that $Q(\mathbf{x}) = \sup \{Q_y(\mathbf{x}) : \mathbf{y} \in [0, 1]^n\} = \inf \{Q^y(\mathbf{x}) : \mathbf{y} \in [0, 1]^n\}$, so that Q belongs to $DM(\mathcal{SQ}_n)$, and this completes the proof. \square

Proposition 6. *The set $DM(\mathcal{SQ}_n)$ is compact in \mathcal{Q}_n with respect to the metric d_∞ .*

Proof. Let Q be an n -quasi-copula in the closure of $DM(\mathcal{SQ}_n)$. Then, there exists a sequence $\{Q_k\}_{k \in \mathbb{N}}$ in $DM(\mathcal{SQ}_n)$ such that $\|Q_k - Q\|_\infty < 1/k$ for every $k \in \mathbb{N}$. Let $\mathbf{x} \in [0, 1]^n$ be fixed. From Lemma 1, for every $k \in \mathbb{N}$, there exists an n -quasi-copula $(Q_k)_x$ such that $(Q_k)_x(\mathbf{x}) = Q_k(\mathbf{x})$ and $(Q_k)_x(\mathbf{y}) \leq Q_k(\mathbf{y})$ for all $\mathbf{y} \in [0, 1]^n$. Then, there exists a subsequence $\left\{ \left(Q_{\sigma(k)} \right)_x \right\}_{k \in \mathbb{N}}$ that converges to Q_x such that $\left(Q_{\sigma(k)} \right)_x(\mathbf{x}) = Q_{\sigma(k)}(\mathbf{x})$ and $\left(Q_{\sigma(k)} \right)_x(\mathbf{y}) \leq Q_{\sigma(k)}(\mathbf{y})$ for all $\mathbf{y} \in [0, 1]^n$. Therefore, we have $Q_x(\mathbf{x}) = Q(\mathbf{x})$, and since $\left(Q_{\sigma(k)} \right)_x(\mathbf{y}) \leq Q(\mathbf{y}) + 1/\sigma(k)$, then $Q_x(\mathbf{y}) \leq Q(\mathbf{y})$ for all $\mathbf{y} \in [0, 1]^n$. Since \mathcal{SQ}_n is closed, then Q_x is supermodular. From Lemma 1, we conclude that $Q \in DM(\mathcal{SQ}_n)$. \square

In what follows, we will need additional notation. For any $Q \in \mathcal{Q}_n$, let $Q_{L,k}$ denote the ordinal sum of Q and the n -quasi-copula Q_L given in Proposition 3, with intervals $]0, 1 - 1/k[$ and $]1 - 1/k, 1[$, where $k \in \mathbb{N}, k \geq 2$.

Lemma 2. *For any $Q \in \mathcal{Q}_n$, we have $Q_{L,k} \notin DM(\mathcal{SQ}_n)$. Moreover, $Q_{L,k}$ converges pointwise to Q as k goes to ∞ .*

Proof. Suppose $Q_{L,k} \in DM(\mathcal{SQ}_n)$. Then, from Lemma 1, for every $\mathbf{x} \in [1 - 1/k, 1]^n$, there exists $Q^x \in \mathcal{SQ}_n$ such that $Q^x(\mathbf{x}) = Q_{L,k}(\mathbf{x})$ and $Q^x(\mathbf{y}) \geq Q_{L,k}(\mathbf{y})$ for all $\mathbf{y} \in [0, 1]^n$. This implies:

$$Q^x \left(\overbrace{\left(\frac{k-1}{k}, \dots, \frac{k-1}{k} \right)}^n \right) = \frac{k-1}{k},$$

i.e., Q^x is an ordinal sum. Thus, the bijection from $\left[0, \frac{k-1}{k}\right]^n$ on $[0, 1]^n$ defined by:

$$(x_1, \dots, x_n) \longrightarrow \left(\frac{kx_1}{k-1}, \dots, \frac{kx_n}{k-1} \right)$$

provides a family Q^x_* of supermodular n -quasi-copulas such that, for a fixed $\mathbf{x} \in [0, 1]^n$, we have $Q^x_*(\mathbf{x}) = Q_{L,k}(\mathbf{x})$ and $Q^x_*(\mathbf{y}) \geq Q_{L,k}(\mathbf{y})$ for all $\mathbf{y} \in [0, 1]^n$. Therefore, we obtain a contradiction and, hence, $Q_{L,k} \notin DM(\mathcal{SQ}_n)$.

On the other hand, from the definition of $Q_{L,k}$, we have:

$$|Q_{L,k}(\mathbf{x}) - Q(\mathbf{x})| \leq \sum_{i=1}^n \min(1 - x_i, x_i/k).$$

This guarantees that $Q_{L,k}$ converges pointwise to Q as k goes to ∞ . \square

We are now in a position to provide the main result about the “size” of the set $DM(\mathcal{SQ}_n)$ in \mathcal{Q}_n .

Theorem 1. *The set $DM(\mathcal{SQ}_n)$ is nowhere dense in \mathcal{Q}_n .*

Proof. Suppose the set $DM(\mathcal{SQ}_n)$ is not nowhere dense in \mathcal{Q}_n . Since, from Proposition 6, $DM(\mathcal{SQ}_n)$ is closed in \mathcal{Q}_n , it must contain an open ball B , and the n -quasi-copula Q is in its interior. For a

sufficiently large k , we have that $Q_{L,k} \in B$. However, this is a contradiction since, from Lemma 2, we have $Q_{L,k} \notin DM(\mathcal{SQ}_n)$. Therefore, $DM(\mathcal{SQ}_n)$ is nowhere dense in \mathcal{Q}_n . \square

Since $\mathcal{C}_n \subset \mathcal{SQ}_n$, it follows that $DM(\mathcal{C}_n)$ is nowhere dense in \mathcal{Q}_n .

In the sequel, we show that $DM(\mathcal{C}_n)$ is nowhere dense in $DM(\mathcal{SQ}_n)$. In order to prove it, we need some preliminary results.

Lemma 3. *Let $Q \in \mathcal{Q}_n$. Then, $Q \in DM(\mathcal{C}_n)$ if, and only if, for every $\mathbf{x} \in [0, 1]^n$, there exist C_x and C^x in \mathcal{C}_n such that $C_x(\mathbf{x}) = C^x(\mathbf{x}) = Q(\mathbf{x})$ and $C_x(\mathbf{y}) \leq Q(\mathbf{y}) \leq C^x(\mathbf{y})$ for all $\mathbf{y} \in [0, 1]$.*

The proof of Lemma 3 is similar to the proof of Lemma 1, and we omit it.

As a consequence of Lemma 3, we have the following result, whose proof is similar to the proof of Proposition 6, and we omit it.

Proposition 7. *The set $DM(\mathcal{C}_n)$ is closed in \mathcal{Q}_n with respect to the metric d_∞ , and hence, it is also compact.*

For any $Q \in \mathcal{Q}_n$, let $Q_{W_n,k}$ denote the ordinal sum of Q and W_n with respect to the intervals $]0, 1 - 1/k[$ and $]1 - 1/k, 1[$, where $k \in \mathbb{N}, k \geq 2$.

Lemma 4. *For any $Q \in \mathcal{Q}_n$, we have $Q_{W_n,k} \notin DM(\mathcal{C}_n)$. Moreover, $Q_{W_n,k}$ converges pointwise to Q as k goes to ∞ .*

Proof. Suppose $Q_{W_n,k} \in DM(\mathcal{C}_n)$. Then, from Lemma 3, there exists $C_k \in \mathcal{C}_n$ such that:

$$C_k \left(\overbrace{\frac{k-1}{k}, \dots, \frac{k-1}{k}}^n \right) = Q_{W_n,k} \left(\overbrace{\frac{k-1}{k}, \dots, \frac{k-1}{k}}^n \right) = \frac{k-1}{k}$$

and $C_k(\mathbf{y}) \leq Q_{W_n,k}(\mathbf{y})$ for all $\mathbf{y} \in [0, 1]^n$, i.e., C_k is an ordinal sum. Thus, there exists n -copulas C, C' such that C_k can be represented as the ordinal sum of (C, C') with respect to the intervals $]0, 1 - 1/k[$ and $]1 - 1/k, 1[$. Since $C_k(\mathbf{y}) \leq Q_{W_n,k}(\mathbf{y})$, it follows that $C' \leq W_n$, which is absurd for $n \geq 3$. Therefore, $Q_{W_n,k} \notin DM(\mathcal{C}_n)$.

In order to prove the convergence of $Q_{W_n,k}$, we just have to follow the same steps as the ones given in the proof of Lemma 2, which completes the proof. \square

We are now in a position to check the “size” of the set $DM(\mathcal{C}_n)$ in $DM(\mathcal{SQ}_n)$. As a consequence of Proposition 7 and Lemma 4, we have the following result, whose proof is similar to the proof of Theorem 1, and we omit it.

Theorem 2. *The set $DM(\mathcal{C}_n)$ is nowhere dense in $DM(\mathcal{SQ}_n)$.*

From the results presented here and mimicking the same arguments used in their proofs, it is possible to show the following two results, whose proofs are omitted.

Theorem 3. *The set \mathcal{C}_n is nowhere dense in \mathcal{SQ}_n .*

For the next result, we recall that a function $F: [0, 1]^n \rightarrow [0, 1]$ is called k -dimensionally-increasing, with $k \in \{1, \dots, n\}$, if any of its k -dimensional sections is k -increasing. Let $\mathcal{DQ}_{n,k}$ denote the class of all k -dimensionally-increasing n -quasi-copulas. In [22], it was shown that $\mathcal{C}_n \subset \mathcal{DQ}_{n,n-1} \subset \mathcal{DQ}_{n,n-2} \subset \dots \subset \mathcal{DQ}_{n,3} \subset \mathcal{SQ}_n \subset \mathcal{Q}_n$. Then, we have:

Theorem 4. *For every $k = 3, \dots, n - 1$, the set $\mathcal{DQ}_{n,k}$ is nowhere dense in $\mathcal{DQ}_{n,k-1}$.*

5. Conclusions

In this paper, we considered the set of quasi-copulas \mathcal{Q}_n equipped with the natural ordering among real functions and the metric d_∞ . As is known, \mathcal{Q}_n is compact, and it is a complete lattice. We consider the subsets given by the class of copulas \mathcal{C}_n and of supermodular quasi-copulas \mathcal{SQ}_n , with $\mathcal{C}_n \subset \mathcal{SQ}_n$. The following results hold:

- \mathcal{C}_n and \mathcal{SQ}_n are compact (see, respectively, [20], Theorem 1.7.7, and Proposition 4);
- \mathcal{C}_n is nowhere dense in \mathcal{SQ}_n (see Theorem 3);
- \mathcal{SQ}_n is nowhere dense in \mathcal{Q}_n (it is an immediate consequence of Theorem 1).

Moreover, by considering the Dedekind–MacNeille completion $\text{DM}(\mathcal{C}_n)$ and $\text{DM}(\mathcal{SQ}_n)$, we have that:

- $\text{DM}(\mathcal{C}_n)$ and $\text{DM}(\mathcal{SQ}_n)$ are compact (see, respectively, Propositions 6 and 7);
- $\text{DM}(\mathcal{C}_n)$ is nowhere dense in $\text{DM}(\mathcal{SQ}_n)$ (see Theorem 2);
- $\text{DM}(\mathcal{SQ}_n)$ is nowhere dense in \mathcal{Q}_n (see Theorem 1).

In brief, both the sets of copulas and supermodular quasi-copulas (and their Dedekind–MacNeille completions) are small, in the sense of the Baire category, in the set of n -quasi-copulas.

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