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Asymptotic analysis of a problem for dynamic thermoelastic shells in normal damped response contact

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Abstract The purpose of this paper is twofold. We first provide the mathematical analysis of a dynamic contact problem in thermoelasticity, when the contact is governed by a normal damped response function and the constitutive thermoelastic law is given by the Duhamel-Neumann relation. Under suitable hypotheses on data and using a Faedo-Galerkin strategy, we show the existence and uniqueness of solution for this problem. We then study the particular case when the deformable body is, in fact, a shell and use asymptotic analysis to study the convergence to a 2D limit problem when the thickness tends to zero.

Keywords Thermodynamics · Asymptotic Analysis · Shells · Membrane · Contact

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1 Introduction

For the last 50 years, asymptotic methods have been used to derive and justify simplified models for three-dimensional solid mechanics problems for beams, plates and shells. The foundation for these methods was established by Lions in [7]. Regarding elastic shells, a complete theory can be found in [3].

Recently, in [1] the asymptotic limit of a dynamic problem for elliptic shells in thermoelasticity has been analyzed. The aim of the present note is to obtain similar results for thermoelastic shells in contact with a foundation. Contact problems abound in industry and many other areas in which mathematical models apply, as can tell the growing number of publications on the mathematical theory of contact (see for example [12] and references therein). The addition of a contact condition introduces a nonlinearity in the problem and, thus the methods and arguments needed will differ considerably from our previous work. Nevertheless, since both problems are cast into the same framework of the asysmptotic analysis of shells in thermoelasticity, the state of the art will not be reviewed here again, and we refer the interested reader to [1] and references therein.

Following the same principle, some notation will not be introduced here, since all required definitions are available in [1], and we will focus in the novelties due to the contact condition and how it affects the subsequent analysis.

The structure of the paper is the following: in Section 2 we shall describe the variational and mechanical formulations of the contact problem in cartesian coordinates in a general domain, and present a result of existence and uniqueness of solution for that problem. In Section 3 we consider the particular case when the deformable body is, in fact, a shell and reformulate the variational formulation in curvilinear coordinates. Then we give the scaled formulation. To do that, we will use a projection map into a reference domain and we will introduce the scaled unknowns and forces as well as the assumptions on coefficients. We also devote this section to recall and derive results that will be needed later. In Section 4 we briefly describe the formal asymptotic analysis which leads to the formulation of limit two-dimensional problems. Then, in Section 5 we prove the existence and uniqueness of solution for the two-dimensional limit problem and then we focus on the elliptic membrane case, for which we provide a rigorous convergence result. Finally, in Section 6 we show that the solution to the re-scaled version of this problem, with true physical meaning, also converges. The paper ends with Section 7, devoted to the conclusions and future work.

2 A three-dimensional dynamic contact problem for thermoelastic bodies. The normal damped response case

Let $\hat{\Omega}^\varepsilon$ be a three-dimensional bounded domain and assume that $\bar{\hat{\Omega}}^\varepsilon$ is the reference configurarion of a deformable body made of an elastic material, which is homogeneous and isotropic, with Lamé coefficients $\hat{\lambda}^\varepsilon \geq 0, \hat{\mu}^\varepsilon > 0$. Let

$\hat{\Gamma}^\varepsilon = \partial\hat{\Omega}^\varepsilon$ denote the boundary of the body, which is divided into three disjoint parts $\hat{\Gamma}_+^\varepsilon$, $\hat{\Gamma}_C^\varepsilon$ and $\hat{\Gamma}_0^\varepsilon$, where the measure of the latter is strictly positive.

The equations for the three-dimensional dynamic thermoelastic frictionless contact problem between a regular three-dimensional solid and a deformable foundation with normal damped response are the following:

Problem 1 Find the stress field $\hat{\boldsymbol{\sigma}}^\varepsilon = (\hat{\sigma}_{ij}^\varepsilon)$, the displacements field $\hat{\boldsymbol{u}}^\varepsilon = (\hat{u}_i^\varepsilon)$ and the temperature field $\hat{\vartheta}^\varepsilon$ verifying

$$\hat{\sigma}_{ij}^\varepsilon = \hat{\lambda}^\varepsilon \hat{e}_{kk}^\varepsilon(\hat{\boldsymbol{u}}^\varepsilon)\delta_{ij} + 2\hat{\mu}^\varepsilon \hat{e}_{ij}^\varepsilon(\hat{\boldsymbol{u}}^\varepsilon) - \hat{\alpha}_T^\varepsilon(3\hat{\lambda}^\varepsilon + 2\hat{\mu}^\varepsilon)\hat{\vartheta}^\varepsilon\delta_{ij} \text{ in } \hat{\Omega}^\varepsilon \times (0,T), \quad (1)$$

$$\hat{\rho}^\varepsilon \ddot{\hat{\boldsymbol{u}}}^\varepsilon - \mathrm{div}\hat{\boldsymbol{\sigma}}^\varepsilon = \hat{\boldsymbol{f}}^\varepsilon \text{ in } \hat{\Omega}^\varepsilon \times (0,T) \quad (2)$$

$$\hat{\beta}^\varepsilon \dot{\hat{\vartheta}}^\varepsilon = \partial_j(\hat{k}^\varepsilon \hat{\partial}_j^\varepsilon \hat{\vartheta}^\varepsilon) - \hat{\alpha}_T^\varepsilon(3\hat{\lambda}^\varepsilon + 2\hat{\mu}^\varepsilon)\hat{e}_{kk}^\varepsilon(\dot{\hat{\boldsymbol{u}}}^\varepsilon) + \hat{q}^\varepsilon \text{ in } \hat{\Omega}^\varepsilon \times (0,T), \quad (3)$$

$$\hat{\boldsymbol{u}}^\varepsilon = \boldsymbol{0} \text{ on } \hat{\Gamma}_0^\varepsilon \times (0,T), \quad (4)$$

$$\hat{\vartheta}^\varepsilon = 0 \text{ on } \hat{\Gamma}_0^\varepsilon \times (0,T), \quad (5)$$

$$\hat{\sigma}^\varepsilon \hat{\boldsymbol{n}}^\varepsilon = \hat{\boldsymbol{h}}^\varepsilon \text{ on } \hat{\Gamma}_+^\varepsilon \times (0,T), \quad (6)$$

$$-\hat{\sigma}_n^\varepsilon = \hat{p}^\varepsilon(\dot{\hat{u}}_n^\varepsilon),\ \hat{\boldsymbol{\sigma}}_t^\varepsilon = (\hat{\sigma}_{ti}^\varepsilon) = \boldsymbol{0} \text{ on } \hat{\Gamma}_C^\varepsilon \times (0,T), \quad (7)$$

$$\hat{k}^\varepsilon \hat{\partial}_j^\varepsilon \hat{\vartheta}^\varepsilon n_j = 0 \text{ on } (\hat{\Gamma}_+^\varepsilon \cup \hat{\Gamma}_C^\varepsilon) \times (0,T), \quad (8)$$

$$\dot{\hat{\boldsymbol{u}}}^\varepsilon(\cdot,0) = \hat{\boldsymbol{u}}^\varepsilon(\cdot,0) = \boldsymbol{0} \text{ in } \hat{\Omega}^\varepsilon, \quad (9)$$

$$\hat{\vartheta}^\varepsilon(\cdot,0) = 0 \text{ in } \hat{\Omega}^\varepsilon. \quad (10)$$

We refer the reader to [1] for the details on the set of equations and conditions (1)–(10),with the exception of the contact condition (7), on which we elaborate now. We consider that the body may enter in contact with a deformable foundation which, initially, is at a known distance (or gap) $\hat{s}^\varepsilon$ measured along the direction of outward normal vector $\hat{\boldsymbol{n}}^\varepsilon = (\hat{n}_i^\varepsilon)$ on $\hat{\Gamma}_C^\varepsilon$, and we assume that the normal response on the contact surface only happens when the surface element is moving towards the foundation, and vanishes when it is moving away. Thus to model contact in the normal direction we are using the so-called normal damped response (see [12] and references therein). Therefore, $\hat{p}^\varepsilon : \mathbb{R} \to \mathbb{R}_+$ is a non negative function which vanishes when its argument (the surface velocity) is nonpositive. Specifically, one may use

$$\hat{p}^\varepsilon(r) = \hat{\kappa}^\varepsilon\, r_+, \quad (11)$$

where $\hat{\kappa}^\varepsilon > 0$ stands for the normal damping coefficient, and we denote $r_+ = \max\{r, 0\}$ for any $r \in \mathbb{R}$. The set of mathematical assumptions for $\hat{p}^\varepsilon(\cdot) : \mathbb{R} \to \mathbb{R}^+$ is detailed below:

$$\begin{cases} \hat{p}^\varepsilon(r) = 0 \text{ if } r \le 0, \\ \text{There exists } L_p > 0 \text{ such that } |\hat{p}^\varepsilon(r_1) - \hat{p}^\varepsilon(r_2)| \le L_p|r_1 - r_2| \\ \qquad\qquad \forall\, r_1, r_2 \in \mathbb{R}, \\ (\hat{p}^\varepsilon(r_1) - \hat{p}^\varepsilon(r_2))(r_1 - r_2) \ge 0 \ \forall\, r_1, r_2 \in \mathbb{R}. \end{cases} \quad (12)$$

In particular, hypotheses (12) are verified by (11). For simplicity, we shall consider that in the reference configuration body and foundation are already in contact, thus $\hat{s}^\varepsilon = 0$. Now, to derive the variational formulation of the problem, let

$$V(\hat{\Omega}^\varepsilon) := \{\hat{\boldsymbol{v}}^\varepsilon = (\hat{v}_i^\varepsilon) \in [H^1(\hat{\Omega}^\varepsilon)]^3; \hat{\boldsymbol{v}}^\varepsilon = \boldsymbol{0} \text{ on } \hat{\Gamma}_0^\varepsilon\},$$
$$S(\hat{\Omega}^\varepsilon) := \{\hat{\varphi}^\varepsilon \in H^1(\hat{\Omega}^\varepsilon); \hat{\varphi}^\varepsilon = 0 \text{ on } \hat{\Gamma}_0^\varepsilon\},$$

which are the Hilbert spaces of admissible displacements and temperatures, respectively. We define the nonlinear map $\hat{P}^\varepsilon \colon [H^1(\hat{\Omega}^\varepsilon)]^3 \to [H^1(\hat{\Omega}^\varepsilon)]^{3'}$ such that

$$\left\langle \hat{P}^\varepsilon(\hat{\boldsymbol{u}}^\varepsilon), \hat{\boldsymbol{v}}^\varepsilon \right\rangle = \int_{\hat{\Gamma}_C^\varepsilon} \hat{p}^\varepsilon(\hat{u}_n^\varepsilon)\hat{v}_n^\varepsilon \, d\hat{\Gamma}^\varepsilon \; \forall \hat{\boldsymbol{u}}^\varepsilon, \hat{\boldsymbol{v}}^\varepsilon \in [H^1(\hat{\Omega}^\varepsilon)]^3.$$

Above and below we use the notation for a duality pair $\langle\cdot,\cdot\rangle$ in $V'(\hat{\Omega}^\varepsilon) \times V(\hat{\Omega}^\varepsilon)$ (also for $S'(\hat{\Omega}^\varepsilon) \times S(\hat{\Omega}^\varepsilon)$).

Then, it is straightforward to obtain the following variational formulation:

Problem 2 Find a pair $t \mapsto (\hat{\boldsymbol{u}}^\varepsilon(\hat{\boldsymbol{x}}^\varepsilon, t), \hat{\vartheta}^\varepsilon(\hat{\boldsymbol{x}}^\varepsilon, t))$ of $[0,T] \to V(\hat{\Omega}^\varepsilon) \times S(\hat{\Omega}^\varepsilon)$ verifying

$$\hat{\rho}^\varepsilon \left\langle \ddot{\hat{u}}_i^\varepsilon, \hat{v}_i^\varepsilon \right\rangle + a^{V,\varepsilon}(\hat{\boldsymbol{u}}^\varepsilon, \hat{\boldsymbol{v}}^\varepsilon) - c^\varepsilon(\hat{\vartheta}^\varepsilon, \hat{\boldsymbol{v}}^\varepsilon) + \left\langle \hat{P}^\varepsilon(\dot{\hat{\boldsymbol{u}}}^\varepsilon), \hat{\boldsymbol{v}}^\varepsilon \right\rangle = \left\langle \hat{J}^\varepsilon(t), \hat{\boldsymbol{v}}^\varepsilon \right\rangle \; \forall \hat{\boldsymbol{v}}^\varepsilon \in V(\hat{\Omega}^\varepsilon), \; a.e. \text{ in } (0,T), \tag{13}$$

$$\hat{\beta}^\varepsilon \left\langle \dot{\hat{\vartheta}}^\varepsilon, \hat{\varphi}^\varepsilon \right\rangle + a^{S,\varepsilon}(\hat{\vartheta}^\varepsilon, \hat{\varphi}^\varepsilon) + c^\varepsilon(\hat{\varphi}^\varepsilon, \dot{\hat{\boldsymbol{u}}}^\varepsilon) = \left\langle \hat{Q}^\varepsilon(t), \hat{\varphi}^\varepsilon \right\rangle \; \forall \hat{\varphi}^\varepsilon \in S(\hat{\Omega}^\varepsilon), \; a.e. \text{ in } (0,T), \tag{14}$$

with $\dot{\hat{\boldsymbol{u}}}^\varepsilon(\cdot,0) = \hat{\boldsymbol{u}}^\varepsilon(\cdot,0) = \boldsymbol{0}$ and $\hat{\vartheta}^\varepsilon(\cdot,0) = 0$.

In favour of simplicity, we are going to assume that the different parameters of the problem (thermal conductivity, thermal dilatation, specific heat coefficient, mass density, Lamé coefficients) are constants.

Theorem 1 *Let us assume that*

$$\begin{cases} \hat{\boldsymbol{f}}^\varepsilon \in H^1(0,T;[L^2(\hat{\Omega}^\varepsilon)]^3), \\ \hat{\boldsymbol{h}}^\varepsilon \in H^2(0,T;[L^2(\hat{\Gamma}_+^\varepsilon)]^3), \; \textit{and } \hat{\boldsymbol{h}}^\varepsilon(\cdot,0) = \boldsymbol{0}, \\ \hat{q}^\varepsilon \in H^1(0,T;L^2(\hat{\Omega}^\varepsilon)). \end{cases}$$

Then, there exists a unique pair $(\hat{\boldsymbol{u}}^\varepsilon(\boldsymbol{x},t), \hat{\vartheta}^\varepsilon(\hat{\boldsymbol{x}},t))$ solution to Problem 2 such that

$$\begin{cases} \hat{\boldsymbol{u}}^\varepsilon \in L^\infty(0,T;V(\hat{\Omega}^\varepsilon)) \\ \dot{\hat{\boldsymbol{u}}}^\varepsilon \in L^\infty(0,T;[L^2(\hat{\Omega}^\varepsilon)]^3) \cap L^\infty(0,T;V(\hat{\Omega}^\varepsilon)), \\ \ddot{\hat{\boldsymbol{u}}}^\varepsilon \in L^\infty(0,T;V'(\hat{\Omega}^\varepsilon)) \cap L^\infty(0,T;[L^2(\hat{\Omega}^\varepsilon)]^3), \end{cases} \tag{15}$$

$$\begin{cases} \hat{\vartheta}^\varepsilon \in L^\infty(0,T;L^2(\hat{\Omega}^\varepsilon)) \cap L^2(0,T;S(\hat{\Omega}^\varepsilon)), \\ \dot{\hat{\vartheta}}^\varepsilon \in L^\infty(0,T;L^2(\hat{\Omega}^\varepsilon)) \cap L^2(0,T;S(\hat{\Omega}^\varepsilon)). \end{cases} \tag{16}$$

Remark 1 The regularity results in (15c) and (16b) imply that the duality products involving $\ddot{\hat{\boldsymbol{u}}}^\varepsilon$ and $\dot{\hat{\vartheta}}^\varepsilon$ in (13) and (14) can be replaced by the usual inner products in $L^2(\hat{\Omega}^\varepsilon)$.

Proof We proceed by following the Faedo-Galerkin method. Let $\{\hat{\boldsymbol{w}}_i\}_{i=1}^\infty$ and $\{\hat{s}_i\}_{i=1}^\infty$ be two sequences of functions such that

$$\begin{cases} \hat{\boldsymbol{w}}_i \in V(\hat{\Omega}^\varepsilon) \quad \forall i, \\ \hat{\boldsymbol{w}}_1, \dots, \hat{\boldsymbol{w}}_m \quad \text{are orthonormal functions and} \quad V_m = \langle \hat{\boldsymbol{w}}_1, \dots, \hat{\boldsymbol{w}}_m \rangle, \forall m \\ V(\hat{\Omega}^\varepsilon) = \overline{\bigcup_{m \geq 1} V_m}. \end{cases} \tag{17}$$

$$\begin{cases} \hat{s}_i \in S(\hat{\Omega}^\varepsilon) \quad \forall i, \\ \hat{s}_1, \dots, \hat{s}_m \quad \text{are orthonormal functions and} \quad S_m = \langle \hat{s}_1, \dots, \hat{s}_m \rangle, \forall m \\ S(\hat{\Omega}^\varepsilon) = \overline{\bigcup_{m \geq 1} S_m}. \end{cases} \tag{18}$$

The approximated solutions $(\hat{\boldsymbol{u}}^m, \hat{\vartheta}^m)$ are defined by the following problem:

Problem 3 Find the functions $\hat{\boldsymbol{u}}^m \colon [0,T] \to V_m$ and $\hat{\vartheta}^m \colon [0,T] \to S_m$ in the form

$$\hat{\boldsymbol{u}}^m(\hat{\boldsymbol{x}}, t) = \sum_{i=1}^m u_i^m(t) \hat{\boldsymbol{w}}_i(\hat{\boldsymbol{x}}),$$

$$\hat{\vartheta}^m(\hat{\boldsymbol{x}}, t) = \sum_{i=1}^m \vartheta_i^m(t) \hat{s}_i(\hat{\boldsymbol{x}}),$$

such that

$$\hat{\rho}^\varepsilon \left\langle \ddot{\hat{\boldsymbol{u}}}^m, \hat{\boldsymbol{v}}^m \right\rangle + a^{V,\varepsilon}(\hat{\boldsymbol{u}}^m, \hat{\boldsymbol{v}}^m) - c^\varepsilon(\hat{\vartheta}^m, \hat{\boldsymbol{v}}^m) + \left\langle \hat{P}^\varepsilon(\dot{\hat{\boldsymbol{u}}}^m), \hat{\boldsymbol{v}}^m \right\rangle = \left\langle \hat{J}^\varepsilon(t), \hat{\boldsymbol{v}}^m \right\rangle, \qquad \forall\, \hat{\boldsymbol{v}}^m \in V_m, \tag{19}$$

$$\hat{\beta}^\varepsilon \left\langle \dot{\hat{\vartheta}}^m, \hat{\varphi}^m \right\rangle + a^{S,\varepsilon}(\hat{\vartheta}^m, \hat{\varphi}^m) + c^\varepsilon(\hat{\varphi}^m, \dot{\hat{\boldsymbol{u}}}^m) = \left\langle \hat{Q}^\varepsilon(t), \hat{\varphi}^m \right\rangle, \qquad \forall\, \hat{\varphi}^m \in S_m. \tag{20}$$

with the initial conditions

$$\hat{\boldsymbol{u}}^m(0) = \dot{\hat{\boldsymbol{u}}}^m(0) = \boldsymbol{0}, \quad \hat{\vartheta}^m(0) = 0. \tag{21}$$

Finding a solution for Problem 3 is equivalent to solving a first order differential equation system

$$\dot{\mathbf{Z}}(t) = \mathbf{F}(t, \mathbf{Z}), \quad \mathbf{Z}(0) = \mathbf{0}.$$

where $\mathbf{Z}(t) = (v_1^m(t), \ldots, v_m^m(t), u_1^m(t), \ldots, u_m^m(t), \vartheta_1^m(t), \ldots, \vartheta_m^m(t))$, with $v_j^m(t) = \dot{u}_j^m(t)$. The Picard-Lindeloff theorem gives a unique absolutely continuous solution in an interval $[0, t_m]$ which depends on the supreme of function $\mathbf{F}$ (which does not depend on time). Then, being the functions F_j uniformly Lipschitz in the variable $\mathbf{Z}$, if we prove that the solution $\mathbf{Z}(t)$ is bounded, we can extend the solution to the whole interval $[0, T]$.
Now the goal is to obtain estimations in appropriate normed spaces for $\hat{\boldsymbol{u}}^m$, $\dot{\hat{\boldsymbol{u}}}^m$, $\hat{\vartheta}^m$ and $\dot{\hat{\vartheta}}^m$.

We can take $\hat{\boldsymbol{v}}^m = \dot{\hat{\boldsymbol{u}}}^m \in V_m$ and $\hat{\varphi}^m = \hat{\vartheta}^m \in S_m$ in (19), (20) respectively, and adding both equations we have that

$$\hat{\rho}^\varepsilon \left\langle \ddot{\hat{\boldsymbol{u}}}^m, \dot{\hat{\boldsymbol{u}}}^m \right\rangle + a^{V,\varepsilon}(\hat{\boldsymbol{u}}^m, \dot{\hat{\boldsymbol{u}}}^m) + \hat{\beta}^\varepsilon \left\langle \dot{\hat{\vartheta}}^m, \hat{\vartheta}^m \right\rangle + a^{S,\varepsilon}(\hat{\vartheta}^m, \hat{\vartheta}^m) + \left\langle \hat{P}^\varepsilon(\dot{\hat{\boldsymbol{u}}}^m), \dot{\hat{\boldsymbol{u}}}^m \right\rangle = \left\langle \hat{J}^\varepsilon(t), \dot{\hat{\boldsymbol{u}}}^m \right\rangle + \left\langle \hat{Q}^\varepsilon(t), \hat{\vartheta}^m \right\rangle,$$

or equivalently

$$\frac{1}{2}\frac{d}{dt}\left\{ \hat{\rho}^\varepsilon \left| \dot{\hat{\boldsymbol{u}}}^m(t) \right|_0^2 + a^{V,\varepsilon}(\hat{\boldsymbol{u}}^m(t), \hat{\boldsymbol{u}}^m(t)) + \hat{\beta}^\varepsilon \left| \hat{\vartheta}^m(t) \right|_0^2 \right\} + a^{S,\varepsilon}\left(\hat{\vartheta}^m, \hat{\vartheta}^m\right) + \left\langle \hat{P}^\varepsilon(\dot{\hat{\boldsymbol{u}}}^m), \dot{\hat{\boldsymbol{u}}}^m \right\rangle = \left\langle \hat{J}^\varepsilon(t), \dot{\hat{\boldsymbol{u}}}^m \right\rangle + \left\langle \hat{Q}^\varepsilon(t), \hat{\vartheta}^m \right\rangle, \tag{22}$$

which, taking into account the monotonicity of $\hat{p}^\varepsilon$, becomes

$$\frac{1}{2}\frac{d}{dt}\left\{ \hat{\rho}^\varepsilon \left| \dot{\hat{\boldsymbol{u}}}^m(t) \right|_0^2 + a^{V,\varepsilon}(\hat{\boldsymbol{u}}^m(t), \hat{\boldsymbol{u}}^m(t)) + \hat{\beta}^\varepsilon \left| \hat{\vartheta}^m(t) \right|_0^2 \right\} + a^{S,\varepsilon}\left(\hat{\vartheta}^m, \hat{\vartheta}^m\right) \leq \left\langle \hat{J}^\varepsilon(t), \dot{\hat{\boldsymbol{u}}}^m \right\rangle + \left\langle \hat{Q}^\varepsilon(t), \hat{\vartheta}^m \right\rangle.$$

Notice that we shall use the notation $|\cdot|_0$ for a (vector or scalar) L^2 norm. The same applies for $\|\cdot\|_1$ to denote a H^1 norm. Integrating in $[0, t]$, taking into account (21), the coercivity of $a^{V,\varepsilon}$, $a^{S,\varepsilon}$, integrating by parts the term in $\hat{\Gamma}_+^\varepsilon$ and using Korn's inequality we get

$$\begin{aligned} &\hat{\rho}^\varepsilon \left| \dot{\hat{\boldsymbol{u}}}^m(t) \right|_0^2 + C \left\| \hat{\boldsymbol{u}}^m(t) \right\|_V^2 + \hat{\beta}^\varepsilon \left| \hat{\vartheta}^m(t) \right|_0^2 + \hat{k}\tilde{C} \int_0^t \left\| \hat{\vartheta}^m(s) \right\|_S^2 ds \\ &\quad \leq \int_0^t \left\{ \left| \hat{\boldsymbol{f}}^\varepsilon(s) \right|_0 \left| \dot{\hat{\boldsymbol{u}}}^m(s) \right|_0 + \left| \dot{\hat{\boldsymbol{h}}}^\varepsilon(s) \right|_{0,\hat{\Gamma}_+^\varepsilon} \left| \hat{\boldsymbol{u}}^m(s) \right|_{0,\hat{\Gamma}_+^\varepsilon} + \left| \hat{q}^\varepsilon(s) \right|_0 \left| \hat{\vartheta}^m(s) \right|_0 \right\} ds. \end{aligned} \tag{23}$$

Above and in what follows, $C, \tilde{C}$ denote positive constants whose specific value may change from line to line, only depending on data. Next, applying Young's inequality to each term in the right side in (23) and the continuity of the trace operator, yields that

$$\begin{aligned} &\left| \dot{\hat{\boldsymbol{u}}}^m(t) \right|_0^2 + \left\| \hat{\boldsymbol{u}}^m(t) \right\|_V^2 + \left| \hat{\vartheta}^m(t) \right|_0^2 + \int_0^t \left\| \hat{\vartheta}^m(s) \right\|_S^2 ds \\ &\quad \leq C(\hat{\boldsymbol{f}}^\varepsilon, \dot{\hat{\boldsymbol{h}}}^\varepsilon, \hat{q}^\varepsilon) + \tilde{C} \int_0^t \left\{ \left| \dot{\hat{\boldsymbol{u}}}^m(s) \right|_0^2 + \left\| \hat{\boldsymbol{u}}^m(s) \right\|_1^2 + \left| \hat{\vartheta}^m(s) \right|_0^2 \right\} ds, \end{aligned} \tag{24}$$

which, applying Gronwall's Lemma, gives

$$\left|\dot{\hat{\boldsymbol{u}}}^m(t)\right|_0^2 + \|\hat{\boldsymbol{u}}^m(t)\|_V^2 + \left|\hat{\vartheta}^m(t)\right|_0^2 \leq C(\hat{\boldsymbol{f}}^\varepsilon, \dot{\hat{\boldsymbol{h}}}^\varepsilon, \hat{q}^\varepsilon) + e^{\tilde{C}T}, \ \forall\, m, \tag{25}$$

from where,

$$\dot{\hat{\boldsymbol{u}}}^m \in L^\infty(0,T,[L^2(\hat{\Omega}^\varepsilon)]^3), \quad \hat{\vartheta}^m \in L^\infty(0,T,L^2(\hat{\Omega}^\varepsilon)), \quad \hat{\boldsymbol{u}}^m \in L^\infty(0,T,V(\hat{\Omega}^\varepsilon)).$$

Further, going back to (24), we have

$$\hat{\vartheta}^m \in L^2(0,T,S(\hat{\Omega}^\varepsilon)),$$

and going back to (22), repeating the process, but keeping the term

$$\int_0^t \left\langle \hat{P}^\varepsilon(\dot{\hat{\boldsymbol{u}}}^m), \dot{\hat{\boldsymbol{u}}}^m \right\rangle dr = \int_0^t \hat{\kappa}^\varepsilon (\dot{\hat{u}}_n^m)_+^2 dr,$$

we find that

$$(\dot{\hat{u}}_n^m)_+ \in L^2(0,T;L^2(\hat{\Gamma}_C^\varepsilon)).$$

Note that all the estimates are independent of m. Then

$$\{\hat{\boldsymbol{u}}^m\}_m \text{ is a bounded subset of } L^\infty(0,T,V(\hat{\Omega}^\varepsilon)), \tag{26}$$

$$\left\{\dot{\hat{\boldsymbol{u}}}^m\right\}_m \text{ is a bounded subset of } L^\infty(0,T,[L^2(\hat{\Omega}^\varepsilon)]^3), \tag{27}$$

$$\left\{\hat{\vartheta}^m\right\}_m \text{ is a bounded subset of } L^\infty(0,T,L^2(\hat{\Omega}^\varepsilon)) \text{ and } L^2(0,T;S(\hat{\Omega}^\varepsilon)), \tag{28}$$

$$\left\{(\dot{\hat{u}}_n^m)_+\right\}_m \text{ is a bounded subset of } L^2(0,T;L^2(\hat{\Gamma}_C^\varepsilon)). \tag{29}$$

We now add equations (19) and (20) and write the result at times $t+h$, with $h>0$ and $0 \leq t \leq T-h$, then subtract the resulting equations to get:

$$\begin{aligned}
&\hat{\rho}^\varepsilon \left\langle \ddot{\hat{u}}_i^m(t+h) - \ddot{\hat{u}}_i^m(t), \hat{v}_i^m \right\rangle + a^{V,\varepsilon}(\hat{\boldsymbol{u}}^m(t+h) - \hat{\boldsymbol{u}}^m(t), \hat{\boldsymbol{v}}^m) - c^\varepsilon(\hat{\vartheta}^m(t+h) - \hat{\vartheta}^m(t), \hat{\boldsymbol{v}}^m) \\
&\quad + \left\langle \hat{P}^\varepsilon(\dot{\hat{\boldsymbol{u}}}^m(t+h)) - \hat{P}^\varepsilon(\dot{\hat{\boldsymbol{u}}}^m(t)), \hat{\boldsymbol{v}}^m \right\rangle + \hat{\beta}^\varepsilon \left\langle \dot{\hat{\vartheta}}^m(t+h) - \dot{\hat{\vartheta}}^m(t), \hat{\varphi}^m \right\rangle \\
&\quad + a^{S,\varepsilon}(\hat{\vartheta}^m(t+h) - \hat{\vartheta}^m(t), \hat{\varphi}^m) + c^\varepsilon(\hat{\varphi}^m, \dot{\hat{\boldsymbol{u}}}^m(t+h) - \dot{\hat{\boldsymbol{u}}}^m(t)) \\
&\quad = \int_{\hat{\Omega}^\varepsilon} (\hat{f}^{i,\varepsilon}(t+h) - \hat{f}^{i,\varepsilon}(t)) \hat{v}_i^m \, d\hat{x}^\varepsilon + \int_{\hat{\Gamma}_+^\varepsilon} (\hat{h}^{i,\varepsilon}(t+h) - \hat{h}^{i,\varepsilon}(t)) \hat{v}_i^m \, d\hat{\Gamma}^\varepsilon + \int_{\hat{\Omega}^\varepsilon} (\hat{q}^\varepsilon(t+h) - \hat{q}^\varepsilon(t)) \hat{\varphi}^m \, d\hat{x}^\varepsilon \\
&\quad \forall \hat{\boldsymbol{v}}^m \in V_m, \ \forall \hat{\varphi}^m \in S_m, \ a.e. \text{ in } (0,T).
\end{aligned}$$

Next we take $\hat{\boldsymbol{v}}^m = \dot{\hat{\boldsymbol{u}}}^m(t+h) - \dot{\hat{\boldsymbol{u}}}^m(t) \in V_m$ and $\hat{\varphi}^m = \hat{\vartheta}^m(t+h) - \hat{\vartheta}^m(t) \in S_m$ to obtain

$$\begin{aligned}
&\hat{\rho}^\varepsilon \left\langle \ddot{\hat{u}}_i^m(t+h) - \ddot{\hat{u}}_i^m(t), \dot{\hat{u}}_i^m(t+h) - \dot{\hat{u}}_i^m(t) \right\rangle + a^{V,\varepsilon}(\dot{\hat{\boldsymbol{u}}}^m(t+h) - \hat{\boldsymbol{u}}^m(t), \dot{\hat{\boldsymbol{u}}}^m(t+h) - \dot{\hat{\boldsymbol{u}}}^m(t)) \\
&\quad + \left\langle \hat{P}^\varepsilon(\dot{\hat{\boldsymbol{u}}}^m(t+h)) - \hat{P}^\varepsilon(\dot{\hat{\boldsymbol{u}}}^m(t)), \dot{\hat{\boldsymbol{u}}}^m(t+h) - \dot{\hat{\boldsymbol{u}}}^m(t) \right\rangle \\
&\quad + \hat{\beta}^\varepsilon \left\langle \dot{\hat{\vartheta}}^m(t+h) - \dot{\hat{\vartheta}}^m(t), \hat{\vartheta}^m(t+h) - \hat{\vartheta}^m(t) \right\rangle + a^{S,\varepsilon}(\hat{\vartheta}^m(t+h) - \hat{\vartheta}^m(t), \hat{\vartheta}^m(t+h) - \hat{\vartheta}^m(t)) \\
&\quad = \int_{\hat{\Omega}^\varepsilon} (\hat{f}^{i,\varepsilon}(t+h) - \hat{f}^{i,\varepsilon}(t))(\dot{\hat{u}}_i^m(t+h) - \dot{\hat{u}}_i^m(t))\, d\hat{x}^\varepsilon + \int_{\hat{\Gamma}_+^\varepsilon} (\hat{h}^{i,\varepsilon}(t+h) - \hat{h}^{i,\varepsilon}(t))(\dot{\hat{u}}_i^m(t+h) - \dot{\hat{u}}_i^m(t))\, d\hat{\Gamma}^\varepsilon \\
&\quad + \int_{\hat{\Omega}^\varepsilon} (\hat{q}^\varepsilon(t+h) - \hat{q}^\varepsilon(t))(\hat{\vartheta}^m(t+h) - \hat{\vartheta}^m(t))\, d\hat{x}^\varepsilon.
\end{aligned}$$

Further, because of the monotonicity of $\hat{p}^\varepsilon$ we have that

$$\begin{aligned}
&\hat{\rho}^\varepsilon \frac{1}{2}\frac{d}{dt}\left\{ \left| \dot{\hat{\boldsymbol{u}}}^m(t+h) - \dot{\hat{\boldsymbol{u}}}^m(t) \right|_0^2 \right\} + \frac{1}{2}\frac{d}{dt}\left\{ a^{V,\varepsilon}(\hat{\boldsymbol{u}}^m(t+h) - \hat{\boldsymbol{u}}^m(t), \hat{\boldsymbol{u}}^m(t+h) - \hat{\boldsymbol{u}}^m(t)) \right\} \\
&\quad + \frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}^\varepsilon} \hat{\beta}^\varepsilon(\hat{\vartheta}^m(t+h) - \hat{\vartheta}^m(t))^2 d\hat{x}^\varepsilon + a^{S,\varepsilon}(\hat{\vartheta}^m(t+h) - \hat{\vartheta}^m(t), \hat{\vartheta}^m(t+h) - \hat{\vartheta}^m(t)) \\
&\quad \leq \int_{\hat{\Omega}^\varepsilon} (\hat{f}^{i,\varepsilon}(t+h) - \hat{f}^{i,\varepsilon}(t))(\dot{\hat{u}}_i^m(t+h) - \dot{\hat{u}}_i^m(t))\, d\hat{x}^\varepsilon + \int_{\hat{\Gamma}_+^\varepsilon} (\hat{h}^{i,\varepsilon}(t+h) - \hat{h}^{i,\varepsilon}(t))(\dot{\hat{u}}_i^m(t+h) - \dot{\hat{u}}_i^m(t))\, d\hat{\Gamma}^\varepsilon \\
&\quad + \int_{\hat{\Omega}^\varepsilon} (\hat{q}^\varepsilon(t+h) - \hat{q}^\varepsilon(t))(\hat{\vartheta}^m(t+h) - \hat{\vartheta}^m(t))\, d\hat{x}^\varepsilon.
\end{aligned}$$

Integrating in time in $[0, t]$ we get:

$$\begin{aligned}
&\frac{1}{2}\hat{\rho}^\varepsilon \left| \dot{\hat{\boldsymbol{u}}}^m(t+h) - \dot{\hat{\boldsymbol{u}}}^m(t) \right|_0^2 - \frac{1}{2}\hat{\rho}^\varepsilon \left| \dot{\hat{\boldsymbol{u}}}^m(h) - \dot{\hat{\boldsymbol{u}}}^m(0) \right|_0^2 \\
&\quad + \frac{1}{2} a^{V,\varepsilon}(\hat{\boldsymbol{u}}^m(t+h) - \hat{\boldsymbol{u}}^m(t), \hat{\boldsymbol{u}}^m(t+h) - \hat{\boldsymbol{u}}^m(t)) - \frac{1}{2} a^{V,\varepsilon}(\hat{\boldsymbol{u}}^m(h) - \hat{\boldsymbol{u}}^m(0), \hat{\boldsymbol{u}}^m(h) - \hat{\boldsymbol{u}}^m(0)) \\
&\quad + \frac{1}{2}\int_{\hat{\Omega}^\varepsilon} \hat{\beta}^\varepsilon(\hat{\vartheta}^m(t+h) - \hat{\vartheta}^m(t))^2 d\hat{x}^\varepsilon - \frac{1}{2}\int_{\hat{\Omega}^\varepsilon} \hat{\beta}^\varepsilon(\hat{\vartheta}^m(h) - \hat{\vartheta}^m(0))^2 d\hat{x}^\varepsilon \\
&\quad + \int_0^t a^{S,\varepsilon}(\hat{\vartheta}^m(r+h) - \hat{\vartheta}^m(r), \hat{\vartheta}^m(r+h) - \hat{\vartheta}^m(r)) dr \\
&\quad \leq \int_0^t \int_{\hat{\Omega}^\varepsilon} (\hat{f}^{i,\varepsilon}(r+h) - \hat{f}^{i,\varepsilon}(r))(\dot{\hat{u}}_i^m(r+h) - \dot{\hat{u}}_i^m(r))\, d\hat{x}^\varepsilon dr \\
&\quad + \int_0^t \int_{\hat{\Gamma}_+^\varepsilon} (\hat{h}^{i,\varepsilon}(r+h) - \hat{h}^{i,\varepsilon}(r))(\dot{\hat{u}}_i^m(r+h) - \dot{\hat{u}}_i^m(r))\, d\hat{\Gamma}^\varepsilon dr \\
&\quad + \int_0^t \int_{\hat{\Omega}^\varepsilon} (\hat{q}^\varepsilon(r+h) - \hat{q}^\varepsilon(r))(\hat{\vartheta}^m(r+h) - \hat{\vartheta}^m(r))\, d\hat{x}^\varepsilon dr.
\end{aligned}$$

Now, dividing the equation by h^2 and having in mind (25), we can take limits when $h \to 0^+$ to have

$$\frac{1}{2}\hat{\rho}^\varepsilon \left|\ddot{\boldsymbol{u}}^m(t)\right|_0^2 - \frac{1}{2}\hat{\rho}^\varepsilon \left|\ddot{\boldsymbol{u}}^m(0)\right|_0^2 + \frac{1}{2}a^{V,\varepsilon}(\dot{\boldsymbol{u}}^m(t), \dot{\boldsymbol{u}}^m(t)) - \frac{1}{2}a^{V,\varepsilon}(\dot{\boldsymbol{u}}^m(0), \dot{\boldsymbol{u}}^m(0))$$
$$+ \frac{1}{2}\int_{\hat{\Omega}^\varepsilon} \hat{\beta}^\varepsilon(\dot{\hat{\vartheta}}^m(t))^2 d\hat{x}^\varepsilon - \frac{1}{2}\int_{\hat{\Omega}^\varepsilon} \hat{\beta}^\varepsilon(\dot{\hat{\vartheta}}^m(0))^2 d\hat{x}^\varepsilon + \int_0^t a^{S,\varepsilon}(\dot{\hat{\vartheta}}^m(r), \dot{\hat{\vartheta}}^m(r))dr$$
$$\leq \int_0^t \int_{\hat{\Omega}^\varepsilon} \dot{\hat{f}}^{i,\varepsilon}(r)\ddot{\hat{u}}_i^m(r)\, d\hat{x}^\varepsilon dr + \int_0^t \int_{\hat{\Gamma}_+^\varepsilon} \dot{\hat{h}}^{i,\varepsilon}(r)\ddot{\hat{u}}_i^m(r)\, d\hat{\Gamma}^\varepsilon dr + \int_0^t \int_{\hat{\Omega}^\varepsilon} \dot{\hat{q}}^\varepsilon(r)\dot{\hat{\vartheta}}^m(r)\, d\hat{x}^\varepsilon dr. \tag{30}$$

Integrating by parts the term on $\hat{\Gamma}_+^\varepsilon$ above and applying Young's inequality, we get

$$\hat{\rho}^\varepsilon|\ddot{\boldsymbol{u}}^m(t)|_0^2 - \hat{\rho}^\varepsilon|\ddot{\boldsymbol{u}}^m(0)|_0^2 + ||\dot{\boldsymbol{u}}^m(t)||_V^2 + \hat{\beta}^\varepsilon|\dot{\hat{\vartheta}}^m(t)|_0^2 - \hat{\beta}^\varepsilon|\dot{\hat{\vartheta}}^m(0)|_0^2 + \int_0^t \|\dot{\hat{\vartheta}}^m(r)\|_S^2 dr$$
$$\leq \tilde{C}(\dot{\boldsymbol{f}}^\varepsilon, \ddot{\boldsymbol{h}}^\varepsilon, \dot{\hat{q}}^\varepsilon) + C\int_0^t \left\{|\ddot{\boldsymbol{u}}^m(r)|_0^2 + ||\dot{\boldsymbol{u}}^m(r)||_1^2 + |\dot{\hat{\vartheta}}^m(r)|_0^2\right\} dr. \tag{31}$$

In order to obtain bounds for $|\ddot{\boldsymbol{u}}^m(0)|_0^2$ and $|\dot{\hat{\vartheta}}^m(0)|_0^2$ we first notice that equations (19) and (20) hold for $t = 0$ due to the compatibility required between initial and boundary conditions. Therefore, taking $t = 0$ and $\hat{\boldsymbol{v}}^m = \ddot{\boldsymbol{u}}^m(0) \in V_m$ in (19) and $\hat{\varphi}^m = \dot{\hat{\vartheta}}^m(0) \in S_m$ in (20), taking into account the initial conditions, and using Young's inequality, we obtain

$$\hat{\rho}^\varepsilon|\ddot{\boldsymbol{u}}^m(0)|_0^2 = \int_{\hat{\Omega}^\varepsilon} \hat{f}^{i,\varepsilon}(0)\ddot{\hat{u}}_i^m(0)\, d\hat{x}^\varepsilon + \int_{\hat{\Gamma}_+^\varepsilon} \hat{h}^{i,\varepsilon}(0)\ddot{\hat{u}}_i^m(0)\, d\hat{\Gamma}^\varepsilon \leq \frac{1}{\delta}C + \delta|\ddot{\boldsymbol{u}}^m(0)|_0^2,$$
$$\hat{\beta}^\varepsilon|\dot{\hat{\vartheta}}^m(0)|_0^2 = \int_{\hat{\Omega}^\varepsilon} \hat{q}^\varepsilon(0)\dot{\hat{\vartheta}}^m(0)\, d\hat{x}^\varepsilon \ \leq \frac{1}{\tilde{\delta}}\tilde{C} + \tilde{\delta}|\dot{\hat{\vartheta}}^m(0)|_0^2,$$

where δ, and $\tilde{\delta}$ are sufficiently small positive constants. Next, applying Korn's inequality and Gronwall's lemma in (31) we find

$$|\ddot{\boldsymbol{u}}^m(t)|_0^2 + ||\dot{\boldsymbol{u}}^m(t)||_V^2 + |\dot{\hat{\vartheta}}^m(t)|_0^2 \leq C.$$

Again, all the estimates are independent of m. Then,

$$\left\{\dot{\boldsymbol{u}}^m\right\}_m \text{ is a bounded subset of } L^\infty(0,T,V(\hat{\Omega}^\varepsilon)), \tag{32}$$
$$\left\{\ddot{\boldsymbol{u}}^m\right\}_m \text{ is a bounded subset of } L^\infty(0,T,[L^2(\hat{\Omega}^\varepsilon)]^3), \tag{33}$$
$$\left\{\dot{\hat{\vartheta}}^m\right\}_m \text{ is a bounded subset of } L^\infty(0,T,L^2(\hat{\Omega}^\varepsilon)). \tag{34}$$

Observe that (26)–(29) and (32)–(34) imply that there exists subsequences of $\hat{\boldsymbol{u}}^m$ and $\hat{\vartheta}^m$, also denoted by $\hat{\boldsymbol{u}}^m$ and $\hat{\vartheta}^m$, and there exist elements $\hat{\boldsymbol{u}}^\varepsilon$, $\dot{\hat{\boldsymbol{u}}}^\varepsilon$, $\ddot{\hat{\boldsymbol{u}}}^\varepsilon$, $\hat{\vartheta}^\varepsilon$, $\dot{\hat{\vartheta}}^\varepsilon$ and χ^ε such that

$$\hat{\boldsymbol{u}}^m \xrightarrow[m\to\infty]{*} \hat{\boldsymbol{u}}^\varepsilon \quad \text{in } L^\infty(0,T;V(\hat{\Omega}^\varepsilon)), \tag{35}$$

$$\dot{\hat{\boldsymbol{u}}}^m \xrightarrow[m\to\infty]{*} \dot{\hat{\boldsymbol{u}}}^\varepsilon \quad \text{in } L^\infty(0,T;[L^2(\hat{\Omega}^\varepsilon)]^3) \cap L^\infty(0,T;V(\hat{\Omega}^\varepsilon)), \tag{36}$$

$$\ddot{\hat{\boldsymbol{u}}}^m \xrightarrow[m\to\infty]{*} \ddot{\hat{\boldsymbol{u}}}^\varepsilon \quad \text{in } L^\infty(0,T;[L^2(\hat{\Omega}^\varepsilon)]^3), \tag{37}$$

$$\hat{\vartheta}^m \xrightarrow[m\to\infty]{*} \hat{\vartheta}^\varepsilon \quad \text{in } L^\infty(0,T;L^2(\hat{\Omega}^\varepsilon)) \cap L^\infty(0,T;S(\hat{\Omega}^\varepsilon)), \tag{38}$$

$$\dot{\hat{\vartheta}}^m \xrightarrow[m\to\infty]{*} \dot{\hat{\vartheta}}^\varepsilon \quad \text{in } L^\infty(0,T;L^2(\hat{\Omega}^\varepsilon)), \tag{39}$$

$$(\dot{\hat{u}}_n^m)_+ \xrightarrow[m\to\infty]{} \chi^\varepsilon \quad \text{in } L^2(0,T;L^2(\hat{\Gamma}_C^\varepsilon)). \tag{40}$$

In order to show that $\chi^\varepsilon = (\dot{\hat{u}}_n^\varepsilon)_+$, we first observe that (36) and (37) imply that

$$\left\{\dot{\hat{\boldsymbol{u}}}^m\right\}_m \text{ is a bounded subset of } [H^1(\hat{\Omega}^\varepsilon \times (0,T))]^3.$$

Since the trace map is a compact operator from $H^1(\hat{\Omega}^\varepsilon \times (0,T))$ to $L^2(\hat{\Gamma}^\varepsilon \times (0,T))$, we can affirm that there exists a subsequence of $\dot{\hat{\boldsymbol{u}}}^m$ (still denoted by $\hat{\boldsymbol{u}}^m$) such that

$$\dot{\hat{\boldsymbol{u}}}^m \to \dot{\hat{\boldsymbol{u}}}^\varepsilon, \text{strongly in } [L^2(\hat{\Gamma}_C^\varepsilon \times (0,T))]^3, \qquad \text{and then } \dot{\hat{\boldsymbol{u}}}^m(\boldsymbol{y}) \to \dot{\hat{\boldsymbol{u}}}^\varepsilon(\boldsymbol{y}) \; a.e. \text{ on } \hat{\Gamma}_C^\varepsilon \times (0,T).$$

Then, being the positive part a continuous function it holds that

$$(\dot{\hat{u}}_n^m)_+ \to (\dot{\hat{u}}_n^\varepsilon)_+ \text{ a.e. on } \hat{\Gamma}_C^\varepsilon \times (0,T). \tag{41}$$

On the other hand, (29) implies that

$$(\dot{\hat{u}}_n^m)_+ \text{ is a bounded subset of } L^2(\hat{\Gamma}_C^\varepsilon \times (0,T)). \tag{42}$$

From (41), (42) and [6, Lema 1.3] it follows that

$$(\dot{\hat{u}}_n^m)_+ \rightharpoonup (\dot{\hat{u}}_n^\varepsilon)_+ \text{ in } L^2(\hat{\Gamma}_C^\varepsilon \times (0,T)).$$

Since (40) also implies that $(\dot{\hat{u}}_n^m)_+ \rightharpoonup \chi^\varepsilon$ in $L^2(\hat{\Gamma}_C^\varepsilon \times (0,T))$, the uniqueness of weak limits implies that $\chi^\varepsilon = (\dot{\hat{u}}_n^\varepsilon)_+$ and

$$(\dot{\hat{u}}_n^m)_+ \xrightarrow[m\to\infty]{} (\dot{\hat{u}}_n^\varepsilon)_+ \text{ in } L^2(0,T;L^2(\hat{\Gamma}_C^\varepsilon)). \tag{43}$$

Consider now $\hat{\boldsymbol{v}}^m = \hat{\boldsymbol{w}}_j$ and $\hat{\varphi}^m = \hat{s}_i$ in equations (19) and (20) fixed:

$$\hat{\rho}^\varepsilon \left\langle \ddot{\hat{\boldsymbol{u}}}^m, \hat{\boldsymbol{w}}_j \right\rangle + a^{V,\varepsilon}(\hat{\boldsymbol{u}}^m, \hat{\boldsymbol{w}}_j) - c^\varepsilon(\hat{\vartheta}^m, \hat{\boldsymbol{w}}_j) + \left\langle \hat{P}^\varepsilon(\dot{\hat{\boldsymbol{u}}}^m), \hat{\boldsymbol{w}}_j \right\rangle = \left\langle \hat{J}^\varepsilon(t), \hat{\boldsymbol{w}}_j \right\rangle, \tag{44}$$

$$\left\langle \dot{\hat{\vartheta}}^m, \hat{s}_i \right\rangle + a^{S,\varepsilon}(\hat{\vartheta}^m, \hat{s}_i) + c^\varepsilon(\hat{s}_i, \dot{\hat{\boldsymbol{u}}}^m) = \left\langle \hat{Q}^\varepsilon(t), \hat{s}_i \right\rangle. \tag{45}$$

Observe that (35) and (36) imply that

$$a^{V,\varepsilon}(\hat{\boldsymbol{u}}^m, \hat{\boldsymbol{w}}_j) \to a^{V,\varepsilon}(\hat{\boldsymbol{u}}^\varepsilon, \hat{\boldsymbol{w}}_j) \text{ and } c^\varepsilon(\hat{s}_i, \dot{\hat{\boldsymbol{u}}}^m) \to c^\varepsilon(\hat{s}_i, \dot{\hat{\boldsymbol{u}}}^\varepsilon) \text{ in } L^\infty(0,T).$$

Analogously, from (38) we can state that

$$a^{S,\varepsilon}(\hat{\vartheta}^m, s_i) \to a^{S,\varepsilon}(\hat{\vartheta}^\varepsilon, s_i) \text{ and } c^\varepsilon(\hat{\vartheta}^m, \hat{\boldsymbol{w}}_j) \to c^\varepsilon(\hat{\vartheta}^\varepsilon, \hat{\boldsymbol{w}}_j) \text{ in } L^\infty(0,T).$$

Now, from (37) and (39) we have:

$$\left\langle \ddot{\hat{\boldsymbol{u}}}^m, \hat{\boldsymbol{w}}_j \right\rangle = (\ddot{\hat{\boldsymbol{u}}}^m, \hat{\boldsymbol{w}}_j) \to (\ddot{\hat{\boldsymbol{u}}}^\varepsilon, \hat{\boldsymbol{w}}_j) \text{ and } \left\langle \dot{\vartheta^m}, s_i \right\rangle = (\dot{\vartheta^m}, s_i) \to (\dot{\hat{\vartheta}}^\varepsilon, s_i) \qquad \text{in } L^\infty(0,T).$$

Also, from (43)

$$\left\langle \hat{P}^\varepsilon(\dot{\hat{\boldsymbol{u}}}^m), \hat{\boldsymbol{w}}_j \right\rangle \to \left\langle \hat{P}^\varepsilon(\dot{\hat{\boldsymbol{u}}}^\varepsilon), \hat{\boldsymbol{w}}_j \right\rangle \text{ in } \mathcal{D}'(0,T).$$

Then, we can take $m \to \infty$ in (44)–(45) obtaining that

$$\hat{\rho}^\varepsilon(\ddot{\hat{\boldsymbol{u}}}^\varepsilon, \hat{\boldsymbol{w}}_j) + a^{V,\varepsilon}(\hat{\boldsymbol{u}}^\varepsilon, \hat{\boldsymbol{w}}_j) - c^\varepsilon(\hat{\vartheta}^\varepsilon, \hat{\boldsymbol{w}}_j) + \left\langle \hat{P}^\varepsilon(\dot{\hat{\boldsymbol{u}}}^\varepsilon), \hat{\boldsymbol{w}}_j \right\rangle = \left\langle \hat{J}^\varepsilon(t), \hat{\boldsymbol{w}}_j \right\rangle, \quad \text{in } \mathcal{D}'(0,T), \forall j \geq 1, \tag{46}$$

$$(\dot{\hat{\vartheta}}^\varepsilon, \hat{s}_i) + a^{S,\varepsilon}(\hat{\vartheta}^\varepsilon, \hat{s}_i) + c^\varepsilon(\dot{\hat{\boldsymbol{u}}}^\varepsilon, \hat{s}_i) = \left\langle \hat{Q}^\varepsilon(t), \hat{s}_i \right\rangle, \quad \text{in } L^\infty(0,T), \forall i \geq 1. \tag{47}$$

Next, from (17), (18), (46) and (47) we conclude that (13) holds in $\mathcal{D}'(0,T)$ while (14) holds in $L^\infty(0,T)$. Let us see now that (13) also holds a.e. in $(0,T)$. Indeed, we have that

$$\left\langle \hat{P}^\varepsilon(\dot{\hat{\boldsymbol{u}}}^\varepsilon), \hat{\boldsymbol{w}}_j \right\rangle = -\hat{\rho}^\varepsilon(\ddot{\hat{\boldsymbol{u}}}^\varepsilon, \hat{\boldsymbol{w}}_j) - a^{V,\varepsilon}(\hat{\boldsymbol{u}}^\varepsilon, \hat{\boldsymbol{w}}_j) + c^\varepsilon(\hat{\vartheta}^\varepsilon, \hat{\boldsymbol{w}}_j) + \left\langle \hat{J}^\varepsilon(t), \hat{\boldsymbol{w}}_j \right\rangle, \quad \text{in } \mathcal{D}'(0,T), \forall j \geq 1.$$

We observe that the left-hand side is in $\mathcal{D}'(0,T)$, while the right-hand side terms are in $L^\infty(0,T)$, from which we deduce that $\hat{P}^\varepsilon(\dot{\hat{\boldsymbol{u}}}^\varepsilon) \in L^\infty(0,T;V')$ and (13) and (14) hold a.e. in $(0,T)$. Besides, since the initial conditions (21) are null, it is trivial that, when $m \to \infty$, the limit functions have null initial conditions as well, which completes the proof for the existence and regularity of the solutions. We focus now on proving the uniqueness.

Let us assume that there exist two solutions $\{\hat{\boldsymbol{u}}^{\varepsilon,1}, \hat{\vartheta}^{\varepsilon,1}\}$ and $\{\hat{\boldsymbol{u}}^{\varepsilon,2}, \hat{\vartheta}^{\varepsilon,2}\}$ for Problem 2. Let us define $\boldsymbol{w}^\varepsilon = \hat{\boldsymbol{u}}^{\varepsilon,1} - \hat{\boldsymbol{u}}^{\varepsilon,2}$ and $\phi^\varepsilon = \hat{\vartheta}^{\varepsilon,1} - \hat{\vartheta}^{\varepsilon,2}$. Now, we consider equations (13)–(14) at time t for $\{\hat{\boldsymbol{u}}^{\varepsilon,i}, \hat{\vartheta}^{\varepsilon,i}\}$, take as test function

$\hat{\boldsymbol{v}}^\varepsilon = \dot{\boldsymbol{w}}^\varepsilon(t)$ and $\hat{\varphi}^\varepsilon = \phi^\varepsilon(t)$ for both $i=1$ and $i=2$, and subtract the resulting equations to find:

$$\int_{\hat{\Omega}^\varepsilon} \hat{\rho}^\varepsilon \ddot{\boldsymbol{w}}^\varepsilon(t)\dot{\boldsymbol{w}}^\varepsilon(t) d\hat{x}^\varepsilon + a^{V,\varepsilon}(\boldsymbol{w}^\varepsilon(t), \dot{\boldsymbol{w}}^\varepsilon(t)) - c^\varepsilon(\phi^\varepsilon(t), \dot{\boldsymbol{w}}^\varepsilon(t))$$
$$+ \int_{\hat{\Gamma}_C^\varepsilon} (\hat{p}^\varepsilon(\dot{\hat{u}}_n^{\varepsilon,1}(t)) - \hat{p}^\varepsilon(\dot{\hat{u}}_n^{\varepsilon,2}(t)))\dot{w}_n^\varepsilon(t)\, d\hat{\Gamma}^\varepsilon = 0, \quad a.e. \text{ in } (0,T),$$
$$\int_{\hat{\Omega}^\varepsilon} \hat{\beta}^\varepsilon \dot{\phi}^\varepsilon(t)\phi^\varepsilon(t) d\hat{x}^\varepsilon + a^{S,\varepsilon}(\phi^\varepsilon(t), \phi^\varepsilon(t)) + c^\varepsilon(\phi^\varepsilon(t), \dot{\boldsymbol{w}}^\varepsilon(t)) = 0, \quad a.e. \text{ in } (0,T).$$

Adding these last two equations we have

$$\int_{\hat{\Omega}^\varepsilon} \hat{\rho}^\varepsilon \ddot{\boldsymbol{w}}^\varepsilon(t)\dot{\boldsymbol{w}}^\varepsilon(t) d\hat{x}^\varepsilon + a^{V,\varepsilon}(\boldsymbol{w}^\varepsilon(t), \dot{\boldsymbol{w}}^\varepsilon(t)) + \int_{\hat{\Gamma}_C^\varepsilon} (\hat{p}^\varepsilon(\dot{\hat{u}}_n^{\varepsilon,1}(t)) - \hat{p}^\varepsilon(\dot{\hat{u}}_n^{\varepsilon,2}(t)))(\dot{\hat{u}}_n^{\varepsilon,1}(t) - \dot{\hat{u}}_n^{\varepsilon,2}(t))\, d\hat{\Gamma}^\varepsilon$$
$$+ \int_{\hat{\Omega}^\varepsilon} \hat{\beta}^\varepsilon \dot{\phi}^\varepsilon(t)\phi^\varepsilon(t) d\hat{x}^\varepsilon + a^{S,\varepsilon}(\phi^\varepsilon(t), \phi^\varepsilon(t)) = 0, \quad a.e. \text{ in } (0,T).$$

Taking into account the monotonicity of p^ε,

$$\frac{1}{2}\frac{d}{dt}\left\{\int_{\hat{\Omega}^\varepsilon} \hat{\rho}^\varepsilon \dot{\boldsymbol{w}}^\varepsilon(t)\dot{\boldsymbol{w}}^\varepsilon(t) d\hat{x}^\varepsilon + a^{V,\varepsilon}(\boldsymbol{w}^\varepsilon(t), \boldsymbol{w}^\varepsilon(t)) + \int_{\hat{\Omega}^\varepsilon} \hat{\beta}^\varepsilon \phi^\varepsilon(t)\phi^\varepsilon(t) d\hat{x}^\varepsilon\right\}$$
$$+ a^{S,\varepsilon}(\phi^\varepsilon(t), \phi^\varepsilon(t)) \leq 0, \quad a.e. \text{ in } (0,T).$$

Integrating in $[0,t]$, and taking into account the initial conditions we obtain:

$$\hat{\rho}^\varepsilon |\dot{\boldsymbol{w}}^\varepsilon(t)|_0^2 + ||\boldsymbol{w}^\varepsilon(t)||_V^2 + \hat{\beta}^\varepsilon |\phi^\varepsilon(t)|_0^2 + \int_0^t a^{S,\varepsilon}(\phi^\varepsilon(r), \phi^\varepsilon(r)) dr \leq 0, \quad a.e. \text{ in } (0,T),$$

from where one easily deduce that $\boldsymbol{w}^\varepsilon = \boldsymbol{0}$ and $\phi^\varepsilon = 0$. □

3 The three-dimensional shell contact problem

In this section we consider the particular case when the deformable body is, in fact, a shell. The reader interested in a detailed exposition of the notation can consult [3] and in the context of contact problems in [10].

Let ω be a bounded domain of $\mathbb{R}^2$,

and let $\boldsymbol{\theta} \in \mathcal{C}^2(\bar{\omega}; \mathbb{R}^3)$ be an injective mapping such that the two vectors $\boldsymbol{a}_\alpha(\boldsymbol{y}) := \partial_\alpha \boldsymbol{\theta}(\boldsymbol{y})$ are linearly independent. These vectors form the covariant basis of the tangent plane to the surface $S := \boldsymbol{\theta}(\bar{\omega})$ at the point $\boldsymbol{\theta}(\boldsymbol{y})$. We then define the contravariant basis vectors $\boldsymbol{a}^\alpha(\boldsymbol{y})$, the first fundamental form $a_{\alpha\beta}$, the second fundamental form $b_{\alpha\beta}$ in covariant or mixed components b_α^β and the Christoffel symbols of the surface S as $\Gamma_{\alpha\beta}^\sigma$.

We then define the three-dimensional domain $\Omega^\varepsilon := \omega \times (-\varepsilon, \varepsilon)$ and its boundary $\Gamma^\varepsilon = \partial\Omega^\varepsilon$ with the boundary partitioned into $\Gamma_+^\varepsilon := \omega \times \{\varepsilon\}$, $\Gamma_C^\varepsilon := \omega \times \{-\varepsilon\}$, $\Gamma_0^\varepsilon := \gamma_0 \times [-\varepsilon, \varepsilon]$, where $\gamma_0 \subseteq \gamma$.

Let $\boldsymbol{\Theta} : \bar{\Omega}^\varepsilon \to \mathbb{R}^3$ be the mapping defined by

$$\boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon) := \boldsymbol{\theta}(\boldsymbol{y}) + x_3^\varepsilon \boldsymbol{a}_3(\boldsymbol{y}) \ \forall \boldsymbol{x}^\varepsilon = (\boldsymbol{y}, x_3^\varepsilon) = (y_1, y_2, x_3^\varepsilon) \in \bar{\Omega}^\varepsilon. \tag{48}$$

By identifying $\hat{\Omega}^\varepsilon = \boldsymbol{\Theta}(\Omega^\varepsilon)$, $\hat{\Gamma}^\varepsilon = \boldsymbol{\Theta}(\Gamma^\varepsilon)$, $\hat{\Gamma}_0^\varepsilon = \boldsymbol{\Theta}(\Gamma_0^\varepsilon)$, etc. we cast this setting into the more general three dimensional framework of the preceding section, as a particular case. Then, we pass to define the covariant and contravariant basis of the tangent space $\boldsymbol{g}_i^\varepsilon$ and $\boldsymbol{g}^{i,\varepsilon}$, respectively, and from them we obtain the covariant and contravariant components of the metric tensor $g_{ij}^\varepsilon, g^{ij,\varepsilon}$, and Christoffel symbols $\Gamma_{ij}^{p,\varepsilon}$. The volume element in the set $\boldsymbol{\Theta}(\bar{\Omega}^\varepsilon)$ is $\sqrt{g^\varepsilon}dx^\varepsilon$ and the surface element in $\boldsymbol{\Theta}(\Gamma^\varepsilon)$ is $\sqrt{g^\varepsilon}d\Gamma^\varepsilon$ where $g^\varepsilon := \det(g_{ij}^\varepsilon)$. Let $\boldsymbol{n}^\varepsilon(\boldsymbol{x}^\varepsilon)$ denote the unit outward normal vector on $\boldsymbol{x}^\varepsilon \in \Gamma^\varepsilon$ and $\hat{\boldsymbol{n}}^\varepsilon(\hat{\boldsymbol{x}}^\varepsilon)$ the unit outward normal vector on $\hat{\boldsymbol{x}}^\varepsilon = \boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon) \in \boldsymbol{\Theta}(\Gamma^\varepsilon)$ (see, [2, p. 41] for the relation between the two). In particular, on Γ_C^ε, it is verified that $\hat{v}_n = (\hat{v}_i^\varepsilon \hat{n}^{i,\varepsilon}) = v_i^\varepsilon n^{i,\varepsilon} = -v_3^\varepsilon$.

We now define the corresponding contravariant components in curvilinear coordinates for the applied forces densities:

$$\hat{f}^{i,\varepsilon}(\hat{\boldsymbol{x}}^\varepsilon)\hat{\boldsymbol{e}}_i\, d\hat{x}^\varepsilon =: f^{i,\varepsilon}(\boldsymbol{x}^\varepsilon)\boldsymbol{g}_i^\varepsilon(\boldsymbol{x}^\varepsilon)\sqrt{g^\varepsilon(\boldsymbol{x}^\varepsilon)}\, dx^\varepsilon, \quad \hat{h}^{i,\varepsilon}(\hat{\boldsymbol{x}}^\varepsilon)\hat{\boldsymbol{e}}_i d\hat{\Gamma}^\varepsilon =: h^{i,\varepsilon}(\boldsymbol{x}^\varepsilon)\boldsymbol{g}_i^\varepsilon(\boldsymbol{x}^\varepsilon)\sqrt{g^\varepsilon(\boldsymbol{x}^\varepsilon)}d\Gamma^\varepsilon,$$

and the covariant components in curvilinear coordinates for the displacements field:

$$\hat{\boldsymbol{u}}^\varepsilon(\hat{\boldsymbol{x}}^\varepsilon) = \hat{u}_i^\varepsilon(\hat{\boldsymbol{x}}^\varepsilon)\hat{\boldsymbol{e}}^i =: u_i^\varepsilon(\boldsymbol{x}^\varepsilon)\boldsymbol{g}^{i,\varepsilon}(\boldsymbol{x}^\varepsilon), \text{ with } \hat{\boldsymbol{x}}^\varepsilon = \boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon).$$

Remark 2 Notice that forces and unknowns above depend also on the time variable $t \in [0, T]$, but we decided to keep it implicit for the sake of readiness, since the subject of the change of variable is the spatial component. The same comment applies in a number of situations below.

We also define $\vartheta^\varepsilon(\boldsymbol{x}^\varepsilon) := \hat{\vartheta}^\varepsilon(\hat{\boldsymbol{x}}^\varepsilon)$ and $q^\varepsilon(\boldsymbol{x}^\varepsilon) := \hat{q}^\varepsilon(\hat{\boldsymbol{x}}^\varepsilon)$. Regarding the normal damped response function, we define $p^\varepsilon(r^\varepsilon) := \hat{p}^\varepsilon(r^\varepsilon)$. Let us define the spaces,

$$V(\Omega^\varepsilon) = \{\boldsymbol{v}^\varepsilon = (v_i^\varepsilon) \in [H^1(\Omega^\varepsilon)]^3; \boldsymbol{v}^\varepsilon = \boldsymbol{0} \text{ on } \Gamma_0^\varepsilon\}, \quad S(\Omega^\varepsilon) = \{\varphi^\varepsilon \in H^1(\Omega^\varepsilon); \varphi^\varepsilon = 0 \text{ on } \Gamma_0^\varepsilon\}.$$

Both are real Hilbert spaces with the induced inner product of $[H^1(\Omega^\varepsilon)]^d$, $d \in \{1, 3\}$. The corresponding norm is denoted by $\|\cdot\|_{1,\Omega^\varepsilon}$ in both cases, since no confusion is possible. With these definitions it is straightforward to derive from the Problem 2 the following variational problem (see [3] for the case in linear elasticity and use similar arguments):

Problem 4 Find a pair $t \mapsto (\boldsymbol{u}^\varepsilon(\boldsymbol{x}^\varepsilon, t), \vartheta^\varepsilon(\boldsymbol{x}^\varepsilon, t))$ of $[0,T] \to V(\Omega^\varepsilon) \times S(\Omega^\varepsilon)$ verifying

$$\begin{aligned}
&\int_{\Omega^\varepsilon} \rho^\varepsilon (\ddot{u}^\varepsilon_\alpha g^{\alpha\beta,\varepsilon} v^\varepsilon_\beta + \ddot{u}^\varepsilon_3 v^\varepsilon_3)\sqrt{g^\varepsilon}\, dx^\varepsilon + \int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e^\varepsilon_{k||l}(\boldsymbol{u}^\varepsilon) e^\varepsilon_{i||j}(\boldsymbol{v}^\varepsilon)\sqrt{g^\varepsilon} dx^\varepsilon \\
&\quad - \int_{\Omega^\varepsilon} \alpha^\varepsilon_T (3\lambda^\varepsilon + 2\mu^\varepsilon)\vartheta^\varepsilon (e^\varepsilon_{\alpha||\beta}(\boldsymbol{v}^\varepsilon) g^{\alpha\beta,\varepsilon} + e^\varepsilon_{3||3}(\boldsymbol{v}^\varepsilon))\sqrt{g^\varepsilon} dx^\varepsilon - \int_{\Gamma^\varepsilon_C} p^\varepsilon(-\dot{u}^\varepsilon_3) v^\varepsilon_3 \sqrt{g^\varepsilon} d\Gamma^\varepsilon \\
&\quad = \int_{\Omega^\varepsilon} f^{i,\varepsilon} v^\varepsilon_i \sqrt{g^\varepsilon} dx^\varepsilon + \int_{\Gamma^\varepsilon_+} h^{i,\varepsilon} v^\varepsilon_i \sqrt{g^\varepsilon} d\Gamma^\varepsilon \quad \forall \boldsymbol{v}^\varepsilon \in V(\Omega^\varepsilon),\ a.e. \text{ in } (0,T), \\
&\int_{\Omega^\varepsilon} \beta^\varepsilon \dot{\vartheta}^\varepsilon \varphi^\varepsilon \sqrt{g^\varepsilon} dx^\varepsilon + \int_{\Omega^\varepsilon} k^\varepsilon (\partial^\varepsilon_\alpha \vartheta^\varepsilon g^{\alpha\beta,\varepsilon} \partial^\varepsilon_\beta \varphi^\varepsilon + \partial^\varepsilon_3 \vartheta^\varepsilon \partial^\varepsilon_3 \varphi^\varepsilon)\sqrt{g^\varepsilon} dx^\varepsilon \\
&\quad + \int_{\Omega^\varepsilon} \alpha^\varepsilon_T (3\lambda^\varepsilon + 2\mu^\varepsilon)\varphi^\varepsilon (e^\varepsilon_{\alpha||\beta}(\dot{\boldsymbol{u}}^\varepsilon) g^{\alpha\beta,\varepsilon} + e^\varepsilon_{3||3}(\dot{\boldsymbol{u}}^\varepsilon))\sqrt{g^\varepsilon} dx^\varepsilon \\
&\quad = \int_{\Omega^\varepsilon} q^\varepsilon \varphi^\varepsilon \sqrt{g^\varepsilon} dx^\varepsilon \quad \forall \varphi^\varepsilon \in S(\Omega^\varepsilon),\ a.e. \text{ in } (0,T),
\end{aligned}$$

with $\dot{\boldsymbol{u}}^\varepsilon(\cdot,0) = \boldsymbol{u}^\varepsilon(\cdot,0) = \boldsymbol{0}$ and $\vartheta^\varepsilon(\cdot,0) = 0$.

Above, $A^{ijkl,\varepsilon} = A^{jikl,\varepsilon} = A^{klij,\varepsilon} \in \mathcal{C}^1(\bar{\Omega}^\varepsilon)$, defined by

$$A^{ijkl,\varepsilon} := \lambda g^{ij,\varepsilon} g^{kl,\varepsilon} + \mu(g^{ik,\varepsilon} g^{jl,\varepsilon} + g^{il,\varepsilon} g^{jk,\varepsilon}), \tag{49}$$

represent the contravariant components of the three-dimensional elasticity tensor, and the functions $e^\varepsilon_{i||j}(\boldsymbol{v}^\varepsilon) = e^\varepsilon_{j||i}(\boldsymbol{v}^\varepsilon) \in L^2(\Omega^\varepsilon)$ that represent the covariant components of the linearized change of metric tensor, or strain tensor, are defined by

$$e^\varepsilon_{i||j}(\boldsymbol{v}^\varepsilon) := \frac{1}{2}(\partial^\varepsilon_j v^\varepsilon_i + \partial^\varepsilon_i v^\varepsilon_j) - \Gamma^{p,\varepsilon}_{ij} v^\varepsilon_p,$$

for all $\boldsymbol{v}^\varepsilon \in [H^1(\Omega^\varepsilon)]^3$, where ∂^ε_i denotes partial derivative with respect to x^ε_i. Note that the following simplifications are verified,

$$\Gamma^{3,\varepsilon}_{\alpha 3} = \Gamma^{p,\varepsilon}_{33} = 0 \text{ in } \bar{\Omega}^\varepsilon, \quad A^{\alpha\beta\sigma 3,\varepsilon} = A^{\alpha 333,\varepsilon} = 0 \text{ in } \bar{\Omega}^\varepsilon, \tag{50}$$

as a consequence of the definition of $\boldsymbol{\Theta}$ in (48). The definitions of the fourth order tensor (49) imply that (see [3, Theorem 1.8-1]) for $\varepsilon > 0$ small enough, there exists a constant $C_e > 0$, independent of ε, such that,

$$\sum_{i,j} |t_{ij}|^2 \le C_e A^{ijkl,\varepsilon}(\boldsymbol{x}^\varepsilon) t_{kl} t_{ij}, \tag{51}$$

for all $\boldsymbol{x}^\varepsilon \in \bar{\Omega}^\varepsilon$ and all $\boldsymbol{t} = (t_{ij}) \in \mathbb{S}^3$ (vector space of 3×3 real symmetric matrices).

Remark 3 We recall that the vector field $\boldsymbol{u}^\varepsilon = (u^\varepsilon_i) : \Omega^\varepsilon \times [0,T] \to \mathbb{R}^3$ solution of Problem 4 has to be interpreted conveniently. The functions $u^\varepsilon_i : \bar{\Omega}^\varepsilon \times [0,T] \to \mathbb{R}^3$ are the covariant, time dependent, components of the "true" displacements field $\boldsymbol{\mathcal{U}}^\varepsilon := u^\varepsilon_i \boldsymbol{g}^{i,\varepsilon} : \bar{\Omega}^\varepsilon \times [0,T] \to \mathbb{R}^3$.

For convenience, we consider a reference domain independent of the small parameter ε. Hence, let us define the three-dimensional domain $\Omega := \omega \times (-1,1)$ and its boundary $\Gamma = \partial\Omega$. We also define the following parts of the boundary,

$$\Gamma_+ := \omega \times \{1\}, \quad \Gamma_C := \omega \times \{-1\}, \quad \Gamma_0 := \gamma_0 \times [-1,1].$$

Let $\boldsymbol{x} = (x_1, x_2, x_3)$ be a generic point in $\bar{\Omega}$ and we consider the notation ∂_i for the partial derivative with respect to x_i. We define the projection map $\pi^\varepsilon : \bar{\Omega} \to \bar{\Omega}^\varepsilon$, such that

$$\pi^\varepsilon(\boldsymbol{x}) = \boldsymbol{x}^\varepsilon = (x_i^\varepsilon) = (x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon) = (x_1, x_2, \varepsilon x_3) \in \bar{\Omega}^\varepsilon,$$

hence, $\partial_\alpha^\varepsilon = \partial_\alpha$ and $\partial_3^\varepsilon = \frac{1}{\varepsilon}\partial_3$. We consider the displacements related scaled unknown $\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon)) : \bar{\Omega} \times [0,T] \to \mathbb{R}^3$ and the scaled vector fields $\boldsymbol{v} = (v_i) : \bar{\Omega} \to \mathbb{R}^3$ defined as

$$u_i^\varepsilon(\boldsymbol{x}^\varepsilon) =: u_i(\varepsilon)(\boldsymbol{x}) \text{ and } v_i^\varepsilon(\boldsymbol{x}^\varepsilon) =: v_i(\boldsymbol{x}) \ \forall \boldsymbol{x} \in \bar{\Omega}, \ \boldsymbol{x}^\varepsilon = \pi^\varepsilon(\boldsymbol{x}) \in \bar{\Omega}^\varepsilon.$$

Besides, we define the scaled temperature $\vartheta(\varepsilon) : \bar{\Omega} \times [0,T] \to \mathbb{R}$ defined as

$$\vartheta(\varepsilon)(\boldsymbol{x}) := \vartheta^\varepsilon(\boldsymbol{x}^\varepsilon) \quad \forall \boldsymbol{x} \in \Omega, \text{ where } \boldsymbol{x}^\varepsilon = \pi^\varepsilon(\boldsymbol{x}) \in \Omega^\varepsilon.$$

For the sake of simplicity, from now on, we are going to assume that the different parameters of the problem (thermal conductivity, thermal dilatation, specific heat coefficient, mass density, Lamé coefficients) are all independent of ε. Also, let the functions, $\Gamma_{ij}^{p,\varepsilon}, g^\varepsilon, A^{ijkl,\varepsilon}$ be associated with the functions $\Gamma_{ij}^p(\varepsilon)$, $g(\varepsilon)$, $A^{ijkl}(\varepsilon)$, defined by

$$\Gamma_{ij}^p(\varepsilon)(\boldsymbol{x}) := \Gamma_{ij}^{p,\varepsilon}(\boldsymbol{x}^\varepsilon), \ g(\varepsilon)(\boldsymbol{x}) := g^\varepsilon(\boldsymbol{x}^\varepsilon), \ A^{ijkl}(\varepsilon)(\boldsymbol{x}) := A^{ijkl,\varepsilon}(\boldsymbol{x}^\varepsilon),$$

for all $\boldsymbol{x} \in \bar{\Omega}$, $\boldsymbol{x}^\varepsilon = \pi^\varepsilon(\boldsymbol{x}) \in \bar{\Omega}^\varepsilon$. For all $\boldsymbol{v} = (v_i) \in [H^1(\Omega)]^3$, let there be associated the scaled linearized strains $(e_{i||j}(\varepsilon)(\boldsymbol{v})) \in [L^2(\Omega)]^{3\times 3}_{sym}$, which we also denote as $(e_{i||j}(\varepsilon; \boldsymbol{v}))$, defined by

$$e_{\alpha||\beta}(\varepsilon; \boldsymbol{v}) := \frac{1}{2}(\partial_\beta v_\alpha + \partial_\alpha v_\beta) - \Gamma_{\alpha\beta}^p(\varepsilon) v_p, \tag{52}$$

$$e_{\alpha||3}(\varepsilon; \boldsymbol{v}) := \frac{1}{2}(\frac{1}{\varepsilon}\partial_3 v_\alpha + \partial_\alpha v_3) - \Gamma_{\alpha 3}^p(\varepsilon) v_p, \tag{53}$$

$$e_{3||3}(\varepsilon; \boldsymbol{v}) := \frac{1}{\varepsilon}\partial_3 v_3. \tag{54}$$

Note that with these definitions it is verified that

$$e_{i||j}^\varepsilon(\boldsymbol{v}^\varepsilon)(\pi^\varepsilon(\boldsymbol{x})) = e_{i||j}(\varepsilon; \boldsymbol{v})(\boldsymbol{x}) \ \forall \boldsymbol{x} \in \Omega.$$

Remark 4 The functions $\Gamma_{ij}^p(\varepsilon), g(\varepsilon), A^{ijkl}(\varepsilon)$ converge in $\mathcal{C}^0(\bar{\Omega})$ when ε tends to zero.

Remark 5 When we consider $\varepsilon = 0$ the functions will be defined with respect to $\boldsymbol{y} \in \bar{\omega}$. Notice the singularities in (53) and (54) for that case. We shall distinguish the three-dimensional Christoffel symbols from the two-dimensional ones associated to S by using $\Gamma^{\sigma}_{\alpha\beta}(\varepsilon)$ and $\Gamma^{\sigma}_{\alpha\beta}$, respectively.

In [3, Theorem 3.3-2] we find an important result which shows that under suitable regularity conditions, take for example $\boldsymbol{\theta} \in \mathcal{C}^2(\bar{\omega};\mathbb{R}^3)$, there exists an $\varepsilon_0 > 0$ such that $A^{ijkl}(\varepsilon)$ is positive-definite, uniformly with respect to $\boldsymbol{x} \in \bar{\Omega}$ and ε, provided that $0 < \varepsilon \leq \varepsilon_0$. Further, the asymptotic behavior of $A^{ijkl}(\varepsilon)$ is detailed. Indeed, it is satisfied that

$$A^{ijkl}(\varepsilon) = A^{ijkl}(0) + \mathcal{O}(\varepsilon) \text{ and } A^{\alpha\beta\sigma 3}(\varepsilon) = A^{\alpha 333}(\varepsilon) = 0,$$

for all ε, $0 < \varepsilon \leq \varepsilon_0$, and

$$A^{\alpha\beta\sigma\tau}(0) = \lambda a^{\alpha\beta}a^{\sigma\tau} + \mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}), \quad A^{\alpha\beta 33}(0) = \lambda a^{\alpha\beta}, \tag{55}$$

$$A^{\alpha 3\sigma 3}(0) = \mu a^{\alpha\sigma}, \quad A^{3333}(0) = \lambda + 2\mu, \quad A^{\alpha\beta\sigma 3}(0) = A^{\alpha 333}(0) = 0. \tag{56}$$

Moreover, and related with (51), there exists a constant $C_e > 0$, independent of the variables and ε, such that

$$\sum_{i,j} |t_{ij}|^2 \leq C_e A^{ijkl}(\varepsilon)(\boldsymbol{x}) t_{kl} t_{ij}, \tag{57}$$

for all ε, $0 < \varepsilon \leq \varepsilon_0$, for all $\boldsymbol{x} \in \bar{\Omega}$ and all $\boldsymbol{t} = (t_{ij}) \in \mathbb{S}^3$.

Notice that the limits are functions of $\boldsymbol{y} \in \bar{\omega}$ only, that is, independent of the transversal variable x_3. We also recall [3, Theorem 3.3-1], which provides the asymptotic behavior of Christoffel's symbols $\Gamma^p_{ij}(\varepsilon)$, $g^{ij}(\varepsilon)$ and $g(\varepsilon)$. Indeed, if $\boldsymbol{\theta} \in \mathcal{C}^3(\bar{\omega};\mathbb{R}^3)$, then

$$\Gamma^{\sigma}_{\alpha\beta}(\varepsilon) = \Gamma^{\sigma}_{\alpha\beta} - \varepsilon x_3 b^{\sigma}_{\beta}|_{\alpha} + \mathcal{O}(\varepsilon^2), \quad \partial_3 \Gamma^p_{\alpha\beta}(\varepsilon) = \mathcal{O}(\varepsilon), \quad \Gamma^3_{\alpha 3}(\varepsilon) = \Gamma^p_{33}(\varepsilon) = 0, \tag{58}$$

$$\Gamma^3_{\alpha\beta}(\varepsilon) = b_{\alpha\beta} - \varepsilon x_3 b^{\sigma}_{\alpha} b_{\sigma\beta}, \quad \Gamma^{\sigma}_{\alpha 3}(\varepsilon) = -b^{\sigma}_{\alpha} - \varepsilon x_3 b^{\tau}_{\alpha} b^{\sigma}_{\tau} + \mathcal{O}(\varepsilon^2), \tag{59}$$

$$g^{\alpha\beta}(\varepsilon) = a^{\alpha\beta} + 2\varepsilon x_3 a^{\alpha\sigma} b^{\beta}_{\sigma} + \mathcal{O}(\varepsilon^2), \quad g^{i3}(\varepsilon) = \delta^{i3}, \quad g(\varepsilon) = a + \mathcal{O}(\varepsilon), \tag{60}$$

for all ε, $0 < \varepsilon \leq \varepsilon_0$, where the order symbols $\mathcal{O}(\varepsilon)$ and $\mathcal{O}(\varepsilon^2)$ are meant with respect to the norm $\|\cdot\|_{0,\infty,\bar{\Omega}}$ defined by

$$\|w\|_{0,\infty,\bar{\Omega}} = \sup\{|w(\boldsymbol{x})|; \boldsymbol{x} \in \bar{\Omega}\},$$

and the covariant derivatives $b^{\sigma}_{\beta}|_{\alpha}$ are defined by

$$b^{\sigma}_{\beta}|_{\alpha} := \partial_{\alpha} b^{\sigma}_{\beta} + \Gamma^{\sigma}_{\alpha\tau} b^{\tau}_{\beta} - \Gamma^{\tau}_{\alpha\beta} b^{\sigma}_{\tau}.$$

The functions $b_{\alpha\beta}, b^{\sigma}_{\alpha}, \Gamma^{\sigma}_{\alpha\beta}, b^{\sigma}_{\beta}|_{\alpha}$ and a are identified with functions in $\mathcal{C}^0(\bar{\Omega})$. Further, there exist constants a_0, g_0 and g_1 such that

$$\begin{aligned} &0 < a_0 \leq a(\boldsymbol{y}) \ \forall \boldsymbol{y} \in \bar{\omega}, \\ &0 < g_0 \leq g(\varepsilon)(\boldsymbol{x}) \leq g_1 \ \forall \boldsymbol{x} \in \bar{\Omega} \text{ and } \forall\, \varepsilon, 0 < \varepsilon \leq \varepsilon_0. \end{aligned} \tag{61}$$

Let the scaled heat source $q(\varepsilon) : \Omega \times (0,T) \to \mathbb{R}$ and scaled applied forces $\boldsymbol{f}(\varepsilon) : \Omega \times (0,T) \to \mathbb{R}^3$ and $\boldsymbol{h}(\varepsilon) : \Gamma_+ \times (0,T) \to \mathbb{R}^3$ be defined by

$$q^\varepsilon(\boldsymbol{x}^\varepsilon) =: q(\varepsilon)(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Omega, \text{ where } \boldsymbol{x}^\varepsilon = \pi^\varepsilon(\boldsymbol{x}) \in \Omega^\varepsilon,$$
$$\boldsymbol{f}^\varepsilon = (f^{i,\varepsilon})(\boldsymbol{x}^\varepsilon) =: \boldsymbol{f}(\varepsilon) = (f^i(\varepsilon))(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Omega, \text{ where } \boldsymbol{x}^\varepsilon = \pi^\varepsilon(\boldsymbol{x}) \in \Omega^\varepsilon,$$
$$\boldsymbol{h}^\varepsilon = (h^{i,\varepsilon})(\boldsymbol{x}^\varepsilon) =: \boldsymbol{h}(\varepsilon) = (h^i(\varepsilon))(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Gamma_+, \text{ where } \boldsymbol{x}^\varepsilon = \pi^\varepsilon(\boldsymbol{x}) \in \Gamma_+^\varepsilon.$$

Regarding the normal damped response function, we define $p(\varepsilon)(r(\varepsilon)) := p^\varepsilon(r^\varepsilon)$. Also, we define the spaces

$$V(\Omega) = \{\boldsymbol{v} = (v_i) \in [H^1(\Omega)]^3; \boldsymbol{v} = \boldsymbol{0} \textit{ on } \Gamma_0\}, \quad S(\Omega) = \{\varphi \in H^1(\Omega); \varphi = 0 \textit{ on } \Gamma_0\},$$

which are Hilbert spaces, with associated norms denoted by $\|\cdot\|_{1,\Omega}$. The scaled variational problem can then be written as follows:

Problem 5 Find a pair $t \mapsto (\boldsymbol{u}(\varepsilon)(\boldsymbol{x},t), \vartheta(\varepsilon)(\boldsymbol{x},t))$ of $[0,T] \to V(\Omega) \times S(\Omega)$ verifying

$$\begin{aligned}
&\int_\Omega \rho(\ddot{u}_\alpha(\varepsilon) g^{\alpha\beta}(\varepsilon) v_\beta + \ddot{u}_3(\varepsilon) v_3)\sqrt{g(\varepsilon)}\, dx + \int_\Omega A^{ijkl}(\varepsilon) e_{k||l}(\varepsilon;\boldsymbol{u}(\varepsilon)) e_{i||j}(\varepsilon;\boldsymbol{v})\sqrt{g(\varepsilon)} dx \\
&\quad - \int_\Omega \alpha_T(3\lambda + 2\mu)\vartheta(\varepsilon)(e_{\alpha||\beta}(\varepsilon;\boldsymbol{v}) g^{\alpha\beta}(\varepsilon) + e_{3||3}(\varepsilon;\boldsymbol{v}))\sqrt{g(\varepsilon)} dx - \frac{1}{\varepsilon}\int_{\Gamma_C} p(\varepsilon)(-\dot{u}_3(\varepsilon)) v_3 \sqrt{g(\varepsilon)} d\Gamma \\
&\quad = \int_\Omega f^i(\varepsilon) v_i \sqrt{g(\varepsilon)} dx + \frac{1}{\varepsilon}\int_{\Gamma_+} h^i(\varepsilon) v_i \sqrt{g(\varepsilon)} d\Gamma \ \forall \boldsymbol{v} \in V(\Omega), \ \textit{a.e.} \text{ in } (0,T),
\end{aligned} \tag{62}$$

$$\begin{aligned}
&\int_\Omega \beta\dot{\vartheta}(\varepsilon)\varphi\sqrt{g(\varepsilon)} dx + \int_\Omega k(\partial_\alpha\vartheta(\varepsilon) g^{\alpha\beta}(\varepsilon)\partial_\beta\varphi + \frac{1}{\varepsilon^2}\partial_3\vartheta(\varepsilon)\partial_3\varphi)\sqrt{g(\varepsilon)} dx \\
&\quad + \int_\Omega \alpha_T(3\lambda + 2\mu)\varphi(e_{\alpha||\beta}(\varepsilon;\dot{\boldsymbol{u}}(\varepsilon)) g^{\alpha\beta}(\varepsilon) + e_{3||3}(\varepsilon;\dot{\boldsymbol{u}}(\varepsilon)))\sqrt{g(\varepsilon)} dx \\
&\quad = \int_\Omega q(\varepsilon)\varphi\sqrt{g(\varepsilon)} dx \quad \forall \varphi \in S(\Omega), \ \textit{a.e.} \text{ in } (0,T),
\end{aligned} \tag{63}$$

with $\dot{\boldsymbol{u}}(\varepsilon)(\cdot,0) = \boldsymbol{u}(\varepsilon)(\cdot,0) = \boldsymbol{0}$ and $\vartheta(\varepsilon)(\cdot,0) = 0$.

Remark 6 Notice that the time-dependent version of the linearized strain tensor above is well posed when we define

$$e_{i||j}(\varepsilon;\boldsymbol{u}(\varepsilon))(t) := e_{i||j}(\varepsilon;\boldsymbol{u}(\varepsilon)(t)).$$

See for example [9]. Further, as commented earlier, we usually omit the explicit time dependence for the sake of a shorter notation.

Remark 7 The unique solvability of Problem 5 for $\varepsilon > 0$ small enough is similar to Problem 4 and the regularity obtained for the solutions is analogue. In particular, we find $\dot{\boldsymbol{u}}(\varepsilon)(\cdot,t) \in V(\Omega)$ and $\dot{\vartheta}(\varepsilon)(\cdot,t) \in S(\Omega)$ *a.e.* in $(0,T)$.

We now present some additional results which will be used in the next section. In [3, Theorem 3.4-1], we find the following useful result:

Theorem 2 *Let ω be a domain in $\mathbb{R}^2$ with boundary γ, let $\Omega = \omega \times (-1, 1)$, and let $g \in L^p(\Omega)$, $p > 1$, be a function such that*

$$\int_\Omega g \partial_3 v dx = 0, \text{ for all } v \in \mathcal{C}^\infty(\bar{\Omega}) \text{ with } v = 0 \text{ on } \gamma \times [-1, 1].$$

Then $g = 0$ a.e in Ω.

We provide here, as a standalone theorem, a result which can be found inside the proof of [3, Theorem 4.4-1].

Theorem 3 *Let $X(\Omega) := \{v \in L^2(\Omega);\ \partial_3 v \in L^2(\Omega)\}$ ($\partial_3 v$ being a derivative in the sense of distributions). Then, the trace $v(\cdot, z)$ of any function $v \in X(\Omega)$ is well defined as a function in $L^2(\omega)$ for all $z \in [-1, 1]$ and the trace operator defined in this fashion is continuous. In particular, there exists a constant $c_1 > 0$ such that*

$$\|v\|_{L^2(\Gamma_+ \cup \Gamma_C)} \leq c_1 \left(|v|^2_{0,\Omega} + |\partial_3 v|^2_{0,\Omega} \right)^{1/2}$$

for all $\boldsymbol{v} \in X(\Omega)$. As consequence there exists a constant $c_2 > 0$ such that

$$\|v_3\|_{L^2(\Gamma_+ \cup \Gamma_C)} \leq c_2 \left(\sum_{i,j} |e_{i||j}(\varepsilon; \boldsymbol{v})|^2_{0,\Omega} \right)^{1/2} \quad \forall \boldsymbol{v} \in V(\Omega). \tag{64}$$

4 Formal asymptotic analysis

In this section we briefly describe the formal procedure to identify possible two-dimensional limit problems, depending on the geometry of the middle surface, the set where the boundary conditions are given, the order of the applied forces and, of paramount interest in this paper, the order of the normal damped response function (the general procedure is described in detail in [3] and was used for shells in unilateral contact in [10] and normal compliance contact in [11]). We consider scaled applied forces and heat source of the form

$$\boldsymbol{f}(\varepsilon)(\boldsymbol{x}) = \varepsilon^m \boldsymbol{f}^m(\boldsymbol{x}),\ q(\varepsilon)(\boldsymbol{x}) = \varepsilon^m q^m(\boldsymbol{x})\, \forall \boldsymbol{x} \in \Omega, \quad \boldsymbol{h}(\varepsilon)(\boldsymbol{x}) = \varepsilon^{m+1} \boldsymbol{h}^{m+1}(\boldsymbol{x})\, \forall \boldsymbol{x} \in \Gamma_+,$$

where m is an integer number that will show the order of the volume, heat source and surface forces, respectively. We also define the scaled normal damped response function $p(\varepsilon)(r(\varepsilon)) = \varepsilon^{m+1} p^{m+1}(r(\varepsilon))$. We substitute in (62) to obtain the following problem:

Problem 6 Find a pair $t \mapsto (\boldsymbol{u}(\varepsilon)(\boldsymbol{x},t), \vartheta(\varepsilon)(\boldsymbol{x},t))$ of $[0,T] \to V(\Omega) \times S(\Omega)$ verifying

$$\begin{aligned}
&\int_\Omega \rho(\ddot{u}_\alpha(\varepsilon) g^{\alpha\beta}(\varepsilon) v_\beta + \ddot{u}_3(\varepsilon) v_3)\sqrt{g(\varepsilon)}\, dx + \int_\Omega A^{ijkl}(\varepsilon) e_{k||l}(\varepsilon;\boldsymbol{u}(\varepsilon)) e_{i||j}(\varepsilon;\boldsymbol{v})\sqrt{g(\varepsilon)} dx \\
&\quad - \int_\Omega \alpha_T(3\lambda+2\mu)\vartheta(\varepsilon)(e_{\alpha||\beta}(\varepsilon;\boldsymbol{v}) g^{\alpha\beta}(\varepsilon) + e_{3||3}(\varepsilon;\boldsymbol{v}))\sqrt{g(\varepsilon)} dx - \int_{\Gamma_C} \varepsilon^m p^{m+1}(-\dot{u}_3(\varepsilon)) v_3 \sqrt{g(\varepsilon)} d\Gamma \\
&\quad = \int_\Omega \varepsilon^m f^{i,m} v_i \sqrt{g(\varepsilon)} dx + \int_{\Gamma_+} \varepsilon^m h^{i,m+1} v_i \sqrt{g(\varepsilon)} d\Gamma \;\; \forall \boldsymbol{v} \in V(\Omega),\;\; a.e. \text{ in } (0,T),
\end{aligned} \tag{65}$$

$$\begin{aligned}
&\int_\Omega \beta \dot{\vartheta}(\varepsilon)\varphi \sqrt{g(\varepsilon)} dx + \int_\Omega k(\partial_\alpha \vartheta(\varepsilon) g^{\alpha\beta}(\varepsilon)\partial_\beta \varphi + \frac{1}{\varepsilon^2}\partial_3\vartheta(\varepsilon)\partial_3\varphi)\sqrt{g(\varepsilon)} dx \\
&\quad + \int_\Omega \alpha_T(3\lambda+2\mu)\varphi(e_{\alpha||\beta}(\varepsilon;\dot{\boldsymbol{u}}(\varepsilon)) g^{\alpha\beta}(\varepsilon) + e_{3||3}(\varepsilon;\dot{\boldsymbol{u}}(\varepsilon)))\sqrt{g(\varepsilon)} dx \\
&\quad = \int_\Omega \varepsilon^m q^m \varphi \sqrt{g(\varepsilon)} dx \quad \forall \varphi \in S(\Omega),\;\; a.e. \text{ in } (0,T),
\end{aligned} \tag{66}$$

with $\dot{\boldsymbol{u}}(\varepsilon)(\cdot,0) = \boldsymbol{u}(\varepsilon)(\cdot,0) = \boldsymbol{0}$ and $\vartheta(\varepsilon)(\cdot,0) = 0$.

Assume that $\boldsymbol{\theta} \in \mathcal{C}^3(\bar{\omega};\mathbb{R}^3)$ and that the scaled unknowns $\boldsymbol{u}(\varepsilon)$, $\vartheta(\varepsilon)$ admit asymptotic expansions of the form

$$\begin{aligned}
\boldsymbol{u}(\varepsilon) &= \boldsymbol{u}^0 + \varepsilon \boldsymbol{u}^1 + \varepsilon^2 \boldsymbol{u}^2 + \ldots, \\
\vartheta(\varepsilon) &= \vartheta^0 + \varepsilon\vartheta^1 + \varepsilon^2\vartheta^2 + \ldots
\end{aligned} \tag{67}$$

where $\boldsymbol{u}^0 \in V(\Omega)$, $\boldsymbol{u}^j \in [H^1(\Omega)]^3$, $\vartheta^0 \in S(\Omega)$, $\vartheta^j \in H^1(\Omega)$, $j \geq 1$. The assumption (67) implies an asymptotic expansion of the scaled linear strain as follows

$$e_{i||j}(\varepsilon) \equiv e_{i||j}(\varepsilon;\boldsymbol{u}(\varepsilon)) = \frac{1}{\varepsilon} e^{-1}_{i||j} + e^0_{i||j} + \varepsilon e^1_{i||j} + \varepsilon^2 e^2_{i||j} + \varepsilon^3 e^3_{i||j} + \ldots$$

where,

$$\begin{cases}
e^{-1}_{\alpha||\beta} = 0, \\
e^{-1}_{\alpha||3} = \frac{1}{2}\partial_3 u^0_\alpha, \\
e^{-1}_{3||3} = \partial_3 u^0_3,
\end{cases}
\qquad
\begin{cases}
e^0_{\alpha||\beta} = \frac{1}{2}(\partial_\beta u^0_\alpha + \partial_\alpha u^0_\beta) - \Gamma^\sigma_{\alpha\beta} u^0_\sigma - b_{\alpha\beta} u^0_3, \\
e^0_{\alpha||3} = \frac{1}{2}(\partial_3 u^1_\alpha + \partial_\alpha u^0_3) + b^\sigma_\alpha u^0_\sigma, \\
e^0_{3||3} = \partial_3 u^1_3,
\end{cases}$$

$$\begin{cases}
e^1_{\alpha||\beta} = \frac{1}{2}(\partial_\beta u^1_\alpha + \partial_\alpha u^1_\beta) - \Gamma^\sigma_{\alpha\beta} u^1_\sigma - b_{\alpha\beta} u^1_3 + x_3(b^\sigma_{\beta|\alpha} u^0_\sigma + b^\sigma_\alpha b_{\sigma\beta} u^0_3), \\
e^1_{\alpha||3} = \frac{1}{2}(\partial_3 u^2_\alpha + \partial_\alpha u^1_3) + b^\sigma_\alpha u^1_\sigma + x_3 b^\tau_\alpha b^\sigma_\tau u^0_\sigma, \\
e^1_{3||3} = \partial_3 u^2_3.
\end{cases}$$

Besides, the functions $e_{i||j}(\varepsilon;\boldsymbol{v})$ admit the following expansion,

$$e_{i||j}(\varepsilon;\boldsymbol{v}) = \frac{1}{\varepsilon}e^{-1}_{i||j}(\boldsymbol{v}) + e^{0}_{i||j}(\boldsymbol{v}) + \varepsilon e^{1}_{i||j}(\boldsymbol{v}) + ...$$

where,

$$\begin{cases} e^{-1}_{\alpha||\beta}(\boldsymbol{v}) = 0, \\ e^{-1}_{\alpha||3}(\boldsymbol{v}) = \frac{1}{2}\partial_3 v_\alpha, \\ e^{-1}_{3||3}(\boldsymbol{v}) = \partial_3 v_3, \end{cases} \qquad \begin{cases} e^{0}_{\alpha||\beta}(\boldsymbol{v}) = \frac{1}{2}(\partial_\beta v_\alpha + \partial_\alpha v_\beta) - \Gamma^\sigma_{\alpha\beta}v_\sigma - b_{\alpha\beta}v_3, \\ e^{0}_{\alpha||3}(\boldsymbol{v}) = \frac{1}{2}\partial_\alpha v_3 + b^\sigma_\alpha v_\sigma, \\ e^{0}_{3||3}(\boldsymbol{v}) = 0, \end{cases}$$

$$\begin{cases} e^{1}_{\alpha||\beta}(\boldsymbol{v}) = x_3 b^\sigma_{\beta|\alpha}v_\sigma + x_3 b^\sigma_\alpha b_{\sigma\beta}v_3, \\ e^{1}_{\alpha||3}(\boldsymbol{v}) = x_3 b^\tau_\alpha b^\sigma_\tau v_\sigma, \\ e^{1}_{3||3}(\boldsymbol{v}) = 0. \end{cases}$$

Upon substitution on (65) and (66), we proceed to characterize the terms involved in the asymptotic expansions by considering different values for m and grouping terms of the same order. In this way, taking in (65) the order $m=-2$ and particular cases of test functions, we reason that $\boldsymbol{f}^{-2} = \boldsymbol{h}^{-1} = \boldsymbol{0}$ and $p^{-1}=0$, which leads to $\partial_3\boldsymbol{u}^0 = 0$. From (66), we reason that $q^{-2}=0$ and find that $\partial_3\vartheta^0 = 0$. Thus the zeroth order terms of both unknowns would be independent of the transversal variable x_3. Particularly, $\boldsymbol{u}^0$ can be identified with a function $\boldsymbol{\xi}^0 \in V(\omega)$, and ϑ^0 can be identified with a function $\zeta^0 \in S(\omega)$ where

$$V(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in [H^1(\omega)]^3; \eta_i = 0 \text{ on } \gamma_0\}, \quad S(\omega) := \{\varphi \in H^1(\omega); \varphi = 0 \text{ on } \gamma_0\}.$$

Taking $m=-1$, and using particular cases of test functions, we reason that $\boldsymbol{f}^{-1} = \boldsymbol{h}^0 = \boldsymbol{0}$ and $p^0 = 0$ and we find that

$$e^0_{\alpha||3} = 0, \quad \lambda a^{\alpha\beta}e^0_{\alpha||\beta} + (\lambda+2\mu)e^0_{3||3} = \alpha_T(3\lambda+2\mu)\vartheta^0, \quad e^0_{\alpha||\beta} = \gamma_{\alpha\beta}(\boldsymbol{\xi}^0),$$

where

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}(\partial_\beta\eta_\alpha + \partial_\alpha\eta_\beta) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3, \tag{68}$$

denote the covariant components of the linearized change of metric tensor associated with a displacement field $\eta_i\boldsymbol{a}^i$ of the surface S. From (66) we reason that $q^{-1}=0$ and find that $\partial_3\vartheta^1 = 0$.

Having these results in mind, for $m=0$, developing $A^{ijkl}(0)$ and taking $\boldsymbol{v} = \boldsymbol{\eta} \in V(\omega)$ and $\varphi \in S(\omega)$ leads to the following two-dimensional problem, to which we may refer as *thermoelastic membrane contact problem with normal damped response*:

Problem 7 Find a pair $t \mapsto (\boldsymbol{\xi}^0(\boldsymbol{y},t), \zeta^0(\boldsymbol{y},t))$ of $[0,T] \to V(\omega) \times S(\omega)$ verifying

$$2\int_\omega \rho(\ddot{\xi}^0_\alpha a^{\alpha\beta}\eta_\beta + \ddot{\xi}^0_3\eta_3)\sqrt{a}dy + \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\xi}^0)\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy - 4\int_\omega \frac{\alpha_T\mu(3\lambda+2\mu)}{\lambda+2\mu}\zeta^0 a^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy$$
$$- \int_{\Gamma_C} p^1(-\dot{\xi}^0_3)\eta_3\sqrt{a}d\Gamma = \int_\omega F^{i,0}\eta_i\sqrt{a}dy \quad \forall \boldsymbol{\eta} = (\eta_i) \in V(\omega),\ a.e. \text{ in } (0,T),$$
$$2\int_\omega \left(\beta + \frac{\alpha_T^2(3\lambda+2\mu)^2}{\lambda+2\mu}\right)\dot{\zeta}^0\varphi\sqrt{a}dy + 2\int_\omega k\partial_\alpha\zeta^0 a^{\alpha\beta}\partial_\beta\varphi\sqrt{a}dy$$
$$+ 4\int_\omega \frac{\alpha_T\mu(3\lambda+2\mu)}{\lambda+2\mu}\varphi a^{\alpha\beta}\gamma_{\alpha\beta}(\dot{\boldsymbol{\xi}}^0)\sqrt{a}dy = \int_\omega Q^0\varphi\sqrt{a}dy \quad \forall \varphi \in S(\omega),\ a.e. \text{ in } (0,T),$$

with $\dot{\boldsymbol{\xi}}^0(\cdot,0) = \boldsymbol{\xi}^0(\cdot,0) = \mathbf{0}$ and $\zeta^0(\cdot,0) = 0$.

Above, we have introduced $F^{i,0} := \int_{-1}^1 f^{i,0}dx_3 + h_+^{i,1}$, with $h_+^{i,1}(\cdot) = h^{i,1}(\cdot,+1)$, and $Q^0 := \int_{-1}^1 q^0 dx_3$. Also, $a^{\alpha\beta\sigma\tau}$ denotes the contravariant components of the fourth order two-dimensional elasticity tensor, defined as follows:

$$a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda+2\mu}a^{\alpha\beta}a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}). \tag{69}$$

The problem above will be analyzed in more detail in the following section. There, we shall study the existence and uniqueness of solution under additional hypotheses of geometric nature and a more suitable set of functional spaces, and provide a rigorous convergence result. To that end, the following ellipticity result for the elasticity tensor will be used. There exists a constant $c_e > 0$ independent of the variables and ε, such that

$$\sum_{\alpha,\beta} |t_{\alpha\beta}|^2 \le c_e a^{\alpha\beta\sigma\tau}(\boldsymbol{y})t_{\sigma\tau}t_{\alpha\beta}, \tag{70}$$

for all $\boldsymbol{y} \in \bar{\omega}$ and all $\boldsymbol{t} = (t_{\alpha\beta}) \in \mathbb{S}^2$ (vector space of 2×2 real symmetric matrices).

5 Elliptic membrane case. Convergence

In what follows, we assume that the family of three-dimensional linearly thermoelastic shells consist of elliptic membrane shells, that is, the middle surface of the shell S is uniformly elliptic and the boundary condition of place is considered on the whole lateral face of the shell, that is, $\gamma_0 = \gamma$. Further, from the formal asymptotic analysis made in the preceding section, we assume the hypotheses which led to Problem 7, namely,

$$\boldsymbol{f}(\varepsilon)(\boldsymbol{x}) = \boldsymbol{f}^0(\boldsymbol{x}),\ q(\varepsilon)(\boldsymbol{x}) = q^0(\boldsymbol{x})\ \forall \boldsymbol{x} \in \Omega,\ \boldsymbol{h}(\varepsilon)(\boldsymbol{x}) = \varepsilon\boldsymbol{h}^1(\boldsymbol{x})\ \forall \boldsymbol{x} \in \Gamma_+,$$
$$p(\varepsilon)(r(\varepsilon)) = \varepsilon p^1(r).$$

Since there is no possible ambiguity, in what follows we drop the superindices indicating the order of the different functions.

We also recall that for elliptic membranes it is verified the following two-dimensional Korn's type inequality (see, for example, [3, Theorem 2.7-3]): there exists a constant $c_M = c_M(\omega, \boldsymbol{\theta}) > 0$ such that

$$\left(\sum_{\alpha} \|\eta_\alpha\|^2_{1,\omega} + |\eta_3|^2_{0,\omega}\right)^{1/2} \leq c_M \left(\sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\boldsymbol{\eta})|^2_{0,\omega}\right)^{1/2} \quad \forall \boldsymbol{\eta} \in V_M(\omega), \quad (71)$$

where

$$V_M(\omega) := H^1_0(\omega) \times H^1_0(\omega) \times L^2(\omega),$$

is the right space for the well-posedness of Problem 7. In this section and in the sequel, C represents a positive generic constant whose specific value may change from line to line, independent of ε and the unknowns. Besides, for the sake of simplicity, we assume that all the parameters involved are constant. Also, the notation $\bar{\boldsymbol{v}}$ stands for the average on x_3, i.e., $\bar{\boldsymbol{v}} := \frac{1}{2}\int_{-1}^{1} \boldsymbol{v}(x_3)dx_3$.

To favour a clearer exposition, let us reformulate Problem 7:

Problem 8 Find a pair $t \mapsto (\boldsymbol{\xi}(\boldsymbol{y},t), \zeta(\boldsymbol{y},t))$ of $[0,T] \to V_M(\omega) \times H^1_0(\omega)$ verifying

$$2\int_\omega \rho(\ddot{\xi}_\alpha a^{\alpha\beta}\eta_\beta + \ddot{\xi}_3\eta_3)\sqrt{a}dy + \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\xi})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy - 4\int_\omega \frac{\alpha_T\mu(3\lambda+2\mu)}{\lambda+2\mu}\zeta a^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy$$
$$- \int_{\Gamma_C} p(-\dot{\xi}_3)\eta_3\sqrt{a}d\Gamma = \int_\omega F^i\eta_i\sqrt{a}dy \quad \forall \boldsymbol{\eta} = (\eta_i) \in V_M(\omega), \ \textit{a.e.} \text{ in } (0,T), \quad (72)$$

$$2\int_\omega \left(\beta + \frac{\alpha_T^2(3\lambda+2\mu)^2}{\lambda+2\mu}\right)\dot{\zeta}\varphi\sqrt{a}dy + 2\int_\omega k\partial_\alpha\zeta a^{\alpha\beta}\partial_\beta\varphi\sqrt{a}dy$$
$$+ 4\int_\omega \frac{\alpha_T\mu(3\lambda+2\mu)}{\lambda+2\mu}\varphi a^{\alpha\beta}\gamma_{\alpha\beta}(\dot{\boldsymbol{\xi}})\sqrt{a}dy = \int_\omega Q\varphi\sqrt{a}dy \quad \forall\varphi \in H^1_0(\omega), \ \textit{a.e.} \text{ in } (0,T), \quad (73)$$

with $\dot{\boldsymbol{\xi}}(\cdot,0) = \boldsymbol{\xi}(\cdot,0) = \boldsymbol{0}$ and $\zeta(\cdot,0) = 0$.

Above, we have used $F^i := \int_{-1}^{1} f^i dx_3 + h^i_+$ with $h^i_+(\cdot) = h^i(\cdot,+1)$ and $Q := \int_{-1}^{1} q dx_3$. The following shows that there is a unique solution for this problem.

Theorem 4 *Let ω be a domain in $\mathbb{R}^2$, let $\boldsymbol{\theta} \in \mathcal{C}^2(\bar{\omega};\mathbb{R}^3)$ be an injective mapping such that the two vectors $\boldsymbol{a}_\alpha = \partial_\alpha\boldsymbol{\theta}$ are linearly independent at all points of $\bar{\omega}$. Let f^i and $q \in H^1(0,T;L^2(\Omega))$, $h^i \in H^2(0,T;L^2(\Gamma_+))$ and assume (12). Then the Problem 8, has a unique solution $(\boldsymbol{\xi},\zeta)$ such that*

$$\boldsymbol{\xi} \in L^\infty(0,T;V_M(\omega)), \ \dot{\boldsymbol{\xi}} \in L^\infty(0,T;[L^2(\omega)]^3) \cap L^\infty(0,T;V_M(\omega)), \ \ddot{\boldsymbol{\xi}} \in L^\infty(0,T;[L^2(\omega)]^3),$$
$$\zeta \in L^\infty(0,T;L^2(\omega)) \cap L^2(0,T;H^1_0(\omega)), \ \dot{\zeta} \in L^\infty(0,T;L^2(\omega)) \cap L^2(0,T;H^1_0(\omega)).$$

Proof. Like in Theorem 1, we will use a Faedo-Galerkin approach to prove the existence part. Then, a proof by contradiction will show uniqueness.

Existence: Since $V_M(\omega)$ is a separable space, there exists a countable base $\{\boldsymbol{v}^m\} \subset V_M(\omega)$ such that

$$V_M(\omega) = \overline{\bigcup_{m\geq 1} V_m}, \quad \text{where } V_m = \text{span}\{\boldsymbol{v}^1, \boldsymbol{v}^2, \ldots, \boldsymbol{v}^m\}.$$

Similarly, there exists a countable base $\{\chi^m\} \subset H_0^1(\omega)$ such that

$$H_0^1(\omega) = \overline{\bigcup_{m\geq 1} S_m}, \quad \text{where } S_m = \text{span}\{\chi^1, \chi^2, \ldots, \chi^m\}.$$

We now formulate Problem 8 for the finite dimensional subspaces:

Problem 9 Find a pair $t \mapsto (\boldsymbol{\xi}^m(\boldsymbol{y},t), \zeta^m(\boldsymbol{y},t))$ of $[0,T] \to V_m \times S_m$ verifying

$$\begin{aligned}
&2\int_\omega \rho(\ddot{\xi}^m_\alpha a^{\alpha\beta}\eta^m_\beta + \ddot{\xi}^m_3\eta^m_3)\sqrt{a}dy + \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\xi}^m)\gamma_{\alpha\beta}(\boldsymbol{\eta}^m)\sqrt{a}dy - 4\int_\omega \frac{\alpha_T\mu(3\lambda+2\mu)}{\lambda+2\mu}\zeta^m a^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\eta}^m)\sqrt{a}dy \\
&\quad - \int_{\Gamma_C} p(-\dot{\xi}^m_3)\eta^m_3\sqrt{a}d\Gamma = \int_\omega F^i\eta^m_i\sqrt{a}dy \quad \forall \boldsymbol{\eta}^m = (\eta^m_i) \in V_m,\ \forall\, t \in [0,T],
\end{aligned} \tag{74}$$

$$\begin{aligned}
&2\int_\omega \left(\beta + \frac{\alpha_T^2(3\lambda+2\mu)^2}{\lambda+2\mu}\right)\dot{\zeta}^m\varphi^m\sqrt{a}dy + 2\int_\omega k\partial_\alpha\zeta^m a^{\alpha\beta}\partial_\beta\varphi^m\sqrt{a}dy \\
&\quad + 4\int_\omega \frac{\alpha_T\mu(3\lambda+2\mu)}{\lambda+2\mu}\varphi^m a^{\alpha\beta}\gamma_{\alpha\beta}(\dot{\boldsymbol{\xi}}^m)\sqrt{a}dy = \int_\omega Q\varphi^m\sqrt{a}dy \quad \forall \varphi^m \in S_m,\ \forall\, t \in [0,T],
\end{aligned} \tag{75}$$

with $\dot{\boldsymbol{\xi}}^m(\cdot,0) = \boldsymbol{\xi}^m(\cdot,0) = \mathbf{0}$ and $\zeta^m(\cdot,0) = 0$.

Now, the classical theory of systems of ordinary differential equations guarantees the existence and uniqueness of solution for Problem 9. Taking $\boldsymbol{\eta}^m = \dot{\boldsymbol{\xi}}^m$ in (74) and $\varphi^m = \zeta^m$ in (75), adding both expressions and integrating the time variable in $[0,t]$ gives

$$\begin{aligned}
&\rho|\dot{\boldsymbol{\xi}}^m(t)|^2_{a,\omega} + \frac{1}{2}\|\boldsymbol{\xi}^m(t)\|^2_{a,\omega} + \left(\beta + \frac{\alpha_T^2(3\lambda+2\mu)^2}{\lambda+2\mu}\right)|\zeta^m(t)|^2_{0,\omega} + 2k\int_0^t |||\zeta^m(r)|||^2_{a,\omega}\,dr \\
&\quad - \int_0^t\int_{\Gamma_C} p(-\dot{\xi}^m_3(r))\dot{\xi}^m_3(r)\sqrt{a}d\Gamma\,dr = \int_0^t\int_\omega Q(r)\zeta^m\sqrt{a}dy\,dr \\
&\quad + \int_0^t\int_\omega\int_{-1}^1 f^i(r)dx_3\dot{\xi}^m_i(r)\sqrt{a}dy\,dr + \int_0^t\int_{\Gamma_+} h^i(r)\dot{\xi}^m_i(r)\sqrt{a}d\Gamma\,dr,
\end{aligned} \tag{76}$$

where we have introduced the following norms:

$$|\boldsymbol{\eta}|^2_{a,\omega} := \int_\omega (\eta_\alpha a^{\alpha\beta}\eta_\beta + (\eta_3)^2)\sqrt{a}dy \ \forall \boldsymbol{\eta} \in [L^2(\omega)]^3,$$

which is equivalent to the usual norm $|\cdot|_{0,\omega}$ because of the ellipticity of $(a^{\alpha\beta})$ and the regularity of $\boldsymbol{\theta}$. Also,

$$\|\boldsymbol{\eta}\|_{a,\omega}^2 := \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\eta})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy \ \forall \boldsymbol{\eta} \in V_M(\omega),$$

which is a norm in $V_M(\omega)$ because of the Korn inequality (71) and the ellipticity of $a^{\alpha\beta\sigma\tau}$ (see (70)). Finally,

$$|||\varphi|||_{a,\omega}^2 := \int_\omega \partial_\alpha\varphi a^{\alpha\beta}\partial_\beta\varphi\sqrt{a}dy,$$

which is a norm in $H_0^1(\omega)$ equivalent to the usual $\|\cdot\|_{1,\omega}$ because of the ellipticity of $(a^{\alpha\beta})$, the regularity of $\boldsymbol{\theta}$ and the Poincaré inequality.

By using the monotonicity of p, then the Hölder inequality in the right-hand side terms of (76), then using Theorem 3 for the terms on Γ_+ followed by the use of Gronwall inequality, we obtain that the following weak convergences take place for subsequences indexed by m as well:

$$\boldsymbol{\xi}^m \xrightarrow[m\to\infty]{*} \boldsymbol{\xi} \text{ in } L^\infty(0,T;V_M(\omega)), \quad \dot{\boldsymbol{\xi}}^m \xrightarrow[m\to\infty]{*} \dot{\boldsymbol{\xi}} \text{ in } L^\infty(0,T;[L^2(\omega)]^3), \tag{77}$$

$$\zeta^m \xrightarrow[m\to\infty]{*} \zeta \text{ in } L^\infty(0,T;L^2(\omega)), \quad \zeta^m \xrightarrow[m\to\infty]{} \zeta \text{ in } L^2(0,T;H_0^1(\omega)), \tag{78}$$

$$p(-\dot{\xi}_3^m) \xrightarrow[m\to\infty]{*} \chi \text{ in } L^\infty(0,T;L^2(\omega)). \tag{79}$$

Notice that (79) is a consequence of the Lipschitz continuity of p, the fact that $p(0)=0$, and the boundedness of its argument. Using these convergences back in (74)–(75), we can formulate the following limit problem:

Problem 10 Find a pair $t \mapsto (\boldsymbol{\xi}(\boldsymbol{y},t), \zeta(\boldsymbol{y},t))$ of $[0,T] \to V_M(\omega) \times H_0^1(\omega)$ verifying

$$\begin{aligned}
&2\int_\omega \rho(\ddot{\xi}_\alpha a^{\alpha\beta}\eta_\beta + \ddot{\xi}_3\eta_3)\sqrt{a}dy + \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\xi})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy - 4\int_\omega \frac{\alpha_T\mu(3\lambda+2\mu)}{\lambda+2\mu}\zeta a^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy \\
&\quad - \int_{\Gamma_C} \chi\eta_3\sqrt{a}d\Gamma = \int_\omega F^i\eta_i\sqrt{a}dy \quad \forall \boldsymbol{\eta} = (\eta_i) \in V_M(\omega), \ a.e. \text{ in } (0,T), \\
&2\int_\omega \left(\beta + \frac{\alpha_T^2(3\lambda+2\mu)^2}{\lambda+2\mu}\right)\dot{\zeta}\varphi\sqrt{a}dy + 2\int_\omega k\partial_\alpha\zeta a^{\alpha\beta}\partial_\beta\varphi\sqrt{a}dy \\
&\quad + 4\int_\omega \frac{\alpha_T\mu(3\lambda+2\mu)}{\lambda+2\mu}\varphi a^{\alpha\beta}\gamma_{\alpha\beta}(\dot{\boldsymbol{\xi}})\sqrt{a}dy = \int_\omega Q\varphi\sqrt{a}dy \quad \forall\varphi \in H_0^1(\omega), \ a.e. \text{ in } (0,T),
\end{aligned}$$

with $\dot{\boldsymbol{\xi}}(\cdot,0) = \boldsymbol{\xi}(\cdot,0) = \mathbf{0}$ and $\zeta(\cdot,0)=0$.

Now we will use an argument of monotonicity (see, for example, [8]). We first define for any given $\phi \in H^1(0,T;L^2(\omega))$, with $\phi(0)=0$, the following quantity:

$$X^m = -\int_0^t \int_{\Gamma_C} \Big(p(-\dot{\xi}_3^m(r)) - p(-\dot{\phi}(r))\Big)(\dot{\xi}_3^m(r) - \dot{\phi}(r))\sqrt{a}d\Gamma\, dr \geq 0.$$

From (76) we find that

$$X^m = \int_0^t \int_\omega F^i(r)\dot{\xi}_i^m(r)\sqrt{a}dy\,dr - \rho|\dot{\boldsymbol{\xi}}^m(t)|^2_{a,\omega} - \frac{1}{2}\|\boldsymbol{\xi}^m(t)\|^2_{a,\omega}$$
$$+ \int_0^t \int_\omega Q(r)\zeta^m\sqrt{a}dy\,dr - \left(\beta + \frac{\alpha_T^2(3\lambda+2\mu)^2}{\lambda+2\mu}\right)|\zeta^m(t)|^2_{0,\omega} - 2k\int_0^t |||\zeta^m(r)|||^2_{a,\omega}\,dr$$
$$- \int_0^t \int_{\Gamma_C} p(-\dot{\xi}_3^m(r))(-\dot{\phi}(r))\sqrt{a}d\Gamma\,dr - \int_0^t \int_{\Gamma_C} -p(-\dot{\phi}(r))(\dot{\xi}_3^m(r) - \dot{\phi}(r))\sqrt{a}d\Gamma\,dr.$$

Thus, on one hand

$$0 \le \limsup_{m\to\infty} X^m \le \int_0^t \int_\omega F^i(r)\dot{\xi}_i(r)\sqrt{a}dy\,dr - \rho|\dot{\boldsymbol{\xi}}(t)|^2_{a,\omega} - \frac{1}{2}\|\boldsymbol{\xi}(t)\|^2_{a,\omega}$$
$$+ \int_0^t \int_\omega Q(r)\zeta\sqrt{a}dy\,dr - \left(\beta + \frac{\alpha_T^2(3\lambda+2\mu)^2}{\lambda+2\mu}\right)|\zeta(t)|^2_{0,\omega} - 2k\int_0^t |||\zeta(r)|||^2_{a,\omega}\,dr$$
$$- \int_0^t \int_{\Gamma_C} \chi(r)(-\dot{\phi}(r))\sqrt{a}d\Gamma\,dr - \int_0^t \int_{\Gamma_C} -p(-\dot{\phi}(r))(\dot{\xi}_3(r) - \dot{\phi}(r))\sqrt{a}d\Gamma\,dr,$$

where he have used the weak upper semicontinuity of various terms. On the other hand, doing in Problem 10 the substitutions $\boldsymbol{\eta} = \dot{\boldsymbol{\xi}}$, $\varphi = \zeta$, then the summation of both equations, followed by the integration in $[0,t]$, and using the resulting identity into the inequality above, we find that

$$0 \le -\int_0^t \int_{\Gamma_C} \chi(r)\dot{\xi}_3(r)\sqrt{a}d\Gamma\,dr - \int_0^t \int_{\Gamma_C} \chi(r)(-\dot{\phi}(r))\sqrt{a}d\Gamma\,dr - \int_0^t \int_{\Gamma_C} -p(-\dot{\phi}(r))(\dot{\xi}_3(r) - \dot{\phi}(r))\sqrt{a}d\Gamma\,dr$$
$$= -\int_0^t \int_{\Gamma_C} (\chi(r) - p(-\dot{\phi}(r)))(\dot{\xi}_3(r) - \dot{\phi}(r))\sqrt{a}d\Gamma\,dr.$$

Therefore, by using arguments adapted from those in [4, p. 55], we deduce that $\chi = p(-\dot{\xi}_3)$. Indeed, this is because we can always take $\phi = \xi_3 - \varsigma\varphi$ with $\varsigma > 0$ and $\varphi \in H^1(0,T;L^2(\omega))$, with $\varphi(0) = 0$, to find

$$0 \le -\int_0^t \int_{\Gamma_C} (\chi(r) - p(-\dot{\xi}_3(r) + \varsigma\dot{\varphi}(r)))\dot{\varphi}(r)\sqrt{a}d\Gamma\,dr,$$

and take $\varsigma \to 0$, thus

$$0 \le -\int_0^t \int_{\Gamma_C} (\chi(r) - p(-\dot{\xi}_3(r)))\dot{\varphi}(r)\sqrt{a}d\Gamma\,dr,$$

from where $\chi = p(-\dot{\xi}_3)$. Therefore, we find that Problem 10 is indeed the same as Problem 8.

We will now prove the additional regularities for $\dot{\boldsymbol{\xi}}$, $\ddot{\boldsymbol{\xi}}$ and $\dot{\zeta}$. First, we add equations (74) and (75) and write the resulting equation at times $\tilde{t} = t + h$

and t, with $h > 0$ and $0 < t \leq T - h$. Then subtract these last two equations and take $\boldsymbol{\eta}^m = \dot{\boldsymbol{\xi}}^m(\tilde{t}) - \dot{\boldsymbol{\xi}}^m(t) \in V_m$ and $\varphi^m = \zeta^m(\tilde{t}) - \zeta^m(t) \in S_m$ to obtain

$$
\begin{aligned}
&2\int_\omega \rho((\ddot{\xi}_\alpha^m(\tilde{t}) - \ddot{\xi}_\alpha^m(t))a^{\alpha\beta}(\dot{\xi}_\beta^m(\tilde{t}) - \dot{\xi}_\beta^m(t)) + (\ddot{\xi}_3^m(\tilde{t}) - \ddot{\xi}_3^m(t))(\dot{\xi}_3^m(\tilde{t}) - \dot{\xi}_3^m(t)))\sqrt{a}dy \\
&\quad + \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\xi}^m(\tilde{t}) - \boldsymbol{\xi}^m(t))\gamma_{\alpha\beta}(\dot{\boldsymbol{\xi}}^m(\tilde{t}) - \dot{\boldsymbol{\xi}}^m(t))\sqrt{a}dy \\
&\quad - \int_{\Gamma_C} (p(-\dot{\xi}_3^m(\tilde{t})) - p(-\dot{\xi}_3^m(t)))(\dot{\xi}_3^m(\tilde{t}) - \dot{\xi}_3^m(t))\sqrt{a}d\Gamma \\
&\quad + 2\int_\omega \left(\beta + \frac{\alpha_T^2(3\lambda + 2\mu)^2}{\lambda + 2\mu}\right)(\dot{\zeta}^m(\tilde{t}) - \dot{\zeta}^m(t))(\zeta^m(\tilde{t}) - \zeta^m(t))\sqrt{a}dy \\
&\quad + 2\int_\omega k\partial_\alpha(\zeta^m(\tilde{t}) - \zeta^m(t))a^{\alpha\beta}\partial_\beta(\zeta^m(\tilde{t}) - \zeta^m(t))\sqrt{a}dy \\
&\quad = \int_\omega (F^i(\tilde{t}) - F^i(t))(\dot{\xi}_i^m(\tilde{t}) - \dot{\xi}_i^m(t))\sqrt{a}dy + \int_\omega (Q(\tilde{t}) - Q(t))(\zeta^m(\tilde{t}) - \zeta^m(t))\sqrt{a}dy, \quad \forall t \in [0, T-h],
\end{aligned}
$$

which, because of the monotonicity of p gives

$$
\begin{aligned}
&\frac{d}{dt}\rho|\dot{\boldsymbol{\xi}}^m(\tilde{t}) - \dot{\boldsymbol{\xi}}^m(t)|_{a,\omega}^2 + \frac{1}{2}\frac{d}{dt}\|\boldsymbol{\xi}^m(\tilde{t}) - \boldsymbol{\xi}^m(t)\|_{a,\omega}^2 + \left(\beta + \frac{\alpha_T^2(3\lambda + 2\mu)^2}{\lambda + 2\mu}\right)\frac{d}{dt}|\zeta^m(\tilde{t}) - \zeta^m(t)|_{0,\omega}^2 \\
&\quad + 2k|||\zeta^m(\tilde{t}) - \zeta^m(t)|||_{a,\omega}^2 \leq \int_\omega (F^i(\tilde{t}) - F^i(t))(\dot{\xi}_i^m(\tilde{t}) - \dot{\xi}_i^m(t))\sqrt{a}dy \\
&\quad + \int_\omega (Q(\tilde{t}) - Q(t))(\zeta^m(\tilde{t}) - \zeta^m(t))\sqrt{a}dy, \quad \forall t \in [0, T-h].
\end{aligned}
$$

Next, we integrate in $[0, t]$ to get

$$
\begin{aligned}
&\rho|\dot{\boldsymbol{\xi}}^m(\tilde{t}) - \dot{\boldsymbol{\xi}}^m(t)|_{a,\omega}^2 - \rho|\dot{\boldsymbol{\xi}}^m(h) - \dot{\boldsymbol{\xi}}^m(0)|_{a,\omega}^2 + \frac{1}{2}\|\boldsymbol{\xi}^m(\tilde{t}) - \boldsymbol{\xi}^m(t)\|_{a,\omega}^2 - \frac{1}{2}\|\boldsymbol{\xi}^m(h) - \boldsymbol{\xi}^m(0)\|_{a,\omega}^2 \\
&\quad + \left(\beta + \frac{\alpha_T^2(3\lambda + 2\mu)^2}{\lambda + 2\mu}\right)|\zeta^m(\tilde{t}) - \zeta^m(t)|_{0,\omega}^2 - \left(\beta + \frac{\alpha_T^2(3\lambda + 2\mu)^2}{\lambda + 2\mu}\right)|\zeta^m(h) - \zeta^m(0)|_{0,\omega}^2 \\
&\quad + 2k\int_0^t |||\zeta^m(r+h) - \zeta^m(r)|||_{a,\omega}^2 dr \leq \int_0^t \int_\omega (F^i(r+h) - F^i(r))(\dot{\xi}_i^m(r+h) - \dot{\xi}_i^m(r))\sqrt{a}\,dy\,dr \\
&\quad + \int_0^t \int_\omega (Q(r+h) - Q(r))(\zeta^m(r+h) - \zeta^m(r))\sqrt{a}\,dy\,dr, \quad \forall t \in [0, T-h],
\end{aligned}
$$

and dividing the equation by h^2 and taking limits when $h \to 0$ we obtain

$$
\begin{aligned}
&\rho|\ddot{\boldsymbol{\xi}}^m(t)|_{a,\omega}^2 + \frac{1}{2}\|\dot{\boldsymbol{\xi}}^m(t)\|_{a,\omega}^2 + \left(\beta + \frac{\alpha_T^2(3\lambda + 2\mu)^2}{\lambda + 2\mu}\right)|\dot{\zeta}^m(t)|_{0,\omega}^2 + 2k\int_0^t |||\dot{\zeta}^m(r)|||_{a,\omega}^2 dr \\
&\quad \leq \rho|\ddot{\boldsymbol{\xi}}^m(0)|_{a,\omega}^2 + \frac{1}{2}\|\dot{\boldsymbol{\xi}}^m(0)\|_{a,\omega}^2 + \left(\beta + \frac{\alpha_T^2(3\lambda + 2\mu)^2}{\lambda + 2\mu}\right)|\dot{\zeta}^m(0)|_{0,\omega}^2 + \int_0^t \int_\omega \dot{F}^i(r)\ddot{\xi}_i^m(r)\sqrt{a}\,dy\,dr \\
&\quad + \int_0^t \int_\omega \dot{Q}(r)\dot{\zeta}^m(r)\sqrt{a}\,dy\,dr, \quad \forall t \in [0, T],
\end{aligned}
$$

from which, by Young's inequality, we obtain

$$\begin{aligned}
&\rho|\ddot{\boldsymbol{\xi}}^m(t)|^2_{a,\omega} + \frac{1}{2}\|\dot{\boldsymbol{\xi}}^m(t)\|^2_{a,\omega} + \left(\beta + \frac{\alpha_T^2(3\lambda+2\mu)^2}{\lambda+2\mu}\right)|\dot{\zeta}^m(t)|^2_{0,\omega} + 2k\int_0^t |||\dot{\zeta}^m(r)|||^2_{a,\omega}dr \\
&\quad \leq \rho|\ddot{\boldsymbol{\xi}}^m(0)|^2_{a,\omega} + \frac{1}{2}\|\dot{\boldsymbol{\xi}}^m(0)\|^2_{a,\omega} + \left(\beta + \frac{\alpha_T^2(3\lambda+2\mu)^2}{\lambda+2\mu}\right)|\dot{\zeta}^m(0)|^2_{0,\omega} \\
&\quad + C(\dot{\boldsymbol{f}},\dot{\boldsymbol{h}},\dot{q}) + \bar{C}\int_0^t \left\{|\ddot{\boldsymbol{\xi}}^m(r)|^2_{0,\omega}dr + \int_0^t |\dot{\zeta}^m(r)|^2_{0,\omega}\right\}dr, \quad \forall t\in[0,T].
\end{aligned} \tag{80}$$

In order to obtain bounds for $|\ddot{\boldsymbol{\xi}}^m(0)|^2_{a,\omega}$ and $|\dot{\zeta}^m(0)|^2_{0,\omega}$ we first notice that equations (74) and (75) hold for $t=0$ due to the compatibility required between initial and boundary conditions. Therefore, taking $t=0$ and $\boldsymbol{\eta}^m = \ddot{\boldsymbol{\xi}}^m(0)\in V_m$ in (74) and $\varphi^m = \dot{\zeta}^m(0)\in S_m$ in (75) and, taking into account the initial conditions, we obtain

$$\begin{aligned}
&\rho|\ddot{\boldsymbol{\xi}}^m(0)|^2_{a,\omega} = \int_\omega F^i(0)\ddot{\boldsymbol{\xi}}^m_i(0)\sqrt{a}dy \leq \frac{1}{\delta}C + \delta|\ddot{\boldsymbol{\xi}}^m(0)|^2_{0,\omega} \\
&\left(\beta + \frac{\alpha_T^2(3\lambda+2\mu)^2}{\lambda+2\mu}\right)|\dot{\zeta}^m(0)|^2_{0,\omega} = \int_\omega Q(0)\dot{\zeta}^m(0)\sqrt{a}\,dy \ \leq \frac{1}{\tilde{\delta}}\tilde{C} + \tilde{\delta}|\dot{\zeta}^m(0)|^2_{0,\omega},
\end{aligned}$$

where δ and $\tilde{\delta}$ are sufficiently small positive constants.
Now, back to (80), taking into account the initial conditions and the bounds above we have

$$\begin{aligned}
&\rho|\ddot{\boldsymbol{\xi}}^m(t)|^2_{a,\omega} + \frac{1}{2}\|\dot{\boldsymbol{\xi}}^m(t)\|^2_{a,\omega} + \left(\beta + \frac{\alpha_T^2(3\lambda+2\mu)^2}{\lambda+2\mu}\right)|\dot{\zeta}^m(t)|^2_{0,\omega} + 2k\int_0^t |||\dot{\zeta}^m(r)|||^2_{a,\omega}dr \\
&\quad \leq C + \bar{C}\int_0^t \left\{|\ddot{\boldsymbol{\xi}}^m(r)|^2_{0,\omega}dr + \int_0^t |\dot{\zeta}^m(r)|^2_{0,\omega}\right\}dr, \quad \forall t\in[0,T].
\end{aligned}$$

Next, we use the equivalence between then norms $|\cdot|_{a,\omega}$ and $|\cdot|_{0,\omega}$ and we apply Gronwall's Lemma to conclude that

$$|\ddot{\boldsymbol{\xi}}^m(t)|^2_{0,\omega} + |\dot{\zeta}^m(t)|^2_{0,\omega} \leq C, \quad \forall t\in[0,T],$$

and further

$$\|\dot{\boldsymbol{\xi}}^m(t)\|^2_{a,\omega} + 2k\int_0^t |||\dot{\zeta}^m(r)|||^2_{a,\omega}dr \leq C \quad \forall t\in[0,T].$$

Therefore, the following weak convergences take place for subsequences still indexed by m.

$$\dot{\boldsymbol{\xi}}^m \xrightarrow[m\to\infty]{*} \dot{\boldsymbol{\xi}} \text{ in } L^\infty(0,T;V_M(\omega)), \quad \ddot{\boldsymbol{\xi}}^m \xrightarrow[m\to\infty]{*} \ddot{\boldsymbol{\xi}} \text{ in } L^\infty(0,T;[L^2(\omega)]^3), \tag{81}$$

$$\dot{\zeta}^m \xrightarrow[m\to\infty]{*} \dot{\zeta} \text{ in } L^\infty(0,T;L^2(\omega)), \quad \dot{\zeta}^m \xrightarrow[m\to\infty]{} \dot{\zeta} \text{ in } L^2(0,T;H^1_0(\omega)). \tag{82}$$

Uniqueness: We proceed by contradiction. We first assume that there exist two solutions $(\boldsymbol{\xi}^1, \zeta^1)$ and $(\boldsymbol{\xi}^2, \zeta^2)$. Define $\bar{\boldsymbol{\xi}} = \boldsymbol{\xi}^1 - \boldsymbol{\xi}^2$ and $\bar{\zeta} = \zeta^1 - \zeta^2$. Now, take $\boldsymbol{\eta} = \dot{\bar{\boldsymbol{\xi}}}$ in the version of (72) for $\boldsymbol{\xi}^1$ and $\boldsymbol{\eta} = -\dot{\bar{\boldsymbol{\xi}}}$ in the version of (72) for $\boldsymbol{\xi}^2$. We then sum both expresions to find that

$$2\int_\omega \rho(\ddot{\bar{\xi}}_\alpha a^{\alpha\beta}\dot{\bar{\xi}}_\beta + \ddot{\bar{\xi}}_3\dot{\bar{\xi}}_3)\sqrt{a}dy + \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\bar{\boldsymbol{\xi}})\gamma_{\alpha\beta}(\dot{\bar{\boldsymbol{\xi}}})\sqrt{a}dy - 4\int_\omega \frac{\alpha_T\mu(3\lambda+2\mu)}{\lambda+2\mu}\bar{\zeta}a^{\alpha\beta}\gamma_{\alpha\beta}(\dot{\bar{\boldsymbol{\xi}}})\sqrt{a}dy$$
$$- \int_{\Gamma_C}(p(-\dot{\xi}_3^1) - p(-\dot{\xi}_3^2))\dot{\bar{\xi}}_3\sqrt{a}d\Gamma = 0.$$

Similarly, take $\varphi = \bar{\zeta}$ in the version of (73) for ζ^1 and $\varphi = -\bar{\zeta}$ in the version of (73) for ζ^2. Then, we sum both expresions to find that

$$2\int_\omega \left(\beta + \frac{\alpha_T^2(3\lambda+2\mu)^2}{\lambda+2\mu}\right)\dot{\bar{\zeta}}\bar{\zeta}\sqrt{a}dy + 2\int_\omega k\partial_\alpha\bar{\zeta}a^{\alpha\beta}\partial_\beta\bar{\zeta}\sqrt{a}dy + 4\int_\omega \frac{\alpha_T\mu(3\lambda+2\mu)}{\lambda+2\mu}\bar{\zeta}a^{\alpha\beta}\gamma_{\alpha\beta}(\dot{\bar{\boldsymbol{\xi}}})\sqrt{a}dy = 0.$$

Then, we add both expressions above and integrate with respect to the time variable in $[0, t]$, to find

$$\rho|\dot{\bar{\boldsymbol{\xi}}}(t)|^2_{a,\omega} + \frac{1}{2}\|\bar{\boldsymbol{\xi}}(t)\|^2_{a,\omega} + \left(\beta + \frac{\alpha_T^2(3\lambda+2\mu)^2}{\lambda+2\mu}\right)|\bar{\zeta}(t)|^2_{0,\omega} + 2k\int_0^t |||\bar{\zeta}(r)|||^2_{a,\omega}\,dr$$
$$= \int_0^t\int_{\Gamma_C}(p(-\dot{\xi}_3^1(r)) - p(-\dot{\xi}_3^2(r)))(\dot{\xi}_3^1(r) - \dot{\xi}_3^2(r))\sqrt{a}d\Gamma\,dr \le 0, \tag{83}$$

where we have used the monotonicity of p. We deduce from (83) that $\bar{\boldsymbol{\xi}} = \mathbf{0}$ and $\bar{\zeta} = 0$, thus showing uniqueness.

□

Now, we present here the main result of this paper, namely that the scaled three-dimensional unknowns $(\boldsymbol{u}(\varepsilon), \vartheta(\varepsilon))$ converge, as ε tends to zero, towards a limit $(\boldsymbol{u}, \vartheta)$ independent of the transversal variable, and that this limit can be identified with the solution $(\boldsymbol{\xi}, \zeta)$ of the Problem 8, posed over the two-dimensional set ω.

In what follows, and for the sake of simplicity, we assume that for each $\varepsilon > 0$ the initial condition for the scaled linear strain is

$$e_{i||j}(\varepsilon)(0, \cdot) = 0, \tag{84}$$

this is, the domain is on its natural state with no strains on it at the beginning of the period of observation.

Theorem 5 *Assume that $\boldsymbol{\theta} \in \mathcal{C}^3(\bar{\omega}; \mathbb{R}^3)$. Consider a family of elastic elliptic shells with thickness 2ε approaching zero and all sharing the same elliptic middle surface $S = \boldsymbol{\theta}(\bar{\omega})$. For all ε, $0 < \varepsilon \le \varepsilon_0$ let $(\boldsymbol{u}(\varepsilon), \vartheta(\varepsilon))$ be the solution of the associated three-dimensional scaled Problem 6 for $m = 0$. Assume also that (12) is satisfied. Then, there exist functions $\vartheta, u_\alpha \in H^1(\Omega)$ satisfying $\vartheta = 0$, $u_\alpha = 0$ on $\gamma \times [-1, 1]$ and a function $u_3 \in L^2(\Omega)$, such that* *(a)* $\vartheta(\varepsilon) \to \vartheta$, $u_\alpha(\varepsilon) \to u_\alpha$ *in* $H^1(\Omega)$ *and* $u_3(\varepsilon) \to u_3$ *in* $L^2(\Omega)$ *when* $\varepsilon \to 0$ *a.e. in* $(0,T)$,

(b) ϑ *and* $\boldsymbol{u} = (u_i)$ *are independent of the transversal variable* x_3.

Furthermore, the pair $(\boldsymbol{u}, \vartheta)$ *can be identified with the solution of Problem 8.*

Proof. We follow the structure of the proof given in [3, Theorem 4.4-1] for the case of elastic elliptic membrane shells. Hence, we shall reference some steps which apply in the same manner and omit some details. Also, for the sake of readability we may use the shorter notations $e_{i||j}(\varepsilon) := e_{i||j}(\varepsilon; \boldsymbol{u}(\varepsilon))$. In addition to that, all references to (65) or (66) have to be considered as for $m = 0$ and drop the superindices. The proof is divided into several parts, numbered from (i) to (vi).

(i) A priori boundedness and extraction of weak convergent sequences. For $\varepsilon > 0$ *sufficiently small, there exist bounded sequences, also indexed by* ε*, and weak limits as specified below:*

$$u_\alpha(\varepsilon) \xrightarrow[\varepsilon\to 0]{*} u_\alpha \text{ in } L^\infty(0,T;H^1(\Omega)), \quad u_3(\varepsilon) \xrightarrow[\varepsilon\to 0]{*} u_3 \text{ in } L^\infty(0,T;L^2(\Omega)),$$

$$\dot{\boldsymbol{u}}(\varepsilon) \xrightarrow[\varepsilon\to 0]{*} \dot{\boldsymbol{u}} \text{ in } L^\infty(0,T;[L^2(\Omega)]^3), \quad e_{i||j}(\varepsilon) \xrightarrow[\varepsilon\to 0]{*} e_{i||j} \text{ in } L^\infty(0,T;L^2(\Omega))$$

$$\vartheta(\varepsilon) \xrightarrow[\varepsilon\to 0]{*} \vartheta \text{ in } L^\infty(0,T;L^2(\Omega)), \quad \partial_\alpha\vartheta(\varepsilon) \xrightarrow[\varepsilon\to 0]{} \vartheta_\alpha \text{ in } L^2(0,T;L^2(\Omega)),$$

$$\varepsilon^{-1}\partial_3\vartheta(\varepsilon) \xrightarrow[\varepsilon\to 0]{} \vartheta_{3,-1} \text{ in } L^2(0,T;L^2(\Omega)).$$

Moreover, $\vartheta, u_\alpha = 0$ *on* Γ_0.

For the proof of this step we take $\boldsymbol{v} = \dot{\boldsymbol{u}}(\varepsilon)$ in (65) (see Remark 7) and $\varphi = \vartheta(\varepsilon)$ in (66) and sum both expressions to find

$$\begin{aligned}
&\int_\Omega \rho(\ddot{u}_\alpha(\varepsilon)g^{\alpha\beta}(\varepsilon)\dot{u}_\beta(\varepsilon) + \ddot{u}_3(\varepsilon)\dot{u}_3(\varepsilon))\sqrt{g(\varepsilon)}\,dx + \int_\Omega A^{ijkl}(\varepsilon)e_{k||l}(\varepsilon)\dot{e}_{i||j}(\varepsilon)\sqrt{g(\varepsilon)}dx \\
&\quad + \int_\Omega \beta\dot{\vartheta}(\varepsilon)\vartheta(\varepsilon)\sqrt{g(\varepsilon)}dx + \int_\Omega k(\partial_\alpha\vartheta(\varepsilon)g^{\alpha\beta}(\varepsilon)\partial_\beta\vartheta(\varepsilon) + \frac{1}{\varepsilon^2}\partial_3\vartheta(\varepsilon)\partial_3\vartheta(\varepsilon))\sqrt{g(\varepsilon)}dx \\
&\quad - \int_{\Gamma_C} p(-\dot{u}_3(\varepsilon))\dot{u}_3(\varepsilon)\sqrt{g(\varepsilon)}d\Gamma \\
&\quad = \int_\Omega f^i\dot{u}_i(\varepsilon)\sqrt{g(\varepsilon)}dx + \int_{\Gamma_+} h^i\dot{u}_i(\varepsilon)\sqrt{g(\varepsilon)}d\Gamma + \int_\Omega q\vartheta(\varepsilon)\sqrt{g(\varepsilon)}dx.
\end{aligned} \tag{85}$$

We now introduce the following norms:

$$|\boldsymbol{v}|^2_{g(\varepsilon),\Omega} := \int_\Omega (v_\alpha g^{\alpha\beta}(\varepsilon)v_\beta + (v_3)^2)\sqrt{g(\varepsilon)}dx \ \forall \boldsymbol{v} \in [L^2(\Omega)]^3,$$

which is equivalent to the usual norm $|\cdot|_{0,\Omega}$ because of the ellipticity of $(g^{\alpha\beta}(\varepsilon))$ and the regularity of $\boldsymbol{\Theta}$. Also,

$$\|\boldsymbol{v}\|^2_{A(\varepsilon),\Omega} := \int_\Omega A^{ijkl}(\varepsilon)e_{k||l}(\varepsilon;\boldsymbol{v})e_{i||j}(\varepsilon;\boldsymbol{v})\sqrt{g(\varepsilon)}dx \ \forall \boldsymbol{v} \in V(\Omega),$$

which is a norm in $V(\Omega)$ because of the Korn inequality (see [3, Theorem 4.4-1]) and the ellipticity of $A^{ijkl}(\varepsilon)$. Finally,

$$|||\varphi|||_{g(\varepsilon),\Omega} := \int_\Omega \partial_\alpha\varphi g^{\alpha\beta}(\varepsilon)\partial_\beta\varphi\sqrt{g(\varepsilon)}dx,$$

which is a seminorm in $S(\Omega)$. Because of the uniform ellipticity of the tensors and matrices involved, and the properties of $g(\varepsilon)$, we are going to be able to use constants independent of ε in the estimates below. Indeed, going back to (85), we obtain

$$\begin{aligned}&\frac{\rho}{2}\frac{d}{dt}\{|\dot{\boldsymbol{u}}(\varepsilon)|^2_{g(\varepsilon),\Omega}\} + \frac{1}{2}\frac{d}{dt}\{\|\boldsymbol{u}(\varepsilon)\|^2_{A(\varepsilon),\Omega}\} + \frac{\beta}{2}\frac{d}{dt}\{|\vartheta(\varepsilon)|^2_{0,\Omega}\} + k|||\vartheta(\varepsilon)|||^2_{g(\varepsilon),\Omega} + \frac{k}{\varepsilon^2}|\partial_3\vartheta(\varepsilon)|^2_{0,\Omega}\\ &= \int_{\Gamma_C} p(-\dot{u}_3(\varepsilon))\dot{u}_3(\varepsilon)\sqrt{g(\varepsilon)}d\Gamma + \int_\Omega f^i\dot{u}_i(\varepsilon)\sqrt{g(\varepsilon)}dx + \int_{\Gamma_+} h^i\dot{u}_i(\varepsilon)\sqrt{g(\varepsilon)}d\Gamma + \int_\Omega q\vartheta(\varepsilon)\sqrt{g(\varepsilon)}dx.\end{aligned}$$

Integrating in $[0,t]$ with respect to the time variable, using the equivalences mentioned above, together with the uniformity with respect to ε of the constants involved in those equivalences, integrating by parts the term with the tractions h^i, using Theorem 3 and Young's inequality, we find that there exist a constant $C>0$ independent of ε such that

$$\begin{aligned}&|\dot{\boldsymbol{u}}(\varepsilon)(t)|^2_{0,\Omega} + |e_{i||j}(\varepsilon)(t)|^2_{0,\Omega} + |\vartheta(\varepsilon)(t)|^2_{0,\Omega} + \int_0^t (|\partial_\alpha\vartheta(\varepsilon)|^2_{0,\Omega} + \frac{1}{\varepsilon^2}|\partial_3\vartheta(\varepsilon)(r)|^2_{0,\Omega})dr\\ &- \int_0^t\int_{\Gamma_C} p(-\dot{u}_3(\varepsilon)(r))\dot{u}_3(\varepsilon)(r)\sqrt{g(\varepsilon)}d\Gamma\, dr \le C(\int_0^t |\dot{\boldsymbol{u}}(\varepsilon)(r)|^2_{0,\Omega}dr + \int_0^t |\vartheta(\varepsilon)(r)|^2_{0,\Omega}dr + \int_0^t |e_{i||j}(\varepsilon)(r)|^2_{0,\Omega}dr\\ &+ \int_0^t |\boldsymbol{f}(r)|^2_{0,\Omega}dr + \int_0^t |q(r)|^2_{0,\Omega}dr + \int_0^t |\dot{\boldsymbol{h}}(r)|^2_{0,\Gamma_+}dr + |\dot{\boldsymbol{h}}(t)|^2_{0,\Gamma_+})\end{aligned}$$

Hence, by using Gronwall's inequality and the three-dimensional Korn's inequality that can be found in [3, Theorem 4.4-1], all the assertions of (i) follow.

(ii) The limits of the scaled unknowns, u_i, ϑ found in Step (i) are independent of x_3.

The part corresponding to u_i is analogous to the Step (ii) in [3, Theorem 4.4-1], so we omit it. Regarding ϑ, its independence on x_3 is a consequence of the boundedness of $\{\varepsilon^{-1}\partial_3\vartheta(\varepsilon)\}$.

(iii) Extracting weakly convergence subsequences on the contact boundary. The norms $|u_3(\varepsilon)|_{0,\Gamma_C}$, $|\dot{u}_3(\varepsilon)|_{0,\Gamma_C}$ are bounded independently of $\varepsilon, 0<\varepsilon\le\varepsilon_1$ almost everywhere in $(0,T)$. Moreover, there exist subsequences, also denoted $(u_3(\varepsilon))_{\varepsilon>0}$ and $(\dot{u}_3(\varepsilon))_{\varepsilon>0}$ such that $u_3(\varepsilon)\overset{*}{\rightharpoonup} u_3$ and $\dot{u}_3(\varepsilon)\overset{*}{\rightharpoonup}\dot{u}_3$ in $L^\infty(0,T;L^2(\Gamma_C))$.

The first part is an straightforward consequence of Step (i) and (64). For $v=u_3(\varepsilon)$ we find that

$$|u_3(\varepsilon)|_{0,\Gamma_C} \le C|e_{i||j}(\varepsilon)|_{0,\Omega} \quad a.e. \text{ in } (0,T).$$

Then, there exists $\psi \in L^\infty(0,T;L^2(\Gamma_C))$ such that for a subsequence keeping the same notation, it holds $u_3(\varepsilon) \stackrel{*}{\rightharpoonup} \psi$ in $L^\infty(0,T;L^2(\Gamma_C))$. Since we are in the conditions of [11, Theorem 3.6], we can identify $\psi = u_3$. For the second part, we first recall that $\dot{\boldsymbol{u}}(\varepsilon) \in V(\Omega)$ and $\dot{\vartheta}(\varepsilon) \in S(\Omega)$ (see Remark 7). Next, we use the technique of incremental coefficients in the time variable, then integrate on $[0,t]$ to obtain the expression similar to (30) in the scaled framework and without tractions. Indeed,

$$\frac{1}{2}\rho\,|\ddot{\boldsymbol{u}}(\varepsilon)(t)|_0^2 - \frac{1}{2}\rho\,|\ddot{\boldsymbol{u}}(\varepsilon)(0)|_0^2 + \frac{1}{2}a^V(\dot{\boldsymbol{u}}(\varepsilon)(t),\dot{\boldsymbol{u}}(\varepsilon)(t)) - \frac{1}{2}a^V(\dot{\boldsymbol{u}}(\varepsilon)(0),\dot{\boldsymbol{u}}(\varepsilon)(0))$$
$$+\frac{1}{2}\int_\Omega \beta(\dot{\vartheta}(\varepsilon)(t))^2dx - \frac{1}{2}\int_\Omega \beta(\dot{\vartheta}(\varepsilon)(0))^2dx + \int_0^t a^S(\dot{\vartheta}(\varepsilon)(r),\dot{\vartheta}(\varepsilon)(r))dr$$
$$\le \int_0^t\int_\Omega \dot{f}^i(r)\ddot{u}(\varepsilon)_i(r)\,dxdr + \int_0^t\int_\Omega \dot{q}(r)\dot{\vartheta}(\varepsilon)(r)\,dxdr. \tag{86}$$

Then, we use Korn's inequality on the left-hand side and apply Gronwall's inequality to obtain that $|e_{i||j}(\dot{\boldsymbol{u}})(\varepsilon)|_{0,\Omega}^2$ is bounded independently of ε. Then can proceed like in the first part using (64) for $v = \dot{u}_3(\varepsilon)$ to show that $\dot{u}_3(\varepsilon) \stackrel{*}{\rightharpoonup} \dot{u}_3$ in $L^\infty(0,T;L^2(\Gamma_C))$.

(iv) *The limits $e_{i||j}$ found in (i) are independent of the variable x_3. Moreover, they are related with the limits $\boldsymbol{u} := (u_i)$ and ϑ by*

$$e_{\alpha||\beta} = \gamma_{\alpha\beta}(\boldsymbol{u}) := \frac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha) - \Gamma^\sigma_{\alpha\beta}u_\sigma - b_{\alpha\beta}u_3,$$
$$e_{\alpha||3} = 0, \tag{87}$$
$$e_{3||3} = \frac{\alpha_T(3\lambda+2\mu)}{\lambda+2\mu}\vartheta - \frac{\lambda}{\lambda+2\mu}a^{\alpha\beta}e_{\alpha||\beta}. \tag{88}$$

Indeed, first considering $\boldsymbol{v} = \boldsymbol{u}(\varepsilon)$ in (52) and $\boldsymbol{\eta} = \boldsymbol{u}$ in (68) (*par abus de langage*, since $\boldsymbol{u}$ is independent of x_3, but actually $\boldsymbol{u} \in [H^1(\Omega)]^2 \times L^2(\Omega)$), taking into account Step (*i*) and the convergences $\Gamma^\sigma_{\alpha\beta}(\varepsilon) \to \Gamma^\sigma_{\alpha\beta}$ and $\Gamma^3_{\alpha\beta}(\varepsilon) \to b_{\alpha\beta}$ in $\mathcal{C}^0(\bar{\Omega})$ given by (58)–(60), we have that

$$e_{\alpha||\beta}(\varepsilon) = \frac{1}{2}(\partial_\beta u_\alpha(\varepsilon) + \partial_\alpha u_\beta(\varepsilon)) - \Gamma^p_{\alpha\beta}(\varepsilon)u_p(\varepsilon) \rightharpoonup e_{\alpha||\beta} = \gamma_{\alpha\beta}(\boldsymbol{u}) \text{ in } L^2(\Omega) \text{ } a.e. \text{ in } (0,T).$$

Moreover, $e_{\alpha||\beta}$ are independent of x_3, as a straightforward consequence of the independence on x_3 of u_i (Step (*ii*)). In addition, let $\boldsymbol{v} \in V(\Omega)$. As a consequence of the definition of the scaled strains in (52)–(54), we find

$$\varepsilon e_{\alpha||\beta}(\varepsilon;\boldsymbol{v}) \to 0 \text{ in } L^2(\Omega), \quad \varepsilon e_{\alpha||3}(\varepsilon;\boldsymbol{v}) \to \frac{1}{2}\partial_3 v_\alpha \text{ in } L^2(\Omega),$$
$$\varepsilon e_{3||3}(\varepsilon;\boldsymbol{v}) = \partial_3 v_3 \text{ in } L^2(\Omega), \text{ for all } \varepsilon > 0.$$

Now, for all $\boldsymbol{v} \in V(\Omega)$, in (65) we can take as test function $\varepsilon \boldsymbol{v} \in V(\Omega)$. Then, taking into account (50), we have

$$\varepsilon \int_\Omega \rho(\ddot{u}_\alpha(\varepsilon) g^{\alpha\beta}(\varepsilon) v_\beta + \ddot{u}_3(\varepsilon) v_3)\sqrt{g(\varepsilon)}\, dx + \varepsilon \int_\Omega A^{ijkl}(\varepsilon) e_{k\|l}(\varepsilon) e_{i\|j}(\varepsilon;\boldsymbol{v})\sqrt{g(\varepsilon)} dx$$
$$- \int_\Omega \alpha_T(3\lambda+2\mu)\vartheta(\varepsilon)(\varepsilon e_{\alpha\|\beta}(\varepsilon;\boldsymbol{v}) g^{\alpha\beta}(\varepsilon) + \varepsilon e_{3\|3}(\varepsilon;\boldsymbol{v}))\sqrt{g(\varepsilon)} dx - \varepsilon \int_{\Gamma_C} p(-\dot{u}_3(\varepsilon)) v_3 \sqrt{g(\varepsilon)} d\Gamma$$
$$= \varepsilon \int_\Omega f^i v_i \sqrt{g(\varepsilon)} dx.$$

Passing to the limit as $\varepsilon \to 0$, decomposing $A^{ijkl}(\varepsilon)$ into the components with different asymptotic behaviour (see (55)–(56)), the properties of $g(\varepsilon)$ (see (61)) and the convergences in Step (i), we obtain the following equality:

$$\int_\Omega \left(2\mu a^{\alpha\sigma} e_{\alpha\|3}\partial_3 v_\sigma + (\lambda+2\mu) e_{3\|3}\partial_3 v_3\right)\sqrt{a} dx + \int_\Omega \lambda a^{\alpha\beta} e_{\alpha\|\beta}\partial_3 v_3 \sqrt{a} dx$$
$$= \int_\Omega \alpha_T(3\lambda+2\mu)\vartheta\partial_3 v_3\sqrt{a} dx \quad \forall \boldsymbol{v} \in V(\Omega), \ \textit{a.e.} \text{ in } (0,T). \tag{89}$$

By taking particular test functions and using Theorem 2, we deduce (87). Then, we go back to (89) and use again Theorem 2 to deduce (88). The independence of $e_{3\|3}$ on x_3 is a consequence of this relation, as well.

(v) We find a limit two-dimensional problem verified by functions $\bar{\boldsymbol{u}} = (\bar{\boldsymbol{u}}_i)$ and ϑ. In particular, since the solution of this problem is unique, the convergences on Step (i) are verified for the whole families $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$ and $(\vartheta(\varepsilon))_{\varepsilon>0}$. We have that $\bar{\boldsymbol{u}}(t) = (\bar{u}_i(t)) \in V_M(\omega)$ and $\vartheta(t) \in S(\Omega)$ a.e. in (0,T).

By using [3, Theorem 4.2-1] (parts (a) and (b)), and Step (*ii*) we find that $\bar{u}_\alpha \in H_0^1(\omega)$ and $\bar{\vartheta} \in H_0^1(\omega)$. Therefore, $\bar{\boldsymbol{u}} \in V_M(\omega)$ a.e. in (0,T). Now, let $\boldsymbol{v} = (v_i) \in V(\Omega)$ be independent of the variable x_3. Then, the asymptotic behaviour of the functions $\Gamma^p_{\alpha\beta}(\varepsilon)$ and $\Gamma^\sigma_{\alpha 3}(\varepsilon)$ (see (58)–(60)) implies the following convergences when $\varepsilon \to 0$ (see (52)–(54)):

$$e_{\alpha\|\beta}(\varepsilon;\boldsymbol{v}) \to \gamma_{\alpha\beta}(\boldsymbol{v}) := \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma^\sigma_{\alpha\beta} v_\sigma - b_{\alpha\beta} v_3 \text{ in } L^2(\Omega), \tag{90}$$

$$e_{\alpha\|3}(\varepsilon;\boldsymbol{v}) \to \frac{1}{2}\partial_\alpha v_3 + b^\sigma_\alpha v_\sigma \text{ in } L^2(\Omega), \quad e_{3\|3}(\varepsilon;\boldsymbol{v}) = 0. \tag{91}$$

Having this in mind, let now $\boldsymbol{v} = (v_i) \in V(\Omega)$ be independent of x_3 in (65) and take the limit when $\varepsilon \to 0$. In the process, we make use of the asymptotic behaviour of $A^{ijkl}(\varepsilon)$ (see (55)–(56)) and $g(\varepsilon)$ (see (61)), take into account the weak convergences $e_{i\|j}(\varepsilon) \stackrel{*}{\rightharpoonup} e_{i\|j}$ in $L^\infty(0,T;L^2(\Omega))$, simplify by using (87) and consider the precise limits of the functions $e_{i||j}(\varepsilon;\boldsymbol{v})$ in (90)–(91). As a result, we obtain the equality

$$\begin{aligned}
&\int_\Omega \rho(\ddot{u}_\alpha a^{\alpha\beta} v_\beta + \ddot{u}_3 v_3)\sqrt{a}dx + \int_\Omega \left(\lambda a^{\alpha\beta}a^{\sigma\tau} + \mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma})\right) e_{\sigma||\tau}\gamma_{\alpha\beta}(\boldsymbol{v})\sqrt{a}dx \\
&\quad + \int_\Omega \lambda a^{\alpha\beta} e_{3||3}\gamma_{\alpha\beta}(\boldsymbol{v})\sqrt{a}dx - \int_{\Gamma_C} \chi v_3\sqrt{a}d\Gamma - \int_\Omega \alpha_T(3\lambda+2\mu)\vartheta a^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{v})\sqrt{a}dx \\
&= \int_\Omega f^i v_i\sqrt{a}dx + \int_{\Gamma_+} h^i v_i \sqrt{a}d\Gamma, \ \textit{a.e.} \text{ in } (0,T),
\end{aligned} \tag{92}$$

where we also used Step (*iii*) and (12) to find that there exists $\chi \in L^\infty(0,T;L^2(\Gamma_C))$ such that $p(-\dot{u}_3(\varepsilon)) \overset{*}{\rightharpoonup} \chi$. Using (88) and since $\boldsymbol{u}$, $\boldsymbol{v}$ and ϑ are all independent of x_3 (see Step (*ii*)), we can identify them with their averages and we obtain from (92) that

$$\begin{aligned}
&2\int_\omega \rho(\ddot{\bar{u}}_\alpha a^{\alpha\beta}\bar{v}_\beta + \ddot{\bar{u}}_3\bar{v}_3)\sqrt{a}dy + \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\bar{\boldsymbol{u}})\gamma_{\alpha\beta}(\bar{\boldsymbol{v}})\sqrt{a}dy - 4\int_\omega \frac{\alpha_T\mu(3\lambda+2\mu)}{\lambda+2\mu}\bar{\vartheta}a^{\alpha\beta}\gamma_{\alpha\beta}(\bar{\boldsymbol{v}})\sqrt{a}dy \\
&\quad - \int_{\Gamma_C}\chi\bar{v}_3\sqrt{a}d\Gamma = \int_\omega\left(\int_{-1}^1 f^i dx_3\right)\bar{v}_i\sqrt{a}dy + \int_{\Gamma_+} h^i\bar{v}_i\sqrt{a}d\Gamma, \ \textit{a.e.} \text{ in } (0,T),
\end{aligned} \tag{93}$$

where $a^{\alpha\beta\sigma\tau}$ denotes the contravariant components of the fourth order two-dimensional tensor defined in (69). Now, given $\boldsymbol{\eta} = (\eta_i) \in [H_0^1(\omega)]^3$, we can define $\boldsymbol{v} = (v_i)$ such that $\boldsymbol{v}(\boldsymbol{y},x_3) = \boldsymbol{\eta}(\boldsymbol{y})$ for all $(\boldsymbol{y},x_3) \in \Omega$. Then $\boldsymbol{v} \in V(\Omega)$ and it is independent of x_3; hence, as a consequence of [3, Theorem 4.2-1], the variational problems above are satisfied for $\bar{\boldsymbol{v}} = \boldsymbol{\eta}$. Since both sides of the equation above are continuous linear forms with respect to $\bar{v}_3 = \eta_3 \in L^2(\omega)$ for any given $\bar{v}_\alpha \in H_0^1(\omega)$, these expressions are valid for all $\boldsymbol{\eta} = (\eta_i) \in V_M(\omega)$, since $H_0^1(\omega)$ is dense in $L^2(\omega)$.
Similarly, let $\varphi \in S(\Omega)$ be independent of x_3 in (66) and take the limit when $\varepsilon \to 0$. We take into account the weak convergences in Step (i), simplify by using the time derivative of (88). As a result, we obtain the equality

$$\begin{aligned}
&2\int_\omega \left(\beta + \frac{\alpha_T^2(3\lambda+2\mu)^2}{\lambda+2\mu}\right)\dot{\bar{\vartheta}}\varphi\sqrt{a}dy + 2\int_\omega k\partial_\alpha\bar{\vartheta}a^{\alpha\sigma}a^{\beta\sigma}\partial_\beta\varphi\sqrt{a}dy \\
&\quad + 4\int_\omega \frac{\alpha_T\mu(3\lambda+2\mu)}{\lambda+2\mu}\varphi a^{\alpha\beta}\gamma_{\alpha\beta}(\dot{\bar{\boldsymbol{u}}})\sqrt{a}dy = \int_\omega Q\varphi\sqrt{a}dy \quad \forall\varphi\in H_0^1(\omega) \ \textit{a.e.} \text{ in } (0,T),
\end{aligned} \tag{94}$$

hence obtaining (73), with ζ identified with $\bar{\vartheta}$.

(vi) The weak convergences are, in fact, strong.

For this step we first consider a case without tractions, that is, we take $\boldsymbol{h} = \boldsymbol{0}$. Then we will show the changes to be made for the case with tractions. In both cases we are using a monotonicity argument. We define the quantity:

$$
\begin{aligned}
\Lambda(\varepsilon) := &\int_\Omega \rho\left((\ddot{u}_\alpha(\varepsilon)-\ddot{u}_\alpha)g^{\alpha\beta}(\varepsilon)(\dot{u}_\beta(\varepsilon)-\dot{u}_\beta)+(\ddot{u}_3(\varepsilon)-\ddot{u}_3)(\dot{u}_3(\varepsilon)-\dot{u}_3)\right)\sqrt{g(\varepsilon)}\,dx\\
&+\int_\Omega A^{ijkl}(\varepsilon)(e_{k||l}(\varepsilon)-e_{k||l})(\dot{e}_{i||j}(\varepsilon)-\dot{e}_{i||j})\sqrt{g(\varepsilon)}dx\\
&-\int_{\Gamma_C}(p(-\dot{u}_3(\varepsilon))-p(-\dot{u}_3))(\dot{u}_3(\varepsilon)-\dot{u}_3)\sqrt{g(\varepsilon)}d\Gamma\\
&+\int_\Omega \beta(\dot{\vartheta}(\varepsilon)-\dot{\vartheta})(\vartheta(\varepsilon)-\vartheta)\sqrt{g(\varepsilon)}dx\\
&+\int_\Omega k\{\partial_\alpha(\vartheta(\varepsilon)-\vartheta)g^{\alpha\beta}(\varepsilon)\partial_\beta(\vartheta(\varepsilon)-\vartheta)+\frac{1}{\varepsilon^2}(\partial_3(\vartheta(\varepsilon)-\vartheta))^2\}\sqrt{g(\varepsilon)}dx.
\end{aligned}
$$

On one hand, we integrate with respect to the time variable in $[0,t]$ and take into account (84) and the initial conditions in Problem 6 to obtain

$$
\begin{aligned}
2\int_0^t \Lambda(\varepsilon)dr = &\int_\Omega \rho\left((\dot{u}_\alpha(\varepsilon)-\dot{u}_\alpha)g^{\alpha\beta}(\varepsilon)(\dot{u}_\beta(\varepsilon)-\dot{u}_\beta)+(\dot{u}_3(\varepsilon)-\dot{u}_3)^2\right)\sqrt{g(\varepsilon)}\,dx\\
&+\int_\Omega A^{ijkl}(\varepsilon)(e_{k||l}(\varepsilon)-e_{k||l})(e_{i||j}(\varepsilon)-e_{i||j})\sqrt{g(\varepsilon)}dx\\
&+2\int_0^t\int_{\Gamma_C}(p(-\dot{u}_3(\varepsilon))-p(-\dot{u}_3))(-\dot{u}_3(\varepsilon)+\dot{u}_3)\sqrt{g(\varepsilon)}d\Gamma\,dr+\int_\Omega \beta(\vartheta(\varepsilon)-\vartheta)^2\sqrt{g(\varepsilon)}dx\\
&+2\int_0^t\int_\Omega k\{\partial_\alpha(\vartheta(\varepsilon)-\vartheta)g^{\alpha\beta}(\varepsilon)\partial_\beta(\vartheta(\varepsilon)-\vartheta)+\frac{1}{\varepsilon^2}(\partial_3(\vartheta(\varepsilon)-\vartheta))^2\}\sqrt{g(\varepsilon)}dxdr,
\end{aligned}
\tag{95}
$$

and as consequence of the monotonicity of p, (57) and (61), we find

$$
\begin{aligned}
\int_0^t \Lambda(\varepsilon)ds \geq\, &C(|\dot{\boldsymbol{u}}(\varepsilon)-\dot{\boldsymbol{u}}|^2_{0,\Omega}+|e_{i||j}(\varepsilon)-e_{i||j}|^2_{0,\Omega}+|\vartheta(\varepsilon)-\vartheta|^2_{0,\Omega}\\
&+\int_0^t|\partial_\alpha\vartheta(\varepsilon)-\partial_\alpha\vartheta|^2_{0,\Omega}ds+\frac{1}{\varepsilon^2}\int_0^t|\partial_3\vartheta(\varepsilon)-\partial_3\vartheta|^2_{0,\Omega}ds.
\end{aligned}
\tag{96}
$$

On the other hand, from the expression of $\Lambda(\varepsilon)$ and making use of (65)–(66) for $\boldsymbol{v} = \dot{\boldsymbol{u}}(\varepsilon)$ and $\varphi = \vartheta(\varepsilon)$, we deduce that

$$\begin{aligned}
\Lambda(\varepsilon) &= \int_\Omega f^i \dot{u}_i(\varepsilon)\sqrt{g(\varepsilon)}dx - \frac{d}{dt}\int_\Omega A^{ijkl}(\varepsilon)e_{k||l}(\varepsilon)e_{i||j}\sqrt{g(\varepsilon)}dx + \int_\Omega A^{ijkl}(\varepsilon)e_{k||l}\dot{e}_{i||j}\sqrt{g(\varepsilon)}dx \\
&- \frac{d}{dt}\int_\Omega \rho\dot{u}_\alpha(\varepsilon)g^{\alpha\beta}(\varepsilon)\dot{u}_\beta\sqrt{g(\varepsilon)}\,dx + \int_\Omega \rho\ddot{u}_\alpha g^{\alpha\beta}(\varepsilon)\dot{u}_\beta\sqrt{g(\varepsilon)}\,dx \\
&- \frac{d}{dt}\int_\Omega \rho\dot{u}_3(\varepsilon)\dot{u}_3\sqrt{g(\varepsilon)}\,dx + \int_\Omega \rho\ddot{u}_3\dot{u}_3\sqrt{g(\varepsilon)}\,dx \\
&+ \int_{\Gamma_C} p(-\dot{u}_3)(\dot{u}_3(\varepsilon)-\dot{u}_3)\sqrt{g(\varepsilon)}d\Gamma + \int_{\Gamma_C} p(-\dot{u}_3(\varepsilon))\dot{u}_3\sqrt{g(\varepsilon)}d\Gamma \\
&+ \int_\Omega q\vartheta(\varepsilon)\sqrt{g(\varepsilon)}dx - \frac{d}{dt}\int_\Omega \beta\vartheta(\varepsilon)\vartheta\sqrt{g(\varepsilon)}dx + \int_\Omega \beta\dot{\vartheta}\vartheta\sqrt{g(\varepsilon)}dx \\
&- \int_\Omega k\partial_\alpha\vartheta g^{\alpha\beta}(\varepsilon)\partial_\beta(\vartheta(\varepsilon)-\vartheta)\sqrt{g(\varepsilon)}dx - \int_\Omega k\partial_\alpha\vartheta(\varepsilon)g^{\alpha\beta}(\varepsilon)\partial_\beta\vartheta\sqrt{g(\varepsilon)}dx \\
&- \frac{1}{\varepsilon^2}\int_\Omega k\partial_3\vartheta\partial_3(\vartheta(\varepsilon)-\vartheta)\sqrt{g(\varepsilon)}dx - \frac{1}{\varepsilon^2}\int_\Omega k\partial_3\vartheta(\varepsilon)\partial_3\vartheta\sqrt{g(\varepsilon)}dx.
\end{aligned} \tag{97}$$

Integrating with respect to the time variable in $[0,t]$ and taking into account the initial conditions given in Problem 6 and (84) we obtain

$$\begin{aligned}
\int_0^t \Lambda(\varepsilon)dr &= \int_0^t\int_\Omega f^i\dot{u}_i(\varepsilon)\sqrt{g(\varepsilon)}dxdr - \int_\Omega A^{ijkl}(\varepsilon)e_{k||l}(\varepsilon)e_{i||j}\sqrt{g(\varepsilon)}dx + \int_0^t\int_\Omega A^{ijkl}(\varepsilon)e_{k||l}\dot{e}_{i||j}\sqrt{g(\varepsilon)}dxdr \\
&- \int_\Omega \rho\dot{u}_\alpha(\varepsilon)g^{\alpha\beta}(\varepsilon)\dot{u}_\beta\sqrt{g(\varepsilon)}\,dx + \int_0^t\int_\Omega \rho\ddot{u}_\alpha g^{\alpha\beta}(\varepsilon)\dot{u}_\beta\sqrt{g(\varepsilon)}\,dxdr \\
&- \int_\Omega \rho\dot{u}_3(\varepsilon)\dot{u}_3\sqrt{g(\varepsilon)}\,dx + \int_0^t\int_\Omega \rho\ddot{u}_3\dot{u}_3\sqrt{g(\varepsilon)}\,dxdr \\
&+ \int_0^t\int_{\Gamma_C} p(-\dot{u}_3)(\dot{u}_3(\varepsilon)-\dot{u}_3)\sqrt{g(\varepsilon)}d\Gamma dr + \int_0^t\int_{\Gamma_C} p(-\dot{u}_3(\varepsilon))\dot{u}_3\sqrt{g(\varepsilon)}d\Gamma dr \\
&+ \int_0^t\int_\Omega q\vartheta(\varepsilon)\sqrt{g(\varepsilon)}dxdr - \int_\Omega \beta\vartheta(\varepsilon)\vartheta\sqrt{g(\varepsilon)}dx + \int_0^t\int_\Omega \beta\dot{\vartheta}\vartheta\sqrt{g(\varepsilon)}dxdr \\
&- \int_0^t\int_\Omega k\partial_\alpha\vartheta g^{\alpha\beta}(\varepsilon)\partial_\beta(\vartheta(\varepsilon)-\vartheta)\sqrt{g(\varepsilon)}dxdr - \int_0^t\int_\Omega k\partial_\alpha\vartheta(\varepsilon)g^{\alpha\beta}(\varepsilon)\partial_\beta\vartheta\sqrt{g(\varepsilon)}dxdr \\
&- \frac{1}{\varepsilon^2}\int_0^t\int_\Omega k\partial_3\vartheta\partial_3(\vartheta(\varepsilon)-\vartheta)\sqrt{g(\varepsilon)}dxdr - \frac{1}{\varepsilon^2}\int_0^t\int_\Omega k\partial_3\vartheta(\varepsilon)\partial_3\vartheta\sqrt{g(\varepsilon)}dxdr.
\end{aligned}$$

Take into account that $\partial_3\vartheta = 0$, and let $\varepsilon \to 0$. Then, because of the weak convergences studied in steps (i), (iii) and (v), the asymptotic behaviour of the functions $A^{ijkl}(\varepsilon)$ and $g(\varepsilon)$ (see (55)–(56) and (61)) and by using the Lebesgue dominated convergence theorem, we find that

$$\lim_{\varepsilon\to 0}\int_0^t \Lambda(\varepsilon)dr = \int_0^t\int_\Omega f^i\dot{u}_i\sqrt{a}dxdr - \int_0^t\int_\Omega \rho\ddot{u}_\alpha a^{\alpha\beta}\dot{u}_\beta\sqrt{a}dxdr - \int_0^t\int_\Omega \rho\ddot{u}_3\dot{u}_3\sqrt{a}dxdr$$
$$- \int_0^t\int_\Omega A^{ijkl}(0)e_{k||l}\dot{e}_{i||j}\sqrt{a}dxdr + \int_0^t\int_{\Gamma_C}\chi\dot{u}_3\sqrt{a}d\Gamma dr + \int_0^t\int_\Omega q\vartheta\sqrt{a}dxdr$$
$$- \int_0^t\int_\Omega \beta\dot{\vartheta}\vartheta\sqrt{a}dxdr - \int_0^t\int_\Omega k\partial_\alpha\vartheta a^{\alpha\beta}\partial_\beta\vartheta\sqrt{a}dxdr. \qquad (98)$$

Moreover, by the expressions of $A^{ijkl}(0)$ (see (55)–(56)) and using (87) we have

$$\int_\Omega A^{ijkl}(0)e_{k||l}\dot{e}_{i||j}\sqrt{a}dx = \int_\Omega \left(\lambda a^{\alpha\beta}a^{\sigma\tau} + \mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma})\right) e_{\sigma||\tau}\dot{e}_{\alpha||\beta}\sqrt{a}dx$$
$$+ \int_\Omega \lambda a^{\alpha\beta}e_{3||3}\dot{e}_{\alpha||\beta}\sqrt{a}dx + \int_\Omega \left(\lambda a^{\sigma\tau}e_{\sigma||\tau} + (\lambda+2\mu)e_{3||3}\right)\dot{e}_{3||3}\sqrt{a}dx.$$

Then, using (88), we find that (98) is actually null, since its expression above coincides with the result of adding (93) for $\bar{\boldsymbol{v}} = \dot{\boldsymbol{u}}$ to (94) for $\varphi = \vartheta$ (both integrated in $[0,t]$). Indeed,

$$\lim_{\varepsilon\to 0}\int_0^t \Lambda(\varepsilon)dr = \int_0^t \Big(\int_\Omega f^i\dot{u}_i\sqrt{a}dx - \int_\Omega \rho\ddot{u}_\alpha a^{\alpha\beta}\dot{u}_\beta\sqrt{a}dx - \int_\Omega \rho\ddot{u}_3\dot{u}_3\sqrt{a}dx - \frac{1}{2}\int_\Omega a^{\alpha\beta\sigma\tau}e_{\sigma||\tau}\dot{e}_{\alpha||\beta}\sqrt{a}dx$$
$$+ \int_{\Gamma_C}\chi\dot{u}_3\sqrt{a}d\Gamma + \int_\Omega q\vartheta\sqrt{a}dx - \int_\Omega \left(\beta + \frac{\alpha_T^2(3\lambda+2\mu)^2}{\lambda+2\mu}\right)\dot{\vartheta}\vartheta\sqrt{a}dx - \int_\Omega k\partial_\alpha\vartheta a^{\alpha\beta}\partial_\beta\vartheta\sqrt{a}dx\Big)dr = 0. \qquad (99)$$

Now, for the case where tractions are not null, in (97) we have an additonal term

$$\int_{\Gamma_+} h^i\dot{u}_i(\varepsilon)\sqrt{g(\varepsilon)}d\Gamma.$$

We integrate (97) in $[0,t]$ and integrate by parts the terms with tractions corresponding to the first two components, which can be displayed as

$$-\int_0^t\int_{\Gamma_+}\dot{h}^\alpha(r)u_\alpha(\varepsilon)(r)\sqrt{g(\varepsilon)}d\Gamma\, dr + \int_{\Gamma_+} h^\alpha(t)u_\alpha(\varepsilon)(t)\sqrt{g(\varepsilon)}d\Gamma + \int_0^t\int_{\Gamma_+} h^3(r)\dot{u}_3(\varepsilon)(r)\sqrt{g(\varepsilon)}d\Gamma\, dr.$$

When passing to the limit $\varepsilon \to 0$, the terms with $u_\alpha(\varepsilon)$ above converge by using compactness arguments, since $u_\alpha(\varepsilon) \in H^1(\Omega\times(0,T))$ and the trace into $L^2(\Gamma\times(0,T))$ is a compact operator (see [8, p. 416]). For the term with $\dot{u}_3(\varepsilon)$, we omit the details for the sake of briefness, refer the interested reader to [1] and provide the following sketch the proof. We proceed like in the second part of Step (*iii*), with the difference that now there are tractions on the right-hand side. Indeed, instead of (86) we obtain

$$\frac{1}{2}\rho\,|\ddot{\boldsymbol{u}}(\varepsilon)(t)|_0^2-\frac{1}{2}\rho\,|\ddot{\boldsymbol{u}}(\varepsilon)(0)|_0^2+\frac{1}{2}a^V(\dot{\boldsymbol{u}}(\varepsilon)(t),\dot{\boldsymbol{u}}(\varepsilon)(t))-\frac{1}{2}a^V(\dot{\boldsymbol{u}}(\varepsilon)(0),\dot{\boldsymbol{u}}(\varepsilon)(0))$$
$$+\frac{1}{2}\int_\Omega\beta(\dot\vartheta(\varepsilon)(t))^2dx-\frac{1}{2}\int_\Omega\beta(\dot\vartheta(\varepsilon)(0))^2dx+\int_0^t a^S(\dot\vartheta(\varepsilon)(r),\dot\vartheta(\varepsilon)(r))dr$$
$$\le\int_0^t\int_\Omega\dot f^i(r)\ddot u(\varepsilon)_i(r)\,dxdr+\int_0^t\int_{\Gamma_+}\dot h^i(r)\ddot u(\varepsilon)_i(r)\,d\Gamma dr+\int_0^t\int_\Omega\dot q(r)\dot\vartheta(\varepsilon)(r)\,dxdr.$$

On the right-hand side we integrate by parts the term on the boundary Γ_+, use Theorem 3 combined with Young's inequality and use Korn's inequality on the left-hand side. Next, apply Gronwall's inequality to obtain that $|e_{i||j}(\dot{\boldsymbol{u}})(\varepsilon)|_{0,\Omega}^2$ is bounded independently of ε. Then we reason like in Step (iii) with Γ_C replaced by Γ_+ and find that $\dot u_3(\varepsilon)\overset{*}{\rightharpoonup}\dot u_3$ in $L^\infty(0,T;L^2(\Gamma_+))$. Besides, we use Lebesgue Theorem where needed, as well. Thus, the limit of the terms with traction is

$$-\int_0^t\int_{\Gamma_+}\dot h^\alpha(r)u_\alpha(r)\sqrt{a}d\Gamma\,dr+\int_{\Gamma_+}h^\alpha(t)u_\alpha(t)\sqrt{a}d\Gamma+\int_0^t\int_{\Gamma_+}h^3(r)\dot u_3(r)\sqrt{a}d\Gamma\,dr.$$

We can undo the integration by parts, then reason like in (99).
The strong convergences $e_{i||j}(\varepsilon)\to e_{i||j}$ in $L^\infty(0,T;L^2(\Omega))$ also imply the strong convergences for $u_i(\varepsilon)$, by following arguments not depending on the particular set of equations, but on arguments of differential geometry and functional analysis which do not differ from those used in [3, Theorem 4.4-1]. Therefore, we just omit them and refer the interested reader to the book.
It only remains to show that $\chi=p(-\dot u_3)$. To do that we can reason like in Step (x) in [11, Theorem 5.3].

□

Remark 8 Notice that unlike what happens in the references [5,8], cited several times in this work, we cannot use compactness arguments for the convergence of all the contact boundary terms, since in our functional framework (that of linearly elliptic membrane shells) we do not have enough regularity to conclude that $u_3(\varepsilon)\in H^1(\Omega\times(0,T))$. Indeed, we have found no uniform upper bounds for $\partial_\alpha u_3(\varepsilon)$. Furthermore, the trace defined in Theorem 3 is not a compact operator.

6 Back to the physical framework

It remains to be proved an analogous result to the previous theorem but in terms of de-scaled unknowns. We shall present the limit problem in a de-scaled form. The scalings in Section 3 suggest the de-scalings $\xi_i^\varepsilon(\boldsymbol{y})=\xi_i(\boldsymbol{y})$ and $\zeta^\varepsilon(\boldsymbol{y})=\zeta(\boldsymbol{y})$ for all $\boldsymbol{y}\in\bar\omega$. This way, from Problem 8 we can derive

Problem 11 Find a pair $t \mapsto (\boldsymbol{\xi}^\varepsilon(\boldsymbol{y},t), \zeta^\varepsilon(\boldsymbol{y},t))$ of $[0,T] \to V_M(\omega) \times H_0^1(\omega)$ verifying

$$2\varepsilon \int_\omega \rho(\ddot{\xi}_\alpha^\varepsilon a^{\alpha\beta}\eta_\beta + \ddot{\xi}_3^\varepsilon \eta_3)\sqrt{a}dy + \varepsilon \int_\omega a^{\alpha\beta\sigma\tau,\varepsilon}\gamma_{\sigma\tau}(\boldsymbol{\xi}^\varepsilon)\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy - \int_{\Gamma_C^\varepsilon} p^\varepsilon(-\dot{\xi}_3^\varepsilon)\eta_3^\varepsilon\sqrt{a}d\Gamma$$
$$- 4\varepsilon \int_\omega \frac{\alpha_T\mu(3\lambda+2\mu)}{\lambda+2\mu}\zeta^\varepsilon a^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy = \int_\omega F^{i,\varepsilon}\eta_i\sqrt{a}dy \;\forall\boldsymbol{\eta} = (\eta_i) \in V_M(\omega),$$
$$2\varepsilon \int_\omega \left(\beta + \frac{\alpha_T^2(3\lambda+2\mu)^2}{\lambda+2\mu}\right)\dot{\zeta}^\varepsilon\varphi\sqrt{a}dy + 2\varepsilon\int_\omega k\partial_\alpha^\varepsilon\zeta^\varepsilon a^{\alpha\beta}\partial_\beta^\varepsilon\varphi\sqrt{a}dy$$
$$+ 4\varepsilon \int_\omega \frac{\alpha_T\mu(3\lambda+2\mu)}{\lambda+2\mu}\varphi a^{\alpha\beta}\gamma_{\alpha\beta}(\dot{\boldsymbol{\xi}}^\varepsilon)\sqrt{a}dy = \int_\omega Q^\varepsilon\varphi\sqrt{a}dy \quad \forall\varphi \in H_0^1(\omega),$$

with $\dot{\boldsymbol{\xi}}^\varepsilon(\cdot,0) = \boldsymbol{\xi}^\varepsilon(\cdot,0) = \mathbf{0}$ and $\zeta^\varepsilon(\cdot,0) = 0$.

Above, we have used $F^{i,\varepsilon} := \int_{-\varepsilon}^{\varepsilon} f^{i,\varepsilon}dx_3^\varepsilon + h_+^{i,\varepsilon}$, with $h_+^{i,\varepsilon}(\cdot) = h^{i,\varepsilon}(\cdot,\varepsilon)$, and $Q^\varepsilon = \int_{-\varepsilon}^{\varepsilon} q^\varepsilon dx_3^\varepsilon$. Moreover, the convergences $u_\alpha(\varepsilon) \to u_\alpha$ in $H^1(\Omega)$ and $u_3(\varepsilon) \to u_3$ in $L^2(\Omega)$ from the Theorem 5 and [3, Theorem 4.2-1] together lead to the following convergences:

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_\alpha^\varepsilon dx_3^\varepsilon \to \xi_\alpha \text{ in } H^1(\Omega), \quad \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_3^\varepsilon dx_3^\varepsilon \to \xi_3 \text{ in } L^2(\Omega), \quad \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \zeta^\varepsilon dx_3^\varepsilon \to \zeta \text{ in } L^2(\Omega)\ a.e. \text{ in } (0,T).$$

Furthermore, we can prove the convergences of the averages of the tangential and normal components of the three-dimensional displacement vector field. To this end, we can use the same arguments as in [3, Theorem 4.6-1].

7 Conclusions and Outlook

We have found and mathematically justified a two-dimensional limit model for thermoelastic shells in contact with a deformable foundation, where the contact is modeled by using a normal damped response function, in the particular case of the so-called elliptic membranes. To this end we used the insight provided by the asymptotic expansion method and we have justified this approach by obtaining convergence theorems. We have also proved existence, uniqueness and regularity results for both three and two-dimensional problems by combining Faedo-Galerking techniques, monotonicity and compacity arguments.

Future work will be devoted to the study of alternative limit contact models, possibly thermoelastic flexural shells, which would be found under a different set of hypotheses for the order of the functions involved or the geometry of the middle surface. Further, we are interested in cases when contact is not frictionless, and further, models where it is coupled with other effects like wear, adhesion or damage.

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