ON MULTI-INDEX FILTRATIONS ASSOCIATED TO WEIERSTRASS SEMIGROUPS

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ABSTRACT. The aim of this paper is to review the main techniques in the computation of Weierstraß semigroup at several points of curves defined over perfect fields, with special emphasis on the case of two points. Some hints about the usage of some packages of the computer algebra software SINGULAR are also given.

1. INTRODUCTION

There are several classical problems in the theory of algebraic curves which are interesting from a computational point of view. One of them is the computation of the Weierstraß semigroup of a smooth projective algebraic curve $\widetilde{\chi}$ at a certain rational point P, together with a rational function $f_m \in \mathbb{F}(\widetilde{\chi})$ regular outside P and achieving a pole at P of order m, for each m in this semigroup. This problem is solved with the aid of the adjunction theory for plane curves, profusely developed by A. von Brill and M. Noether in the 19th century (see [3], [20]) so that we assume the knowledge of a singular plane birational model χ for the smooth curve $\widetilde{\chi}$.

Given a smooth projective algebraic curve χ (over a perfect field $\mathbb{F}$) and a set $P_1, \ldots, P_r$ of (rational) points of χ, we consider the family of finitely dimensional vector subspaces of $\mathbb{F}(\chi)$ given by $\mathscr{L}(\underline{m}P) = \mathscr{L}(m_1 P_1 + m_2 P_2 + \ldots + m_r P_r)$, where $\underline{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r$. This family gives rise to a $\mathbb{Z}^r$-multi-index filtration on the $\mathbb{F}$-algebra A of the affine curve $\chi \setminus \{P_1, \ldots, P_r\}$, since one has $A = \bigcup_{\underline{m} \in \mathbb{Z}^r} \mathscr{L}(\underline{m}P)$. This multifiltration is related to Weierstraß semigroups (with respect to several points in general, see Delgado [10]) and, in case of finite fields, to the methodology for trying to improve the Goppa estimation of the minimal distance of algebraic-geometrical codes, see for instance Carvalho and Torres [9]. A connection of that filtration with global geometrical-topological aspects in a particular case was shown by Campillo, Delgado and Gusein-Zade [6]. Poincaré series associated to this filtrations in particular cases were studied by the author in [19]

Thus, a natural question is to provide a computational method in order to estimate the values of $\dim_{\mathbb{F}} \mathscr{L}(\underline{m}P) = \ell(\underline{m}P)$ for $\underline{m} \in \mathbb{Z}^r$. More precisely, it would be convenient to estimate and compute values of type $\ell((\underline{m}+\underline{\varepsilon})\underline{P}) - \ell(\underline{m}P)$ where $\underline{\varepsilon} \in \mathbb{Z}^r$ is a vector whose components are 0 or 1. This can be done by extending the method developed by Campillo and Farrán [7] in the case $r = 1$, based on the knowledge of a plane model $\widetilde{\chi}$ for χ (with

singularities) and representing the global regular differentials in terms of adjoint curves to $\widetilde{\chi}$.

The paper is organised as follows: Sections 2 and 3 are devoted to fix the algebraic-geometrical prerequisites. Section 4 deals with the study of more specific questions concerning to our purpose, namely the adjunction theory of curves, with the remarkable Brill-Noether Theorem. In Section 5 we define the Weierstraß semigroup at several points and describe two methods to compute values of the form $\ell((\underline{m} + \underline{\varepsilon})\underline{P}) - \ell(\underline{mP})$. The last section is devoted to show and explain some procedures implemented in SINGULAR based on Section 5.

Notice the practical relevance of these ideas in view of the algebraic-geometric coding theory: the Weierstraß semigroup plays an important role in the decoding procedure of Feng and Rao, see e.g. Campillo and Farrán [8], or Høholdt, van Lint and Pellikaan [15].

2. TERMINOLOGY AND NOTATIONS

Let $\mathbb{F}$ be a perfect field, and let $\overline{\mathbb{F}}$ a fixed algebraic closure of $\mathbb{F}$. Let χ be an absolutely irreducible projective algebraic curve defined over $\mathbb{F}$. We distinguish three types of points on χ, namely the geometric points, i.e. those with coordinates on $\overline{\mathbb{F}}$; the rational points, i.e. those with coordinates on $\mathbb{F}$; and the closed points, which are residue classes of geometric points under the action of the Galois group of the field extension $\overline{\mathbb{F}}/\mathbb{F}$, namely

$$P := \{\sigma(p) : \sigma \in \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})\},$$

where p is a geometric point. Notice that closed points correspond one to one to points on the curve χ viewed as an $\mathbb{F}$-scheme which are closed for the Zariski topology. Every closed point has an associated residue field $\mathbb{F}'$ which is a finite extension of $\mathbb{F}$. The degree of a closed point P is defined as the cardinal of its conjugation class, which equals the degree of the extension $\mathbb{F}'/\mathbb{F}$. In particular, P is rational if and only if $\deg P = 1$.

Let us assume χ to be non-singular (or, equivalently, smooth, since $\mathbb{F}$ is perfect). Let $\overline{\mathbb{F}}(\chi)$ be the field of rational functions of χ. Let P be a closed point on χ. The local ring $\mathcal{O}_{\chi,P}$ of χ at P with maximal ideal $\mathfrak{m}_{\chi,P}$ is therefore a discrete valuation ring with associated discrete valuation v_P. An element $f \in \mathcal{O}_{\chi,P}$ is said to vanish at P (or to have a zero at P) if $f \in \mathfrak{m}_{\chi,P}$. A rational function f such that $f \notin \mathcal{O}_{\chi,P}$ is said to have a pole at P. The order of the pole of f at P is given by $|v_P(f)|$.

A *rational divisor* D over $\mathbb{F}$ is a finite linear combination of closed points $P \in \chi$ with integer coefficients n_P, that is, $D = \sum_P n_P P$. If $n_P \geq 0$ for all P, then D is called *effective*. We define the *degree* of D as $\deg D := \sum_P n_P \deg P$, and the *support* of D as the set $\mathrm{supp}(D) = \{P \in \chi \text{ closed} \mid n_P \neq 0\}$. The set of rational divisors on $\mathbb{F}$ form an abelian group $\mathscr{D}(\mathbb{F})$. Rational functions define *principal* divisors, namely divisors of the form

$$(f) := \sum_P \mathrm{ord}_P(f) P.$$

A rational divisor $D = \sum n_P P$ defines a $\mathbb{F}$-vector space

$$\mathscr{L}(D) = \left\{ f \in \mathbb{F}(\chi)^* \mid (f) \geq -D \right\} \cup \{0\},$$

that is, the set of rational functions f with poles only at the points P with $n_P \geq 0$ (and, furthermore, with the pole order of f at P must be less or equal than n_P), and if $n_P < 0$ such functions must have a zero at P of order greater or equal than n_P. The dimension $\ell(D) := \dim_{\mathbb{F}} \mathscr{L}(D)$ is finite. Two elements $f, g \in \mathscr{L}(D)$ satisfy $(f) + D = (g) + D$ if and only if $f = \lambda g$, $\lambda \in \mathbb{F}$, i.e., if and only if $f = \lambda g$ for a constant $\lambda \in \mathbb{F}$. Therefore the set $|D|$ of effective divisors equivalent to D can be identified with the projective space $\mathbb{P}_{\mathscr{L}(D)}$ of dimension $\ell(D) - 1$. The set $|D|$ is called a *complete linear system* of D.

Let $\Omega_{\mathbb{F}}(\mathbb{F}(\chi))$ be the module of differentials on $\mathbb{F}(\chi)$. A differential form $\omega \in \Omega_{\mathbb{F}}(\mathbb{F}(\chi))$ defines a divisor $(\omega) := \sum_P \mathrm{ord}_P(\omega)P$, called a canonical divisor. A rational divisor D defines again a $\mathbb{F}$-vector space

$$\Omega(D) := \{\omega \in \Omega_{\mathbb{F}}(\mathbb{F}(\chi))^* \mid (\omega) \geq D\} \cup \{0\}.$$

of finite dimension, denoted by $i(D)$. It is a central result in the theory of algebraic curves the interplay of the dimensions $\ell(D)$ and $i(D)$. The dimension $\ell(D)$ is bounded in the following sense:

Proposition 2.1 (Riemann's inequality)**.** *There exists a nonnegative integer g such that*

$$\ell(D) \geq \deg D + 1 - g.$$

for any rational divisor D on χ.

Definition 2.2. *The smallest integer g satisfying the Riemann's inequality is called the* genus *of χ.*

Riemann's inequality tells us that if D is a large divisor, $\mathscr{L}(D)$ is also large. But we can be a bit more precise by using $i(D)$:

Theorem 2.3 (Riemann-Roch)**.** *Let D be a rational divisor. Then:*

$$\ell(D) - i(D) = \deg D + 1 - g.$$

3. RATIONAL PARAMETRIZATIONS

Let $\mathbb{F}$ be a perfect field, and let χ be an absolutely irreducible algebraic plane curve defined over $\mathbb{F}$. Let P be a closed point on χ. Let us consider the local ring $\mathscr{O} := \mathscr{O}_{\chi,P}$ with maximal ideal $\mathfrak{m}$, and write $\overline{\mathscr{O}}$ for the semilocal ring of the normalisation of χ at P. Finally, let $\hat{\mathscr{O}}$ be the completion of $\mathscr{O}$ with respect to the $\mathfrak{m}$-adic topology. Each maximal ideal of $\overline{\mathscr{O}}$ (or, equivalently, every minimal prime ideal $\mathfrak{p}$ of $\hat{o}o$) is said to be a *branch* of χ at P.

Let us now choose an affine chart containing P so that the curve χ has an equation $f(X,Y) = 0$, and set $A := \mathbb{F}[X,Y]/(f(X,Y))$ as the affine coordinate ring. Notice that $\mathscr{O} = A_P$. Hence

$$\mathbb{F} \subseteq \mathbb{F}[X,Y]/(f(X,Y)) = A \subseteq A_P = \mathscr{O}.$$

Since $\mathbb{F}$ is perfect, we can apply Hensel's lemma to find a finite field extension $K/\mathbb{F}$ such that $K \subseteq \hat{A}_P = \hat{\mathscr{O}}$ is a coefficient field for $\hat{\mathscr{O}}$. Moreover, K is the integral closure of $\mathbb{F}$ in $\hat{\mathscr{O}}$.

Since $\hat{\mathscr{O}} \subseteq \overline{\hat{\mathscr{O}}} \cong \hat{\overline{\mathscr{O}}}$, one has

$$K \subseteq \hat{\mathscr{O}}/\mathfrak{p} \subseteq \overline{\hat{\mathscr{O}}/\mathfrak{p}} = \overline{\hat{\mathscr{O}}}_{\mathfrak{m}},$$

and we can apply Hensel's lemma again to obtain a finite extension K'/K which is a coefficient field for the local ring $\overline{\hat{\mathscr{O}}}_{\mathfrak{m}}$. Without loss of generality we can consider P as the ideal (X,Y) in $K[\![X,Y]\!]$ so that $\hat{\mathscr{O}} \cong K[\![X,Y]\!]/(f(X,Y))$. This implies the existence of natural morphisms

$$K[\![X,Y]\!]/(f(X,Y)) \cong \hat{\mathscr{O}} \longrightarrow \hat{\mathscr{O}}/\mathfrak{p} \longrightarrow K'[\![t]\!] \cong \overline{\hat{\mathscr{O}}}_{\mathfrak{m}}$$

for any local uniformizing parameter $t \in \mathfrak{m} \setminus \mathfrak{m}^2$. Notice that K can be considered isomorphic to the residue field at P. Preserving these notations, a *parametrization* of the curve χ at the point P related to the coordinates X,Y is a K-algebra morphism $\rho : K[\![X,Y]\!] \longrightarrow K'[\![t]\!]$ being continuous for the (X,Y)-adic and t-adic topologies and satisfying $\mathrm{Im}(\rho) \not\subseteq K'$ and $\rho(f) = 0$. This is equivalent to give formal power series $x(t), y(t) \in K'[\![t]\!]$ with $x(t) \neq 0$ or $y(t) \neq 0$ such that $f(x(t), y(t)) \equiv 0$.

Consider parametrizations $\rho : K[\![X,Y]\!] \to K'[\![t]\!]$ and $\sigma : K[\![X,Y]\!] \to K''[\![t]\!]$ of the same rational branch. The parametrization σ is said to be *derived* from ρ if there is a formal power series $\tau(u) \in K''[\![u]\!]$ with positive order and a continuous K-algebra morphism $\alpha : K'[\![t]\!] \to K''[\![u]\!]$ with $\alpha(t) = \tau(u)$ such that $\sigma = \alpha \circ \rho$. We write $\sigma \succ \rho$. The relation $\succ$ is a partial preorder. Two parametrizations σ and ρ are called *equivalent* if $\sigma \succ \rho$ and $\rho \succ \sigma$. Those parametrizations being minimal with respect to $\succ$ up to equivalence are called *primitive*. Equivalent primitive parametrizations are called *rational*. They always exist and are invariant under the action of the Galois group of the extension $\overline{K}/K$. Rational parametrizations are in one to one correspondence with rational branches of the curve (cf. Campillo and Castellanos [5]).

4. BRILL-NOETHER THEORY FOR CURVES

This section contains a summary of the classic adjunction theory of curves, started by Riemann [21] and developed by M. Noether and A. von Brill in the 19th century.

Let P be a closed point. Let $\mathscr{C}_P$ be the annihilator of the $\mathscr{O}$-module $\overline{\mathscr{O}}/\mathscr{O}$, i.e.

$$\mathscr{C}_P = \mathscr{C}_{\overline{\mathscr{O}}/\mathscr{O}} = \{\varphi \in \overline{\mathscr{O}} \mid \varphi\overline{\mathscr{O}} \subseteq \mathscr{O}\}.$$

This set is the largest ideal in $\mathscr{O}$ which is also an ideal in $\overline{\mathscr{O}}$, and it is called the *conductor ideal* of the extension $\overline{\mathscr{O}}/\mathscr{O}$. Since $\overline{\mathscr{O}}$ is a semilocal Dedekind domain with maximal ideals $\overline{\mathfrak{m}}_{Q_1}, \ldots, \overline{\mathfrak{m}}_{Q_d}$ (where Q_i denote the rational branches of χ at P), the conductor ideal has a unique factorisation

$$\mathscr{C}_P = \prod_{i=1}^{d} \overline{\mathfrak{m}}_{Q_i}^{d_{Q_i}}$$

as ideal in $\overline{\mathscr{O}}$. The exponents d_{Q_i} can be easily computed by means of the Dedekind formula (see Zariski [23]): if $(x_i(t_i), y_i(t_i))$ is a rational parametrisation of Q_i one has

$$d_{Q_i} = \mathrm{ord}_{t_{Q_i}}\left(\frac{f_Y(X(t_{Q_i}), Y(t_{Q_i}))}{X'(t_{Q_i})}\right) = \mathrm{ord}_{t_{Q_i}}\left(\frac{f_X(X(t_{Q_i}), Y(t_{Q_i}))}{Y'(t_{Q_i})}\right). \qquad (4.1)$$

Let $n : \widetilde{\chi} \to \chi$ ne the normalisation morphism of χ. Notice that $\widetilde{\chi}$ is nonsingular with $\mathbb{F}(\widetilde{\chi}) = \mathbb{F}(\chi)$. Let $\mathscr{O} = \mathscr{O}_{\chi,P}$ and $\overline{\mathscr{O}}$ its normalisation. Let $Q \in n^{-1}(\{P\})$. Since Q is nonsingular, it is $\mathscr{C}_P \cdot \mathscr{O} = \mathfrak{m}_Q^{d_Q}$ for a nonnegative integer d_Q. We define the effective divisor

$$\mathscr{A} := \sum_P \sum_{Q \in n^{-1}(\{P\})} d_Q \cdot Q$$

which is called the *adjunction divisor* of χ. Notice that $\mathscr{A}$ is a well-defined divisor on $\widetilde{\chi}$ (in fact, if P is nonsingular, there is only one $Q \in n^{-1}(\{P\})$ and in this case $d_Q = 0$). This implies in particular that the support of $\mathscr{A}$ consists of all rational branches of χ at singular points. Moreover, by setting $n_P := \dim_{\mathbb{F}} \overline{\mathscr{O}} / \mathscr{C}_P$ we have

$$n_P = \sum_{Q \in n^{-1}(\{P\})} d_Q$$

for every P on χ. Therefore $\deg \mathscr{A} = \sum_{P \in \chi} n_P$ (cf. Arbarello et al. [1, Appendix A]; also Tsfasman and Vlăduţ [22, 2.5.2]).

Let $F := F(X_0, X_1, X_2)$ be a homogeneous (absolutely irreducible) polynomial of degree d over $\mathbb{F}$ which defines the projective plane curve χ. Let $\mathscr{F}_d$ be the set of all homogeneous polynomials in three variables of degree d. Let $i : \chi \to \mathbb{P}_{\mathbb{F}}^2$ be the embedding of χ into the projective plane and $\mathbf{N} : \widetilde{\chi} \to \mathbb{P}_{\mathbb{F}}^2$ be the natural morphism given by $\mathbf{N} = i \circ n$. A rational divisor D on $\mathbb{P}_{\mathbb{F}}^2$ such that χ is not contained in $\mathrm{supp}(D)$ is called an *adjoint divisor* of χ if the pull-back divisor $\mathbf{N}^*D$ satisfies $\mathrm{supp}(\mathscr{A}) \subseteq \mathrm{supp}(\mathbf{N}^*D)$ for $\mathscr{A}$ the adjunction divisor of χ. We can consider the analogous notion at the level of homogeneous polynomials. For $H \in \mathscr{F}_d$ with $F \nmid H$ one can consider the pull-back $\mathbf{N}^*H$, which is actually the intersection divisor on $\widetilde{\chi}$ cut out by the plane curve defined by H on $\mathbb{P}_{\mathbb{F}}^2$, namely

$$\mathbf{N}^*H = \sum_{Q \in \widetilde{\chi}} r_Q \cdot Q, \tag{4.2}$$

with $r_Q = \mathrm{ord}_Q(h)$ for $h \in \mathscr{O}_{\chi, n(Q)}$ being a local equation of the curve defined by H at the point $n(Q)$. If H satisfies additionally $\mathbf{N}^*D \geq \mathscr{A}$, then it will be called an *adjoint form* on χ, and the curve defined by H will be called an *adjoint curve* to χ. Notice that adjoint curves there always exists (take for instance the polars of the curve, cf. Brieskorn and Knörrer [2], p. 599).

Let $d := \deg \chi$. The differentials gob ally defined at χ are in one to one correspondence with adjoint curves on $\widetilde{\chi}$ of degree $d - 3$:

Theorem 4.1. *Let $\mathscr{A}_n$ be the set of adjoints of degree n of the curve χ embedded in $\mathbb{P}_{\mathbb{F}}^2$, let $K_{\widetilde{\chi}}$ be a canonical divisor on $\widetilde{\chi}$ and set $d := \deg \chi$. For $n = d - 3$ there is an $\mathbb{F}$-isomorphism of complete linear systems*

$$\begin{array}{ccc} \mathscr{A}_n & \longrightarrow & |K_{\widetilde{\chi}}| \\ D & \longmapsto & \mathbf{N}^*D - \mathscr{A}. \end{array}$$

The key idea is to realise that the map is injective since $n = d - 3 < d$; see Gorenstein [11, p. 433] or [22, 2.2.1] for further details.

In practice, we know a priori the equation of the plane curve χ (defined over a perfect field $\mathbb{F}$) given by the form $F \in \mathscr{F}_d$ and the data of a certain divisor $R = \sum_{Q'} r_{Q'} \cdot Q'$ (for finitely many points Q' on $\widetilde{\chi}$) which is effective and rational over $\mathbb{F}$, involving a finite number of rational branches Q of χ and their corresponding coefficients. Moreover, we are able to compute the adjunction divisor of χ, $\mathscr{A} = \sum_Q d_Q \cdot Q$. Our aim is to interpret the condition of being an adjoint form—called *adjoint condition*—given by (4.2) in terms of equations. More generally, we are interesting in finding some *adjoint* form $H \in \mathbb{F}[X_0, X_1, X_2]$ satisfying

$$\mathbf{N}^* H \geq \mathscr{A} + R. \tag{4.3}$$

This process is known as *computing adjoint forms with base conditions* (see [7], §4).

First of all, we choose a positive integer $\widetilde{n} \in \mathbb{N}$ in such a way that there exists an adjoint of degree $\widetilde{n}$ not being a multiple of F and satisfying (4.3). A bound for $\widetilde{n}$ can be found in Haché [13]. Take then also a form $H \in \mathscr{F}_{\widetilde{n}}$ in a general way, what is nothing else but taking a homogeneous polynomial in three variables of degree $\widetilde{n}$ with its coefficients as indeterminates (that is, $H(X_0, X_1, X_2) = \sum_{i+j+k=\widetilde{n}} \lambda_{i,j,k} X_0^i X_1^j X_2^k$). Second we compute a rational primitive parametrization $(X(t), Y(t))$ of χ at every branch involved in the support of the adjunction divisor $\mathscr{A}$ and the divisor R. Next we get the support of the adjunction divisor $\mathscr{A}$ from the conductor ideal via the Dedekind formula (4.1). Last we consider the coefficient r_Q of the divisor R at Q, and thus the local condition at Q imposed on H by (4.3) is given by

$$\operatorname{ord}_t h(X(t), Y(t)) \geq d_Q + r_Q, \tag{4.4}$$

with h the local affine equation of H at Q. The inequality (4.4) expresses a linear condition (given by a linear inequation) on the coefficients $\lambda_{i,j,k}$ of h.

The required linear equations are a consequence of the vanishing of those terms, and when Q takes all the possible values, i.e., all the possible branches of the singular points on χ and of the support of R, we get the linear equations globally imposed by the condition (4.3). An easy reasoning reveals that the number of such adjoint conditions is equal to

$$\frac{1}{2} \deg \mathscr{A} + \deg R = \frac{1}{2} \sum_{P \in \chi} n_P + \deg R = \sum_{P \in \chi} \delta_P + \deg R. \tag{4.5}$$

Example 4.2. *Let χ be the projective plane curve over the finite field of two elements $\mathbb{F}_2$ given by the equation $F(X, Y, Z) = X^3 - Y^2 Z$. The only singular point of χ is $P_1 = [0 : 0 : 1]$. Let be the point $P_2 = [0 : 1 : 0]$ and the effective divisor $R = 0P_1 + P_2$. The adjunction divisor of χ is $\mathscr{A} = 2P_1$. A local equation of χ with $P_1 = (0, 0)$ is $f(x, y) = x^3 - y^2$. A parametrization of χ at P_1 is given by*

$$X_1(t_1) = t_1^2$$
$$Y_1(t_1) = t_1^3$$

Take a form $H \in \mathscr{F}_{4-3=1}$, $H(X, Y, Z) = aX + bY + cZ$. First we want to express the adjoint conditions in terms of the coefficients

$$\mathbf{N}^* H \geq \mathscr{A} + R = 2P_1 + P_2.$$

To this end we consider first a local equation for H at P_1, namely

$$h(x,y) = H(x,y,1) = ax + by + c.$$

Then $h(X_1(t_1), Y_1(t_1)) = h(t^2, t^3) = at_1^2 + bt_1^3 + c$. So if we wish to have

$$\mathrm{ord}_{t_1}(h(X_1(t_1), Y_1(t_1))) = \mathrm{ord}_{t_1}(bt_1^3 + at_1^2 + c) \geq 2$$

(since 2 is the coefficient for P_1 and $(X_1(t_1), Y_1(t_1))$ is a parametrization at P_1), then this is possible if and only if $c = 0$. Thus $c = 0$ is one of the required linear *adjoint conditions.*

Now consider a local equation for χ at P_2. This is $f'(x,z) = F(x,1,z) = x^3 - z$, and admits a parametrization

$$X_2(t_2) = t_2$$
$$Z_2(t_2) = t_2^3$$

Consider the local equation for H at P_2

$$h'(x,z) = H(x,1,z) = ax + b + cz.$$

Hence the adjoint conditions imposed by $N^*H \geq \mathscr{A} + R = 2P_1 + P_2$ at P_2 come from considering $h'(X_2(t_2), Z_2(t_2)) = h'(t_2, t_2^3) = at_2 + b + ct_2^3$ and they impose

$$\mathrm{ord}_{t_2}(h'(X_2(t_2), Z_2(t_2))) = \mathrm{ord}_{t_2}(ct_2^3 + at_2 + b) \geq 1.$$

This inequality holds whenever $b = 0$. Hence $b = 0$ is another linear *equation taking part in the set of adjoint conditions contained in $N^*H \geq \mathscr{A} + R$. We have obtained two adjoint conditions, as we had hoped by (4.5), since $\frac{1}{2}\deg\mathscr{A} + \deg R = \frac{1}{2}\cdot 2 + 1 = 2$.*

We conclude this section with two remarkable results. Let χ be an absolutely irreducible projective plane curve defined over a perfect field $\mathbb{F}$ and given by an equation $F(X_0, X_1, X_2) = 0$, where $F \in \mathscr{F}_d$. One application of the adjoint forms is the following result, due to Max Noether (he stated it of course not in this way; our version may be found in Haché and Le Brigand [14], Theorem 4.2, and Le Brigand and Risler [18], §3.1):

Theorem 4.3 (Max Noether). *Let χ, χ' be curves as above given by homogeneous equations $F(X_0, X_1, X_2) = 0$ and $G(X_0, X_1, X_2) = 0$ respectively and such that χ' does not contain χ as a component. Then, if we consider another such a curve given by $H(X_0, X_1, X_2) = 0$ with $N^*H \geq \mathscr{A} + N^*G$ (where $\mathscr{A}$ is the adjunction divisor on χ), there exist forms A, B with coefficients in $\mathbb{F}$ such that $H = AF + BG$.*

This theorem has great importance, and, for instance, allows us to prove the Brill-Noether theorem, which gives a basis for the vector spaces $\mathscr{L}(D)$. Readers are referred to [14], Theorem 4.4, for further details. A short remark about notation is needed. For any non effective divisor D we will write $D = D_+ - D_-$ with D_+ and D_- effective divisors of disjoint support.

Theorem 4.4 (Brill-Noether). *Let χ be an adjoint curve as above with normalization $\widetilde{\chi}$. Let $\mathscr{A}$ be its adjunction divisor and let D be a divisor on $\widetilde{\chi}$ rational over $\mathbb{F}$. Moreover,*

consider a form $H_0 \in \mathscr{F}_{\widetilde{n}}$ defined over $\mathbb{F}$, not divisible by F and satisfying $N^*H_0 \geq \mathscr{A} + D_+$. Then

$$\mathscr{L}(D) = \left\{ \frac{h}{h_0} \mid H \in \mathscr{F}_{\widetilde{n}}, F \nmid H \text{ and } N^*H + D \geq N^*H_0 \right\} \cup \{0\},$$

where $h, h_0 \in \mathbb{F}(\chi)$ denote respectively the rational functions H, H_0 restricted on χ.

Remark 4.5. *Such a form $H_0 \in \mathscr{F}_{\widetilde{n}}$ exists whenever $\widetilde{n} > \max\left\{ d-1, \frac{d-3}{2} + \frac{\deg(\mathscr{A}+D_+)}{d} \right\}$ (see Haché and Le Brigand [14] for details).*

5. THE WEIERSTRASS SEMIGROUP AT SEVERAL POINTS

Let χ be an absolutely irreducible projective algebraic plane curve defined over a perfect field $\mathbb{F}$. Let $\underline{P}$ denote a set of r different points $P_1, \ldots, P_r$ on χ. Furthermore, the perfect field $\mathbb{F}$ must have cardinality greater or equal to r: $\sharp\mathbb{F} \geq r$. Let $\widetilde{\chi}$ be the normalization of χ.

Our purpose is to compute the dimensions of the so-called *Riemann-Roch quotients*:

$$0 \leq \dim_{\mathbb{F}} \frac{\mathscr{L}(\underline{mP})}{\mathscr{L}((\underline{m}-\underline{1})\underline{P})} \leq r$$

by choosing functions in $\mathscr{L}(\underline{mP}) = \mathscr{L}(m_1P_1 + \ldots + m_rP_r)$ but not in $\mathscr{L}((\underline{m}-\underline{1})\underline{P}) = \mathscr{L}((m_1-1)P_1 + \ldots + (m_r-1)P_r)$, that is, achieving at the P_i poles of order m_i. We are going to restrict to the case $m_i \in \mathbb{N}$, for all $i = 1, \ldots, r$. Such dimensions will be determined by the previous calculus of the *Riemann-Roch quotients with respect to P_i*:

$$0 \leq \dim_{\mathbb{F}} \frac{\mathscr{L}(\underline{mP})}{\mathscr{L}((\underline{m}-\varepsilon_i)\underline{P})} \leq 1,$$

where ε_i denotes the vector in $\mathbb{N}^r$ with 1 in the i-th position and 0 in the other ones.

Summarizing, this section deals with the following topics:

(a) How to compute $\dim_{\mathbb{F}} \frac{\mathscr{L}(\underline{mP})}{\mathscr{L}((\underline{m}-\varepsilon_i)\underline{P})}$ and an associated function belonging to this quotient vector space when such a dimension is 1.
(b) How to compute $\dim_{\mathbb{F}} \frac{\mathscr{L}(\underline{mP})}{\mathscr{L}((\underline{m}-\underline{1})\underline{P})}$ (deducing bounds).
(c) How to compute the Weierstrasß semigroup at two points.

All the statements and proofs of this section can be found in [9], §2.

Consider a finite set of nonsingular points $P_1, \ldots, P_r$ on χ and a divisor $m_1P_1 + \ldots + m_rP_r$ for $m_i \in \mathbb{N} \; \forall \; i = 1, \ldots r$. We will denote $\underline{P} = \{P_1, \ldots, P_r\}$, $\underline{mP} = m_1P_1 + \ldots + m_rP_r$, $\underline{m} = (m_1, \ldots, m_r)$, $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ and $\underline{1} = (1, \ldots, 1)$.

Definition 5.1. *For $\underline{P} \in \chi$ we define*

$$\Gamma_{\underline{P}} := \left\{ -(\mathrm{ord}_{P_1}(f), \ldots, \mathrm{ord}_{P_r}(f)) \mid f \in \mathbb{F}(\chi)^* \text{ regular at } \chi \setminus \underline{P} \right\}.$$

Obviously Γ_P is a subsemigroup of $(\mathbb{N},+)$. Notice that, for $\underline{m}P = m_1P_1 + m_2P_2$, the fact that $f \in \mathscr{L}(\underline{mP})$ is equivalent to the inequalities

$$(\star) \begin{cases} \mathrm{ord}_{P_1}(f) \geq -m_1 \\ \mathrm{ord}_{P_2}(f) \geq -m_2. \end{cases}$$

This means: the set of possible orders $(\star)$ which can be taken by the function f is represented by the shadowed area in the figure (each axis represents one of the two branches):
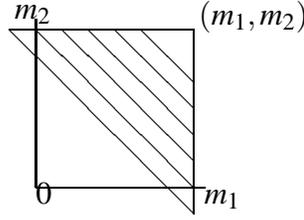
Definition 5.2. *An element $\underline{m} \in \mathbb{N}^r$ is called a* non-gap *of $\underline{P}$ if and only if $\underline{m} \in \Gamma_P$. Otherwise $\underline{m}$ is called a* gap.

A very important characterization for the non-gaps is given by the following (see [10], p. 629):

Lemma 5.3. *If $\underline{m} \in \mathbb{Z}^r$ then one has:*

$$\underline{m} \in \Gamma_P \quad \text{if and only if} \quad \ell(\underline{m}P) = \ell((\underline{m} - \varepsilon_i)\underline{P}) + 1 \ \forall \ i = 1, \ldots, r.$$

For every $i = 1, \ldots, r$ and $\underline{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r$, we set

$$\nabla_i(\underline{m}) := \left\{ (n_1, \ldots, n_r) \in \Gamma_P \mid n_i = m_i \text{ and } n_j \leq m_j \ \forall j \neq i \right\}.$$

Then the two conditions proven to be equivalent in Lemma 5.3 are indeed also equivalent to $\nabla_i(\underline{m}) \neq 0$ for every $i \in \{1, \ldots, r\}$.

A gap $\underline{m}$ satisfying $\ell(\underline{m}P) = \ell((\underline{m} - \varepsilon_i)\underline{P})$ for all $i \in \{1, \ldots, r\}$ (or, equivalently, such that $\nabla_i(\underline{m}) = \emptyset$ for all $i \in \{1, \ldots, r\}$) is called *pure*. It is easily seen: if $\underline{m}$ is a pure gap, then m_i is a gap for Γ_{P_i} for every $i \in \{1, \ldots, r\}$. Furthermore, if $\underline{1} \in \Gamma_P$, then there are no pure gaps. The converse does not hold, as Example 5.19 will show.

A basic tool on Weierstraß semigroups is the following

Theorem 5.4 (Weierstraß gap theorem). *Let $\widetilde{\chi}$ be a curve of genus $g \geq 1$. Let P be a rational branch on $\widetilde{\chi}$. Then there are g gaps $\gamma_1, \ldots, \gamma_g$ such that*

$$1 = \gamma_1 < \ldots < \gamma_g \leq 2g - 1.$$

Proposition 5.5. *Let $\underline{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r$. If $\underline{m}$ is a gap, then there exists a regular differential form ω on $\widetilde{\chi}$ with $(\omega) \geq \underline{m} - \varepsilon_i$ and a zero at P_i of order $m_i - 1$ for some $i \in \{1, \ldots, r\}$.*

Proof. After Riemann-Roch theorem it is clear that

$$\begin{aligned} \ell(\underline{m}P) - i(\underline{m}P) &= m_1 + m_2 + \ldots + m_r + 1 - g \\ \ell((\underline{m} - \varepsilon_i)\underline{P}) - i((\underline{m} - \varepsilon_i)\underline{P}) &= m_1 + m_2 + \ldots + m_r - 1 + 1 - g. \end{aligned}$$

By adding both equations we have

$$\underbrace{[\ell(\underline{m}P) - \ell((\underline{m} - \varepsilon_i)\underline{P})]}_{=\varphi(\underline{m})} - \underbrace{[i(\underline{m}P) - i((\underline{m} - \varepsilon_i)\underline{P})]}_{=\psi(\underline{m})} = 1,$$

for every $i = 1, \ldots, r$, where $0 \le \varphi(\underline{m}) \le 1$ and $-1 \le \psi(\underline{m}) \le 0$, and therefore

$$\varphi(\underline{m}) = 1 \Leftrightarrow \ell(\underline{m}P) - \ell((\underline{m} - \varepsilon_i)\underline{P}) = 1 \Leftrightarrow \ell(\underline{m}P) = \ell((\underline{m} - \varepsilon_i)\underline{P}) + 1 \Leftrightarrow \underline{m} \in \Gamma_{\underline{P}} \Leftrightarrow \psi(\underline{m}) = 0.$$

Hence if $\underline{m} \notin \Gamma_{\underline{P}}$ then $\dim_{\mathbb{F}} \left(\frac{\Omega((\underline{m}-\varepsilon_i)\underline{P})}{\Omega(\underline{m}P)} \right) = 1$ and so there exists a regular differential form ω on $\widetilde{\chi}$ with $(\omega) \ge \underline{m} - \varepsilon_i$ and $\mathrm{ord}_{P_i}(\omega) = m_i - 1$ for *some* $i \in \{1, \ldots, r\}$. $\qquad\square$

Proposition 5.6. *Let χ be a plane curve of genus g, let $\underline{P}$ be a set of r closed points on χ and set $\underline{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r$. If $\underline{m}$ is a gap, then $m_1 + \ldots + m_r < 2g$.*

Proof. Denote by $D_{2g,\underline{P}}$ a divisor with degree $2g$ and support $\underline{P}$, and by $D_{2g-1,\underline{P}}$ a divisor with degree $2g - 1$ and support $\underline{P}$. If $m_1 + \ldots + m_r \ge 2g - 1$ then $m_1 + \ldots + m_r \ge 0$ as a consequence of Riemann-Roch, and for every $4i = 1, \ldots, r$

$$\ell(D_{2g,\underline{P}}) = 2g + 1 - g = g + 1 \ne g = 2g - 1 + 1 - g = \ell(D_{2g-1,\underline{P}}),$$

which implies that $\underline{m}$ is a non-gap, i.e., $\underline{m} \in \Gamma_{\underline{P}}$. So, if $\underline{m} \notin \Gamma_{\underline{P}}$, then $m_1 + \ldots + m_r < 2g$. $\quad\square$

Notice that, for divisors of the form $\underline{m}P = m_1 P_1 + m_2 P_2$, the plane $\mathbb{N} \times \mathbb{N}$ is divided in three parts by the line $m_1 + m_2 = 2g$ as in the figure, namely
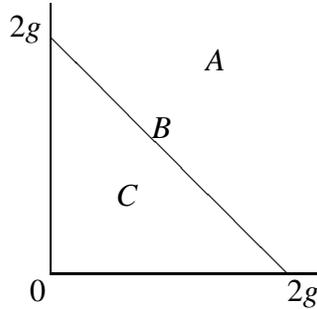
$$A := \{(m_1, m_2) \mid m_1 + m_2 > 2g, \ m_1 > 0, \ m_2 > 0\}$$
$$B := \{(m_1, m_2) \mid m_1 + m_2 = 2g, \ m_1 > 0, \ m_2 > 0\} \cup \{(m_1, 0), \ m_1 > 2g\} \cup \{(0, m_2), \ m_2 > 2g\}$$
$$C := \{(m_1, m_2) \mid m_1 + m_2 < 2g, \ m_1 \ge 0, \ m_2 \ge 0\}.$$

All the points lying on A and B correspond to values in $\Gamma_{\underline{P}}$, but nothing can be a priory said about the points on C.

5.1. **Dimension of the Riemann-Roch quotients with respect to P_i and associated functions.** We start by computing the dimension of the Riemann-roch quotients associated to the points P_i.

Proposition 5.7. *Let $\underline{m} \in \mathbb{N}^r$ such that $\sum_{i=1}^{r} m_i < 2g$. Then, for $i \in \{1, \ldots, r\}$ we have:*
  a) $\dim_{\mathbb{F}}[\Omega((\underline{m} - \varepsilon_i)\underline{P}) \setminus \Omega(\underline{m}P)] = 1$ *if and only if $\exists$ a homogeneous polynomial H_0 of degree $d - 3$ with $N^* H_0 \ge \mathscr{A} + (\underline{m} - \varepsilon_i)\underline{P}$ such that P_i is not in the support of the effective divisor $N^* H_0 - \mathscr{A} - (\underline{m} - \varepsilon_i)\underline{P}$.*

b) $\exists \, \underline{m}' \geq \underline{m}$ with $\dim_{\mathbb{F}}[\Omega((\underline{m}' - \varepsilon_i)\underline{P}) \setminus \Omega(\underline{m}'\underline{P})] = 1$ if and only if $\exists$ a homogeneous polynomial H_0 of degree $d - 3$ such that $\mathbf{N}^*H_0 \geq \mathscr{A} + (\underline{m} - \varepsilon_i)\underline{P}$.

Proof.

a) If $\dim_{\mathbb{F}}[\Omega((\underline{m} - \varepsilon_i)\underline{P}) \setminus \Omega(\underline{m}\underline{P})] = 1$, then this is equivalent to $\underline{m} \notin \Gamma_{\underline{P}}$ and also to the existence of an index i with $\ell(\underline{m}\underline{P}) = \ell((\underline{m} - \varepsilon_i)\underline{P})$, or, in other words, to the existence of an index i with $i((\underline{m} - \varepsilon_i)\underline{P}) = i(\underline{m}\underline{P}) + 1$; that is, there exists a homogeneous polynomial H_0 of degree $d - 3$ such that $\mathbf{N}^*H_0 \geq \mathscr{A} + (\underline{m} - \varepsilon_i)\underline{P}$.

b) If there is $\underline{m}' \geq \underline{m}$ with $\dim_{\mathbb{F}}[\Omega((\underline{m}' - \varepsilon_i)\underline{P}) \setminus \Omega(\underline{m}'\underline{P})] = 1$ then there exists an adjoint H_0 of degree $d - 3$ whose divisor is $\geq (\underline{m}' - \varepsilon_i)\underline{P}$ outside $\mathscr{A}$, i.e., $\mathbf{N}^*H_0 - \mathscr{A} \geq (\underline{m}' - \varepsilon_i)\underline{P} \geq (\underline{m} - \varepsilon_i)\underline{P}$. Conversely, if there is H_0 of degree $d - 3$ with $\mathbf{N}^*H_0 \geq \mathscr{A} + (\underline{m} - \varepsilon_i)\underline{P}$ then there exists $\omega \neq 0$ differential form such that $(\omega) = \mathbf{N}^*H_0 - \mathscr{A} \geq (\underline{m} - \varepsilon_i)\underline{P}$. Assume that $\underline{m}' - \varepsilon_i$ are the orders of the zeros of ω at $\underline{P}$. Thus, $\underline{m}' - \varepsilon_i \geq \underline{m} - \varepsilon_i$, what implies $\underline{m}' \geq \underline{m}$ and $\omega \in \Omega((\underline{m} - \varepsilon_i)\underline{P}) \setminus \Omega(\underline{m}\underline{P})$. $\square$

The following corollary yields a way to relate the adjunction theory and the computation of the Weierstraß semigroup at several points:

Corollary 5.8. *Let $\underline{m} \in \mathbb{N}^r$ with $\sum_{i=1}^{r} m_i < 2g$. For a given form H of degree $d - 3$ and $i \in \{1, \ldots, r\}$ there exists a condition imposed by the inequality $\mathbf{N}^*H \geq \mathscr{A} + \underline{m}\underline{P}$ at P_i which is independent of the conditions imposed by $\mathbf{N}^*H \geq \mathscr{A} + (\underline{m} - \varepsilon_i)\underline{P}$ at P_i if and only if*

$$\dim_{\mathbb{F}} \frac{\Omega((\underline{m} - \varepsilon_i)\underline{P})}{\Omega(\underline{m}\underline{P})} = 1.$$

The second step is the computation of the rational functions associated to the nongaps of the Weierstraß semigroup. Note that, if $\dim_{\mathbb{F}} \frac{\Omega((\underline{m} - \varepsilon_i)\underline{P})}{\Omega(\underline{m}\underline{P})} = 0$, then $\dim_{\mathbb{F}} \frac{\mathscr{L}(\underline{m}\underline{P})}{\mathscr{L}((\underline{m} - \varepsilon_i)\underline{P})} = 1$ and so there is a rational function $f_{i,\underline{m}} \in \frac{\mathscr{L}(\underline{m}\underline{P})}{\mathscr{L}((\underline{m} - \varepsilon_i)\underline{P})}$ with a pole of order m_i at P_i. In order to compute such a function, we base on Brill-Noether Theorem 4.4:

Algorithm 5.9. *Preserving notations as above, we obtain a function $f_{i,\underline{m}} \in \frac{\mathscr{L}(\underline{m}\underline{P})}{\mathscr{L}((\underline{m} - \varepsilon_i)\underline{P})}$ with a pole of order m_i at P_i by following these steps:*

- *Compute a homogeneous polynomial H_0 not divisible by F of large enough degree n in the sense of Remark 4.5 satisfying $\mathbf{N}^*H_0 \geq \mathscr{A} + \underline{m}\underline{P}$.*
- *Calculate $R_{\underline{m}}$, which is the effective divisor such that $\mathbf{N}^*H_0 = \mathscr{A} + \underline{m}\underline{P} + R_{\underline{m}}$. Obviously $R_{\underline{m} - \varepsilon_i} = R_{\underline{m}} + P_i$.*
- *Find a form $H_{\underline{m}}$ of degree n not divisible by F such that $\mathbf{N}^*H_{\underline{m}} \geq R_{\underline{m}}$ but not satisfying $\mathbf{N}^*H_{\underline{m}} \geq R_{\underline{m} - \varepsilon_i} = R_{\underline{m}} + P_i$.*
- *Output: $f_{i,\underline{m}} = \frac{h_{\underline{m}}}{h_0}$, where $h_{\underline{m}}, h_0$ are the restricted forms on χ for $H_{\underline{m}}$ and H_0 respectively.*

Example 5.10. *Let χ be the curve given by the equation $F(X,Y,Z) = X^3Z + X^4 + Y^3Z + YZ^3$ and consider the points $P_1 = [0:1:1]$ and $P_2 = [0:1:0]$ and $\underline{m} = (1,2)$. We want to compute $\dim_{\mathbb{F}} \frac{\mathscr{L}(\underline{m}\underline{P})}{\mathscr{L}((\underline{m} - \varepsilon_1)\underline{P})}$ and $\dim_{\mathbb{F}} \frac{\mathscr{L}(\underline{m}\underline{P})}{\mathscr{L}(\underline{m} - \varepsilon_2)\underline{P}}$.*

A local parametrization of F at P_1 is given by

$$X_1(t_1) = t_1$$
$$Y_1(t_1) = t_1^3 + t_1^4 + t_1^9 + t_1^{10} + t_1^{11} + t_1^{12} + \dots$$

with local equation $f_1(x,y) = y^2 + y^3 + x^3 + x^4$. Analogously at P_2

$$X_2(t_2) = t_2$$
$$Z_2(t_2) = t_2^4 + t_2^7 + t_2^{10} + t_2^{12} + t_2^{13} + t_2^{16} + \dots$$

with local equation $f_2(x,z) = z + z^3 + x^3 z + x^4$.

First, we calculate the adjunction divisor: this is $\mathscr{A} = 2P_1$.

Search a form H of degree $d - 3 = 4 - 3 = 1$, that is, a linear form $H(X,Y,Z) = aX + bY + cZ$. At P_1 H admits the equation $h_1(x,y) = H(X, Y-1, 1) = ax + by + b + c$. At P_2 H admits the equation $h_2(x,z) = H(X,1,Z) = ax + b + cz$. Then:

$$h_1(X_1(t_1), Y_1(t_1)) = at_1 + b(t_1^3 + t_1^4 + t_1^9 + \dots) + b + c = (b+c) + at_1 + bt_1^3 + bt_1^4 + bt_1^9 + \dots.$$
$$h_2(X_2(t_2), Z_2(t_2)) = b + at_2 + ct_2^4 + ct_2^7 + ct_2^{10} + \dots.$$

In order to compute $\dim_{\mathbb{F}} \frac{\mathscr{L}(m\underline{P})}{\mathscr{L}((\underline{m}-\varepsilon_1)\underline{P})}$ we impose the systems of equations with the adjunction conditions at P_1:

$$\begin{cases} N^*H \geq \mathscr{A} + (m_1 - 1)P_1 = 2P_1 \\ N^*H \geq \mathscr{A} + m_1 P_1 = 3P_1, \end{cases}$$

or, in other words

$$\begin{cases} \mathrm{ord}_{t_1}(h_1(X_1(t_1), Y_1(t_1))) \geq 2 \Longrightarrow b + c = a = 0 \\ \mathrm{ord}_{t_1}(h_1(X_1(t_1), Y_1(t_1))) \geq 3 \Longrightarrow b + c = a = 0 \end{cases}$$

So the second system does not add any independent condition to the first one; this means, by Corollary 5.8, that $\dim_{\mathbb{F}} \frac{\mathscr{L}(m\underline{P})}{\mathscr{L}((\underline{m}-\varepsilon_1)\underline{P})} = 1$.

In order to compute $\dim_{\mathbb{F}} \frac{\mathscr{L}(m\underline{P})}{\mathscr{L}((\underline{m}-\varepsilon_2)\underline{P})}$ the systems of equations with the adjunction conditions at P_2 are

$$\begin{cases} N^*H \geq (m_2 - 1)P_2 = P_2 \\ N^*H \geq m_2 P_2 = 2P_2, \end{cases}$$

that is,

$$\begin{cases} \mathrm{ord}_{t_2}(h_2(X_2(t_2), Z_2(t_2))) \geq 1 \Rightarrow b = 0 \\ \mathrm{ord}_{t_2}(h_2(X_2(t_2), Z_2(t_2))) \geq 2 \Rightarrow a = 0 = b. \end{cases}$$

Notice that, in this case, the adjunction divisor does not appear in the inequalities since P_2 does not belong to its support. The second system adds one independent condition to the first one and this means that $\dim_{\mathbb{F}} \frac{\mathscr{L}(m\underline{P})}{\mathscr{L}((\underline{m}-\varepsilon_2)\underline{P})} = 0$ again by Corollary 5.8. □

Example 5.11. *Consider the previous example but with $\underline{m} = (4,6)$. As $m_1 + m_2 = 4 + 6 = 10 > 2g$, we know without calculations $\underline{m} \in \Gamma_{\underline{P}}$, i.e., that $\dim_{\mathbb{F}} \frac{\mathscr{L}(\underline{m}\underline{P})}{\mathscr{L}((\underline{m}-\varepsilon_i)\underline{P})} = 1$ for $i = 1,2$. So we will look for the corresponding functions $f_{i,\underline{m}}$ with poles at P_i of order m_i for $i = 1,2$.*

First of all, we search $\widetilde{n} > \max\left\{3, \frac{2}{4} + \frac{12}{4}\right\} = \max\left\{3, \frac{14}{4}\right\}$. Let us take $\widetilde{n} = 5$.

Then we look for a form H_0 of degree $\widetilde{n} = 5$ such that $N^*H_0 \geq \mathscr{A} + \underline{m}\underline{P}$. In this case $N^*H_0 \geq 4P_1 + 6P_2 + 2P_3$, since $\mathscr{A} = 2P_3$, where $P_3 = [0:0:1]$. After some computations we find $H_0 = X^4Z$.

In order to compute N^*H_0, we have to calculate $N^*(X)$, $N^*(Y)$ and $N^*(Z)$. Intersection points between $\{F = 0\}$ and $\{X = 0\}$ are $P_1 = [0:1:1]$, $P_2 = [0:1:0]$ and $P_3 = [0:0:1]$ with multiplicities 1, 1 and 2 respectively. So $N^*(X) = P_1 + P_2 + 2P_3$. Intersection points between $\{F = 0\}$ and $\{Y = 0\}$ are $P_3 = [0:0:1]$ and $P_4 = [1:0:1]$ such that $N^*(Y) = 3P_3 + P_4$. And the only point lying in the intersection between $\{F = 0\}$ and $\{Z = 0\}$ is $P_2 = [0:1:0]$ with multiplicity 4, therefore $N^*(Z) = 4P_2$.

Thus $N^*H_0 = 4N^*(X) + N^*(Z) = 4P_1 + 8P_2 + 8P_3$. The residue divisor $R_{\underline{m}} = N^*H_0 - \mathscr{A} - \underline{m}\underline{P} = 2P_2 + 6P_3$. Following the algorithm described above, we have to find a form H_{ε_1} such that $N^*H_{\varepsilon_1} \geq R_{\underline{m}}$ but $N^*H_{\varepsilon_1} \not\geq R_{\underline{m}} + P_1$. For instance we take $H_{\varepsilon_1} = Y^2Z^3$, since

$$N^*H_{\varepsilon_1} = 12P_2 + 6P_3 + 2P_4 \geq 2P_2 + 6P_3$$
$$N^*H_{\varepsilon_1} \not\geq P_1 + 2P_2 + 6P_3.$$

So $f_{1,\underline{m}} = \frac{Y^2Z^3}{X^4Z} = \frac{Y^2Z^2}{X^4} \in \frac{\mathscr{L}(\underline{m}\underline{P})}{\mathscr{L}((\underline{m}-\varepsilon_1)\underline{P})}$.

Now we have to find a form H_{ε_2} such that $N^*H_{\varepsilon_2} \geq R_{\underline{m}}$ but $N^*H_{\varepsilon_2} \not\geq R_{\underline{m}} + P_2$. For instance we take $H_{\varepsilon_2} = X^2Y^3$, since

$$N^*H_{\varepsilon_2} = 2P_1 + 2P_2 + 13P_3 + 3P_4 \geq 2P_2 + 6P_3$$
$$N^*H_{\varepsilon_2} \not\geq 3P_2 + 6P_3.$$

Thus $f_{2,\underline{m}} = \frac{X^2Y^3}{X^4Z} = \frac{Y^3}{X^2Z} \in \frac{\mathscr{L}(\underline{m}\underline{P})}{\mathscr{L}((\underline{m}-\varepsilon_2)\underline{P})}$. $\square$

Algorithm 5.12. *There is an alternative way of calculating these functions $f_{i,\underline{m}}$, computationally more effective:*

(1) *Take a basis of $\mathscr{L}(\underline{m}\underline{P})$, say $\{h_1, \ldots, h_s\}$.*
(2) *Calculate the pole orders at P_i, $\{-\text{ord}_{P_i}(h_1), \ldots, -\text{ord}_{P_i}(h_s)\}$.*
(3) *Order these pole orders increasing, in such a way that $-\text{ord}_{P_i}(h_s) = m_i$. We can assume this, as otherwise, if $-\text{ord}_{P_i}(h_s) = k_i > m_i$ we can replace m_i by k_i, since $\mathscr{L}(m_1P_1 + \ldots + m_iP_i + \ldots + m_rP_r) = \mathscr{L}(m_1P_1 + \ldots + k_iP_i + \ldots + m_rP_r)$.*
(4) *The function h_s has pole order m_i at P_i, but other functions could also have the same property. So, for any h_j satisfying $-\text{ord}_{P_i}(h_j) = m_i$, there exists $\lambda_j \neq 0$ in $\mathbb{F}$ such that $h_j = \lambda_j h_s$, that is, $-\text{ord}_{P_i}(h_j - \lambda_j h_s) < m_i$. So we change h_j by $g_j := h_j - \lambda_j h_s$, and $g_k := h_k$ for $k \neq j$.*

(5) *Now we have a set of functions* $\{g_1, \ldots, g_s\}$ *where* $g_s = f_{i,\underline{m}}$, *and* $\{g_1, \ldots, g_{s-1}\}$ *is a basis for the vector space* $\mathcal{L}((\underline{m} - \varepsilon_i)\underline{P})$.

Example 5.13. *We present a worked example in* SINGULAR *for computing functions as above. First we import the library* `brnoeth.lib` *and another one* `several.lib` *in which we have programmed the procedure* `ordRF` *that computes the pole orders of a rational function:*

```
> LIB "brnoeth.lib";
> LIB "several.lib";
> int plevel=printlevel;
> printlevel=-1;
```

We define the ring and the curve:

```
> ring s=2,(x,y),lp;
> list C=Adj_div(x3y+y3+x);
 ==>The genus of the curve is 3
```

The list of computed places is

```
> C=NSplaces(1,C);
> C[3];
 -->[1]:
 -->    1,1
 -->[2]:
 -->    1,2
 -->[3]:
 -->    1,3
```

The base point of the first place of degree 1 *is, in homogeneous coordinates:*

```
> def SS=C[5][1][1];
> setring SS;
> POINTS[1];
 -->[1]:
 -->    0
 -->[2]:
 -->    1
 -->[3]:
 -->    0
> setring s;
```

We define the divisor `G=4C[3][1]+4C[3][3]`:

```
> intvec G=4,0,4;
> def R=C[1][2];
```

A basis `LG` *for* $\mathcal{L}(\underline{m}\underline{P})$ *is supplied by the Brill-Noether algorithm:*

```
> setring R;
> list LG=BrillNoether(G,C);
 -->Vector basis successfully computed
> int lG=size(LG);
```

The pole orders for the rational functions in `LG` *are*

```
> int j;
> intvec h;
```

```
> for (j=1;j<=lG;j=j+1){
. h[j]=ordRF(LG[j],SS,1)[1]; . }
> h;
 -->0,-1,2,-2,-3,-4
```

And the desired rational function is

```
> LG[lG];
 -->_[1]=xyz2+y4
 -->_[2]=x4
> printlevel=plevel;
```

5.2. **Dimension of the Riemann-Roch quotients.** Computing the dimension of $\frac{\mathscr{L}(m P)}{\mathscr{L}((\underline{m}-\underline{1})P)}$
is an easy task by Corollary 5.8:

Proposition 5.14.

$$\frac{\mathscr{L}(\underline{m}P)}{\mathscr{L}((\underline{m}-\underline{1})P)} = \frac{\mathscr{L}(\underline{m}P)}{\mathscr{L}((\underline{m}-\varepsilon_1)P)} \oplus \frac{\mathscr{L}((\underline{m}-\varepsilon_1)P)}{\mathscr{L}((\underline{m}-\varepsilon_1-\varepsilon_2)P)} \oplus \ldots \oplus \frac{\mathscr{L}((\underline{m}-\varepsilon_1-\varepsilon_2-\ldots-\varepsilon_{r-1})P)}{\mathscr{L}((\underline{m}-\underline{1})P)}.$$

Proof. It is just to define the map $(f_1,\ldots,f_r) \mapsto f_1 + \ldots + f_r$. $\qquad\qquad\square$

Note 5.15. *Notice that the map $(f_1,\ldots,f_r) \mapsto f_1 \cdot \ldots \cdot f_r$ cannot work, because products do not preserve poles. It is also important the fact that the P_i are different, otherwise the statement does not hold: take for example $f(z) = \frac{1}{z}$ and $h(z) = z - \frac{1}{z}$ in $\mathbb{C}$. The sum $f(z) + h(z) = \frac{1}{z} + z - \frac{1}{z} = z$ has no poles, however $f(z)$ and $h(z)$ have both a simple pole at 0.*

5.3. **Computing the Weierstraß semigroup at several points.** Preserving notations, let $\Gamma_{\underline{P}}$ be the Weierstraß semigroup at the points $P_1, P_2, \ldots, P_r$ and Γ_{P_i} the Weierstraß semigroups corresponding to the points P_i for $i = 1, \ldots, r$. Write $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ and $\underline{m_i} := \underline{m} - m_i \varepsilon_i$.

Proposition 5.16. *Let $\underline{m} \in \mathbb{N}^r$, $i \in \{1,\ldots,r\}$ and $\underline{m_i} \in \mathbb{N}^r \setminus \Gamma_{\underline{P}}$. Let*

$$m := \min\left\{ n \in \mathbb{N}^* \mid \underline{m_i} + n\varepsilon_i \in \Gamma_{\underline{P}} \right\}.$$

Then any vector $\underline{n} = (n_1,\ldots,n_r) \in \mathbb{N}^r$ belongs to $\mathbb{N}^r \setminus \Gamma_{\underline{P}}$ whenever $n_i = m$, and $n_j = m_j = 0$ or $n_j < m_j$ for $j \neq i$. In particular, m is a gap at P_i.

Define the usual partial order $\preceq$ over $\mathbb{N}^r$, that is, for $\underline{m}, \underline{n} \in \mathbb{N}^r$:

$$(m_1,\ldots,m_r) \preceq (n_1,\ldots,n_r) \iff m_i \leq n_i \text{ for all } i = 1,\ldots,r.$$

Proposition 5.17. *Let $i \in \{1,\ldots,r\}$ and $\underline{m} = (m_1,\ldots,m_r)$ be a minimal element of the set*

$$\left\{ (n_1,\ldots,n_r) \in \Gamma_{\underline{P}} \mid n_i = m_i \right\}$$

with respect to the partial order $\preceq$. Assume that $n_i > 0$ and the existence of one $j \in \{1,\ldots,r\}$, $j \neq i$ with $m_j > 0$. Then:

  a) $\underline{m_i} \in \mathbb{N}^r \setminus \Gamma_{\underline{P}}$.
  b) $m_i = \min\{n \in \mathbb{N}^* \mid \underline{m_i} + n\varepsilon_i \in \Gamma_{\underline{P}}\}$; *in particular, m_i is a gap at P_i.*

Propositions 5.16 and 5.17 determine a surjective map

$$\varphi_i : \quad \left\{\underline{m}_i \in \mathbb{N}^r \mid \underline{m}_i \in \mathbb{N}^r \setminus \Gamma_{\underline{P}}\right\} \quad \longrightarrow \quad \mathbb{N} \setminus \Gamma_{P_i}$$

$$\underline{m}_i \quad \mapsto \quad \min\left\{m \in \mathbb{N}^* \mid \underline{m}_i + m\varepsilon_i \in \Gamma_{\underline{P}}\right\}.$$

For $r = 2$ this is in fact a bijection between the set of gaps at P_1 and the set of gaps at P_2:

$$m_1 \in \mathbb{N} \setminus \Gamma_{P_1} \Leftrightarrow (m_1, 0) \in \mathbb{N}^2 \setminus \Gamma_{\underline{P}} \mapsto \beta_{m_1} := \varphi_2((m_1, 0)) \in \mathbb{N} \setminus \Gamma_{P_2}.$$

Furthermore, $m_1 = \min\left\{n \in \mathbb{N}^* \mid (n, \beta_{m_1}) \in \Gamma_{\underline{P}}\right\}$. More details can be found in Homma and Kim [16] and Kim [17].

We summarize some remarkable facts for the case of two points ($r = 2$), which will be useful from the computational point of view:

(i) All the gaps at P_1 and at P_2 are also gaps at P_1, P_2.

(ii) By the Corollary 5.17, for any gap m_1 at P_1, one has that (m_1, β_{m_1}) are gaps at P_1, P_2 for $\beta_{m_1} = 0, 1, \ldots, l_{m_1}$, until certain $0 \leq l_{m_1} \leq 2g - 1$, with g the genus of the curve and where l_{m_1} satisfy that $l_{m_1} + 1$ is a gap at P_2. The point $(m_1, l_{m_1} + 1)$ is an element of $\Gamma_{\underline{P}}$, which we will call the *minimal (non-gap) at m_1*. We will refer to the set of the minimal non-gaps at every gap at P_1 (they will be g, since the number of gaps at P_1 is precisely g) as the set of *minimal non-gaps at P_1*.

(iii) The gaps obtained of that form, this is, the set

$$\left\{(m_1, \beta_{m_1}) \in \mathbb{N}^2 \setminus \Gamma_{\underline{P}} \mid m_1 \in \mathbb{N} \setminus \Gamma_{P_1} \text{ and } \beta_{m_1} = 0, 1, \ldots, l_{m_1} \text{ with } l_{m_1} + 1 \in \mathbb{N} \setminus \Gamma_{P_2}\right\}$$

will be called the *set of gaps* respect to P_1.

(iv) Similarly, for any gap m_2 at P_2, one has that (α_{m_2}, m_2) are gaps at P_1, P_2 for $\alpha_{m_2} = 0, 1, \ldots, l_{m_2}$, until some $0 \leq l_{m_2} \leq 2g - 1$, with g being the genus of the curve and where l_{m_2} satisfy that $l_{m_2} + 1$ is a gap at P_1. The point $(l_{m_2} + 1, m_2)$ is an element of $\Gamma_{\underline{P}}$, which we will call the *minimal (non-gap) at m_2*. The set of the minimal non-gaps for every gap at P_2 will be called the set of *minimal non-gaps at P_2*. The cardinality of such a set is g, since g is the number of gaps at P_2.

(v) The set of gaps

$$\left\{(\alpha_{m_2}, m_2) \in \mathbb{N}^2 \setminus \Gamma_{\underline{P}} \mid m_2 \in \mathbb{N} \setminus \Gamma_{P_2} \text{ and } \alpha_{m_2} = 0, 1, \ldots, l_{m_2} \text{ with } l_{m_2} + 1 \in \mathbb{N} \setminus \Gamma_{P_1}\right\}$$

is called the *set of gaps* respect to P_2.

(vi) The intersection between the set of gaps respect to P_1 and respect to P_2 is not necessarily empty. In fact, the gaps in the intersection are just the *pure gaps* at P_1, P_2.

The minimal non-gaps at P_1 and P_2 provide enough information in order to deduce the Weierstraß semigroup at P_1, P_2. Recall that we have already described algorithms to compute the dimension (and associated functions, when is possible) of the Riemann-Roch quotients $\frac{\mathscr{L}(m\underline{P})}{\mathscr{L}((\underline{m} - \varepsilon_i)\underline{P})}$ for given $\underline{m}$, $i \in \{1, 2\}$ and two rational points P_1, P_2 on an absolutely irreducible projective algebraic plane curve χ (see Algorithm 5.9 and Algorithm 5.12). An algorithm computing the set of minimal non-gaps at P_i, for $i = 1, 2$ is the following:

Algorithm 5.18. *Write* $\dim(\underline{m}, P, C, i)$ *for the procedure calculating the dimension of the quotient vector space* $\frac{\mathcal{L}(\underline{m}P)}{\mathcal{L}((\underline{m}-\varepsilon_i)\underline{P})}$*:*

  INPUT*: points* P_1, P_2*, an integer* $i \in \{1, 2\}$ *and a curve* χ*.*

  OUTPUT*: the set of minimal non-gaps at* P_i*.*

- *let L be a empty list and g be the genus of* χ*;*
- *let* W_1 *and* W_2 *be the lists of gaps of* χ *at* P_1 *and* P_2*, respectively;*
- FOR $k = 1, \ldots, g$; $k = k + 1$;
    - IF $i = 1$ THEN
        * *j=size of* W_2;
        * WHILE $\Big(\dim((W_1[k], W_2[j]), P, \chi, i) = 1$ AND $\dim((W_1[k], W_2[j] - 1), P, \chi, i) = 1)$ OR $j = 0 \Big)$ DO
            · $j = j - 1$;
        * $L = L \cup \{(W_1[k], W_2[j])\}$;
        * $W_2 = W_2 \setminus \{j\}$;
    - ELSE
        * *j=size of* W_1;
        * WHILE $\Big(\dim((W_1[j], W_2[k]), P, \chi, i) = 1$ AND $\dim((W_1[j] - 1, W_2[k]), P, \chi, i) = 1)$ OR $j = 0 \Big)$ DO
            · $j = j - 1$;
        * $L = L \cup \{(W_1[j], W_2[k])\}$;
        * $W_1 = W_1 \setminus \{j\}$;
- RETURN*(L);*

Example 5.19. *Let* χ *be the curve over* $\mathbb{F}_2$ *given by the equation* $F(X, Y, Z) = X^3 Z + X^4 + Y^3 Z + YZ^3$*. Consider the points* $P_1 = [0 : 1 : 1]$ *and* $P_2 = [0 : 1 : 0]$ *on* χ*. Then*

$$\mathbb{N}^2 \setminus \Gamma_{\{P_1, P_2\}} = \Big\{ (0, 1), (0, 2), (1, 0), (1, 2), (2, 0), (2, 1) \Big\},$$

as shown in the figure (the black points are the elements of $\Gamma_{\underline{P}}$*, the other ones are the gaps at* P_1, P_2*):*
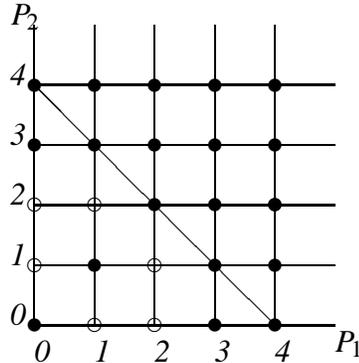
As an illustration of the Corollary 5.17, for instance let $i = 1$*,* $\underline{m} = (m_1, m_2) = (2, 2) \in \Gamma_{\underline{P}}$ *and the set* $\Big\{ (n_1, n_2) \in \Gamma_{\underline{P}} \mid n_1 = m_1 \Big\} = \Big\{ (2, n) \text{ for } n \geq 2 \Big\}$*. A minimal element for*

this set is $(2,2)$, *and*

$$\underline{m}_i = \underline{m} - m_1 \varepsilon_1 = (2,2) - 2(1,0) = (0,2)$$

is a gap at P_1, P_2. *We compute*

$$\min\left\{n \in \mathbb{N}^* \mid (n,2) \in \Gamma_{\underline{P}}\right\} = 2 = m_1,$$

and $m_1 = 2$ *is actually a gap at* P_1.

In this example we can also see the bijection between the gaps at P_1 and the gaps at P_2. Preserving notations as above, take now $n_1 = 1$ as a gap at P_1. Then $(1,0)$ is a gap at P_1, P_2 and

$$\varphi_2((1,0)) = \min\left\{n \in \mathbb{N}^* \mid (1,0) + (0,n) \in \Gamma_{\underline{P}}\right\} = \min\left\{n \in \mathbb{N}^* \mid (1,n) \in \Gamma_{\underline{P}}\right\} = 1,$$

with 1 being a gap at P_2. Moreover, $n_1 = 1 = \min\left\{n \in \mathbb{N}^* \mid (n, 1 = \varphi_2((1,0))) \in \Gamma_{\underline{P}}\right\}$.

Now take $p_1 = 2$ as the other gap at P_1. Then $\varphi_2((2,0)) = 2$, which is a gap at P_2. Indeed $p_1 = 2 = \min\left\{n \in \mathbb{N}^* \mid (n, \varphi_2((2,0))) \in \Gamma_{\underline{P}}\right\}$. The same happens to the gaps at P_2.

6. COMPUTATIONAL ASPECTS USING SINGULAR

We are interested in explaining the most important procedures implemented in SINGU-LAR and to give examples to show how to work with them.

More precisely, in subsection 6.1) we give some hints of use of the library `brnoeth.lib`, since our procedures are based on most of the algorithms contained in it. Then, in Subsection 6.2 we present the procedures which pretend generalize the computation of the Weierstraß semigroup to the case of several points, i.e., a set of procedures which try to:

- compute $\dim_{\mathbb{F}} \frac{\mathscr{L}(\underline{m}\underline{P})}{\mathscr{L}((\underline{m}-\varepsilon_i)\underline{P})}$ and a function $f_{\underline{m},i} \in \mathscr{L}(\underline{m}\underline{P}) \setminus \mathscr{L}((\underline{m}-\varepsilon_i)\underline{P})$ if possible.
- compute the set of minimal non-gaps at a point P_i, for $i \in \{1,2\}$.

6.1. **Hints of usage of brnoeth.lib.** The purpose of the library `brnoeth.lib` of SINGU-LAR is the implementation of the Brill-Noether algorithm for solving the Riemann-Roch problem and some applications in Algebraic Geometry codes, involving the computation of Weierstraß semigroups for one point.

A first warning: `brnoeth.lib` accepts only prime base fields and absolutely irreducible planes curves, although this is not checked.

Curves are usually defined by means of polynomials in two variables, that is, by its local equation. It is possible to compute most of the concepts concerning to the curve with the procedure `Adj_div`. We defined the procedure (previously we must have defined the ring, the polynomial f and have charged the library `brnoeth.lib`):

```
> list C=Adj_div(f);
```

The output consist of a list of lists as follows:

- The first list contains the affine and the local ring.
- The second list has the degree and the genus of the curve.

- Each entry of the third list corresponds to one closed place, that is, a place and all its conjugates, which is represented by two integer, the first one the degree of the point and the second one indexing the conjugate point.
- The fourth one has the conductor of the curve.
- The fifth list consists of a list of lists, the first one, namely `C[5][d][1]` being a (local) ring over an extension of degree d and the second one (`C[5][d][2]`) containing the degrees of base points of places of degree d.

Furthermore, inside the local ring `C[5][d][1]` we can find the following lists:

- `list POINTS`: base points of the places of degree d.
- `list LOC_EQS`: local equations of the curve at the base points.
- `list BRANCHES`: Hamburger-Noether expressions of the places.
- `list PARAMETRIZATIONS`: local parametrizations of the places.

Now we explain how the different kinds of common objects must be treated in `Singular`.

Affine points P are represented by a standard basis of a prime ideal, and a vector of integers containing the position of the places above P in the list supplied by `C[3]`; if the point lies at the infinity, the ideal is replaced by an homogeneous irreducible polynomial in two variables.

A **place** is represented by the four list previously cited: a base point (`list POINTS` of homogeneous coordinates); a local equation (`list LOC_EQS`) for the curve at the base point; a Hamburger-Noether expansion of the corresponding branch (`list BRANCHES`); and a local parametrization (`list PARAMETRIZATIONS`) of such a branch.

A **divisor** is represented by a vector of integers, where the integer at the position i means the coefficient of the i-th place in the divisor.

Rational functions are represented by ideals with two homogeneous generators, the first one being the numerator of the rational function, and the second one being the denominator.

Furthermore, we can compute a complete list containing all the non-singular affine (closed) places with fixed degree d just by using the procedure `NSplaces` in this way:

```
> C=NSplaces(1..d,C);
```

Closer to our aim is the procedure `Weierstrass`, which computes the non-gaps of the Weierstraß semigroup at one point and the associated functions with poles. It contains three inputs:

- an *integer* indicating the rational place in which we compute the semigroup;
- an *integer* indicating how many non-gaps we want to calculate;
- the curve given in form of a *list* `C=Adj_div(f)` for some polynomial f representing the local equation of the curve at the point given in the first entry.

This procedure needs to be called from the ring `C[1][2]`. Moreover, the places must be necessarily *rational*.

6.2. **Procedures generalizing to several points.** We present now a main procedure to compute the dimension of the so-called Riemann-Roch vector spaces of the form $\mathscr{L}(\underline{m}\underline{P}) \setminus \mathscr{L}((\underline{m}-\varepsilon_i)\underline{P})$. If this dimension is equal to 1, the procedure is also able to compute a rational function belonging to the space.

The technique developed here is not by using the adjunction theory directly, as we have developed theoretically in the Chapter 3 (Algorithm 5.9), because of its high cost, but we use the Algorithm 5.12, or, more properly speaking, a slight variant of it: we order the poles in a vector from the biggest one to the smallest one (in absolute value) and we take the first in such a vector.

```
proc RRquot (intvec m, list P, list CURVE, int chart)
"USAGE:RRquot( m, P, CURVE, ch );  m,P intvecs, CURVE a list and
ch an integer. RETURN:   an integer 0 (dimension of
L(m)\L(m-e_i)), or a list with three entries:
  @format
  RRquot[1] ideal (the associated rational function)
  RRquot[2] integer (the order of the rational function)
  RRquot[3] integer (dimension of L(m)\L(m-e_i))
  @end format
NOTE:     The procedure must be called from the ring CURVE[1][2],
          where CURVE is the output of the procedure @code{NSplaces}.
@*        P represents the coordinates of the place CURVE[3][P].
@*        Rational functions are represented by
          numerator/denominator
          in form of ideals with two homogeneous generators.
WARNING:  The place must be rational, i.e., necessarily
CURVE[3][P][1]=1. @* SEE ALSO: Adj_div, NSplaces, BrillNoether
EXAMPLE:  example RRquot; shows an example " {
  // computes a basis for the quotient of Riemann-Roch vector spaces L(m)\L(m-e_i)
  // where m=m_1 P_1 + ... + m_r P_r and m-e_i=m_1P_1+...+(m_i-1)P_i+...+m_r P_r,
  // a basis for the vector space L(m-e_i) and the orders of such functions, via
  //   Brill-Noether
  // returns 2 lists : the first consists of all the pole orders in
  //   increasing order and the second consists of the corresponding rational
  //   functions, where the last one is the basis for the quotient vector space
  // P_1,...,P_r must be RATIONAL points on the curve.

  def BS=basering;
  def SS=CURVE[5][1][1];
  intvec posinP;
  int i,dimen;
  setring SS;
  //identify the points P in the list CURVE[3]
  int nPOINTS=size(POINTS);
  for(i=1;i<=size(m);i=i+1)
   {
     posinP[i]=isPinlist(P[i],POINTS);
   }
```

```
//in case the point P is not in the list CURVE[3]
  if (posinP==0)
    {
      ERROR("The given place is not a rational place on the curve");
    }
  setring BS;
  //define the divisor containing m in the right way
  intvec D=zeroes(m,posinP,nPOINTS);
  list Places=CURVE[3];
  intvec pl=Places[posinP[chart]];
  int dP=pl[1];
  int nP=pl[2];

  //check that the points are rational
  if (dP<>1)
  {
    ERROR("The given place is not defined over the prime field");
  }
  int auxint=0;
  ideal funcion;
  funcion[1]=1;
  funcion[2]=1;

  // Brill-Noether algorithm
  list LmP=BrillNoether(D,CURVE);
  int lmP=size(LmP);
  if (lmP==1)
  {
    dimen=0;
    return(dimen);
  }
  list ordLmP=list();
  list sortpol=list();
    for (i=1;i<=lmP;i=i+1)
    {
      ordLmP[i]=orderRF(LmP[i],SS,nP)[1];
    }
    ordLmP=extsort(ordLmP);
    if (D[posinP[chart]] <> -ordLmP[1][1])
      {
    dimen=0;
        return(dimen);
      }
    LmP=permute_L(LmP,ordLmP[2]);
    funcion=LmP[1];
    dimen=1;
    return(list(funcion,ordLmP[1][1],dimen));
} example
```

```
 {
  "EXAMPLE:"; echo=2;
  int plevel=printlevel;
  printlevel=-1;
  ring s=2,(x,y),lp;
  poly f=y2+y3+x3+x4;
  list C=Adj_div(f);
  C=NSplaces(1,C);
  def pro_R=C[1][2];
  setring pro_R;
  intvec m=4,6;
  intvec P1=0,1,1;
  intvec P2=0,1,0;
  list P=P1,P2;
  int chart=1;
  RRquot(m,P,C,chart);
  printlevel=plevel;
 }
```

Let us see an example:

```
> example RRquot;
// proc RRquot from lib brnoeth.lib
EXAMPLE:
  int plevel=printlevel;
  printlevel=-1;
  ring s=2,(x,y),lp;
  poly f=y2+y3+x3+x4;
  list C=Adj_div(f);
The genus of the curve is 2
  C=NSplaces(1,C);
  def pro_R=C[1][2];
  setring pro_R;
  intvec m=4,6;
  intvec P1=0,1,1;
  intvec P2=0,1,0;
  list P=P1,P2;
  int chart=1;
  RRquot(m,P,C,chart);
Vector basis successfully computed
-->[1]:
   _[1]=x3+yz2
   _[2]=xyz+xz2
-->[2]:
   -4
-->[3]:
   1
  printlevel=plevel;
```

This procedure needs also the following auxiliar procedures:

As RRquot reads off the point through its homogeneous coordinates we need to localize that point in the list POINTS and make the correspondence between such a point and its position in the list of points contained in the third output of the procedure Adj_div. This is done by mean of the routine isPinlist. Its inputs are the point P in homogeneous coordinates, that is, a vector of integers, and the list L of points from Adj_div. The output is an integer being zero if the point is not in the list or a positive integer indicating the position of P in L. Look at the example:

```
> example isPinlist;
// proc isPinlist from lib brnoeth.lib
EXAMPLE:
  ring r=0,(x,y),ls;
  intvec P=1,0,1;
  list POINTS=list(list(1,0,1),list(1,0,0));
  isPinlist( P,POINTS);
-->1
```

We need also a procedure for ordering a list of integers. This is partially solved by the procedure sort from general.lib. But sort is not able to order lists of negative numbers, so we have extended this algorithm to extsort. The extsort procedure needs to permute a vector of integers by the instructions given by another similar vector. This is actually done for lists of integers (permute_L in brnoeth.lib), but not for vectors of integers. This lack is covered by the procedure perm_L, whose entries are a pair of vectors, the second vector fixing the permutation of the first one. The output consists of the permutated vector, as the following example shows:

```
> example extsort;
// proc extsort from lib brnoeth.lib
EXAMPLE:
  ring r=0,(x,y),ls;
  list L=10,9,8,0,7,1,-2,4,-6,3,0;
  extsort(L);
-->[1]:
   -6,-2,0,0,1,3,4,7,8,9,10
-->[2]:
   9,7,4,11,6,10,8,5,3,2,1
```

Finally, it was interesting to fix the system for reading off the data of the divisor needed in the BrillNoether procedure. Our routine zeroes takes two vectors of integers m and pos, and an integer siz and it builds up a vector of size siz, with the values contained in m set in the places given by pos and zeroes in the other places. This algorithm is called zeroes:

```
> example zeroes;
// proc zeroes from lib brnoeth.lib
EXAMPLE:
  ring r=0,(x,y),ls;
  intvec m=4,6;
  intvec pos=4,2;
  zeroes(m,pos,5);
-->0,6,0,4,0
```

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