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Characterizing Student Engagement in a Post-Secondary Developmental Mathematics Class and Exploring the Reflexivity between Social and Sociomathematical Norms

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To my first math teachers, James and Jean Fifty

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ABSTRACT

Characterizing Student Engagement in a Post-Secondary Developmental Mathematics Class and Exploring the Reflexivity between Social and Sociomathematical Norms

by

David Fifty

University of New Hampshire, December, 2020

Traditionally, post-secondary developmental mathematics courses aspire to equip students with mathematical content knowledge needed to succeed in calculus and subsequent STEM courses. The literature shows that this goal alone is insufficient, as the emphasis on content acquisition often comes at the expense of developing higher-order skills such as argumentation, reasoning, and flexibility in mathematics problem solving (Chiaravalloti, 2009; Partanen & Kaasila, 2014; Star et al., 2015). Redesigning curricula with these additional objectives in mind requires providing students with opportunities to engage with mathematics in ways that may contrast with their past experiences or expectations. It requires changing patterns of classroom engagement and development of different classroom norms.

This mixed methods research study incorporated a semester-long teaching experiment that aimed to support students' development of higher-order skills by negotiating productive classroom norms. One of the primary interventions was a sequence of "Multiple Solutions Activities" that required groups of students to analyze and critique unfamiliar or erroneous mathematical solutions. The overarching goal of the research was to study students' engagement during these activities across the semester by characterizing the nature of specific types of classroom norms. Social norms describe the classroom participation structure, while sociomathematical norms focus on aspects of student activity that are inherently mathematical, such as what constitutes an acceptable mathematical solution (Yackel & Cobb, 1996). Because of a reflexive relationship between norms and beliefs, students' social and mathematical beliefs were also of interest to characterize the influence of the teaching experiment; these beliefs were assessed by a pre- and post-course questionnaire.

The results paint a complex picture of student engagement and values. Despite quantitative analysis suggesting encouraging improvements in students' mathematical engagement, qualitative analysis highlighted that this change was not homogenous. In particular, the analysis revealed variations in students' perceptions of the value of multiple solutions and in the nature of the norms developed in student groups. Consequently, the study highlights the lasting impact of classroom norms on students' beliefs, and vice versa, which may hinder the development of alternative norms in subsequent classes. The results of the project also expand upon Yackel and Cobb's (1996) Interpretive Framework for characterizing classroom engagement by suggesting a reflexive relationship exists between social and sociomathematical norms. The data analysis describes concurrent development and mutual influence between the participation structure of a group and their taken-as-shared mathematical beliefs. In all, the project shows that deliberate attention towards negotiating productive classroom norms and students' in-class engagement can positively affect students' attitudes towards multiple solutions.

Chapter 1. Statement of the Problem

Incoming college students who are low-achieving in mathematics or have underdeveloped mathematics backgrounds are often placed in developmental mathematics courses such as College Algebra or Precalculus. In the context of post-secondary mathematics education, a *developmental* course refers to any college mathematics course that is part of the high school core curriculum (Hagedorn, 1999). Often, such courses are disproportionally populated by students of color[1], students of low-socioeconomic status, and first generation college students (Hodara, 2019), so providing high quality developmental mathematics courses is necessary for supporting diverse and equitable access to STEM.

Research over recent decades describes a variety of difficulties associated with teaching and learning in developmental mathematics courses. This includes students' poor conceptions of the nature of mathematics and their own mathematical capabilities (Stage & Kloosterman, 1991); inferior methods of instruction and lack of faculty dedicated to developmental mathematics courses (Boyer et al., 2007); ineffectiveness of developmental mathematics courses on student performance (Lagerlof & Seltzer, 2009); and decreased student persistence and success over extended developmental mathematics sequences (Ngo & Kosiewicz, 2017). . There is also a significant positive correlation between success in developmental mathematics classes and socioeconomic status (SES) (Hagedorn, 1999), meaning that students from higher SES perform better in these courses while students from lower SES tend to perform poorly in these courses. Unsurprisingly, completion, retention, and graduation rates of students enrolled

[1] American Indian/Alaska Native, Black/African American, and Hispanic/Latino students

in developmental mathematics courses have been areas of concern (Boyer et al., 2007; Bahr, 2013; Kirp, 2017).

One traditional shortcoming in particular had been unsuitable course objectives, as remediation was often viewed merely as an attempt to bring a student up to a passing grade to get through the course (Treisman, 1985). Simply providing students with repeated exposure to remedial content is typically insufficient in preparing them for subsequent classes if students are not developing more productive mathematical capabilities (Goudas & Boylan, 2013). Repeated exposure alone may not require students to change the mathematical practices and habits that contributed towards their need for remediation, since it does not address students' abilities to learn new mathematics (Carlson et al., 2010). This struggle to learn new mathematics may contribute to the long-term frustrations and high attrition rates that students who enroll in developmental mathematics classes experience in their mathematics sequence (Carlson et al., 2010; Thompson et al., 2007).

To improve achievement amongst those taking developmental mathematics courses, some researchers have suggested that educators need to focus on improving students' argumentation skills, reasoning strategies, and flexible knowledge (see section 2.8) (Chiaravalloti, 2009; Partanen & Kaasila, 2014; Star & Rittle-Johnson, 2008). Many upper-secondary students have deep-rooted struggles in these areas, preventing them from productively engaging in mathematics learning (Wismath & Worrall, 2015; Kirp, 2017). One attributing cause may be that many students, especially those with poor mathematics skills, prefer a dependent learning style with a procedural focus towards mastering algorithms (Chiaravalloti, 2009; Partanen & Kaasila, 2014).

Dependent learning styles do not support the development of *autonomy*, which characterizes students' mathematical independence from a source of authority, such as a teacher or textbook. When students lack autonomy, their mathematical activity is typically characterized by an emphasis on reproducing algorithms from these sources of authority to arrive at answers. This pursuit of mechanistic reproduction of algorithms leads to a rigid understanding of mathematics because it is reliant on memorization and is difficult to adapt to new circumstances; this rigid understanding later serves as an ill-formed prerequisite for new mathematical conceptions.

Any reform efforts for improving students learning outcomes in developmental mathematics need to surpass considerations to revise the curriculum. As Bonham and Boylan (2011, p. 6) state, "Redesigning the curriculum content is necessary but not sufficient to stem the crises of failure and noncompletion in developmental mathematics." Additional considerations need to be placed on students' mathematical activity inside the classroom to help shape how students are engaging with mathematics, as practice does not make perfect, only *proper* practice does. Otherwise, students' unproductive engagement may circumvent developing the deeper mathematical reasoning skills needed to succeed in subsequent mathematics courses and STEM fields (Kazemi & Stipek, 2008/2009).

The challenges associated with the teaching and learning in developmental courses, described above, highlight a gap in the literature that warrants research: a need to investigate how educators can develop classroom cultures that foster students' higher order skills, which are necessary for students' success in and beyond developmental mathematics classes. There is also a need to understand how to ensure productive and mathematically meaningful student engagement, as well as what barriers hinder or prevent such engagement. By attempting to study these issues, this research study seeks to generate theoretical and practical knowledge to assist educators in structuring learning environments that help foster students' flexible knowledge and reasoning skills by focusing on establishing productive student engagement.

At the University of New Hampshire, MATH 418: Analysis and Application of Functions (typically referred to as "Precalculus" within the department) is the only developmental mathematics course that is offered. This has traditionally been a challenging course for instructors and students – who normally are, or aspire to be, STEM majors. Despite efforts to improve the Precalculus course, instructors and students experience struggles similar to those discussed in the introduction. This course provides an opportunity to conduct a teaching experiment to study its influence on students' mathematical engagement.

Chapter 2. Theoretical Perspective / Conceptual Framework

Before detailing the research questions, I describe the theoretical backdrop of this study.

2.1 Student Engagement

This project views student engagement as “the in-the-moment relationship between someone and her immediate environment, including the tasks, internal states, and others with whom she interacts” (Middleton el al., 2017, p. 667). As a consequence, student engagement incorporates both individual/cognitive and communal/social components. The following sections depict the theoretical framework used by this study to characterize students’ engagement.

2.2 The Emergent Perspective

The emergent perspective, which was first introduced in seminal papers by Paul Cobb and Erna Yackel (Cobb & Yackel, 1996; Yackel & Cobb, 1996), coordinates constructivism (von Glasersfeld, 1995) and interactionism (Blumer, 1969) to account for individual and communal mathematical activity (Partanen & Kaasila, 2014). This duality acknowledges both psychological and sociological factors of learning in the classroom.

The psychological considerations of the emergent perspective focus on the individual who constructs their own unique understanding (von Glasersfeld, 1996). Learning is characterized as cognitive self-organization, and naturally encompasses the Piagetian conceptions of assimilation and accommodation. Consequently, a major catalyst of cognitive development is the reorganization of individual activity to eliminate cognitive perturbations that the individual experiences.

Meanwhile, the emergent perspective complements this cognitive focus with the interactionist perspective, which considers the interpersonal nature of education (Bauersfeld, 1980; Blumer, 1969). Interactionism asserts that communication is a process of mutual adaptation where individuals negotiate meanings by continually modifying their interpretations (Cobb & Yackel, 1998). Accordingly, learning is characterized by "the subjective reconstruction of societal means and models through negotiation of meaning in social interaction" (Bauersfeld, Krummheuer, and Voigt, 1988, p. 39).

The duality of cognitive constructivism and interactionism inherently links psychological and sociological factors in the emergent perspective through a reflexive relationship (Yackel & Cobb, 1996; Fukawa-Connelly, 2012). One way to understand this reflexivity is to consider how social interaction naturally gives rise to conflicts in individual students' mathematical interpretations; this is to be expected because of the uniqueness of and differences in students' individual conceptions and personal meanings. Thus, the intellectual struggle of assuaging the conflict students' experience in recognizing these differences can be seen to precipitate individual mathematical learning, as students experience cognitive restructuring (Cobb & Yackel, 1996).

In other words, social interaction often explicates differences in individual students' mathematical interpretations; the intellectual struggle of reconciling conceptual conflicts generated by social interaction can be seen to motivate individual mathematical learning. Furthermore, the development of individual student's conceptions can be seen to influence their participation in the mathematical classroom community.

In summary, as students participate in a mathematics classroom community they are implicitly reorganizing their own cognitive structures and beliefs. Reorganizations of these cognitive aspects elicit changes to how students participate in the community. Thus, the emergent perspective draws on both psychological and sociological perspectives, which are reflexively related, to provide a framework that allows analyzing "the development of individual minds [as well as] the evolution of the local social [communities within] which those minds participate" (Cobb, 1995, p. 10).

2.3 Norms and Microcultures

Before describing a framework for analyzing both individual and communal activity at the classroom level, it is necessary to understand norms and microcultures. *Norms* characterize mutually established and regulated activity or behavior amongst a collective (Cobb et al., 2001). In a class, norms are not pre-made rules for students to follow but are rather developed through continual student and teacher interaction, either explicitly or implicitly. Even though teachers typically initiate the negotiation of norms, norms are usually based on mutual expectations that are formed as both students and teachers interact with one another (Yackel et al., 2000).

Norms, the mathematical classroom community, the learning environment, any social interactions, and the construction of mathematical meaning all contribute to the formation of a *microculture* (Voigt, 1995; Guven & Dede, 2017). Particular types of norms, social and sociomathematical norms, can be used to characterize mathematics classroom microcultures (Cobb et al., 2001). It is important to note that microcultures are not transportable. Different classes will establish and negotiate different microcultures. But, the emergent perspective holds that norms and activities will influence students' individual beliefs and mathematical practices, which may persist to subsequent classes.

2.4 Interpretive Framework

In conjunction with the emergent perspective, Yackel and Cobb (1996) designed an interpretive framework to analyze individual and communal activity at the classroom level by coordinating the reflexive relationship between social and psychological components of learning, which are expressed in Table 1. For example, this framework suggests that the development and negotiation of sociomathematical norms in the classroom, to be defined and described shortly, guides or shapes the reorganization of students' individual mathematical beliefs or values. Additionally, the reorganization of individual mathematical beliefs or values influences how students negotiate sociomathematical norms in the classroom. This example shows that the framework not only describes individual and communal activity inside the classroom, but also expresses reflexivity between social and psychological constructs. In the following sections, components will be described as well as the relationship between a sociological component and its psychological correlate. This project focuses on the first two rows of the Interpretive Framework, since, as the authors themselves admit, the last row is the most underdeveloped part of the framework (Cobb et al., 1997).

Figure 1

The Interpretive Framework

Social Perspective		Psychological Perspective
Classroom Social Norms	↔	Beliefs about one's own role, others' roles, and the general nature of mathematical activity in school
Sociomathematical Norms	↔	Mathematical values and beliefs
Classroom Mathematical Practices	↔	Mathematical interpretations and activity

Note: (Yackel & Cobb, 1996)

2.5 Classroom Social Norms and Beliefs of Roles

Two interrelated entities that influence students' participation in classroom interactions are *social norms*, which are a sociological construct, and their psychological counterpart, *individual students' role and activity beliefs*. Social norms characterize accepted patterns of behavior and are jointly established and negotiated by both teachers and students in the classroom community (Cobb & Yackel, 1996; Fukawa-Connelly, 2012). Social norms govern the classroom participation structure and regulate interaction in the microculture (Rumsey & Langrall, 2016).

By contributing to the negotiation of social norms in the classroom microculture, students reorganize their own individual beliefs about their own role in the class or as a learner, other microculture members' roles, and the overall activity of the classroom (Cobb et al., 2001). Accordingly, these individual beliefs influence how students negotiate norms in the classroom.

For example, in a classroom microculture, a teacher might initiate the social norm of listening to one's peers' solutions or collaborating on classwork. These social norms influence the participation structure of the class. Participating in this negotiation might cause students to see their peers' roles develop from a classmate or bystander to a "co-learner" or someone who could be an intellectual resource. Thus, the students' *individual beliefs about their own role and the role of their peers* will concurrently develop in conjunction with the evolution of the social norm.

2.6 Sociomathematical Norms and Mathematical Values

Another key component of a mathematics class microculture is *sociomathematical norms,* which are norms specific to mathematical aspects of students' activity (Yackel & Cobb, 1996; Kazemi & Stipek, 2008/2009). One sociomathematical norm of particular importance to this project is the sociomathematical norm of what constitutes an acceptable mathematical solution. Like social norms, sociomathematical norms are social constructs that are negotiated amongst members of the microculture.

It is important to this project to be able to distinguish between social and sociomathematical norms, so an example is provided. Trying to understand the solutions of others is an example of a social norm. The development of this norm characterizes the participation structure of a class, but is not restricted to the characterization of a mathematics class. On the other hand, sociomathematical norms describe what constitutes an acceptable or different mathematical solution, as these norms are inherently linked to the mathematics context of students' activity and engagement. Further, the sociomathematical norms of a microculture shape how students interact with mathematics, including how they interpret and solve mathematical problems (Voigt, 1995) and reason and justify their thinking (Inglis & Ramos, 2009). This helps to analyze the level of intellectual autonomy of the students in the microculture, by revealing if students are relying on mathematical reasoning or a source of authority, such as a textbook or an instructor.

The emergent perspective and the interpretive framework explain concurrent development of sociomathematical norms and students’ individual mathematical values and beliefs. For example, as members of the microculture collectively negotiate the characterization of mathematical difference, individual members reorganize their internalized conceptions of what it means for mathematical objects or solutions to be different. Because of this reflexive relationship, students' mathematical values and beliefs are characterized as the *psychological correlate* of sociomathematical norms (Table 1).

2.7 Prior Usage of the Interpretive Framework to Study Undergraduate Mathematics Courses

Studies have utilized the interpretive framework in a variety of ways over the past two decades and with various post-secondary classes. To better understand the components and relationships within the interpretive framework, as well as its contributions to the study of student engagement, it is helpful to discuss several examples.

Some studies characterized individual components of the framework. For example, Roy et al. (2014) investigated social and sociomathematical norms that were established and re-established in a mathematics content course for prospective elementary teachers. One unique aspect of the study is its focus on the role of content in the persistence of norms. In particular, this work identifies shifts in sociomathematical norms when the content changed from whole number concepts and operations to those with rational numbers. Most notably, during this transition, the prospective elementary teachers reverted to familiar, but poorly understood, procedures that they remembered from their childhood. As a consequence, the sociomathematical norm of what constitutes an acceptable solution needed to be re-established and reinforced, as norms may not be sustained alone by an introduction and/or limited discussion. This study describes the evolution of sociomathematical norms, representing a focus on a singular cell of the interpretive framework.

Other studies investigate the reflexivity of the emergent perspective by analyzing within-row relationships between social and psychological constructs (Table 1). In one of these studies, Yackel & Rasmussen (2002) investigated the within-row relationship between social norms and their psychological correlate, students' beliefs about their role and about what constitutes mathematical activity (the highlighted arrow in Table 2) in an undergraduate differential equations class. This project explains that students may enter a class with beliefs that contrast with the expectations that underpin inquiry instruction. The authors continue to describe that although an instructor may negotiate norms by explicating their expectations for the class's activity, the students also participate in the constitution of norms. For example, as students act in accordance with expectations, they are contributing to the ongoing constitution of these expectations. This developing pattern of interaction in the class influences and perpetuates the expectations on which they are based, and thus ultimately sustains individual participants' beliefs. This represents social norms and individual beliefs as working together in a dynamic system where both mutually evolve, as each acts as a backdrop to study and understand the other.

Figure 2

Highlighting Within-Row Reflexivity in the Interpretive Framework (Yackel & Cobb, 1996)

Social Perspective		Psychological Perspective
Classroom Social Norms	⟺	Beliefs about one's own role, others' roles, and the general nature of mathematical activity in school

Note: Adapted from Yackel & Cobb (1996)

Recently, diagonal column-row relationships within the interpretive framework have been studied (Table 3). The relationship between social norms and students' individual conceptions was studied in a graduate level mathematics course on chaos and fractals (Rasmussen et al., in press). This study concluded that engaging with another's reasoning supported change in one's own reasoning. Engaging with another's reasoning relates to the social norm of listening to and trying to make sense of another's thinking. The emergent perspective details the mutual evolution of the development of social norms (e.g. supporting engagement with others' arguments) and students' beliefs about the general nature of mathematical activity (e.g. students' understanding of what mathematics looks like). However, this work extends the influence of developing social norms, asserting that support of engagement with others' arguments (i.e. the social norm researched) also deepened students' own mathematical reasoning (i.e. mathematical conceptions and activity).

Figure 3:

Highlighting a Diagonal Column-Row Relationship in the Interpretive Framework

Social Perspective		Psychological Perspective
Classroom Social Norms		Beliefs about one's own role, others' roles, and the general nature of mathematical activity in school
Sociomathematical Norms		Mathematical values and beliefs
Classroom Mathematical Practices		Mathematical conceptions and activity

Note: Adapted from (Yackel & Cobb, 1996)

2.7.1 Relationship Between Sociological Constructs

Existing literature latently describes a connection between social constructs, a *within-column relationship* of the interpretive framework. For example, Yackel and Cobb (1996) described a teaching experiment in which students were prompted to share different solutions, but most students would simply repeat their own, even if it had already been discussed by another peer. The class then negotiated the sociomathematical norm of mathematical difference (between solution methods). This negotiation focused on the idea that "difference" should apply to mathematical concepts used in the solution, not to the language that describes it. This required students to reflect on their solutions as well as those of others; thus, solutions themselves became objects of reflection. By empowering students to scrutinize solutions for themselves, responsibility for mathematical learning is devolved to students, enabling them to become a community of validators (Rumsey & Langrall, 2016). Students in this class began challenging peers' solutions which they believed were already discussed. This depicts the negotiation of a sociomathematical norm influencing the development of social norms and the participation structure.

Another classroom, depicted by Roy et al. (2014), shows that sociomathematical norms may influence social norms. Much like the class discussed above, the presentation of only mathematically different solutions became a sustained social norm, as students started to present different solutions without being prompted by the teacher. But the negotiation of mathematical difference also influenced students' general activity of the class, as students began to anticipate different possibilities for how a task might be solved. Moreover, finding alternative solutions became an inherent part of any posed task.

These examples depict that there may be relationships between social constructs, or within-column relationships (see Table 1). However, such relationships are not explicitly discussed or sufficiently researched. This represents a gap in the literature about the connectivity of various facets of students' engagement.

2.8 Didactical Contracts

The theoretical construct of didactical contracts will be used as an explanatory backdrop to help clarify the development of classroom activity. A *didactical contract* is composed of a set of behaviors of the teacher that are expected by students and a set of behaviors of the students that are expected by the teacher (Yoon et al., 2011; Pierce et al., 2010). The behaviors most useful in this study, are those with respect to the uptake of mathematical knowledge and engagement with in-class activities.

An important aspect of the didactical contract is the usage of resources, termed the *milieu*, in the classroom (Pierce et al., 2010). The milieu includes texts, writing utensils, the white board, as well as course materials, mathematical problems posed, and class activities. In particular, these resources factor into the didactical contract by how they are expected to be used in class.

One key focus of productive didactical contracts is the devolution of responsibility for students' knowledge from the teacher to each student (Yoon et al., 2011). Ideally, teachers aim to develop students' autonomy, making them the primary authority for their own learning. One challenge with this devolution is that teachers may only delegate responsibility to students, with respect to new knowledge, when the milieu is endowed with feedback potential (Pierce et al., 2010). This creates a delicate balance between allowing for productive student struggle and providing immediate explicit feedback.

Didactical contracts provide means to explore perceived expectations of members in classroom microcultures, which is a fundamental aspect to the negotiation, development, and sustainment of norms. In particular, breaches in norms typically coincide with conflicts in expectations, which represent violations in didactical contracts. Additionally, there is a natural correspondence between beliefs and expectations (whether individual or taken-as-shared), which further connects the Interpretive Framework and the didactical contract. The didactical contract will be used as an explanatory mechanism to help describe the negotiations of norms as well as the barriers to developing productive norms, without laying fault to members of the microculture or previous microcultures that students were members of.

2.9 Flexible Knowledge

Recent policy documents advocate for the importance of students developing flexible knowledge in mathematics problem solving, or *flexibility*, which refers to the ability to generate, use, and evaluate multiple solution methods for given problems (Common Core State Standards Initiative, 2010; National Council of Teachers of Mathematics, 2006; Star et al., 2015). In addition to improving students' conceptual and procedural knowledge, developing flexibility often coincides with providing opportunities for students to practice reasoning skills (Star & Rittle-Johnson, 2009). Developing this flexibility may require a form of engagement that students are not accustomed to, which may even contrast with their own preferences (as discussed in Section 1). Thus, to develop flexibility in developmental mathematics classes, it is important for instructors to negotiate norms and practices that encourage engagement that focuses on utilizing these skills.

Chapter 3. The Setting of the Study

At the University of New Hampshire, MATH 418: Analysis and Application of Functions (typically referred to as "Precalculus" within the department) is the only offered developmental mathematics course. This has traditionally been a challenging course for instructors and students. Despite efforts to improve the course, instructors and students still experience struggles similar to those discussed in the introduction. This provides an opportunity to study and influence a "typical" developmental mathematics course, whose students normally are, or aspire to be, STEM majors.

This convergent mixed methods[2] project aims to better understand the microculture of a MATH 418 (Precalculus) class while conducting a teaching experiment as the instructor of the course (Fetters et al., 2013). Being the course instructor allowed me to explicitly initiate the negotiation of norms that I believed would help foster the development of students' reasoning skills, flexible knowledge, and autonomy.

The teaching experiment took place in Spring 2019. The course structure followed a traditional lecture/recitation format; all students attended the same lecture on Mondays, Wednesdays, and Fridays, and were registered for one of three smaller recitations (or labs) on Tuesday and Thursday.

This format of the labs allowed for small groups of students to be studied; charactering the nature of norms and practices negotiated and sustained within these groups allowed for understanding students' in-class engagement. The setting of the course provides a unique

[2] Qualitative and quantitative data are collected in parallel and analysis for integration occurs after data collection has been completed.

opportunity to gather data from multiple, and differently composed, student groups in an effort to see wider patterns and relationships between these types of norms. In particular, this study investigates the relationships and connections between social components of students' engagement. Additionally, this study examines the relationship between social and sociomathematical norms, and explores changes to students' individual beliefs and values.

Chapter 4. Research Questions

The following are the research questions investigated by this dissertation study:

The first set of research questions focuses on characterizing social factors among small groups of students in MATH 418 (Precalculus) during Multiple Solutions Activities (to be described in Section 5.1.2.2).

1a.) **Sociomathematical Norms**: What is the nature of the sociomathematical norms developed amongst groups in the classroom microculture? How do the characterizations of these norms compare and contrast amongst the groups?

1b.) **Social Norms**: What is the nature of the social norms developed amongst groups in the classroom microculture? How do the characterizations of these norms compare and contrast amongst different groups?

The second research question investigates how social factors influence one another, thus exploring possible within-column relationships of the interpretive framework.

2.) **Relationships Amongst Social Components of a Microculture**: In what ways do social and sociomathematical norms influence one another's development within groups? How, if at all, do these components co-develop?

The third research question investigates changes to psychological components of the Interpretive Framework: students' mathematical values and beliefs; and, students' beliefs about their role, others' roles, and the general nature of classroom activity.

3a.) Mathematical Values and Beliefs: How do students' mathematical values and beliefs change, if at all, over the course of the semester?

3b.) Students' Beliefs and Values: How do students' beliefs about role and the general nature of classroom activity change, if at all, over the course of the semester?

Chapter 5. Methodology

This dissertation study was conducted in two stages with the same research questions; this included a pilot study (hereafter Stage 1) and a teaching experiment (Stage 2). Stage 1 was completed during the Spring 2018 semester. The intention of the first round of data collection was to garner information about a typical MATH 418 (Precalculus) microculture and to pilot a sequence of constructed instructional activities (*Multiple Solutions Activities*, described below). Stage 1 primarily consisted of daily observations of MATH 418, administering questionnaires, conducting interviews, and providing the course instructors with instructional activities that aimed to explicate information about the development of social and sociomathematical norms (see section 5.1.2.2 and Appendix A).

The extent of my involvement in MATH 418 was more comprehensive during Stage 2, for which I conducted a teaching experiment and studied the development of in-class student engagement. In Stage 2, I was the sole instructor of record of the course, and intentionally initiated the negotiation of productive norms, aiming to aid in enriching the development of students' individual beliefs and practices. With the help of teaching and learning assistants, we attempted to establish a learning environment that would foster students' mathematical understanding, including their flexible knowledge, and their autonomy. This stage was conducted during the Spring 2019 semester.

5.1 Stage 1 – Spring 2018 Semester

Efforts and analyses in Stage 1 helped structure and direct Stage 2. Collecting and analyzing data from Stage 1 allowed me to learn about norms and practices typically developed in MATH 418. This awareness equipped me with perspective that helped guide my data collection and analysis, instruction, and my negotiation of norms as instructor during the teaching experiment. Importantly, Stage 1 was needed to pilot my research instruments for the second round of data collection.

In Stage 1, each of the three sections of MATH 418 was taught by a graduate student five days a week; each section was populated by less than 15 students. Because of the size of each section and the presence of the same instructor for every class meeting, I conjectured that the norms and practices of a section would be more accessible to study. I chose to more closely examine one of the three sections offered during the Spring 2018 semester, which I refer to as the "focus section." Both the instructor and students of the focus section participated in extensive aspects of the study.

5.1.1 Stage 1 Participants

During the Spring 2018 semester, 25 students enrolled in MATH 418 (Precalculus) took part in the study. Typically, students who enroll in MATH 418 during the spring semester have either already taken MATH 418 but did not earn a sufficient grade for their major, did not reach the necessary score on the placement exam to enroll in MATH 425 (Calculus I), are non-traditional students (e.g., part-time students, students who serve in the military, and/or are students returning to school after a hiatus).

The instructor of the focus section, Ethan[3], was a first-year graduate student in the Mathematics Education PhD program at UNH. The Spring 2018 semester was his first semester as an instructor of record; but during the Fall 2017 semester, he was a teaching assistant for MATH 418. His undergraduate degree was in Secondary Education in Mathematics and then he

[3] Names used reflect pseudonyms for participants of the study.

received a master's degree in Mathematics Education. Ethan had previous teaching experience at his undergraduate university such as holding office hours for an undergraduate introduction to proof course and student teaching for 10-12th grade classes. In addition, Ethan was a long-term substitute teacher when pursuing his master's degree. Ethan was asked to participate in the study because of his background in mathematics education and because of his enthusiasm and passion for teaching.

5.1.2 Instruments

There are five main data sources: (1) a beginning and end of the semester questionnaire about mathematical values and beliefs, (2) class video recordings and group work video recordings, (3) instructional activities, (4) field notes, and (5) student and instructor interviews.

5.1.2.1 Questionnaire. Given the reflexive relationship described by the emergent perspective, one way to assess the impact of norms developed over the semester would be to assess the changes of their psychological correlates. To determine any changes in students' individual beliefs and values, 25 students across all Spring 2018 sections of MATH 418 took a beliefs questionnaire twice, at the beginning and end of the semester. This five-choice Likert scale questionnaire was created and validated by Wismath and Worrall (2015). Although the questionnaire was anonymous, a non-identifying code was generated by each student to pair their beginning and end of semester questionnaires.

Several lessons were learned from piloting the questionnaire. First, the low rate of student response represented a problem. A sufficient sample size is needed for inferential statistical analysis; typically, a power of 0.8 is needed, which often requires a sample size larger than 30. Given the desire to analyze data by using a paired *t*-test to measure differences between beginning and end of semester questionnaire responses, changes needed to be made for Stage 2 to increase the response rate.

Secondly, analyses of the questionnaire responses demonstrated a disproportionate selection of the middle choice for the five-point Likert scale, which was seen to dilute the data. Having a neutral option may have provided participants with an opportunity to advance through a question without serious reflection.

Lastly, despite the fact that the questionnaire used in Stage 1 was validated, I determined that many items did not measure the specific type of beliefs that were important and of interest to this study. For example, the item, "I would recommend taking mathematics courses to my friends," did not provide productive insights into students' mathematical or social beliefs. Based on the limitations of Stage 1 questionnaire, I made revisions to the questionnaire used in Stage 2 with assistance from the UNH Survey Center.

5.1.2.2 Instructional Activities. Typically, norms are widely abided through unconscious acceptance, which can make them difficult to study (Braswell, 2014). To elicit information about social norms, Garfinkel (1967) introduced the idea of *breaching experiments*. This methodology can be characterized by a researcher attempting to violate conjectured social norms. The idea behind this method is that although it may be difficult to perceive some social norms, it is easy to recognize when social norms are violated. This study adapts aspects of this idea in the form of an instructional activity for students, which was designed to help uncover some of the sustained social and sociomathematical norms negotiated within the microculture.

Additionally, research shows that asking students to compare and contrast solutions methods is an effective way to help foster students' flexibility (Star et al., 2015; Star & Rittle-

Johnson, 2008; Rittle-Johnson & Star, 2007). Furthermore, recent standards call for opportunities for students to critique the arguments of others (NGA & CCSSO, 2010). Such practices intend to benefit the development of students' reasoning and argumentation skills. Thus, these sources, in addition to Garfinkel's work, served as inspiration for the formation of the instructional activities.

Each instructional activity, hereafter referred to as *Multiple Solutions Activities*, has three phases. The first phase, *Problem Solving and Rubric Development,* involves students solving a problem in groups of three to four students and creating a grading rubric for the problem. The problem is reflective of the content of the class and is not meant to be overly challenging for students, nor obvious; instead, the problem acts to situate mathematical discussion within each group. Since group work activities in the course are intended to be cooperative, formulating a solution for this problem helps display some of the social norms related to problem solving adopted by the group. Developing a grading key for the problem provides information about the developing sociomathematical norms of each group. For example, by formulating a grading key, groups express and perhaps even further negotiate what constitutes an acceptable mathematical solution. Students are asked to resolve any differences in order to unanimously agree on a rubric; consequently, the solution and grading key produced by the groups should ideally reflect mutually accepted patterns of mathematical and social behavior negotiated within the microculture.

For the next phase, *Evaluating Sample Work*, groups are provided with three samples of fictitious students' work (i.e. "sample solutions"), which they need to cooperatively evaluate with their grading rubric. The sample solutions may represent methods that contrast those typically discussed in class, thus breaching students' expectations for approaching the problem. This aspect of the activities helps foster students' flexible knowledge by exposing them to different ways to approach the problem. These sample solutions aim to elicit the degree to which groups value aspects of mathematical solutions, which represent sociomathematical norms. Other sample solutions may skip steps or explanations, utilize informal mathematical notation, or include minor mathematical errors as additional ways to examine how students measure and perceive mathematical justification and reasoning. Concurrently, interpreting and critiquing the sample solutions provides valuable learning opportunities for the students.

The third and final phase of this activity, *Group and Class Reflection*, is composed of students responding to reflection questions followed by a whole class discussion. Reflection questions (e.g. "What are some components of a quality solution?") are provided to students to compare the relative efficacy, efficiency, and clarity of each approach, a vital component to developing flexible knowledge (e.g. Rittle-Johnson & Star, 2007). Another intention of these questions is to prepare students for a class discussion by having students reflect with their small groups first. The class discussion allows the instructor to explicitly initiate and negotiate social and sociomathematical norms. For instance, given the variety of approaches for the fictitious students' work, the instructor can negotiate the sociomathematical norms of mathematical difference, what constitutes an acceptable solution, and what constitutes an efficient solution.

During the Spring 2018 semester, two groups were video recorded using 360 degree cameras during three instructional activities. Using several instructional activities over the course of the semester allows for the negotiation of norms to be tracked. Thus, the three instructional activities were evenly spaced across the semester. The activities were on the topics of finding the vertex of a quadratic, finding the inverse of an exponential function, and finding the area of a triangle by using the Law of Sines (see Appendix A). Stage 1 provided an opportunity to clarify the language in the directions of the activity.

In addition to video recording the instructional activities, students' work was also collected, including grading rubrics and their evaluations of the sample solutions. Having students' work helped to clarify students' activity and negotiations in conjunction with the video recordings. Students' work provided important information and context about their in-class engagement, as much of the written mathematical work during the activity may not be seen clearly through the video camera.

One major obstacle revealed during Stage 1 was the low and inconsistent student attendance. As a primary data source for investigating student engagement, it was problematic that students sporadically attended class to participate in the activities. For those that did attend class, many came late, which interrupted a flow of the activities. These problems were successfully addressed in Stage 2.

5.1.2.3 Interviews. Semi-structured interviews utilize pre-determined questions which are posed systematically, but the participant and interviewer are expected to digress and probe beyond the questions (Clement, 2000). This type of probing can be effective at garnering additional insights and allows the participant and interviewer to ask clarifying questions to assure mutual understanding. With respect to this study, semi-structured interviews allow for better understating the participants' individual beliefs and values, the psychological constructs of the interpretive framework. Furthermore, repeated interviews allow for tracing the development of these aspects over the semester.

In Stage 1, four students and the instructor participated in two semi-structured individual interviews, once near mid-semester and once towards the end. Two more students participated in one interview: one near mid-semester and the other at the end of the semester. The interview protocol is included in Appendix C.

Analyzing Stage 1 interview data helped determine what information about students' individual beliefs and values would be useful to collect in Stage 2. For ethical considerations, as the instructor, I did not conduct interviews for Stage 2. But, this Stage 1 data helped inform updates to questions included on the Stage 2 questionnaire.

The Stage 1 interview data was also used to make instructional decisions regarding the course structure in Stage 2. Primarily, students reported a disconnectedness between various facets of the course, such as the ALEKS online homework assignments and the written assessments for the course. Consequently, ALEKS was not used in Stage 2. Additionally, students shared the struggle of being distracted by other students' tardiness to class.

5.1.2.4 Class Recordings. Both lectures and group work were video recorded. Lectures were recorded with a digital camcorder from the back of the class. Since most of the lectures were representative of a traditional format, with the instructor presenting mathematical content to the students, the camcorder focused on the teacher. This helped better understand how students were being exposed to the content and captured some instances of the instructor initiating and negotiating norms.

During group work, which included the instructional activities and other worksheets selected by the instructor, two groups of students were recorded using 360° Cameras. These cameras allowed for 360-degree recording of the activities while being only minimally intrusive (some of the students even reported that they would forget that the camera was there). This key data source allowed for the investigation of the development of classroom norms and relationships between these norms, which responds to the research questions.

As discussed in Section 5.1.2.2, brief analysis of the Stage 1 class recordings surfaced a social norm that developed: inconsistent attendance and tardiness was acceptable. The analysis further exemplified the effects of this norm on students' learning opportunities, the progression through the curriculum, and from developing more productive classroom norms. Consequently, this helped motivate the attendance policy for Stage 2.

5.1.2.5 Field Notes. Since I regularly attended the focus section's class, I kept a detailed record of classroom activity and notes on social or mathematical behavior. These field notes served to supplement the class recordings and artifacts collected during the instructional activities.

5.2 Stage 2 - Spring 2019 Semester

In Stage 2, I conducted a teaching experiment to garner more control over structure of the course. As the sole instructor of record, I was able to utilize suggestions made in the literature and findings from the analysis of Stage 1 data to attempt to improve students' in-class engagement and learning outcomes.

The second round of data collection took place during the Spring 2019 semester. This round needed to be significantly adjusted due to changes in the MATH 418 (Precalculus) course structure and an unforeseen spike in enrollment in the course – from expected 35 students to 80. The department changed the course structure to a traditional lecture-recitation format, with lectures on Monday, Wednesday, and Friday, and three smaller recitations on Tuesday and Thursday. All students had the same instructor and teaching assistant.

Because of the unanticipated spike in enrollment, two learning assistants were recruited to help during the lecture periods. The teaching assistant (TA) for the course was a PhD candidate in the mathematics education program, who also had prior experience teaching the course and volunteered to participate in the study. The recitations were either led by the TA or both he and I would co-teach.

5.2.1 Changes to the Course Structure and Setting

Analysis of Stage 1 data provided an opportunity to understand the role of homework in the course, especially in relation to the development of norms and practices. Analysis of interviews revealed discrepancies between the homework's targeted knowledge and acquired knowledge. One student in Stage 1 distinguished between his practices for the ALEKS homework and those for other aspects of the class. His practices differed so widely that he decided to maintain two different notebooks, one for class and one for ALEKS. He also described that when he worked on his ALEKS homework, he was able to draw from technological resources to aid his practices, such as the use of a calculator or the program DESMOS (Desmos Graphing Calculator, 2015). The use of these resources in class were not approved by his instructor. The tension that the student described between usage of ALEKS and other facets of the course was echoed by other interview participants. This represents a myriad of conflicting influences on classroom norms and practices.

Consequently, for Stage 2, I decided not to integrate ALEKS into the course structure. Instead, written homework was collected at the beginning of each lecture. Students were encouraged to use technological resources, like DESMOS, to assist with the completion of the homework. Links or references to DESMOS were often included in the homework postings. Additionally, one of the design purposes of assigning more frequent, but smaller, assignments was to respond to attendance issues experienced in Stage 1 (as described in Section 5.1.2.2). Although the frequency of these assignments was a source of frustration for some students, attendance during lecture was nearly full on a daily basis.

Furthermore, students' participation was assessed during labs for Stage 2. Half of the allotted points were for timely attendance; students were told that to earn this credit, they only needed to show up to class on time. The other portion of the credit was for active engagement during class. This was primarily a deterrent for cellphone misuse in class.

To initiate the negotiation for higher-order skills, it is important to provide students with resources that mirror these values (see section 2.8). Consequently, an online book was chosen for its focus on conceptual competencies and its approachable language and presentation. Another factor that motivated this decision was the consideration that developmental mathematics courses are disproportionately populated by students of low-socioeconomic status. The book was integrated into the course; links to the appropriate section were provided in announcements to correspond with the content covered in class. Homework assignments included reading assignments from the online book.

5.2.2 Participants

The class was composed of 80 students, with more than half enrolling in the course for the second time. The participants during the Spring 2019 semester shared the same characteristics as the students enrolled during the Spring 2018 semester.

A critical research component of Stage 2 was the use of four groups of four students working on a sequence of instructional activities. These groups were randomly assigned and held consistent throughout the semester. Table 1 shows the composition of the groups and the students' pseudonyms.

Table 1

Group Composition by Previous Enrollment

	Students who had previously taken MATH 418	**Students who were taking MATH 418 for the first time**
Group 1	Harry	Albert, Dwayne, Gordon
Group 2	Chad, Molly, Peter, Steve	
Group 3	Herbert, Ted, Wes	Cullen
Group 4	Meghan	Julia, Paul*, Ron**

*Transfer student who enrolled in a developmental mathematics class at former institution
**First semester of post-secondary education

5.2.3 Instruments

Several data sources used during Stage 2 paralleled those in Stage 1: instructional activities, class video recordings, and group work video recordings. The beliefs questionnaire used in Stage 1 was modified (see section 5.1.2.1).

5.2.3.1 Questionnaire. For Stage 2, I removed the neutral response option, creating a four-point Likert scale, which is within the "optimum" range of reliability and validity (Lozano et al., 2008). In addition, Garland (1991) suggests that removing the median of a five point Likert scale minimizes social desirability bias (respondents' desire to please or help the interviewer). Yet, adapting a validated and reliable instrument poses risks, such as skewing data more negatively (Garland, 1991) or positively (Worcester & Burns, 1975).

The wording and inclusion of some of the items in the instrument were further adapted (in consultation with UNH's Survey Center) to provide more insight into students' mathematical beliefs, beliefs about role and general mathematics activity, and students' mathematical habits and practices. The questionnaire was administered to students in the first and last week of the semester, during class time. The questionnaire contained 27 questions and took between 5-10 minutes to complete (Appendix B). Similar to Stage 1, the first item in the questionnaire asked students to form a non-identifying code (see Appendix B), which was used to match students' pre- and post- questionnaires while preserving the confidentiality of students' response.

5.2.3.2 Instructional Activities. The *Multiple Solutions Activities* were also used in Stage 2 with the same three-phase structure as in Stage 1 (see section 5.1.2.2). Overall, in Stage 2, there were four Multiple Solution Activities spread across the semester. The mathematical topics were: function domain and interval notation, vertex of a quadratic function, inverse of an exponential function, and inverse trigonometric functions. Each activity contained three hypothetical students' solutions, which the study participants were asked to analyze during the activities. Below, I describe these solutions and the rationale in their design (see Appendix A for complete activity handouts). It is important to note some of the terminology used below. "*Solution*" refers to the written problem solving process of arriving at an "*answer*," or conclusion. Here, a solution includes any written work, whether it is descriptive or computational. An answer is part of a solution.

5.2.3.2.1 Activity 1 – Functions Domain and Interval Notation.

The Problem/Expectations

The problem for this activity was:

Find the domain of the following function. Express your answer in interval notation.

$$F(x) = \frac{\sqrt{x}+1}{2\sqrt{1-3x}}$$

It was expected that students would evaluate the intersection of the domain restriction in the numerator (e.g. $x \geq 0$), and the denominator (e.g. $1 - 3x > 0$). Students were familiar with both types of natural domain restrictions involved: the input of square roots being non-negative and avoiding division by 0. The students were also previously assessed on interval notation.

One of the primary objectives of this activity was for students to experience the importance and usefulness of adhering to formal notation. Thus, sample solutions utilized improper or informal notation, which lead to mathematical errors or misinterpretations of the results. Such solutions acted as context for the instructors to negotiate the importance of notation as means to communicate one's understandings and ideas with others.

Tom's Solution

This approach explicitly notates work that applies to the denominator and the numerator. First, the work in the denominator shows the whole denominator not-equal to 0:

$$2\sqrt{1-3x} \neq 0$$

This is an inconvenient conceptualization of finding the domain of a function, as it is analyzing the output of a root instead of the input. Nevertheless, the approach continues to isolate x, but then switches to an inequality:

$$\frac{1}{3} \neq x$$

$$x > \frac{1}{3}$$

Even though the input of the square root could be set greater than 0 to adhere to the domain restriction, because of the negative coefficient of the linear term, this should instead yield $x < 1/3$.

The work for the numerator makes a similar conceptual argument by also focusing on the output of the square root, then switching to an inequality:

$$\left(\sqrt{x}\right)^2 \neq 0^2$$

$$x > 0$$

Both inequalities have arrows drawn to a statement: "Domain is the smallest value, so $D_F: x > 0$." This incorrectly addresses a common, informal phrase used when describing the union of two intervals. The solution did not provide an answer in interval notation.

This solution illustrates the importance of adhering to conventional or formal notation. In my experience, it is common for Precalculus students to utilize informal notation with inequalities, making similar mistakes when interpreting the results, as above.

Andrea's Solution

This approach starts by addressing the domain restriction: "Cannot take the square root of a negative value." Then, similar to Tom's approach, the solution splits the work for the numerator and denominator. The work in the denominator starts as such:

For $\sqrt{1-3x}$

$$1 - 3x \geq 0$$

One flaw here is that it does not address the domain restriction of dividing by 0. But, the work continues to correctly isolate x. The resulting inequality then has an arrow that points to the inaccurate statement: “This tells us that all numbers less than 1/3 are in the domain.” This does not appropriately address the “less than or equal to” inequality.

Similar to the work in the denominator, the work in the numerator states: $x \geq 0$, and has an arrow that points to the incorrect statement: “This tells us that all numbers greater than zero are in the domain.” The two statements suggest that the union of the two resulting intervals provide the domain, instead of the intersection. A number line is then drawn with rays for both inequalities, concluding with a claim that the domain of the function is $(-\infty, \infty)$.

Several characteristics of this solution were intended to provide opportunities for initiating the negotiation of productive classroom norms. For example, solutions should communicate one’s understanding, and consequently, should be written so that others can understand. Andrea’s solution was used as context by the instructors to advocate for using written sentences to provide insight into what the writer is thinking, making it easier for another to interpret and follow their work. Also, using different representations, like number lines, can be helpful for both the writer and the reader of the solution.

Brody’s Solution

This solution utilizes informal notation (see Appendix A), making it difficult to interpret. Yet, despite the informal notation, the solution does yield correct inequalities for the numerator and denominator. The concluding line tries to summarize this informally:

$$x < \frac{1}{3} + x \geq 0 = \left[0, \frac{1}{3}\right]$$

In addition to the unconventional notation, the interval incorrectly includes 1/3.

The inclusion of informal notation was motivated by students’ lack of adherence to conventional notation, particularly with inequalities and interval notation. By providing a solution that was rather extreme in its use of unconventional notation, the instructors would be able to use this as context to negotiate the sociomathematical norm that solutions should utilize conventional notation throughout the solution.

5.2.3.2.2 Activity 2 – Vertex of a Quadratic Function.

The Problem/Expectations

The problem for this activity was:

Find the vertex of the following function: $$f(x) = -2x^2 - \frac{1}{3}x + \frac{2}{3}$$

It was anticipated that students would solve this problem by completing the square, a method explored in class and typically emphasized in traditional Precalculus classes. Two of the sample solutions used approaches that were novel to students, and one used completing the square.

Frodo's Solution

This approach finds the zeros of the given quadratic by factoring and obtains the x-coordinate of the vertex by taking the average of the two zeros. The y-coordinate of the vertex is found by evaluating the function at the x-coordinate. Frodo's solution contains an error in the notation of the final answer: Frodo "boxes" the y-coordinate as the answer, and does not report the vertex as a point with both coordinates. The importance of this notation was previously stressed in class; so this provides an opportunity to further negotiate the importance of adhering to conventional notation in solutions, which represents a sociomathematical norm. Frodo's solution also does not contain any written descriptions, other than "vertex @ $x = -1/12$." This absence was used to highlight the usefulness of written descriptions in solutions.

Furthermore, the solution is written clearly and sequentially. This represents a novel solution that yields the correct answer (in the wrong format).

Kennedy's Solution

The approach starts with listing the quadratic formula, and substituting in it the coefficients of the quadratic: $ax^2 + bx + c$. On the next line, $\frac{\frac{1}{3}}{2\cdot(-2)}$ is circled, and an arrow with the word vertex is pointed to the third line, where "x = $\frac{\frac{1}{3}}{-4}$" is incorrectly simplified as -4/3. The approach finds the x-coordinate of the vertex by evaluating "$-\frac{b}{2a}$". Since "b" is a fraction, I incorporated an intentional mistake in simplifying the quotient, a common error amongst students in the course. The y-coordinate is found by evaluating the quadratic at the errant x-coordinate. The answer is written as a point. Other than the mathematical error in simplifying the x-coordinate (the error simplifying $-\frac{b}{2a}$), the approach would yield the correct answer.

This solution represents a correct approach, novel to students, which leads to the incorrect answer, due to the intentional simplification error. The solution also lacks any explanations or clarifications. Inclusion of this solution provided an opportunity for the instructors to negotiate norms that an answer should not determine the validity of the approach and a mathematical solutions should include explanations and clarifications to help the reader interpret the solution.

Andrea's Solution

This approach starts by re-writing the quadratic and setting it equal to 0 - a misleading practice common with students in MATH 418. The approach utilizes the completing the square algorithm, where terms are added to both sides to create a perfect square on the left-hand side. An error is included in this solution when adding terms to both sides:

$$f(x) = -2\left(x^2 + \frac{1}{6}x\right) = -2/3$$

$$f(x) = -2\left(x^2 + \frac{1}{6}x + \left(\frac{\frac{1}{6}}{2}\right)^2\right) = -\frac{2}{3} + \left(\frac{\frac{1}{6}}{2}\right)^2$$

On the left-hand side, the value $-2 \cdot \left(\frac{\frac{1}{6}}{2}\right)^2$ is added, whereas only $\left(\frac{\frac{1}{6}}{2}\right)^2$ is added to the right-hand side.

Eventually, the quadratic is reported in vertex-form and the answer is written as a point with the correct x-coordinate and an incorrect y-coordinate. This solution can be characterized as following a familiar approach, but yielding a partially incorrect answer, as one of the two coordinates is correct. Additionally, there is no descriptive language included in the solution, other than the answer being labeled as "Vertex."

In addition to developing procedural competencies, this solution was included as a means to contrast familiar and unfamiliar solution methods. Accordingly, the instructors were able to negotiate that any valid approach, regardless of familiarity, should be considered acceptable – a sociomathematical norm.

5.2.2.2.3 Activity 3 – Inverse of an Exponential Function.

The Problem/Expectations

The problem for this activity was:

Find the inverse of the following function:

$$f(x) = \frac{9}{4}3^{8x} - \frac{5}{2}$$

In the week leading up to the activity, the students were learning about logarithms and revisiting inverse functions. The students previously worked on similar problems to this one in class and for homework. It was anticipated that the students would follow the traditional algorithm of "swapping" x and y ("$f(x)$") and solving for y. Each of the students' solutions below utilized a logarithm with a different base. This offers opportunities to develop flexibility by having students explore different bases, and consequently different properties of logarithms. Additionally, examining the use of different bases allows students to compare which base provides the more efficient approach. The variety of approaches provided the instructors with the opportunity to negotiate that any valid approach should be considered an acceptable solution – a sociomathematical norm.

Lincoln's Solution

Lincoln's solution largely follows the aforementioned familiar algorithm, yet includes several key differences. First, the solution includes taking a square root of both sides:

$$x + 5 = \frac{9}{4} \cdot 3^{8y}$$

$$\sqrt{x + \frac{5}{2}} = \sqrt{\frac{9}{4} \cdot 3^{8y}}$$

This step is unneeded but was included to provoke discussion about distinguishing between correctness and efficiency. This solution is designed to communicate that although the square-root breaches the standard algorithm, this step does not invalidate the solution.

A base-three logarithm is used in the solution, appropriately utilizing properties of inverse functions to isolate y. The instructors previously emphasized to students that the inverse of a function should be appropriately named f^{-1} if the original function was f. Lincoln's solution breaches this notion by labelling the final answer as "y."

Alexander's Solution

Alexander's solution does not "swap" x and y, but rather isolates x. The solution utilizes a natural logarithm and rules of logarithms to isolate x. The final answer is in terms of y, and is appropriately labeled as "$f^{-1}(y)$."

This solution distinctly simplifies the equation in ways that students may perceive as atypical. First, both sides are multiplied by 4 to obtain integer coefficients. Since the coefficient of the exponential function is also a power of three, the two exponents are added: $3^2 \cdot 3^{8x} = 3^{8x+2}$.

Informal notation and several errors were included in this solution:

$$\frac{\ln(4y+10)}{\ln 3} = \frac{(8x+2)\ln 3}{\ln 3}$$

$$-2, \div 8$$

$$\frac{1}{8}\ln(4y+10) - 2 = x$$

Unconventional notation was purposefully included in this solution to demonstrate that it can be difficult to interpret and can lead to mathematical errors. For example, the "-2" should also be divided by 8 in the following line. Additionally, the "ln 3" disappeared in the last line. This provided the instructor and teaching assistant with context to negotiate the importance of using of conventional notation.

Andrea's Solution

This solution "switches" x and y, and even explicitly states this. The exponential function is rewritten as a power of 9 from a power of 3: $3^{8y} = 9^{4y}$. Next, the solution multiplies both sides of the equation by 4, but incorrectly does not multiply $\frac{5}{2}$ by 4. Then, powers of 9 are combined: $9 \cdot 9^{4y} = 9^{4y+1}$. The solution then utilizes the inverse properties of logarithms and exponential functions in an atypical way:

$$4x + \frac{5}{2} = 9^{4y+1}$$

$$9^{\log_9\left(4x+\frac{5}{2}\right)} = 9^{4y+1}$$

$$4y + 1 = \log_9\left(4x + \frac{5}{2}\right)$$

It was anticipated that this use of inverse functions (between the first two lines) would breach students' expectations. Furthermore, in reference to the last step, students previously explored equations involving a one-to-one function that had different inputs. This intended to provide an opportunity for students to recall prior knowledge.

This solution represents another unfamiliar but valid approach to help the instructors negotiate what constitutes an acceptable solution. Additionally, the solution appropriately simplified and isolated y, but the final answer was not labeled with functional notation: $f^{-1}(x)$. This provided context for the instructors to remind students about the importance of mathematical labelling and adhering to conventional notation.

5.2.2.2.4 Activity 4 – Inverse Trigonometric Functions.

The Problem/Expectations

The problem for this activity was:

Evaluate the following:

$$\tan\left(sin^{-1}\left(\frac{1}{2}\right)\right)$$

It was anticipated that students would evaluate the inner function, $\sin^{-1}(\frac{1}{2})$, and then evaluate the resulting outer function, $\tan\left(\frac{\pi}{6}\right) = 1/\sqrt{3}$. Despite exploring how to evaluate inverse trigonometric functions in class, we did not expect students to verify conditions inherent with the evaluations, such as $-\frac{\pi}{2} \leq \sin^{-1}\left(\frac{1}{2}\right) \leq \frac{\pi}{2}$ or explicitly noting $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$. We expected that most students would express $\tan(\frac{\pi}{6})$ as $\frac{\sin\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right)}$ in order to evaluate.

Jennifer's Solution

Given the expectation that most students would solve this problem by direct evaluation, Jennifer's solution demonstrates that the problem can be solved without evaluating the inner function. The solution (see Appendix #) starts by labeling the inner value as the angle $u = \sin^{-1}(\frac{1}{2})$ and notes that this angle is within the first quadrant of the unit circle. The next line specifies that $\sin(u) = \frac{1}{2}$. Using this information, a right triangle is drawn to depict a ratio of two sides ("opposite over hypotenuse") with the given reference angle u. The Pythagorean Theorem is used to find the third side. An error is included within this step, yielding the incorrect third side:

$$a^2 = 2^2 - 1^2$$

$$a^2 = (2-1)^2$$

$$a = 1$$

Using this triangle, with the incorrect third side, the ratio for $\tan\left(sin^{-1}\left(\frac{1}{2}\right)\right) = \tan(u)$ is found. The answer is incorrect because of the included error. This represents a novel approach that yields an incorrect answer.

Dan's Solution

This solution follows an algorithm that was shown in class. First, the solution starts by naming the inside angle: $\sin^{-1}\left(\frac{1}{2}\right) = A$. Two conditions are then specified, that $\sin(A) = 1/2$ and $-\frac{\pi}{2} < A < \frac{\pi}{2}$. Note that the inequality is incorrectly exclusive of the endpoints. The solution then includes a written explanation that concludes A must be $\frac{\pi}{6}$.

In the following lines, two errors are included:

$$\tan\left(\frac{\pi}{6}\right)$$

$$= \frac{sin}{cos}(\pi/6)$$

$$= \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

First, informal notation is used in the second line, as each individual function is not given its own input. Many students in the class were accustomed to excluding inputs for trigonometric functions, or using informal notation. Secondly, $\sin\left(\frac{\pi}{6}\right)$ and $\cos\left(\frac{\pi}{6}\right)$ are incorrectly calculated (the correct answer should be the reciprocal). Thus, the approach yields the incorrect answer but follows a familiar approach.

Andrea's Solution

This solution represents another novel approach that does not require evaluation of $\sin^{-1}\left(\frac{1}{2}\right)$. First, the solution expresses $\tan\left(\sin^{-1}\left(\frac{1}{2}\right)\right)$ as $\frac{\sin\left(\sin^{-1}\left(\frac{1}{2}\right)\right)}{\cos\left(sin^{-1}\left(\frac{1}{2}\right)\right)}$. Then, the solution labels $w = \sin^{-1}\left(\frac{1}{2}\right)$, and utilizes the identity $\sin^2(w) + \cos^2(w) = 1$ to solve for $\cos(w)$. Next, $\frac{\sin\left(\sin^{-1}\left(\frac{1}{2}\right)\right)}{\cos\left(sin^{-1}\left(\frac{1}{2}\right)\right)}$ is written as equivalent to $\frac{1/2}{\cos(w)}$, and appropriately simplified to $\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$ and $\frac{1}{\sqrt{3}}$. The solution does not contain any errors and yields the correct answer, but also does not provide any written explanation.

Chapter 6. Data Analysis

Often for teaching experiments, it is natural to consider grounded theory and the constant comparative method to analyze the data while informing instructional decisions (Glaser & Strauss, 1967; Creswell, 2013). This project adapts these ideas and uses a modified grounded theory approach, which is detailed below. After describing the data analysis for the 360° video recordings for the instructional activities, examples are provided for further illustration. This chapter also includes a synopsis of how questionnaire data were quantitatively analyzed.

Because of the use of both qualitative and quantitative methods, this study is considered to follow a mixed methods design approach (Fetters et al., 2013). While the quantitative methodology may reveal overarching patterns amongst a larger group of participants, the qualitative methodology allows for surfacing nuance that would not otherwise be captured. Additionally, each component may act as an explanatory backdrop to the other, which may provide more context to understand findings during the data analysis. In this study, such a relationship between these methods results in better understanding students' classroom engagement.

6.1 Grounded Theory Approach

Grounded theory provides a structure to methodically and flexibly analyze qualitative data (Charmaz, 2014). This approach allows for the generation of theory from the data itself by means of iterations of data collection and analysis, also known as the constant comparative method (Glaser & Strauss, 1967). The researcher constantly compares new data to existing conjectures in a systematic and chronological way, ultimately developing increasingly stable explanatory constructs (McClain, 2002). That is, the explanatory constructs co-develop with data collection and analysis (Cobb & Whitenack, 1996).

Grounded theory research is typically used without a pre-existing theoretical basis (Creswell, 2013). However, this study aims to generate explanations and relationships within the adopted theoretical basis: the interpretive framework; thus, a modified grounded theory approach is used to analyze video data. The modifications to grounded theory are manifested in developing initial conjectures from the theoretical framework and especially the interpretive framework (see Section 2.4), instead of purely from the data.

6.1.1 Coding Overview

Qualitative data analysis in this study consists of three phases of coding (Dey, 1999; Creswell, 2013). The first phase of coding focuses on the construction and refinement of *categories* of important and relevant information that characterize patterns in the data. These categories are classified under the two studied social components of the interpretive framework: social norms and sociomathematical norms (see section 6.2.1.1 for an example). Thus, these categories help respond to research question 1 by identifying the emergence of classroom norms.

Additionally, research question 2 seeks to investigate the relationship between social constructs; this requires another phase of analysis, *axial* coding, which focuses on making connections between categories. Thus, categories are compared to investigate possible relationships, such as between behavior coded under social and sociomathematical norms.

Whereas the first two phases can be thought of as organizing the data into categories and connections, the last phase of coding is characterized by synthesizing and analyzing the evolution within these organized constructs. Thus, this phase included generating conjectures for norms and describing the nature of norms (e.g. characterizing the nature of what constitutes an acceptable solution, a sociomathematical norm). Inherent in these efforts, various influences and effects are surfaced that help explain the evolution of the classroom norms. Coding and analysis will be further detailed in the subsequent sections, including explicit examples, to help clarify these general remarks.

6.2 Analysis of Instructional Activity Video Data

The 360° video data from the Multiple Solutions Activities were a main data source of this study. Each camera captured the video and audio of one group. Overall, I collected videos for each of the four Multiple Solutions Activity for all four groups, which totaled approximately 16 hours of video data. The unit of analysis was chosen as an episode of homogenous activity or engagement within the group and during the whole class discussions (Derry et al., 2010). Consequently, episode durations vary as they are dependent on various factors including the type of engagement, levels of collaboration, or even group members' attentiveness. Typically, episode durations ranged from 2 to 5 minutes.

6.2.1 Construction and Refinement of Categories

The first step of the analysis was to use constant comparative method to code data in Excel spreadsheets by describing each student's action or utterance related to their engagement with the Multiple Solutions Activity. This included notating why these actions or utterance were significant, such as what each accomplished or represented. As the coding progressed, these notations aided in constructing finer-subcategories (e.g. "Attempting to Understand the Solution" or "The Role of the Answer in a Solution") from more general ones, like “social norms” or “sociomathematical norms.” These categories and subcategories were continuously refined and revised, until the spreadsheet was stabilized through subsequent iterations.

The categories in the logs represent accounts that detail the evolution of specific sociomathematical and social regularities in the classroom, thus representing “succinct yet empirically grounded chronologies” (Cobb et al., 2001, p. 128). Consequently, the categories should not be viewed as just a means to organize the data, but as data themselves. These chronologies, the categories, form the basis for generating and refining conjectures of norms. Consistent with Park (2015), Guven and Dede (2017), and Sfard (2008), patterns or conjectures were generally considered norms for a particular student group if they were observed during at least three different activities and were supported by a majority of the group members. This definition accounts for longitudinal behavior or activity that is enacted or supported within a unit (i.e. group). This study incorporated a sequence of four activities to determine how the characterizations of norms evolve over the course of the semester; thus, I was interested not just in the presence of norms, but in how they changed over time.

As conjectures for norms are refined, the data must be revisited for instances of violations of the conjectured norms (Cobb et al., 2001; Guven & Dede, 2017). Delegitimized violations provide further support and evidence for the conjectures. Alternatively, ambivalence to norm violations on behalf of the participants, or affirming the violations of the norms, motivate the need to revisit or revise the conjectures to attend to the cases of dissonance.

For example, a conjecture was made that Group 1 characterized an acceptable solution as one that followed any valid approach (see Section 7.2.1.2.2). This conjectured norm was violated when a student in another group suggested during a class discussion that a solution must follow a specific familiar method. This violation was delegitimized when Harry verbally rebuked the suggestion, which provided further evidence for the conjecture.

6.2.1.1 Responding to Research Question 1 – An Example of Coding. The first aspect of the coding scheme was to provide succinct descriptive information about each episode and to describe what this activity shows or why it is important. Figure 4 shows one example (students' initials are used as shorthand).

Figure 4

A Piece of the Coding Spreadsheet Displaying the Video #, Duration of the Episode ("Time Stamp"), the Participants, and the Description of One Episode

General info about the lesson:		Ron, Meghan, Julia, Paul
Video #	Time Stamp	Desription (non evaluative - in a few words describe what happened in the interval)
351	1:54 - 3:50	The group tries to determine the issues with their solution. P unsuccessfully uses an app (still relying on a source of authority). Meg still expresses frustration, but still works on identifying the issue. R still analyzes. J still sits/stares/zones out. Mike comes by the group, but does not offer help or guidance.

Figure 5 provides an example of the coding of "social norms" for this episode. When starting coding, the only column is "social norms." Group members' actions and utterances were described, followed by the significance of what the action or utterance accomplished (in brackets), and occasionally preceded by the associated timestamp. The significance of each action was used in subsequent analyses to help surface patterns of behavior/engagement.

These patterns were then classified and organized as categories, and each category was given a sub-column under “social norms.” Then, relevant codes were moved into the appropriate category from a generic column. For example, Figure 5 includes a sub-column for the category “Attempting to Understand the Solution.” Originally, cells in this column included descriptions of instances of the group attempting (or not attempting) to understand a solution during the activity. As discussed in section 6.2.1, individual entries do not represent social norms in the group but rather, repeated codes across activities provide support for evolving conjectures of norms.

Figure 5

An Example of Social Norm Categories and their Codes

		"Social Norm" Contributions
SN: Questioning, Listening, and Making Sense of others' contributions (S-S, S-T)	Attempting to Understand the solution (S-M)	Attentiveness/Distractions
(epsd) The group isn't collaboratively working on the solution, or discussing what they are doing/thinking [Not using peers as resource]	(155) Meg continues to work on the problem, asking J for her sheet so Meg can check something. P/R separately seem to be using their phones to check the solution/answer. [Attempting to understand solution separately]	J is on her phone, it does not appear she is using it for class purposes - she puts the phone down when Mike stops by (233). [J inattentive - phone use known to violate instructor's expectations]

When noteworthy behaviors did not fit into any existing sub-columns, they were placed in a sub-column titled “Other” (see Figure 6). Subsequent iterations of coding reduced the number of codes within this sub-column. A sub-column, “Comments,” was included to denote important memos that are constructive towards clarifying the significance of the behavior in the episode or repeated activity that is noticed, as well as ideas to improve engagement in the course.

Figure 6

An Example of Another Category (Negativity), a Sub-Column for Codes that Do Not Fit Under Current Categories ("Other"), and a Sub-Column for Comments

		Comments on Role Beliefs/Social Norms
Negativity	Other	
(230) Mike checks in with the group. P/Meg express frustration. Mike asks if they're confident, Meg says, "Well, I'm right." [Conflict in responsibility results in negativity]	The group discusses that they have multiple apps that can do out math probs [Conflict in responsibility - instructors assert the need to check work, students defer to apps]	It isn't clear what strategies the group is trying to employ to check their work - besides using Mathway. The group isn't naturally developing these strategies, or exploring them. Guided modeling of some sort might have helped.

The last row of the spreadsheet was denoted "Summary," and contained brief characterizations of the patterns observed in each category for a single activity. This row helped to identify patterns across activities. Figure 7 shows the summary characterization of students' "Attempts to understand the solutions."

Figure 7

A Summary for the Category "Attempting to Understand the Solution"

Video #	Time Stamp	Attempting to Understand the solution (S-M)
S U M M A R Y		The group's efforts are not always productive. Instead of determining ways to mathematically verify their answer, they spent time appealing to a source of authority (and wasted much time trying to use the software to determine this answer). The group does not see it as their responsibility to validate/verify their answer. (Breach in DC) The group spends little time, if any, investigating the solutions. Often this is done in isolation.

6.2.2 Axial Coding and Selective Coding

Axial coding helped to respond to the second research question: determining relationships between social and sociomathematical constructs (see Chapter 4). By concentrating on a central category, such as a particular sociomathematical norm, axial coding allowed constructing an explanatory model that connects that category to categories that characterize social norms. The final step was selective coding within a narrative. It included developing propositions, which describe the relationship between the categories of the constructed explanatory model.

6.2.2.1 Responding to Research Question 2 – An Example of Coding. Axial coding focused on analyzing the relationship between columns (i.e. categories and sub-categories). Of particular interest to this project is the relationship between categories under social norms with those under sociomathematical norms. One prevalent relationship that was observed was the connection between categories characterizing the sociomathematical norm of "what constitutes a mathematical solution," and the characterization of the social norm of "attempting to understand a solution." For example, one group's limited acts of attempting to understand solutions in the instructional activities constrained the development of their conceptualization of what constitutes an acceptable solution. By comparing activity and patterns amongst these two categories, axial coding revealed that they shared an apparent interdependence in their codevelopment. This will be further discussed and clarified in Section 8.3. Figure 8 depicts summary descriptions of both categories for the second Multiple Solutions Activity.

Figure 8

Summary Descriptions for Two Categories in one Activity

General info about the lesson:			"Social Norm" Contributic
Video #	Time Stamp	What Constitutes a Mathematical Solution	Attempting to Understand the solution (S-M)
	S U M M A R Y	The group expresses value for neatness, but this is not incorporated in their rubric. Specifics to content aren't made - more of a holistic approach focused on work shown and an accurate answer. Given the group's limited acts of interpretation, their evaluation of work mainly revolved around the presence of work.	The group largely does not try to interpret the sample solutions. The group largely follows the first evaluation that is shared. Often this is Meg, who dismisses the approaches, often very quickly. No strategies are employed to understand the solutions, and there is little dialogue.

6.3 Analysis of Supporting Data – The Role of Written Student Work in the Analysis

The data from different sources were triangulated. Specifically, the data from written student work: students' solutions, grading keys, and their evaluation of the sample solutions, were used in two ways. It provided context for students' utterances and discussions, which helped to better understand students' in-class engagement and tracking of longitudinal changes. Second, student written work supported the establishment and refinement of categories with the analysis of 360° videos of the instructional activities.

6.4 Questionnaire Analysis

To respond to the third research question, I analyzed data from questionnaires to determine changes to students' individual values and beliefs. Student responses to the pre- and post-questionnaires were paired using a non-identifying code (see Section 5.1.2.3), which students created when completing the questionnaires. Pre-course questionnaires that did not have a matching code in the post-course questionnaire pool, and vice versa, were not included in the analysis. In total, this yielded 42 questionnaires, of which 26 students reported taking the course in a previous semester.

Paired t-tests (JMP refers to these as *Matched Pairs tests*) were performed to ascertain if the teaching experiment was effective at influencing students' beliefs (JMP, 2020). The paired *t*-test was chosen, since it is considered robust for Type I error with Likert data (Derrick & White, 2017). In addition, according to Normal (2010, p. 631), "parametric statistics can be used with Likert data … with no fear of 'coming to the wrong conclusion,'" which supports treating Likert data as continuous. Additionally, the paired scores were independent of one another and the data contained no outliers. Lastly, the sample (n=42) satisfies the assumption that the data is approximately normally distributed, as sample sizes that exceed 30 are traditionally considered sufficient for the Central Limit Theorem to hold. Thus, all of the necessary assumptions were fulfilled.

Lower sample sizes can be used to perform the paired *t*-tests on smaller samples, but due to insufficient power, these tests may struggle to recognize smaller effects; consequently, many results with smaller sample sizes will register as inconclusive. With these considerations, paired *t*-tests were used on subsets of the sample, such as groups of students with and without prior MATH 418 enrollment.

Two-sample *t*-tests (referred to as *pooled t-tests* in JMP) were used to compare item means, within a single questionnaire, between students who previously enrolled in the course (n=26) and those that were enrolled for the first time (n=16). Similarly, the Likert data were considered continuous, and the two samples independent. Normality was not an issue as both sample sizes were at least 15 (Minitab LLC, 2019). But, given these sizes, and the fact that they are unequal, it was necessary to determine that they do not have unequal variances (Boneau, 1960). Consequently, an *F-test for unequal variances* was used to determine if the sample variances were significantly different. The two samples were considered to have unequal variances for $p<0.05$. If the F-test resulted in statistically significant differences in variances, *Welch's t-test for unequal variances* was utilized. Otherwise, the pooled *t*-test for equal variances was used. The latter is preferred because equal variance tests provide more power to detect significant differences or effects, which is challenging with small sample sizes.

In addition, the questionnaires were analyzed by compiling "Post – Pre Data" ("Post minus Pre Data"), to investigate the differences reported between each student's Post- and Pre-Questionnaires. For a single student and for a single item, a positive "Post – Pre Change" represents a student's post-response being higher than their pre-response, and a negative Post – Pre Change represents a higher pre-response. These data were analyzed with the same method described above for the two sample *t*-tests. This allowed for determining if changes experienced by these two pools of students, those with and without prior MATH 418 experience, were statistically significant or noticeable. The *p*-values less than 0.05 are referred to as statistically "significant," whereas p-values in between 0.05 and 0.10 are referred to as "noticeable."

Lastly, the data were also analyzed by means of descriptive statistics, which allowed for the analysis of basic features of the data, such as the form of the distribution and skew. This also includes analyzing the paired data to find significant correlations among item-wise differences between post-questionnaire and pre-questionnaire results.

6.4.1 Variable of Prior Enrollment in MATH 418

It was natural to question if any underlying variables influenced students' classroom engagement. Through my observations of classroom behavior, I saw differences in student engagement which seemed influenced by "prior enrollment in the course." Consequently, I decided to investigate this variable. Preliminary analyses surfaced interesting differences, so I further explored the variable's influence in the qualitative and quantitative analyses. Otherwise, because of the student population, there were not sufficient sample sizes to explore the role of traditional aspects of identity (e.g. race, ethnicity, gender, etc.), as the class was overwhelmingly white and male.

Chapter 7. Results

7.1 Quantitative Results

Questionnaires provide information about students' beliefs and values, which are the psychological correlates of social and sociomathematical norms (Figure 1). Since beliefs and norms develop concurrently, repeated questionnaires illuminate the development of norms. Sections 7.1.1 and 7.1.2 detail questionnaire results on students' mathematical beliefs and values (the psychological correlate of sociomathematical norms), and students' beliefs about their role, others' roles, and the general nature of activity in the classroom (the psychological correlate of social norms). Section 7.1.3 details significant correlations between changes in items (Post minus Pre) including items that relate students' mathematical and social beliefs. As a reminder, a four-point Likert scale was used with the options: (1) Disagree, (2) Slightly Disagree, (3) Slightly Agree, and (4) Agree, or, (1) Not Important, (2) Slightly Important, (3) Moderately Important, and (4) Very Important.

7.1.1 Students' Mathematical Beliefs and Values (Sociomathematical Norms)

Table 2 shows the results of two questionnaire items that assessed students' beliefs associated with flexible knowledge and what constitutes an acceptable solution. The statistically significant decrease in mean scores suggests that students came to assign less value to following specific procedures, and viewed this as having less influence on receiving full credit for their work. This also suggests improved openness to learning multiple solution approaches.

Table 2

Necessity of Following a Specific Method, Paired t-test Results

Questionnaire Item	*Pre-Mean*	*Post-Mean*	*Difference (SE)*	*p-value*
The most valid ways of solving a problem are the ones discussed in class.	2.881	2.548	-0.333 (0.126)	0.006
To receive full credit, my solution must use the same methods used in class.	2.146	1.830	-0.317 (0.146)	0.018

Note: n=42, 1- Disagree, 4- Agree

This shift can be seen in the skew of the data. On the pre-questionnaire, these two items were differently skewed: the first item had a slight negative skew in the pre-questionnaire, whereas the latter item had a noticeable positive skew (see Figure 9). The two items both experience shifts towards positive skew, as the number of responses disagreeing with the statements (i.e. responses of 1 or 2) shifted from 21.4% in the pre-questionnaire for the first item to 50.0% in the post-questionnaire, and from 61.0% to 75.6% in the second item.

Figure 9

Necessity of Following a Specific Method, Pre- and Post-Questionnaire Distributions

	The most valid ways of solving a problem are the ones discussed in class.		*To receive full credit, my solution must use the same methods used in class.*	
Likert Responses	Pre-Questionnaire	Post-Questionnaire	Pre-Questionnaire	Post-Questionnaire
# of 1	3	3	14	18
# of 2	6	18	11	13
# of 3	26	16	12	9
# of 4	7	5	4	1

The analysis of the variable of prior enrollment indicated that both pools of students, those with and without prior enrollment in the course, showed a decreased mean. Yet, only students without prior MATH 418 experience had a significant decrease in mean on the first question (p=0.001), and only students with prior experience had significant mean decrease in mean on the second question (p=0.015, see Table 3). The differences in the first item were noticeably stronger (p=0.078) for students without prior MATH 418 experience than for students with prior MATH 418 experience.

Table 3

Necessity of Following a Specific Method, Split Between Students with and without Prior MATH 418 Experience

Questionnaire Item	*No Prior MATH 418 (n=16)*			*Prior MATH 418 (n=26)*		
	Pre-Mean	Post-Mean	Difference (SE)	Pre-Mean	Post-Mean	Difference (SE)
The most valid ways of solving a problem are the ones discussed in class.	3.000	2.438	-0.563*** (0.157)	2.808	2.615	-0.192 (0.176)
To receive full credit, my solution must use the same methods used in class.	1.938	1.750	-0.188 (0.262)	2.269	1.880	-0.400** (0.173)

***p*<0.05, ****p*<0.01

Note: n=42, 1- Disagree, 4- Agree

Another item that addressed the perceived value of flexibility was: “I find it helpful to learn several different ways to solve a math problem.” This item’s mean did not demonstrate a statistically significant change (Table 4), but further analysis revealed that the means of students with and without prior MATH 418 experience exhibited changes in different directions (see Table 5). The mean for students’ without prior MATH 418 experience decreased significantly (p=0.034) while the mean for students retaking MATH 418 increased slightly and insignificantly. This suggests that students taking MATH 418 for the first time reported that it was less helpful to them, in general, to learn different ways to solve a problem.

Table 4

Helpfulness of Multiple Approaches

Questionnaire Item	*Pre-Mean*	*Post-Mean*	*Difference (SE)*
I find it helpful to learn several different ways to solve a math problem.	3.190	3.119	-0.071 (0.138)

Note: n=42, 1- Disagree, 4- Agree

Table 5

Helpfulness of Multiple Approaches, Split Between Students with and without Prior MATH 418 Experience, n = 42 (1- Disagree, 4- Agree)

Questionnaire Item	*Student Pool*	*Pre-Mean*	*Post-Mean*	*Row Difference (SE)*
I find it helpful to learn several different ways to solve a math problem.	Prior MATH 418 (n=26)	3.115	3.269	0.154 (0.164)
	No Prior MATH 418 (n=16)	3.3125	2.875	-0.438** (0.223)
Column Difference: (SE Difference)		-0.197 (0.257)	0.394 (0.305)	

***p*<0.05

Note: n=42, 1- Disagree, 4- Agree

Further analysis demonstrates that the difference in the change of mean reported by the two pools of students is statistically significant (p=0.018, See Table 6).

Table 6

Helpfulness of Multiple Approaches, Difference in Post-Pre Data

Questionnaire Item	*Post-Pre Mean*		
	Prior MATH 418 (n=26)	*No Prior MATH 418 (n=16)*	*Difference (SE)*
I find it helpful to learn several different ways to solve a math problem	0.154	-0.438	-0.591** (0.272)

**$p<0.05$

This suggests that the variable, prior enrollment in MATH 418, had a significant role in changes to students' perception on how helpful it was to learn about different solution methods. At the end of the semester, a majority of the first-time students found it helpful, with a rating of a 3 or 4 (62.5%, or 10/16), but this was a decrease from 87.5% or 14/16 from the beginning of the semester. Meanwhile, the number of returning students who found it helpful slightly increased from 77% (20/26) to 81% (21/26). Though, this it is important to note that 73% of returning students, and 62.5% of first-time students, did not report any change in the helpfulness (Figure 10).

Figure 10

Helpfulness of Multiple Approaches, Comparing Pre- and Post-Questionnaire Distributions

I find it helpful to learn several different ways to solve a math problem.		
Item Comparison	No Prior 418 (n=16)	Prior 418 (n=26)
Post > Pre	1	5
No Change	10	19
Pre < Post	5	2

The next questionnaire item intended to investigate whether students found more value in accurate computation or in understanding each step of a solution. The item's mean yielded a significant increase from the pre- to post questionnaire (p=0.035) (see Table 7). This suggests that students' beliefs generally shifted towards valuing computational accuracy over conceptual understanding. Though, the post-mean of 2.548 on the 4-point scale indicates a balance of value between the two (i.e. responses of 1 and 2 indicate more value for conceptual understanding and responses of 3 and 4 indicate more value for correctly performing steps of a solution).

Upon further analysis (Table 8), the mean's increase for this item is largely attributed to students without prior MATH 418 enrollment; this mean significantly increased from 1.875 to 2.438 (p=0.035). The results also demonstrate a notable difference (p=0.054) between the pre-questionnaire mean of students with and without prior MATH 418 experience, as students without prior enrollment had a lower mean (p=0.054). Yet, the two post-means were much closer. One possible explanation for this is the acculturation of the students without prior MATH 418 experience into the course with significantly more students with prior enrollment, and into STEM in general. The students new to MATH 418 may have quickly learned to value mathematics as a toolset to provide results, especially as many pursued applied degrees, like engineering.

Table 7

Accuracy versus Understanding, Paired t-test Results

Questionnaire Item	*Pre-Mean*	*Post-Mean*	*Difference (SE)*	*P-Value*
It is more important to correctly perform the steps of a solution than to understand each one of them.	2.214	2.548	0.333 (0.179)	0.035

Note: n=42, 1- Disagree, 4- Agree

Table 8

Accuracy versus Understanding, Split Between Students with and without Prior MATH 418 Experience

Questionnaire Item	*Student Pool*	*Pre-Mean*	*Post-Mean*	*Row Difference (SE)*
It is more important to correctly perform the steps of a solution than to understand each one of them.	Prior MATH 418 (n=26)	2.423	2.615	0.192 (0.229)
	No Prior MATH 418 (n=16)	1.875	2.438	0.563** (0.288)
Column Difference: (SE Difference)		0.548* (0.334)	0.179 (0.318)	

**p<0.05, *p<0.10

Note: n=42, 1- Disagree, 4- Agree

Similarly, students reported that they would rather focus on learning how to use formulas than learn where they come from (Table 9). In Table 9, the means between pools of students with and without prior MATH 418 experience were rather sustained from the pre-questionnaire to the post-questionnaire, as the respective differences for each pool of students were not found to be significant. Students with prior MATH 418 enrollment had a higher mean

(3.500) than students' without prior enrollment (3.000), but due to the significant differences in variance, Welch's test was used, which was unable to determine significance. This result may be consistent with the student population's pursuit of applied degrees in STEM, as they might perceive developing into "users" of mathematics instead of developing a deeper understanding of the content.

Table 9

Formulas versus Derivations, Split Between Students with and without Prior MATH 418 Experience

Questionnaire Item	*Student Pool*	*Pre-Mean*	*Post-Mean*	*Row Difference (SE)*
I prefer to focus on learning how to use formulas instead of spending time on where they come from.	Prior MATH 418 (n=26)	3.462	3.500	0.038 (0.141)
	No Prior MATH 418 (n=16)	3.133	3.000	-0.133 (0.322)
Column Difference: (SE Difference)		0.329 (0.255)	0.500^{T} (0.267)	

^{T} Variances statistically different (*p*=0.046), means not significantly different under Welch's Test

Note: n=42, 1- Disagree, 4- Agree

Table 10 shows an item that did not have any statistically significant changes in mean, but yielded an interesting change in distribution amongst students without prior MATH 418 experience (as shown in Figure 11). For students without prior experience, despite the mean being sustained, the median and mode shifted from "2" to "3." Furthermore, the number of students who agreed with the statement (responses of "3" or "4") increased from 5 (33.3%) to 9 (60%). These results suggest that although the means represent little consensus (since the means hover around 2.5), there may be nuanced changes in students' expectation (or lack thereof) for written explanations in solutions.

Table 10

Equations as Self-Explanatory, Item Split Between Students with and without Prior MATH 418 Experience

Questionnaire Item	*Total Class (n=41)*		*No Prior MATH 418 (n=15)*		*Prior MATH 418 (n=26)*	
	Pre-Mean	Post-Mean	Pre-Mean	Post-Mean	Pre-Mean	Post-Mean
Solutions written with formulas or equations are self-explanatory. They do not require written explanations.	2.585	2.537	2.467	2.467	2.654	2.577

Note: n=42, 1- Disagree, 4- Agree

Figure 11

Equations as Self-Explanatory, Questionnaire Distributions for Students without Prior Enrollment in MATH 418

	Solutions written with formulas or equations are self-explanatory. They do not require written explanations.	
Likert Responses	Pre-Questionnaire	Post-Questionnaire
# of 1	2	3
# of 2	8	3
# of 3	1	8
# of 4	4	1

Note: n=15

Table 11 displays results on items assessing students' views on two common, unproductive beliefs about mathematics: that mathematics is a discipline rooted in memorization, and generally lacks discussion because in mathematics one is either right or wrong. The items' means significantly decreased which suggests that students' beliefs regarding procedural nature of mathematics and students' value for discussions were positively impacted. The changes in means from the pre- to the post-questionnaire for students with and without prior MATH 418 experience were consistent, as both demonstrated similar shifts (see Table 12).

Table 11

Beliefs about Memorization and Discussion, Paired t-test Results

Questionnaire Items	*Pre-Mean*	*Post-Mean*	*Difference (SE)*	*P-Value*
Mathematics is a set of rules and procedures that need to be memorized.	3.225	2.925	-0.300 (0.120)	0.008
There is no place in mathematics for discussions - you are either right or wrong.	1.976	1.690	-0.286 (0.157)	0.038

Note: n=42, 1- Disagree, 4- Agree

Table 12

Beliefs about Memorization and Discussion, Split Between Students with and without Prior MATH 418 Experience

Questionnaire Item	*No Prior MATH 418 (n=16)*			*Prior MATH 418 (n=26)*		
	Pre-Mean	Post-Mean	Difference (SE)	Pre-Mean	Post-Mean	Difference (SE)
Mathematics is a set of rules and procedures that need to be memorized.	3.267	2.933	-0.333** (0.159)	3.200	2.920	-0.280* (0.169)
There is no place in mathematics for discussions - you are either right or wrong.	1.875	1.688	-0.188 (0.306)	2.038	1.692	-0.346** (0.175)

**p<0.05, *p<0.10

Note: n=42, 1- Disagree, 4- Agree

Table 13 presents items with insignificant changes in mean related to students' mathematical beliefs. Regardless of prior enrollment in MATH 418, students reported that memorization, creativity, and being able to determine the correctness of a peer's solution are "slightly important." Despite the mean remaining stable, for the item "How important is getting the right answer to receiving credit for a math problem?," the mode noticeably shifted from "4" ("Agree") to a "3" ("Slightly Agree"). The variance of these three items remained relatively low (Variance < 0.65) and diminished from the pre-questionnaire to the post-questionnaire (see Figure 12). These results suggest that although the means were sustained, the class may have had a convergent understanding about the importance of these aspects to mathematics problems and solutions.

Table 13

Sustained Questionnaire Items Regarding Mathematical Beliefs

Questionnaire Item	*Total Class (n=42)*		*No Prior MATH 418 (n=16)*		*Prior MATH 418 (n=26)*	
	Pre-Mean	Post-Mean	Pre-Mean	Post-Mean	Pre-Mean	Post-Mean
How important is memorization to solving math problems?	3.143	3.000	3.125	3.000	3.154	3.000
How important is getting the right answer to receiving credit for a math problem?	3.262	3.167	3.188	3.250	3.308	3.115
How important is it for you to be able to determine if a peer's solution is correct?	2.881	2.905	3.063	2.875	2.769	2.923
How important is it for you to be creative when solving a mathematical problem?	2.333	2.381	2.438	2.375	2.269	2.385

Note: n=42, 1- Not Important, 4- Very Important

Figure 12

Distribution and Variance of Three Questionnaire Items Regarding Mathematical Beliefs

	How important is memorization to solving math problems?		*How important is getting the right answer to receiving credit for a math problem?*		*How important is it for you to be able to determine if a peer's solution is correct?*	
Likert Responses	Pre-Questionnaire	Post-Questionnaire	Pre-Questionnaire	Post-Questionnaire	Pre-Questionnaire	Post-Questionnaire
# of 1	1	1	0	0	2	1
# of 2	6	7	8	6	10	11
# of 3	21	25	15	23	21	21
# of 4	14	9	19	13	9	9
Variance	0.564	0.488	0.588	0.435	0.644	0.576

Note: n=42, 1- Not Important, 4- Very Important

On the other hand, the item "How important is it for you to be creative when solving a mathematical problem?," resulted in a mean just below 2.5 for the pre- and post-questionnaires, for both students with and without prior MATH 418 enrollment. This mean indicates a very slight disagreement with the item. Yet, this question had a noticeably higher variance in responses which was independent of students' prior enrollment (see Figure 13).

Figure 13

Importance of Creativity, Distribution and Variance, n = 42 (1- Disagree, 4- Agree)

	How important is it for you to be creative when solving a mathematical problem?					
	Whole Class (n=42)		No Prior 418 (n=16)		Prior 418 (n=26)	
Likert Responses	Pre-Questionnaire	Post-Questionnaire	Pre-Questionnaire	Post-Questionnaire	Pre-Questionnaire	Post-Questionnaire
# of 1	10	8	3	3	7	5
# of 2	13	15	5	6	8	9
# of 3	14	14	6	5	8	9
# of 4	5	5	2	2	3	3
Variance	0.959	0.876	0.929	0.917	1.005	0.886

Note: 1- Not Important, 4- Very Important

Lastly, students had a notable increase (p=0.055) in the item, "How important is it to you to write a solution that your peers could understand?" This change is interesting, as the item, "How important is it for you to determine if a peer's solution is correct?" did not show a notable change. This may suggest that students' values regarding writing solutions were impacted more than their value for interpreting solutions.

Table 14

Writing a Solution to be Understood, Paired t-test Results

Questionnaire Item	*Pre-Mean*	*Post-Mean*	*Difference (SE)*	*P-Value*
How important is it to you to write a solution that your peers could understand?	2.857	3.119	0.262 (0.160)	0.055

Note: n=42, 1- Not Important, 4- Very Important

7.1.2 Students' Beliefs about their Role, Others' Roles, and the General Nature of Activity in the Classroom (Social Norms)

One item relating to students' role beliefs is: "When it comes to math, I would rather try to figure out my own questions or confusion than ask for help." The item's statistically significant decrease in mean (see Table 15) suggests that students grew more comfortable asking for help with their questions. This item does not capture whom students ask for help, whether an instructor or a peer. This result was similar between students' with and without prior enrollment in MATH 418, but was much more pronounced (and significant) in the latter group (see Table 16).

Also corresponding with changes in students' activity in the classroom, students reported less direct copying of the instructor's board writing (see Table 15). Further analysis revealed that this change was mainly restricted to the pool of students with prior MATH 418 experience (p=0.013), than students without prior MATH 418 enrollment, who showed no overall change in mean (see Table 16). The changes experienced by these two pools of students were noticeably different (p=0.090).

Table 15

Pursuing Help and Copying the Board, Paired t-test Results

Questionnaire Item	*Pre-Mean*	*Post-Mean*	*Difference (SE)*	*P-Value*
When it comes to math, I would rather try to figure out my own questions or confusion than ask for help.	2.524	2.262	-0.262 (0.137)	0.031
In typical math lectures, I write down everything that the instructor writes on the board.	3.049	2.738	-0.293 (0.165)	0.042

Note: n=42, 1- Disagree, 4- Agree

Table 16

Pursing Help and Copying the Board, Split Between Students with and without Prior MATH 418 Experience

Questionnaire Item	*No Prior MATH 418 (n=16)*			*Prior MATH 418 (n=26)*		
	Pre-Mean	Post-Mean	Difference (SE)	Pre-Mean	Post-Mean	Difference (SE)
When it comes to math, I would rather try to figure out my own questions or confusion than ask for help.	2.5625	2.1875	-0.375* (0.221)	2.500	2.308	-0.192 (0.176)
In typical math lectures, I write down everything that the instructor writes on the board.	3.000	3.000	0.000 (0.293)	3.077	2.615	-0.462** (0.194)

**p<0.05, *p<0.10

Note: n=42, 1- Disagree, 4- Agree

Students with no prior MATH 418 experience showed a significant increase in mean (p=0.0481) in their use of graphing technology to understand what unfamiliar functions/equations look like (Table 17). Meanwhile, students with prior experience in the course had no change (in mean). Additionally, the differences in these gains were noticeably different among the two pools of students (p=0.096).

Table 17

Using Graphing Technology, Split Between Students with and without Prior MATH 418 Experience

Questionnaire Item	*No Prior MATH 418 (n=16)*			*Prior MATH 418 (n=26)*		
	Pre-Mean	Post-Mean	Difference (SE)	Pre-Mean	Post-Mean	Difference (SE)
I use graphing technology to understand what an unfamiliar function/ equation looks like.	3.067	3.400	0.333** (0.187)	3.308	3.308	0.000 (0.157)

***p*<0.05

Note: n=42, 1- Disagree, 4- Agree

Despite the shift to using an online, open-source textbook (as discussed in Section 5.2.2), no significant change was found in students' use of this type of textbook (Table 18). Even with further analysis (Table 19), neither students with or without prior enrollment in the course experienced significant change in mean.

Table 18

Using Textbooks, Paired t-test Results

Questionnaire Item	*Pre-Mean*	*Post-Mean*	*Difference (SE)*
I usually don't find math textbooks helpful and prefer not to use them.	2.585	2.561	-0.0244 (0.196)

Note: n=42, 1- Disagree, 4- Agree

Table 19

Using Textbooks, Split Between Students with and without Prior MATH 418 Experience

Questionnaire Item	*No Prior MATH 418 (n=16)*			*Prior MATH 418 (n=26)*		
	Pre-Mean	Post-Mean	Difference (SE)	Pre-Mean	Post-Mean	Difference (SE)
I usually don't find math textbooks helpful and prefer not to use them.	2.400	2.400	0.000 (0.324)	2.692	2.654	-0.039 (0.251)

Note: n=42, 1- Disagree, 4- Agree

Table 20 shows a list of results, with no significant changes in mean from the pre- to post-course questionnaires, related to social norms of the class. The first four results suggest that students recognize the benefits of explaining their work or reasoning to others, refer to their notes when completing work outside of class, view math classes as places to learn new content, and strongly agree (means greater than 3.7) that it is their responsibility to ask for help when they do not fully understand something. The following three items suggest that students perceive that it is the instructors' responsibility to prepare them for assessments and for teaching how to write accepted solutions, and that students see value in learning new ways to solve problems by working with peers. Yet, more interestingly in these three items, there exists noticeable and significant differences between the means of students with and without prior MATH 418 enrollment.

Table 20

Questionnaire Items Regarding Role and Classroom Activity

Questionnaire Item	*Total Class (n=42)*		*No Prior MATH 418 (n=16)*		*Prior MATH 418 (n=26)*	
	Pre-Mean	Post-Mean	Pre-Mean	Post-Mean	Pre-Mean	Post-Mean
In math, explaining my work or reasoning to others helps me learn.	3.500	3.548	3.438	3.375	3.538	3.654
When completing homework, I actively refer to my notes from class.	3.286	3.214	3.125	3.1875	3.385	3.231
The purpose of math class is to learn new math content.	3.381	3.309	3.438	3.188	3.346	3.385
It is my responsibility to ask for help when I do not fully understand something.	3.829	3.786	3.733	3.733	3.885	3.846
It is the instructor's role to prepare me for quizzes and exams.	3.190	3.238	3.000	3.000	3.308	3.385
The instructors and TAs are responsible for teaching me how to write a solution that would receive full credit.	3.452	3.595	3.313	3.375	3.538	3.731
Working with peers helps me learn about new ways of thinking about a problem.	3.429	3.405	3.188	3.188	3.577	3.538

Note: n=42, 1- Disagree, 4- Agree

Tables 21-23 (below) show differences between the means of items of students with and without prior enrollment in MATH 418. Table 21 indicates that a noticeable difference between the pools on the pre-questionnaire grew even more by the end of the semester, as students with prior enrollment in MATH 418 had a significantly higher mean than students without prior enrollment (p=0.046). This suggests that students with prior experience in MATH 418, in general, more strongly believed that the instructors and TAs were responsible for teaching them how to write a solution that would receive full credit.

Similarly in Table 22, at the beginning of the semester, students with prior enrollment had a noticeably higher mean that students without prior enrollment (p=0.097) for the item that states it is the instructor's responsibility to prepare students for quizzes and exams. This difference increased and was significant by the end of the semester (p=0.032).

In Table 23, students with prior enrollment reported a higher mean for the item: *Working with peers helps me learn about new ways of thinking about a problem.* This difference was noticeable at the beginning (p=0.085) of the semester, but because of the statistically different variances, Welch's Test was used for the post-questionnaire, and did not provide evidence of effect.

Table 21

Instructors' Role to Teach How to Write a Solution, Split Between Students with and without Prior MATH 418 Experience

Questionnaire Item	*Student Pool*	*Pre-Mean*	*Post-Mean*	*Row Difference (SE)*
The instructors and TAs are responsible for teaching me how to write a solution that would receive full credit.	Prior MATH 418 (n=26)	3.538	3.731	0.192 (0.176)
	No Prior MATH 418 (n=16)	3.313	3.375	0.063 (0.232)
Column Difference: (SE Difference)		0.226 (0.246)	0.356** (0.267)	

**$p<0.05$

Note: n=42, 1- Disagree, 4- Agree

Table 22

Instructors' Role to Prepare Students for Assessment, Split Between Students with and without Prior MATH 418 Experience

Questionnaire Item	*Student Pool*	*Pre-Mean*	*Post-Mean*	*Row Difference (SE)*
It is the instructor's role to prepare me for quizzes and exams.	Prior MATH 418 (n=26)	3.308	3.385	0.077 (0.123)
	No Prior MATH 418 (n=16)	3.000	3.000	0.000 (0.204)
Column Difference: (SE Difference)		0.308* (0.233)	0.385** (0.202)	

**p<0.05, *p<0.10

Note: n=42, 1- Disagree, 4- Agree

Table 23

Working with Peers, Split Between Students with and without Prior MATH 418 Experience

Questionnaire Item	*Student Pool*	*Pre-Mean*	*Post-Mean*	*Row Difference (SE)*
Working with peers helps me learn about new ways of thinking about a problem.	Prior MATH 418 (n=26)	3.577	3.538	0.039 (0.130)
	No Prior MATH 418 (n=16)	3.188	3.188	0.000 (0.183)
Column Difference: (SE Difference)		0.389* (0.279)	0.351[T] (0.230)	

*p<0.10

[T] Variances statistically different (p=0.046), means not significantly different until Welch's Test

Note: n=42, 1- Disagree, 4- Agree

7.1.3 Significant Correlations

The Pairwise Correlation Method (in JMP) allows for calculating the correlation between the respondents' Post – Pre Change between multiple questionnaire items. A positive correlation between two items suggests that students' responses between the two items experience similar change (i.e. both either experience positive or negative Post – Pre Change). A negative correlation suggests that the two items experience opposite change (i.e. one item experiences positive Post – Pre Change and the other negative).

For example, one significant positive correlation (r=0.579, $p<0.001$) was between the items, "Mathematics is a set of rules and procedures that need to be memorized," and, "How important is memorization to solving math problems?" As seen in Table 24, 10 of the 12 students that had a negative Post – Pre Change in the latter question also experienced a negative Post – Pre change in the former question. Thus, as shown in Table 24, the positive correlation manifests by having more entries along the green diagonal than the red diagonal.

Table 24

Comparing Characterizations of Negative, Neutral, and Positive Changes from Pre- to Post-Questionnaire Between Two Questionnaire Items

		How important is memorization to solving math problems?			
	Post – Pre Change	Negative	Neutral	Positive	
Mathematics is a set of rules and procedures that need to be memorized.	Negative	**10**	4	1	15
	Neutral	2	15	3	20
	Positive	0	3	2	5
		12	22	6	

Note: n=40, 1- Disagree/ Not Important, 4- Agree/Important

Table 25 reports on some significant correlations amongst in Post – Pre change among questionnaire items (the full list can be found in Appendix C). The correlations in rows 1-4 relate multiple aspects of mathematical beliefs. The positive correlation in row 1 suggests that students who had an increase/decrease in finding it helpful to learn different ways to solve a math problem tended to also have an increase/decrease in recognizing the importance of writing a solution that peers could understand. This correlation connects students' reported helpfulness of learning multiple approaches with their perceived value of clarity and communication of one's solution, both of which relate to students' personal conceptions of acceptable solutions. The positive correlation in row 2 expressed a connection between the value of discussion and the value for written explanation; thus, those that experienced an increase/ decrease in value for discussions tended to experience an increase/decrease in valuing written explanations in solutions.

Similarly, in row 3, students that tended to report an increased/decreased need for written explanations also had an increase/decrease in permissibility of using different methods than those learned in class. This result connects two aspects of what constitutes an acceptable solution. Furthermore, as students begin to tolerate the acceptability of different approaches, it may be natural to expect explanation of these approaches. The positive correlation in row 4 connects two items that describe mathematical beliefs about the purpose of solutions; this correlation relates the importance of writing a solution that others understand with importance to correctly interpret the validity of another's solution.

The correlations in row 5 connect mathematical beliefs with social beliefs. The changes in the items, "The most valid ways of solving a problem are the ones discussed in class," and, "The instructors and TAs are responsible for teaching me how to write a solution that would receive full credit," are negatively correlated. One way to interpret this is that as students develop value in flexibility and the permissibility of different solution methods, they more strongly believed that educators are responsible for teaching them how to write proper solutions. Thus, students in the class might have been more attentive to notation to be able to express their own approaches and wanted the instructors to guide them in adhering to appropriate conventions.

Table 25

Significant Correlations with Respect to Change in Post–Pre Data (correlations obtained by Pairwise Method)

#	Change in Question A	Change in Question B	Correlation, r-value (Significance, p-value)
1	I find it helpful to learn several different ways to solve a math problem.	How important is it to you to write a solution that your peers could understand?	0.494 (p=0.002)
2	There is no place in mathematics for discussions - you are either right or wrong.	Solutions written with formulas or equations are self-explanatory. They do not require written explanations.	0.374 (p=0.021)
3	To receive full credit, my solution must use the same methods used in class.	Solutions written with formulas or equations are self-explanatory. They do not require written explanations.	0.344 (p=0.034)
4	How important is it to you to write a solution that your peers could understand?	How important is it for you to be able to determine if a peer's solution is correct?	0.390 (p=0.016)
5	The most valid ways of solving a problem are the ones discussed in class.	The instructors and TAs are responsible for teaching me how to write a solution that would receive full credit.	-0.332 (p=0.042)

7.1.4 Summary of Quantitative Results

Quantitative results surfaced several important characteristics about students' mathematical beliefs. These results demonstrate an improved openness to learning about multiple solution approaches, which is key to fostering flexibility. The results also show positive changes in students' beliefs about memorization and discussion in mathematics. Upon further analysis, the variable of prior enrollment highlighted more prominent differences from the pre- to post-questionnaire; this includes the results that students without prior enrollment found it less helpful at the end of the semester to learn different solution methods, and that they more strongly valued correctly performing steps than understanding each one of them.

The results also show several changes in students' social beliefs. This includes significant changes to students' willingness to seek help from others. The variable of prior enrollment also demonstrates significant differences in students' role beliefs at the end of the semester: students with prior enrollment felt more strongly than students without prior enrollment that the instructors are responsible for teaching them how to write a solution that would receive full credit and for preparing students for assessments.

7.2 Qualitative Results

This section details the qualitative results of this project by each of the four groups of four students. Note that all names used are pseudonyms. Following each group's section, there is a table which lists the surfaced social and sociomathematical norms and summarizes the characterizations of each norm.

7.2.1 Group 1 - Albert, Dwayne, Gordon, and Harry

7.2.1.1 Social Norms.

7.2.1.1.1 Peers as Resources. Within Group 1, a normative pattern that developed through the semester was that group members treated one another as a resource. This manifested in a variety of explicit and implicit ways across the sequence of four activities.

For example, the group exhibited comfort admitting to one another when they were confused or wanted help. This ranged from asking one another procedural and conceptual questions. Members demonstrated eagerness to help one another, and this help progressed through the group. For example, in one activity, Albert taught Dwayne about dividing fractions, and then later Dwayne taught Harry to do the same.

Members also sought validation from each other, from conferring about approaches to comparing and checking answers. But there was also a layer of looking out for one another and holding each other accountable. This demonstrated responsibility included correcting peers' language, voicing concern over perceived errors in one another’s solutions, and correcting conceptual errors.

Despite the value in the group for their peers’ feedback, each member's did not appear equally valued. For example, the group would occasionally talk over Harry or disregard some of his questions. On the other hand, Albert's feedback and opinions garnered explicit and full attention.

Lastly, the group was a resource for emotional support. When a member vented about homework or exam scores, the rest of the group tried to comfort the member. The group was responsive and supportive towards frustration, whether it was towards the activity, interpreting solutions, the class, or otherwise.

7.2.1.1.2 Peers as Collaborators. The group members collaborated with one another during the first phase of the Multiple Solutions Activities (i.e. problem solving and forming the rubric). This behavior is similar to utilizing peers as resources, but a key distinction is that the group engaged in problem solving that involved contributions from various members, where the flow of ideas was not one-sided. When solving a problem, the group's conversation was inclusive, as members checked in with one another, engaged in dialogue, and shared perspectives. Harry in particular often attempted to facilitate collaboration in the group. For example, in the third activity, he tried spurring conversation by saying, "Someone go through your thought process with me."

In some cases, members would initially take some time to look over the problem and brain-storm individually. This was especially true with Albert, who would occasionally tell groupmates that he was not done analyzing and consequently not ready to discuss yet. But eventually the group tended to verify solutions and point out mistakes to one another. They discussed their solutions, for example, by suggesting why taking a certain approach was not helpful.

Forming the grading rubric was typically a similarly collaborative venture. Harry, Gordon, and Dwayne often discussed initial thoughts and values, while Albert typically worked alone until he completed a draft of the rubric. But differences in pacing, learn styles, and problem solving preferences occasionally produced disparities to group unity. Some members, like Dwayne, typically worked faster than the others, whereas Harry worked more slowly. Thus, there were cases where rubrics (and solution evaluations) were not identical or where a member copied the rubric of another. But, in most cases, the group ramified disparities in their rubrics.

7.2.1.1.3 Analyzing and Interpreting Solutions. One of the most evident social norms that developed within this group across the sequence of Multiple Solutions Activities, was the importance of interpreting and understanding others' solutions. As the semester progressed, the students spent increasing effort to analyze the provided solutions to understand and evaluate novel approaches and to find errors in them. The group became more proficient at unraveling and decomposing solutions to understand them. Even when the group initially criticized novel approaches, which included comments that a solution looked "really [messed] up" or "really wrong," this did not detract from their efforts to interpret a new method. Furthermore, the group even started to connect different approaches, explaining why solutions were equivalent.

Another feature that developed in this group was the importance of all group members' participation in collaboratively discussing each solution and their evaluation of it. The group exhibited a shared responsibility to explain what they understood about each solution and to help clarify confusion to each other when possible. When analyzing novel solutions, group members would verbally share their confusion with one another. Naturally, not all group members were uniformly vocal. To accommodate Albert's introverted demeanor, the group often asked for his opinion on the solutions, or initially provided him with some space to formulate an opinion, to integrate him into the group discussions. The group demonstrated that they valued each other's concerns, questions, and suggestions about the sample solutions.

7.2.1.1.4 Distractions Hindered Progress. In every activity throughout the semester, the group faced several forms of distraction. The first, and most prevalent, was the return of classwork/homework, which interrupted productive conversations, hindered progress, and prevented focus on the activity.

In particular, this was exacerbated by Dwayne. In every activity, Dwayne would complain and vent about returned work to his group, ask the instructor questions, or focus on reviewing the returned work. Dwayne also caused distractions in a variety of other ways. This included extreme tardiness during one activity and elongated breaks (such as to buy a beverage). It is worth noting that phone usage was not a major distractor for the group. There was only one occurrence in the four activities where a student (Dwayne) used his phone for a non-mathematical purpose. Phones were occasionally used as a mathematical resource.

7.2.1.1.5 A Note on the Heterogeneousness of Attentiveness During Class Discussions and Responding to Reflection Questions. The attentiveness within the group during class discussions was heterogeneous but rather consistent throughout the semester. Each member has their own unique behavior and approach to such discussions.

Despite being a source of distraction for the group, Dwayne contributed to every class discussion and was mostly attentive during the discussions. There were occasions where he did not pay attention to the discussion, but looked through the activity worksheets, took notes on the reflection questions, or discussed a conceptual question with Gordon. Similarly, Gordon largely paid attention in class discussions, but this attentiveness clearly wavered over longer durations. Meanwhile, Harry was engaged throughout every activity. He was fully attentive to the instructors, as he nodded and verbally agreed when appropriate. He also contributed to class discussions. On the other hand, Albert typically focused on the worksheets by writing thoughtful and detailed responses to evaluations and reflection questions during the class discussions. He judiciously participated and followed along when the instructor provided explicit directives, such as finding a mathematical mistake in a solution on the board.

The differences in the approaches aligned with the group's individual work on the reflection questions. Reflection questions (see section 5.1.2.2) were not treated as a means of collective review, but instead as an individual task or occasionally as a way to vent about underlying frustrations. As a consequence, contributions during the class discussion were not always reflective of the whole group, but of individual students. Nevertheless, collectively, the group was more attentive during the class discussions than the others studied.

7.2.1.1.6 Several Cases of Relying on Authority. In two of the activities, there was evidence that a couple members were reliant on sources of authority. Hence, this was not a normative pattern of the group. Nevertheless, it is important for later comparisons to note that this group did occasionally demonstrate intellectual heteronomy.

For example, during the second activity, Gordon expressed uncertainty about his answer, but did not do anything to verify or check the answer himself. Dismissing responsibility for verifying, Gordon attempted to use a smartphone application, "PhotoMath," to do so. Ultimately, he was unable to get the app to work as he intended.

In the same activity, Gordon asked the instructor to check his work. Instead of providing feedback, the instructor asked Gordon to explain his solution, then suggested that Gordon ask a groupmate to explore the approach with him. Later on, Gordon vented his frustration when his peers found a solution they were comfortable with:

Gordon: Is that right? Do you guys agree?

Dwayne: Yeah, because the nice..." (gets cut off)

Gordon: Okay, so I have to redo this then.

Gordon was frustrated at the amount of effort he put into one problem and that he did not receive more prescriptive feedback from the instructor.

In at least one case, the TA did validate approaches for the students. During the third activity, Dwayne asked the TA to look over his solution. The TA supported the solution and pointed out a notation error.

7.2.1.2 Sociomathematical Norms.

7.2.1.2.1 An Acceptable Solution Must Use Proper Notation. A sociomathematical norm that developed in this group was that an acceptable solution must use proper notation. The group was apt to criticize solutions that utilized incorrect notation. Notation was valued within the group for the meaning that it conveys, not as a superficial component to a solution.

Like other groups in the class, this group was keen to note when answers were not represented with the proper notation. This included errors with interval notation, expressing a vertex as a point of two coordinates, and using functional notation to express an inverse function. The group valued the use of proper notation, even if an answer was incorrect. For example, in the first activity, the group suggested credit for using correct interval notation in the answer, regardless of the accuracy of the answer.

Moreover, their attention to notation was not limited to just the answer. The group ensured that notation usage adhered to its proper meaning. For example, in the first activity, the group criticized Tom's usage of "not-equal" signs instead of inequalities. The group was even precise to note when strict inequalities were needed instead of inclusive ones. Members even explored notational aspects of approaches. In the third activity, the group discussed how switching "x's" and "y's" was not a necessary component to determining an inverse function. Upon seeing a solution that did not switch the variables, the group was reflective about role of switching them and what it accomplished. Dwayne pointed out that not switching and isolating "x" was functionally the same as their procedure (switching and isolating "y").

Yet, the group did not always note informal or short-hand notation. For example, in Brody's solution in the first activity, the group did not discuss the informal notation but they did note the incorrect inclusion of an end point. As expected, content knowledge also influenced the group's recognition of notation errors. In the first activity, the group was sharp to criticize inequality errors in the solutions. Yet, in the last activity, the group did not note errant inequalities from domain restrictions until prompted by the instructor.

7.2.1.2.2 An Acceptable Solution May Follow Any Valid Approach. The group's value of different solutions developed through the sequence of Multiple Solutions Activities. Ultimately, a sociomathematical norm emerged within the group, and was sustained: an acceptable solution is one that follows any mathematically valid approach.

In the first activity, the group initially created a grading rubric for adhering to their own specific procedure for solving the problem. The group did not demonstrate any averseness towards novel approaches, but did not seem to anticipate seeing different ways to solve the problem. Yet, in the following activity, members of the group solved the problem in different ways. This spurred conversation and consideration about the viability of different approaches. This conveniently served as a way for the group to acknowledge and ultimately appreciate flexibility. In one instance, Dwayne explained an approach to his group and concluded, "There are other ways to find x and y ... it would be easy to do it this way, the way you did it like this [referring to his own solution], but that's not the only way to find the answer. It doesn't tell us that we have to do it that way." As a consequence, the group acknowledged the need to have a "flexible" grading rubric, one that was broad and amenable to different approaches.

By the second activity, the group was already defending the permissibility of alternative approaches to others in the class. During the class discussion, a student outside of the group suggested that "Frodo" should have done the problem "in normal way." This violated the norm established in the group, and consequently, Harry responded by asserting that they "can't discriminate" against unfamiliar approaches.

In the two subsequent activities, the group continued to be open-minded towards novel approaches, as the group demonstrated that an acceptable solution was one that followed any mathematically valid approach. This expanded their engagement with the activity. During the third activity for example, prior to receiving the sample solutions, the group reflected on generating multiple ways to solve the problem.

Overall, the group did not discriminate against unfamiliar approaches - instead of condemning solutions that deviated from their own method, the group contemplated the viability of these approaches. For example, this was seen in the group's analysis of Lincoln's solution. The group did not dismiss the solution for taking a square-root - eventually concluding "it is unnecessary but it is right." This also represents how the group even began characterizing solutions. Additionally, some approaches were characterized as "roundabout" or "interesting." In this sense, solutions themselves became objects of reflection.

Furthermore, the group demonstrated an openness and drive to understand the solution, and not pre-determine its accuracy by looking at the final answer or its adherence to a particular method. Members expressed acceptance and support for solutions that drastically differed from their own methods. As a consequence, in both of the last two activities, the group began to investigate why an answer was wrong, instead of using the inaccuracy as a judgement to condemn an approach as invalid (as some other groups).

This also demonstrates a relationship between the sociomathematical norm that an acceptable solution is one that utilizes any valid approach, and the social norm of investigating solutions. This will be further detailed section 7.2.1.2.4.

7.2.1.2.3 Work is Valued for the Meaning It Carries. The group also discussed the role of showing work in the solutions. In the first activity, the group debated/discussed whether to award credit for solutions that simply have work, in effect rewarding effort, or for awarding credit for work that is constructive to the solution. Initially, the group was rather split on this idea. Eventually, Albert, Harry, and Gordon reached a consensus that work should be valued for contributing to the solution:

Gordon: I'm going to give him three, because he shows his work, he does it out, he finds the correct numbers, he just doesn't put the answer together.

Albert: He shows his work, but it's incorrectly done.

Harry: Yep, he shows work.

Albert: But it's done in a way that's incorrect and gives him the wrong answer.

Harry: I mean, yeah.

Gordon: Ah-okay, yeah. (Later) His work and answer are incorrect, so. Okay.

The whole group did not reach consensus on this during the first activity, as later on, Dwayne said that he awarded credit simply because, "it is work." Yet, Dwayne's view on work started to align with his group members' view that work should be evaluated for meaning. In the following activity, Dwayne suggested that work should be analyzed for the reasoning it displays and that the group should still interpret sample solutions despite any errors. Thus, the whole solution was not tarnished as invalid or incorrect for minor mistakes, such as errant computations. Furthermore, in the last activity, Dwayne described that work needs to act as justification.

7.2.1.3 A Sociomathematical Norm Influenced by a Social Norm. The following vignette helps to explicate the relationship between the group's sociomathematical norm that an acceptable solution is one that follow's any mathematically valid method and their social norm of investigating and interpreting solutions. During the last Multiple Solutions Activity of the semester, on the topic of inverse trigonometry, as Group 1 formed their grading rubric, they explicitly expressed awareness that there are different ways to solve the problem besides their chosen method. Harry described reluctance to form a rubric that would be limited to only one familiar way of solving:

Harry: I don't know if there is another way to solve it, so I don't want to write [grading] rules.

As they looked at the sample solutions, the group was initially dismissive of "Jennifer's" solution, which utilized right triangle trigonometry with the angle $u = \sin^{-1}\left(\frac{1}{2}\right)$. This represented a novel approach that the group was unfamiliar with.

Gordon: This person is doing some weird math.

Dwayne: What did you do here? What kind of [stuff] is this? How the [heck] did you get to that?

Their lack of familiarity with her solution was obviously discomforting to them, but, despite these initial reactions, the group continued to investigate.

Gordon: [Jennifer] didn't find the inverse sine, so. They never even solved for u.

Harry: She's saying this is sine of u, this triangle, so then tangent would be opposite over adjacent, so one over one. That's what she's saying … she just didn't do it right.

Gordon: Right, because this should be one half, square root of three over two, and one (pointing to the triangle, and referring to a common right triangle).

Gordon's remark suggested that when using trigonometry, the triangle must have a hypotenuse of one. Gordon did not seem to understand how Jennifer formed her triangle, but, as Dwayne asked questions about Jennifer's approach, he was able to clarify Gordon's misconception.

Dwayne: "a" squared plus "b" squared" is "c" squared. How did [she] get two? (Pointing to the hypotenuse). Oh! [She] did one over two. That's correct though. That's just a different proportion. That is right.

This insight helped Gordon, who eventually located the exponent mistake in Jennifer's solution. After he explained the mistake to the group, he noted:

Gordon: If she did her math right, she actually would have got it, because "a" would have come out as square root of three.

Dwayne: So her process is right … but she just made one mistake. And technically her tangent work is correct for the work.

This particular example demonstrates how the mutual influence of the sociomathematical norm that an acceptable solution is one that utilizes any valid approach and the social norm of collaborative analysis and interpretation of solutions. Because the group deemed any valid approach viable, even those that were novel, the group would interpret the solution. Concurrently, as the group investigated solutions, this developed an appreciation and acceptance for solutions, even those that were originally unfamiliar.

Table 26

A Summary of the Norms and their Characterizations, Developed by Group 1

<table>
<tr><th>7.2.1.1 Social Norms</th><th>7.2.1.2 Sociomathematical Norms</th></tr>
<tr><td><u>7.2.1.1.1 Peers as Resources.</u>
-Comfortable asking one another for help
-Conferred about approaches to checking answers
-Accountable for language, errors in each other's solutions, and mathematical understanding
-Source of emotional support</td><td><u>7.2.1.2.1 An Acceptable Solution Must Use Proper Notation.</u>
-Answers must be represented with the proper notation as well as the rest of the approach
-Notation valued within the group for the meaning that it conveys</td></tr>
<tr><td><u>7.2.1.1.2 Peers as Collaborators.</u>
Problem solving:
-Inclusive conversation
-Flow of ideas not one-sided, dialogue to share perspectives
-Attempts to facilitate collaboration
Rubrics:
-Despite typical unity, differences in pacing, learning styles, and problem solving preferences occasionally produced disparities</td><td rowspan="2"><u>7.2.1.2.2 An Acceptable Solution May Follow Any Valid Approach.</u>
-Expressed the need for flexible rubric that is amenable to different approaches
-Voiced permissibility and acceptance alternative approaches, as the group was open-minded towards novel approaches
-Led to solutions becoming objects of reflection, as the group characterized approaches (ex: "roundabout" or "interesting")
-Motivated students to understand the solution, and not pre-determining accuracy by looking at the answer or its adherence to a particular method
-Drove students to investigate why an answer was wrong, instead of using the inaccuracy as a judgement to condemn an approach</td></tr>
<tr><td><u>7.2.1.1.3 Analyzing and Interpreting Solutions.</u>
-Important to make sense of sample solutions and find errors, despite being initially critical of some novel approaches
-Eventually started to connect different approaches, and explain why some were equivalent
-Analysis was collaborative, as there was a shared responsibility to explain what was understood and to clarify confusion</td></tr>
<tr><td><u>7.2.1.1.4 Distractions Hindered Progress.</u>
-Distracted by the returning of classwork, but phone usage not a major distractor</td><td><u>7.2.1.2.3 Work is Valued for the Meaning It Carries.</u>
-Work in a solution must convey meaning and contribute to the reasoning or justification within the solution
-Work is not valued for effort or presence</td></tr>
</table>

7.2.2 Group 2 – Chad, Molly, Peter, and Steve

7.2.2.1 Social Norms.

7.2.2.1.1 Using Informal Language. One pattern that emerged during the first three activities in particular was students' use of informal language. Students' in the group used non-technical, imprecise language when talking with one another about mathematical content. This language was not clarified amongst the group, and was assumed to be mutually understood. This informal language usage was not only unquestioned and uncorrected by group members, but its use was perpetuated, as members started using one another's language. For example, during the first activity, Molly used the term "limitation" instead of "domain restriction." About five minutes later, Steve used the same term.

The group's informal language usage was often vague and held various meanings. One prevalent example was the group's use of the word "formula." In the first activity alone, three members of the group used the word "formula" in a variety of informal and improper ways. The word's usage was deictic, as the informal meaning of the term varied on context. The word continued to be used through the first three activities to describe various entities, including problem-solving algorithms, equations, and non-specific mathematical expressions.

7.2.2.1.2 Steve Determines Mathematical Validity. Across all four activities, Steve was treated as an authority to determine mathematical validity for the sample solutions. The group trusted his determinations and did not question him.

Steve's treatment as a source of authority dissuaded development of autonomy in the group, as collaboration was not always productive. Instead of mutually trying to understand the solutions, the group exhibited trust in Steve to do so for them. The rest of the group rarely sought to understand why Steve determined that a solution was wrong, or what corrections should be made. Consequently, members of the group circumvented opportunities to analyze mathematical arguments, missing chances to develop reasoning skills and flexible knowledge.

For example, in the second activity, Steve told the group that Andrea used “the wrong formula” because she subtracted 2/3 instead of adding. This assertion is incorrect, but the rest of the group believed him. Molly, Peter and Chad did not try to understand the “error.” Additionally, in the following activity, when looking at Alexander’s solution, Steve noted, “Well he multiplied by four right away, which is automatically wrong.” This incorrect determination dissuaded the rest of the group from trying to interpret the solution, and instead the group again moved on.

7.2.2.1.3 Aversion to Interpreting Solutions. Across the four activities, the group often avoided interpreting sample solutions. In this context, “interpreting” signifies trying to make sense of and understand the underlying mathematics of the solutions. In this group, it was very rare for anyone to try to understand where answers came from or question why solutions yielded answers that were characterized as correct or incorrect.

When someone did notice an error, no one else in the group seemed to try to understand the error. This was manifested in two different ways. First, there were cases where someone tried to explain the error to others, but their efforts were dismissed or disregarded. For example, in the second activity, Peter attempted to explain to Steve where “Kennedy” made a mistake, but Steve showed disinterest and instead moved onto the next solution. Secondly, there were instances like those described in the previous section, where the group trusted and noted others’ judgements, and moved on. The only discussion was about how many points to “take away.” Both cases exhibit a general disinterest in trying to understand and interpret the solution.

Another connection to this aversion to interpret is the group’s determination of what is mathematically valid. For example, when interpreting “Frodo’s” sample solution in the second activity, Molly first noted that the solution arrives at their answer. Despite not understanding the solution, Steve concludes that the solution is valid because it arrives at the right answer (this sociomathematical norm will be detailed in section 7.2.2.2.2). This demonstrates that the sociomathematical norm of what constitutes an acceptable solution restrained the group’s efforts to interpret the solution.

The group’s efforts to analyze solutions did not align with the intentions of the activity. Instead of interpreting solutions, group members looked for where solutions deviated from their own. There were instances where the group recognized that a solution used a different approach. In the third activity, the group acknowledged that various solutions used different logarithms, but the group did not explore this further. The group did not try to understand how the logarithms were used differently. Instead, they characterized the usages as “wrong,” and discontinued their investigation. It is also important to note that these behaviors were not due to a lack of time available. The group had ample time and finished early in each activity.

In summary, the group demonstrated an aversion to mathematically interpreting and trying to understand the sample solutions. Instead, the group focused on the alignment of a solution to their own, or determined that methods that arrived at their answer were valid, and those that did not, were invalid. Also, when members had useful perspectives to share, other members were often dismissive and instead focused on continuing through the activity.

7.2.2.1.4 Appealing to Authority. Consistent with some of the patterns described already, the group also demonstrated a reliance on authority for making mathematical determinations. In addition to the group's reliance on Steve to make these determinations, there were several other sources of authority that the group appealed to. During the first activity, after looking at "Tom's" sample solution, Steve determined that, "We had the denominator right though." In this case, Steve and the group discussed that Tom's solution verified their own – treating the sample solution itself as a source of authority. The group also appealed to the instructors when they do not feel confident in their work. One such example occurs in the third activity:

Peter: I think I got it right, but I have no idea.

Steve: I'm not sure if this is right though.

Peter: Yeah, it's either really gross or completely wrong.

After this exchange, the group asked the TA to confirm the group's solution, which he did. This demonstrated that the group was dismissing responsibility for verifying solutions to the instructor. In addition to verifying solutions, the group also relinquished responsibility for interpreting solutions to the instructors. During the same activity, after Peter raised a question about a sample solution depicting a novel approach to the group:

Molly: Did he do it right then?

Peter: Let's wait and see.

This exchange describes their willingness to wait and let others, the instructors, make the determination.

In general, the group members do not exercise mathematical autonomy. They often rely on sources of authority to make such mathematical determinations for them, particularly, when verifying their own work or when new approaches are involved.

7.2.2.1.5 Inattentiveness During Class Discussions. Across the four activities, the group remained generally inattentive, particularly during class discussions. Despite instances where a member participated during the class discussion, the group's attentiveness was not durable, and was quickly subsiding. For example, during the last activity of the semester, the group did not remain engaged during the class discussion. When the TA was leading this discussion, the instructor noticed that Chad was on his phone and tapped Chad's shoulder, causing him to put his phone away. Shortly later, Chad put his head down. Molly also frequently checked her phone, but eventually switched to drawing on her paper. Peter eventually closed his eyes and Steve put his head down. Then, Steve started packing up early and even collected the group's work early. Seeing this, the instructor asked the group to listen to the discussion. These patterns were common across the four activities.

In addition, the group was distracted by their cell phones, which were rarely used as a mathematical resource. The group was also distracted from the activities whenever graded work was passed back to students.

7.2.2.1.6 Peers as Collaborators During the First Phase. During the first phase of the activities (i.e. solving the problem and forming a grading rubric), there were several patterns in the group's collaboration on the first phase of the activities that were sustained across the semester. In all four activities, the group demonstrated the same approach to solving problems. First, members of the group worked individually to solve the given problem; during this time, there was no collaboration, outside of possible arithmetic verification amongst the group. After someone solved the problem, the group began discussing and tried to reach consensus about the correct answer. During this time, solutions were shared as the group constructively provided feedback about mistakes they saw in one another's solutions and asked questions about their approaches. One key component in this was that the group volunteered information to receive feedback. This included showing one's solution to the group, or announcing a question to the group.

Members did not share responsibility for integrating others into conversations or the activity in general. This was especially clear in the case of Chad. In all four activities, Chad did not contribute to any mathematical discussion. He did not volunteer any information or perspective. Chad often "peeked" at other members' papers to copy down their work. On two occasions, Chad asked others in the group what the final answer was, and each time the group provided a detailed description of the procedure used to obtain the answer. The group held responsibility for responding to questions within the group during this phase of the activity, but did not hold responsibility for ensuring everyone was comfortable with solving the problem.

The group did not put forth the same effort towards the formation of the grading rubric. In all of the activities, the majority of the group was not involved in the formation of the rubric, as typically just Molly and/or Steve formed the rubric. Yet, the group maintained uniformity during this phase, as those who were not involved in the formation of the rubric, copied it from those that were involved. When the group asked to move on the next phase of evaluating sample solutions, the instructor would ask the group if everyone agreed on one rubric. The group would answer affirmatively, despite the lack of full collaboration, and in one case, any collaboration.

7.2.2.2 Sociomathematical Norms.

7.2.2.2.1 An Acceptable Solution Needs Formal Notation Only in the Answer. One characterization that was maintained across the semester was that formal notation is only needed in the final answer of a solution. This was seen in varying degrees across all four activities, but especially surfaced in the first three activities. In these activities, the group was conscientious about adhering to formal notation for each answer, whether it was proper use of interval notation in the first activity, listing the vertex as a point of two coordinates in the second, or utilizing proper functional notation for inverse functions in the third.

Yet, the group often avoided attending to and critiquing notation in the rest of the solution. One reason for this was explicated in the first activity when Steve said to the group, "He also kind of used the formula wrong. Because he did the whole way it can't equal until the end where he just rewrote it. Should we take off a point for using the formula incorrectly?" Steve was referencing “Tom’s” solution, where Tom utilized “≠” until the last step, where he switched to using inequalities. Molly responded by suggesting that it was not consequential, “as long as you have it right … at the end.” Steve and the group concurred and moved on.

As a consequence, the group was not averse to informal or unconventional notation. Molly once even went so far as to praise Brody’s notation, despite its pervasive informal and questionable notation, by saying: “It explains everything and is straight to the point, and like he circles his answer so you know right where to look, I guess.” Not only was the group not bothered by the extreme informality of the solution, but this comment was also suggestive of the weight the group gives to the answer, as Molly’s comment demonstrated the diminished importance for the rest of the solution.

The group also expressed that solutions do not need annotations or clarifications and that it is instead the role of the grader to make the necessary interpretations. For example, Steve commented, “Andrea wrote a lot of notes, which are unnecessary. Since we’re the grader, we should know what we’re doing.”

The group did deem it necessary to utilize specific notation if it was perceived to be an integral part to a procedure. This was evident in the third activity, when the group deemed it necessary to switch “x” and “y” (the independent and dependent variables). Solutions that did not adhere to this, were condemned. One solution that did not switch the two variables, correctly labeled the solution as $f^{-1}(y)$. The group also condemned this, and noted that the answer should have read “$f^{-1}(x)$.”

7.2.2.2.2 An Acceptable Solution Must Follow a Familiar Approach or Arrive at the Correct Answer. The group recognized the instructors’ attempts to negotiate the sociomathematical norm that an acceptable solution was one that followed any mathematically valid approach, not just a familiar one. There were even instances where this acknowledgement surfaced. In the second activity, Molly originally suggested to the group that they should take off a point in their evaluation of Frodo’s solution because it did not follow their procedure. After deliberation, Steve shared that the question did not require a specific procedure, which the group agreed to.

Yet, the group struggled adhering to this notion; they condemned solutions that were unfamiliar, as Steve and Molly noted during the same activity:

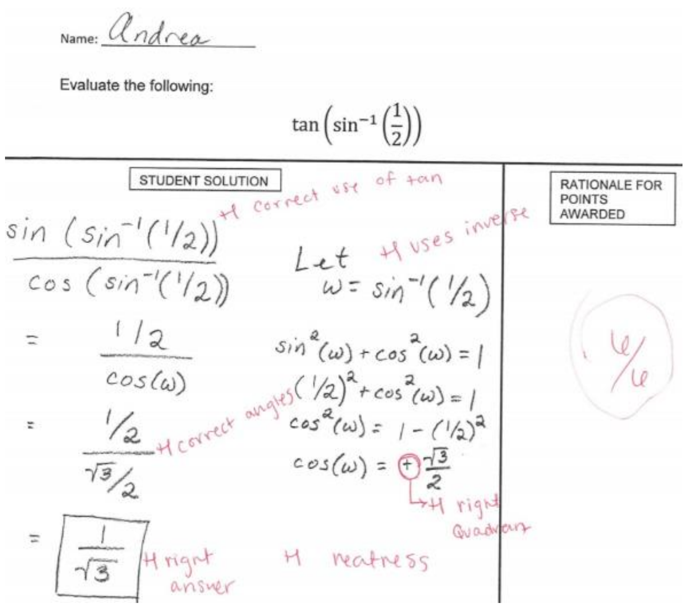
Steve: For execution, only give them one point, they used the formula wrong.

Molly: Frodo’s [solution] is neat and it states a clear answer, but he did it in a really weird way.

This developed into an important characteristic defining what constitutes an acceptable solution within the group: An acceptable solution to a problem is one that uses a familiar approach or leads to the correct answer. This norm persisted across the semester, and was especially evident in the last activity, which is described below.

Figure 14

Molly's Grading of Andrea's Solution



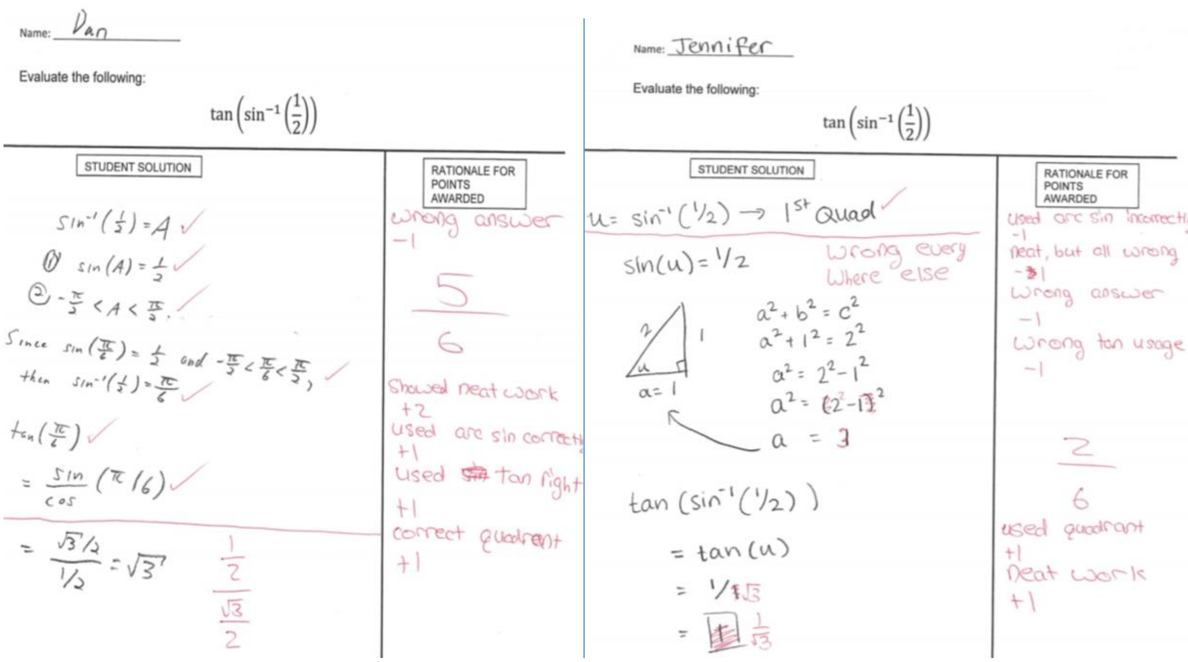
In the last activity of the semester, the group evaluated three sample solutions: Andrea's solution, which used an unfamiliar approach but resulted in the correct answer, Dan's solution, which followed a method shown in class but had a wrong answer because of an intentionally included error, and Jennifer's solution, which was both unfamiliar and also yielded an incorrect answer.

The group favored Andrea's solution (Figure 14), which yielded a correct answer, although it used an unfamiliar method. The group concluded that Andrea's solution was "interesting" and viable, since it "got them the right answer." The students relied on the authority of the answer to determine whether or not the approach was valid, but without thoughtful investigation.

The group was also receptive towards Dan's solution (Figure 15-a), but for a different reason. Dan's solution resembled the approach the instructor modeled for similar problems; thus, it was familiar to the group members. Eventually both Molly and Steve concluded that: "He has everything right except the answer."

Figure 15 a & b

Steve's Grading of Dan (a) and Jennifer's (b) Solutions



When students were familiar with a procedure, they were able to recognize patterns and locate errors, unlike in novel solutions like Jennifer's. Jennifer's solution (Figure 15-b) used an unfamiliar approach and resulted in an incorrect answer. The group had a scathing first response towards the solution:

Steve: Oh God, this already looks bad. Oh yeah, this is real bad. 0 out of 6 … I hope this is not a real student, I really hope.

The only discussion in the group was to determine if Jennifer should earn points for neatness or for “getting the quadrant right.” The group did not notice the arithmetic mistakes until the instructor pointed it out to them.

In general, this group did not develop the sociomathematical norms that the instructors advocated and negotiated for. Instead, they chose to focus on the correct answer, as in Andrea’s solution (Figure 14), or a familiar procedure, as in Dan’s solution (Figure 15-a). The group’s affinity towards familiar approaches coincides with their adopted social norm of aversion to exploring novel solutions (as discussed in Section 7.2.2.1.3).

Group 2 expressed the role their prior experience in MATH 418 had on their adherence to specific procedures, as well as their frustration with the current course, which had a drastically different approach towards learning mathematics:

Steve: Last semester they constantly drilled in our head that there was only one way to do it.

Molly: Yeah. So that's why I feel like a lot of us, or at least personally why I'm struggling.

Steve: It's a lot different.

Molly: I don't have a set rule to follow.

These comments may represent the lingering effects of norms of previous courses and the obstacle this provides for improving engagement by negotiating contrasting norms.

Table 27

A Summary of the Norms and their Characterizations, Developed by Group 2

<table>
<tr><th>7.2.2.1 Social Norms</th><th>7.2.2.2 Sociomathematical Norms</th></tr>
<tr><td><u>7.2.2.1.1 Using Informal Language.</u>
-Used and perpetuated informal language, which was not questioned or corrected
-Assigned various informal meanings to mathematical terms, depending on context</td><td rowspan="3"><u>7.2.2.2.1 An Acceptable Solution Needs Formal Notation Only in the Answer.</u>
-Formal notation was only needed in the answer, not throughout the solution
-Accepting of informal or unconventional notation
-Annotations and clarifications are unnecessary
-Specific notation deemed necessary if it was perceived to be integral to a procedure (ex: switching "x" and "y" for finding an inverse)</td></tr>
<tr><td><u>7.2.2.1.2 Steve Determines Mathematical Validity.</u>
-Steve was treated as a source of authority to make mathematical determinations for the group, which were typically unquestioned</td></tr>
<tr><td><u>7.2.2.1.3 Aversion to Interpreting Solutions.</u>
-Rarely tried understanding where answers came from or why solutions yielded answers that were characterized as incorrect
-Collaboration was rare as attempts to share perspectives were dismissed, as mathematical judgement was often blindly trusted
-Analyzed solutions by noting that solutions deviated from their own, instead of interpreting the approaches
-Because solutions that followed different approaches were considered invalid, they were not interpreted</td></tr>
<tr><td><u>7.2.2.1.4 Appealing to Authority.</u>
-Relied on additional sources of authority, like the answers to sample solutions and the instructors
-Dismissed responsibility for verifying and interpreting solutions to the instructors</td><td rowspan="3"><u>7.2.2.2.2 An Acceptable Solution Must Follow a Familiar Approach or Arrive at the Correct Answer.</u>
-The group deemed solutions acceptable if they utilized a familiar approach or yielded the correct answer
-The group expressed the role of their prior course on their taken-as-shared beliefs about adhering to a familiar or specific approach</td></tr>
<tr><td><u>7.2.2.1.5 Inattentiveness During Class Discussions.</u>
-Avoided engagement during class discussions
-Distracted by sleeping, drawing, and primarily, using phones</td></tr>
<tr><td><u>7.2.2.1.6 Peers as Collaborators During the First Phase.</u>
-Began solving problem individually, followed by discussing and comparing solutions
-Participation based on volunteering perspectives or questions within the group
-Held responsibility for responding to problem-solving questions, but not for ensuring everyone was comfortable with solving the problem
-Rarely any productive collaboration on the formation of the rubric, most members just copied from another</td></tr>
</table>

7.2.3 Group 3 – Ted, Wes, Cullen, and Herbert

7.2.3.1 Social Norms.

7.2.3.1.1 Struggling to Interpret Solutions. Over the duration of the semester, the group exhibited conceptual unpreparedness to interpret the sample solutions. Their lack of interpreting solutions was not due to a lack of effort, but was influenced by several factors.

One such factor was the students' underdeveloped conceptual understanding of both the mathematical content of the Multiple Solution Activities and the prerequisite content for the course. The group struggled with fundamental ideas, like simplifying fractions and utilizing laws of exponents. This frequently prevented them from fully grasping novel approaches. In some instances, this struggle with prerequisite material led the group to dismiss certain solutions. For example, in the third activity, the group was unable to understand that $(3^{8y})^{\frac{1}{2}} = 3^{4y}$, as the group commented that the square-root of 8 is not 4. Consequently, the group dismissed this solution (Lincoln's).

One of the consequences of struggling with the content was the groups' assumption that the solutions, particularly those with novel approaches, were wrong. When encountering unfamiliar approach, the group coped by speculating about the mindset of the fictitious students and what these fictitious students did not understand. For example, in the fourth activity, when discussing the reflection questions, the group admitted that they never tried to make sense of Jennifer's solution, because it was "bad." They were unable to understand the connection between right-triangle trigonometry and the unit-circle. When the group revisited the solution, Wes speculated that Jennifer just "dropped the sine inverse" or that she "didn't know what tan was," Cullen suggested that Jennifer "just mixed up what sine was," and Ted suggested that "she just solved a random triangle." They expressed that there was no reasoning behind the solution, and did not try to make sense of it.

Another aspect of the groups' struggle with interpreting solutions was inability to transfer problem-solving strategies to new situations. For example, in the first activity, after determining the domain of the function, the group verified their result by evaluating the denominator at various values. When the group began to analyze the sample solutions, they saw that the numerator was involved in the solutions and expressed hesitancy and confusion about the role of the numerator in the domain of the function. Despite exhibiting techniques to check the denominator, the group did not think to do so for the numerator. Thus, it was not that the group was unable to determine the domain of a function, but they struggled with the idea of transferring previously used strategies to new situations.

7.2.3.1.2 Longitudinally Diminished Attentiveness. The attentiveness that this group exhibited across the four activities noticeably shifted during the semester. In the first two activities, the group was attentive to the task and their peers. The group was largely attentive to the tasks. When they were distracted, the group quickly refocused, or a member would regroup the others to attend to the activity. In addition, during the first two activities, the group members followed along with the discussion, which could be seen as the members flipped to the appropriate pages and took notes. During the first activity alone, three of the members volunteered to contribute to the class discussion. However, this attentiveness was not uniform and gradually faded, as occasionally a member started to draw on their paper or temporarily close their eyes.

These patterns became more pervasive in the latter two activities. During the first phases, the group was more distracted, as members were seen goofing around or having off-topic conversations. Members of the group were less attentive during the class discussion and more frequently participated in distracted behaviors like drawing or zoning-out. This shift aligned with the group's declining understanding of the content, as the semester progressed. As group members struggled more with understanding the material, they were less attentive during the activity, particularly the class discussion. This shows an unsurprising relationship between content understanding and attentiveness.

One student, Wes, showed notable exception to this behavior during the class activities. From conversations within the group, Wes demonstrated a deeper understanding of the content in the last two activities than the others in the group, and was actively engaged as he intently followed along during the class discussions.

7.2.3.1.3 Appealing to Authority. The group appealed to authority during the sequence of activities. Their appeals to authority were not as pervasive as that of Groups 2 and 4, but met the conditions to be considered normative. Primarily, the group appealed to authority by asking the instructor to verify the group's answer. The instructor tried to devolve this responsibility back to the group, but they instead moved on with the activity. The group also appealed to the authority of the answers in the sample solutions as a means to verify the correctness of their own solution. If the group saw their answer represented in the sample solutions, they were apt to believe that they solved the problem correctly. Whereas in the first activity, the group expressed great worry about their solution because none of the sample solutions shared their answer.

The group also occasionally dismissed responsibility for understanding content and the solutions. In one activity, Ted responded to a peer's question by saying, "I don't know, I'm not a teacher," and in another activity, Wes shared this idea by stating, "What are you asking me? I don't know." In another case, the group appealed to mathematical formulas that they did not understand, trusting that it would lead them to the correct answer.

7.2.3.1.4 Peers as Resources. Within the group, members treated one another as a mathematical resource. When someone was confused or uncertain, they were comfortable posing questions to their peers in the group. This included checking that their answers to the problem match and are fully simplified. Though, the group did not treat this verification as sufficient, and often sought further validation, as described in the previous section.

Members typically exhibited a responsibility for each other's understanding. This was especially evident when members did not share the same answer. During the first activity, Cullen discussed with Herbert about which inequality should be used. Cullen was persistent and patient to explain his thought process to Herbert, even though Herbert did not quickly understand. Cullen began to provide examples to Herbert to explain the conflict, eventually showing Herbert the error. The only instances of unresponsiveness to questions were when other group members did not know the answer or were not equipped to properly help.

7.2.3.1.5 Peers as Collaborators. In general, the participation structure of the group was built on volunteering perspectives, sharing questions and confusion, asking for help, and rebuking ideas when applicable. The group did not typically inquire about one another's thoughts or opinions. This was rarely an issue, as members regularly verbalized their thoughts.

Both during the problem solving and the evaluating solution phases, the group tried to understand the contributions of others. But, one difference in the participation structure between these two phases was in the drive for consensus. When solving the problem, the group exerted effort and expressed value for getting a matching answer. But when forming the grading rubric and evaluating the sample solutions, the group did not always press for uniformity.

The group attempted to hold each other accountable, both with respect to content and attentiveness. This included holding each other accountable for the mathematical language used. For example, during the first activity, Ted politely pointed out to the group that the term that needed to be used was "inequality" instead of "equality," which they had been using. On another occasion, Wes called over the instructor to help the group understand the question, thereby holding the group accountable for learning, instead of just focusing on how to get an answer. As described earlier (section 7.2.3.1.2), there were instances where members of the group tried to hold one another accountable for being engaged in the activity.

7.2.3.2 Sociomathematical Norms.

7.2.3.2.1 An Acceptable Solution Must Use Formal Notation. Several patterns emerged in the group's characterization of the role of notation in acceptable solutions. Primarily, the group determined that answers needed formal notation. The group was attentive to the use of notation, but this was ultimately constrained by the group's conceptual understanding.

The group's attention to notation went through an apparent evolution in the first activity, which seemed to persist through subsequent activities. At the beginning of the first activity, members expressed varied perspectives on the role of notation. For example, Cullen did not use inequalities in his solution, but used equal-signs. When Wes asked about the need for inequalities, Herbert said, "You don't need to make a big deal about it." Wes accepted this suggestion, but specified that he wanted to learn to use inequalities, "just in case it matters on the exam." He asked the instructor for help in understanding inequalities, and the instructor facilitated a discussion amongst the group about it. Through this activity, the group came to reject Cullen's practice, and established the need for using inequalities. The group developed attentiveness to its usage, and criticizes solutions that did not use this notation.

Additionally, the group became attentive to other notation throughout the sequence of activities, especially in answers. For example, they criticized solutions that did not: properly use interval notation, list the vertex as a point, or utilize functional notation for inverse functions. In contrast, the group did not explicitly condemn informal notation, such as Brody's solution in the first activity, although they expressed that the solution was hard to follow. They specified, several times during the semester, value for further labeling and notating.

The group's attention to notation was restricted by their conceptual understanding. For example, in the fourth activity, the group struggled with trigonometry and inverse trigonometry; consequently they did not notice several notation errors, including ones with inequalities that they were attentive to in the first activity.

In summary, the group's value for notation evolved and was largely sustained through the semester, but was ultimately restricted by their conceptual understanding. The group valued the role of notation and the meaning that it conveyed, delegitimizing improper usage of notation. The data revealed that the group had a developing sense of formality in solutions as well as coming to understand notation as a means to more easily interpret a solution.

7.2.3.2.2 An Acceptable Solution Must Show Sufficient Work to Be Understood. Across the semester, the group extensively stated their value for solutions that “show work.” The meaning of this was dynamic and was ultimately dependent on their own understanding. The group explicitly discussed the role of showing work in mathematical solutions. For example, in the first activity, the group debated if it was necessary to explicitly detail arithmetic or algebraic steps, such as "adding ‘3x’ to both sides" or "dividing both sides by 3." The discussion revolved around whether solutions needed to show comprehensive work that demonstrated every step or just sufficient work for the solution to be understood. Ultimately, the group sided with the latter of the two and determined that arithmetic did not need to be shown in the solution.

In other activities, the group found it hard to comprehend sample solutions that did not show these same steps. For example, the group struggled with fundamental properties of exponents (as mentioned in section 7.2.3.1.1), and members did not immediately notice arithmetic errors included in some sample solutions. As a consequence, the group often reiterated that "work needed to be shown." Wes once added, “He’s not getting any points … I don’t know what he’s doing.” Thus, laying blame on the lacking explanatory nature of solutions was a way that the group inadvertently avoid interpreting the solutions.

This shows that the groups’ meaning of "showing work" was dynamic and not dependent on the content but their own ability to interpret the work. In the example described above, when the group understood how arithmetic/algebra was used, these steps were deemed unnecessary, especially in their own solutions. Yet, the exclusion of these steps from solutions contributed to misunderstandings and the inability to interpret some solutions.

7.2.3.2.3 An Acceptable Solution Must Follow a Familiar Approach. Across the semester, the group expressed value for flexibility. Yet in practice, the group struggled to implement this value, and inadvertently and implicitly determined than an acceptable answer is one that used a familiar approach.

The group typically decomposed their own solution to construct grading rubrics that distributed points for adhering to their approach. This often included utilizing specific components of their approach, such as needing to express a quadratic in vertex form. Yet, in several activities, the group eventually decided that a holistic grading rubric might be better. The group noted that creating a rubric based on one method, "makes it harder grading everyone." Cullen noted that, "I think that our rubric might be a little too specific to how we did it." Group members acknowledged that there are different ways to solve problems, and expressed openness towards new approaches.

In practice, there was a struggle to implement these values, particularly when it came to novel solutions. The group was often quick to discredit unfamiliar approaches. When a sample solution deviated from the group's approach in an unexpected way, the whole sample solution was viewed as tarnished. In one case, members of the group declared such a novel solution as incorrect within 20 seconds of looking at it. As another example, during the last activity, the group quickly dismissed Jennifer's novel approach by asserting that "everything's wrong," that she "was confused on solving tangent," and that her solution was devoid of any reasoning.

Content knowledge certainly played a role in the struggle to implement the expressed value for flexibility. In the third activity, Cullen noted multiple times that solutions used the "wrong log," referring to logarithms that had a different base than the exponential function in the problem. Here, his misunderstanding of logarithms impeded his ability to make sense of the new approach. As discussed in Section 7.2.3.1.1, the group struggled with applying problem solving strategies to new situations, thus affecting the group's ability to interpret novel solutions which consequently influenced the nature of what constituted an acceptable solution.

7.2.3.2.4 An Acceptable Solution Must Arrive at the Correct Answer. Similar to the discrepancies between the expressed value and treatment of unfamiliar approaches, there was inconsistency in the group's treatment of answers. The group often suggested the approach was worth more than the answer, and awarded only minimal credit for the correct answer in their grading rubrics. But in practice, the answer represented a way for legitimizing the approach. Thus, the group developed the characterization that an acceptable solution was one with the correct answer.

For example, during the last activity of the semester, the group quickly noted that Andrea’s solution yielded the correct answer. The group looked over the solution, but they were unable to interpret it due to their limited understanding of the content. Nevertheless, they awarded her solution full credit. Similarly, solutions without the right answer, especially unfamiliar ones, were typically glanced over. For example, Kennedy's solution in activity 2 represented a novel approach that had the wrong answer; this solution was called "trash" by the group and was not interpreted. In both these examples, the group’s limited conceptual understanding influenced their inability to interpret novel solutions (see section 7.2.3.1.1).

These descriptions provide context for the relationship between the group’s inability to interpret solutions (a social norm) and that an acceptable solution is one with the correct answer (a sociomathematical norm). Additionally, the group’s characterization of this sociomathematical norm is consistent with the group’s use of the answer as a source of authority to validate their own approach (see section 7.2.3.1.3).

Table 28

A Summary of the Norms and their Characterizations, Developed by Group 3

<table>
<tr><th>7.2.3.1 Social Norms</th><th>7.2.3.2 Sociomathematical Norms</th></tr>
<tr><td><u>7.2.3.1.1 Struggling to Interpret Solutions.</u>
-Struggle to interpret solutions was influenced by underdeveloped conceptual understanding of the course content and prerequisite content
-Coped by speculating the mindset of fictitious students and what they did not understand
-Difficulty transferring problem-solving strategies to new situations</td><td><u>7.2.3.2.1 An Acceptable Solution Must Use Formal Notation.</u>
-Attentiveness to using formal notation, particularly in final answers
-Informal notation was not condemned
-Attention to notation was restricted by conceptual understanding</td></tr>
<tr><td><u>7.2.3.1.2 Longitudinally Diminished Attentiveness.</u>
-Attentiveness during the activities and classroom discussions diminished over the course of the semester
-Decline in attentiveness aligned with student understanding of content</td><td rowspan="2"><u>7.2.3.2.2 An Acceptable Solution Must Show Sufficient Work to be Understood.</u>
-Determined that solutions needed to show sufficient work for the solution to be understood
-The meaning of "showing work" was dynamic and dependent on content and ability to immediately understand the solution
-Claimed “insufficient explanation” and inadvertently avoided interpreting the solutions</td></tr>
<tr><td><u>7.2.3.1.3 Appealing to Authority.</u>
-Appealed to sources of authority, such as the instructors and answers in the sample solutions, for verifying their answer
-Dismissed responsibility for understanding content and solutions</td></tr>
<tr><td><u>7.2.3.1.4 Peers as Resources.</u>
-Members were comfortable posing questions and checking answers with others
-When asked for help, the group persisted in helping members overcome confusion</td><td><u>7.2.3.2.3 An Acceptable Solution Must Follow a Familiar Approach.</u>
-Expressed openness towards new approaches
-In practice, struggled to implement these values, and quickly denounced unfamiliar approaches</td></tr>
<tr><td><u>7.2.3.1.5 Peers as Collaborators.</u>
-Collaboration built on members volunteering contributions, not seeking other's perspectives
-Members attempted to hold one another accountable for using proper mathematical language, understanding content, and attentiveness</td><td><u>7.2.3.2.4 An Acceptable Solution Must Arrive at the Correct Answer.</u>
-Solutions were legitimized by the correct answer
-Norm developed in concert with the group's inability to interpret solutions</td></tr>
</table>

7.2.4 Group 4 - Meghan, Ron, Paul, and Julia

7.2.4.1 Social Norms.

7.2.4.1.1 Viewing Activity Completion as its Primary Purpose. Throughout the four Multiple Solutions Activities, the group developed normative patterns indicating that they perceived the purpose of the activity to be its completion. This manifested in several ways and influenced the collaboration and discussion within the group. For example, the group typically accepted the first offered contribution within the group. This contribution was often accepted without deep discussion, or any at all, which elevated finishing instead of engaging or learning.

In particular, Meghan's dominant personality helped sustain this practice of focusing on task completion across the semester, as other members of the group adhered to her implicit negotiations. One of the ways that this was conveyed was that feedback which did not advance moving on with the activity was not valued. For example, this can be seen during the group's grading of Frodo's solution during the second activity. When Paul questioned the grade given by other members of the group and expressed reasons for his reservation, it was clear that Meghan, and the others, were not attentive or willing to engage. Since these concerns did not advance the completion of the activity, they were not valued.

As a consequence to the group's rushing, there were missed opportunities to pursue productive collaboration within the group. Despite occasional collaboration about mathematical activity, more frequently, students' different approaches to solving problems went undiscussed or were not attended to by the rest of the group (this will be further explored in the next section). During the fourth activity, the group had productive discussion around the review questions. The questions provided means to shift the group's focus away from completion and towards attending to aspects of the solutions, contributing to more productive mathematical engagement.

7.2.4.1.2 Aversion to Interpreting Solutions. Consistent with the group's focus on quickly completing the activities, the group was averse to interpret the sample solutions. Across all activities, the group's efforts seldom aligned with the expectations of the instructors, as the group rarely attempted to investigate the mathematics of the solutions. For example, during the second activity, the group did not investigate or try to make sense of the solutions. Kennedy's solution was not discussed by the group, it was not clear that anyone even tried to analyze the approach. Instead, when they did critique solutions, they mostly focused on evaluating subjective criteria like the quantity of work shown rather than the mathematical validity of an approach. No strategies were employed by the group to understand the solutions. Oftentimes, group members made rushed judgements. For example, in one case, Meghan delegitimized a solution within 20 seconds of turning to the page and criticized the approach.

The group's evaluations of the sample solutions were often determinations about how the solution aligned with their own; suggestions that the fictitious student "didn't do any of the steps right" were very common. At times, a group member expressed what confused them, but no one in the group attempted to find the answer. Even when the group read through a solution, they would not employ any strategies to comprehend the steps.

It is quite clear that the limited content knowledge impaired the group's investigations and contributed to the development of these patterns. Ron once responded to seeing a sample solution by saying, "I don't know, the work doesn't make any sense."

Additionally, this was evident during a conversation that the TA had with the group:

TA: What'd you think of Jennifer[‘s solution]? What'd you give her, 4?

Paul: Yeah, we were feeling generous.

Ron: Probably deserved less, because she was way off.

TA: Well at what point, how much of it was right?

Paul: None of it.

TA: None of it?

Paul: Well, I mean, I mean this part (points to top), she had the right idea.

TA: Okay.

Paul: … She's in the ballpark.

TA: And this part, what happened here?

Paul: Well, I think she just, I'm not sure.

(The TA reiterates the step)

Paul: Yeah, I didn't actually look at this part actually … But I know you can do this stuff, I just don't think she did it right, because she didn't get the right answer.

TA: Yeah, you probably can't do that then, right? (Then points out the exponent error to Paul and Ron).

(Paul then admits to randomly putting an X there, but not knowing the error)

This example shows how several factors influenced the group’s circumvention of interpreting the solution: Paul expressed a lack of understanding of the content, the group lacked self-regulation skills to persist in investigating, and the group was dissuaded from interpreting the solution because it had the correct answer. This last factor represents the sociomathematical norm which developed in the group, that an acceptable solution is one that yields the correct answer (which is discussed in section 7.2.4.2.1). As a consequence of these factors, the group was not developing the necessary reasoning skills to make sense of novel solutions.

Because their engagement was often devoid of any conceptual involvement, the group bypassed developing constructive understanding of the content. For example, the third activity aimed to have students analyze the relationship between logarithmic and exponential functions, but the group focused on symbolic manipulations instead:

Paul: Is the 8y when you put it in a log still an exponent?

Meghan: The 3's cancel out.

Paul: The 3's go away.

Meghan: It goes down and it's just 8y.

Paul: So does the log go away too?

Meghan: Yeah, just on one side though.

The group was also averse to making mathematical determinations, and expressed diminished value for mathematical reasoning. There were instances where the group explicitly rejected utilizing mathematical reasoning in favor of memorization-based intuition. In one case, during the first activity, Ron asked the group if the inequality needed to be "flipped" when multiplied by a negative; Meghan's suggested that they should have, but that it "wouldn't have worked out," "because it has to be greater than" in the answer. Thus, the group did not change the direction of the inequality, consistent with their general unwillingness or inability to mathematically analyze solutions, even their own.

7.2.4.1.3 Appealing to Authority. Consistent with their aversion to engage with the Multiple Solutions Activities, the group spurned responsibility for verifying the correctness of their answers. As a result, throughout all four activities, the group grew more reliant on sources of authority such as the instructors and a mathematics phone application, "Mathway."

Verifying their own solution developed into a source of frustration within the group. The instructor intentionally avoided being a source of authority and tried to devolve the responsibility for verifying back to the students. The instructor tried to turn such instances into learning opportunities, yet, this caused the group to experience and express discomfort. At one point, Paul bluntly told the TA that all he wanted from the TA was his confirmation that he was right.

Another instance of this frustration occurred during the second activity when the group explicitly asked the instructor to check their work. The instructor tried encouraging them to think about ways that they could check and verify their own work and answer. Paul even determined one way to check their answer, which the instructor then validated. Yet, this resulted in expressed negativity by the group:

Meghan: I like how I asked him to check our work and he just didn't. Love that.

Paul: He said nah.

Ron: He told us to check it ourselves (laughs)

Meghan: But this isn't a learning opportunity, I just want you to f***ing tell me.

…

Meghan: Well, if our f****** professor would check our work, it wouldn't be a problem.

This vehement reaction signifies a breach in group's expectations and provides evidence of the students' taken-as-shared belief that it is the instructor's role to check the work. This also signifies the disparity between the expectations of the students for the instructor and those that the instructor has for the students (i.e. that students should verify their answer).

Consequently, the group often turned to another source of authority in order to verify their answer: the phone app Mathway, software that is advertised as a mathematics problem solver that shows step-by-step solutions for problems the users enter. The group's usage of Mathway often wasted significant time, as the students struggled to use the software and were uncertain of the validity of the answers they received from it. In many cases, the application never yielded any helpful information for them, only further confusion.

The app also provoked a dilemma when the group perceived differences between the instructor's and the app's answers:

Meghan: I mean, he said my answer was good, but that's not the answer on Mathway. Do I trust the teacher or do I trust Mathway?

Ron: I wouldn't trust him.

Paul: I'd trust Mathway.

Meghan: I trust Mathway.

When the group perceived their answer to be different from those in the sample solutions, the group continued to question the TA about their answer, who already supported and validated their answer (in contrast to the instructor's practice). Meghan referenced this support as defense during the activity, and ended up question Mathway, "Why is Mathway lying?"

It is clear that content knowledge constrained the group's participation and incited usage of Mathway. This was particularly evident during the beginning of the fourth activity. Both Julia and Paul expressed that they did not know how to do the problem, and Ron expressed that he only knew "a little bit." Meghan worked on the problem, but eventually consulted Mathway, and copied down the app's solution. Julia and Paul tried to copy their peers' work, and Meghan eventually turned her paper so that they could do so.

7.2.4.1.4 Aversion to Advance Through Activity Phases. Another normative pattern within the group was the relinquishment of responsibility for navigating through the phases of the activities. Throughout the sequence of activities, once the group completed phase one of the activity, they did not alert the instructor in order to receive the sample solutions for the next phase (as clearly indicated in the instructions). Instead, all of the group members sat quietly at their table and looked at their phones, out the window, or just idly stared. This continued until the instructor noticed their inactivity, checked-in with the group, and provided materials to work on the next phase of the activity. In one case, the group was inactive for nearly eight minutes until receiving guidance from the instructor. Similar bouts of inactivity occurred when the group was collectively at an impasse with solving the original problem.

This normative pattern contrasted with the group's development and sustainment of pattern of focusing on completing the activities as quickly as possible. Yet, alternatively, this corresponds to general patterns of inattentiveness, to the activity, the instructors, and their peers (described in section 7.2.4.1.8).

7.2.4.1.5 Expressing Frustration in the Nature of the Activity. The group expressed frequent frustration about the activities not representing "mathematics," expressing that they missed the procedural learning that they were accustomed to in mathematics classes:

Meghan: Oh my f***ing [gosh], I literally hate these. Why can't we just do actual math? We don't do math in this class?

Paul: I just want a packet of math problems that aren't fractions, and then you can say just do it all day. I love doing that.

Meghan: I just want this class to be over.

Paul: When you think about it, it's actually a fast class. It's 50 minutes of nothing. I do feel like we do nothing in this class though.

Meghan: We do nothing.

Paul: I miss math.

Meghan in particular frequently voiced frustration, as the activity clearly contrasted with her expressed preferences and expectations for a more traditional mathematics class. Her expressions of negativity invited those by her groupmates, which often derailed the group's work:

Meghan: This is just too much work, we need to be doing just math, like what the f*** is happening?

Ron: I just want to go back to sleep.

Julia: Me too … I wish I was at the parade.

This negativity especially manifested when the group faced adversity in the activities. When the group struggled with the task; it was common to see reactions like:

Meghan: We should've but it wouldn't work, because it has to be greater than, so I just f***ing left it. I really don't care, I don't give a f***. So.

Unfortunately, Meghan was not the sole inciter. Each member expressed negativity across the first three activities, clearly indicating that these activities did not represent mathematics to them. This was consistent with the perception that it was not their responsibility to investigate or understand the solutions in the activity.

It is important to note that the group was not as negative about the fourth activity, especially compared to the previous three. In fact, Paul even expressed value for the activity, and that he did not want to pass in the worksheets so that he could further learn from them.

7.2.4.1.6 Peers as Resources. Despite a lack of productive mathematical collaboration and the pursuit of quickly completing the activity, the group sustained the pattern of communicating what they do not know to one another and that peers are resources for writing a solution. During the first phase of the activities, as the group inspected the original problem, they typically communicated to each other what they did not understand. These reactions were responded to by other members, though usually not until someone in the group composed their own solution. For example, during the last activity, Julia and Paul expressed their struggle with the material when each member was working on their own solution. Eventually, Ron and Meghan offered them considerate and patient feedback and guidance. Meghan allowed Julia to copy her work, but explained the solution to her in great detail. Meghan and Ron were also especially patient and attentive to answer Paul's questions, and supported his work.

This type of responsibility towards peers in the group, during the first phase, was seen in all four activities, as members clearly demonstrated that they view one another as a resource during this first phase. Though, it is important to distinguish and contrast this behavior during the first phase with that during the second phase, when the group was evaluating other sample solutions, which is discussed in section 7.2.4.1.1.

7.2.4.1.7 Inattentiveness During Class Discussions. During all four activities, the group did not appropriately attend to the class discussion, including contributions made by peers outside of the group and the instructors. In one instance, as the discussion began, some members of the group did not turn towards the board or their peers, but disengaged: Julia laughed with a friend across the room; Ron flipped through the pages of his activity; Meghan packed up her belongings despite there being several minutes left in class, and even put on her coat. Although Paul tried to contribute to the discussion, but it was clear he was not attentive to the contributions of his peers, as he unknowingly repeated a peer’s earlier contribution.

Sometimes this inattentiveness was not as passive. During the second activity, Dwayne (Group 1) shared a mistake made in Kennedy’s solution, a mistake that this group did not notice during their own analysis. Instead of listening to Dwayne, the group talked amongst themselves as to whether “Kennedy” was a “guy’s name” or a “girl’s name.” Later, Meghan talked over the instructor to suggest to the group that the hands of the clocks were not moving (the clock was in fact broken), and then asked the group whether or not the activity was going to be handed in. In general, the group did not listen to or try to understand the contributions of others.

Besides the inattentiveness and the premature preparations to depart the class, one member in the group demonstrated a lack of value for the class discussions by admitting to intentionally attempting to derail it. In this class discussion, Paul asked the instructor questions about whose handwriting was used in the activity, but later admitted to Meghan that he was just trying to waste the remaining minutes of class. When the instructor instead deflected the question to talk about the mathematics of the solution, Paul shook his head expressing negativity.

7.2.4.1.8 Distractive Cell Phone Usage. One pervasive component of the group's inactivity and inattentiveness during the activities was cell phone usage, which was an issue in all four activities. For example, within minutes of the start of the class discussion in the second activity, all group members were on their phones. This pattern was not limited to class discussions but represented a growing pattern of acceptability within the group to use their phones in class, which intensified as the semester progressed. During the first activity, Julia hid her phone when the instructor was near; but in the last activity, Meghan and Julia were on their phones while the TA was standing at their table, trying to engage with the group. The TA persisted in trying to explain a concept to the group, but nobody was attentive or tried to engage; eventually Paul sarcastically responded by saying, "Neat." This example shows a pattern of the group members actively choosing to be on their phones instead of engaging with one another or the instructors.

Phone usage varied amongst members of the group, but was most prevalent for Julia. During the fourth activity, for example, she was consistently on her phone, even when her groupmates were working. Julia did not contribute throughout the activity. Her primary engagement with the group was to copy their work, or to question them about the content of assessments, for example, "Wait, do we have to, so we have to know the unit circle for this s***?" In a way, it was not the phone usage that interrupted Julia's mathematical engagement, but the mathematical engagement interrupted her phone usage.

7.2.4.2 Sociomathematical Norms.

7.2.4.2.1 An Acceptable Solution Must Arrive at the Correct Answer. In every activity, the group suggested that an acceptable solution is one that arrives at the correct answer. This notion was pervasive amongst the group, but Meghan succinctly emphasized the necessity of this facet as the group graded a solution during the first activity:

Meghan: This is just a fat zero. Because the answer is wrong.

Instead of noting characteristics of or errors in solutions, the group focused their evaluations on reporting that an answer was "wrong." On every sample solution in the third activity, in the “Rationale for Points Awarded” column, the group mostly repeated their evaluations. For example, two of three of Meghan’s evaluations are, “They showed their work but had the wrong answer,” and the third was very similar.

This norm is found to be intricately tied to the social norm of investigating solutions. The group often focused on the answer when evaluating solutions. This allowed the group to make quick determinations about solutions without investigating them. For example, in the second activity, Ron and Meghan quickly reported that Kennedy’s solution “sucks” after looking at the answer. On another occasion, when Paul obtained the sample solutions from the instructor, his first action was to look at all the answers and quickly determine, “they got the whole [darn] thing wrong.” Thus, incorrect answers tarnished the entire solution and often deterred the group from interpreting the approach. Solutions that had the right answer were not always investigated either, as the group trusted that the approach was correct since it yielded the correct answer (an applicable example is shared in section 7.2.4.2.3). This demonstrates how a sociomathematical norm mediates social norms within the group.

7.2.4.2.2 An Acceptable Solution Follows a Familiar Approach. The group also demonstrated that an acceptable solution was one that follows a familiar or prescribed procedure. The group frequently indicated support for the notion that there was one correct way to solve a problem. For example:

Instructor: What could have Frodo done better to help his reader understand?

Paul: Actually done it the normal way.

Instructor: Be careful. There's not one normalized way, there's not one right way, right?

Paul: No, there is. If you're taught one way, you should do it that way.

The sustainment of this norm acted as a barrier to developing flexibility within the group. In another episode, the TA discussed with the group the role of “switching x and y” when finding an inverse function. Paul characterized not switching as “lazy,” whereas the TA described it as another way to solve the problem. Yet, later on during the activity, the group still condemned Alexander’s solution for not “switching x and y”. Paul again expressed this as a mistake during the class discussion, and the instructor again noted that this is not a mistake, but described it as another way to solve the problem.

While evaluating sample solutions, the group often characterized novel solutions as "unable to be followed," whereas those that followed a familiar approach “could be followed.” In one activity, for example, the group characterized that all of the solutions as unable to be followed, or that they all included mistakes. Yet, they never noted what these mistakes were:

Paul: (of Andrea’s solution) Started off strong, then I just don’t know what you were thinking.

Ron: They were all wrong and have the wrong work.

Meghan: Yeah, we could just say that they all showed their steps but they all made mistakes.

Paul: They all did it completely differently.

Meghan: They all took different approaches to the problem, yet they all managed to mess up along the way, and get the wrong answer.

Paul: Lincoln doesn’t know what to do.

Here, it is clear that the social norm of not investigating solutions influenced their characterization of what constitutes an acceptable solution: an acceptable solution needed to follow their procedure and arrive at their representation of the answer.

7.2.4.2.3 A Case Combining the Two Prior Characterizations. The example below shows the two prior characterizations of what constitutes an acceptable solution: an acceptable solution must arrive at the correct answer (section 7.2.4.2.1) and follow a familiar approach (7.2.4.2.2). It also shows how these characterizations of a sociomathematical norm mediate the development of the social norm of not interpreting solutions.

During Activity 4, the group encountered a solution that did not yield the correct answer, breaching the groups’ sociomathematical norm that an acceptable mathematical solution is one with the correct answer. This caused the group to delegitimize the whole solution:

Paul: I just don't think she did it right, because she didn't get the right answer.

Solutions that did not follow a familiar approach and did not have the correct answer were characterized as “wrong”:

Meghan: Jennifer just used the wrong approach.

Ron: Yeah, I just wrote that she took the wrong steps.

Meanwhile, solutions that yielded the correct answer were deemed as worthy of full credit:

Meghan: Okay, what's all this over here?

Paul: More math. Let w equal arcsine of 1/2. Well that's true.

Meghan: That's just not how you get the answer, but they still got the right answer.

Paul: Yeah it is, they did the.

Ron: They used one of the identities.

Paul: Yeah the identity.

Meghan: Oh dear, I don't want to look at that, okay so 6 then?

In addition to displaying the two aforementioned characterizations of what constitutes an acceptable solution, this example provides a clear illustration of how the sociomathematical norms mediated the development of social norms. In this last excerpt, since an acceptable solution was one that is characterized as having the correct answer, the group determined that they did not need to investigate the sample solution that yielded the correct answer.

7.2.4.2.4 Solutions Must Have Work Present. In addition to having a correct answer, the group stressed the need for the solution to "show work," and expressed value by including this in their holistic grading rubric. However, this notion was not described or articulated. As a consequence, the group expressed point values for varying subjective amounts of work such as "minimal work," "some things missing," and "missing a small step." Despite making these distinctions, the group never discussed them when evaluating the sample solutions.

In general, showing work was a secondary consideration. If a sample solution did not yield the correct answer or use a familiar approach, the group generally considered the work shown in the solution. But, given the group's limited acts of interpretation, their evaluation of work mainly revolved around the acknowledgement of its presence in a given solution, instead of quantifying or qualifying the work. For example, Meghan suggested awarding "3-4 points" for Andrea's solution because, "they showed work, but it just doesn't make any sense."

When questioned about awarding credit for showing work, Meghan defended the idea by suggesting that was what the instructors would want. This is interesting because it represents an instance of students aligning their activity and values with those they perceive/expect of their instructors. Yet, their enacting of these values did not mirror those of the instructors. For example, when considering Lincoln's solution in Activity 3, the group deemed his work unnecessary, and consequently, incorrect. They expressed frustration when the instructor did not share this characterization during the class discussion.

7.2.4.2.5 An Acceptable Solution Does Not Need Formal Notation. In all four activities, the group expressed that formal notation was not needed in the solution. Formal notation was often seen as an obstacle instead of as a mechanism to facilitate and convey understanding. Additionally, the group did not condemn informal or vague notation, such as that used in Brody's solution (Activity 1, Section 5.2.2.2.1).

The group disregarded, or even scoffed at, the instructors' attempts to negotiate the value and importance of adhering to conventional notation. For example, during the last class discussion, the instructor explicated the need for adhering to formal notation. In particular, he noted the inequality error in Dan's solution (using $<$ instead of $\leq$) and expressed concern in Dan's informal use of functional notation (not providing each function with its own input; Activity 4, Section 5.2.2.2.4). The group disagreed with the instructor's assessment:

Meghan: it's not that big a deal (laughs).

Julia: That's gotta be a joke.

In their own solution, the group would misuse this notation, such as not-equal signs and inequalities.

The group's apathy towards notation had its exceptions. In the first two activities, the group expressed the need for appropriately expressing final answers, such as utilizing interval notation. The group aptly criticized solutions that did not express the answer in interval notation, as the problem asked. In another case, the group criticized Frodo's solution (Activity 2) for reporting the vertex as the y-coordinate instead of as a point. The group struggled to discuss the error with correct vocabulary; Meghan, for example, noted that, “The vertex is two points and he only put one.”

On another occasion, the group fervently defended the notion that a final answer should not have square-roots in the denominator. The group expressed frustration towards the instructors’ ambivalence towards this practice:

Paul: You can't have a square-root in the denominator of a fraction, and if this was my class, that would be wrong. But apparently that's okay.

Ron: But they wouldn't care. (Pointing towards instructors)

The TA later conversed with the group about this, classifying it as an “aesthetic choice.” He described that rationalizing the denominator does not change the value, “it’s still a number.” Yet, even later in the class, Meghan expressed disbelief:

Meg: They take points off for everything but not when we don't rationalize the denominator.

This represents a situation where students' prior beliefs influenced the development of norms within the group, and how they may act as barriers towards more productive engagement.

Yet, even this attention towards notation in the final answer was inconsistent. In the third activity, the TA suggested to the group that it was errant to label the final answer as "y" instead of using functional notation, "$f^{-1}(x)$." The TA defended this notion by expressing the need for appropriate labeling. Despite spending several minutes with the group discussing this, the group did not consider this notation in the sample solutions.

Table 29

A Summary of the Norms and their Characterizations, Developed by Group 4

7.2.4.1 Social Norms	7.2.4.2 Sociomathematical Norms
7.2.4.1.1 Viewing Activity Completion as its Primary Purpose. -Typically accepted the first contribution that was offered -Feedback that did not advance completion of the activity was not valued, which deterred opportunities for productive collaboration *7.2.4.1.2 Aversion to Interpreting Solutions.* -Did not attempt to interpret solutions, which was influenced by underdeveloped content understanding and persistence -Evaluation of sample solutions instead focused on subjective criteria like the quantity of work shown or alignment with their own solution -Bypassed conceptual considerations to focus on symbolic manipulations -Avoided making mathematical determinations: memorization was prioritized over mathematical reasoning, even when it explicitly conflicted with reasoning	*7.2.4.2.1 An Acceptable Solution Arrives at the Correct Answer.* -Obtaining the correct answer was necessary for a solution to be acceptable -Evaluations were repetitive, as the group focused on reporting that answers were wrong -This norm was intricately tied to the social norm of not interpreting solutions
7.2.4.1.3 Appealing to Authority. -Dismissed responsibility for verifying their answer -Reliant on sources of authority for verifying answers, particularly instructors and the phone application "Mathway" -Influenced by incommensurate content knowledge *7.2.4.1.4 Aversion to Advance Through Activity Phases.* -Group members did not alert instructors when ready for the next phase of the activity or when they were at an impasse with problem-solving -Instead, they sat idly and waited for the instructors to notice their inactivity	*7.2.4.2.2 An Acceptable Solution Must Follow a Familiar Approach.* -Expressed that there is only one correct way to solve a problem. -Flexibility did not develop within this group -Characterized novel approaches as "unable to be followed" or errant, which they never investigated -This norm was intricately tied to the social norm of not interpreting solutions
7.2.4.1.5 Expressing Frustration in the Nature of the Activity. -Expressed frustration over the activities and missed procedural learning -Members were not as negative about the fourth, some even expressing value for the activity *7.2.4.1.6 Peers as Resources.* -Communicated what they did not understand to each other -Supported each other in forming a solution, but typically after one member composed their own solution	*7.2.4.2.4 An Acceptable Solution Must Have Work Present.* -As a secondary consideration to the answer and familiarity of the approach, the group would consider work shown -Evaluation of work did not quantify or qualify the work, but only acknowledge its existence
7.2.4.1.7 Inattentiveness During Class Discussions. -Inattentive during all class discussions -Passive inattentiveness included members zoning out or flipping through the pages of the activity -Active inattentiveness included members socializing or intentionally trying to derail the class discussion *7.2.4.1.8 Distractive Cell Phone Usage.* -Pervasive phone usage for non-mathematical purposes hindered the group's engagement; this frequency increased as the semester continued	*7.2.4.2.5 An Acceptable Solution Does Not Need Formal Notation.* -Formal notation treated as unnecessary in solutions, except sometimes in the final answer -Informal and vague notation was not condemned

Chapter 8. Discussion

The following sections respond to the research questions (See Chapter 4) then discuss other aspects of the research. Sections 8.1 and 8.2 respond to Research Questions 1a and 1b by describing, comparing, and contrasting the characterizations of social and sociomathematical norms that developed in the class and amongst the groups. Section 8.3 responds to Research Question 2 by discussing relationships that surfaced between social and sociomathematical norms. Then, Section 8.4 is dedicated to discussing how the construct of didactical contracts is a useful explanatory mechanism to describe the evolution of classroom engagement. Section 8.5 describes implications that this research has for educators. Lastly, Section 8.6 is devoted to detailing limitations of the study as well as suggestions for future research.

8.1 Social Norms

8.1.1 Interpreting Solutions

A key difference amongst the four groups was how they approached the evaluation of the sample solutions in phase two of the Multiple Solutions Activities. "Analyzing the solutions" meant different things for various groups, and there were several factors that influenced students' ability to do so.

In Group 1, "analyzing solutions" represented trying to interpret the solutions. Despite expressing skepticism about the viability of some approaches, the group persisted in trying to make sense of them. These efforts further evolved through the semester, as the group eventually started to connect different approaches and explored their equivalence.

On the other hand, Groups 2 and 4 rarely made efforts to make sense of the solutions, as they typically viewed different approaches as invalid (see also Section 8.2.1). Instead of trying to understand the solutions, the groups developed a norm of analyzing the solution's adherence to the method used by the group. To these groups, “analyzing a solution” meant determining its alignment to a specific procedure or how the solution deviated from their own. This norm hindered the usefulness of the Multiple Solutions Activities in fostering students’ higher order skills.

Some groups’ attempts to interpret the solutions were influenced by an underdeveloped conceptual understanding of the course content and prerequisite content. This was especially evident with Groups 3 and 4. There were instances where the instructor overestimated students’ content understanding (e.g. section 7.5.3). The discrepancy between students’ actual understanding and the content of the activities rendered some of the activities ineffective and unintentionally supported unproductive norms as described above.

The instructor’s assumptions of students’ preparedness to engage with the activities were based on the types of problems given in homework assignments the week prior to the activities. The instructor chose problems for the activity that paralleled the ones in the homework, believing that this level of difficulty would be appropriate for students. However, homework submissions were not a valid source for determining students’ content understanding. The qualitative analysis showed that Group 4 tended to use phone applications to solve problems, and admitted that they use such applications on their homework assignments. Consequently, there were instances where at the beginning of activities, members of Group 4 expressed to one another that they did not know how to start problems that should have been familiar (e.g. Activity 3).

In addition, some groups had underdeveloped prerequisite understandings. Group 3 tried to interpret the solutions, but could not make much progress due to limited knowledge of simplifying fractions and exponent rules. As the semester continued, the group's understanding of the content diminished, and they were rarely able to interpret the solutions. In short, content knowledge impacted the quality of engagement students were capable of having during these activities.

Quantitative analysis corroborates the qualitative results described above with respect to some of the unproductive norms related to "analyzing solutions." For example, the foundational aspect of the Multiple Solutions Activities was critiquing and makings sense of the work of others. However, students' pre- and post- means on the item "How important is it for you to be able to determine if a peer's solution is correct?," suggest that students give less than "slight importance" to being able to determine if a peer's solution is correct (2.881 and 2.905 respectively, Table 13). If students did not perceive value in determining if others' solutions were correct, they may not have been motivated to interpret them.

Additionally, the data show a significant mean increase on the item: "It is more important to correctly perform the steps of a solution than to understand each one of them" ($p<0.05$, Table 7). An increased focus on correctly performing steps helps to explain why some groups characterized "analyzing solutions" as determining the solutions' adherence to a particular method. Furthermore, despite students with and without prior MATH 418 enrollment having a significant pre-mean difference (0.548), by the end of the semester, students without prior MATH 418 enrollment's mean increased significantly to close the gap (post difference: 0.179, Table 9). Possible reasons for this include the influence of: assessments that were techniques oriented, progress through their major, or working with peers who previously enrolled in MATH 418.

8.1.2 Role of Peers

8.1.2.1 Peers as Collaborators. Qualitative analysis revealed a key difference in the participation structures amongst the groups: inclusivity. For example, Group 1 put forth effort to facilitate conversation amongst all group members, including Albert, who exhibited an introverted demeanor. This concerted effort developed into a social norm within the group, that every member should be included in discussions. Yet, in other groups, mainly Groups 2 and 3, the participation structure was built upon the social norm that members are expected to volunteer their thoughts, without necessarily seeking everyone's perspective. Consequently, members like Connor in Group 2 were rarely involved in group conversations and rarely contributed.

Ideally, group members would seek out one another’s perspectives and thoughts, as this may lead to learning opportunities for all members in the group. This was clearly the case in Group 1, as members valued Albert’s contributions despite his generally reserved demeanor. This was the intended vision of the Multiple Solutions Activities, and the purpose behind asking students to work cooperatively and to come to consensus. Unfortunately, as some groups demonstrated, particularly Groups 2 and 4, a primary goal for the activity was simply to complete it. When completion becomes the goal of engagement, some students (like Connor) get overlooked and the activity’s benefits are limited and circumvented.

Although there are no questionnaire items that directly measure students’ beliefs about inclusivity or volunteering, there are some items that relate to and support the qualitative findings. For example, the means of responses to the item, “In math, explaining my work or reasoning to others helps me learn,” reveal that explaining work or reasoning to others helps students learn (pre- and post- means of 3.500 and 3.548 respectively, Table 20). This supports a participation structure built on volunteering. On the other hand, students did not strongly feel that it was important to determine if a peer’s solution was correct (pre- and post-means of 2.881 and 2.905 respectively, Table 13). This might suggest that students did not view it as important or helpful to understand the perspectives and contributions of others. This supports the observation that students were less apt to seek other’s perspectives than to volunteer their own.

8.1.2.2 Peers as a Problem Solving Resource. In all four groups, members acted and treated others as a mathematical resource. Nevertheless, there were differences in how this manifested amongst the groups. In Groups 1 and 3, members would often pose questions to one another. In Group 1, members acted as validators that often conferred about approaches taken and answers obtained. Similarly, in Group 3, the students were comfortable asking each other questions, and there was a sense of responsibility for helping one another overcome confusion. Groups 2 and 4 developed a slightly different normative pattern: students first solved the problem individually, and then compared or discussed answers. Though, members of Group 4 would occasionally first share what they did not understand prior to supporting one another.

In general, it seemed that students viewed one another as problem solving resources. This idea was also supported in the quantitative data with significant mean decreases on items such as, “There is no place in mathematics for discussions – you are either right or wrong”

($p<0.05$, Table 11), and, “When it comes to math, I would rather try to figure out my own questions or confusion than ask for help” ($p<0.05$, Table 15). Students also expressed sustained value for explaining their work to others (pre- and post-means greater than or equal to 3.500, Table 20). These results express value for discussions, receiving help, and explaining work.

8.1.2.3 Accountability for Language Usage. The qualitative analysis surfaced two discernable patterns within groups. In Groups 1 and 3, members held each other accountable for language usage, and saw upholding proper usage as their responsibility. This primarily manifested in correcting improper or inappropriate mathematical language. Yet, the norm that developed in Group 2 was much different; the group consistently used informal language (see section 7.2.2.1.1), which was not questioned or corrected. Using each other's informal language seemed to help members find common ground for discussions. Group 2's usage of informal language surfaced a problem: the mathematical terms began to take on various informal meanings dependent on shifting contexts. For example, terms like "formula" were used so frequently and in so many contexts that they lost meaning.

This is one area that warrants more attention. Mathematics as a discipline has nuanced language, which is important to explicate, especially in developmental mathematics classes, at the beginning of a mathematics sequence. Students in Groups 1 and 3 may develop a deeper, more nuanced, understanding of the content by having a better grasp of the language used to describe it. Unfortunately, there were no quantitative data that measured students' beliefs in language usage or in being accountable/responsible for one another’s language usage.

8.1.3 Reliance on Authority

Qualitative analysis revealed patterns of Groups 2, 3, and 4 relying on sources of authority through the sequence of Multiple Solutions Activities. This pattern contrasted with Group 1's experience, who only occasionally pursued confirmation from a source of authority, and was consequently not considered normative (see Section 6.2.1 for details on how patterns were determined to be normative).

One social norm that was sustained in Groups 2, 3, and 4 was that it is the instructor and TA's role and responsibility to make requested mathematical determinations for students. Consequently, the groups relied on these outward sources of authority, indicating intellectual heteronomy. This was especially evident when students wanted to verify their answers; instead of doing so themselves, the groups delegated this responsibility to the instructors. When the instructor tried to help facilitate the group's work, in an effort to develop their autonomy, the group expressed frustration, as this breached the students' expectations of the instructor. The groups also appealed to several other resources that they viewed as authorities: mathematics phone applications (e.g. "Mathway"), the sample solutions themselves, and even a member of the group (e.g. Steve in Group 2).

Quantitative data present a picture of students having high and varied expectations of the instructor and TA, particularly among students with prior MATH 418 experience. Students expressed that it was the instructor's role to prepare them for quizzes and exams (pre- and post-means greater than 3.100, Table 20) and to teach students how to write a solution that would receive full credit (pre- and post-means greater than 3.400, Table 20). Post-survey means of both items were significantly higher amongst students with prior MATH 418 enrollment than among students without prior enrollment ($p<0.05$, Tables 21 and 22).

Additionally, the data show a significant decrease (from a mean of 2.524 to 2.262, $p<0.05$) on the item, "I would rather try to figure out my own questions or confusion than ask for help" (Table 15). Although this item does not directly refer to instructors, it may contribute to the notion that students developed more reliance on receiving help. The mean decrease on this item can also be attributed to the increasing difficulty of course content; as students became less confident in their mathematical abilities and content understanding, they may have started to rely more on authority.

The teaching experiment, which is the focus of this dissertation, intended to foster development of students' higher order skills such as problem solving capacity, and autonomy. Instead, many groups developed *heteronomy*, a reliance on authority to make mathematical determinations. This reliance on authority allowed groups to circumvent opportunities inherent in the activities to develop higher-order skills.

8.1.4 Inattentiveness

8.1.4.1 Class Discussions. Three of the four groups displayed patterns of inattentiveness during the class discussion phase of the Multiple Solutions Activities. This inattentiveness can be characterized in two ways: passive and active. Passive inattentiveness represents actions where students exhibited quiescent behavior, such as staring into space, not turning to the board or speaker, or closing their eyes. On the other hand, active inattentiveness represents explicit and sometimes fervent rejection of engaging. This latter classification includes students packing up early and putting on their coat during the midst of the class discussion, using cell phones, engaging in non-mathematical conversations (see 7.2.4.1.7), or trying to derail the class discussion by voicing non-pertinent remarks.

Students' attentiveness also seemed to erode over time: both within a class discussion and across the sequence of class discussions. Additionally, Group 3's attentiveness in the class discussions diminished across the semester as they became increasingly challenged by the course material.

Class discussions typically took up the last ten minutes of class, and student attentiveness was not always durable across this timespan. This is problematic because without attending to summaries, which provide opportunities to explicitly compare solution methods, students may experience diminished learning gains. Research shows that students benefit from explicit opportunities to identify similarities and differences in methods where students may consider the efficiency of the approaches, as well as the affordances and constraints of each strategy (Star et al., 2015). Open-ended questions posed in the lesson summary, are intended to summarize key ideas and support the instructional aim of developing flexibility; especially discussions that explicate the nuances comparing various solution methods.

Additionally, class discussions of Multiple Solutions Activities represented a key opportunity to negotiate productive social norms, by modeling and expressing value for interpreting novel solutions, and sociomathematical norms, such as an acceptable solution is one that utilizes any viable approach. Summarizing discussions were also key to providing insights on the content, from understanding basic algorithms to important notation convention usage. Pervasive inattentiveness could have hindered the development of flexibility, and productive norms, and compromised content understanding.

8.1.4.2 Phones as Distractors. One aspect contributing to student inattentiveness during the activities was the use of cell phones. Using cell phones during class developed into a norm for Groups 2 and 4. The instructor and the TA wanted to be amenable to students using cell phones as a mathematical resource, but asked students to step outside of the class if they had to use them for other purposes. Yet, the analysis showed that students did not adhere to these expectations, as several students in the groups used the phones primarily for socializing. The course policy assigned a grade to each student for each recitation, to ensure timely attendance and active participation, especially to dissuade cell phone usage. The former was strictly enforced but the latter was not. Often times it was difficult for the instructors to determine if phones were being used for mathematical purposes or not. When it was obvious that the phones were a distraction, as described in Section 7.2.2.1.5, the instructors would often ask students to put phone away without further penalty. The analysis shows that this did not dissuade students from future phone usage; in fact, as described in Section 7.2.4.1.8, students used their phones more as the semester continued.

8.1.5 Frustration

One social norm that developed, in various ways and extremes within Groups 1, 2, and 4, was the permissibility of venting frustration during class. In Group 1, members were emotionally supportive towards one another, whether it was towards the activity or not, and offered comfort to quell the frustration. Yet, in the other two groups, particularly in Group 4, students would foment and perpetuate frustration. This was especially evident when groups perceived breaches of expectations between them and the instructor. For example, the group grew frustrated with what they perceived to be a lack of direction from the instructors, by not being given a prescribed solution method to follow. The instructor tried to facilitate the group's engagement with the activity instead of telling them what to do. This breach in expectations was summarized by Paul as, “I hate when people answer questions with questions ... If I knew, I wouldn’t be asking.”

In general, the groups perceived the Multiple Solutions Activities as a breach of their expectations of engagement in mathematics class. Both Groups 2 and 4 expressed frustration that they were not “doing math” in the class, or rather that the activities were not the procedural ones that they had been used to and expecting. Interestingly, in the last activity of the semester, Group 4 expressed value for the activity. This aligns with quantitative results that show significant decreases in students viewing mathematics as procedural (see Table 12).

8.2 Sociomathematical Norms

The following subsections describe, compare, and contrast various characterizations of sociomathematical norm of what constitutes an acceptable solution.

8.2.1 Familiar Approach vs Any Valid Approach / Openness to Multiple Solutions

One goal of the teaching experiment was to promote and foster students' flexibility. The implementation of the Multiple Solutions Activities was one instructional choice used to facilitate the development of flexibility. Consequently, the instructors utilized these activities as opportunities to negotiate that any mathematically valid approach should constitute an acceptable solution, not just a familiar one. Thus, students were given opportunities to analyze unfamiliar solutions during these activities. Similarly, the questionnaire contained items assessing possible changes in students’ beliefs about multiple solution methods and about tendency towards procedural learning. The quantitative and qualitative results depict nuanced development in the norms and beliefs developed by students regarding different methods and openness towards learning about multiple solutions.

Quantitative analysis reveals the influence of the instructors' negotiations on openness towards multiple solution methods. Items that had statistically significant decreases in mean, like, "The most valid ways of solving a problem are the ones discussed in class" (p<0.01, Table 2), and, "To receive full credit, my solution must use the same methods used in class" (p<0.05, Table 2), show that the negotiations initiated by the instructors were received and sustained by students.

As expected, seeing the discipline as less procedural coincided with this increased flexibility, as students showed a significant decrease in characterizing mathematics as procedural and that it needed to be memorized (p<0.01, Table 11). As mentioned in the Results section, this should not be seen as an increased appreciation for conceptual mathematics; there was no evidence to support change in the item's mean, "I prefer to focus on learning how to use formulas instead of spending time on where they come from" (Table 9).

Similarly, the item, "It is more important to correctly perform the steps of a solution than to understand each one of them," had a significant increase in mean (p<0.05, Table 7). Analyses showed that despite having a significantly lower pre-mean, this increase was attributed to students without prior MATH 418 enrollment. Given the results in the previous paragraphs, this was an unexpected finding. Yet, when considering that these new students are entering not only the culture of the course, but of the discipline of their pursued major, this increase may not be so surprising. The culture of their programs may teach students to be "users" of the mathematics that they learn; thus, mathematics is used as a tool within the discipline. This analysis suggests that although students may have developed an openness towards multiple valid approaches, this openness does not compete with valuing the accuracy of the methods over understanding them. This same consideration could explain the large difference in means between students with and without prior experience in MATH 418 in the questionnaire item, “I prefer to focus on learning how to use formulas instead of spending time on where they come from” (Table 9).

Another unexpected result, was that students without prior enrollment in MATH 418 showed a statistically significant decrease in mean for the item, “I find it helpful to learn several different ways to solve a math problem” (from 3.313 to 2.875, $p<0.05$, Table 5). Furthermore, this change was significantly different than that reported by students with prior enrollment (which was a slight increase). One possible explanation for this outcome is that since students with prior enrollment already had familiarity with one approach, they may have had an advantage on procedural questions compared to their peers without prior MATH 418 enrollment. Students new to the class may not have had familiarity with any approaches, and could have felt overwhelmed by being exposed to several approaches without yet being comfortable with one.

Qualitative data depicts students widely agreeing with the notion that an acceptable solution could follow any viable method yet struggling to implement this in practice. As a consequence, norms diverged into two radically different paths: an acceptable solution was one that utilized a familiar approach (Groups 2, 3, and 4) or used any viable method (Group 1).

As discussed in section 7.2.1.2.2, Group 1's characterization of what constitutes an acceptable solution mediated the development of higher cognitive engagement with the activities. The group began to pre-emptively contemplate alternative approaches, compared how solutions were similar, and characterized different approaches. This suggests that the group regarded the solutions as objects of reflection rather than a sequence of steps.

The other groups were quick to discredit unfamiliar solutions, and any recognized deviations from their own solutions were sometimes thought to tarnish the entire solution. Thus, their engagement with the sample solutions faltered into low-cognitively demanding tasks of determining whether or not a particular solution deviated from their own method or whether it yielded the correct answer or not.

It is important to investigate why students recognized value for flexibility but did not implement this value into their practice. Qualitative analysis suggests two primary factors: insufficient conceptual/content understanding and the persistence of unproductive beliefs.

As discussed previously, students' limited conceptual understanding and prerequisite content knowledge hindered their ability to interpret the solutions. As a means of coping with this underdeveloped understanding, students could only compare the sample solutions to familiar procedures. This contributes to the relationship between the social norm of interpreting solutions and characterizing what an acceptable solution represents, which will be discussed in Section 8.3.

Another major factor in the sustainment of this sociomathematical norm was students' unproductive persistent beliefs. Both members from Groups 2 and 4 expressed this influence, either implicitly or explicitly. For example, Paul in Group 4 articulated that, "if you were taught one way, you should do it that way." Meanwhile, Steve and Molly described that in their experience in MATH 418 the semester prior, they felt that they had to solve each problem one way. Despite explicit interventions and negotiations by the instructor and TA, the students' beliefs and practices went unchanged.

Given that most of the existing literature on student engagement is conducted in earlier grades, the post-secondary setting may explain why the persistence of unproductive beliefs was more prominent in this study. It may be easier to renegotiate roles and student activity amongst younger student populations as student beliefs may be less ingrained. Meanwhile, in this study, despite the reported differences between the prior semester and the semester of this study, some students explicitly refused to change their practices, or expressed extreme frustration at the perceived violations in expectations which then hindered their own engagement. This was characterized by Steve in a homework reflection assignment for the course: "Throughout the semester my studying habits have not changed, I have continued the same strategy that I used since the beginning, but my grade has started get worse and worse, but I do not believe that it [is] due on my part." Because of the persistent nature of beliefs, the renegotiation of norms in post-secondary developmental mathematics classes is a gradual and complex process.

8.2.2 Correct Answer

Another characterization that developed within Groups 2, 3 and 4 was that an acceptable solution needed to have the correct answer. This was most clearly demonstrated by Meghan's comment (Group 4): "This is just a fat zero. Because the answer is wrong."

This norm was both explicit and pervasive. To varying degrees throughout the sequence of activities, the groups judged the appropriateness of a solution by the answer. Incorrect answers tarnished students' perception of entire solutions and often deterred the group from interpreting the approach altogether (see section 8.3 for more details).

Interestingly, when creating grading rubrics for evaluating sample solutions, some students occasionally acknowledged that the approach should be worth more than the conclusion, in particular, Group 3. However, in practice, when students analyzed sample solutions, they relied on the final answer as a way to legitimize or delegitimize the solution. That is, students utilized the answer to determine the value of the approach. For example, a solution would be graded lower if it resulted in an incorrect answer, even if the approach in the solution was valid and the mistake relatively minor.

Overall, quantitative analysis reflected that students upheld importance for the answer. In the questionnaire, students were asked, “How important is getting the right answer to receiving credit for a math problem?” On this item, the class had pre- and post-means above 3.000, which reflects “moderate importance.” Although inferential analysis was unable to find evidence of any significant effect, descriptive analysis shows that despite the mean remaining rather stable (an insignificant decrease in mean), the mode shifted from “very important” to “moderately important” (Figure 12).

The origin of the importance of obtaining the correct answer is not difficult to imagine. If students experienced years of assessments that were graded based on the correctness of answers, including numerous state tests, then students may have developed deep-rooted beliefs about the importance of the correct answer. This valuing of the answer may also reflect students' role as "users" of mathematics; as discussed earlier, most students in the course were pursuing degrees in applied STEM disciplines, such as engineering.

However, the focus on obtaining the correct answer is unproductive when students use the correctness of the answer to justify the solution method. Again, if students had previously been assessed only on the correctness of their answers, this would be a natural connection for students to make. But this hinders the development of higher order mathematics skills, such as flexibility and reasoning. Relying on an answer to determine mathematical validity is indicative of intellectual heteronomy, as students depend on this authority.

8.2.3 Work Shown

Both quantitative and qualitative analyses reveal that students have various understandings of what it means to “show their work.” For example, quantitative analysis revealed that students did not have a uniform understanding of whether written explanations were needed in solutions, as both pre- and post- means were approximately 2.500, the middle of the 4-point Likert scale (Table 10). Similarly, the pre- and post-means for, “How important is it for you to be able to determine if a peer's solution is correct?,” were both approximately 2.900, indicating less than “moderate importance,” and did not show yield a significant change (Table 13). Yet, the students had a notable increase in the item, "How important is it to you to write a solution that your peers could understand?" (Table 14). This suggests that students’ values regarding writing clear solutions were impacted more than their value for interpreting solutions of others. Although these results may implicitly demonstrate beliefs about work needing to be shown, this does not capture what characterizes the notion of required work.

Qualitative analysis revealed interesting patterns amongst three groups' determinations about what work needs to be shown in solutions. Across the sequence of Multiple Solutions

Activities, Group 4 merely indicated that work needed to be shown; their evaluations of work did not involve any means of quantification or qualification, just acknowledging its existence.

Group 3 quantified the amount of work by stipulating that "sufficient" work needs to be shown for the solution to be understood. The significance of this characterization is that the group began to associate work shown with understanding a solution. Thus, "work" is not just a byproduct of obtaining an answer, but rather, the means to communicate one's understanding. Yet, the group did not have any qualification of what work needed to be shown, and consequently, the meaning of "show work" was dynamic and depended on their ability to interpret solutions. Thus, when the students struggled with the content, they blamed the explanatory nature of the solutions, and inadvertently avoided interpreting solutions.

Group 1's characterization differed from the previous two groups, as they qualitatively stipulated what work needed to be shown. The group determined that work must convey meaning and contribute to the reasoning or justification with the solution. This contrasted from the previous groups, particularly Group 4, as work was not valued strictly for its presence.

This spectrum of characterizations surface the notion that students may be processing solutions at different levels. In Sfard's seminal work (1991), she explains that students typically perceive mathematics operationally before structurally. Consequently, students that are still in the operational stage may be regarding solutions as a sequential process, and may not be at a developmental level to contemplate or qualify the structure of the work. As a result, students may simply expect all of the work to be shown or for sufficiently many "steps" to be shown, as can be seen in the characterizations of Group 3 and 4. Thus, one way to explain the mixed quantitative results is that students in the class are at various developmental mathematical stages. Alternatively, this can be an expression of students' interpretation of their role in the classroom microculture: as producers of work that is evaluated by instructors more than evaluators of the work of others.

8.2.4 The Role of Notation in the Solution

Although the four groups developed different norms for notation in solutions, their norms had some shared characteristics. For example, all four groups, to varying degrees, were not averse to informal or unconventional notation; the groups rarely condemned its usage in the Multiple Solutions Activities. Additionally, all four groups concurred that formal/conventional notation was needed in the final answer and were keen to note when the answers were not represented with the proper notation.

Yet, the characterization of the role of notation diverged when considering notation outside of the answer. Groups 2 and 4 sustained the norm that formal notation was not needed in the solution, other than the answer. Molly (Group 2) expressed this succinctly by saying notation was not needed, "As long as you have it right ... at the end." Meanwhile, Group 4 treated formal notation as an obstacle instead of as a mechanism to facilitate and convey their understanding, and scoffed at the instructor and TA who stressed the importance of notation.

Groups 1 and 3 were attentive to notation and that its usage adhered to its proper meaning. In Group 3, this attention was motivated by concern over losing points on class assessments, as one member expressed that it was important to learn "just in case it matters on the exam." Both groups' recognition of notational errors were understandably dependent on content, particularly as the content became increasingly difficult. Consequently, it is important to note in later activities, students in these groups were not ambivalent about adhering to notation conventions, but rather experienced conceptual shortcomings.

The disparity between these two set of groups (Groups 2 and 4, and Groups 1 and 3) may involve students' role beliefs or the persistence of unproductive beliefs. For example, as noted in Section 7.3.2.1, Steve (Group 2) suggested that it was the grader's job to interpret, not the student's job to explain or clarify their work. Such a viewpoint might correlate with not viewing solutions as a means to communicate one's understanding, resulting in Steve and members of his group, devaluing notation. Another example is Group 4’s expressed frustration with the instructor and TA's tolerance of square roots in the denominator, in the last activity (see section 7.5.2.5). The instructor and the TA’s explanations did not appease this frustration, and students’ views on square roots in the denominator remained unchanged. The group’s passivity and disregard for using proper notation in solutions could be the result of ingrained beliefs and experiences from previous mathematics classes.

Only one questionnaire item tangentially related to the role of notation: “Solutions written with formulas or equations are self-explanatory. They do not require written explanations.” The class's pre- and post- means were 2.585 and 2.537 respectively, which represents a neutral position towards whether or not written explanations are needed for solutions written with formulas or equations (Table 10). The neutral means are fitting, given the qualitative variance described above.

8.3 Relationship Between Social and Sociomathematical Norms

The emergent perspective (Yackel & Cobb, 1996) describes within-row relationships in the Interpretive Framework between social and psychological constructs. The results of this study expand the relationships depicted by the Interpretive Framework by showing that there *also* exists a *reflexive, within-column relationship* between social and sociomathematical norm, which suggests that the two mutually influence each other by developing in tandem.

Figure 16

Within-Column Reflexivity in the Interpretive Framework

Social Perspective		Psychological Perspective
Classroom Social Norms	↔	Beliefs about one's own role, others' roles, and the general nature of mathematical activity in school
Sociomathematical Norms	↔	Mathematical values and beliefs

Note: Adapted from Yackel & Cobb (1996)

In the student groups, different social norms of engaging with the Multiple Solutions Activities reinforced different understandings of what constitutes an acceptable solution. This describes the concurrent development and mutual influence between the participation structure of a group and their taken-as-shared mathematical beliefs. Note, that the latter are not individual beliefs, but rather social constructs, as beliefs that fit together constitute norms (Cobb & Yackel, 1998).

Groups that developed characterizations of acceptable solutions as those that adhered to a familiar approach or yielded the correct answer, also developed patterns of avoiding interpreting solutions (Groups 2 and 4) or being unable to interpret solutions (Group 3). These patterns were seen to be mutually supportive, and helped to sustain one another. As the groups further sustained the idea that an acceptable solution was one that utilized a familiar approach or yielded the correct answer (sociomathematical norm), the groups began to critique solutions based on their adherence to a specific method or judged the viability of the approach upon its final answer (social norm). At the same time, as students were judging the viability of the approach on these qualities, they were simultaneously negotiating the notion of what constitutes an acceptable solution.

On the other hand, Group 1's characterization of an acceptable solution as one that followed any viable method concurrently developed with their efforts to analyze and interpret the solutions. As this group expressed permissibility for alternative approaches they simultaneously sustained activity of analyzing solutions to determine their mathematical viability. This process is bidirectional: as students investigated novel solutions to determine their viability, they were sustaining the notion that any viable approach, familiar or not, was an acceptable solution. This concurrent development of social and sociomathematical norms characterizes the within-column reflexivity (see left-hand side arrow in Figure 16).

Just as the emergent perspective characterizes the mutual evolution of within-row social and psychological pairs in the Interpretive Framework (see Section 2.4), in the reflexive relationship between social and sociomathematical norms, neither construct is given primacy. The co-development of the two types of norms should be regarded as a simultaneous and mutually-sustaining, not as a cause-and-effect relationship.

The effects of this within-column relationship between social and sociomathematical norms also suggest a within-column relationship between two types of individual beliefs: the beliefs related to classroom social norms (i.e. individual beliefs about role and the general nature of mathematical activity in the classroom) and mathematical beliefs (see the right-hand side arrow in Figure 16). This is a consequence of composing the within-column relationship between social and sociomathematical norms revealed by this study, with the within-row relationships between social and psychological constructs, described by the emergent perspective (Yackel & Cobb, 1996). In particular, the results presented above show that students' individual conceptions of what a mathematics solution should embody are reflexively related to their beliefs of what classroom activity should look like. In short, the within-column reflexivity described under a social lens may also be seen under a psychological lens as well.

Quantitative analyses also surfaced this relationship between mathematical and social beliefs. As discussed in section 7.1.3, quantitative correlations between changes in mathematical and social beliefs suggested a connection (a significant negative correlation) between what counts as an acceptable solution: "The most valid ways of solving a problem are the ones discussed in class" and a role belief: "The instructors and TAs are responsible for teaching me how to write a solution that would receive full credit" ($r=-0.332$, $p<0.05$, Table 25). It is important to note that correlation does not imply causation, and consequently, this result should be interpreted differently than the qualitative data above. However, this result provides an interesting insight that should be explored further. In particular, this suggests that as students develop more value for flexibility and an openness towards other solution methods, they more strongly expect instructors to teach how to write solutions to receive more credit.

8.4 Didactical Contract

The construct of didactical contract (see Section 2.8) can provide a useful perspective on the observed patterns of negotiation of classroom norms in this study, particularly the struggle to develop and sustain productive norms of engagement. As a reminder, a didactical contract is composed of a set of behaviors of the teacher that are expected by students and a set of behaviors of the students that are expected by the teacher (Yoon et al., 2011; Pierce et al., 2010). That is both students and teacher shave mutual expectations about the nature of the engagement in class (i.e., classroom norms), and about their roles in classroom interactions. While classroom practices and norms decapitate after the conclusion of a particular course, the didactical contract suggest that students develop general expectations about the how mathematical classrooms should feel and look like, and what constitutes "normative" mathematics classroom. These expectations constitute a didactical contract sustained across a variety of instructional contexts.

The results of this study suggest that as the instructional changes introduced by to the MATH 418 course, and the types of social and sociomathematical norms the instructors tried to negotiate, violated students' existing didactical contract. In particular, students' preexisting didactical contract seem to include two key elements: (1) that the instructor's role is to provide a method on how to solve problems given to the students; and (2) that the instructor should verify students' answers. As described in Section 8.1.5, when the instructor instead tried to facilitate the group's engagement and devolved the responsibility for checking answers back to the students, the students expressed frustration, indicating the breach in mutual expectations and the violation of the didactical contract.

This violation in the didactical contract helps to explain the mismatching role beliefs of both the students and the instructor. It also helps to explain why some of the efforts to change classroom norms, to improve student engagement and help students develop greater flexibility and high-order mathematical skills were less successful than expected. Such changes require more than explicit efforts to provide rich learning opportunities through novel instructional activities. Bringing effective reform to developmental mathematics course, requires changing the didactical contract of what it means to engage in mathematics class for both students and instructors alike.

8.5 Implications for Education

The following sections detail suggestions for practice, particularly in post-secondary developmental mathematics classes. These suggestions surfaced from experiences during the teaching experiment and from the results of the study.

8.5.1 Assessment Structure

This study demonstrates an important lesson for educators: utilizing reform pedagogy requires utilizing reform assessment. This alignment is important, as the assessment structure of a course represents an implicit negotiation of what should be valued. The assessments used in this teaching experiment included items assessing conceptual understanding as well as questions requiring use of traditional algorithms. The grading weight of these procedural questions may have motivated students' focus on developing procedural competencies, and hindered the effects of the interventions integrated for the teaching experiment (e.g., Multiple Solutions Activities).

Educators also need to ensure that homework, and its grading, accurately reflect students' understandings. If students are able to utilize online resources (e.g. "Mathway") to circumvent engaging with these assignments they may not develop sufficient understanding to productively engage in other instructional activities. Moreover, grading and providing feedback on these assignments may turn into a fruitless endeavor that drains instructional resources.

Instead, homework and other assessments could be utilized to help negotiate social norms to foster student autonomy. For example, students may be asked, as a part of their assignments, to verify their answers, to “grade” their own work, or to assess fictitious (or real) work of others. This reinforces the expectation in the didactical contract that this is the students’ responsibility to verify the correctness of mathematical work, and provides them with opportunities to practice and receive feedback on doing so.

Lastly, despite the move to an open-sourced, online textbook, quantitative analysis found no evidence in change of students’ use of this resource over the course of the semester (Tables 18 and 19). In reflection homework assignments, many students admitted to never using the book. Despite efforts to integrate the book into the course structure (see Section 5.2.1), further steps could be taken, such as incorporating a reading comprehension question from the book into the quizzes. This again might foster students’ assuming responsibility for their own learning and support the development of a new didactical contract.

8.5.2 Class Discussions

The literature reports on the importance of class discussions to crystalize the content and purpose of class activities. For example, simple exposure alone may be inadequate to develop flexibility (Rittle-Johnson & Star, 2008). Without class discussions or opportunities to explicitly compare solution methods, implementation of such reform pedagogy results in unproductive show-and-tell sessions. This study contributes to confirming the importance of class discussions by showing that inattentiveness during class activities hindered the development of more productive classroom norms and higher-order skills, such as flexibility.

Information about how to conduct productive class discussions is less articulated and prevalent in the research literature at the post-secondary level (cf. Smith & Stein, 2011 for secondary level). The following paragraphs include suggestions for educators for facilitating more productive classroom discussions from lessons surfaced during this teaching experiment.

As described in the previous section, value is implicitly negotiated through the assessment structure. One way to motivate attentiveness during class discussions is to assign students grades for participating in class discussions. In this study, since student participation in these discussions was not explicitly included in the assessment structure of the course, this may have implicitly negotiated less importance or value than other aspects of the course.

Additionally, violations of productive social norms need to be delegitimized. In this study, the instructors did not penalize students for non-mathematical cell phone use (which was a major source of distraction and inattentiveness during class discussions) despite asserting so in the syllabus. Not responding to these violations implicitly negotiates acceptance of them. Thus, not only do class rules need to be articulated to students, but they also need to be sustained by the instructional staff.

Within the Multiple Solutions Activities, the reflection questions comparing across the solutions, were not always treated by students as a means of collective review or reflection. For example, even more collaborative groups like Group 1 attended to the questions individually. This lack of small group review may not provide all students with sufficient comfort to verbally participate and engage in whole class discussions. Instead of separating the reflection questions and whole class discussions, the discussions could instead incorporate opportunities for small group reflection to ensure that groups collaboratively evaluate the questions, which may elicit improved engagement.

Lastly, some students were more attentive when they were provided with more explicit directives, especially with content written on the board (e.g. finding a mistake in a solution on the board). In addition to further scaffolding the reflection, students seemed more engaged when provided a visual aid, like seeing the solutions on the board. This might support and reinforce the importance of using formal mathematical language coupled with the aid of visual images, especially in developmental mathematics classes. With instructional activities like the Multiple Solutions Activities, visuals provide a way for students to explicitly see instructor point to the aspects of solutions being discussed, and may help students develop comfort with language that is concurrently being used by the instructor.

8.5.3 Explicitly Discussing Grading Rubrics

As discussed in Section 8.3, the significant negative correlation between Post-Pre changes in the items: "The most valid ways of solving a problem are the ones discussed in class," and, "The instructors and TAs are responsible for teaching me how to write a solution that would receive full credit" (r=-0.332, $p<0.05$, Table 25), suggests that as students develop more value for flexibility and openness towards other solution methods, students more strongly expect instructors to teach them how to write solutions that receive full credit.

Thus, one way to help promote flexibility in developmental mathematics classes is to dedicate class time to explicitly discuss the structure and characteristics of an acceptable solution, and to articulate how solutions are going to be graded. If students have a clear understanding of the grading rubric for a problem, they might feel less anxiety and more freedom to explore different ways to solve a problem. This shifts focus from memorizing instructor-approved methods to using and adapting instructor-approved solution structures.

8.6 Limitations and Suggestions for Future Research

This study uses a methodology of teaching experiment. Thus, its findings are inherently contextualized to the time, location, and individual differences of its participants, and should be interpreted as such. One specific limitation of study arises from the relatively small sample size used in quantitative analysis, which affects the power of inferential tests, and may result in effects going unnoticed. This is especially relevant with the results between students with and without prior MATH 418 enrollment. Additionally, the smaller sample sizes contributed towards different variances in some cases between the two pools of students, which did not allow for the use of more powerful tests (as described in section 6.4).

The data analysis revealed the effect of one particular variable: previous MATH 418 enrollment. Given the increasing importance and prevalence of research on identity, future research regarding student engagement should explore the influence of other variables, such as gender, race, age, and their intersectionality, as it relates to mathematics. Doing so would require a sample that is more diverse than the one used in this study, and of larger size to be able to notice effects. Given the increased difficulty of negotiating productive norms with larger enrollments, it may be prudent to instead focus on increasing the response rate of questionnaires.

By nature of a teaching experiment, the qualitative analysis surfaced norms that were not explicitly measured in items in the questionnaire. For example, the norm that formal notation was only needed in the answer instead of the whole solution was not measured in the questionnaire. This can be construed as a strength of using mixed methods inquiry: qualitative analysis surfaces nuance that would not otherwise be captured by quantitative analysis.

Consequently, it is expected that the quantitative instruments will not measure the pervasiveness of every result surfaced in the qualitative analysis, as the quantitative and qualitative analyses should not be expected to completely align. Thus, this study, by nature of its methodology, cannot make claims about the pervasiveness of specific norms or beliefs amongst all groups or students in the class.

Learning assistants (LAs) were not originally planned to be used in the course, but were eventually integrated because of the unexpectedly high enrollment. Due to the late implementation, LAs were not incorporated in this study and consequently, their influence on the class was not measured or studied. This is especially relevant to this study, as LAs are members of the microculture that students may have regarded as another source of authority. As more institutions incorporate LAs into introductory and developmental mathematics courses, studying the influence of LAs on the microculture becomes more important. Additionally, future research should concurrently study changes in LAs' beliefs. Adopting a new role may influence beliefs that had previously been ingrained by years of experience as a student.

Chapter 9. Conclusion

To conclude, this research project surfaces several ideas that warrant explication. First, the norms developed and sustained across groups are not uniform, as the norms that developed amongst several groups varied, sometimes in important ways. Research on social and sociomathematical norms prevalent in the existing literature often examines norms that characterize patterns within an entire classroom. The contribution of this research project is that it identified that there are different layers to a single mathematics classroom microculture. Given the growing transition to inquiry-based or student-centered classrooms, it may be necessary to also transition from thinking about classroom norms to group norms. This research exemplifies the significance of studying norms within this smaller unit of analysis.

Second, the results of this study expand upon Yackel and Cobb's (1996) Interpretive Framework by suggesting that reflexivity also exists between social and sociomathematical norms, as well as between corresponding types of individual beliefs. As described above, in the student groups, different social norms of engaging with the Multiple Solutions Activities reinforced different understandings of what constitutes an acceptable solution. This describes the concurrent development and mutual influence between the participation structure of a group and their taken-as-shared mathematical beliefs. The significance of this within-column relationship in the Interpretive Framework is that it complements the within-row relationships described by the emergent perspective, and suggests more intricate relationships between social norms, sociomathematical norms, beliefs about role and the general nature of classroom activity, and mathematical beliefs.

Third, the persistence of students' unproductive beliefs represents a deep rooted conflict that may explain the origin of norms that developed in contrast to the instructors' negotiations and expectations. As seen in this study, some students explicitly refused to change their practices, or expressed extreme frustration at the perceived violations in expectations. Consequently, it is reasonable to conclude that in the post-secondary setting, students' beliefs may be more deeply ingrained and more difficult to renegotiate than that seen in the existing research on student engagement, which has been primarily conducted in earlier grades.

Lastly, in addition to being a theoretical framework, the emergent perspective also acts as a cautionary tale. As framed within this project, the emergent perspective depicts that norms have lasting impacts, not just on the engagement within the current class, but students' subsequent classes as well. Quantitative analyses found that the variable of prior enrollment in MATH 418 produced significant effects. Therefore, this study demonstrated that some of the norms of previous classes, and possibly of the other earlier mathematical experiences, influenced students' beliefs, which, in turn, hindered the development of more productive norms. Thus, it is reasonable to suggest that fostering productive norms could benefit students both in their current class, but in future classes as well, by supporting changes to the didactical contract of what it means to productively participate in mathematical class.

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APPENDIX A

STAGE 2 MULTIPLE SOLUTIONS ACTIVITIES

Activity 1

Math 418 Activity 2/5/19

Directions

1.) On the following piece of paper, under the section labeled "Example Solution(s)," write a solution to the given problem. Share, compare, and contrast different possible approaches amongst your group members.

2.) Afterwards, create a grading rubric for the problem. A grading rubric should contain a detailed classification of how to award points to students' solutions. For this problem, determine how to distribute 6 points of credit (i.e. the problem is scored on a range of 0 to 6 points). On the next page, under the "Points Given" section, write when points should be awarded followed by a brief statement of why points are awarded.

3a.) After creating the grading rubric, ask your instructor for the sample of students' work. This packet will contain several examples of students' solutions. With your group members, *try to understand how the students approached the problem*. Then, using the grading rubric you just constructed, grade and evaluate each student's solution. Your group must agree on the points awarded to each solution. This shouldn't just include where points are awarded, but a brief statement of *why* as well.

3b.) As you evaluate student solutions, you may want to edit your rubric by altering how to award points or to account for additional details. Write these details and/or changes at the bottom of the following page.

4.) On students' solutions, explicitly write your suggestions for how the students' could improve their solution.

5.) Compare and contrast the students' solutions. For example: What were some of the ways that students solved these problems, and how were they different? What were the benefits or shortcomings of each method?

Find the domain of the following function. Express your answer in interval notation.

$$F(x) = \frac{\sqrt{x} + 1}{2\sqrt{1 - 3x}}$$

EXAMPLE SOLUTION(S)	POINTS GIVEN/ WHY

EDITS/ ADDITIONAL DETAILS

Name: Tom

Find the domain of the following function. Express your answer in interval notation.

$$F(x) = \frac{\sqrt{x}+1}{2\sqrt{1-3x}}$$

STUDENT SOLUTION

RATIONALE FOR POINTS AWARDED

Denominator

$\cancel{2}\sqrt{1-3x} \neq 0$

$(\sqrt{1-3x})^2 \neq (0)^2$

$1-3x \neq 0$

$1 \neq 3x$

$\frac{1}{3} \neq x$

$x > 1/3$

Numerator

$(\sqrt{x})^2 \neq 0^2$

$x > 0$

↓

Domain is the smallest value

So $D_F: x > 0$

Name: Andrea

Find the domain of the following function. Express your answer in interval notation.

$$F(x) = \frac{\sqrt{x}+1}{2\sqrt{1-3x}}$$

STUDENT SOLUTION

RATIONALE FOR POINTS AWARDED

Cannot take the sq. root of a neg. value.

So: For $\sqrt{1-3x}$

$1-3x \geq 0$

$-3x \geq -1$

$x \leq 1/3$

↓

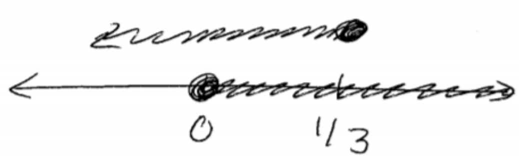
This tells us that all numbers less than 1/3 are in the domain.

For $\sqrt{x}$

$x \geq 0$

↓

This tells us that all numbers greater than zero are in the domain



So the domain is $(-\infty, \infty)$

Name: Brady

Find the domain of the following function. Express your answer in interval notation.

$$F(x) = \frac{\sqrt{x}+1}{2\sqrt{1-3x}}$$

STUDENT SOLUTION	RATIONALE FOR POINTS AWARDED
$\sqrt{1-3x}$ { ≥ 0, $\neq 0$ $= > 0$ $1-3x > 0$ $-3x > -1$ $x < \frac{1}{3}$ $\sqrt{x}+1$ { ≥ 0 $x \geq 0$ $x < \frac{1}{3} + x \geq 0 = [0, \frac{1}{3}]$	

5.) Compare and contrast the students' solutions.

a.) What are some of the things that you liked about Tom's solution? What are some of the things that you didn't like?

b.) What are some of the things that you liked about Andrea's solution? What are some of the things that you didn't like?

c.) What are some of the things that you liked about Brody's solution? What are some of the things that you didn't like?

d.) Were any of these solutions difficult to understand? How come?

e.) What are some components of a quality solution?

Activity 2

Math 418 Activity 2/19/19

Directions

1.) On the following piece of paper, under the section labeled “Example Solution(s),” write a solution to the given problem. Share, compare, and contrast different possible approaches amongst your group members.

2.) Afterwards, create a grading rubric for the problem. A grading rubric should contain a detailed classification of how to award points to students' solutions. For this problem, determine how to distribute 6 points of credit (i.e. the problem is scored on a range of 0 to 6 points). On the next page, under the “Points Given” section, write when points should be awarded followed by a brief statement of why points are awarded.

3a.) After creating the grading rubric, ask your instructor for the sample of students' work. This packet will contain several examples of students' solutions. With your group members, *try to understand how the students approached the problem*. Then, using the grading rubric you just constructed, grade and evaluate each student's solution. Your group must agree on the points awarded to each solution. This shouldn't just include where points are awarded, but a brief statement of *why* as well.

3b.) As you evaluate student solutions, you may need to edit your rubric by altering how to award points or to account for additional details. Write these details at the bottom of the following page.

4.) On students' solutions, explicitly write your suggestions for how the students' could improve their solution.

5.) Compare and contrast the students' solutions. For example: What were some of the ways that students solved these problems, and how were they different? What were the benefits or shortcomings of each method?

Find the vertex of the following function:

$$f(x) = -2x^2 - \frac{1}{3}x + \frac{2}{3}$$

EXAMPLE SOLUTION(S)

POINTS GIVEN/ WHY

EDITS/ ADDITIONAL DETAILS

Name: Frodo

Find the vertex of the following function:

$$f(x) = -2x^2 - \frac{1}{3}x + \frac{2}{3}$$

STUDENT SOLUTION	RATIONALE FOR POINTS AWARDED
$-2x^2 - \frac{1}{3}x + \frac{2}{3} = 0$	
$-2x^2 - \frac{4}{3}x + x + \frac{2}{3} = 0$	
$-2x(x + \frac{2}{3}) + 1(x + \frac{2}{3}) = 0$	
$(x + \frac{2}{3})(-2x + 1) = 0$	
$x = -\frac{2}{3}, \frac{1}{2}$	
vertex @ $x = \frac{-\frac{2}{3} + \frac{1}{2}}{2} = \frac{1}{2}(-\frac{4}{6} + \frac{3}{6}) = \frac{1}{2}(-\frac{1}{6}) = -\frac{1}{12}$	
$f(-\frac{1}{12}) = -2(-\frac{1}{12})^2 - \frac{1}{3}(-\frac{1}{12}) + \frac{2}{3}$	
$-2(\frac{1}{144})_{72} + \frac{1}{36} \cdot \frac{2}{2} + \frac{2}{3} \cdot \frac{24}{24}$	
$\frac{-1}{72} + \frac{2}{72} + \frac{48}{72}$	
$\boxed{\frac{49}{72}}$	

Name: Kennedy

Find the vertex of the following function:

$$f(x) = -2x^2 - \frac{1}{3}x + \frac{2}{3}$$

STUDENT SOLUTION	RATIONALE FOR POINTS AWARDED
$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$ $\frac{\frac{1}{3} \pm \sqrt{\frac{1}{3}^2 - 4(-2)(\frac{2}{3})}}{2\cdot(-2)}$ Vertex $X = \frac{\frac{1}{3}}{-4} = -\frac{4}{3}$ $Y = -2(-\frac{4}{3})^2 - \frac{1}{3}(-\frac{4}{3}) + \frac{2}{3}$ $Y = -2\left(\frac{16}{9}\right) + \frac{4}{9} + \frac{2\cdot3}{3\cdot3}$ $Y = \frac{-32}{9} + \frac{4}{9} + \frac{6}{9} = \frac{-22}{9}$ $\left(-\frac{4}{3}, -\frac{22}{9}\right)$	

Name: Andrea

Find the vertex of the following function:

$$f(x) = -2x^2 - \frac{1}{3}x + \frac{2}{3}$$

STUDENT SOLUTION	RATIONALE FOR POINTS AWARDED

$f(x) = -2x^2 - \frac{1}{3}x + \frac{2}{3} = 0$

$\quad -\frac{2}{3} \quad -\frac{2}{3}$

$f(x) = (-2x^2 - \frac{1}{3}x) = \frac{-2}{3}$

$f(x) = -2(x^2 - \frac{1/3}{-2}x) = \frac{-2}{3}$

$f(x) = -2(x^2 + \frac{1}{6}x) = \frac{-2}{3}$

$f(x) = -2(x^2 + \frac{1}{6}x + \left(\frac{1/6}{2}\right)^2) = \frac{-2}{3} + \left(\frac{1/6}{2}\right)^2$

$f(x) = -2(x^2 + \frac{1}{6}x + \left(\frac{1}{12}\right)^2) = \frac{-2}{3} + \left(\frac{1}{12}\right)^2$

$f(x) = -2(x + \frac{1}{12})(x + \frac{1}{12}) = \frac{-2 \cdot 144}{3 \cdot 144} + \frac{1}{144} \cdot \frac{3}{3}$

$f(x) = -2(x + \frac{1}{12})^2 = \frac{-288}{432} + \frac{3}{432}$

$144 \cdot 3 = 432$

$f(x) = -2(x + \frac{1}{12})^2 = -\frac{285}{432}$

$f(x) = -2(x + \frac{1}{12})^2 + \frac{285}{432}$

Vertex $= \left(\frac{-1}{12}, \frac{285}{432}\right)$

5.) Compare and contrast the students' solutions.

a.) What are some of the things that you liked about Frodo's solution? What are some of the things that you didn't like?

b.) What are some of the things that you liked about Kennedy's solution? What are some of the things that you didn't like?

c.) What are some of the things that you liked about Andrea's solution? What are some of the things that you didn't like?

d.) Were any of these solutions difficult to understand? How come?

e.) What are some components of a quality solution?

Activity 3

Math 418 Activity 4/2/19

Directions

1.) On the following piece of paper, under the section labeled "Example Solution(s)," write a solution to the given problem. Share, compare, and contrast solutions amongst your group members.

2.) Afterwards, create a grading rubric for the problem. A grading rubric should contain a detailed classification of how to award points to students' solutions. For this problem, determine how to distribute 6 points of credit (i.e. the problem is scored on a range of 0 to 6 points). On the next page, under the "Points Given" section, write when points should be awarded followed by a brief statement of why points are awarded.

3a.) After creating the grading rubric, ask your instructor for the sample of students' work. This packet will contain several examples of students' solutions. With your group members, *try to understand how the students approached the problem*. Then, using the grading rubric you just constructed, grade and evaluate each student's solution. Your group must agree on the points awarded to each solution. This shouldn't just include where points are awarded, but a brief statement of *why* as well.

3b.) As you evaluate student solutions, you may want to edit your rubric by altering how to award points or to account for additional details. Write these details and/or changes at the bottom of the following page.

4.) On students' solutions, explicitly write your suggestions for how the students' could improve their solution.

5.) Compare and contrast the students' solutions. For example: What were some of the ways that students solved these problems, and how were they different? What were the benefits or shortcomings of each method?

Find the inverse of the following function:

$$f(x) = \frac{9}{4}\,3^{8x} - \frac{5}{2}$$

EXAMPLE SOLUTION(S)	POINTS GIVEN/ WHY

EDITS/ ADDITIONAL DETAILS

Name: Lincoln

Find the inverse of the following function:

$$f(x) = \frac{9}{4} 3^{8x} - \frac{5}{2}$$

STUDENT SOLUTION	RATIONALE FOR POINTS AWARDED

$$y = \frac{9}{4} \cdot 3^{8x} - \frac{5}{2}$$

$$x = \frac{9}{4} \cdot 3^{8y} - \frac{5}{2}$$

$$x + \frac{5}{2} = \frac{9}{4} \cdot 3^{8y}$$

$$\sqrt{x + \frac{5}{2}} = \sqrt{\frac{9}{4} \cdot 3^{8y}}$$

$$\sqrt{x + \frac{5}{2}} = \frac{3}{2} \cdot 3^{4y}$$

$$\frac{2}{3}\sqrt{x + \frac{5}{2}} = 3^{4y}$$

$$\log_3\left(\frac{2}{3}\sqrt{x + \frac{5}{2}}\right) = \cancel{\log_3} 3^{4y}$$

$$\boxed{\frac{\log_3\left(\frac{2}{3}\sqrt{x + \frac{5}{2}}\right)}{4} = y}$$

Name: Alexander

Find the inverse of the following function:

$$f(x) = \frac{9}{4} 3^{8x} - \frac{5}{2}$$

STUDENT SOLUTION	RATIONALE FOR POINTS AWARDED

$y = \frac{9}{4} 3^{8x} - 5/2$

$4y = 3^2 \cdot 3^{8x} - 10$

$+10 \qquad +10$

$4y+10 = 3^{8x+2}$

$\ln(4y+10) = \ln(3^{8x+2})$

$\frac{\ln(4y+10)}{\ln 3} = \frac{(8x+2)\ln 3}{\ln 3}$

$-2, \div 8$

$\frac{1}{8}\ln(4y+10) - 2 = x$

$f^{-1}(y) = \frac{1}{8}\ln(4y+10) - 2$

Name: Andrea

Find the inverse of the following function:

$$f(x) = \frac{9}{4}\, 3^{8x} - \frac{5}{2}$$

STUDENT SOLUTION	RATIONALE FOR POINTS AWARDED

$$y = \frac{9}{4} \cdot 3^{8x} - \frac{5}{2}$$

* Switch x & y

$$x = \frac{9}{4} \cdot 3^{8y} - \frac{5}{2}$$

$$x + \frac{5}{2} = \frac{9}{4} \cdot 9^{4y}$$

$$4x + \frac{5}{2} = 9^{4y+1}$$

$$9^{\log_9 (4x + \frac{5}{2})} = 9^{4y+1}$$

$$4y + 1 = \log_9 \left(4x + \frac{5}{2}\right)$$

$$4y = \log_9 \left(4x + \frac{5}{2}\right) - 1$$

$$y = \frac{\log_9 \left(4x + \frac{5}{2}\right) - 1}{4}$$

5.) Compare and contrast the students' solutions.

a.) What were some of the ways that Lincoln's, Alexander's, and Andrea's solutions differed?

b.) Did you notice any benefits or shortcomings for any of these approaches?

c.) Were any of these solutions difficult to understand? How come?

Activity 4

Math 418 Activity 4/30/19

Directions

1.) On the following piece of paper, under the section labeled “Example Solution(s),” write a solution to the given problem. Share, compare, and contrast solutions amongst your group members.

2.) Afterwards, create a grading rubric for the problem. A grading rubric should contain a detailed classification of how to award points to students' solutions. For this problem, determine how to distribute 6 points of credit (i.e. the problem is scored on a range of 0 to 6 points). On the next page, under the “Points Given” section, write when points should be awarded followed by a brief statement of why points are awarded.

3a.) After creating the grading rubric, ask your instructor for the sample of students' work. This packet will contain several examples of students' solutions. With your group members, *try to understand how the students approached the problem*. Then, using the grading rubric you just constructed, grade and evaluate each student's solution. Your group must agree on the points awarded to each solution. This shouldn't just include where points are awarded, but a brief statement of *why* as well.

3b.) As you evaluate student solutions, you may want to edit your rubric by altering how to award points or to account for additional details. Write these details and/or changes at the bottom of the following page.

4.) On students' solutions, explicitly write your suggestions for how the students' could improve their solution.

5.) Compare and contrast the students' solutions. For example: What were some of the ways that students solved these problems, and how were they different? What were the benefits or shortcomings of each method?

Evaluate the following:

$$\tan\left(sin^{-1}\left(\frac{1}{2}\right)\right)$$

EXAMPLE SOLUTION(S)	POINTS GIVEN/ WHY

EDITS/ ADDITIONAL DETAILS

Name: Jennifer

Evaluate the following:

$$\tan\left(\sin^{-1}\left(\frac{1}{2}\right)\right)$$

STUDENT SOLUTION

$u = \sin^{-1}(1/2) \rightarrow 1^{st}$ Quad

$\sin(u) = 1/2$

$a^2 + b^2 = c^2$

$a^2 + 1^2 = 2^2$

$a^2 = 2^2 - 1^2$

$a^2 = (2-1)^2$

$a = 1$

$\tan(\sin^{-1}(1/2))$

$= \tan(u)$

$= 1/1$

$= \boxed{1}$

RATIONALE FOR POINTS AWARDED

Name: Dan

Evaluate the following:

$$\tan\left(\sin^{-1}\left(\frac{1}{2}\right)\right)$$

STUDENT SOLUTION	RATIONALE FOR POINTS AWARDED

$\sin^{-1}\left(\frac{1}{2}\right) = A$

① $\sin(A) = \frac{1}{2}$

② $-\frac{\pi}{2} < A < \frac{\pi}{2}$.

Since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and $-\frac{\pi}{2} < \frac{\pi}{6} < \frac{\pi}{2}$, then $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$\tan\left(\frac{\pi}{6}\right)$

$= \frac{\sin}{\cos}(\pi/6)$

$= \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$

Name: Andrea

Evaluate the following:

$$\tan\left(\sin^{-1}\left(\frac{1}{2}\right)\right)$$

STUDENT SOLUTION	RATIONALE FOR POINTS AWARDED

$$\frac{\sin(\sin^{-1}(1/2))}{\cos(\sin^{-1}(1/2))}$$

$$= \frac{1/2}{\cos(\omega)}$$

$$= \frac{1/2}{\sqrt{3}/2}$$

$$= \boxed{\frac{1}{\sqrt{3}}}$$

Let $\omega = \sin^{-1}(1/2)$

$\sin^2(\omega) + \cos^2(\omega) = 1$

$(1/2)^2 + \cos^2(\omega) = 1$

$\cos^2(\omega) = 1 - (1/2)^2$

$\cos(\omega) = + \frac{\sqrt{3}}{2}$

5.) Compare and contrast the students' solutions.

a.) What were some of the notational mistakes in Jennifer's, Dan's, and/or Andrea's solutions?

b.) What were some of the mathematical or computational mistakes in these solutions?

c.) Were any of these solutions difficult to understand? How come?

APPENDIX B

STAGE 2 QESTIONNAIRE

Mathematical Beliefs Questionnaire Spring 19 (End of Semester)

1 Please only take this survey if you have signed the consent form and have allowed for your responses to be analyzed for research purposes. If you do not wish to have your responses analyzed, **close this survey and instead complete one using the link that David emailed you.**

If you added the course late and have not yet seen the consent form, please do not take this survey and email David at dri36@wildcats.unh.edu.

This survey is anonymous and your identity is protected. After completing the survey, you will be automatically redirected to a link to enter your name for credit. **Your name will not be linked to your survey response.**

2 Enter the following six characters without spaces: (1-3) The first three letters of the city/town where you went to High School, (4-5) Your birth month (for example, if you are born in January, please enter "01" instead of "1"), and (6) your middle initial (if you do not have a middle name, use "X").

(Example: MAN09R).

____

3 For the following questions: a **solution** refers to the written process/work to reach a conclusion or answer.

4 For the following questions, indicate whether you strongly disagree, disagree, neither agree or disagree, agree, or strongly agree with the given statement.

5 Mathematics is a set of rules and procedures that need to be memorized.

- ○ Disagree (1)
- ○ Slightly Disagree (2)
- ○ Slightly Agree (3)
- ○ Agree (4)

6 There is no place in mathematics for discussions - you are either right or wrong.

- ○ Disagree (1)
- ○ Slightly Disagree (2)
- ○ Slightly Agree (3)
- ○ Agree (4)

7 In math, explaining my work or reasoning to others helps me learn.

- ○ Disagree (1)
- ○ Slightly Disagree (2)
- ○ Slightly Agree (3)
- ○ Agree (4)

8 It is the instructor's role to prepare me for quizzes and exams.

- ○ Disagree (1)
- ○ Slightly Disagree (2)
- ○ Slightly Agree (3)
- ○ Agree (4)

9 I use graphing technology to understand what an unfamiliar function/equation looks like.

- Disagree (1)
- Slightly Disagree (2)
- Slightly Agree (3)
- Agree (4)

10 For the following questions, indicate whether you strongly disagree, disagree, neither agree or disagree, agree, or strongly agree with the given statement.

11 Working with peers helps me learn about new ways of thinking about a problem.

- Disagree (1)
- Slightly Disagree (2)
- Slightly Agree (3)
- Agree (4)

12 The solution to a math problem must contain a check of my work or a way to verify my answer.

- Disagree (1)
- Slightly Disagree (2)
- Slightly Agree (3)
- Agree (4)

13 The most valid ways of solving a problem are the ones discussed in class.

- Disagree (1)
- Slightly Disagree (2)
- Slightly Agree (3)
- Agree (4)

14 When completing homework, I actively refer to my notes from class.

○ Disagree (1)

○ Slightly Disagree (2)

○ Slightly Agree (3)

○ Agree (4)

15 It is more important to correctly perform the steps of a solution than to understand each one of them.

○ Disagree (1)

○ Slightly Disagree (2)

○ Slightly Agree (3)

○ Agree (4)

16 For the following questions, indicate whether you strongly disagree, disagree, neither agree or disagree, agree, or strongly agree with the given statement.

17 The instructors and TAs are responsible for teaching me how to write a solution that would receive full credit.

○ Disagree (1)

○ Slightly Disagree (2)

○ Slightly Agree (3)

○ Agree (4)

18 The purpose of math class is to learn new math content.

○ Disagree (1)

○ Slightly Disagree (2)

○ Slightly Agree (3)

○ Agree (4)

19 To receive full credit, my solution must use the same methods used in class.

- ○ Disagree (1)
- ○ Slightly Disagree (2)
- ○ Slightly Agree (3)
- ○ Agree (4)

20 It is my responsibility to ask for help when I do not fully understand something.

- ○ Disagree (1)
- ○ Slightly Disagree (2)
- ○ Slightly Agree (3)
- ○ Agree (4)

21 I prefer to focus on learning how to use formulas instead of spending time on where they come from.

- ○ Disagree (1)
- ○ Slightly Disagree (2)
- ○ Slightly Agree (3)
- ○ Agree (4)

22 For the following questions, indicate whether you strongly disagree, disagree, neither agree or disagree, agree, or strongly agree with the given statement.

23 Solutions written with formulas or equations are self-explanatory. They do not require written explanations.

- Disagree (1)
- Slightly Disagree (2)
- Slightly Agree (3)
- Agree (4)

24 I usually don't find math textbooks helpful and prefer not to use them.

- Disagree (1)
- Slightly Disagree (2)
- Slightly Agree (3)
- Agree (4)

25 When it comes to math, I would rather try to figure out my own questions or confusion than ask for help.

- Disagree (1)
- Slightly Disagree (2)
- Slightly Agree (3)
- Agree (4)

26 In typical math lectures, I write down everything that the instructor writes on the board.

- Disagree (1)
- Slightly Disagree (2)
- Slightly Agree (3)
- Agree (4)

27 I find it helpful to learn several different ways to solve a math problem.

- Disagree (1)
- Slightly Disagree (2)
- Slightly Agree (3)
- Agree (4)

28 The following questions will ask how important certain aspects of math are to you.

29 How important is it to you to write a solution that your peers could understand?

- Not important (1)
- Slightly important (2)
- Moderately important (3)
- Very important (4)

30 How important is memorization to solving math problems?

- Not important (1)
- Slightly important (2)
- Moderately important (3)
- Very important (4)

31 How important is getting the right answer to receiving credit for a math problem?

- Not important (1)
- Slightly important (2)
- Moderately important (3)
- Very important (4)

32 How important is it for you to be creative when solving a mathematical problem?

- ○ Not important (1)
- ○ Slightly important (2)
- ○ Moderately important (3)
- ○ Very important (4)

33 How important is it for you to be able to determine if a peer's solution is correct?

- ○ Not important (1)
- ○ Slightly important (2)
- ○ Moderately important (3)
- ○ Very important (4)

34 I have taken MATH 418 before this semester.

- ○ Yes (1)
- ○ No (2)

35 I am taking this course because my major requires me to take Calculus (MATH 425).

- ○ Yes (1)
- ○ No (2)

APPENDIX C

STAGE 2 QUESTIONNAIRE RESULTS

Table 30

Full List of Pre- and Post-Questionnaire Means and Standard Deviations

Item	*Pre-Mean*	*Post-Mean*	*Pre-Standard Deviation*	*Post-Standard Deviation*
Mathematics is a set of rules and procedures that need to be memorized.	3.225	2.952	0.660	0.825
There is no place in mathematics for discussions - you are either right or wrong.	1.976	1.690	0.998	0.749
In math, explaining my work or reasoning to others helps me learn.	3.500	3.548	0.634	0.632
It is the instructor's role to prepare me for quizzes and exams.	3.190	3.238	0.740	0.656
I use graphing technology to understand what an unfamiliar function/equation looks like.	3.220	3.310	0.759	0.715
Working with peers helps me learn about new ways of thinking about a problem.	3.429	3.405	0.888	0.734
The solution to a math problem must contain a check of my work or a way to verify my answer.	3.095	2.854	0.759	0.989
The most valid ways of solving a problem are the ones discussed in class.	2.881	2.548	0.772	0.803
When completing homework, I actively refer to my notes from class.	3.286	3.214	0.835	0.951
It is more important to correctly perform the steps of a solution than to understand each one of them.	2.213	2.548	1.071	0.993

The instructors and TAs are responsible for teaching me how to write a solution that would receive full credit.	3.452	3.595	0.772	0.665
The purpose of math class is to learn new math content.	3.381	3.310	0.661	0.517
To receive full credit, my solution must use the same methods used in class.	2.133	1.829	1.002	0.863
It is my responsibility to ask for help when I do not fully understand something.	3.829	3.786	0.381	0.415
I prefer to focus on learning how to use formulas instead of spending time on where they come from.	3.341	3.310	0.794	0.841
Solutions written with formulas or equations are self-explanatory. They do not require written explanations.	2.585	2.548	0.948	0.968
I usually don't find math textbooks helpful and prefer not to use them.	2.585	2.548	0.974	0.889
When it comes to math, I would rather try to figure out my own questions or confusion than ask for help.	2.524	2.262	0.862	0.857
In typical math lectures, I write down everything that the instructor writes on the board.	3.049	2.738	0.999	0.939
I find it helpful to learn several different ways to solve a math problem.	3.190	3.119	0.804	0.968
How important is it to you to write a solution that your peers could understand?	2.857	3.119	0.872	0.803
How important is memorization to solving math problems?	3.143	3.000	0.751	0.698
How important is getting the right answer to receiving credit for a math problem?	3.262	3.167	0.767	0.660

How important is it for you to be creative when solving a mathematical problem?	2.333	2.381	0.979	0.936
How important is it for you to be able to determine if a peer's solution is correct?	2.881	2.905	0.803	0.759
I have taken MATH 418 before this semester.	26 Yes, 16 No		-	-
I am taking this course because my major requires me to take Calculus (MATH 425).	37 Yes, 5 No	33 Yes, 9 No	-	-

Note: n=42, 1- Disagree/ Not Important, 4- Agree/ Very Important

Table 31

Pre- and Post-Questionnaire Means, Split Between Those with and without Prior Math 418 Enrollment

Questionnaire Item	*No Prior 418 (n=16)*		*Prior 418 (n=26)*	
	Pre Mean	*Post Mean*	*Pre Mean*	*Post Mean*
Mathematics is a set of rules and procedures that need to be memorized.	3.267	2.933	3.200	2.920
There is no place in mathematics for discussions - you are either right or wrong.	1.875	1.688	2.038	1.692
In math, explaining my work or reasoning to others helps me learn.	3.438	3.375	3.538	3.654
It is the instructor's role to prepare me for quizzes and exams.	3.000	3.000	3.308	3.385
I use graphing technology to understand what an unfamiliar function/equation looks like.	3.067	3.400	3.308	3.308
Working with peers helps me learn about new ways of thinking about a problem.	3.188	3.188	3.577	3.538
The solution to a math problem must contain a check of my work or a way to verify my answer.	3.188	2.938	3.040	2.800
The most valid ways of solving a problem are the ones discussed in class.	3.000	2.438	2.808	2.615
When completing homework, I actively refer to my notes from class.	3.125	3.188	3.385	3.231
It is more important to correctly perform the steps of a solution than to understand each one of them.	1.875	2.438	2.423	2.615
The instructors and TAs are responsible for teaching me how to write a solution that would receive full credit.	3.313	3.375	3.538	3.731
The purpose of math class is to learn new math content.	3.438	3.188	3.346	3.385

To receive full credit, my solution must use the same methods used in class.	1.938	1.750	2.280	1.880
It is my responsibility to ask for help when I do not fully understand something.	3.733	3.733	3.885	3.846
I prefer to focus on learning how to use formulas instead of spending time on where they come from.	3.133	3.000	3.462	3.500
Solutions written with formulas or equations are self-explanatory. They do not require written explanations.	2.467	2.467	2.654	2.577
I usually don't find math textbooks helpful and prefer not to use them.	2.400	2.400	2.692	2.654
When it comes to math, I would rather try to figure out my own questions or confusion than ask for help.	2.563	2.188	2.500	2.308
In typical math lectures, I write down everything that the instructor writes on the board.	3.000	3.000	3.077	2.615
I find it helpful to learn several different ways to solve a math problem.	3.313	2.875	3.115	3.269
How important is it to you to write a solution that your peers could understand?	2.938	3.125	2.808	3.115
How important is memorization to solving math problems?	3.125	3.000	3.154	3.000
How important is getting the right answer to receiving credit for a math problem?	3.188	3.250	3.308	3.115
How important is it for you to be creative when solving a mathematical problem?	2.438	2.375	2.269	2.385
How important is it for you to be able to determine if a peer's solution is correct?	3.063	2.875	2.769	2.923

Note: n=42, 1- Disagree/ Not Important, 4- Agree/ Very Important

Table 32

Significant Correlations with Respect to Change in Post–Pre Data (Correlations obtained by Pairwise Method)

Question A	*Question B*	*Correlation, r-value (Significance, p-value)*
The most valid ways of solving a problem are the ones discussed in class.	*In typical math lectures, I write down everything that the instructor writes on the board.*	-0.428 (*p*=0.007)
Mathematics is a set of rules and procedures that need to be memorized.	*How important is memorization to solving math problems?*	0.579 (*p*=0.0001)
To receive full credit, my solution must use the same methods used in class.	*I use graphing technology to understand what an unfamiliar function/equation looks like.*	0.468 (*p*=0.003)
I find it helpful to learn several different ways to solve a math problem.	*How important is it to you to write a solution that your peers could understand?*	0.494 (*p*=0.001)
The solution to a math problem must contain a check of my work or a way to verify my answer.	*How important is it for you to be able to determine if a peer's solution is correct?*	0.329 (*p*=0.044)
There is no place in mathematics for discussions - you are either right or wrong.	*Solutions written with formulas or equations are self-explanatory. They do not require written explanations.*	0.374 (*p*=0.021)
The solution to a math problem must contain a check of my work or a way to verify my answer.	*When completing homework, I actively refer to my notes from class.*	0.324 (*p*=0.047)
The most valid ways of solving a problem are the ones discussed in class.	*The instructors and TAs are responsible for teaching me how to write a solution that would receive full credit.*	-0.332 (*p*=0.042)
The purpose of math class is to learn new math content.	*It is my responsibility to ask for help when I do not fully understand something.*	0.343 (*p*=0.035)
To receive full credit, my solution must use the same methods used in class.	*Solutions written with formulas or equations are self-explanatory. They do not require written explanations.*	0.344 (*p*=0.034)

In typical math lectures, I write down everything that the instructor writes on the board.	*How important is getting the right answer to receiving credit for a math problem?*	0.349 (*p*=0.032)
How important is it to you to write a solution that your peers could understand?	*How important is it for you to be creative when solving a mathematical problem?*	0.358 (*p*=0.027)
How important is it to you to write a solution that your peers could understand?	*How important is it for you to be able to determine if a peer's solution is correct?*	0.390 (*p*=0.016)

APPENDIX D

IRB APPROVAL AND MODIFICATION APPROVALS

University *of* New Hampshire

Research Integrity Services, Service Building
51 College Road, Durham, NH 03824-3585
Fax: 603-862-3564

01-Feb-2018

Fifty, David
Mathematics Dept, Kingsbury Hall
4A Mathes Cove Rd
Durham, NH 03824

IRB #: 6858
Study: Socio-mathematical Norms of a Pre-calculus Class
Approval Date: 26-Jan-2018

The Institutional Review Board for the Protection of Human Subjects in Research (IRB) has reviewed and approved the protocol for your study as Exempt as described in Title 45, Code of Federal Regulations (CFR), Part 46, Subsection 101(b). Approval is granted to conduct your study as described in your protocol.

Researchers who conduct studies involving human subjects have responsibilities as outlined in the document, *Responsibilities of Directors of Research Studies Involving Human Subjects.* This document is available at http://unh.edu/research/irb-application-resources. Please read this document carefully before commencing your work involving human subjects.

Upon completion of your study, please complete the enclosed Exempt Study Final Report form and return it to this office along with a report of your findings.

If you have questions or concerns about your study or this approval, please feel free to contact me at 603-862-2003 or Julie.simpson@unh.edu. Please refer to the IRB # above in all correspondence related to this study. The IRB wishes you success with your research.

For the IRB,

Julie F. Simpson
Director

cc: File
,
Buchbinder, Orly

University *of* New Hampshire

Research Integrity Services, Service Building
51 College Road, Durham, NH 03824-3585
Fax: 603-862-3564

01-Feb-2018

Fifty, David
Mathematics Dept, Kingsbury Hall
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Upon completion of your study, please complete the enclosed Exempt Study Final Report form and return it to this office along with a report of your findings.

If you have questions or concerns about your study or this approval, please feel free to contact me at 603-862-2003 or Julie.simpson@unh.edu. Please refer to the IRB # above in all correspondence related to this study. The IRB wishes you success with your research.

For the IRB,

Julie F. Simpson

Julie F. Simpson
Director

cc: File
,
Buchbinder, Orly

University *of* New Hampshire

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51 College Road, Durham, NH 03824-3585
Fax: 603-862-3564

14-Dec-2018

Fifty, David
Mathematics Dept, Kingsbury Hall
4A Mathes Cove Rd
Durham, NH 03824

IRB #: 6858
Study: Socio-mathematical Norms of a Pre-calculus Class
Study Approval Date: 26-Jan-2018
Modification Approval Date: 14-Dec-2018
Modification: Changes per 12/10/2018 request

The Institutional Review Board for the Protection of Human Subjects in Research (IRB) has reviewed and approved your modification to this study, as indicated above. Further changes in your study must be submitted to the IRB for review and approval prior to implementation.

Researchers who conduct studies involving human subjects have responsibilities as outlined in the document, *Responsibilities of Directors of Research Studies Involving Human Subjects.* This document is available at http://unh.edu/research/irb-application-resources or from me.

Note: IRB approval is separate from UNH Purchasing approval of any proposed methods of paying study participants. Before making any payments to study participants, researchers should consult with their BSC or UNH Purchasing to ensure they are complying with institutional requirements. If such institutional requirements are not consistent with the confidentiality or anonymity assurances in the IRB-approved protocol and consent documents, the researcher may need to request a modification from the IRB.

If you have questions or concerns about your study or this approval, please feel free to contact Melissa McGee at 603-862-2005 or melissa.mcgee@unh.edu. Please refer to the IRB # above in all correspondence related to this study.

For the IRB,

Julie F. Simpson
Director

cc: File
McCrone, Sharon
Buchbinder, Orly

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25-Jan-2019

Fifty, David
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4A Mathes Cove Rd
Durham, NH 03824

IRB #: 6858
Study: Socio-mathematical Norms of a Pre-calculus Class
Study Approval Date: 26-Jan-2018
Modification Approval Date: 25-Jan-2019
Modification: Change procedure to anonymize survey responses (MOD 3)

The Institutional Review Board for the Protection of Human Subjects in Research (IRB) has reviewed and approved your modification to this study, as indicated above. Further changes in your study must be submitted to the IRB for review and approval prior to implementation.

Researchers who conduct studies involving human subjects have responsibilities as outlined in the document, *Responsibilities of Directors of Research Studies Involving Human Subjects.* This document is available at http://unh.edu/research/irb-application-resources or from me.

Note: IRB approval is separate from UNH Purchasing approval of any proposed methods of paying study participants. Before making any payments to study participants, researchers should consult with their BSC or UNH Purchasing to ensure they are complying with institutional requirements. If such institutional requirements are not consistent with the confidentiality or anonymity assurances in the IRB-approved protocol and consent documents, the researcher may need to request a modification from the IRB.

If you have questions or concerns about your study or this approval, please feel free to contact Melissa McGee at 603-862-2005 or melissa.mcgee@unh.edu. Please refer to the IRB # above in all correspondence related to this study.

For the IRB,

Julie F. Simpson
Director

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McCrone, Sharon
Buchbinder, Orly