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On the Continuous Dependence of the Stationary Distribution of a Piecewise Deterministic Markov Process on its Jump Intensity

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Abstract. We examine a piecewise deterministic Markov process, whose whole randomness stems from the jumps, which occur at the random time points according to a Poisson process, and whose post-jump locations are attained by randomly selected transformations of the pre-jumps states. Between the jumps, the process is deterministically driven by a continuous semiflow. The aim of the paper is to establish the continuous dependence of the invariant measure of this process on the jump intensity.

INTRODUCTION

Piecewise deterministic Markov processes (PDMPs) may successfully serve as stochastic models for various phenomena in the real world, such as those appearing in resource allocation and service provisioning (cf. [1]), biology (cf. [3]), as well as population dynamics ([2]). The fundamentals of asymptotic properties of transition semigroups associated with PDMPs have attracted considerable attention. Most common results concentrate, however, on processes evolving on a locally compact space (see e.g. [4]). The theory for the general case of a non-locally compact Polish state space has been less developed so far (cf. [5, 6, 7]).

Here, we examine a special case of the PDMP described in [5, 6], whose deterministic motion between jumps depends on a single continuous semiflow, and any post-jump location is attained by a continuous transformation of the pre-jump state, randomly selected (with a place-dependent probability) among all possible ones. It is assumed that the jump times of the PDMP coincide with those of a homogeneous Poisson process. Random dynamical systems of this type constitute a mathematical framework for certain particular biological models, such as those for gene expression [3] or cell division [7]. The aim of the paper is to establish the continuous (in the Fortet-Mourier distance) dependence of the invariant probability measures of both the aforementioned PDMP and the Markov chain given by its post-jump locations on the jump intensity.

Let $(H, \|\cdot\|)$ be a separable Banach space, and let X be an arbitrary closed subset of H, equipped with the σ-field $\mathcal{B}(X)$ of all its Borel subsets. By $(BM(X), \|\cdot\|_\infty)$ we denote the Banach space of all bounded Borel measurable functions $f : X \to \mathbb{R}$ endowed with the supremum norm. Further, let $BC(X)$ and $BL(X)$ stand for the subspaces of $BM(X)$ consisting of continuous and Lipschitz-continuous functions, respectively. Moreover, define $\|\cdot\|_{BL}$ as $\|f\|_{BL} := \max\{\|f\|_\infty, |f|_{Lip}\}$, $f \in BL(X)$, with $|f|_{Lip}$ being the minimal Lipschitz constant of f, and note that $\|\cdot\|_{BL}$ defines a norm in $BL(X)$, for which it is a Banach space.

We will write $(\mathcal{M}_{sig}(X), \|\cdot\|_{TV})$ for the Banach space of all finite signed Borel measures on X, endowed with the total variation norm $\|\cdot\|_{TV}$, which can be expressed as $\|\mu\|_{TV} := |\mu|(X) = \sup\{|\langle f, \mu\rangle| : f \in BM(X), \|f\|_\infty \leq 1\}$ for any $\mu \in \mathcal{M}_{sig}(X)$, where $\langle f, \mu\rangle := \int_X f(x)\,\mu(dx)$, and $|\mu|$ stands for the absolute variation of μ. The symbol $\mathcal{M}_1(X)$ will be used to denote the subset of $\mathcal{M}_{sig}(X)$, consisting of all probability measures. Moreover, we will write $\mathcal{M}_{1,1}(X)$ for the set $\{\mu \in \mathcal{M}_1(X) : \langle \|\cdot\|, \mu \rangle < \infty\}$.

Now, for any $\mu \in \mathcal{M}_{sig}(X)$, we define the linear functional $I_\mu : BL(X) \to \mathbb{R}$ by $I_\mu(f) = \langle f, \mu \rangle$ for $f \in BL(X)$. One may prove that $I_\mu \in BL(X)^*$ for every $\mu \in \mathcal{M}_{sig}(X)$, where $BL(X)^*$ stands for the dual space of $(BL(X), \|\cdot\|_{BL})$ with the operator norm $\|\cdot\|_{BL}^*$ given by $\|\varphi\|_{BL}^* := \sup\{|\varphi(f)| : f \in BL(X), \|f\|_{BL} \leq 1\}$ for any $\varphi \in BL(X)^*$.

According to [8, Lemma 6] the map $\mathcal{M}_{sig}(X) \ni \mu \mapsto I_\mu \in BL(X)^*$ is injective, whence $(\mathcal{M}_{sig}(X), \|\cdot\|_{TV})$ may be embedded into $(BL(X)^*, \|\cdot\|_{BL}^*)$. This, in turn, allows for identifying any measure $\mu \in \mathcal{M}_{sig}(X)$ with the functional $I_\mu \in BL(X)^*$. In view of this, $\|\cdot\|_{BL}^*$ induces a norm on $\mathcal{M}_{sig}(X)$. Such a norm is called the Fortet-Mourier (or dual bounded Lipschitz) norm (cf. [9]), and it will be denoted by $\|\cdot\|_{FM}$, so that

$$\|\mu\|_{FM} := \left\|I_\mu\right\|_{BL}^* = \sup\{|\langle f, \mu \rangle| : f \in BL(X), \|f\|_{BL} \leq 1\} \quad \text{for any} \quad \mu \in \mathcal{M}_{sig}(X).$$

Let us also indicate that $\|\mu\|_{FM} \leq \|\mu\|_{TV}$ for any $\mu \in \mathcal{M}_{sig}(X)$, and the equality holds if μ is non-negative.

It can be shown that $\mathrm{cl}\mathcal{M}_{sig}(X)$, endowed with the appropriate restriction of $\|\cdot\|_{BL}^*$, is a separable Banach space (cf. [10, Corollary 2.3.10]). Moreover, from [10, Theorem 2.3.22] it follows that each member κ of the dual space $(\mathrm{cl}\mathcal{M}_{sig}(X))^*$ can be represented by some $f \in BL(X)$, so that $\kappa(\mu) = \langle f, \mu \rangle$ for any $\mu \in \mathcal{M}_{sig}(X)$. These observations make the norm $\|\cdot\|_{BL}^*$ convenient for integrating (in the Bochner sense) functions with values in $\mathcal{M}_{sig}(X)$ (viewed as maps into the closure of this space). The main reason for this is that, in view of the above, the Pettis measurability theorem (cf. [10, Proposition 3.2.2]) allows for a relatively easy verification of strong measurability of such functions. In addition to this, [10, Proposition 3.2.5] provides a simple condition for integrability, as well as for ensuring that the resulting integral is a member of $\mathcal{M}_{sig}(X)$.

Finally, it is well-known (cf. [8, Theorem 18]) that, for any measures $\mu_n, \mu \in \mathcal{M}_1(X)$, $n \in \mathbb{N}$, the limit $\lim_{n\to\infty} \|\mu_n - \mu\|_{FM} = 0$ holds if and only if $\mu_n \xrightarrow{w} \mu$, as $n \to \infty$, that is $\lim_{n\to\infty}\langle f, \mu_n \rangle = \langle f, \mu \rangle$ for each $f \in BC(X)$.

In the end, let us recall a few basic definitions from the theory of Markov operators. A map $\Pi : X \times \mathcal{B}(X) \to [0, 1]$ is called a stochastic kernel if, for any fixed $A \in \mathcal{B}(X)$, $x \mapsto \Pi(x, A)$ is a Borel measurable map on X, and, for any fixed $x \in X$, $A \mapsto \Pi(x, A)$ is a probability Borel measure on X. Given such a kernel Π, we can define the corresponding regular Markov operator $P : \mathcal{M}_{sig}(X) \to \mathcal{M}_{sig}(X)$ by $P\mu(A) = \int_X \Pi(x, A)\,\mu(dx)$ for $\mu \in \mathcal{M}_{sig}(X)$, $A \in \mathcal{B}(X)$.

A regular Markov operator P is said to be Feller if the map $x \mapsto \langle f, P\delta_x \rangle$ is continuous for any $f \in BC(X)$. A measure $\mu^* \in \mathcal{M}_1(X)$ is said to be invariant for P whenever $P\mu^* = \mu^*$. We will say that the operator P is exponentially ergodic in the Fortet-Mourier distance if it admits a unique invariant probability measure μ^* and there exists $q \in (0, 1)$ such that, for any $\mu \in \mathcal{M}_1(X)$ and some constant $C(\mu)$, we have $\|P^n\mu - \mu^*\|_{FM} \leq C(\mu)q^n$ for all $n \in \mathbb{N}$.

A regular Markov semigroup $(P(t))_{t\geq 0}$ is a family of regular Markov operators $P(t) : \mathcal{M}_{sig}(X) \to \mathcal{M}_{sig}(X)$, $t \geq 0$, which form a semigroup (under composition) with the identity transformation $P(0)$ as the unity element. Provided that $P(t)$ is Feller for every $t \geq 0$, the semigroup $(P(t))_{t\geq 0}$ is said to be Feller, too. If, for some $\mu^* \in \mathcal{M}_1(X)$, we have $P(t)\mu^* = \mu^*$ for every $t \geq 0$, then we call μ^* an invariant measure of $(P(t))_{t\geq 0}$.

A STOCHASTIC MODEL FOR GENE EXPRESSION

Let $(\Theta, \mathcal{B}(\Theta), \vartheta)$ stand for a topological measure space with a σ-finite Borel measure ϑ. For any $\lambda > 0$, we consider a PDMP that evolves on X through random jumps, occuring at the jump times of a homogeneous Poisson process with intensity λ, which is defined as follows. Any post-jump state is attained by a transformation drawn from a given collection $\{w_\theta : \theta \in \Theta\}$ of maps from X to itself. The choice of w_θ depends on the pre-jump state, say $x \in X$. More specifically, it is determined by a probability density function $\theta \mapsto p(x, \theta)$. We require both functions $(x, \theta) \mapsto p(x, \theta)$ and $(x, \theta) \mapsto w_\theta(x)$ to be continuous. Between the jump times, the process is deterministically driven by a continuous semiflow $S : [0, \infty) \times X \to X$.

Let us now describe the model more formally. For any $\mu \in \mathcal{M}_1(X)$ and any $\lambda > 0$, let us define, on a suitable probability space $(\Omega, \mathcal{A}, \mathbb{P}_\mu)$, a stochastic proces $(X_n)_{n\in\mathbb{N}_0}$ with initial distribution μ, by setting

$$X_{n+1} = w_{\theta_{n+1}}\left(S\left(\Delta t_{n+1}, X_n\right)\right) \quad \text{with} \quad \Delta t_{n+1} = t_{n+1} - t_n \quad \text{for every} \quad n \in \mathbb{N}_0,$$

where $(t_n)_{n\in\mathbb{N}_0}$ and $(\theta_n)_{n\in\mathbb{N}}$ are the sequences of random variables, with values in $[0, \infty)$ and Θ, respectively, defined in such a way that $(t_n)_{n\in\mathbb{N}_0}$ is strictly increasing, $t_0 = 0$, $\lim_{n\to\infty} t_n = \infty$ $\mathbb{P}_\mu$-a.s., and

$$\mathbb{P}_\mu\left(\Delta t_{n+1} \leq t \,|\, V_n\right) = 1 - e^{-\lambda t}, \quad \mathbb{P}_\mu\left(\theta_{n+1} \in B \,|\, S\left(\Delta t_{n+1}, X_n\right) = x,\ V_n\right) = \int_B p(x, \theta)\,\vartheta(d\theta) \quad \text{for} \quad t \geq 0,\ x \in X,\ B \in \mathcal{B}(\Theta),$$

with $V_0 := X_0$ and $V_n := (V_0, t_1, \ldots, t_n, \theta_1, \ldots, \theta_n)$ for any $n \in \mathbb{N}$. Assuming that Δt_{n+1} and θ_{n+1} are conditionally independent given V_n, it is easy to verify that $(X_n)_{n\in\mathbb{N}_0}$ is a time-homogenous Markov chain with transition law

$$\Pi_\lambda(x,A) = \int_0^\infty \lambda e^{-\lambda t} \int_\Theta p(S(t,x),\theta)\, \delta_{w_\theta(S(t,x))}(A)\, \vartheta(d\theta)\, dt \quad \text{for} \quad x \in X,\ A \in \mathcal{B}(X),$$

that is $\Pi_\lambda(X_n, A) = \mathbb{P}_\mu(X_{n+1} \in A \,|\, X_n)$ for any $A \in \mathcal{B}(X)$, $n \in \mathbb{N}_0$. By P_λ we will denote the Markov operator corresponding to Π_λ.

Further, on the same probability space, we define the Markov process $(X(t))_{t\geq 0}$ by setting $X(t) = S\,(t - t_n, X_n)$ for any $t \in [t_n, t_{n+1})$, $n \in \mathbb{N}_0$. Let $(P_\lambda(t))_{t\geq 0}$ denote the transition semigroup of the process $(X(t))_{t\geq 0}$, which means that, for any $t \geq 0$, $P_\lambda(t)$ is the Markov operator corresponding to the stochastic kernel $\Pi_\lambda(t)$ satisfying

$$\Pi_\lambda(t)(x,A) = \mathbb{P}_\mu\left(X(s+t) \in A \,|\, X(s) = x\right) \quad \text{for any} \quad A \in \mathcal{B}(X),\ x \in X,\ s \geq 0.$$

In addition to this, we assume that there exist a point $\bar{x} \in X$, a Borel measurable function $J : X \to [0,\infty)$ and constants $\alpha \in \mathbb{R}$, $L, L_w, L_p, \underline{\lambda}, \overline{\lambda}, \beta > 0$, such that

$$LL_w + \frac{\alpha}{\lambda} < 1 \quad \text{for each} \quad \lambda \in [\underline{\lambda}, \overline{\lambda}], \tag{1}$$

and, for any $x, y \in X$, the following conditions hold:

$$\sup_{x\in X} \int_0^\infty e^{-\underline{\lambda} t} \int_\Theta p\,(S\,(t,x),\theta)\, \|w_\theta\,(S\,(t,\bar{x}))\|\, \vartheta(d\theta)\, dt < \infty, \tag{2}$$

$$\|S\,(t,x) - S\,(t,y)\| \leq Le^{\alpha t}\|x - y\| \quad \text{for} \quad t \geq 0, \tag{3}$$

$$\|S\,(t,x) - S\,(s,x)\| \leq (t-s)e^{\max\{\alpha s, \alpha t\}} J(x) \quad \text{for} \quad 0 \leq s \leq t, \tag{4}$$

$$\int_\Theta p(x,\theta)\, \|w_\theta(x) - w_\theta(y)\|\, \vartheta(d\theta) \leq L_w \|x-y\|, \quad \int_\Theta |p(x,\theta) - p(y,\theta)|\, \vartheta(d\theta) \leq L_p \|x-y\|, \tag{5}$$

$$\int_{\Theta(x,y)} \min\{p(x,\theta), p(y,\theta)\}\, \vartheta(d\theta) \geq \beta, \quad \text{where} \quad \Theta(x,y) := \{\theta \in \Theta : \|w_\theta(x) - w_\theta(y)\| \leq L_w \|x-y\|\}. \tag{6}$$

Let us now present a few facts concerning the properties of the Markov operator P_λ.

Lemma 1 *Suppose that conditions* (4)-(5) *hold. Then, for any $\lambda > 0$ and any $\mu \in \mathcal{M}_{sig}(X)$ satisfying $\langle J, |\mu|\rangle < \infty$, where J is given in* (4)*, the map $t \mapsto e^{-\lambda t} \int_X \int_\Theta p(S(t,x),\theta)\, \delta_{w_\theta(S(t,x))}\, \vartheta(d\theta)\, \mu(dx)$ is Bochner integrable on* $[0,\infty)$*, and*

$$P_\lambda \mu = \int_0^\infty \lambda e^{-\lambda t} \left(\int_X \int_\Theta p\,(S\,(t,x),\theta)\ \delta_{w_\theta(S(t,x))}\, \vartheta(d\theta)\, \mu(dx) \right) dt.$$

Willing to prove Lemma 1, it suffices to apply [10, Proposition 3.2.2], which allows one to verify the strong measurability of the given map, and further use [10, Proposition 3.2.5].

Lemma 2 *Let $f \in BL(X)$. Upon assuming* (3) *and* (5) *with constants satisfying* (1)*, we have*

$$\left\| \int_X \int_\Theta p\,(S\,(t,x),\theta)\ \delta_{w_\theta(S(t,x))}\, \vartheta(d\theta)\, \mu(dx) \right\|_{FM} \leq \left(1 + \left(L_w + L_p\right) Le^{\alpha t}\right) \|\mu\|_{FM} \quad \textit{for any} \quad \mu \in \mathcal{M}_{sig}(X),\ t \geq 0.$$

In order to prove Lemma 2, one only needs to observe that the integrand of the integral with respect to μ, say $x \mapsto h(x)$, satisfies $\|h\|_\infty \leq 1$ and $|h|_{Lip} \leq (L_w + L_p)Le^{\alpha t}$.

Further, Lemmas 1 and 2 allow us to prove the following statement.

Lemma 3 *Let $\mathcal{M}_{sig}(X)$ be endowed with the norm $\|\cdot\|_{FM}$, and suppose that conditions* (3)-(5) *hold with constants satisfying* (1)*. Then, the map $(\max\{0,\alpha\},\infty) \times \mathcal{M}_{sig}(X) \ni (\lambda,\mu) \mapsto P_\lambda\, \mu \in \mathcal{M}_{sig}(X)$ is jointly continuous.*

Finally, let us also summarize the results on existence and uniqueness of invariant measures for P_λ and $(P_\lambda(t))_{t\geq 0}$.

Theorem 4 *Suppose that conditions* (2), (3) *and* (5),(6) *hold with constants satisfying* (1)*. Then, P_λ possesses the unique invariant measure $\mu_\lambda^* \in \mathcal{M}_{1,1}(X)$, and there exists $q \in (0,1)$ such that, for any $\mu \in \mathcal{M}_{1,1}(X)$, we can choose $C(\mu) < \infty$ for which $\|P_\lambda^n \mu - \mu_\lambda^*\|_{FM} \leq C(\mu) q^n$ for every $\lambda \in [\underline{\lambda}, \overline{\lambda}]$. In particular, the convergence $\|P_\lambda^n \mu - \mu_\lambda^*\|_{FM} \to 0$ is then uniform with respect to λ.*

Note that, in view of [5, Theorem 4.1], it is sufficient to prove that the constants $C(\mu)$ and q do not depend on the jump rate λ. This follows simply by observing that $P_\lambda = \overline{P}_{\overline{\lambda}}$, where $\overline{P}_{\overline{\lambda}}$ is the transition operator of the instance of our model with the semiflow $\overline{S}(t,x) := S(\overline{\lambda}\lambda^{-1}t, x)$ in place of S.

Theorem 5 ([5, Theorem 4.4]) *Let ϑ be a finite Borel measure on Θ. Then, for any $\lambda > 0$, there is a one-to-one correspondence between invariant measures for P_λ and those for $(P_\lambda(t))_{t\geq 0}$. Moreover, if $\mu_\lambda^* \in \mathcal{M}_1(X)$ is an invariant measure of P_λ, then $\nu_\lambda^* := G_\lambda \mu_\lambda^*$, where $G_\lambda \mu(A) = \int_X \int_0^\infty \lambda e^{-\lambda t} \delta_{S(t,x)}(A)\, dt\, \mu(dx)$ for $\mu \in \mathcal{M}_1(X)$ and $A \in \mathcal{B}(X)$, is an invariant measure of $(P_\lambda(t))_{t\geq 0}$.*

MAIN RESULTS

We are now in a position to state the main results of this paper, together with the sketches of their proofs.

Theorem 6 *Suppose that conditions (2)-(6) hold with constants satisfying (1). Moreover, for any $\lambda \in [\underline{\lambda}, \overline{\lambda}]$, let μ_λ^* denote the unique invariant probability measure of P_λ. Then, for every $\lambda_0 \in [\underline{\lambda}, \overline{\lambda}]$, we have $\mu_\lambda^* \xrightarrow{w} \mu_{\lambda_0}^*$, as $\lambda \to \lambda_0$.*

Proof. Let $\lambda_0 \in [\underline{\lambda}, \overline{\lambda}]$. Note that, according to Theorem 4, for every $\mu \in \mathcal{M}_1(X)$ and any $\lambda \in [\underline{\lambda}, \overline{\lambda}]$, we have $\|P_\lambda^n \mu - \mu_\lambda^*\|_{FM} \to 0$, as $n \to \infty$. Moreover, the convergence is uniform with respect to λ. Lemma 3, in turn, yields that the map $(\lambda, \mu) \mapsto P_\lambda \mu$ is jointly continuous, and therefore, for any $\mu \in \mathcal{M}_1(X)$ and any $n \in \mathbb{N}_0$, we obtain $\|P_\lambda^n \mu - P_{\lambda_0}^n \mu\|_{FM} \to 0$, as $\lambda \to \lambda_0$. Finally, according to [11, Theorem 7.11], we get

$$\lim_{\lambda\to\lambda_0} \mu_\lambda^* = \lim_{\lambda\to\lambda_0} \lim_{n\to\infty} P_\lambda^n \mu = \lim_{n\to\infty} \lim_{\lambda\to\lambda_0} P_\lambda^n \mu = \lim_{n\to\infty} P_{\lambda_0}^n \mu = \mu_{\lambda_0}^*,$$

where the limits are taken in $(\mathcal{M}_{sig}, \|\cdot\|_{FM})$. This, together with [8, Theorem 18], gives the desired conclusion. □

Theorem 7 *Let ϑ be a finite Borel measure on Θ. Further, suppose that conditions (2)-(6) hold with constants satisfying (1), and, for any $\lambda \in [\underline{\lambda}, \overline{\lambda}]$, let ν_λ^* denote the unique invariant probability measure of $(P_\lambda(t))_{t\geq 0}$. Then, for any $\lambda_0 \in [\underline{\lambda}, \overline{\lambda}]$, we have $\nu_\lambda^* \xrightarrow{w} \nu_{\lambda_0}^*$, as $\lambda \to \lambda_0$.*

Proof. Let $\lambda_0 \in [\underline{\lambda}, \overline{\lambda}]$. Using Theorem 5, for any $\lambda \in [\underline{\lambda}, \overline{\lambda}]$ and any $f \in BL(X)$ satisfying $\|f\|_{BL} \leq 1$, we obtain

$$\left|\left\langle f, \nu_\lambda^* - \nu_{\lambda_0}^* \right\rangle\right| = \left|\left\langle f, G_\lambda \mu_\lambda^* - G_{\lambda_0} \mu_{\lambda_0}^* \right\rangle\right| \leq \int_0^\infty \left|\lambda e^{-\lambda t} - \lambda_0 e^{-\lambda_0 t}\right| dt + \left|\int_0^\infty \lambda_0 e^{-\lambda_0 t} \left\langle f \circ S(t,\cdot), \mu_\lambda^* - \mu_{\lambda_0}^* \right\rangle dt\right|. \quad (7)$$

Observe that, upon assuming (3), $f \circ S(t,\cdot) \in BL(X)$ and $\|f \circ S(t,\cdot)\|_{BL} \leq 1 + Le^{\alpha t}$, whence

$$\left|\int_0^\infty \lambda_0 e^{-\lambda_0 t} \left\langle f \circ S(t,\cdot), \mu_\lambda^* - \mu_{\lambda_0}^* \right\rangle dt\right| \leq \left\|\mu_\lambda^* - \mu_{\lambda_0}^*\right\|_{FM} \int_0^\infty \lambda_0 e^{-\overline{\lambda} t}\left(1 + Le^{\alpha t}\right) dt = \left\|\mu_\lambda^* - \mu_{\lambda_0}^*\right\|_{FM} \left(1 + \frac{L\lambda_0}{\lambda_0 - \alpha}\right).$$

Finally, a suitable estimation of the first integral in (7) leads to $\|\nu_\lambda^* - \nu_{\lambda_0}^*\|_{FM} \leq |\lambda - \lambda_0|(\lambda^{-1} + \lambda_0^{-1}) + c\|\mu_\lambda^* - \mu_{\lambda_0}^*\|_{FM}$ with $c := 1 + L\lambda_0(\lambda_0 - \alpha)^{-1}$, which, combined with the assertion of Theorem 6, completes the proof. □

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