Pseudospectral frequency-domain analysis of rectangular waveguides filled by dielectrics whose permittivity varies continuously along the broad dimension

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Abstract

The calculation of dispersion diagrams and field patterns of metallic rectangular waveguides filled with an inhomogeneous dielectric whose permittivity varies continuously along the broad size of the guide is considered. In general, this problem has no exact solution, thus numerical techniques should be used. In this letter, the pseudospectral frequency-domain (PSFD) method is proposed to address the problem. Starting from the Helmholtz equation, a matrix eigenvalue problem is obtained by applying the collocation technique with Chebyshev polynomials as basis functions. The results obtained are compared with those calculated by the conventional finite-difference frequency-domain method showing that the PSFD technique provides an excellent accuracy.

KEYWORDS

rectangular waveguide, inhomogeneous dielectric, pseudospectral frequency-domain method, Chebyshev polynomials, finite-difference frequency-domain method

1 INTRODUCTION

The recently introduced transformation electromagnetics (also referred to as transformation optics) is a powerful technique for designing novel microwave and optical devices such as electromagnetic cloaks, field concentrators and rotators, planar focusing antennas, waveguide bends and couplers, etc.[1,2] The implementation of transformation electromagnetics devices requires the use of spatially inhomogeneous materials which has boost a renewed interest in artificial materials.[3]

The characterization of waveguiding structures is a fundamental problem in microwave engineering. In this work, we consider the analysis of uniform metallic rectangular waveguides filled with an inhomogeneous dielectric material whose permittivity varies continuously along the transverse direction x, as shown in Figure 1. In general, this problem has no exact solution.[4,5] Hence, numerical techniques are needed to obtain the propagation constants and field patterns of inhomogeneously filled rectangular waveguides.

The numerical analysis of the waveguide depicted in Figure 1 has previously been addressed by means of the Galerkin method.[6,7] A drawback of this approach is that the integrals arising in the formulation should be recalculated for each permittivity profile. Furthermore, only cut-off frequencies are computed. Besides, the formulation used involves the numerical searching for the zeros of a matrix determinant of large dimension, which is a complicated numerical task.

To overcome the abovementioned limitations, in this letter we propose an alternative numerical technique for the analysis of the rectangular waveguide shown in Figure 1. This technique is based on the pseudospectral frequency-domain (PSFD) method, which has been successfully applied to several microwave and optical problems.[8,9] The PSFD method is a global collocation technique that provides high accuracy and flexibility while being free of integral calculations. In addition, not only cut-off frequencies but also dispersion diagrams and field patterns are calculated. For comparison purposes, the results obtained by the PSFD method are compared with those computed by the finite-difference frequency-domain (FDFD) method.[10]

As an alternative to the frequency domain, the dispersion characteristics of uniform waveguiding structures can be analysed by using the pseudospectral time-domain method.[11] However, due to the eigenvalue (resonant) nature of the problem, the spectral analysis of the transient response of the waveguide cross section becomes a difficult task.[12]

2 THEORY

2.1 Differential problem

Consider a metallic rectangular waveguide of dimensions $a \times b$, filled with an inhomogeneous dielectric material whose permittivity varies continuously along the x direction, as shown in Figure 1. The waveguide structure is assumed to be uniform along the propagation direction z. Thus, the time-harmonic electromagnetic fields can be expressed as

$$\boldsymbol{E}(x,y,z) = \boldsymbol{e}(x,y)\exp(-\gamma z)$$
$$\boldsymbol{H}(x,y,z) = \boldsymbol{h}(x,y)\exp(-\gamma z)$$

where $\gamma = \alpha + j\beta$ is the propagation constant, with α and β being the attenuation and phase constants, respectively.

Since the permittivity varies along the x direction only, the waveguide under study supports two different sets of modes: the longitudinal section electric (LSE) modes and the longitudinal section magnetic (LSM) modes.[7] The LSE modes are characterized by $E_x = 0$, and the LSM modes by $H_x = 0$. The LSE modes with no field variation along the y direction reduce to standard transverse electric (TE^z) modes.
The solution for the LSE modes can be obtained by solving the scalar Helmholtz equation for e_y:

$$\frac{\partial^2 e_y}{\partial x^2}+\frac{\partial^2 e_y}{\partial y^2}+\left[k_0^2\varepsilon_r(x)+\gamma^2\right]e_y=0$$

where k_0 is the free-space wavenumber and $\varepsilon_r(x)$ the relative permittivity. By applying the method of separation of variables, e_y is found to be of the form

$$e_y(x,y)=X(x)\cos\left(\frac{n\pi}{b}y\right)$$

with $n = 0,1,2,3,\ldots$, and where $X(x)$ is the solution of

$$\frac{d^2X(x)}{dx^2}+\left[k_0^2\varepsilon_r(x)-\left(\frac{n\pi}{b}\right)^2+\gamma^2\right]X(x)=0 \quad (1)$$

subjected to the boundary condition $X(0)=X(a)=0$.

The solution for the LSM modes can be determined by solving the scalar Helmholtz equation for h_y. Analogously to the LSE case, after applying the method of separation of variables, h_y can be expressed as

$$h_y(x,y)=X(x)\sin\left(\frac{n\pi}{b}y\right)$$

with $n = 1,2,3,\ldots$, and where $X(x)$ is the solution of

$$\frac{d^2X(x)}{dx^2}-\frac{d\ln\varepsilon_r(x)}{dx}\frac{dX(x)}{dx}+\left[k_0^2\varepsilon_r(x)-\left(\frac{n\pi}{b}\right)^2+\gamma^2\right]X(x)=0$$

subjected to the boundary conditions

$$\frac{dX(0)}{dx}=\frac{dX(a)}{dx}=0.$$

It is worth noting that if the permittivity varies with both transverse coordinates, i.e. $\varepsilon_r=\varepsilon_r(x,y)$, the waveguide solutions are no longer LSE and LSM modes, in general. Consequently, in such a case, the vector Helmholtz equation (instead of the scalar one) need to be solved.[10]

2.2 Numerical solution: the PSFD method

To illustrate the application of the Chebyshev PSFD method to the present problem, we consider the LSE case. Since the Chebyshev polynomials are defined for $x\in[-1,1]$, we initially consider this mathematical interval as the domain of solution.
The Chebyshev PSFD method is based on approximating the unknown function $X(x)$ in (1) as a linear combination of Chebyshev polynomials as

$$X(x)=\sum_{p=0}^{N}a_pT_p(x) \quad (2)$$

with $x\in[-1,1]$, where a_p are unknown coefficients and $T_p(x)=\cos\left[p\cos^{-1}(x)\right]$ is the pth-order Chebyshev polynomial.[13]

The solution interval $[-1,1]$ is discretized by considering $N+1$ collocation points defined as

$$x_i = \cos\left(\frac{i\pi}{N}\right)$$

with $i = 0,1,...,N$. These points are known as Chebyshev-Gauss-Lobatto points. They are composed of the extrema of $T_N(x)$ along with the endpoints -1 and 1.

An alternative and equivalent way to express (2) is as a linear combination of Chebyshev cardinal basis functions:

$$X(x) = \sum_{i=0}^{N} X_i C_i(x) \tag{3}$$

where

$$C_i(x) = (-1)^{i+1} \frac{(1-x^2)}{c_i N^2 (x-x_i)} \frac{dT_N}{dx}$$

with $c_0 = c_N = 2$ and $c_i = 1$ for $i = 1,2,...,N-1$. The unknown coefficients X_i in (3) are the values of the function X at the collocation points, i.e. $X_i \equiv X(x_i)$.

By using (3), the first derivative of X at the collocation point x_k is calculated simply as

$$\left.\frac{dX(x)}{dx}\right|_{x=x_k} = \sum_{i=0}^{N} X_i \left.\frac{dC_i(x)}{dx}\right|_{x=x_k}$$

Then, the first-order derivative of X at the whole set of collocation points can be expressed in matrix-vector form as the product $\mathbf{D}_x\mathbf{X}$, where $\mathbf{X} = [X_0, X_1,..., X_N]^T$ and $\mathbf{D}_x$ is the first-order Chebyshev differentiation matrix of dimension $(N+1)\times(N+1)$. The elements of $\mathbf{D}_x$ are the derivatives of the cardinal functions at the grid points

$$\mathbf{D}_x(k,i) = \left.\frac{dC_i(x)}{dx}\right|_{x=x_k}$$

which are given by[11]

$$\mathbf{D}_x(k,i) = \begin{cases} \frac{2N^2+1}{6} & k = i = 0 \\ -\frac{2N^2+1}{6} & k = i = N \\ -\frac{x_k}{2(1-x_k^2)} & 0 < k = i < N \\ \frac{(-1)^{i+k} c_k}{c_i (x_k - x_i)} & k \neq i \end{cases}$$

where $c_i = 2$ for $i = 0, N$ and $c_i = 1$ otherwise. The second-order differentiation matrix can be calculated simply as $\mathbf{D}_{xx} = \mathbf{D}_x\mathbf{D}_x$.

Now, the discretization of (1) is carried out by replacing each term of this equation by its pseudospectral counterpart, which leads to the following ordinary eigenvalue problem for γ^2:

$$\mathbf{AX} = -\gamma^2 \mathbf{X} \tag{4}$$

with

$$\mathbf{A} = \left(\frac{2}{a}\right)^2 \mathbf{D}_{xx} + k_0^2 \mathbf{E}_r - \left(\frac{n\pi}{b}\right)^2 \mathbf{U}$$

where $\mathbf{E}_r = \mathrm{diag}[\varepsilon_r(x_0), \varepsilon_r(x_1), ..., \varepsilon_r(x_N)]$ is a diagonal matrix with the relative permittivity values and $\mathbf{U}$ is the $(N+1)\times(N+1)$ identity matrix. Note that the solution interval for the physical problem is $[0,a]$. Thus, the differentiation matrix $\mathbf{D}_x$ calculated on $[-1,1]$ has been scaled by the factor $2/a$.

The boundary conditions $X_0 = X_N = 0$ can easily be imposed by simply removing the first and last rows and columns of (4), which leads to

$$\overline{\mathbf{A}}\overline{\mathbf{X}} = -\gamma^2 \overline{\mathbf{X}}$$

where $\overline{\mathbf{A}}$ denotes the restricted $\mathbf{A}$ matrix and $\overline{\mathbf{X}}$ contains the elements of $\mathbf{X}$ at the interior grid points only.
The direct computing of the cut-off frequencies can be done by simple letting $\gamma = 0$ in (4) and rewriting the resulting expression as a generalized eigenvalue problem for k_0^2:

$$\left[\left(\frac{2}{a}\right)^2 \mathbf{D}_{xx} - \left(\frac{n\pi}{b}\right)^2 \mathbf{U}\right]\mathbf{X} = -k_0^2 \mathbf{E}_r \mathbf{X} \tag{5}$$

Since we have forced the condition $\gamma = 0$, k_0 in (5) should be understood as the cut-off wavenumber.

3 RESULTS

To illustrate the accuracy of the PSFD method, the calculation of the cut-off wavelength, λ_c , of the TE_{10}, TE_{50}, $TE_{10,0}$ and $TE_{15,0}$ modes of an empty rectangular waveguide is firstly considered. Figure 2 shows a log-log graph of the relative error in the cut-off wavelength as a function of N. As it can be seen, the PSFD errors decrease exponentially until they reach values around $10^{-13}\%$. Beyond this level, round-off errors are dominant and convergence curves become nearly flat. For comparison purposes, the results obtained by using the FDFD method[10] are also shown in Figure 2. As expected, the FDFD method exhibits a linear convergence rate with slope -2, which is a much poorer behaviour than the one obtained with the PSFD method.

Secondly, we consider a WR75 rectangular waveguide filled with a dielectric material whose relative permittivity varies linearly with the position as $\varepsilon_r(x) = 1 - d(x/a)$, where d is a parameter ranging form -1 to 2. Note that $d = 0$ corresponds to the empty case and for $d > 1$ the waveguide is partially filled with a negative permittivity. Figure 3 depicts the normalized cut-off wavelength, λ_c / a, against the parameter d for the first LSE_{mn} modes. As expected, λ_c decreases as d increases. The field pattern for the E_y component of the LSE_{10} (TE_{10}) mode is shown in Figure 4 for $d = 2$. In this case, the permittivity is positive for $x/a < 0.5$

and negative for $x/a > 0.5$. It can be seen that the electric field tends to concentrate in the region with positive permittivity.

As a third example, the dispersion curves for the first LSE_{mn} modes of a WR75 rectangular waveguide filled by a dielectric with parabolic permittivity profile $\varepsilon_r(x) = 1 - (x/a)^2$ is shown in Figure 5. The normalized phase constant, β / k_0, is plotted for propagating modes and the normalized attenuation constant, α / k_0, for modes under cut-off. The results obtained by the PSFD method with $N = 14$ are compared with those calculated by the FDFD formulation with 80 cells. Although both methods provide the same results within the scale of the plot, if, for instance, we focus on the LSE_{40} mode at 15 GHz, it is found that the PSFD method computes its attenuation constant providing 6 exact figures $(\alpha = 607.766 \text{ m}^{-1})$ while the FDFD method only provides 3 of them.

Finally, we consider a WR75 rectangular waveguide filled with a dielectric with Gaussian permittivity profile

$$\varepsilon_r(x) = \varepsilon_{r2} + (\varepsilon_{r1} - \varepsilon_{r2}) e^{-10^2[(x/a)-0.5]^2}$$

where $\varepsilon_{r1} = 1$ and $\varepsilon_{r2} = 9$. Figure 6 illustrates the relative error in the cut-off wavelength of the TE_{10} and the TE_{20} modes as a function of N. For the sake of comparison, the results obtained by the PSFD and the FDTD methods are both shown. The relative error has been calculated by using the cut-off wavelength computed by the PSFD method with $N = 50$ as *exact* value. Even though we are now dealing with a non polynomial permittivity profile, it can be seen that the convergence curves exhibit the same behaviour as in the homogeneous case shown in Figure 2 and discussed above.

4 CONCLUSION

The pseudospectral frequency-domain (PSFD) method has been successfully applied to the analysis of rectangular waveguides filled with an inhomogeneous dielectric whose permittivity varies continuously. Several permittivity profiles such as linear, parabolic and Gaussian profiles have been considered. Starting from the Helmholtz equation, a matrix eigenvalue problem has been obtained for computing cut-off frequencies, dispersion diagrams and field patterns of the waveguide problem. The results obtained have been compared with those calculated by the conventional second-order FDFD method showing that the PSFD technique provides excellent convergence and accuracy.

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Figure Captions

Figure 1. Uniform metallic rectangular waveguide filled with an inhomogeneous dielectric.

Figure 2. Cut-off wavelength relative error as a function of the number of collocation points, N , for several TE_{m0} modes of an empty WR75 waveguide.

Figure 3. Normalized cut-off wavelength, λ_c / a , against the parameter d for the first LSE_{mn} modes of a WR75 waveguide filled with a linear relative permittivity.

Figure 4. Field pattern for the E_y component of the LSE_{10} mode for the case $d = 2$ in Figure 3.

Figure 5. Dispersion curves for the first LSE_{mn} modes of a WR75 waveguide filled with a parabolic relative permittivity. The positive part of the y axis represents β / k_0 and the negative part represents $-\alpha / k_0$.

Figure 6. Cut-off wavelength relative error as a function of the number of collocation points, N , for a WR75 waveguide filled with a Gaussian permittivity profile.

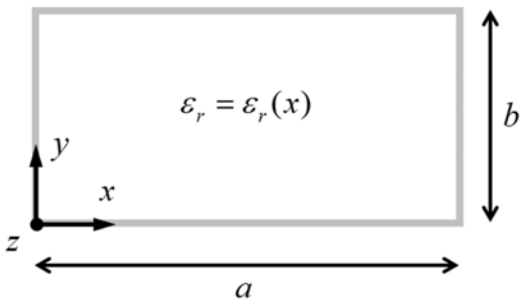

Figure 1.

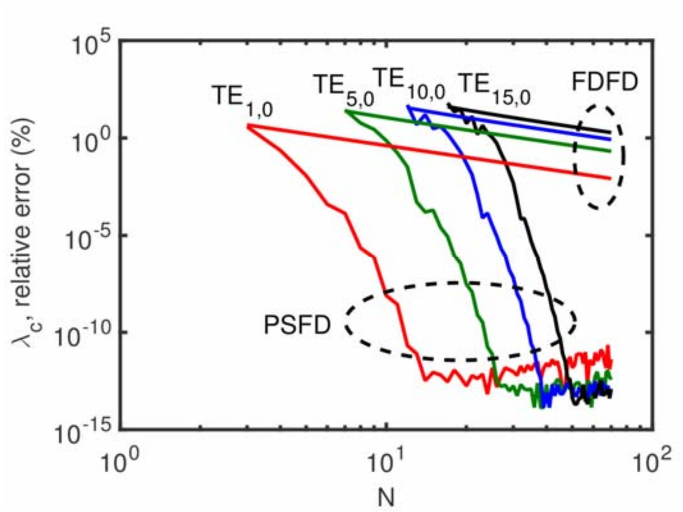

Figure 2.

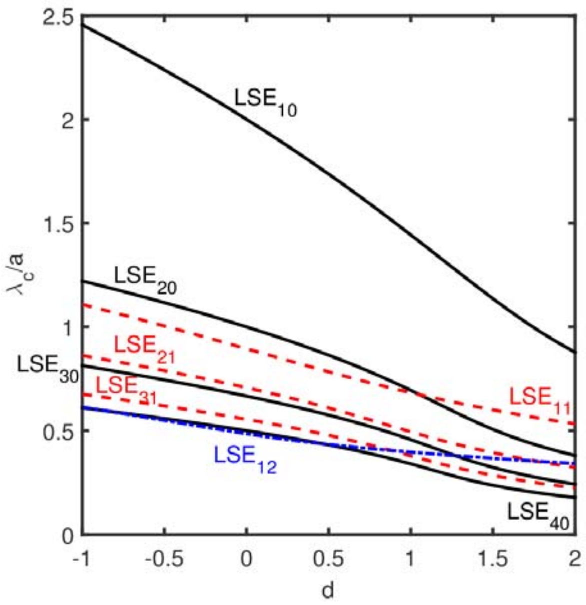

Figure 3.

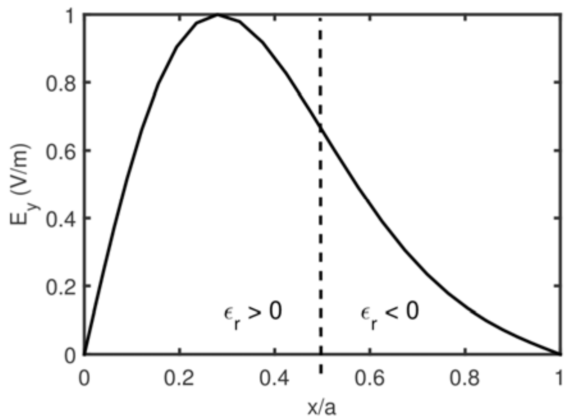

Figure 4.

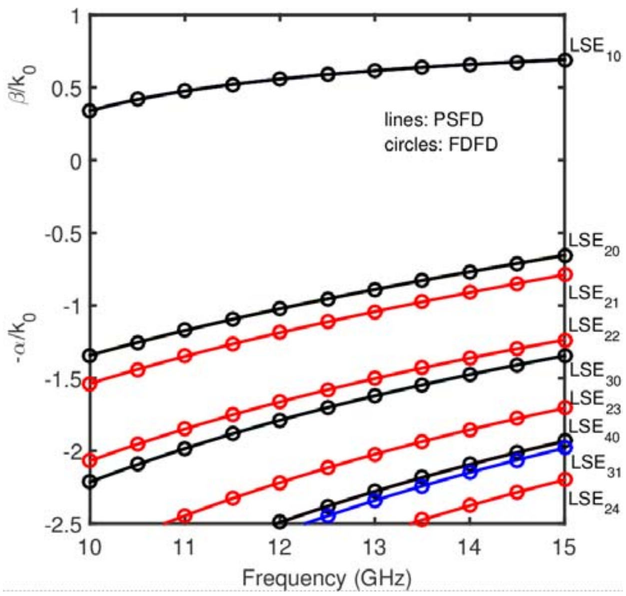

Figure 5.

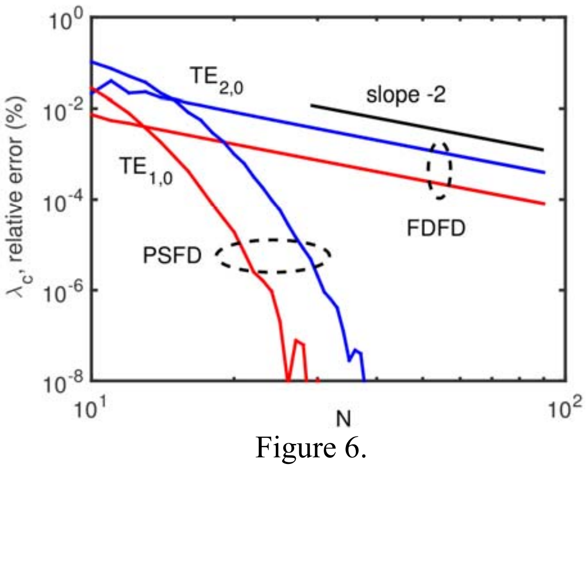

Figure 6.