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Classification of Almost Norden Golden Manifolds

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Abstract

Almost Norden manifolds were classified by means of the Levi–Civita connection by Ganchev and Borisov and by means of the canonical connection by Ganchev and Mihova. This canonical connection was obtained using the potential tensor of the Levi–Civita of the twin metric. We recall that this canonical connection is the well-adapted connection, obtaining its explicit expression for some classes and being able to obtain two equivalent classifications of almost Norden golden manifolds.

Keywords Almost Norden manifold · Almost Norden golden manifold · Well-adapted connection · Classification

Mathematics Subject Classification 53C15 · 53C05 · 53C07

1 Introduction

Almost complex and almost golden structures on a manifold are polynomial structures of degree 2, i.e., structures given by a tensor field of type $(1, 1)$ satisfying a polynomial equation of degree 2 (see [12]).

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Definition 1 Let M be a manifold and let Id be the identity tensor field of type $(1, 1)$ on M.

(i) A polynomial structure J of degree 2 on M satisfying $J^2 = -\mathrm{Id}$ is called an almost complex structure. In this case, (M, J) is an almost complex manifold.
(ii) A polynomial structure φ of degree 2 on M satisfying $\varphi^2 = \varphi - \frac{3}{2}\mathrm{Id}$ is called an almost complex golden structure. In this case, (M, φ) is an almost complex golden manifold.

Almost complex golden structures were introduced by Crasmareanu and Hreţcanu in [2]. This kind of structures is the analogue of almost golden structures in the complex case (also introduced in the quoted paper). An almost golden structure on a manifold is also a polynomial structure φ of degree satisfying $\varphi^2 = \varphi + \mathrm{Id}$. The characteristic polynomials of almost golden and almost complex golden structures are $x^2 - x - 1$ and $x^2 - x - \frac{3}{2}$, respectively, whose roots are $\phi = \frac{1+\sqrt{5}}{2}$ and $\bar{\phi} = \frac{1-\sqrt{5}}{2}$ in the golden case, and $\phi_c = \frac{1+\sqrt{5}i}{2}$ and $\bar{\phi}_c = \frac{1-\sqrt{5}i}{2}$ in the complex golden case. Recall that the numbers ϕ and $\bar{\phi}_c$ are called the golden ratio and the complex golden ratio, respectively. These numbers explain the name of both kind of polynomial structures. Since then, almost golden and almost complex golden structures have become in active research field (see, e.g., [1,3,6,10,11] and the references therein).

Almost complex and almost complex golden structures are closely related as it is shown in two of the above-mentioned references.

Proposition 2 [1,11] *Let M be a manifold.*

(i) *Let φ be an almost complex golden structure on M. Then*

$$J_\varphi = \frac{1}{\sqrt{5}}(\mathrm{Id} - 2\varphi) \tag{1}$$

is an almost complex structure on M.
(ii) *Let J be an almost complex structure on M. Then, $\varphi_J = \frac{1}{2}(\mathrm{Id} - \sqrt{5}J)$ is an almost complex golden structure on M.*

We say that J_φ is the almost complex structure induced by φ and φ_J is the almost complex golden structure induced by J.

The previous result has two direct consequences. The first of them is that an almost complex golden manifold has even dimension. The second one and more importantly to this work is that there exists a 1:1 correspondence between almost complex and almost complex golden structures on M because $\varphi_{J_\varphi} = \varphi$ and $J_{\varphi_J} = J$.

Every almost complex structure J induces two almost complex golden structures

$$\varphi_\pm = \frac{1}{2}(\mathrm{Id} \pm \sqrt{5}J),$$

and every almost complex golden structure φ induces two almost complex structures

$$J_\pm = \pm\frac{1}{\sqrt{5}}(\mathrm{Id} - 2\varphi) \tag{2}$$

(see the above-mentioned references). Starting from an almost complex golden φ on M, there is no reason to choose the pair (J_φ, φ_J) as (J_+, φ_-) instead of (J_-, φ_+) in Proposition 2. In both cases, the previous highlighted consequences are true.

In [1,11], the authors also studied almost complex golden structures which admit compatible pseudo-Riemannian metrics. Compatible metrics on almost complex golden manifolds are introduced in the same way that Norden metrics on almost complex manifolds. We recall both notions below.

Definition 3 Let M be a manifold and let g be a pseudo-Riemannian metric on M.

(i) Let (M, J) be an almost complex manifold. The metric g is called a Norden metric on M if it satisfies one of the two equivalent conditions

$$g(JX, Y) = g(X, JY), \quad g(JX, JY) = -g(X, Y),$$

for all vector fields X, Y on M. In this case, (J, g) is an almost Norden structure on M and (M, J, g) is an almost Norden manifold.

(ii) Let (M, φ) be an almost complex golden manifold. The metric g is called a Norden golden metric on M if it satisfies one of the two equivalent conditions

$$g(\varphi X, Y) = g(X, \varphi Y), \quad g(\varphi X, \varphi Y) = g(\varphi X, Y) - \frac{3}{2} g(X, Y),$$

for all vector fields X, Y on M. In this case, (φ, g) is an almost Norden golden structure on M and (M, φ, g) is an almost Norden golden manifold.

Given a pseudo-Riemannian metric g on an almost complex golden manifold (M, φ), identity (1) allows us to set out easily the next equivalence

$$g(\varphi X, Y) = g(X, \varphi Y) \Longleftrightarrow g(J_\varphi X, Y) = g(X, J_\varphi Y), \tag{3}$$

for all vector fields X, Y on M. Then, (M, φ, g) is an almost Norden golden manifold if and only if (M, J_φ, g) is an almost Norden manifold. Thus, if (φ, g) is an almost Norden golden structure on a $2n$-dimensional manifold M, then the metric g has signature (n, n). We say that (J_φ, g) is the almost Norden structure induced by the almost Norden golden structure (φ, g).

Moreover, as direct consequence of Proposition 2 and identity (3), it is possible to extend the 1:1 correspondence between almost complex and almost complex golden structures to the metric case, i.e., one can claim that there exists an 1:1 correspondence between almost Norden and almost Norden golden structures on a manifold.

An almost complex golden manifold (M, φ) is called an integrable complex golden manifold if the Nijenhuis tensor of φ vanishes, i.e., if

$$N_\varphi(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, Y] = 0,$$

for all vector fields X, Y on M. Lemma 22 allows us to claim that the Nijenhuis tensor of the almost complex golden structure φ vanishes if and only if the Nijenhuis tensor of the almost complex structure J_φ induced by φ vanishes too. Thus, (M, φ) is an integrable complex golden manifold if and only if (M, J_φ) is a complex manifold.

In [11], the authors studied holomorphic Norden golden manifolds, which are almost Norden golden manifolds such that the Levi–Civita connection of the pseudo-Riemannian metric parallelizes the almost complex golden structure. They proved that (M, φ, g) is a holomorphic Norden golden manifold if and only if the Levi–Civita connection of g parallelizes the almost complex structure J_φ, i.e., if and only if (M, J_φ, g) is a Kähler Norden manifold (see [11, Prop. 4.3]). Hereunder, we will call Kähler Norden golden manifolds to holomorphic Norden golden manifolds.

These two classes of almost Norden golden manifolds, integrable golden Norden and Kähler Norden golden manifolds, show that one can classify this kind of manifolds using the classification of almost Norden manifolds obtained by Ganchev and Borisov in [7] and the above 1:1 correspondence between almost Norden golden and almost Norden manifolds as follows.

Definition 4 Let (M, φ, g) be an almost Norden golden manifold. We say that the manifold (M, φ, g) belongs to certain class according to the classification of [7] if the almost Norden manifold (M, J_φ, g) belongs to this class of manifolds.

Theorem 9 collects defining conditions of all classes of almost Norden manifolds. As a result, one can claim that (M, J, g) and $(M, -J, g)$ belong to the same class of almost Norden manifolds. Therefore, given an almost Norden golden manifold (M, φ, g), the choice of the pair (J_φ, φ_J) as (J_+, φ_-) or (J_-, φ_+) does not change its classification according to the above definition because $J_- = -J_+$ [see identity (2)].

Our first challenge is to classify almost Norden golden manifolds. We will use the classification shown in [7] to obtain a systematic classification of these manifolds as established in Definition 4.

All classes of almost Norden manifolds, analogously to the classification of almost Hermitian, almost para-Hermitian and almost product Riemannian manifolds obtained by Gray and Hervella, Naveira and Gadea and Masqué in [13], [14] and [8], respectively, have characteristic conditions described by means of tensors and 1-forms defined from the Levi–Civita connection of the manifold. Then, roughly speaking, the Levi–Civita connection allows to classify this kind of manifolds.

The classification of almost Norden manifolds obtained by Ganchev and Borisov is not the unique classification of this kind of manifolds by means of a connection. In [9], Ganchev and Mihova classify these manifolds using tensors and 1-forms defined from the canonical connection of an almost Norden manifold, which is an adapted connection.

Let (M, J, g) be an almost Norden manifold. A connection ∇ on M is an adapted connection to (J, g) if parallelizes the almost complex structure J and the pseudo-Riemannian metric g, i.e., $\nabla J = 0$ and $\nabla g = 0$. The canonical connection of (M, J, g) is the unique adapted connection ∇^{w} to (J, g) satisfying the following condition:

$$g(T^{\mathrm{w}}(X, Y), Z) - g(T^{\mathrm{w}}(Z, Y), X) = g(T^{\mathrm{w}}(JX, Y), JZ) - g(T^{\mathrm{w}}(JZ, Y), JX), \tag{4}$$

for all vector fields X, Y, Z on M, being T^{w} the torsion tensor of ∇^{w} (see [9, Sec. 3]). In [5, Theor. 4.4], this connection is called the well-adapted connection of (M, J, g).

Adapted connections have been studied on almost golden Riemannian manifolds (see [2,6]). However, the notion of adapted connection on almost Norden golden manifolds has not yet been introduced. We will introduce this notion analogously to the notion of adapted connection on almost Norden manifolds. Let (M, φ, g) be an almost Norden golden manifold. A connection ∇ on M is an adapted connection to (φ, g) if it satisfies $\nabla\varphi = 0$ and $\nabla g = 0$. We will prove that a connection ∇ is adapted to (φ, g) if and only if ∇ is an adapted connection to its induced almost Norden structure (J_φ, g). This result allows us to introduce the well-adapted connection of an almost golden Norden manifold.

Note that the structure (φ, g) and its induced almost Norden structure (J_φ, g) share the well-adapted connection. Because of this fact, starting from the classification of the almost Norden manifolds obtained by Ganchev and Mihova in [9], we will characterize every class of almost Norden golden manifolds by means of conditions over the torsion tensor of the well-adapted connection. This classification constitutes our second challenge.

The organization of the paper is as follows:

Section 2 is devoted to almost Norden manifolds. First, we will recall the definition of tensors and 1-forms that are involved in the defining conditions of the different classes of this kind of manifolds. We will complete this section showing the classification of almost Norden manifolds obtained in [7] (Theorem 9). We will also recall the classification of this kind of manifolds obtained in [9] using the well-adapted connection (Theorem 11).

In Sect. 3, we will focus on the well-adapted of an almost Norden manifold. We will obtain the expression of different tensors and 1-forms defined from the Levi–Civita connection of the metric by means of the well-adapted connection that are not showed in [9]. We will finish this section obtaining the formula of the well-adapted connection in the integrable and quasi-Kähler cases (Remarks 15 and 16).

Section 4 is devoted to adapted connections on almost Norden golden manifolds. We will prove that a connection is an adapted connection to an almost Norden golden structure (φ, g) if and only if it is an adapted connection to (J_φ, g) (Proposition 19). This result will allow us to introduce a distinguished adapted connection on an almost Norden golden manifold (M, φ, g) as the well-adapted connection of the almost Norden manifold (M, J_φ, g) (Definition 21).

In the last section, we will focus on the classification of almost Norden golden manifolds. First, we will use the classification of almost Norden manifolds previously recalled to classify almost Norden golden manifolds according to Definition 4 (Theorem 24). Later, the characteristic conditions of all classes of almost Norden manifolds expressed in terms of the torsion tensor and the torsion form of its well-adapted connection also will allow us to obtain defining conditions of all classes of almost Norden golden manifolds using the above-mentioned adapted connection share by the structures (φ, g) and (J_φ, g) (Theorem 25).

We will consider smooth manifolds and operators being of class C^∞. The module of vector fields and the set of tensors fields of type $(1, 2)$ on a manifold M will be denoted by $\mathfrak{X}(M)$ and $\mathscr{T}_2^1(M)$, respectively. The Levi–Civita connection of a pseudo-Riemannian metric g will be denoted by ∇^{g}.

2 Classification of almost Norden manifolds

As we said in Introduction, the classification of almost Norden manifolds was obtained by Ganchev and Borisov in [7]. In this section, first we will recall the definitions of tensors and 1-forms that help to obtain these classification, and later, we will recall the mentioned classification.

Let (M, J, g) be an almost Norden manifold. The fundamental tensor Φ is the tensor field of type $(0, 2)$ given by

$$\Phi(X, Y) = g(JX, Y), \quad \forall X, Y \in \mathfrak{X}(M).$$

The tensor $\nabla^{\mathrm{g}} J$ is the tensor field of type $(1, 2)$ defined as follows:

$$(\nabla^{\mathrm{g}}_X J)Y = \nabla^{\mathrm{g}}_X JY - J\nabla^{\mathrm{g}}_X Y, \quad \forall X, Y \in \mathfrak{X}(M).$$

Using the above one, it is easy to obtain the expression of the covariant derivative of the tensor Φ.

Proposition 5 *Let (M, J, g) be an almost Norden manifold. Then,*

$$(\nabla^{\mathrm{g}}_X \Phi)(Y, Z) = g((\nabla^{\mathrm{g}}_X J)Y, Z), \quad \forall X, Y, Z \in \mathfrak{X}(M). \tag{5}$$

There are two tensor fields of type $(1, 2)$ on almost Norden manifolds that allow to characterize some classes of this kind of manifolds. These two tensor fields are given by

$$\begin{aligned} N_J(X, Y) &= [X, Y] + [JX, JY] - J[JX, Y] - J[X, JY], \\ \widetilde{N}_J(X, Y) &= (\nabla^{\mathrm{g}}_X J)JY + (\nabla^{\mathrm{g}}_{JX} J)Y + (\nabla^{\mathrm{g}}_Y J)JX + (\nabla^{\mathrm{g}}_{JY} J)X, \quad \forall X, Y \in \mathfrak{X}(M). \end{aligned} \tag{6}$$

The first one is the tensor Nijenhuis of J, which also can be expressed using the Levi–Civita of g as follows:

$$N_J(X, Y) = (\nabla^{\mathrm{g}}_X J)JY + (\nabla^{\mathrm{g}}_{JX} J)Y - (\nabla^{\mathrm{g}}_Y J)JX - (\nabla^{\mathrm{g}}_{JY} J)X, \quad \forall X, Y \in \mathfrak{X}(M). \tag{7}$$

The tensor field of type $(0, 3)$ defined by the covariant derivative of the fundamental tensor field obtained in (5) allows to introduce a 1-form, which is the other key, jointly with the above tensor fields, of the classification of almost Norden manifolds.

Definition 6 [7, Eq. (3)] Let (M, J, g) be an almost Norden manifold. Let $p \in M$ and let $(v_1, \ldots, v_{2n})$ be a basis of $T_p(M)$. The 1-form associated with the tensor Φ is defined as follows:

$$\delta\Phi(v) = \sum_{i,j=1}^{2n} g^{ij} g((\nabla^{\mathrm{g}}_{v_i} J)v_j, v), \quad v \in T_p(M),$$

where $(g^{ij})_{i,j=1}^{2n}$ is the inverse matrix of $(g_{ij})_{i=1}^{2n} = (g_p(v_i, v_j))_{i,j=1}^{2n}$. We will call it the codifferential of Φ.

It is obvious that the codifferential of Φ can be locally defined as follows:

$$\delta\Phi(X) = \sum_{i,j=1}^{2n} g^{ij} g((\nabla^{\mathrm{g}}_{X_i} J)X_j, X), \quad \forall X \in \mathfrak{X}(M),$$

where $(X_1, \dots, X_{2n})$ is a local basis of TM and $(g^{ij})_{i,j=1}^{2n}$ is the inverse matrix of $(g(X_i, X_j))_{i,j=1}^{2n}$.

The G-structure defined by (J, g) over M allows to choose local basis of TM that simplifies the above expression of $\delta\Phi$.

Lemma 7 [5, Prop. 3.3] *Let (M, J, g) be an almost Norden manifold. Then, for every $p \in M$ there exist a local basis $(X_1, \dots, X_n, Y_1, \dots, Y_n)$ of TM such that*

$$JX_i = Y_i, \quad g(X_i, X_j) = g(Y_i, Y_j) = 0, \quad g(X_i, Y_j) = \delta_{ij}, \quad \forall i, j = 1, \dots, n.$$

A local basis satisfying the above conditions is called an adapted local basis to (J, g).

Lemma 8 *Let (M, J, g) be an almost Norden manifold. The 1-form $\delta\Phi$ can be locally expressed by means of an adapted local basis $(X_1, \dots, X_n, Y_1, \dots, Y_n)$ to (J, g) as follows:*

$$\delta\Phi(X) = \sum_{i=1}^{n} (g((\nabla^{\mathrm{g}}_{X_i} J)Y_i, X) + g((\nabla^{\mathrm{g}}_{Y_i} J)X_i, X)), \quad \forall X \in \mathfrak{X}(M). \tag{8}$$

Now we recall the classification of almost Norden manifolds.

Theorem 9 [7] *Let (M, J, g) be a $2n$-dimensional almost Norden manifold. Then, one has the following classes of this kind of manifolds:*

(i) *The class $\mathcal{W}_0$ or Kähler Norden manifolds characterized by the condition*

$$\nabla^{\mathrm{g}}\Phi = 0.$$

(ii) *The class $\mathcal{W}_1$ characterized by the condition*

$$\begin{aligned}(\nabla^{\mathrm{g}}_X \Phi)(Y, Z) &= \frac{1}{2n}(g(X, Y)\delta\Phi(Z) + g(X, Z)\delta\Phi(Y)) \\ &\quad + \frac{1}{2n}(g(X, JY)\delta\Phi(JZ) + g(X, JZ)\delta\Phi(JY)),\end{aligned}$$

for all vector fields X, Y, Z on M.

(iii) *The class* $\mathscr{W}_2$ *characterized by the conditions*

$$N_J = 0, \delta\Phi = 0.$$

(iv) *The class* $\mathscr{W}_3$ *or quasi-Kähler Norden manifolds characterized by the condition*

$$\widetilde{N}_J = 0.$$

(v) *The class* $\mathscr{W}_1 \oplus \mathscr{W}_2$ *or integrable Norden manifolds characterized by the condition*

$$N_J = 0.$$

(vi) *The class* $\mathscr{W}_2 \oplus \mathscr{W}_3$ *characterized by the condition*

$$\delta\Phi = 0.$$

(vii) *The class* $\mathscr{W}_1 \oplus \mathscr{W}_3$ *characterized by the condition*

$$\underset{XYZ}{\mathfrak{S}} (\nabla^{\mathrm{g}}_X \Phi)(Y, Z) = \frac{1}{n} \underset{XYZ}{\mathfrak{S}} (g(X, Y)\delta\Phi(Z) + g(X, JY)\delta\Phi(JZ)),$$

for all X, Y, Z *vector fields on* M*, where* $\underset{XYZ}{\mathfrak{S}}$ *denotes the cyclic sum by* X, Y, Z.

(viii) *The class* $\mathscr{W}$ *or the whole class of almost Norden manifolds.*

To obtain the classification of almost Norden manifolds using the well-adapted connection, in [9, Sec. 5], Ganchev and Mihova introduced the next 1-form defined from its torsion tensor.

Definition 10 Let (M, J, g) be an $2n$-dimensional almost Norden manifold. Let $p \in M$ and let $(v_1, \dots, v_{2n})$ be a basis of $T_p(M)$. The 1-form associated with the torsion tensor T^{w} of the connection ∇^{w} introduced in (4) is defined as follows:

$$t^{\mathrm{w}}(v) = \sum_{i,j}^{2n} g^{ij} g(T^{\mathrm{w}}(v, v_i), v_j), \quad v \in T_p(M), \tag{9}$$

where $(g^{ij})_{i,j=1}^{2n}$ is the inverse matrix of $(g_{ij})_{i=1}^{2n} = (g_p(v_i, v_j))_{i,j=1}^{2n}$. We will call it the torsion form of the well-adapted connection ∇^{w}.

The new defining conditions of all classes of almost Norden manifolds obtained by Ganchev and Mihova using the well-adapted connection are collected in the next result.

Theorem 11 [9] *Let* (M, J, g) *be a* $2n$*-dimensional almost Norden manifold. The classes given in Theorem 9 can be characterized by means of the torsion tensor and the torsion form of the well-adapted connection as follows:*

(i) *The class $\mathscr{W}_0$ or Kähler Norden manifolds characterized by the condition*

$$T^{\mathrm{w}}(X,Y)=0, \quad \forall X,Y\in\mathfrak{X}(M).$$

(ii) *The class $\mathscr{W}_1$ characterized by the condition*

$$T^{\mathrm{w}}(X,Y)=\frac{1}{2n}(t^{\mathrm{w}}(X)Y-t^{\mathrm{w}}(Y)X+t^{\mathrm{w}}(JX)JY-t^{\mathrm{w}}(JY)JX), \quad \forall X,Y\in\mathfrak{X}(M).$$

(iii) *The class $\mathscr{W}_2$ characterized by the conditions*

$$T^{\mathrm{w}}(JX,JY)-T^{\mathrm{w}}(X,Y)=0, t^{\mathrm{w}}(X)=0, \quad \forall X,Y\in\mathfrak{X}(M).$$

(iv) *The class $\mathscr{W}_3$ or quasi-Kähler Norden manifolds characterized by the condition*

$$T^{\mathrm{w}}(JX,Y)+JT^{\mathrm{w}}(X,Y)=0, \quad \forall X,Y\in\mathfrak{X}(M). \tag{10}$$

(v) *The class $\mathscr{W}_1\oplus\mathscr{W}_2$ or integrable Norden manifolds characterized by the condition*

$$T^{\mathrm{w}}(JX,JY)-T^{\mathrm{w}}(X,Y)=0, \quad \forall X,Y\in\mathfrak{X}(M). \tag{11}$$

(vi) *The class $\mathscr{W}_2\oplus\mathscr{W}_3$ characterized by the condition*

$$t^{\mathrm{w}}(X)=0, \quad \forall X\in\mathfrak{X}(M).$$

(vii) *The class $\mathscr{W}_1\oplus\mathscr{W}_3$ characterized by the condition*

$$T^{\mathrm{w}}(JX,Y)+JT^{\mathrm{w}}(X,Y)=\frac{1}{n}(t^{\mathrm{w}}(JY)X-t^{\mathrm{w}}(Y)JX), \quad \forall X,Y\in\mathfrak{X}(M).$$

(viii) *The class $\mathscr{W}$ or the whole class of almost Norden manifolds.*

3 The well-adapted connection of an almost Norden manifold

Let (M,J,g) be an almost Norden manifold. It is well known that one can introduce another pseudo-Riemannian metric $\widetilde{g}$ on M, called the twin metric of g, as follows:

$$\widetilde{g}(X,Y)=g(JX,Y), \quad \forall X,Y\in\mathfrak{X}(M).$$

Let $\widetilde{\nabla}^{\widetilde{\mathrm{g}}}$ be the Levi–Civita connection of the metric $\widetilde{g}$. In [9, Sec. 3], by means of the potential tensor of $\widetilde{\nabla}^{\widetilde{\mathrm{g}}}$ with respect to ∇^{g}, $\widetilde{\nabla}^{\widetilde{\mathrm{g}}}-\nabla^{\mathrm{g}}$, the authors distinguished, among the set of connections on M satisfying $\nabla g=0$ and $\nabla\widetilde{g}=0$, the well-adapted connection as the unique connection that verifies identity (4). Note that $\nabla\widetilde{g}=0$ and $\nabla J=0$ are equivalent conditions if ∇ parallelizes the metric g. They did not obtain their classification of almost Norden manifolds by means of the well-adapted connection directly from the classification previously recalled in Theorem 9. They need another classification based in the difference tensor of both Levi–Civita connections. This technique avoids analyzing the relation between the Levi–Civita connection of g and the well-adapted connection, making it difficult to obtain useful relations between key tensors of the classification of these manifolds and the torsion tensor of the well-adapted connection and the explicit expression of this adapted connection in some classes of almost Norden manifolds.

We will devote this section highlighting and exploiting the relation between the Levi–Civita and the well-adapted connection of an almost Norden manifold.

The next result characterizes the set of adapted connections to an almost Norden structure.

Lemma 12 [5, Lemma 4.3] *Let (M, J, g) be an almost Norden manifold. The set of adapted connections to (J, g) is the following*

$$\left\{\nabla^{\mathrm{g}}+S : S\in\mathscr{T}_2^1(M),\ \begin{array}{l} JS(X,Y)-S(X,JY)=(\nabla^{\mathrm{g}}_X J)Y, \\ g(S(X,Y),Z)+g(S(X,Z),Y)=0, \end{array} \quad \forall X,Y,Z\in\mathfrak{X}(M)\right\}.$$

Given X, Y, Z vector fields on M, if S^{w} denotes the potential tensor of ∇^{w} with respect to ∇^{g}, i.e.,

$$S^{\mathrm{w}}(X,Y)=\nabla^{\mathrm{w}}_X Y-\nabla^{\mathrm{g}}_X Y,$$

then one has

$$T^{\mathrm{w}}(X,Y)=\nabla^{\mathrm{w}}(X,Y)-\nabla^{\mathrm{w}}(Y,X)-[X,Y]=S^{\mathrm{w}}(X,Y)-S^{\mathrm{w}}(Y,X), \tag{12}$$

and as ∇^{w} is an adapted connection, one also has

$$\begin{aligned} g(T^{\mathrm{w}}(X,Y),Z)&=g(S^{\mathrm{w}}(X,Y),Z)-g(S^{\mathrm{w}}(Y,X),Z),\\ g(T^{\mathrm{w}}(Y,Z),X)&=g(S^{\mathrm{w}}(Y,Z),X)-g(S^{\mathrm{w}}(Z,Y),X)\\ &=-g(S^{\mathrm{w}}(Y,X),Z)+g(S^{\mathrm{w}}(Z,X),Y),\\ g(T^{\mathrm{w}}(Z,X),Y)&=g(S^{\mathrm{w}}(Z,X),Y)-g(S^{\mathrm{w}}(X,Z),Y)\\ &=g(S^{\mathrm{w}}(Z,X),Y)+g(S^{\mathrm{w}}(X,Y),Z), \end{aligned}$$

therefore one gets the next identity summing up the above ones

$$g(S^{\mathrm{w}}(X,Y),Z)=\frac{1}{2}(g(T^{\mathrm{w}}(X,Y),Z)-g(T^{\mathrm{w}}(Y,Z),X)+g(T^{\mathrm{w}}(Z,X),Y)). \tag{13}$$

Bearing in mind the first defining condition of the set of adapted connections and the above identity, one obtains the next equality that involves the covariant derivative of the fundamental tensor of (M, J, g)

$$\begin{aligned} g((\nabla^{\mathrm{g}}_X J)Y,Z)&=g(S^{\mathrm{w}}(X,Y),JZ)-g(S^{\mathrm{w}}(X,JY),Z)\\ &=\frac{1}{2}(g(T^{\mathrm{w}}(X,Y),JZ)-g(T^{\mathrm{w}}(Y,JZ),X)+g(T^{\mathrm{w}}(JZ,X),Y)) \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}(g(T^{\mathrm{w}}(X, JY), Z) - g(T^{\mathrm{w}}(JY, Z), X) + g(T^{\mathrm{w}}(Z, X), JY))\\
&= g(T^{\mathrm{w}}(JY, X), Z) - g(T^{\mathrm{w}}(Z, X), JY)\\
&+\frac{1}{2}(g(T^{\mathrm{w}}(JY, Z), X) - g(T^{\mathrm{w}}(Y, JZ), X)), \qquad (14)
\end{aligned}
$$

because as direct consequence of (4) one has

$$
g(T^{\mathrm{w}}(JZ, X, Y) - g(T^{\mathrm{w}}(Y, X), JZ) = -g(T^{\mathrm{w}}(Z, X), JY) + g(T^{\mathrm{w}}(JY, X), Z).
$$

The next results provide key identities that allow us to link both classifications of almost Norden manifolds recalled in Theorems 9 and 11 .

Lemma 13 *Let (M, J, g) be an almost Norden manifold. The next identity holds:*

$$
\delta\Phi(X) = 2t^{\mathrm{w}}(JX), \quad \forall X \in \mathfrak{X}(M). \qquad (15)
$$

Proof Starting from (9) and given an adapted local basis $(X_1, \ldots, X_n, Y_1, \ldots Y_n)$ to (J, g) and a vector field X on M, it is obvious that one gets the following

$$
t^{\mathrm{w}}(X) = \sum_{i=1}^{n}(g(T^{\mathrm{w}}(X, X_i), Y_i) + g(T^{\mathrm{w}}(X, Y_i), X_i)). \qquad (16)
$$

Bearing in mind (14) one has

$$
g((\nabla^{\mathrm{g}}_{X_i} J)Y_i, X) = \frac{3}{2}g(T^{\mathrm{w}}(X, X_i), X_i) + \frac{1}{2}g(T^{\mathrm{w}}(JX, Y_i), X_i), \qquad (17)
$$

$$
g((\nabla^{\mathrm{g}}_{Y_i} J)X_i, X) = -\frac{3}{2}g(T^{\mathrm{w}}(X, Y_i), Y_i) + \frac{1}{2}g(T^{\mathrm{w}}(JX, X_i), Y_i), \qquad (18)
$$

and taking into account (4) one also has

$$
\begin{aligned}
&g(T^{\mathrm{w}}(X, X_i), X_i) - g(T^{\mathrm{w}}(X_i, X_i), X) = g(T^{\mathrm{w}}(JX, X_i), Y_i) - g(T^{\mathrm{w}}(Y_i, X_i), JX),\\
&-g(T^{\mathrm{w}}(X, Y_i), Y_i) + g(T^{\mathrm{w}}(Y_i, Y_i), X) = g(T^{\mathrm{w}}(JX, Y_i), X_i) - g(T^{\mathrm{w}}(X_i, Y_i), JX).
\end{aligned}
$$

Then, summing up (17) and (18), the above identities yield the following equality

$$
g((\nabla^{\mathrm{g}}_{X_i} J)Y_i, X) + g((\nabla^{\mathrm{g}}_{Y_i} J)X_i, X) = 2(g(T^{\mathrm{w}}(JX, X_i), Y_i) + g(T^{\mathrm{w}}(JX, Y_i), X_i)),
$$

therefore, (8), (16) and the previous one carry to the next identity

$$
\delta\Phi(X) = 2t^{\mathrm{w}}(JX).
$$

□

Lemma 14 *Let (M, J, g) be an almost Norden manifold. The next identities hold:*

$$N_J(X, Y) = 2(T^{\mathrm{w}}(X, Y) - T^{\mathrm{w}}(JX, JY)), \tag{19}$$

$$g(\widetilde{N}_J(X, Y), Z) = 4(g(T^{\mathrm{w}}(Z, JX) + JT^{\mathrm{w}}(Z, X), Y)), \quad \forall X, Y \in \mathfrak{X}(M). \tag{20}$$

Proof Given X, Y, Z vector fields on M, starting from (14) one obtains the following equalities

$$\begin{aligned} g(\nabla^{\mathrm{w}}_X J)JY, Z) + g((\nabla^{\mathrm{w}}_{JX} J)Y, Z) &= g(T^{\mathrm{w}}(X, Y), Z) - g(T^{\mathrm{w}}(JX, JY), Z) \\ &\quad + g(T^{\mathrm{w}}(Z, X), Y) - g(T^{\mathrm{w}}(Z, JX), JY) \\ &\quad + \frac{1}{2}(g(T^{\mathrm{w}}(JY, Z), JX) - g(T^{\mathrm{w}}(Y, Z), X)) \\ &\quad - \frac{1}{2}(g(T^{\mathrm{w}}(JY, JZ), X) + g(T^{\mathrm{w}}(Y, JZ), JX)), \end{aligned} \tag{21}$$

$$\begin{aligned} g(\nabla^{\mathrm{w}}_Y J)JX, Z) + g((\nabla^{\mathrm{w}}_{JY} J)X, Z) &= g(T^{\mathrm{w}}(Y, X), Z) - g(T^{\mathrm{w}}(JY, JX), Z) \\ &\quad + g(T^{\mathrm{w}}(Z, Y), X) - g(T^{\mathrm{w}}(Z, JY), JX) \\ &\quad + \frac{1}{2}(g(T^{\mathrm{w}}(JX, Z), JY) - g(T^{\mathrm{w}}(X, Z), Y)) \\ &\quad - \frac{1}{2}(g(T^{\mathrm{w}}(JX, JZ), Y) + g(T^{\mathrm{w}}(X, JZ), JY)), \end{aligned} \tag{22}$$

and bearing in mind (7) one can conclude

$$\begin{aligned} g(N_J(X, Y), Z) &= 2g(T^{\mathrm{w}}(X, Y), Z) - 2g(T^{\mathrm{w}}(JX, JY), Z) \\ &\quad + \frac{1}{2}(g(T^{\mathrm{w}}(Z, X), Y) - g(T^{\mathrm{w}}(Z, JX), JY)) \\ &\quad + \frac{1}{2}(g(T^{\mathrm{w}}(Y, Z), X) - g(T^{\mathrm{w}}(JY, Z), JX)) \\ &\quad - \frac{1}{2}(g(T^{\mathrm{w}}(JY, JZ), X) - g(T^{\mathrm{w}}(JX, JZ), Y)) \\ &\quad - \frac{1}{2}(g(T^{\mathrm{w}}(Y, JZ), X) - g(T^{\mathrm{w}}(X, JZ), JY)) \\ &= 2g(T^{\mathrm{w}}(X, Y), Z) - 2g(T^{\mathrm{w}}(JX, JY), Z), \end{aligned}$$

because as direct consequence of (4) one has

$$\begin{aligned} &- g(T^{\mathrm{w}}(X, Z), Y) + g(T^{\mathrm{w}}(JX, Z), JY) \\ &\quad = -g(T^{\mathrm{w}}(Y, Z), X) + g(T^{\mathrm{w}}(JY, Z), JX), \\ &g(T^{\mathrm{w}}(JY, JZ), X) - g(T^{\mathrm{w}}(JX, JZ), Y) \\ &\quad = -g(T^{\mathrm{w}}(Y, JZ), JX) + g(T^{\mathrm{w}}(X, JZ), JY). \end{aligned}$$

Given X, Y, Z vector fields on M, one easily gets the following equality

$$J(\nabla^{g}_{X} J)Y = J(\nabla^{g}_{X} JY - J\nabla^{g}_{X} Y) = J\nabla^{g}_{X} JY - \nabla^{g}_{X} Y = -(\nabla^{g}_{X} J)JY,$$

then taking into account (7) one gets the next two identities

$$\begin{aligned}
N_J(JX, Y) &= (\nabla^{g}_{JX} J)JY - (\nabla^{g}_{X} J)Y + (\nabla^{g}_{Y} J)X - (\nabla^{g}_{JY} J)JX,\\
JN_J(X, Y) &= (\nabla^{g}_{X} J)Y - (\nabla^{g}_{JX} J)JY - (\nabla^{g}_{Y} J)X + (\nabla^{g}_{JY} J)JX,
\end{aligned}$$

and thus, one obtains $N_J(JX, Y) = -JN_J(X, Y)$. If one combines the last equality and (19), one can conclude

$$T^{w}(JX, Y) + T^{w}(X, JY) = -JT^{w}(X, Y) + JT^{w}(JX, JY).$$

As direct consequence of the previous identity, one gets

$$-g(T^{w}(JX, JY), Z) = g(T^{w}(JX, Y), JZ) + g(T^{w}(X, JY), JZ) - g(T^{w}(X, Y), Z).$$

Bearing in mind (4) and the above one, identities (21) and (22) read as follows:

$$\begin{aligned}
g(\nabla^{w}_{X} J)JY, Z) + g((\nabla^{w}_{JX} J)Y, Z) &= g(T^{w}(X, Y), Z) - g(T^{w}(JX, JY), Z)\\
&\quad + g(T^{w}(Z, X), Y) - g(T^{w}(Z, JX), JY)\\
&\quad - g(T^{w}(Y, Z), X) + g(T^{w}(JY, Z), JX),\\
g(\nabla^{w}_{Y} J)JX, Z) + g((\nabla^{w}_{JY} J)X, Z) &= g(T^{w}(Y, X), Z) - g(T^{w}(JY, JX), Z)\\
&\quad + g(T^{w}(Z, Y), X) - g(T^{w}(Z, JY), JX)\\
&\quad - g(T^{w}(X, Z), Y) + g(T^{w}(JX, Z), JY),
\end{aligned}$$

and thus, identities (4), (6) and the previous one allow us to claim

$$\begin{aligned}
g(\widetilde{N}_J(X, Y), Z) &= 2(g(T^{w}(Z, X), Y) - g(T^{w}(Z, JX), JY))\\
&\quad - 2(g(T^{w}(Y, Z), X) - g(T^{w}(JY, Z), JX))\\
&= 4(g(T^{w}(Z, X), Y) - g(T^{w}(Z, JX), JY))\\
&= 4(g(T^{w}(Z, X), Y) - g(JT^{w}(Z, JX), Y)).
\end{aligned}$$

□

The equivalence between the defining conditions of the classes $\mathscr{W}_2$, $\mathscr{W}_3$, $\mathscr{W}_1 \oplus \mathscr{W}_2$ and $\mathscr{W}_2 \oplus \mathscr{W}_3$ shown in Theorems 9 and 11 are direct consequence of identities (15), (19) and (20). Furthermore, the mentioned equivalences of integrable and quasi-Kähler classes also allow us to obtain explicit expressions of the well-adapted connection as it is shown in the below remarks.

But first, we should remember that almost Norden manifolds are $(J^2 = \pm 1)$-metric manifolds in the case $(\alpha, \varepsilon) = (-1, -1)$ and $(J^2 = \pm 1)$-metric manifolds of class $\mathscr{G}_1$ such that $(\alpha, \varepsilon) = (-1, -1)$ are quasi-Kähler Norden manifolds (see [4,5]).

Remark 15 In [5, Prop. 6.5], the authors claim that if (M, J, g) is an almost Norden manifold and then

$$S^{\mathrm{w}}(X, Y) = \frac{1}{2}(\nabla^{\mathrm{g}}_X J)JY \Longleftrightarrow T^{\mathrm{w}}(JX, JY) = T^{\mathrm{w}}(X, Y),$$

for all vector fields X, Y on M. Thus, identity (11) allows us to claim that (M, J, g) is an integrable Norden manifold if and only if

$$\nabla^{\mathrm{w}}_X Y = \nabla^{\mathrm{g}}_X Y + \frac{1}{2}(\nabla^{\mathrm{g}}_X J)JY, \quad \forall X, Y, \in \mathfrak{X}(M). \tag{23}$$

In this case, the torsion tensor of the well-adapted connection satisfies

$$T^{\mathrm{w}}(X, Y) = \frac{1}{2}\left((\nabla^{\mathrm{g}}_X J)JY - (\nabla^{\mathrm{g}}_Y J)JX\right), \quad \forall X, Y \in \mathfrak{X}(M). \tag{24}$$

Thus, identity (23) is an explicit expression of the well-adapted connection in the case of almost Norden manifolds of the classes $\mathscr{W}_1$, $\mathscr{W}_2$ and $\mathscr{W}_1 \oplus \mathscr{W}_2$.

Remark 16 In [4, Coroll. 5.8], the authors prove the characterization (10) of quasi-Kähler Norden manifolds in a different manner from that used in [9]. The proof of the quoted corollary also allows us to claim that the well-adapted connection is given by

$$\nabla^{\mathrm{w}}_X Y = \nabla^{\mathrm{g}}_X Y + \frac{1}{2}(\nabla^{\mathrm{g}}_X J)JY + \frac{1}{4}((\nabla^{\mathrm{g}}_Y J)JX - (\nabla^{\mathrm{g}}_{JY} J)X), \quad \forall X, Y \in \mathfrak{X}(M), \tag{25}$$

and its torsion tensor is given by

$$T^{\mathrm{w}}(X, Y) = \frac{1}{4} N_J(X, Y), \quad \forall X, Y \in \mathfrak{X}(M), \tag{26}$$

if the (M, J, g) is a quasi-Kähler Norden manifold.

Reciprocally, given X, Y vector fields on M, if the well-adapted connection satisfies (25) and taking into account that $N_J(JX, Y) = -JN_J(X, Y)$, then one obtains that (M, J, g) is of class $\mathscr{W}_3$ because of

$$T^{\mathrm{w}}(JX, Y) = \frac{1}{4} N_J(JX, Y) = -\frac{1}{4} JN_J(X, Y) = -JT^{\mathrm{w}}(X, Y).$$

The above identities allow us to claim that (M, J, g) is a quasi-Kähler Norden manifold if and only if the well-adapted connection satisfies (25). Thus, that identity is an explicit expression of the well-adapted connection in the case of almost Norden manifolds of the class $\mathscr{W}_3$.

Moreover, if the well-adapted connection of (J, g) satisfies (10), then also satisfies

$$T^{\mathrm{w}}(JX, JY) = -T^{\mathrm{w}}(X, Y), \quad \forall X, Y, \in \mathfrak{X}(M). \tag{27}$$

Reciprocally, if the well-adapted connection satisfies the above condition then (M, J, g) is a quasi-Kähler Norden manifold (see [4, Prop. 5.3]). Thus, identity (27) is another characterization of the class $\mathcal{W}_3$ manifolds of almost Norden manifolds.

As ∇^{w} is an adapted connection, the condition shown in (27) makes that the well-adapted connection being the analogous to the Chern connection of an almost Hermitian manifold on almost quasi-Kähler Norden manifolds (see [4]).

Identities (12) and (13) allow to claim that the potential and the torsion tensors of the well-adapted connection mutually define each other. Then, the well-adapted connection verifies (23) or (25) if and only if its torsion tensor satisfies (24) or (26), respectively. Thus, we can establish new characterizations of integrable and quasi-Kähler Norden manifolds as follows.

Corollary 17 *Let (M, J, g) be an almost Norden manifold.*

(i) *The manifold (M, J, g) is of class $\mathcal{W}_1 \oplus \mathcal{W}_2$ if and only if*

$$T^{\mathrm{w}}(X, Y) = \frac{1}{2}\left((\nabla^{\mathrm{g}}_X J)JY - (\nabla^{\mathrm{g}}_Y J)JX\right), \quad \forall X, Y \in \mathfrak{X}(M).$$

(ii) *The manifold (M, J, g) is of class $\mathcal{W}_3$ if and only if*

$$T^{\mathrm{w}}(X, Y) = \frac{1}{4}N_J(X, Y), \quad \forall X, Y \in \mathfrak{X}(M).$$

4 Adapted connections on almost Norden golden manifolds

Now we will introduce the notion of adapted connection to an almost Norden golden structure in a similar way to adapted connections to almost Norden structures. We will study briefly this kind of connections.

Definition 18 Let (M, φ, g) be an almost Norden golden manifold and let ∇ be a connection on M. We say that ∇ is adapted to the almost Norden golden structure (φ, g) if $\nabla\varphi = 0$ and $\nabla g = 0$.

Given X, Y vector fields on an almost Norden golden manifold (M, φ, g), identity (1) and the definition of the tensor field $\nabla^{\mathrm{g}}\varphi$ carry to next ones

$$(\nabla^{\mathrm{g}}_X J_\varphi)Y = \nabla^{\mathrm{g}}_X J_\varphi Y - J_\varphi(\nabla^{\mathrm{g}}_X Y) = -\frac{2}{\sqrt{5}}(\nabla^{\mathrm{g}}_X \varphi Y - \varphi(\nabla_X Y)) = -\frac{2}{\sqrt{5}}(\nabla_X \varphi)Y, \tag{28}$$

$$g(X, J_\varphi Y) = \frac{1}{\sqrt{5}}g(X, Y) - \frac{2}{\sqrt{5}}g(X, \varphi Y). \tag{29}$$

Identity (28) allows us to set out the below result that links the notions of adapted connections on almost Norden and almost Norden golden manifolds.

Proposition 19 *Let (M, φ, g) be an almost Norden golden manifold, and let ∇ be a connection on M. The connection ∇ is adapted to the almost Norden golden structure (φ, g) if and only if ∇ is adapted to its induced almost Norden structure (J_φ, g).*

The set of adapted connections to an almost Norden golden structure can also be characterized in a similar way as the set of adapted connections to an almost Norden structure.

Lemma 20 *Let (M, φ, g) be an almost Norden golden manifold. The set of adapted connections to (φ, g) is the following*

$$\left\{ \nabla^{\mathrm{g}} + S : S \in \mathscr{T}_2^1(M), \begin{array}{l} \varphi S(X,Y) - S(X,\varphi Y) = (\nabla^{\mathrm{g}}_X \varphi)Y, \\ g(S(X,Y),Z) + g(S(X,Z),Y) = 0, \end{array} \quad \forall X, Y, Z \in \mathfrak{X}(M) \right\}.$$

Proof According to Lemma 12, it is sufficient to prove the first defining condition of the above set. Recalling that a connection ∇ is adapted to (φ, g) if and only if is adapted to its induced almost Norden structure (J_φ, g), and taking into account (1) and (28), given and adapted connection $\nabla = \nabla^{\mathrm{g}} + S$ to (φ, g), the tensor S satisfies

$$\begin{aligned} J_\varphi S(X,Y) - S(X, J_\varphi Y) &= (\nabla^{\mathrm{g}}_X J_\varphi)Y \\ \frac{1}{\sqrt{5}}(S(X,Y) - 2\varphi S(X,Y)) - S\left(X, \frac{1}{\sqrt{5}}(Y - 2\varphi Y)\right) &= -\frac{2}{\sqrt{5}}(\nabla_X \varphi)Y, \\ \varphi S(X,Y) - S(X,\varphi Y) &= (\nabla^{\mathrm{g}}_X \varphi)Y, \quad \forall X, Y \in \mathfrak{X}(M). \end{aligned}$$

□

Proposition 19 and the adapted connection to an almost Norden structure recalled in (4) allow us to introduce a distinguished adapted connection on almost Norden golden manifolds.

Definition 21 Let (M, φ, g) be an almost Norden golden manifold. The well-adapted connection ∇^{w} of (M, φ, g) is the unique adapted connection to (J_φ, g) satisfying

$$g(T^{\mathrm{w}}(X,Y),Z) - g(T^{\mathrm{w}}(Z,Y),X) = g(T^{\mathrm{w}}(J_\varphi X, Y), J_\varphi Z) - g(T^{\mathrm{w}}(J_\varphi Z, Y), J_\varphi X), \tag{30}$$

for all vector fields X, Y, Z on M.

Given X, Y, Z vector fields on M, identities (1) and (29) allow to obtain another expression of the well-adapted connection as follows:

$$\begin{aligned} g(T^{\mathrm{w}}(J_\varphi X, Y), J_\varphi Z) = &\frac{1}{5}(g(T^{\mathrm{w}}(X,Y),Z) - 2g(T^{\mathrm{w}}(\varphi X, Y), Z)) \\ &+ \frac{2}{5}(-g(T^{\mathrm{w}}(X,Y,)\varphi Z) + 2g(T^{\mathrm{w}}(\varphi X, Y), \varphi Z)), \end{aligned}$$

$$g(T^{\mathrm{w}}(J_\varphi Z, Y), J_\varphi X) = \frac{1}{5}(g(T^{\mathrm{w}}(Z, Y), X) - 2g(T^{\mathrm{w}}(\varphi Z, Y), X)) + \frac{2}{5}(-g(T^{\mathrm{w}}(Z, Y,)\varphi X) + 2g(T^{\mathrm{w}}(\varphi Z, Y), \varphi X)),$$

and then, according to (30), one has

$$g(T^{\mathrm{w}}(X, Y), Z) - g(T^{\mathrm{w}}(Z, Y), X) = g(T^{\mathrm{w}}(\varphi X, Y), \varphi Z) - g(T^{\mathrm{w}}(\varphi Z, Y), \varphi X) + \frac{1}{2}(g(T^{\mathrm{w}}(\varphi Z, Y), X) + g(T^{\mathrm{w}}(Z, Y), \varphi X)) - \frac{1}{2}(g(T^{\mathrm{w}}(\varphi X, Y), Z) + g(T^{\mathrm{w}}(X, Y), \varphi Z)).$$

Given X, Y vector fields on M, identity (28) allows us to get the next relations between tensors defined from J_φ and φ

$$(\nabla^{\mathrm{g}}_X J_\varphi) J_\varphi Y = -\frac{2}{5}(\nabla^{\mathrm{g}}_X \varphi)\,(Y - 2\varphi Y) = -\frac{2}{5}(\nabla^{\mathrm{g}}_X \varphi)Y + \frac{4}{5}(\nabla^{\mathrm{g}}_X \varphi)\varphi Y, \tag{31}$$

$$(\nabla^{\mathrm{g}}_{J_\varphi X} J_\varphi)Y = -\frac{2}{5}(\nabla^{\mathrm{g}}_{X-2\varphi X}\varphi)Y = -\frac{2}{5}(\nabla^{\mathrm{g}}_X \varphi)Y + \frac{4}{5}(\nabla^{\mathrm{g}}_{\varphi X}\varphi)Y. \tag{32}$$

We will finish this section obtaining the equation that links the Nijenhuis tensors of φ and J_φ, N_φ and N_{J_φ}.

Lemma 22 *Let (M, φ, g) be an almost Norden golden manifold. The following relation holds:*

$$N_{J_\varphi}(X, Y) = \frac{4}{5} N_\varphi(X, Y), \quad \forall X, Y \in \mathfrak{X}(M). \tag{33}$$

Proof Given X, Y vector field on M, one has the following equalities

$$[J_\varphi X, J_\varphi Y] = \frac{1}{5}\left([X, Y] - 2[\varphi X, Y] - 2[X, \varphi Y] + 4[\varphi X, \varphi Y]\right),$$

$$J_\varphi[J_\varphi X, Y] = \frac{1}{5}\left([X, Y] - 2[\varphi X, Y] - 2\varphi[X, Y] + 4\varphi[\varphi X, Y]\right),$$

$$J_\varphi[X, J_\varphi Y] = \frac{1}{5}\left([X, Y] - 2[X, \varphi Y] - 2\varphi[X, Y] + 4\varphi[X, \varphi Y]\right),$$

and then, taking into account that $\frac{4}{5}\varphi^2 = \frac{4}{5}\varphi - \frac{6}{5}\mathrm{Id}$, one obtains

$$\begin{aligned} N_{J_\varphi}(X, Y) &= -[X, Y] + [J_\varphi X, J_\varphi Y] - J_\varphi[J_\varphi X, Y] - J_\varphi[X, J_\varphi Y] \\ &= \frac{4}{5}\varphi[X, Y] - \frac{6}{5}[X, Y] + \frac{4}{5}([\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]) \\ &= \frac{4}{5} N_\varphi(X, Y). \end{aligned}$$

□

5 Classification of almost Norden golden manifolds

According to Definition 4, we are going to classify almost Norden golden manifolds using the classes of almost Norden manifolds obtained by Ganchev and Borisov recalled in Theorem 9. Let (M, φ, g) be an almost Norden golden manifold, we need to rewrite the characteristic conditions of the different classes of almost Norden manifolds (M, J_φ, g) in terms of tensors and forms defined from φ.

Starting from the expression (8) of the codifferential $\delta\Phi$ and using adapted local basis to (J_φ, g), we introduce the next 1-form defined from φ.

Definition 23 Let (M, φ, g) be an almost Norden golden manifold. Then, there exist a 1-form $\delta\varphi$, called the codifferential of φ, which locally can be expressed by means of an adapted local basis $(X_1, \ldots, X_n, Y_1, \ldots, Y_n)$ to (J_φ, g) as follows:

$$\delta\varphi(X) = \sum_{i=1}^{n}(g((\nabla^{\mathrm{g}}_{X_i}\varphi)Y_i, X) + g((\nabla^{\mathrm{g}}_{Y_i}\varphi)X_i, X)), \quad \forall X \in \mathfrak{X}(M).$$

Identities (8) and (28) allow to establish easily the relation between the 1-forms $\delta\Phi$ and $\delta\varphi$

$$\delta\varphi(X) = -\frac{\sqrt{5}}{2}\delta\Phi(X), \quad \forall X \in \mathfrak{X}(M).$$

Bearing in mind (1), (28), (29), (31), (32) and (33) and the above equality, one obtain the below classification according with Theorem 9 by means of the tensors $\nabla^{\mathrm{g}}\varphi$ and N_φ and the 1-form $\delta\varphi$.

Theorem 24 *Let (M, φ, g) be an $2n$-dimensional almost Norden golden manifold. Then, one has the following classes of this kind of manifolds:*

(i) *The class $\mathscr{W}_0$ or Kähler Norden golden manifolds characterized by the condition*

$$\nabla^{\mathrm{g}}\varphi = 0.$$

(ii) *The class $\mathscr{W}_1$ characterized by the condition*

$$\begin{aligned} g((\nabla^{\mathrm{g}}_X\varphi)Y, Z) = {} & \frac{1}{5n}(g(X, Y)\delta\varphi(3Z - \varphi Z) + g(X, Z)\delta\varphi(3Y - \varphi Y)) \\ & + \frac{1}{5n}(g(X, \varphi Y)\delta\varphi(2\varphi Z - Z) + g(X, \varphi Z)\delta\varphi(2\varphi Y - Y)), \end{aligned}$$

for all vector fields X, Y, Z on M.

(iii) *The class $\mathscr{W}_2$ characterized by the conditions*

$$N_\varphi = 0, \delta\varphi = 0.$$

(iv) *The class $\mathscr{W}_3$ or quasi-Kähler Norden golden manifolds characterized by the condition*

$$(\nabla^{\mathrm{g}}_X\varphi)\varphi Y + (\nabla^{\mathrm{g}}_{\varphi X}\varphi)Y + (\nabla^{\mathrm{g}}_Y\varphi)\varphi X + (\nabla^{\mathrm{g}}_{\varphi Y}\varphi)X = (\nabla^{\mathrm{g}}_X\varphi)Y + (\nabla^{\mathrm{g}}_Y\varphi)X,$$

for all X, Y *vector fields on* M.

(v) *The class* $\mathscr{W}_1 \oplus \mathscr{W}_2$ *or integrable Norden golden manifolds characterized by the condition*

$$N_\varphi = 0.$$

(vi) *The class* $\mathscr{W}_2 \oplus \mathscr{W}_3$ *characterized by the condition*

$$\delta\varphi = 0.$$

(vii) *The class* $\mathscr{W}_1 \oplus \mathscr{W}_3$ *characterized by the condition*

$$\underset{XYZ}{\mathfrak{S}} (\nabla^{\mathrm{g}}_X \varphi)(Y, Z) = \frac{2}{5n} \underset{XYZ}{\mathfrak{S}} (g(X, Y)\delta\varphi(3Z - \varphi Z) + g(X, \varphi Y)\delta\varphi(2\varphi Z - Z)),$$

for all X, Y, Z *vector fields on* M.

(viii) *The class* $\mathscr{W}$ *or the whole class of almost Norden golden manifolds.*

Taking as starting-point Theorem 11, the above-mentioned identities that have allowed us to rewrite the defining conditions of the classes of almost Norden manifolds to characterize the classes of almost Norden golden manifolds by means of tensors and 1-forms defined from the structure (φ, g), combined with the defining condition of the well-adapted connection of the manifold (M, φ, g), given in (30), and its torsion form t^{w}, allow us to obtain the classification of almost Norden golden manifolds by means of the well-adapted connection below. Note that the second characteristic condition of the class $\mathscr{W}_3$ has been obtained starting from condition (27).

Theorem 25 *Let* (M, φ, g) *be a* $2n$*-dimensional almost Norden golden manifold. The classes given in Theorem 24 can be characterized by means of the torsion tensor and the torsion form of the well-adapted connection as follows:*

(i) *The class* $\mathscr{W}_0$ *or Kähler golden Norden manifolds characterized by the condition*

$$T^{\mathrm{w}}(X, Y) = 0, \quad \forall X, Y \in \mathfrak{X}(M).$$

(ii) *The class* $\mathscr{W}_1$ *characterized by the condition*

$$T^{\mathrm{w}}(X, Y) = \frac{1}{5n}(t^{\mathrm{w}}(X)(3Y - \varphi Y) - t^{\mathrm{w}}(Y)(3X - \varphi X)) + \frac{1}{5n}(t^{\mathrm{w}}(\varphi X)(2\varphi Y - Y) - t^{\mathrm{w}}(\varphi Y)(2\varphi X - X)), \quad \forall X, Y \in \mathfrak{X}(M).$$

(iii) *The class* $\mathscr{W}_2$ *characterized by the conditions*

$$2(T^{\mathrm{w}}(\varphi X, \varphi Y) - T^{\mathrm{w}}(X, Y)) = T^{\mathrm{w}}(\varphi X, Y) + T^{\mathrm{w}}(X, \varphi Y), t^{\mathrm{w}}(X) = 0, \quad \forall X \in \mathfrak{X}(M).$$

(iv) *The class $\mathscr{W}_3$ or quasi-Kähler Norden golden manifolds characterized by one the two equivalent conditions*

$$T^{\mathrm{w}}(\varphi X, Y) + \varphi T^{\mathrm{w}}(X, Y) = T^{\mathrm{w}}(X, Y), \quad \forall X, Y \in \mathfrak{X}(M),$$
$$2T^{\mathrm{w}}(\varphi X, \varphi Y) + 3T^{\mathrm{w}}(X, Y) = T^{\mathrm{w}}(\varphi X, Y) + T^{\mathrm{w}}(X, \varphi Y), \quad \forall X, Y \in \mathfrak{X}(M).$$

(v) *The class $\mathscr{W}_1 \oplus \mathscr{W}_2$ or integrable Norden golden manifolds characterized by the condition*

$$2(T^{\mathrm{w}}(\varphi X, \varphi Y) - T^{\mathrm{w}}(X, Y)) = T^{\mathrm{w}}(\varphi X, Y) + T^{\mathrm{w}}(X, \varphi Y), \quad \forall X, Y \in \mathfrak{X}(M).$$

(vi) *The class $\mathscr{W}_2 \oplus \mathscr{W}_3$ characterized by the condition*

$$t^{\mathrm{w}}(X) = 0, \quad \forall X \in \mathfrak{X}(M).$$

(vii) *The class $\mathscr{W}_1 \oplus \mathscr{W}_3$ characterized by the condition*

$$T^{\mathrm{w}}(X, Y) - T^{\mathrm{w}}(\varphi X, Y) - \varphi T^{\mathrm{w}}(X, Y) = \frac{1}{n}(t^{\mathrm{w}}(Y)\varphi X - t^{\mathrm{w}}(\varphi Y)X), \quad \forall X \in \mathfrak{X}(M).$$

(viii) *The class $\mathscr{W}$ or the whole class of almost Norden golden manifolds with null trace.*

Using the expressions of the well-adapted connection of an almost Norden manifold in the integrable and quasi-Kähler cases shown in Remarks 15 and 16 , one obtains the expression of well-adapted connection of an almost Norden golden manifold in both cases as below.

Remark 26 Identities (23) and (31) allow us to obtain the expression of the well-adapted connection of almost Norden golden manifolds of the classes $\mathscr{W}_1$, $\mathscr{W}_2$ and $\mathscr{W}_1 \oplus \mathscr{W}_2$ as follows:

$$\nabla^{\mathrm{w}}_X Y = \nabla^{\mathrm{g}}_X Y - \frac{1}{5}(\nabla^{\mathrm{g}}_X \varphi)Y + \frac{2}{5}(\nabla^{\mathrm{g}}_X \varphi)\varphi Y, \quad \forall X, Y, \in \mathfrak{X}(M).$$

Analogously, in the case of quasi-Kähler Norden golden manifolds, identities (25), (31) and (32) allow us to obtain the next expression of the well-adapted connection

$$\nabla^{\mathrm{w}}_X Y = \nabla^{\mathrm{g}}_X Y + \frac{2}{5}(\nabla^{\mathrm{g}}_X \varphi)\varphi Y + \frac{1}{5}((\nabla^{\mathrm{g}}_Y \varphi)\varphi X - (\nabla^{\mathrm{g}}_{\varphi Y} \varphi)X - (\nabla^{\mathrm{g}}_X \varphi)Y), \quad \forall X, Y \in \mathfrak{X}(M).$$

Finally, as direct consequence of Corollary 17 and identities (31) and (33) we can obtain the next characterizations of integrable and quasi-Kähler Norden golden manifolds using the explicit expression of the torsion tensor of the well-adapted connection.

Corollary 27 *Let (M, φ, g) be an almost Norden golden manifold.*

(i) *The manifold* (M, φ, g) *is of class* $\mathscr{W}_1 \oplus \mathscr{W}_2$ *if and only if*

$$T^{\mathrm{w}}(X, Y) = \frac{2}{5}\left((\nabla^{\mathrm{g}}_X \varphi)\varphi Y - (\nabla^{\mathrm{g}}_Y \varphi)\varphi X\right) - \frac{1}{5}\left((\nabla^{\mathrm{g}}_X \varphi)Y - (\nabla^{\mathrm{g}}_Y \varphi)X\right),$$

for all X, Y *vector fields on* M.

(ii) *The manifold* (M, φ, g) *is of class* $\mathscr{W}_3$ *if and only if*

$$T^{\mathrm{w}}(X, Y) = \frac{1}{5} N_\varphi(X, Y),$$

for all X, Y *vector fields on* M.

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References

1. Bilen, L., Turanli, S., Gezer, A.: On Kähler–Norden–Codazzi golden structures on pseudo-Riemannian manifolds. Int. J. Geom. Methods Mod. Phys. **15**, 1850080 (2018)
2. Crasmareanu, M., Hreţcanu, C.E.: Golden differential geometry. Chaos Solitons Fractals **38**, 1229–1238 (2008)
3. Erdem, S.: On product and golden structures and harmonicity. Turk. J. Math. **42**, 444–470 (2018)
4. Etayo, F., deFrancisco, A., Santamaría, R.: The Chern connection of a ($J^2 = \pm 1$)-metric manifold of class $\mathscr{G}_1$. Mediterr. J. Math. **15**, 157 (2018)
5. Etayo, F., Santamaría, R.: The well adapted connection of a ($J^2 = \pm 1$)-metric manifold. RACSAM **111**, 355–375 (2017)
6. Etayo, F., Santamaría, R., Upadhyay, A.: On the geometry of almost golden Riemannian manifolds. Mediterr. J. Math. **14**, 187 (2017)
7. Ganchev, G.T., Borisov, A.V.: Note on the almost complex manifolds with a Norden metric. C.R. Acad. Bulg. Sci. **39**, 31–34 (1986)
8. Gadea, P.M., Masqué, J.M.: Classification of almost parahermitian manifolds. Rend. Mat. Roma **11**, 377–396 (1991)
9. Ganchev, G., Mihova, V.: Canonical connection and the canonical conformal group on an almost complex manifold with B-metric. Ann. Univ. Sofia Fac. Math. Inform. **81**, 195–206 (1987)
10. Gezer, A., Cengiz, N., Salimov, A.: On integrability of golden Riemannian structures. Turk. J. Math. **37**, 693–703 (2013)
11. Gezer, A., Karaman, C.: Golden-Hessian structures. Proc. Natl. Acad. Sci. India Sect. A Phys. Sci. **86**, 41–46 (2016)
12. Goldberg, S.I., Yano, K.: Polynomial structures on manifolds. Kodai Math. Sem. Rep. **22**, 199–218 (1970)
13. Gray, A., Hervella, L.M.: The sixteen classes of almost Hermitian manifolds and their linear invariants. Ann. Mat. Pura Appl. **123**, 35–58 (1980)
14. Naveira, A.M.: A classification of Riemannian almost-product manifolds. Rend. Mat. Roma **3**, 577–592 (1983)