June 4-6, 2018, Ponta Delgada, Azores, Portugal

Path planning trajectories in fluid environments

Luís Machado
Institute of Systems and Robotics
Univ. Coimbra
and
Department of Mathematics
Univ. Trás-os-Montes e Alto Douro
Vila Real, Portugal
Email: lmiguel@utad.pt

Fátima Silva Leite
Institute of Systems and Robotics
and
Department of Mathematics
Univ. Coimbra
Coimbra, Portugal
Email: fleite@mat.uc.pt

Maria Teresa T. Monteiro
ALGORITMI Research Centre
Department of Production and Systems,
Univ. Minho
Braga, Portugal
Email: tm@dps.uminho.pt

***Abstract*—This paper deals with the problem of finding a mission path that minimizes acceleration and drag while a vehicle moves from an initial position to a final target in fluid environments. A variational problem will be formulated in the general context of manifolds, where the energy functional depends on acceleration and drag forces. The corresponding Euler-Lagrange equations will be derived. Questions regarding the integrability of the Euler-Lagrange equations are the main challenge of this problem even when the geometry of the configuration space is not taken into consideration. This is mainly due to the fact that the power needed to overcome the drag forces is proportional to the cube of the speed. A numerical optimization approach will be presented in order to obtain approximate solutions for the problem in some particular configuration spaces.**

I. Introduction

In path planning problems of autonomous marine or aerial vehicles, it is frequently required to find a mission path that minimizes acceleration and drag while the vehicle moves from an initial position to a final target passing through a series of waypoints [12]. The resistance that the fluid environment offers to the moving vehicle is characterized by the drag force. To be more precise, drag is a mechanical force generated by the interaction and contact of a solid body with a fluid (gas or liquid). It is obtained by the difference in velocity between the solid object and the fluid and acts in the opposite direction to the motion of the object. Drag forces can be found in several daily life situations. For instance, it is clear the difficulty of walking in water because of the much greater resistance it offers to motion when compared with air. Also, when we extend our arms out of the window of a moving car one can feel the strong push of the wind. Therefore it is clear that in some applications it is important to minimize the drag forces. This is extremely related to the reduction of the fuel consumption in automobiles, submarines and aircrafts and also to improve safety and durability of structures subject to strong winds. In other cases, the drag force produces very beneficial effects and it is therefore required to maximize it. It is because of the drag force that is possible to parachute, for pollen to fly for distant locations and also to promote the beauty of the ocean waves [3]. The magnitude of the drag, which depends on the density of the fluid, the speed and the size, shape and orientation of the body is, typically, proportional to the square of the speed. In this case, the power needed to overcome the drag force is proportional to the cube of the speed.

978-1-5386-5346-3/18/$31.00 © 2018 IEEE

In this paper we are interested in determine optimal trajectories of a vehicle moving in a fluid environment that minimize not only the power needed to overcome changes in velocity but also the drag forces. Therefore, a variational problem where the energy functional depends on the acceleration and drag, is formulated on the general context of manifolds and the corresponding Euler-Lagrange equations are derived. As we will see, the presence of the drag term increases substantially the complexity of the problem even when the problem is only formulated on a Euclidean space. Algebraic integrability properties of the Euler-Lagrange equations in Euclidean spaces have been partially studied in [16] using the theory of Darboux polynomials.

In the absence of drag, the problem boils down to the classical problem of finding geometric cubic polynomials prescribing initial and final positions and velocities [14], [5]. Even in this simpler case, a major issue that remains unsolved is finding closed form solutions for the Euler-Lagrange equations. This drawback motivated several authors to look for alternative approaches. We refer to [4], [5], [9], [11] and references therein to mention a few.

To overcome the integrability issues raised from the variational problem proposed in the paper, a numerical algorithm to find approximate solutions for this highly nonlinear optimization problem is proposed and some of the numerical illustrations will be shown for some particular curved spaces.

II. Problem's formulation

A. Preliminaries and notations

Let M be a Riemannian manifold of dimension n. If $p \in M$, T_pM and TM denote respectively the tangent space of M at p and the tangent bundle of M. Also denote by ∇ the unique affine connection on M that is compatible with the Riemannian metric $\langle \cdot, \cdot \rangle$. If $x : [0, T] \to M$ is a smooth curve on M, then the covariant derivative of a vector field Y along x with respect to ∇ will be denoted by $\frac{DY}{dt}$. For the particular case when the manifold is embedded in some Euclidean space, the covariant derivative at t is simply obtained by projecting the usual derivative $\frac{dY}{dt}(t)$ onto the tangent space of M at $x(t)$.

Define higher-order covariant derivatives of x by

$$\frac{D^k x}{dt^k} = \frac{D}{dt}\Big(\frac{D^{k-1}x}{dt^{k-1}}\Big), \quad k \geq 2,$$

where, for convenience, $\frac{Dx}{dt}$ is used to denote the velocity vector field of the curve x, $\dot{x} = \frac{dx}{dt}$.

The curvature tensor, denoted by R, is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

for smooth vector fields X, Y and Z and, among others, it satisfies the property

$$\big\langle R(X,Y)Z, W\big\rangle = \big\langle R(W,Z)Y, X\big\rangle. \tag{1}$$

Applying the Riemannian connection ∇ to the curvature tensor R, one can define a new tensor field ∇R by

$$\begin{aligned}(\nabla_W R)(X,Y)Z = \nabla_W\big(R(X,Y)Z\big) - R(\nabla_W X, Y)Z \\ - R(X, \nabla_W Y)Z - R(X,Y)\nabla_W Z\end{aligned} \tag{2}$$

For more details about fundamental concepts of differential geometry we refer to the classical books on the subject [8], [6], [10], [13].

B. Description of the problem and derivation of the Euler-Lagrange equations

Denote by Ω the class of all piecewise smooth paths $x : [0,T] \to M$ such that $x(0)$, $x(T)$, $\dot{x}(0)$ and $\dot{x}(T)$ are fixed, and consider the following minimization problem

$$\min_{x\in\Omega} \int_0^T \Big\langle \frac{D^2x}{dt^2}, \frac{D^2x}{dt^2}\Big\rangle + \tau\Big\langle \frac{dx}{dt}, \frac{dx}{dt}\Big\rangle^{\frac{3}{2}}\, dt, \tag{$\mathcal{P}$}$$

where τ denotes a nonnegative real parameter. The parameter τ can be interpreted as a control parameter that somehow will measure the effect of the drag force in the optimal path x going from the initial position $x(0)$ to the final target $x(T)$ while prescribing initial and final velocities.

Next we present the first order necessary optimality conditions for problem ($\mathcal{P}$). Since the proof follows similar proofs that appeared already in the literature (see, for instance, [5]), we only present a sketch.

Theorem 1. *A necessary condition for a curve $x : [0,T] \to M$ to be a solution for problem ($\mathcal{P}$) is that*

$$\frac{D^4x}{dt^4} + R\Big(\frac{D^2x}{dt^2}, \frac{dx}{dt}\Big)\frac{dx}{dt} - \frac{3}{2}\tau\frac{D}{dt}\Big[\Big\langle\frac{dx}{dt}, \frac{dx}{dt}\Big\rangle^{\frac{1}{2}}\frac{dx}{dt}\Big] = 0. \tag{3}$$

Proof. (Sketch) Denote the energy functional by

$$\mathcal{J}(x) = \int_0^T \Big\langle \frac{D^2x}{dt^2}, \frac{D^2x}{dt^2}\Big\rangle + \tau\Big\langle \frac{dx}{dt}, \frac{dx}{dt}\Big\rangle^{\frac{3}{2}}\, dt.$$

Let ε be a positive real number and

$$\begin{array}{rccc}\alpha: & (-\varepsilon,\varepsilon)\times[0,T] & \longrightarrow & M \\ & (s,t) & \longmapsto & \alpha(s,t)\end{array}$$

be a one parameter variation of x such that $\alpha(0,t) = x(t)$. Denote by $W(t) = \dfrac{\partial\alpha}{\partial s}(0,t)$ the variational vector field associated to the variation α.

In order for x to be a solution for problem ($\mathcal{P}$) one must have

$$\frac{d}{ds}J(\alpha_s)\Big|_{s=0} = 0,$$

for all variations α.

Using the fact that if V is a vector field along the parameterized surface α, then

$$\frac{D}{\partial s}\Big(\frac{DV}{\partial t}\Big) = \frac{D}{\partial t}\Big(\frac{DV}{\partial s}\Big) + R\Big(\frac{\partial\alpha}{\partial s}, \frac{\partial\alpha}{\partial t}\Big)V,$$

together with property (1) of the curvature tensor, it is possible to write

$$\begin{aligned}&\frac{d}{ds}J(\alpha_s)\\ &= 2\int_0^T \Big\langle \frac{D^4\alpha}{\partial t^4} + R\Big(\frac{D^2\alpha}{\partial t^2}, \frac{\partial\alpha}{\partial t}\Big)\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial s}\Big\rangle dt\\ &\quad - 3\tau\int_0^T \Big\langle \frac{D}{\partial t}\Big[\Big\langle\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial t}\Big\rangle^{\frac{1}{2}}\frac{\partial\alpha}{\partial t}\Big], \frac{\partial\alpha}{\partial s}\Big\rangle dt\\ &\quad + 2\Big\langle \frac{D}{dt}\Big(\frac{\partial\alpha}{\partial s}\Big), \frac{D^2\alpha}{\partial t^2}\Big\rangle\Big|_0^T\\ &\quad - 2\Big\langle \frac{D^3\alpha}{\partial t^3} - \frac{3}{2}\tau\Big\langle\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial t}\Big\rangle^{\frac{1}{2}}\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial s}\Big\rangle\Big|_0^T.\end{aligned}$$

Now, setting $s = 0$ in the above and using the fact that $W(0) = W(T) = 0$ and also $\frac{DW}{dt}(0) = \frac{DW}{dt}(T) = 0$, one gets

$$\begin{aligned}&\frac{d}{ds}J(\alpha_s)\Big|_{s=0}\\ &= 2\int_0^T \Big\langle \frac{D^4x}{dt^4} + R\Big(\frac{D^2x}{dt^2}, \frac{dx}{dt}\Big)\frac{dx}{dt}, W\Big\rangle dt\\ &\quad - 3\tau\int_0^T \Big\langle \frac{D}{dt}\Big[\Big\langle\frac{dx}{dt}, \frac{dx}{dt}\Big\rangle^{\frac{1}{2}}\frac{dx}{dt}\Big], W\Big\rangle dt.\end{aligned}$$

The result follows by considering appropriate vector fields W. □

Remark 1. In the absence of drag forces, the Euler-Lagrange equation (3) reduces to

$$\frac{D^4x}{dt^4} + R\Big(\frac{D^2x}{dt^2}, \frac{dx}{dt}\Big)\frac{dx}{dt} = 0,$$

which represents the differential equation that characterizes the geometric cubic polynomials on M appearing for the first time in the literature in [14].

Motivated by the invariants along a geometric cubic polynomial derived in [2] and [1], in the next two propositions we obtain the expressions for two invariants along a curve x satisfying the differential equation (3). The second invariant is derived under the assumption that the Riemannian manifold M is locally symmetric. In this case, it can be proved that the curvature tensor is parallel, i.e., $\nabla R = 0$ (see [8]).

Proposition 1. *The quantity*

$$I_1 = \Big\langle \frac{D^3x}{dt^3}, \frac{dx}{dt} \Big\rangle - \frac{1}{2}\Big\langle \frac{D^2x}{dt^2}, \frac{D^2x}{dt^2} \Big\rangle - \tau \Big\langle \frac{dx}{dt}, \frac{dx}{dt} \Big\rangle^{\frac{3}{2}}, \tag{4}$$

is invariant along a curve x satisfying (3).

Proof. The proof closely follows the one given in [2] for geometric cubic polynomials.

Take the inner product on both sides of the Euler-Lagrange equation (3) with $\frac{dx}{dt}$, and note that

$$\frac{3}{2}\Big\langle \frac{D}{dt}\Big[\Big\langle \frac{dx}{dt}, \frac{dx}{dt}\Big\rangle^{\frac{1}{2}} \frac{dx}{dt}\Big], \frac{dx}{dt}\Big\rangle = \frac{d}{dt}\Big\langle \frac{dx}{dt}, \frac{dx}{dt}\Big\rangle^{\frac{3}{2}}.$$

$\square$

Proposition 2. *If M is a locally symmetric Riemannian manifold, then the quantity*

$$\begin{aligned} I_2 = \Big\langle \frac{D^3x}{dt^3}, \frac{D^3x}{dt^3} \Big\rangle + \Big\langle R\Big(\frac{D^2x}{dt^2}, \frac{dx}{dt}\Big)\frac{dx}{dt}, \frac{D^2x}{dt^2} \Big\rangle \\ - 3\tau \Big\langle \frac{dx}{dt}, \frac{dx}{dt} \Big\rangle^{\frac{1}{2}} \Big\langle \frac{D^3x}{dt^3}, \frac{dx}{dt} \Big\rangle + \frac{9}{4}\tau^2 \Big\langle \frac{dx}{dt}, \frac{dx}{dt} \Big\rangle^2, \end{aligned}$$

is preserved along any curve x that satisfies the Euler-Lagrange equations (3).

Proof. The proof is similar to the one given in [1] for Riemannian cubic polynomials.

Note that since M is locally symmetric, the tensor ∇R, defined by (2), vanishes identically.

The expression for I_2 is therefore obtained by taking the inner product on both sides of the Euler-Lagrange equation (3) with $\frac{D^3x}{dt^3}$. In this case, some computations yield

$$\begin{aligned} &\Big\langle \frac{D}{dt}\Big[\Big\langle \frac{dx}{dt}, \frac{dx}{dt}\Big\rangle^{\frac{1}{2}} \frac{dx}{dt}\Big], \frac{D^3x}{dt^3}\Big\rangle \\ &= \frac{d}{dt}\Big[\Big\langle \frac{dx}{dt}, \frac{dx}{dt}\Big\rangle^{\frac{1}{2}} \Big\langle \frac{D^3x}{dt^3}, \frac{dx}{dt}\Big\rangle - \frac{3}{4}\tau \Big\langle \frac{dx}{dt}, \frac{dx}{dt}\Big\rangle^2\Big], \end{aligned}$$

and

$$\Big\langle R\Big(\frac{D^2x}{dt^2}, \frac{dx}{dt}\Big)\frac{dx}{dt}, \frac{D^3x}{dt^3}\Big\rangle = \frac{1}{2}\frac{d}{dt}\Big\langle R\Big(\frac{D^2x}{dt^2}, \frac{dx}{dt}\Big)\frac{dx}{dt}, \frac{D^2x}{dt^2}\Big\rangle.$$

$\square$

As it can be seen, equation (3) is highly nonlinear and in the next section we start its analysis for the particular case of the Euclidean spaces.

1) Particular case of Euclidean spaces: For the case when $M = \mathbb{R}^n$, that is, assuming that the configuration space is flat, the curvature tensor vanishes everywhere and the covariant derivative reduces to the usual derivative. In this case, the Euler-Lagrange equation becomes

$$\ddddot{x} - \tfrac{3}{2}\tau\langle \dot{x}, \dot{x}\rangle^{-\frac{1}{2}}\langle \ddot{x}, \dot{x}\rangle \dot{x} - \tfrac{3}{2}\tau\langle \dot{x}, \dot{x}\rangle^{\frac{1}{2}}\ddot{x} = 0, \tag{5}$$

or alternatively

$$\frac{d}{dt}\Big(\dddot{x} - \tfrac{3}{2}\tau\langle \dot{x}, \dot{x}\rangle^{\frac{1}{2}}\dot{x}\Big) = 0, \tag{6}$$

meaning that the quantity

$$I = \dddot{x} - \tfrac{3}{2}\tau\langle \dot{x}, \dot{x}\rangle^{\frac{1}{2}}\dot{x},$$

is an invariant along an optimal trajectory x.

The analogous to the invariants I_1 and I_2 given in propositions 1 and 2 are given, respectively, by

$$I_1 = \big\langle \dddot{x}, \dot{x}\big\rangle - \tfrac{1}{2}\big\langle \ddot{x}, \ddot{x}\big\rangle - \tau\big\langle \dot{x}, \dot{x}\big\rangle^{\frac{3}{2}},$$

and

$$I_2 = \big\langle \dddot{x}, \dddot{x}\big\rangle - 3\tau\big\langle \dot{x}, \dot{x}\big\rangle^{\frac{1}{2}}\big\langle \dddot{x}, \dot{x}\big\rangle + \tfrac{9}{4}\tau^2\big\langle \dot{x}, \dot{x}\big\rangle^2.$$

Remark 2. The invariant I_1 comes directly from Noether's symmetry Theorem [7], when the system is conservative. In fact, when the Lagrangian L does not depend on time explicitly, the quantity

$$\Big\langle \frac{d}{dt}\Big(\frac{\partial L}{\partial \ddot{x}}\Big), \dot{x}\Big\rangle - \Big\langle \frac{\partial L}{\partial \ddot{x}}, \ddot{x}\Big\rangle - \Big\langle \frac{\partial L}{\partial \dot{x}}, \dot{x}\Big\rangle + L$$

is preserved along each extremal of problem $(\mathcal{P})$.

Based on some of the invariants given above, authors in [16] provided a very interesting study of the Euler-Lagrange equation for Euclidean spaces using the theory of Darboux polynomials. However, this study was inconclusive and our purpose here is to show the behavior of the solutions of the proposed optimization problem for certain Riemannian manifolds.

In order to get some insight about the solutions of the Euler-Lagrange equation, we start to consider the one-dimensional case, that is $M = \mathbb{R}$. In this case, the Euler-Lagrange equation (5) reduces to

$$\ddddot{x} - 3\tau\dot{x}\ddot{x} = 0, \tag{7}$$

which is still nonlinear (unless $\tau = 0$) and only approximate solutions can be obtained. In Fig. 1 it is shown an approximate solution of (7) for several values of τ and the corresponding velocity is sketched in Fig. 2. When $\tau = 0$ the solution will be, as expected, the cubic polynomial satisfying the required boundary conditions. From the analysis of Fig. 2, as long as the parameter τ increases one can notice abrupt changes on the velocity of the curve.

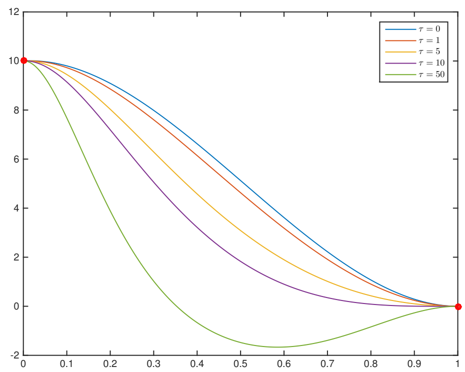

Fig. 1. Approximate solution for the Euler-Lagrange equation (7) with boundary conditions $x(0) = 10$, $x(1) = 0$, $\dot{x}(0) = 1$ and $\dot{x}(1) = 0$.

2018 13th APCA International Conference on Automatic Control and Soft Computing (CONTROLO)
June 4-6, 2018, Ponta Delgada, Azores, Portugal

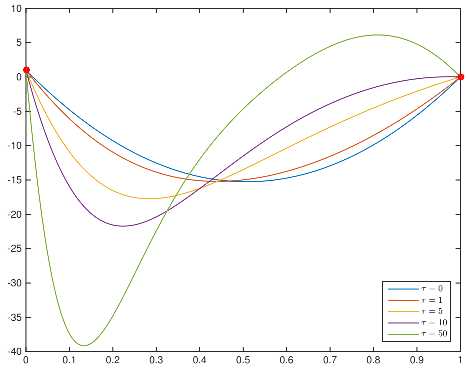

Fig. 2. Approximate solution for the velocity of the optimal solutions represented in Fig. 1.

Due to the high nonlinearity of the Euler-Lagrange equation, we propose in the next section a numerical algorithm to obtain approximate solutions for the optimization problem $(\mathcal{P})$ for Euclidean spaces and Euclidean spheres.

III. Numerical optimization algorithm

In this section, we consider the cases when M is the Euclidean space $\mathbb{R}^n$ or the unit $n-$sphere S^n, and present a numerical algorithm that generates approximate solutions for the optimization problem $(\mathcal{P})$. As we already mentioned for the Euclidean case, the covariant derivatives reduce to the usual derivatives. Since S^n is an embedded manifold of the Euclidean space $\mathbb{R}^{n+1}$, the covariant derivatives at each time t are obtained by the projection of the corresponding usual derivatives at t onto the tangent space of S^n at $x(t)$. In particular,

$$\frac{D^2x}{dt^2} = \ddot{x} - \langle \ddot{x}, x\rangle x,$$

and, in this case, the optimization problem $(\mathcal{P})$ can be written as

$$\min_{x\in\Omega} \int_0^T \langle \ddot{x}, \ddot{x}\rangle - 2\langle \ddot{x}, x\rangle^2 + \langle \ddot{x}, x\rangle^2 \langle x, x\rangle + \tau\langle \dot{x}, \dot{x}\rangle^{\frac{3}{2}}\ dt.$$

The strategy that we propose to solve the above problem consists in implementing a discretization procedure for the derivatives $\dot{x}$ and $\ddot{x}$ in the interval $[0, T]$ using finite differences (forward, central and backward to the lower, interior and upper bound instants of time, respectively). This discretization procedure originates a set of k vectors in a Euclidean space, $x(t_1), \dots, x(t_k)$, which are the unknown variables of the problem. For convenience of implementation, we consider the discretization step $h = t_{i+1} - t_i$, $i = 1, \dots, k-1$ a constant value. The objective function is therefore implemented via numerical integration, more precisely using the trapezium rule for the k variables $x(t_1), \dots, x(t_k)$. Finally, the nonlinear constrained optimization problem $(\mathcal{P})$ is solved using a classical optimization routine based in the Sequential Quadratic Programming (SQP) technique [15]. Its solution is the set of values $x(t_1), \dots, x(t_k)$ that estimate the whole curve $t \mapsto x(t)$, $t \in [0, T]$.

Next, we describe the high level Algorithm 1 that solves numerically the proposed nonlinear optimization problem $(\mathcal{P})$ and whose solutions estimate the optimal solutions of the problem. This algorithm was implemented using the MATLAB code. Several routines from MATLAB toolboxes were used. The `trapz` routine was used to estimate the value of the integral function and the `fmincon` routine was used to find a local minimum for the objective function.

Algorithm 1

1: select the discretization step h and the number of variables k

2: discretize $\dot{x}$ and $\ddot{x}$ using finite differences

Forward finite differences:

$$\dot{x}(t_1) \approx \frac{1}{h}\left[x(t_2) - x(t_1)\right]$$

$$\ddot{x}(t_1) \approx \frac{1}{h^2}\left[x(t_3) - 2x(t_2) + x(t_1)\right]$$

Central finite differences:

$$\dot{x}(t_i) \approx \frac{1}{2h}\left[x(t_{i+1}) - x(t_{i-1})\right]$$

$$\ddot{x}(t_i) \approx \frac{1}{h^2}\left[x(t_{i+1}) - 2x(t_i) + x(t_{i-1})\right]$$

Backward finite differences:

$$\dot{x}(t_k) \approx \frac{1}{h}\left[x(t_k) - x(t_{k-1})\right]$$

$$\ddot{x}(t_k) \approx \frac{1}{h^2}\left[x(t_k) - 2x(t_{k-1}) + x(t_{k-2})\right]$$

3: estimate the integral

$$\int_a^b f(t)dt \approx \frac{h}{2}[f(t_1) + 2f(t_2) + 2f(t_3) + \cdots + 2f(t_{k-2}) + 2f(t_{k-1}) + f(t_k)]$$

4: solve the optimization problem to find the variables $x(t_1), \dots, x(t_k)$

5: design the curve x

Numerical experiments on $\mathbb{R}^3$ and S^2 are presented in figures 3 and 4, respectively. Both simulations have been performed for several values of τ and the number of variables considered was $k = 51$. In both cases, we have assumed equally spaced times and the value of the discretization step used was $h = 0.02$.

Fig. 3 illustrates approximate solutions for problem $(\mathcal{P})$ in $\mathbb{R}^3$ for several values of τ and boundary conditions $x(0) = (0, 0, 1)$, $x(T) = \left(\frac{1}{2}, \frac{1}{2}, -1\right)$, $\dot{x}(0) = \left(2, 2, 2\right)$ and $\dot{x}(T) = (0, 1, 0)$. Approximate solutions for problem $(\mathcal{P})$ in S^2 for several values of τ are represented in Fig. 4 for boundary conditions $x(0) = (0, 0, 1)$, $x(T) = \left(\frac{1}{2}, \frac{1}{2}, -\frac{\sqrt{2}}{2}\right)$, $\dot{x}(0) = \left(0, 0, 0\right)$ and $\dot{x}(T) = (1, 1, \sqrt{2})$.

IV. Conclusion

We formulated a variational problem on a Riemannian manifold in order to minimize acceleration together with drag forces while a vehicle is moving in a fluid environment from an initial position to a final target. The Euler-Lagrange equations associated to this problem were derived in Theorem 1. Due to the high nonlinearity of the Euler-Lagrange equations, this problem represents a source of many challenging questions regarding numerical integration on manifolds. To overcome

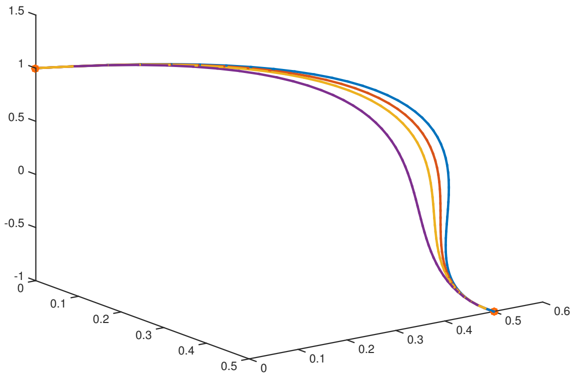

Fig. 3. Approximate solutions for problem $(\mathcal{P})$ in $\mathbb{R}^3$ for $\tau = 0, 1, 2, 5$.

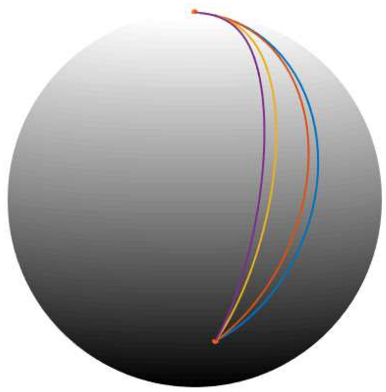

Fig. 4. Approximate solutions for problem $(\mathcal{P})$ in S^2 for $\tau = 0, 1, 10, 20$.

this difficulty, we proposed in Section III a possible numerical algorithm to obtain approximate solutions for the variational problem in the particular cases of the Euclidean spaces and the unit spheres. We believe that similar algorithms for solving the problem in other curved spaces can be designed and this will be our future purpose.

V. Acknowledgements

The first two authors acknowledge “Fundação para a Ciência e a Tecnologia” (FCT−Portugal) and COMPETE 2020 Program for financial support through project UID-EEA-00048-2013.

The third author was supported by ALGORITMI R&D Center under the project PEst-UID/CEC/00319/2013 financed by “Fundação para a Ciência e a Tecnologia” (FCT−Portugal).

References

[1] L. Abrunheiro and M. Camarinha, *Riemannian cubic polynomials*, Rend. Sem. Mat. Univ. Pol. Torino - Vol. 63, no. 4, Control Theory and Stabil. I (2005), 297–303.
[2] M. Camarinha, *The geometry of cubic polynomials on Riemannian manifolds*, Ph. D. Thesis, University of Coimbra, Portugal, 1996.
[3] Y. Cengel and J. Cimbala, *Fluid Mechanics Fundamentals and Applications*, McGraw-Hill Science/Engineering/Math, 3^{rd} edition, 2013.
[4] P. Crouch, G. Kun and F. Silva Leite, *The De Casteljau algorithms on Lie groups and spheres*, Journal of Dynamical and Control Systems, 5 (1999), No. 3, 397–429.
[5] P. Crouch and F. Silva Leite, *The Dynamic Interpolation Problem: on Riemannian Manifolds, Lie Groups and Symmetric Spaces*, Journal of Dynamical and Control Systems, 1 (1995), No. 2, 177–202.
[6] M. P. do Carmo, *Riemannian Geometry*, Mathematics: Theory and Applications, Birkäuser, Boston, MA, 1992.
[7] I. M. Gelfand and S. V. Fomin, *Calculus of variations*, Revised English edition translated and edited by Richard A. Silverman, Prentice Hall, Englewood Cliffs, NJ, 1963.
[8] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, London, 1978.
[9] K. Hüper and F. Silva Leite, *On the geometry of rolling and interpolation curves on S^n, $SO(n)$ and Grassmann manifolds*, Journal of Dynamical and Control Systems, 13 (2007), No. 4, 467–502.
[10] J. Jost, *Riemannian Geometry and Geometric Analysis*, 6^{th} edition, Universitext, Springer, 2011.
[11] P. E. Jupp and J. T. Kent, *Fitting smooth paths to spherical data*, Applied Statistics, 36 (1987), No. 1, 34–46.
[12] D. Kruger, R. Stolkin, A. Blum and J. Briganti, *Optimal AUV path planning for extended missions in complex, fast-flowing estuarine environments*, 2007 IEEE International Conference on Robotics and Automation (ICRA), (2007), 4265–4270.
[13] J. W. Milnor, *Morse Theory*, Princeton University Press, Princeton, New Jersey, 1963.
[14] L. Noakes, G. Heinzinger and B. Paden, *Cubic splines on curved spaces*, IMA J. Math. Control Inform., 6 (1989), No. 4, 465–473.
[15] J. Nocedal, S.J. Wright, *Numerical Optimization*, second ed., in: Springer Series in Operations Research, Springer-Verlag, New York, 2006.
[16] I. Yudin and F. Silva Leite, *Algebraic integrability for minimum energy curves*, Kybernetika, 51 (2015), No. 2, 321–334.